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ABSTRACT

This paper considers smooth invariant manifolds of global solutions of retarded Functional Differential Equations in R^n. The persistence, under small perturbations, of such manifolds where the flow is given by an Ordinary Differential Equation in R^n is studied. The novelty of the present approach lies on the use of the dynamics of the flow on the manifolds, instead of their attractivity properties.

1. Introduction

If an Ordinary Differential Equation (ODE) defined by a Lipschitz vector field on R^n is considered as a retarded Functional Differential Equation (FDE) on the phase space $C = C([-r,0]; R^n)$, then the set of points in C which are initial data for global solutions of the equation is a C^1-manifold. It was noted by Kurzweil [1] that, under small perturbations of the retarded type, the set of initial data for global solutions remains a C^1-manifold where the flow is given by an ODE, provided the functions giving the ODE and the perturbations are bounded and differentiable, with the derivatives satisfying a "modulus of continuity" condition. The results on another paper of Kurzweil [2] apply to the case when the unperturbed equation is not an ODE, but the set of initial data for global solutions is a C^1-manifold with high normal rates, when compared with the rates of the flow in the manifold. These results were established by a contraction mapping argument applied to functions defined on the manifold of initial data for global solutions of the unperturbed equation, in such a way that the fixed point maps onto the set of initial data for global solutions of the perturbed equation.

In the present paper, the existence of C^1-manifolds where the flow of retarded equations is given by an ODE is established by determining directly the ODE giving the flow in such manifolds. This is accomplished using a nonlinear variation of constants formula. The novelty of this approach is on the use of the dynamics of the flow on the manifolds, instead of their attractivity properties.

The equations considered are retarded FDEs of the form

$$\dot{x}(t) = f(x_t) + g(x_t) \qquad (1g)$$

where f is a C^2 Lipschitz function from $C = C([-r,0]; R^n)$ into R^n , g is a C^1 Lipschitz function from C into R^n , and $x_t(\theta) = x(t + \theta)$, $\theta \in [-r,0]$. The solution of (1g) in C which satisfies $x_0 = \phi$ is denoted by $x(\phi,g)$ and its value at time t by $x(t;\phi,g)$. The subset of C consisting of initial data for global solutions (solutions defined for $t \in (-\infty,\infty)$) of (1g) is denoted by A_g . It is an invariant set under (1g).

Definition

Given a retarded FDE with phase space C and an integer $k \geqslant 1$, a manifold $B \subset C$ is said to be a manifold where the flow is given by a C^k ODE in R^n , if there exists $F \in C^k(R^n, R^n)$ such that the ODE $\dot{x}(t) = F(x(t))$ has unique solutions for each arbitrary initial condition $x(0) = a \in R^n$, all solutions of this ODE are also solutions of the given FDE having orbits in B , and vice versa, the solutions of the given FDE through initial data $\phi \in B$ at $t = 0$ are also solutions of the ODE for $t \geqslant -r$.

Remark

The preceding definition implies that such manifolds are invariant under the given FDE and are diffeomorphic to R^n (thus C^1 and n-dimensional). In particular, they are included in the set of initial data for global solutions of the FDE. Besides, the map $\pi:\phi \to \phi(0)$ from

B to R^n must then be one-to-one and onto, and $f(\phi) = F(\phi(0))$ for $\phi \in B$, $F(a) = f(\pi^{-1}a)$ for $a \in R^n$.

The maximal solution of an ODE $\dot{x}(t) = F(x(t))$ having unique solutions for all initial conditions $x(0) = a \in R^n$ is denoted by $t \to \xi(t;a,F)$. The following notation is also used:

$$H_F(t,a) = \frac{\partial \xi}{\partial a}(t;a,F) \quad .$$

Let X be any Banach spaces of differentiable Lipschitz functions from C into R^n. Given Banach spaces Z, W, let $\underline{BC}^1(Z,W)$ denote the Banach space of bounded C^1 functions from Z into W which have bounded first derivative, and taken with the uniform C^1-norm. For $\gamma > 0$, consider the Banach space

$$Y_\gamma = \{p: R^n \times [-r,0] \to R^n \text{ such that } p \text{ is continuous and } \|p\|_\gamma < \infty\}$$

with norm

$$\|p\|_\gamma = \sup_{\substack{a \in R^n \\ \theta \in [-r,0]}} |e^{-\gamma|a|}p(a,\theta)|$$

2. Perturbations of Ordinary Differential Equations

In order to illustrate the technique introduced in the present paper, we begin with the case where the unperturbed equation is given by an ODE, namely, we consider (1g) with $f(x_t) = F(x(t))$ with F being a C^k $(k \geq 2)$ Lipschitz function from R^n into R^n. Then, the set of points in C which consists of segments of solutions of the ODE $\dot{x}(t) = F[x(t)]$, is clearly a manifold where the flow of (1g) is given by a C^k $(k \geq 2)$ ODE.

Theorem 1

If F is a C^k $(k \geq 2)$ LIpschitz function from R^n into R^n and

$$B_0 = \{\xi(.;a,F) \in C: a \in R^n\} ,$$

then, for $g \in X$ sufficiently small, there exists a manifold B_g where the flow of

$$\dot{x}(t) = F[x(t)] + g(x_t) \tag{2}$$

(1g) is given by a C^k ODE. Furthermore, B_g and the associated ODE depend continously on g .

Proof.

The proof consists in constructing a function $K_g: R^n \to R^n$ so that B_g consists on segments of solutions of the ODE $\dot{x}(t) = [F + K_g](x(t))$. The solutions of this ODE must also satisfy the FDE (2) . It follows that K_g must satisfy

$$K_g[\xi(t;b,F + K_g)] = g\ [\xi(t + .;b,F + K_g)] \tag{3}$$

On the other hand, we expect to relate B_0 to B_g through a map $h_g: R^n \to C$ so that $\phi \in B_0 \to \phi + h_g(\phi(0)) \in B_g$. Thus, we want to have

$$\xi(\theta;a,F + K_g) = \xi(\theta;a,F) + h_g(a)(\theta) \quad , \quad \theta \in [-r,0] \tag{4}$$

and then, also

$$K_g(b) = g\ [\xi(.;b,F) + h_g(b)(.)] \tag{5}$$

To make this construction precise we need to establish the existence of the function h_g with the above properties. For the purpose, and motivated by the variation of constants formula for (2), we apply the Implicit Function Theorem to the function $G:\underline{B}C^1(R^n;C) \times X \to Y_\gamma$ given by

$$G(h,g)(a,\theta) = h(a)(\theta) - \int_0^\theta H_F[\theta - s,\xi(s;a,F) + h(a)(s)] \,.$$

$$.g\,[\xi(.;\xi(s;a,F) + h(a)(s),F) + h(\xi(s;a,F) + h(a)(s))(.)]\,ds \,. \qquad (6)$$

The function G maps into Y_γ because g is a global Lipschitz function, and it is continuously differentiable becase $f,g \in C^1$ and $\partial H_F(t,a)/\partial a$ is a solution of a nonhomogeneous linear variational equation for F with $F \in C^2$. On the other hand, $G(0,0) = 0$ and $\frac{\partial G}{\partial h}(0,0) = I$. By the Implicit Function Theorem, we get a C^1 function h^*, defined for g small, with $h^*(g) \in BC^1(R^n;C)$, $G(h^*(g),g) = 0$ and $h^*(0) = 0$.

We can now define $K_g: R^n \to R^n$ by equation (5) with $h_g = h^*(g)$ and consider, as suggested above, the set

$$B_g = \{\xi(.;a,F + K_g) \in C : \; a \in R^n\} \,.$$

Let $y(\theta) = \xi(\theta;a,F) + h^*(g)(a)(\theta)$, $\theta \in [-r,0]$. We have

$$y(\theta) = \xi(\theta;a,F) + \int_0^\theta H_F(\theta - s,y(s))K_g(y(s))ds$$

which is the nonlinear variation of constants formula for

$$\dot{x}(t) = F[x(t)] + K_g[x(t)] \,.$$

It follows that equation (4) holds, and

$$\dot{\xi}(t;a,F + K_g) = F[\xi(t;a,F + K_g)] + K_g[\xi(t;a,F + K_g)] =$$

$$= F[\xi(t;a,F + K_g)] + g\, [\xi(.;\xi(t;a,F + K_g),F + K_g)] =$$

$$= F[\xi(t;a,F + K_g)] + g\, [\xi(t + .;a,F + K_g)] \,.$$

Therefore $\xi(t;a,F + K_g)$ is a global solution of (1g) and the set B_g is invariant under (1g).

Clearly h^* depends continuously on g and, consequently, also K_g and B_g do. Q.E.D.

In general, the C^1-manifold B_g established in the preceding theorem is not the whole set A_g of initial data for global solutions (even if B_0 is), as the following example shows.

Example 1:

Consider the following scalar linear equation

$$\dot{x}(t) = -x(t) + \varepsilon\, x(t - r) \tag{7ε}$$

on C, with $\varepsilon > 0$ small. This is an equation of type (1g) with $f(\phi) = -\phi(0)$ and $g(\phi) = g_\varepsilon(\phi) = \varepsilon\, \phi\, (-r)$. One of the characteristic values of (7ε), say $\lambda_{-1}(\varepsilon)$, approaches the point -1 as $\varepsilon \to 0$, but there exist infinitely many characteristic values whose real parts decrease to $-\infty$ as $\varepsilon \to 0$. These last characteristic values give characteristic solutions which are global solutions and have initial conditions that do not approach the set of initial data for global solutions of (7_0),

$$A_0 = \{ce^{-\cdot} \in C : c \in R\} \,,$$

as $\varepsilon \to 0$.

Actually, $B_{g_\varepsilon} = \{ce^{\lambda_{-1}(\varepsilon)\cdot} \in C: c \in R\}$ and $B_{g_\varepsilon} \subsetneq A_{g_\varepsilon}$.

The set A_g of initial data for global solutions of (1g) is not always a C^1-manifold. However, under certain conditions, it is possible to establish the equality $A_g = B_g$, thus establishing that A_g is an n-dimensional C^1-manifold (actually diffeomorphic to R^n) where the flow is given by a C^k $(k \geq 1)$ ODE. A simple condition implying the equality $A_g = B_g$ is the following

Theorem 2

Assume (1_0) is a C^2 ODE. Then for $g \in BC^1(C; R^n)$ sufficiently small, the manifold B_g , established in Theorem 1, is equal to A_g .

Proof

By a simple application of the variation of constants formula for perturbations of the ODE $\dot{x} = (F + K_g)(x)$, established in theorem 1, it is possible to prove that all the orbits of (1g) lie within a certain fixed distance of B_g for $t \geq r$, and B_g is uniformly asymptotically stable under (1g). More precisely, there exists $\alpha > 0$ such that

$$\inf_{\psi \in B_g} |x_t(\phi,g) - \psi| \leq e^{-\alpha t} , \qquad t \geq r , \phi \in C .$$

Now, assume $\phi \in A_g$ and $\phi \notin B_g$. Let $\delta = \inf_{\psi \in B_g} |\phi - \psi| > 0$. Take $T > 0$

so large that $e^{-\alpha T} < \delta$. Since A_g is the set of initial data for global solutions of (1g) , there exists a point $\xi \in A_g$ such that $x_T(\xi,g) = \phi$. Thus, $\inf_{\psi \in B_g} |\phi - \psi| < \delta$, contradicting the assumption. This proves $A_g \subset B_g$ and consequently, by Theorem 1, $A_g = B_g$ Q.E.D.

As a corollary, one obtains the result of Kurzweil mentioned in the introduction to this paper.

Theorem 3 (Kurzweil):

If (1_0) is a C^2 ODE and $g \in BC^1(C; R^n)$ is sufficiently small, then the set of initial data for global solutions of (1g) is a C^1-manifold diffeomorphic to R^n .

2. General Case

The general case corresponds to equation (1_0) possibly involving retarded terms, but having an invariant manifold where the flow is given by a C^k $(k \geq 2)$ ODE in R^n $\dot{x}(t) = F[x(t)]$. We then need to refer to the linear operator $L:\underline{B}C^1(R^n,C) \to Y_\gamma$ defined by

$$(Lv)(a)(\theta) = \int_0^\theta H_F[\theta - s,\xi(s;a,F)]f'(\xi(s + .;a,F))v(\xi(s;a,F))(.)ds$$

where $a \in R^n$, $\theta \in [-r,0]$.

Theorem 4

If B_0 is a manifold where the flow of (1_0) is given by a C^k $(k \geq 2)$ ODE in R^n , $\dot{x}(t) = F[x(t)]$ with F, f and f' globally Lipshitzian, and if the spectrum of L does not contain the unity, then, for $g \in X$ sufficiently small, there exists a manifold B_g where the flow of (1g) is also given by a C^k ODE. Furthermore, B_g and the associated ODE depend continuously on g.

Proof.

Under the hypothesis in the theorem, $B_0 = \{\xi(.;a,F) \in C : a \in R^n\}$. On the other hand, the equation (1g) can be written as

$$\dot{x}(t) = F[x(t)] + [(f + g)(x_t) - F(x(t))] .$$

Now the proof follows the steps in the proof of Theorem 1. The *Implicit* Function Theorem is now applied to the function

$$G(h,g)(a,\theta) = h(a)(\theta) - \int_0^\theta H_F\,[\theta - s,\xi(s;a,F) + h(a)(s)] \,.$$

$$.\{(f + g)[\xi(.;\xi(s;a,F) + h(a)(s),F) + h(\xi(s;a,F) + h(a)(s))(.)] -$$

$$- f[\xi(.;\xi(s;a,F) + h(a)(s),F]\}ds ,$$

and, thus $\frac{\partial G}{\partial h}(0,0) = I - L$. Since the spectrum of L does not contain 1, the Implicit Function Theorem can be applied to get a function h^*, defined for g small, with properties as in the proof of Theorem 1.

The proof now proceeds in exactly the same way as for Theorem 1.

Examples of equations that satisfy the conditions of theorem 4 and are not ODEs can be readily given.

Example 2:

Let us consider the equation

$$\dot{x}(t) = x(t) - x(t - r) \tag{8}$$

with $r > 0$. The manifold $B_0 \subset C([-r,0];R^n)$ consisting of the constant functions in $[-r,0]$, is a manifold where the flow of (8) is given by the ODE $\dot{x} = F(x) = 0$. Then $H_f(t,a) = 1$ for all $t,a \in R$. Thus

$$(Lv)(a)(\theta) = \theta[v(a)(o) - v(a)(-r)] .$$

The operator L admits 1 as one of its eigenvalues if and only if there exists $u \neq 0$ such that

$$u(a)(\theta) = \theta[u(a)(o) - u(a)(-r)] \quad , \quad \theta \in [-r,0] \ , \ a \in R .$$

If $r = 1$, the above equation admits solutions

$$u(a)(\theta) = -\theta c \quad , \quad \theta \in [-r,o] ,$$

with c being an arbitrary constant, thus implying that 1 is an eigenvalue of L. However, if $r \neq 1$, the only function satisfying the equation is $u = 0$, implying that $\ker(I - L) = \{0\}$. On the other hand, given $w \in \underline{B}C^1(R^n,C)$, with

$$v(a)(\theta) = w(a)(\theta) - \theta w(a)(o) + \theta[w(a)(-r) + rw(a)(o)]/(1 + r)$$

we have $(I - L)v = w$, implying that $(I - L)$ is a surjection. Consequently, when $r \neq 1$, 1 does not belong to the spectrum of L and, therefore, the hypothesis of theorem 4 is satisfied.

Other examples can be constructed with the aid of the following lemma.

Lemma

If $F:R^n \to R^n$ is C^2, $f:C \to R^n$ is C^1, and both are globally Lipshitzian with

$$\alpha = \sup\{|F^1(a)|:a \in R^n\} \ , \ \beta = \sup\{\|f'(\phi)\|:\phi \in C\} \ ,$$

then $\|L\| < \beta(e^{\alpha r} - 1)/\alpha$.

Proof

The principal matrix solution $H_F(t,a)$ of $\dot{x} = F(x)$, which appears in the definition of L, satisfies the linear variational equation for this ODE and has for initial condition at $t = 0$ the identity. Consequently, $|H_F(t,a)| \leq e^{\alpha|t|}$ and, thus,

$$\|L\| \leq \max_{\theta\varepsilon[-r,o]} [\int_\theta^o e^{\alpha(s-\theta)}\beta \, ds] \leq \beta[e^{\alpha r} - 1]/\alpha \ .$$

Following this lemma, equations satisfying the hypothesis of theorem 4 can be found by determining equations $\dot{x} = f(x_t)$ with a manifold where the flow is given by a C^k $(k \geq 2)$ ODE in R^n, $\dot{x} = F(x)$, with α and β defined as in the Lemma being finite and satisfying $\beta[e^{\alpha r} - 1]/\alpha < 1$.

Example 3:

Let us consider the scalar FDE

$$x(t) = (1 - e^{-r})^{-1} \int_{-r}^0 x(t + \theta)d\theta \ .$$

For this equation, $B_o = \{\phi \in C:\phi(\theta) = e^\theta\}$ is an invariant manifold where the flow is given by the ODE $\dot{x} = x$. For this case, $\alpha = 1$ and $\beta = r/(1 - e^{-r})$. Thus, the hypothesis of theorem 4 is satisfied provided $r(e^r - 1)/(1 - e^{-r}) < 1$, since by the previous lemma this implies $\|L\| < 1$. This inequality holds provided $re^r < 1$, showing that theorem 4 can be applied to the given FDE whenever r belongs to the interval $[0,r_0)$ where $r_0 e^{r_0} = 1$.

Remarks:

1. The preceding results indicate that, for FDEs which satisfy the conditions of Theorem 4, the existence of an invariant C^1-manifold where the flow is given by an ODE in R^n , implies strong hyperbolicity of that manifold. The rates of the flow in the manifold have to be slower than the rates of the normal flow. It would be of interest to establish this fact by a direct argument.

2. The ideas used in the preceding results can be extended to the case where the FDE is defined in a compact manifold $M \subset R^n$ instead of the whole space R^n . One then considers invariant manifolds where the flow is given by an ODE in M . The perturbed manifold B_g established as in Theorem 1 is then necessarily equal to the set of initial data for global solutions, A_g . In fact, the boundedness of the perturbation g follows from the compactness of the manifold M , and the estimates of theorem 2 can always be carried out.

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