Russell's Theory of Logical Types
and the Atomistic Hierarchy of Sentences

Russell's philosophical views underwent a number of changes throughout his life, and it is not always well appreciated that views he held at one time came later to be rejected; nor, similarly, that views he rejected at one time came later to be accepted. It is not well known, for example, that the theory of logical types Russell described in his later or post-PM philosophy is not the same as the theory originally described in PM in 1910–13; nor that some of the more important applications that Russell made of the theory at the earlier time cannot be validated or even significantly made in the framework of his later theory. What is somewhat surprising, however, is that Russell himself seems not to have realized that he was describing a new theory of logical types in his later philosophy, and that as a result of the change some of his earlier logical constructions, including especially his construction of the different kinds of numbers, were no longer available to him.

In the original framework, for example, propositional functions are independently real properties and relations that can themselves have properties and relations of a higher order/type, and all talk of classes, and thereby ultimately of numbers, can be reduced to extensional talk of properties and relations as "single entities," or what Russell in POM had called "logical subjects." The Platonic reality of classes and numbers was replaced in this way by a more fundamental Platonic reality of propositional functions as properties and relations. In Russell's later philosophy, however, "a propositional function is nothing but an expression. It does not, by itself, represent anything. But it can form part of a sentence which does say something, true or false" ([MPD], p. 69). Surprisingly, Russell even insists that this was what he meant by a propositional function in PM. "Whitehead and I thought of a propositional function as an expression containing an undetermined variable and becoming an ordinary sentence as soon as a value is assigned to the variable: 'x is human,' for example, becomes an ordinary sentence as soon as we substitute a proper name for 'x.' In this view...the propositional function is a method of making a bundle of such sentences" ([MPD], p. 124). Russell does realize that some sort of change has come about, however, for he admits that

"I no longer think that the laws of logic are laws of things; on the contrary, I now regard them as purely linguistic" (ibid., p. 102).

How an uncountable reality of classes and numbers can be reduced to a countable reality of "linguistic conveniences," Russell never explained; but it is clear that he thought that such a reduction was already accomplished in PM, i.e., that PM could sustain a nominalistic construal of propositional functions. Now whether or not PM can sustain such an interpretation is not our concern here (though, given the axioms of reducibility and infinity, we think it cannot); for what Russell failed to see was that the theory of types he described and was committed to in his later philosophy was but a fragment of the theory described in PM, and that in fact the analysis of classes and numbers given in PM cannot be given in this fragment. This new theory of types was dictated by what Russell later called "the technical form of the principle of atomicity," namely, the thesis that "all propositions are either atomic, or molecular, or generalizations of molecular propositions; or at least, that a language of which this is true, and into which any statement is translatable, can be constructed" (IMT, pp. 250f.). The "logical language" in question here is what Russell called "the atomistic hierarchy of sentences" (ibid., p. 187), and it amounts, as we shall see, to but a fragment of second-order predicate logic. Russell does also allow for a hierarchy of "languages" constructed on the basis of the atomistic hierarchy, but this additional hierarchy, as we shall also see, turns out to be essentially a nominalistic construal of ramified second-order logic. That is, ramified second-order logic is all that is left in Russell's later philosophy of his original theory of types. This system is not only much weaker than Russell's original logic, but, even worse, on grammatical grounds alone it cannot sustain Russell's analysis of classes and numbers. For, despite Russell's misleading notation otherwise (in his 1925 introduction to PM), propositional functions (construed as expressions) cannot occur as (higher-order) abstract singular terms in ramified second-order logic, and yet it is precisely their occurrence as such that is essential to Russell's analysis of classes and numbers.

Now it is not whether PM can sustain a nominalistic interpretation that is our concern in this essay, as we have said, but rather how it is that Russell came to be committed in his later philosophy to the atomistic hierarchy and the nominalistic interpretation of propositional functions as expressions generated in a ramified second-order hierarchy of languages based on the atomistic hierarchy. We shall pursue this question by beginning with a discussion of the difference between Russell's 1908 theory of types and that presented in PM in 1910. This will be followed by a brief summary of the ontology that Russell took to be implicit in PM, and that he described in various publications between 1910 and 1913. The central notion in this initial discussion is what Russell in his early philosophy called the notion of a logical subject, or equivalently that of a "term" or "single entity." (In PM, this notion was redescribed as the systematically ambiguous notion of an

"object.") As explained in Cocchiarella (1980), this notion provides the key to the various problems that led Russell in his early philosophy to the development of his different theories of types, including that presented in PM. This remains true, moreover, even when we turn to Russell's later philosophy, i.e., to his post-PM views, except that then it is described as the notion of what can and cannot be named in a logically perfect language. The ontology of these later views is what Russell called logical atomism, and it is this ontology that determines what Russell described as the atomistic hierarchy of sentences. In other words, it is the notion of what can and cannot be named in the atomistic hierarchy that explains how Russell, however unwittingly, came to replace his earlier theory of logical types by the theory underlying the atomistic hierarchy of sentences as the basis of a logically perfect language.

1. The 1910 versus the 1908 Theory of Logical Types

An important fact that is commonly overlooked in most of the literature on the theory of logical types is that the theory Russell described in PM in 1910 is not the same as the theory he described in "Mathematical Logic as Based on the Theory of Types" (1908) — unless, that is, one assumes that both propositions (as objective truths and falsehoods) and propositional functions are for Russell "single entities" in both theories. Russell did not assume this, however, and in fact while propositions are single entities in "Mathematical Logic," propositional functions, or so Russell then thought, were nonentities. Two years later, in PM, propositional functions are reckoned as single (nonlinguistic) entities, and propositions are reconstrued by Russell as not being single entities after all. The difference, apparently, was the result of Russell's shifting from a propositional theory of belief or judgment to his famous multiple-relations theory (which he later rejected in 1913 as a result of criticisms by Wittgenstein). Thus, according to Russell in PM, "what we call 'a proposition' (in the sense in which this is distinguished from the phrase expressing it) is not a single entity at all," and in fact "the phrase which expresses a proposition is what we call an 'incomplete' symbol" (p. 44). We should note, incidentally, that being a single entity is what Russell also means by being capable of being a logical subject.

To see what this difference between the two theories comes to, let us turn to Alonzo Church's formal characterization of Russell's ramified types, hereafter called r-types, and orders.[1]

(1)   There is an r-type i to which all and only individuals belong, and whose order is stipulated to be 0.

(2)   If $m \in \omega$, $n \in \omega - \{0\}$, and $\beta_1, \ldots, \beta_m$ are given r-types, then there is an r-type $(\beta_1, \ldots, \beta_m)/n$ to which belong all and only m-ary propositional functions of level n and with arguments of r-types $\beta_1, \ldots, \beta_m$, respec-

tively; and the order of such a function is $N + n$, where N is the greatest of the orders corresponding to the types $\beta_1, \ldots, \beta_m$ (and $N = 0$ if $m = 0$).

The notion of the *level* of a propositional function ϕ of r-type $(\beta_1, \ldots \beta_m)/n$ is needed here, it should be noted, as a counterpart to Russell's nonsyntactical use (in 1910) of the notion of an apparent (or "bound") variable. Thus if N is the greatest of the orders corresponding to $\beta_1, \ldots, \beta_m$, and k is the greatest of the orders of the apparent variables occurring in ϕ (in Russell's nonsyntactical sense), then $n = 1$ if $k \leq N$, and $n = k + 1$ if $N < k$. Since ϕ is said to be *predicative*, according to Russell, when "it is of the lowest order compatible with its having the arguments it has" (PM, p. 53), then in terms of the notion of level, it follows that ϕ is *predicative* if, and only if, $n = 1$.[2]

Now the preceding definition recognizes both propositions and propositional functions as single entities. Propositions of order n, for example, are represented here as 0-ary propositional functions of level n, i.e., as propositional functions, of r-type $(\ )/n$, where "$(\ )$" represents the null sequence. This of course is merely a convenience of terminology, since propositions are really not propositional functions in the intended sense. That both propositions and propositional functions are "single entities" is acknowledged here in the fact that both can occur as arguments of propositional functions, or as "logical subjects" of the resulting propositions. For example, an individual can stand to a propositional function of r-type $(\beta_1, \ldots, \beta_m)/n$ in a predicative relation of r-type $(i, (\beta_1, \ldots, \beta_m)/n)/1$; and where belief is a predicative relation between an individual and a proposition of order n, belief will be a propositional function of r-type $(i, (\ )/n)/1$.

Church is not unaware that Russell rejected propositions as single entities in 1910, and that he did so on the basis of his multiple-relations theory of belief. Church claims, however, that the "fragmenting of propositions" that is involved in the multiple-relations theory also requires the "fragmenting of propositional functions" (1976, p. 748), and therefore if propositional phrases are to be analyzed as incomplete symbols then phrases for propositional functions must also be so analyzable. The result would mean that the only category or type that was really fundamental with respect to quantification was that of the individuals, since only individuals would then remain as real single entities. The result, in other words, would mean that the theory of logical types was reducible to first-order logic. Such a reduction was certainly not intended by Russell, and in any case, or so Church argues, "it is probable that the contextual definitions [i.e., analyses of phrases for propositions and propositional functions as incomplete symbols] would not stand scrutiny" (ibid.).

Actually, Church is not correct in thinking that the fragmenting of propositional functions that is involved in Russell's multiple-relations theory means that propositional functions are ultimately to be eliminated as single entities. For although the propositional functions that occur in a belief or judgment on this theory

are indeed "fragmented" in the sense of analysis, nevertheless each propositional function, as well as the "fragments" of that function that result upon analysis, retains its status as a single entity in the belief or judgment complex. Consider, for example, the judgment that all men are mortal as made by some person A. The truth of this judgment, according to Russell, is a "second-order truth" (PM, p. 45), and that this is so can be seen in the following analysis:

   Judges(A, $(x)[\hat{\phi}!x \supset \hat{\psi}!x]$, $\hat{x}$ is a man, $\hat{x}$ is mortal).

We assume in this analysis that $\hat{x}$ *is a man* and $\hat{x}$ *is mortal* are predicative propositional functions of r-type $(i)/1$ and that $(x)[\hat{\phi}!x \supset \hat{\psi}!x]$ is a propositional function of r-type $((i)/1, (i)/1)/1$, and therefore of order 2. The judgment is said to have second-order truth because 2 is the maximum of the orders of the propositional functions occurring in its analysis. Note that the propositional phrase "all men are mortal" does not occur in this analysis as a singular term even though it may appear to so occur (when appended to "that") in the English sentence, "A judges that all men are mortal." This is what Russell meant by saying that propositions are no longer to be reckoned as single entities. The phrases for the propositional functions that result from the analysis of "A judges that all men are mortal," on the other hand, all occur as singular terms in the final analysis, and it is for this reason that the propositional functions they represent must be reckoned as single entities. Indeed, without including propositional functions among the single entities combined in a judgment or belief complex, there would simply be no multiple-relations theory of belief at all.

   This is not to say that the multiple-relations theory of belief is a viable theory after all (or at least not without serious reconstruction). Our point rather is that as far as Russell was concerned in 1910–13, propositional functions are single entities (of different r-types and orders) and can be quantified over as such, but that the same cannot be said of propositions. That is, propositions (in the sense of objective truths and falsehoods) are not single entities according to the Russell of 1910, and therefore they cannot be quantified over as such. This means modifying Church's characterization of r-types by excluding all r-types of the form ( )/n, where $n \in \omega - \{0\}$; or, in other words, by requiring in clause (2) of the definition of r-type that $m \in \omega - \{0\}$ as well.

   Note that rejecting propositions while retaining propositional functions in no way affects Russell's logical reconstruction of mathematics. For it is propositional functions, and not propositions, that are essential to that reconstruction. This is so because a statement about a class, i.e., a statement in which an expression for a class occurs as a singular term, is to be analyzed, according to Russell, as a statement about the extension of a propositional function; and the latter, assuming that propositional functions can be single logical subjects, is in turn to be analyzed as a statement about some (or preferably *any*) propositional function materially equivalent to the propositional function in question. Thus, reading "$\hat{z}(\psi z)$" as "the

class defined by ψ," Russell gives the following contextual analysis for statements in which a class appears as a single logical subject (PM, p. 188):

$$f\{\hat{z}(\psi z)\} \;=\; df \,(\exists\phi)((x)[\phi!x \;\equiv\; \psi x] \;\&\; f\{\phi!\hat{z}\}).$$

In this analysis, needless to say, it is essential that a propositional function can occur as a single logical subject of the analysans. In Russell's 1908 theory, on the other hand, it is propositions and not propositional functions that are reckoned as single entities; and in that regard Russell's logical reconstruction of mathematics is very much in question, since it is not r-types of the form $(\;)/n$ that are then to be excluded but rather all r-types for propositional functions that are not of this form.

Russell's pre-1910 rejection of propositional functions as single entities goes back as far as 1903, incidentally, when, as a result of his paradox, Russell was led to claim that "the ϕ in ϕx is not a separate and distinguishable entity: it lives in the propositions of the form ϕx and cannot survive analysis" (POM, p. 88). Thus, since being a separable entity is the same for Russell as having the capacity of being a logical subject, there can be no propositions of the form $\psi(\phi)$, and therefore none of the form $\phi(\phi)$ or $\sim\phi(\phi)$ as well, on this earlier view of Russell's. In other words, it was by "the recognition that the functional part of a propositional function is not an independent entity" (ibid.) that Russell sought to avoid the contradiction that would otherwise result when his paradox was applied to propositional functions as single entities. (See Cocchiarella [1980], section 4, for a fuller discussion of this point.)

Despite his rejection in POM of propositional functions as single entities, Russell still found it "impossible to exclude variable propositional functions altogether" (POM, p. 104); that is, he still admitted quantification with respect to such variables. This was because on Russell's view "wherever a variable class or variable relation [in extension] occurs, we have admitted a variable propositional function which is thus essential to assertions about every class or about every relation" (ibid.). This view was later developed by Russell into his famous "no classes" theory, first in the form of the substitutional theory of 1906, then in the form of the 1908 theory of types, and finally in PM. (See Cocchiarella [1980], sections 6–8 for a fuller discussion of this development.) It was only in the 1910 theory of logical types, however, that Russell was finally able to give a coherent account of his "no classes" theory, for it is only in the 1910 theory that quantification over propositional functions as independently real entities is finally recognized.

2. Propositional Functions as Properties and Relations in Russell's 1910–1913 PM-Ontology

"Pure mathematics," Russell wrote in 1911, "is the sum of everything that we can know, whether directly or by demonstration, about certain universals"

([1911], p. 293). The certain universals in question here are the independently real propositional functions that occur as "single entities" in the analyses Russell gave in PM of our talk of classes and numbers. "Logic and mathematics force us, then," according to this Russell of 1911, "to admit a kind of realism in the scholastic sense, that is to say, to admit that there is a world of universals and of truths which do not bear directly on such and such a particular existence. This world of universals must *subsist*, although it cannot *exist* in the same sense as that in which particular data exist" (ibid.).

Propositional functions, accordingly, are universals for Russell in his 1910-13 PM ontology, and as such they may also be called properties and relations (in intension). This was already suggested by Russell in "The Regressive Method of Discovering the Premises of Mathematics" (1907; reprinted in 1973a, p. 281), where two of his "principles" for mathematical logic are as follows:

> Any propositional function of x is equivalent to one assigning a property to x.
> Any propositional function of x and y is equivalent to one asserting a relation between x and y.

But these two "principles" were said by Russell in "The Regressive Method" to be "less evident" than the others he listed there for mathematical logic; and, as already indicated, Russell attempted to do without them completely in "Mathematical Logic" (1908). Nevertheless, regardless of his earlier hesitancy, and sometimes outright rejection, it is clear that Russell did assume these "principles" in his 1910-13 PM ontology.

Another assumption that Russell also made in his 1910-13 ontology, albeit only implicitly, was that some properties and relations are simple while others are complex. This assumption goes back as far as POM where it is described as the distinction between properties and relations that are or are not logically analyzable in terms of other properties and relations. That is, if properties and relations "have been analyzed as far as possible, they must be simple terms, incapable of expressing anything except themselves" (POM, p. 446); and if they are otherwise analyzable, then they must be complex. Of course, for Russell, throughout the period in which he was a logical realist, logical analysis is the same as ontological analysis; i.e., "where the mind can distinguish elements [in a logical analysis], there must *be* different elements to distinguish" (ibid.).

This assumption is not itself a consequence of the comprehension principle for properties and relations, incidentally; for despite the validity of the latter in PM, where propositional functions are properties and relations, nothing follows about properties and relations being themselves complex if they are specified in an instance of that principle by a complex expression for a propositional function. In other words, the complex/simple distinction is not essential to the validation of the comprehension principle (as is all too frequently assumed in the literature).

Nevertheless, it is a sufficient condition if we also assume that the language of PM is "a logically perfect language" in the sense that complex expressions for propositional functions represent (onto)logical analyses of those propositional functions as independently real universals. This in fact was what Russell assumed in 1910-13, at least implicitly, and, as we shall see, it is not unrelated to the nominalistic validation of the comprehension principle in his later philosophy where the complexity of a propositional function is none other than its syntactical complexity as an expression. Of course properties and relations will then be distinguished from propositional functions, and in fact Russell will then in general speak of them only as simple.

The comprehension principle, incidentally, really has two forms that are valid in PM, but only one that is valid in the theory of logical types of Russell's later philosophy. These are

$$(\exists f)(x_1)\ldots(x_m)[f(x_1,\ldots,x_m) \equiv \phi],$$

and

$$(\exists f)\, f = \phi(\hat{x}_1,\ldots,\hat{x}_1),$$

where f is a variable of r-type $(\beta_1,\ldots,\beta_m)/n$ that does not occur free in $\phi, x_1, \ldots,$ x_m are variables of r-types $\beta_1, \ldots, \beta_m$, respectively, and the bound variables in ϕ are all of an order less than the order of f. Here the second form implies the first, but not conversely. The second form, given Russell's analysis of identity, is an abbreviation of

$$(\exists f)(\psi)[\psi!\{f\} \equiv \psi!\{\phi(\hat{x}_1,\ldots,\hat{x}_m)\}],$$

which requires that propositional functions be "single logical subjects"; and this form is not even meaningful in ramified second-order logic where all propositional functions are of r-types of the form $(i,\ldots,i)/n$, for arbitrary "level" n; i.e., where propositional functions (of arbitrary "level") have only individuals as arguments. In other words, strictly speaking, only the first form remains "significant" in Russell's later philosophy.

Nothing comparable can be said of Russell's analysis of classes, on the other hand. That is, there is no form of that analysis that remains significant in Russell's later philosophy. This is because expressions for classes are to occur as singular terms, and Russell's analysis, as described in section 1, requires that expressions for propositional functions must then also occur as singular terms; and yet it is precisely that type of occurrence that is not "significant" in ramified second-order logic.

3. Russell's 1910-1913 Commitment to Abstract Facts

In 1910-11, Russell described his ontology as consisting simply of "an ultimate dualism" of universals and particulars. That is, "the disjunction, 'universal-

particular' includes all objects. We might also call it the disjunction 'abstract-concrete' " (1910–11, p. 214). The particulars of this dualism are "particular exis-tents" or "entities which can only be subjects or terms of relations, and cannot be predicates or relations" (1911–12, p. 109). (A "predicate" for Russell at this time was always a property or quality, or what he also called a concept.) A universal, on the other hand, is "anything that is a predicate or a relation" (ibid.), but which may also be a "subject" or "term" of a relation.

Particulars, incidentally, are the individuals of PM; i.e., they are the objects of r-type i. This terminology differs from that Russell used earlier in POM where the word "individual" was taken as synonymous with "term" and "entity," or hav-ing the capacity of being a logical subject (cf. POM, p. 43). This means that universals were also construed as individuals in POM, since as logical subjects they were also "terms." In PM, on the other hand, particulars and only particulars are individuals, i.e., are of r-type i, which is not to say that universals have lost their individuality or capacity to occur as "terms" of other universals. The matter is really terminological, in other words, for the word used in PM to cover the individuality of both particulars and universals is "object." That is, both particu-lars and universals are "objects" in PM, though of course they are objects of "es-sentially different types" (PM, p. 24). As propositional functions, moreover, universals are also of different types among themselves, since some may be argu-ments or "terms" of others. "The division of objects into types," according to Rus-sell, "is necessitated by the vicious circle fallacies which otherwise arise" (ibid., p. 161).

Among particulars Russell included not just "existents" but "all complexes of which one or more constituents are existents, such as this-before-that, this-above-that, the-yellowness-of-this," etc. ([1910–11], p. 213). In 1912, Russell some-times called these complex particulars events, and other times facts. For example, my seeing the sun and my desiring food are "events" that happen in my mind (PP, p. 49); and when "I am acquainted with my acquaintance with the sense-datum representing the sun,...the whole *fact* with which I am acquainted is, 'Self-acquainted-with-sense-datum' " (ibid., pp. 50–51). Note that one of the ways that we can have knowledge of such a complex particular is "by means of *acquain-tance* with the complex fact itself, which may (in a large sense) be called percep-tion, though it is by no means confined to objects of the senses" (ibid., p. 136).

The importance of events or facts as complex particulars in Russell's 1910–13 ontology is that they provide the basis of his new theory of truth; that is, the theory in which truth and falsehood are no longer properties of propositions as indepen-dently real single entities, but are rather "properties of beliefs and statements" (PP, p. 121). (Note that a *statement* for Russell is always the overt expression of a judgment or belief.) For example, in the case of a simple statement, such as that a has the relation R to b, "we may define *truth*...as consisting in the fact that there is a complex *corresponding* to the discursive thought which is the judgment.

That is, when we judge '*a* has the relation *R* to *b*,' our judgment is said to be *true* when there is a complex '*a*-in-the-relation-*R*-to-*b*,' and is said to be *false* when this is not the case" (PM, p. 43).

We should note that truth and falsehood are no more univocal in Russell's new theory than they were in his earlier 1908 theory when they were properties of propositions of different orders. In particular, although beliefs or statements are themselves particular complex occurrences (and therefore are particulars), the kind of truth or falsehood each will have will depend on the highest order of the propositional functions occurring as "terms" in the belief or statement complex. For example, a statement of "this is red" is said to have *elementary truth or false-hood*, while a statement of "all men are mortal," as already explained in section 1, will have *second-order truth*. Similarly, a statement of "Napoleon had all the (predicative) properties of a great general" will have *third-order truth or false-hood* (cf. Cocchiarella [1980], p. 104); and of course, there can be statements or beliefs with *fourth-order truth or falsehood*, and so on. Instead of a hierarchy of propositions as abstract entities that may be true or false, in other words, Russell's 1910–13 framework has only a hierarchy of truth and falsehood as proper-ties of particular occurrences of beliefs and statements.

The hierarchy of truth and falsehood as properties of beliefs and statements as complex particulars fits in well with Russell's 1910–11 "ultimate dualism" of universals and particulars; i.e., with his claim that "*the disjunction 'universal-particular' includes all objects*" (1910–11, p. 214, emphasis added). By 1912, however, Russell came to realize that not all of the facts he needed in his "realism in the scholastic sense" could be construed as events or complex particulars. That is, with respect to Russell's "abstract-particular" disjunction, which he had origi-nally identified in 1910 with the "universal-particular" disjunction, there are *ab-stract facts* as well as concrete facts (events). These are "facts about universals," and, according to Russell, "they may be known by acquaintance to many different people" (PP, p. 137). For example, "the statement, 'two and two are four' deals exclusively with universals" (ibid., p. 105), and therefore the complex that makes it true is not an event or complex particular but an abstract fact. The statement itself, to be sure, as a statement made by someone at some particular time, is an event or complex particular, and as a statement about classes of classes of in-dividuals (or rather about predicative propositional functions of predicative propositional functions of individuals) the truth it has is a property of third-order (the order of the identity relation in this case). But still, since the fact that makes this statement true "deals exclusively with universals," i.e., with objects that "*sub-sist* or *have being*, where 'being' is opposed to 'existence' as being timeless" (ibid., p. 100), then the fact itself must subsist and belong to "the world of being" (ibid.).

Russell's commitment to abstract facts, it should be emphasized, cannot be brushed aside here as something that can be avoided, as though his original "ulti-mate dualism" of universals and particulars might suffice after all. Consider, for

example, comparing Russell's ontology of universals and particulars, and now abstract facts as well, with an event ontology that is combined with the ontological commitments of one or another set theory. In the latter, there are no facts other than events (if the identification of concrete facts with events is to be retained at all), and, in particular, there are no set-theoretical facts regarding pure sets (i.e., sets whose transitive closure contains no elements other than the empty set). Nor are any set-theoretical facts really needed, moreover, to account for the truth of statements of membership in a set. For a set, at least on the iterative concept, has its being in its members, and in that regard a set's existence (or "being," as Russell would say) is all that is needed to account for the truth or falsehood of statements ascribing membership in that set. That is, the being of a set consists in its having just the members that it has, and therefore no fact over and above the being of the set itself is needed to account for membership in the set. A property or relation (in intension), on the other hand, does not have its being in its instances, and for that reason its being cannot alone account for the truth of statements ascribing that property or relation to its instances. The usual gambit logical realists make here to account for such truth is to posit propositions as objective truths and falsehoods in themselves, i.e., as independently real single entities. Russell, however, had deliberately removed that option, and in consequence, he was forced to fall back on abstract facts as an additional category of his ontology beyond the events or concrete facts that make up the world of existence. In his later philosophy, when "all the propositions of mathematics and logic are assertions as to the correct use of a certain small number of words" ("Is Mathematics Purely Linguistic?" in *Essays*, p. 306), these abstract facts are replaced by truths that are "purely linguistic." What Russell failed to see, however, was that such a replacement did not result in an equipollent system of logic.

4. Logical Atomism and the Doctrine of Logical Types

The status of facts as "objects" or complex single entities that can be named was important to Russell in the 1910–13 framework, and in general he went out of his way to use only perfect nominalizations of sentences (where the verb has been completely deactivated) to refer to such facts. That is why he used examples like "my seeing the sun," "my desire for food," "*a* in the relation R to b," and "*a* having the quality q," as opposed to the imperfect nominalizations "that I see the sun," "that I desire food," etc. Indeed, he sometimes even used hyphens to emphasize the perfect nominalization in question, such as in "this-before-that," "this-above-that," and "the-yellowness-of-this." Such nominalizations were necessary, according to Russell, insofar as we can be directly acquainted with the facts in question. For acquaintance is a binary relation and is to be represented by a two-place predicate expression taking only singular terms as argument expressions.

A problem does arise here regarding the logical status of facts, however. For example, insofar as the concrete fact denoted by "this-before-that" is a particular,

it must be of the *r*-type *i* of individuals. But as a complex, a fact has a logical structure, and according to Russell the complexity of that structure must somehow be represented in its logical type, which for us is its *r*-type. That is, as a complex a fact must have an *r*-type other than *i*; and therefore, assuming that no object is of more than one *r*-type, facts really cannot be particulars after all. This argument, or at least one with the same conclusion, was apparently forced on Russell sometime in 1913 by Ludwig Wittgenstein. Whatever his initial reluctance, and it must have been great since it meant giving up completing the manuscript of his 1913 *Theory of Knowledge*, Russell came to accept the conclusion by the end of 1913.

Now it is significant that in accepting this conclusion Russell went on to claim that "only particulars can be named" (PLA, p. 267) and therefore that facts cannot be named at all. For example, in January 1914, Russell wrote that although "an observed fact...does not differ greatly from a simple sense-datum as regards its function in giving knowledge," nevertheless "its *logical* structure is very different...from that of sense: *sense* gives acquaintance with particulars, and is thus a two-term relation in which the object can be *named* but not asserted,...whereas the observation of a complex fact, which may be suitably called perception, is not a two-termed relation, but involves the propositional form on the object-side, and gives knowledge of a truth, not mere acquaintance with a particular" ([1914c], p. 147). In other words, it was no longer even meaningful, no less true, for Russell that "the complex '*a*-in-the-relation-*R*-to-*b*' may be capable of being perceived...as one object" (PM, p. 43). That is, this perception was no longer "a relation of two terms, namely, '*a*-in-the-relation-*R*-to-*b*,' and the percipient" (ibid.). And the reason, Russell claims, is that the *logical structure* of a fact precludes its being the sort of entity that can be named, i.e., the sort of entity that can stand as a "term" in a relation. "You cannot name a fact.... You can never put the sort of thing that makes a proposition to be true or false in the position of a logical subject" (PLA, p. 188).

Before taking up this rather extraordinary claim, let us note that as so far defined no *r*-type is the *r*-type of a fact, and therefore in a trivial sense no fact can be a "logical subject" according to the theory of *r*-types. The reason why this is so is that every *r*-type other than *i* is the *r*-type of a propositional function, and facts are not propositional functions. Resurrecting propositional *r*-types of the form ( )/n for facts will not do, moreover, since these fail to indicate both the number and the *r*-types of the constituents of a fact.

We can rectify this situation, however, if we assume along with the Russell of the 1910–13 ontology that every fact consists of some one relation actually relating the remaining constituents of that fact. (This assumption of Russell's goes as far back as the *Principles*; cf. POM, p. 52.) For example, the relation that Russell calls "formal implication" (and represented by $(x)[\hat{\phi}!x \supset \hat{\psi}!x]$ in section 1) is the relating relation of a general fact (cf. PM, p. 138), and "the asymmetrical

relation of predication" is the relating relation of a fact corresponding to a true subject-predicate sentence (cf. 1911–12). Following Russell's later usage (adopted from Wittgenstein's *Tractatus*), we shall call the relating relation of a fact the *component* relation of that fact, whereas the "terms" or "logical subjects" of that relation will be called simply the *constituents* of the fact. We can now revise the definition of r-type given in section 1 by supplementing that definition with the following clause:

> (3) If $m, n \in m - \{0\}$, $m \geq 2$, and $\beta_1, \ldots, \beta_m$ are given r-types, then there is an r-type $[(\beta_1, \ldots, \beta_m)/n]$ to which belong all and only facts whose component relation is of r-type $(\beta_1, \ldots, \beta_m)/n$ and whose constituents are of r-type $\beta_1, \ldots, \beta_m$ respectively; and the order of such a fact is the order of its component relation.

Of course it now follows that no concrete fact is a particular, since particulars are all of r-type i and no fact is of r-type i. But that—as far as the theory of logical types as the theory of r-types is concerned—has nothing to do with Russell's new or post-PM claim that you cannot name a fact, or that a fact cannot be a logical subject.

There is no reason, for example, at least as far as the theory of r-types is concerned, why there cannot be different types of relations of acquaintance, just as there are on Russell's multiple-relations theory different types of relations of belief. Only one of these acquaintance relations will in fact be a relation between individuals; others will be relations between individuals and facts (of a given r-type) or between individuals and universals (of a given r-type). (Russell also called acquaintance with universals *conceiving*; cf. 1910–11, p. 212.) Also, as far as the theory of r-types is concerned, there is no reason why all and only individuals should be particulars, i.e., why particulars should constitute a distinct logical category. After all, if concrete facts can be logical subjects, then why shouldn't they be called complex particulars, just as Russell in fact did so call them in his 1910–13 ontology?

It is not just facts that Russell now says cannot be named, it should be noted, but anything that suggests "the form of a proposition" (PLA, p. 205). A property or quality, for example, cannot be named by a nominalized predicate, since "a predicate," according to Russell's new or post-PM view, "can never occur except as a predicate. When it seems to occur as a subject [i.e., as a nominalized predicate], the phrase wants amplifying and explaining, unless, of course, you are talking about the word itself" (ibid.). Similarly, "a relation can never occur except as a relation, never as a subject" (ibid., p. 206). "All propositions in which an attribute or a relation *seems* to be the subject," in other words, "are only significant if they can be brought into a form in which the attribute is attributed or relation relates. If this were not the case, there would be significant propositions in which an attribute or a relation would occupy a position appropriate to a sub-

stance, which would be contrary to the doctrine of types, and would produce contradictions" (LA, pp. 337–38).

Note that here we have another extraordinary claim: *facts and universals cannot be logical subjects, and therefore cannot be named, because that would be contrary to the doctrine of types and would produce contradictions.* What is so extraordinary about this is that facts and universals are logical subjects in the theory of *r*-types, and as a theoretical account of the doctrine of logical types, the theory of *r*-types was designed explicitly so as not to produce contradictions. Apparently, Russell has somehow replaced his earlier version of the doctrine of logical types with a new or much restricted version, and it is not at all clear how well aware he was of the consequences of this move.

Note also that on this new version of the doctrine of types, Russell must reject his multiple-relations theory of belief, as well of course as the view that we can be acquainted with facts and universals. Thus, besides "the impossibility of treating the proposition believed as an independent entity, entering as a unit into the occurrence of the belief" (PLA, p. 226), there is now also "the impossibility of putting the subordinate verb on a level with its terms as an object term in the belief" (ibid.). "That is a point," Russell observed, "in which I think that the theory of judgment which I set forth once in print some years ago was a little unduly simple, because I did then treat the object verb as if one could put it as just an object like the terms, as if one could put, 'loves' on a level with Desdemona and Cassio as a term for the relation 'believe' " (ibid.) in the case of Othello believing that Desdemona loves Cassio. (Note that Russell uses "verb" to stand for both the word and the attribute or relation the word stands for; cf. PLA, p. 217.)

It is clear, as these observations indicate, that Russell has changed or seriously modified his 1910–13 ontology, and that somehow the change involves a new version of the doctrine of types. Thus, in 1924 Russell writes that "the doctrine of types leads to. . .a more complete and radical atomism than any that I conceived to be possible twenty years ago" (LA, p. 333), which in this case includes the 1910–13 ontology. This complete and radical atomism is of course Russell's form of logical atomism, the justification of which he claims is none other than "the justification of analysis" (PLA, p. 270). On this view, "you can get down in theory, if not in practice, to ultimate simples, out of which the world is built, and. . .those simples have a kind of reality not belonging to anything else" (ibid.); i.e., each simple has a kind of reality or mode of being that is unique to the entities of that kind (and which is the same as its logical type). "Simples. . .are of an infinite number of sorts. There are particulars and qualities and relations of various orders, a whole hierarchy of different sorts of simples" (ibid.). Aside from simples, "the only other sort of object you come across in the world" is facts (ibid.). That is, in the ontology of logical atomism, there are only simples and facts. Everything else is what Russell called a "logical fiction" (cf. PLA, pp. 253f).

The hierarchy of different sorts of simples that is in question here, it should

be noted, is not the hierarchy of r-types (where properties and relations can be logical subjects). For by the "order" of a relation Russell means in this context only the degree or adicity of that relation (cf. PLA, pp. 206f). That is, he does not mean "order" in the sense defined in section 1. In that regard, the r-types of the hierarchy of simples, i.e., of particulars, qualities, and relations of various "orders" now intended by Russell, can be indicated as follows, namely: i, $(i)/1$, $(i, i)/1$, $(i, i, i)/1$, and so on ad infinitum. (Note that as simples, no quality or relation has a "level" higher than 1; i.e., each is "predicative" in the sense of section 1). It is only first-order properties and relations, in other words, and even then only *simple* first-order properties and relations, that are involved in the ontology of logical atomism. Of course, quantifiers that "refer" to these simple properties and relations are no less significant than quantifiers that refer to particulars as individuals, which means that some restricted form of second-order logic is needed for the representation of this ontology. Indeed, the sentences of this restricted form of second-order logic are precisely what Russell later called the *atomistic hierarchy of sentences*; i.e., the hierarchy of sentences obtained from atomic sentences by the three operations of *substitution*, *combination*, and *generalization* (cf. IMT, p. 187).

Note that by the operation or principle of *substitution* Russell only means that an atomic sentence $R_n(a_1, \ldots, a_n)$ "remains significant if any or all of the names are replaced by any other names, and R_n is replaced by any other n-adic relation" (ibid., p. 185). Truth-functional or molecular compounds of atomic sentences are then obtained by iterated application of the stroke-operation of combination (having the truth-table of "either not. . .or not. . ."). Finally, "given any sentence containing either a name 'a' or a word 'R' denoting a relation or predicate, we can construct a new sentence in two ways" (ibid.), according to Russell, by the operation of *generalization*; i.e., quantification is significant with respect to both the subject and relation or predicate positions of atomic sentences. The resulting "hierarchy" of sentences, needless to say, consists only of second-order sentences; that is, it consists of sentences that are significant in second-order logic where there are no higher-order universals of an r-type $(\beta_1, \ldots, \beta_m)/n$, where β_i, for some i, is the r-type of a property or relation. This means that expressions for the higher-order universals that Russell took numbers to be in his 1910–13 ontology are no longer significant in his new theory of types. And yet, according to Russell's version of the principle of atomicity, "Everything we wish to say can be said in sentences belonging to the 'atomistic hierarchy' " (IMT, p. 160).

5. Propositional Functions as Linguistic Conveniences

In considering whether the atomistic hierarchy of sentences "can constitute an 'adequate' language, i.e., one into which any statement in any language can be translated" (IMT, p. 187), Russell asks if we can "be content with names, predicates, dyadic relations, etc., as our only variables, or do we need variables of

other kinds?" (ibid.). This question, we are told, "is concerned with generaliza-
tion and is relevant in solving the paradoxes" (ibid.). The other kinds of variables
Russell has in mind here are propositional variables and propositional function
variables (or what Russell also called variable propositions and variable
functions).

By a proposition in his post-PM view Russell means not an objective truth or
falsehood but "a sentence in the indicative" (PLA, p. 185), i.e., "a sentence assert-
ing something" (ibid.). In other words, "a proposition is just a symbol," and in
particular "it is a complex symbol in the sense that it has parts which are also sym-
bols" (ibid.). (Russell sometimes also means by a proposition an image-
proposition [cf. "On Propositions" (1919) and AMi]; but we shall ignore image-
propositions here since they correspond only to atomic sentences and do not
contain propositional functions.) A propositional function, similarly, "is simply
*any expression containing an undetermined constituent, or several undetermined
constituents, and becoming a proposition as soon as the undetermined consti-
tuents are determined*" (ibid., p. 230). Of course, as an expression that can be
mentioned and talked about as such, a propositional function is a "single entity."
But being mentioned is not the same as being used, and as for its use in logical
syntax "the only thing really that you can do with a propositional function is to
assert either that it is always true, or that it is sometimes true, or that it is never
true" (ibid.); that is, otherwise than being referred to as an expression, "a proposi-
tional function is nothing" (ibid.). This means that as an expression that is being
used rather than mentioned, a propositional function cannot occur as the gram-
matical subject of a proposition. This is why Russell in his 1925 introduction to
the second edition of PM claims that "there is no logical matrix of the form
$f!(\phi!\hat{z})$. The only matrices in which $\phi!\hat{z}$ is the only argument are those containing
$\phi!a$, $\phi!b$, $\phi!c$, . . . , where $a, b, c, \ldots$, are constants" (p. xxxi), and of course these
are matrices in which $\phi!\hat{z}$ does not occur as a singular term or logico-grammatical
subject. Indeed, this is precisely what Russell means by his new fundamental as-
sumption that "*a function can only appear in a matrix through its values*" (p. xxix).

Note that by a matrix Russell means in this context any propositional function
(expression) that "has elementary propositions as its values" (p. xxii), where an
elementary proposition is either an atomic proposition or a truth-functional com-
pound of atomic propositions (p. xvii). Note also that an n-adic relation symbol
R_n "cannot occur in a atomic proposition $R_m(x_1, \ldots, x_m)$ unless $n = m$, and then
can only occur as R_m occurs, not as $x_1, x_2, \ldots, x_m$ occur" (ibid.); i.e., relation
symbols are not allowed to occur as singular terms (the way they were allowed
to occur, e.g., in Russell's 1910–13 multiple-relations theory of belief or in what
he then called *higher-order matrices*). Finally, note that what "$\phi!a$" stands for,
according to Russell, is any elementary proposition that contains an atomic sen-
tence of the form $R_n(a, b_1, \ldots, b_{n-1})$. That is, $\phi!a$ is an elementary proposition
in which, strictly speaking, $\phi!\hat{z}$ does not occur as an "argument" at all once we

are given the predicate and relation symbols upon which any application of Russell's logical syntax is to be based. This is why Russell says in his 1925 introduction to PM that the "peculiarity of functions of second and higher-order is arbitrary" (p. xxxii), and that in fact by adopting predicate and relation variables (i.e., predicate variables of different adicities) we can avoid the notation for propositional functions altogether (ibid.). In other words, no new variables are really needed, according to Russell, beyond those already occurring in the atomistic hierarchy of sentences.

Because "the logic of propositions, and still more of general propositions concerning a given argument, would be intolerably complicated if we abstained from the use of variable functions" (ibid.), Russell does go on to include propositional variables and function variables in his new logical syntax. But, despite appearances to the contrary, these new variables all belong to ramified second-order logic; i.e., they are not allowed to occur as singular terms or logico-grammatical subjects of the new sentences formed by their addition to the atomistic hierarchy. Russell's notation can be deceptive in this regard, however, for even though "there is no logical matrix of the form $f!(\phi!\hat{z})$" (p. xxxi), i.e., a matrix where f is a second-order variable of r-type $(i/1)/1$, nevertheless, according to Russell, there are logical matrices of the form $f!(\phi!\hat{z}, x_1, x_2, \ldots, x_n)$, where "we call f a 'second-order function' because it takes functions among its arguments" (ibid.). A matrix of this form, however, "is always derived from a stroke-function

$$F(p_1, p_2, p_3, \ldots, p_n)$$

by substituting $\phi!x_1, \phi!x_2, \ldots, \phi!x_n$ for $p_1, p_2, \ldots, p_n$. *This is the sole method for constructing such matrices*" (ibid., emphasis added). Note that the propositional variables $p_1, \ldots, p_n$ do not occur in a stroke-function as singular terms, but as "arguments" of a sentential connective (viz., the stroke connective having the truth-table of "either not...or not..."). This means that the substitution of $\phi!x_1, \phi!x_2, \ldots, \phi!x_n$ for $p_1, p_2, \ldots, p_n$ in a stroke-function does not result in a proposition in which $\phi!\hat{z}$ occurs as a singular term, despite appearances to the contrary in Russell's way of representing this substitution as $f!(\phi!\hat{z}, x_1, x_2, \ldots, x_n)$. In other words, despite appearances, f is not occurring in this matrix as an $(n + 1)$-ary second-order variable of r-type $((i)/1, i, \ldots, i)/1$, but as an n-ary second-order variable of r-type $(i, \ldots, i)/2$. This is why Russell says that "*since ϕ can only appear through its values it must appear in a logical matrix with one or more variable arguments*" (ibid., emphasis added).[3]

Now in regard to generalization and the ramification of propositional functions, note that according to Russell "when we have a general proposition $(\phi).F\{\phi!\hat{z}, x, y, \ldots\}$, the only values ϕ can take are matrices, so that functions containing apparent variables are not included" (ibid., p. xxxiii). However, "we can, if we like, introduce a new variable to denote not only functions such as $\phi!\hat{x}$, but also such as

$$(y).\phi!(\hat{x}, y), (y, z).\phi!(\hat{x}, y, z),\ldots, (\exists y).\phi!(\hat{x}, y),\ldots;$$

in a word, all such functions of one variable as can be derived by generalization from matrices containing only individual variables" (ibid.). For this purpose, Russell introduces the variables ϕ_1, ψ_1, χ_1, etc.; i.e., "the suffix 1 is intended to indicate that the values of the functions may be first-order propositions, resulting from generalization in respect of individuals" (ibid.). "Theoretically," according to Russell, "it is unnecessary to introduce such variables as ϕ_1, because they can be replaced by an infinite conjunction or disjunction" (ibid.).

Of course, "when the apparent variable is of higher-order than the argument, a new situation arises. The simplest cases are

$$(\phi).f!(\phi!\hat{z}, x), (\exists\phi).f!(\phi!\hat{z}, x).$$

These are functions of x [where f is of r-type $(i)/2$, and not of r-type $((i)/1, i)/1$ as might appear from Russell's notation], but are obviously not included among the values for $\phi!x$ (where ϕ is the argument)" (ibid., p. xxxiv). Russell's original reason for this restriction of the values of $\phi!x$ was that paradoxes would otherwise ensue, including in particular his own paradox of predication. But that reason assumes that f is of r-type $((i)/1, i)/1$ in the preceding examples, and not of r-type $(i)/2$, as is required in Russell's new "atomistic" theory. That is, given Russell's fundamental new assumption that "a function can only appear in a matrix through its values," his own paradox is not even formulable, since it depends on propositional functions being expressions that can occur as singular terms of second-order matrices (or, as in Frege's *Grundgesetze*, on propositional functions having certain abstracts as their singular term counterparts). In other words, no paradox would be forthcoming in Russell's new or restricted logical syntax even if we were to allow the "values" of $\phi!x$ to include propositional functions in which ϕ occurs as a bound variable. (This of course is just the situation that obtains in standard impredicative second-order logic.)

There is a reason, nevertheless, why the "values" of $\phi!x$ should not include propositional functions in which ϕ or another function variable has a bound occurrence, and that is Russell's new or post-PM nominalistic construal of propositional functions. For "in the language of the second-order, variables denote symbols, not what is symbolized" (IMT, p. 192), and in that regard, of course, they cannot themselves be among the symbols they "denote." That is, bound propositional function variables are to be given a substitutional and not an "objectual" interpretation (as they were in Russell's original 1910–13 theory). On this interpretation, to attempt to make the "values" of $\phi!x$ include propositional functions that contain bound occurrences of ϕ "is like attempting to catch one's own shadow. It is impossible to obtain one variable which embraces among its values all possible functions of individuals" ([1925b], p. xxxiv).

Of course, "we can adopt a new variable ϕ_2 which is to include functions in

which $\phi!\hat{z}$ can be an apparent variable" (ibid.), but then "we shall obtain new func-
tions" (ibid.), and in this way go on to adopt new variables ϕ_3, ϕ_4, etc. Each of
the new variables will belong to a language one order higher than the language
whose propositional functions are the substituends or "values" of those variables,
and therefore of course none of the substituends or "values" of these new variables
can contain bound occurrences of those variables themselves. But all of these
variables, it should be emphasized, will be variables of ramified second-order
logic; i.e., they will have as substituends only propositional functions of in-
dividuals, albeit functions of higher and higher "levels," and in that sense of
higher and higher "orders" as well. For, as defined in section 1, the order of an
m-ary propositional function of r-type $(i, \ldots, i)/n$ will be the same as its level,
and of course that is why the languages generated by the addition of the new vari-
ables will be one order higher than the language whose propositional functions
are the substituends or "values" of those variables. This means that the higher-
order languages of Russell's later philosophy are not the higher-order languages
of the simple theory of types, and indeed this is why according to Russell, "my
hierarchy of languages is not identical with Carnap's or Tarski's" (IMT, p. 60).
For on Russell's "atomistic" view, "what is necessary for significance is that every
complete asserted proposition should be derived from a matrix by generalization,
and that, in the matrix, the substitution of constant values for the variables should
always result, ultimately, in a stroke-function of atomic propositions. We say 'ul-
timately,' because, when such variables as $\phi_2\hat{z}$ are admitted, the substitution of
a value for ϕ_2 may yield a proposition still containing apparent variables, and in
this proposition the apparent variables must be replaced by constants before we
arrive at a stroke-function of atomic propositions. *We may introduce variables
requiring several such stages, but the end must always be the same: a stroke-
function of atomic propositions*" (ibid., p. xxxv, emphasis added). In other
words, ultimately, according to Russell, "there is . . .no reason to admit as fun-
damental any variables except name-variables and relation-variables (in inten-
sion)" (IMT, p. 192), where the latter cannot themselves occur as singular terms.
That is, in the end, according to Russell, a proposition is significant only if it can
be translated into the atomistic hierarchy of sentences.

It is in this sense, accordingly, that propositional functions are merely linguis-
tic conveniences in Russell's later philosophy. And, indeed, as a claim about the
reducibility of the truth-conditions of ramified second-order logic to the truth-
conditions of the atomistic hierarchy of sentences, such a view is completely un-
problematic. Where Russell errs in his later philosophy is in thinking that every-
thing that could be said in his original theory of types can also be said in the
atomistic hierarchy, or what comes to the same thing, that his earlier theory of
types is equipollent to ramified second-order logic. In particular, Russell's own
analysis of classes in terms of propositional functions is no longer available to him
in his later philosophy; and apparently the reason he failed to see this was his new

way of representing a logical matrix. Russell's claim, accordingly, that "truth in pure mathematics is syntactical" ([MPD], p. 220) and that "numbers are classes of classes, and classes [as propositional functions] are symbolic conveniences" (ibid., p. 102), cannot be justified, since in order to talk of numbers as classes of classes, we must first be able to use expressions for classes as singular terms, which in Russell's framework ultimately means that we must be able to use propositional functions as abstract singular terms, and not merely as expressions that can be asserted as being always true, or sometimes true, or never true.

6. Russell's Weakened Form of the Principle of Atomicity

As originally formulated by Wittgenstein, the principle of atomicity is the thesis that "every statement about complexes can be analyzed into a statement about their constituent parts, and into those propositions which completely describe the complex" (*Tractatus*, 2.0201). For Russell, the technical form of this principle became, as we have said, the thesis that every significant sentence can be translated into the atomistic hierarchy of the sentences of an ideal or logically perfect language (whose logical syntax turns out to be that of ramified second-order logic). But since, according to Russell, all complexes are facts and facts cannot be named, it follows that the names of such a language can only denote simple particulars.

This makes the practical application of such a language very difficult, if not impossible, it should be noted, since Russell himself always maintained that what we take to be a simple particular may in the end really be complex and susceptible to further analysis. Indeed, by 1940, Russell came to the conclusion that "everything that there is in the world I call a 'fact.' The sun is a fact; Caesar's crossing the Rubicon was a fact; if I have a toothache, my toothache is a fact" (HK, p. 43). Facts in this sense, it should be noted, "are to be conceived as occurrences" (IMT, p. 268), i.e., as events.

Events, from 1914 to 1940, were the original simple particulars of Russell's atomist ontology, with ordinary physical objects being somehow analyzed as complexes consisting of a "compresence" of events (cf. LA, p. 341). That analysis is very much in doubt, however, insofar as complexes cannot be named and ascribed properties and relations in Russell's atomistic hierarchy. That is, just as Russell's earlier analysis of classes and numbers is no longer significant in his new logical syntax, so too his analysis of physical objects as a series of events is at least problematic, if not also nonsignificant in its allowing such complexes to have properties and stand in various relationships to one another. In addition, most, if not all, events will have an internal complexity of their own, and so they will not really be the simple particulars of an atomist ontology after all.

Russell never doubted the adequacy (or availability within the atomistic hierarchy) of his analysis of physical objects as complexes of events, it must be said; but he did agree, at least from 1940 on, that most events, notwithstanding their

status as particulars, were themselves complexes. Their constituents, it turned out, or at least so Russell proposed, were simple qualities. Thus, from 1940 on, events were no longer the simple particulars of Russell's atomist ontology, but were reconstrued as complexes of simple qualities. For Russell, this meant that words for qualities, such as "red," "blue," "hard," "soft," etc., are "names in the syntactical sense" (IMT, p. 89) of his ideal language. For example, according to Russell, "wherever there is, for common sense, a 'thing' having the quality *C*, we should say, instead, that *C* itself exists in that place, and that the 'thing' is to be replaced by the collection of qualities existing in the place in question. Thus '*C*' becomes a name, not a predicate" (ibid., p. 93). This does not mean that properties and relations in general can now be named; for Russell continued to insist right until the end that "relation-words ought only to be employed as actually relating and that sentences in which such words appear as subjects are only significant when they can be translated into sentences in which the relation-words perform their proper function of denoting a relation between terms. Or as it may be put in other words: verbs are necessary, but verbal nouns are not" ([MPD], p. 173).

What is important about this modification in Russell's ontology is that simple qualities are not the only particulars there are. That is, in Russell's ideal language of the atomistic hierarchy, at least from 1940 on, names may denote not only simple qualities but complexes of such as well (cf. HK, p. 84). This in fact is what Russell means by the weakened form of the principle of atomicity; that is, the form in which the principle "is not to be applied to everything that is in fact complex, but only to things named by complex names" (IMT, p. 251). "A name *N* may be in fact the name of a complex, but may not itself have any logical complexity, i.e. any parts that are symbols. This is the case with all names that actually occur. Caesar was complex, but 'Caesar' is logically simple, i.e., none of its parts are symbols" (ibid.). On the other hand, "though 'Caesar' is simple, 'the death of Caesar' is complex" (ibid.), and according to the principle of atomicity, it is to be analyzed into a statement about its constituent parts. In other words, although facts in the sense of events can be named in Russell's later philosophy after all, complex names of facts must still be analyzed and are not allowed to occur as names in the logical language based on the atomistic hierarchy of sentences (cf. IMT, p. 309).

This weakening of the principle of atomicity does allow Russell to translate sentences about physical objects into the logical language of his atomistic hierarchy, even though physical objects are ultimately themselves complexes of events, which in turn are complexes of compresent simple qualities. The translation, however, must never be such as to syntactically represent physical objects both as particulars and as complexes, since statements about complexes as single entities or "logical subjects" will have no counterparts in the atomistic hierarchy. How satisfactory a resolution of the problem of the practical application of Rus-

sell's ideal language this comes to in the end, we shall not attempt to assess here. For it still remains true in any case that Russell's original analysis of classes and numbers in terms of propositional functions as single entities will have no counterpart in his atomistic hierarchy of sentences.

Notes

1. Cf. Church, "Comparison of Russell's Resolution of the Semantical Antinomies with that of Torski," *Journal of Symbolic Logic*, 41 (1976), pp. 747–60. We take ω to be the set of natural numbers; thus, "$m \in \omega$" is read "m is a natural number," and "$n \in \omega - \{0\}$" is read "n is a natural number other than 0." We assume, incidentally, that the definition applies to expressions as well as to what the expressions stand for.

2. This notion of "level" should not be confused with Frege's. It corresponds, though not exactly, to Russell's notion of "order" in PM. We have retained Church's terminology here, since we are after all using his characterization of the *r*-types. We should note, however, that we use the phrase "ramified second-order logic" with its now standard meaning (as described, e.g., in Church [1956], section 58), i.e., as referring to the system of all the propositional functions that have *r*-types of the form $(i,\ldots, i)/n$, for arbitrary "level" n. These functions have only individuals as arguments, and therefore, as defined earlier their "order" is the same as their "level." This means that functions of every "order" are among the functions of ramified *second-order* logic, even though they always have only individuals as their arguments. I believe, incidentally, that a confusion of the different notions of order and level in part explains why Russell failed to see that the theory of types in his later philosophy was not the same as the theory he described in PM.

3. We should note, incidentally, that the use of the exclamation mark following the variable f no longer means that f is "predicative" in the sense defined in section 1. Rather, in Russell's 1925 introduction to PM, it simply means that the function has elementary propositions as its values (see p. xxviii).