POINTWISE ACCURACY OF A STABLE PETROV-GALERKIN APPROXIMATION TO STOKES PROBLEM

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Pointwise Accuracy of a Stable Petrov-Galerkin Approximation to Stokes Problem

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Abstract: The finite element approximation of the Stokes problem due to Hughes, Balestra and Franca [8] is analyzed in maximum norm. The method consists of modifying the usual bilinear form associated with the saddle-point structure to become coercive over the finite element space. Exploiting the enhanced stability, so getting rid of the inf-sup condition, quasi-optimal L^∞-error estimates are derived for both velocity and pressure.

AMS(MOS) Subject Classifications: Primary 65N15, 65N30, 76D07
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1. Introduction

The simplest flow of a viscous incompressible fluid is governed by the Stokes problem

$$
(1.1)\qquad \begin{aligned} -\Delta \underset{\sim}{u} + \nabla p &= \underset{\sim}{f}\,, && \text{in } \Omega \\ \operatorname{div} \underset{\sim}{u} &= 0\,, && \text{in } \Omega \\ \underset{\sim}{u} &= \underset{\sim}{0}\,, && \text{on } \partial\Omega\,, \end{aligned}
$$

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where $\underset{\sim}{u}$ indicates velocity and p pressure; $\underset{\sim}{f}$ denotes a given external force. Here, $\Omega \subset R^2$ is assumed to be bounded and regular enough. This system can be written in a weak form giving rise to the so-called velocity-pressure formulation: find $\underset{\sim}{u} \in \underset{\sim}{X} := [H_0^1(\Omega)]^2$ and $p \in M := L_0^2(\Omega)$ such that

$$(1.2) \qquad \begin{cases} \langle \nabla \underset{\sim}{u}, \nabla \underset{\sim}{v} \rangle - \langle \operatorname{div} \underset{\sim}{v}, p \rangle = \langle \underset{\sim}{f}, \underset{\sim}{v} \rangle , & \forall \underset{\sim}{v} \in \underset{\sim}{X} \\ \langle \operatorname{div} \underset{\sim}{u}, q \rangle = 0 , & \forall q \in M, \end{cases}$$

where $\langle \cdot , \cdot \rangle$ denotes the inner product in $L^2(\Omega)$ and $L_0^2(\Omega)$ is the space of L^2-functions having mean value zero. Existence, uniqueness and regularity are well known for (1.2), [7, 11].

Let τ_h be a regular and quasi-uniform partition of Ω into finite elements T [5, p. 132, 140]; h stands for the mesh-size. Given finite element subspaces $\underset{\sim}{X}_h \subset \underset{\sim}{X}$ and $M_h \subset M$ we can, in principle, think of an approximation scheme as follows: seek $\underset{\sim}{u}_h \in \underset{\sim}{X}_h$ and $p_h \in M_h$ such that

$$(1.3) \qquad \begin{cases} \langle \nabla \underset{\sim}{u}_h, \nabla \underset{\sim}{v} \rangle - \langle \operatorname{div} \underset{\sim}{v}, p_h \rangle = \langle \underset{\sim}{f}, \underset{\sim}{v} \rangle, & \forall \underset{\sim}{v} \in \underset{\sim}{X}_h \\ \langle \operatorname{div} \underset{\sim}{u}_h, q \rangle = 0 , & \forall q \in M_h . \end{cases}$$

However, simple choices like $P_1 - P_1$ fail to be stable due to the <u>saddle point structure</u> of (1.3) which requires a compatibility to hold between the discrete spaces $\underset{\sim}{X}_h$ and M_h. This constraint can be expressed by the celebrated <u>inf-sup condition</u>

$$(1.4) \qquad \inf_{q \in M_h} \sup_{\underset{\sim}{v}_h \in \underset{\sim}{X}_h} \frac{\langle \operatorname{div} \underset{\sim}{v}, q \rangle}{\|q\|_M \|\underset{\sim}{v}\|_X} > \beta > 0 ;$$

where β is independent of h (see Brezzi [2]). Moreover, since the underlying problem is linear, stability is equivalent to having optimal order error estimates in energy norms [2].

Many attempts have been made to construct families which pass the stability test (1.4); we refer to Brezzi-Fortin [4] and Girault-Raviart [7] for examples and references. One of the leading ideas was to enrich the velocity space $\underset{\sim}{X}_h$ or to diminish the pressure space M_h. However, most of the resulting spaces involve interpolation patterns which are computationally inconvenient and so difficult to find in existing engineering codes. Sharp error estimates are available in energy norms [2, 4, 7] as well as in maximum norm (see Durán-Nochetto-Wang [6]).

Another possibility consists of modifying the discrete equations (1.3). The method proposed by Hughes, Balestra and Franca reads as follows [8]: find $\underset{\sim}{u}_h \in \underset{\sim}{X}_h$ and $p_h \in M_h$ $(\subset L_0^2(\Omega) \cap H^1(\Omega))$ such that

$$
\begin{aligned}
&\langle \nabla \underset{\sim}{u}_h, \nabla \underset{\sim}{v} \rangle - \langle \operatorname{div} \underset{\sim}{v}, p_h \rangle = \langle \underset{\sim}{f}, \underset{\sim}{v} \rangle , \qquad \forall \underset{\sim}{v} \in \underset{\sim}{X}_h \\
(1.5)\qquad &\langle \operatorname{div} \underset{\sim}{u}_h, q \rangle + \alpha h^2 [\langle \nabla p_h, \nabla q \rangle - \sum_{T \in \tau_h} \langle \Delta \underset{\sim}{u}_h, \nabla q \rangle_T] \\
&\qquad = \alpha h^2 \langle \underset{\sim}{f}, \nabla q \rangle , \ \forall q \in M_h ,
\end{aligned}
$$

where the subscript T indicates that the inner product is to be extended over T only. We assume that $\underset{\sim}{X}_h|_T \supset \mathbf{P}_k(T)$ and $M_h|_T \supset \mathbf{P}_{k-1}(T)$ for every $T \in \tau_h$ where k is a given positive integer. The constant $\alpha > 0$ is to be chosen small enough so that the resulting bilinear form is coercive over $\underset{\sim}{X}_h \times M_h$ with respect to the norm $[\|\underset{\sim}{u}\|^2_{H^1} + h^2 \|\nabla p\|^2_{L^2}]^{1/2}$, [3,8]. Due to the coerciveness any choice for the discrete spaces can be made and the corresponding scheme can be easily implemented; in particular the pair $P_1 - P_1$ becomes stable.

The discrete scheme (1.5) was introduced within the framework of Petrov-Galerkin approximations, thus enhancing stability without upsetting consistency. Consequently, the accuracy of the scheme is only dictated by the interpolant polynomials used. Moreover, (1.5) is not a penalty method because the continuous solution satisfies the equations.

The accuracy of scheme (1.5) was analyzed in energy norms in [3,8], where optimal order error estimates were proven. The main object of this paper is to demonstrate quasi-optimal accuracy in L^∞. To this end, we make use of the method of weighted Sobolev norms introduced by Natterer [9] and Nitsche [10].

The rest of the paper is organized as follows. In section 2 notation and properties about weights are reminded and some weighted a priori estimates are proven. Section 3 is devoted to prove quasi-optimal L^∞-error estimates; so logarithmic factors occur. The proof is split into several steps: we first prove an estimate for gradients, then a bound for velocity and pressure and finally we assemble all these results.

2. Weighted a priori estimates

In this section we introduce the usual weight function along with its major properties and prove some preliminary results.

The weight function σ is defined by

$$\sigma(\underset{\sim}{x}) := \left(|\underset{\sim}{x} - \underset{\sim}{x}_0|^2 + \theta^2\right)^{1/2}, \qquad \text{for } \underset{\sim}{x}, \underset{\sim}{x}_0 \in \Omega, \tag{2.1}$$

where $\theta > h$ is a parameter to be determined later on. The following properties about σ are well known [5, 10]:

$$(2.2) \qquad \max_{\underset{\sim}{x} \in T} \sigma(\underset{\sim}{x}) \leq C \min_{\underset{\sim}{x} \in T} \sigma(\underset{\sim}{x}) , \qquad \forall T \in \tau_h ,$$

$$(2.3) \qquad |D^j \sigma^\alpha(\underset{\sim}{x})| \leq C(j,\alpha)\sigma^{\alpha-j}(\underset{\sim}{x}) , \qquad \forall \underset{\sim}{x} \in \Omega ,$$

where $\alpha \in \mathbf{R}$ and $D^j f$ denotes the tensor of derivatives of order j of f, and

$$(2.4) \qquad \int_\Omega \sigma^{-(2+\alpha)} \leq \begin{cases} C\,\theta^{-\alpha} , & \text{if } \alpha > 0 \\ C|\log \theta| , & \text{if } \alpha = 0 \end{cases}$$

for θ small enough. For $\alpha \in \mathbf{R}$ and $j \in \mathbf{N}$, the weighted Sobolev seminorms are defined by

$$\|D^j q\|^2_{\sigma^\alpha} := \sum_{|\beta| = j} \int_\Omega |\partial^\beta q|^2 \sigma^\alpha , \qquad q \in H^j(\Omega)$$

and the same notation will be used for vector valued functions.

Given $q \in M$ (or $\underset{\sim}{v} \in \underset{\sim}{X}$) the symbol $\hat{q} = I_h q \in M_h \oplus \mathbf{R}$ (or $\hat{\underset{\sim}{v}} = I_h \underset{\sim}{v} \in \underset{\sim}{X}_h$) indicates its <u>local</u> average interpolant [7, p. 109]. Due to this local character, I_h is an optimal order interpolant operator in L^p $(1 \leq p \leq \infty)$ as well as in weighted Sobolev norms. Moreover, the following <u>superapproximation property</u> holds

$$(2.5) \qquad \|\sigma^{-2} q - I_h(\sigma^{-2} q)\|_{\sigma^2} \leq C \frac{h}{\theta} \|q\|_{\sigma^{-2}} ,$$

$$(2.6) \qquad \|\nabla[\sigma^{-2} q - I_h(\sigma^{-2} q)]\|_{\sigma^2} \leq C \frac{h}{\theta}(\|q\|_{\sigma^{-4}} + \|\nabla q\|_{\sigma^{-2}}) ,$$

for all $q \in M_h \oplus \mathbb{R}$; similar bounds are valid for all $\underset{\sim}{v} \in \underset{\sim}{X}_h$. These estimates result from an application of the Bramble-Hilbert lemma together with the properties (2.2) and (2.3) about σ.

Let us now turn our attention to the regularity of the following generalized Stokes problem:

$$(2.7) \qquad \begin{cases} -\Delta \underset{\sim}{v} + \nabla q = \underset{\sim}{b} \,, & \text{in } \Omega \\ \text{div } \underset{\sim}{v} = g \,, & \text{in } \Omega \\ \underset{\sim}{v} = \underset{\sim}{0} \,, & \text{on } \partial\Omega \end{cases}$$

with g verifying the compatibility condition

$$(2.8) \qquad \int_\Omega g(x)dx = 0 \,.$$

As a by-product of the inf-sup condition, there exists a unique solution $(\underset{\sim}{v},q)$ which satisfies

$$(2.9) \qquad \|\underset{\sim}{v}\|_{H^1(\Omega)} + \|q\|_{L_0^2(\Omega)} \le C(\|\underset{\sim}{b}\|_{H^{-1}(\Omega)} + \|g\|_{L_0^2(\Omega)}).$$

Moreover, since $\partial\Omega$ is assumed to be regular enough (say, $\partial\Omega \in C^{k+1}$), the pair $(\underset{\sim}{v},q)$ also satisfies

$$(2.10) \qquad \|\underset{\sim}{v}\|_{W^{k+1,s}(\Omega)} + \|q\|_{W^{k,s}(\Omega)} \le Cs(\|\underset{\sim}{b}\|_{W^{k-1,s}(\Omega)} + \|g\|_{W^{k,s}(\Omega)}),$$

where $2 \le s < \infty$ and $C > 0$ is a constant independent of s. This a priori estimate is a consequence of the results in Agmon et al. [1] (see also Teman

[11, p. 33]); the dependence on s follows from tracing constants in the singular integrals involved.

We are now in a position to prove some weighted a priori estimates. The method of proof is based on Nitsche [10] (see also Ciarlet [5, p. 148, 160]) and was first used in this context by Durán-Nochetto-Wang [6].

Lemma 2.1 *There exists a constant* $C > 0$ *such that*

$$(2.11)\qquad \|D^2\underset{\sim}{v}\|_{\sigma^2} + \|\nabla q\|_{\sigma^2} \le C\Big(\frac{|\log h|^{1/2}}{\theta}\|\underset{\sim}{b}\|_{\sigma^4} + \|\nabla g\|_{\sigma^2} + \|g\|_{L^2(\Omega)}\Big).$$

Proof: It suffices to deal with the components $\mu_j = x_j - x_j^0$ $(j = 1,2)$ of $\underset{\sim}{x} - \underset{\sim}{x}_0$ rather than σ [5, p. 148]. Since no confusion is possible, we remove the subscript j. It is easily seen that

$$\|\mu\, D^2\underset{\sim}{v}\|_{L^2} + \|\mu\,\nabla q\|_{L^2} \le C\big(\|D^2(\mu\underset{\sim}{v})\|_{L^2} + \|\nabla(\mu q)\|_{L^2} + \|\nabla\underset{\sim}{v}\|_{L^2} + \|q\|_{L^2}\big).$$

The first two terms on the right hand side can be bounded by making use of the a priori estimate (2.10), with $k = 1$ and $s = 2$. Indeed, since $(\mu\underset{\sim}{v},\mu q)$ satisfy

$$-\Delta(\mu\underset{\sim}{v}) + \nabla(\mu q) = \mu\underset{\sim}{b} - 2\nabla\mu\cdot\nabla\underset{\sim}{v} + q\nabla\mu$$

$$\operatorname{div}(\mu\underset{\sim}{v}) = \mu g + \nabla\mu\cdot\underset{\sim}{v}\ ,$$

(2.10) together with the fact that μ is linear yields

$$\|D^2(\mu\underset{\sim}{v})\|_{L^2} + \|\nabla(\mu q)\|_{L^2}$$

$$\le C\big(\|\mu\underset{\sim}{b}\|_{L^2} + \|\mu\nabla g\|_{L^2} + \|g\|_{L^2} + \|\nabla\underset{\sim}{v}\|_{L^2} + \|q\|_{L^2}\big)\ .$$

To proceed further, let us estimate the last two terms. This will be a consequence of (2.9) as soon as we get a bound for $\|\underset{\sim}{b}\|_{H^{-1}}$. So, given $\underset{\sim}{\phi} \in [H_0^1(\Omega)]^2$, we have

$$|\langle \underset{\sim}{b}, \underset{\sim}{\phi} \rangle| < |\langle \underset{\sim}{b}, \underset{\sim}{\phi} - \hat{\underset{\sim}{\phi}} \rangle| + |\langle \underset{\sim}{b}, \hat{\underset{\sim}{\phi}} \rangle|$$

$$< C\left(h\|\underset{\sim}{b}\|_{L^2} + |\log h|^{1/2}\|\underset{\sim}{b}\|_{L^1}\right)\|\underset{\sim}{\phi}\|_{H^1} ,$$

and therefore

$$\|\underset{\sim}{b}\|_{H^{-1}} < C\left(h\|\underset{\sim}{b}\|_{L^2} + |\log h|^{1/2}\|\underset{\sim}{b}\|_{L^1}\right) .$$

Here we have used a well known 2-D inverse inequality between H^1 and L^∞ for finite element subspaces. A straightforward calculation based on (2.4) finally gives the desired estimate

$$\|\underset{\sim}{b}\|_{H^{-1}} < C \frac{|\log h|^{1/2}}{\theta} \|\underset{\sim}{b}\|_{\sigma^4} .$$

At this stage, it only remains to combine the equality

$$\|D^2\underset{\sim}{v}\|_{\sigma^2}^2 = \theta^2\|D^2\underset{\sim}{v}\|_{L^2}^2 + \sum_{j=1}^{2} \|\mu_j\, D^2\underset{\sim}{v}\|_{L^2}^2 ,$$

the analogous one for q and previous estimates to obtain the assertion. ◇

3. Error Analysis

This section is concerned with the pointwise accuracy of the proposed finite element approximation. The present analysis is in the spirit of the former one by Nitsche [10] (see also Ciarlet [5, §3.3]). So, taking advantage of the enforced coerciveness, we first derive an error bound for gradients of velocity $\underset{\sim}{u}$ and pressure p in terms of errors for $\underset{\sim}{u}$ and p. Then, we manipulate these two unknowns separately and get error estimates depending on the gradients. Finally, we assemble these results to achieve error estimates in weighted norms and the corresponding L^∞-analogues.

Let us start by writing the error equations, namely

$$(3.1)\qquad \langle \nabla(\underset{\sim}{u}-\underset{\sim}{u}_h), \nabla \underset{\sim}{v}\rangle - \langle \mathrm{div}\, \underset{\sim}{v}, p-p_h\rangle = 0 \quad , \quad \forall \underset{\sim}{v} \in \underset{\sim}{X}_h$$

$$(3.2)\qquad \langle \mathrm{div}(\underset{\sim}{u}-\underset{\sim}{u}_h), q\rangle + \alpha h^2 [\langle \nabla(p-p_h), \nabla q\rangle - \sum_{T \in \tau_h} \langle \Delta(\underset{\sim}{u}-\underset{\sim}{u}_h), \nabla q\rangle_T] = 0$$
$$\forall q \in M_h \oplus \mathbb{R}.$$

The main results of this paper are summarized as follows.

Theorem. There exist a small constant $\alpha > 0$ taking place in (1.5) and a constant $C > 0$ depending on α but not on h such that

$$(3.3)\qquad |\log h|^{-1/2} \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_{L^\infty(\Omega)} + h \|\nabla(\underset{\sim}{u}-\underset{\sim}{u}_h)\|_{L^\infty(\Omega)} + h\|p-p_h\|_{L^\infty(\Omega)}$$
$$\leq C\, h |\log h| \left(\|\nabla(\underset{\sim}{u}-\hat{\underset{\sim}{u}})\|_{L^\infty(\Omega)} + \|p-\hat{p}\|_{L^\infty(\Omega)}\right)$$
$$+ C\, h^2 |\log h| \left(\max_{T \in \tau_h} \|\Delta(\underset{\sim}{u}-\hat{\underset{\sim}{u}})\|_{L^\infty(T)} + \|\nabla(p-\hat{p})\|_{L^\infty(\Omega)}\right) .$$

The proof of this theorem will be carried out into three steps. Let us first introduce some further notation:

(3.4) $$E_h := \|p-\hat{p}\|^2_{\sigma^{-2}} + h^2\|\nabla(p-\hat{p})\|^2_{\sigma^{-2}} + \|\nabla(\underset{\sim}{u}-\hat{\underset{\sim}{u}})\|^2_{\sigma^{-2}} + \|\underset{\sim}{u}-\hat{\underset{\sim}{u}}\|^2_{\sigma^{-4}} + h^2 \sum_{T\in\tau_h} \|\Delta(\underset{\sim}{u}-\hat{\underset{\sim}{u}})\|^2_{\sigma^{-2},T} ;$$

(3.5) $$\underset{\sim}{e}_u := \underset{\sim}{u} - \underset{\sim}{u}_h , \quad \hat{\underset{\sim}{e}}_u = \hat{\underset{\sim}{u}} - \underset{\sim}{u}_h , \quad e_p := p - p_h \quad \hat{e}_p = \hat{p} - p_h ;$$

$C > 0$ denotes a generic constant which may vary at different occurrences but is always independent of h.

3.1. Error estimates for gradients

The present goal is to demonstrate the following error bound in weighted norms:

(3.6) $$\|\nabla\underset{\sim}{e}_u\|^2_{\sigma^{-2}} + \alpha h^2\|\nabla e_p\|^2_{\sigma^{-2}} \le C[\frac{1}{\varepsilon} E_h + \frac{1}{\varepsilon}\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + (\varepsilon + \frac{h}{\theta})\|e_p\|^2_{\sigma^{-2}}].$$

Hereafter, $\varepsilon > 0$ indicates a small constant to be specified later on. Moreover, since we shall make the choice $\theta \gg h$, the constant in front of $\|e_p\|_{\sigma^{-2}}$ is small enough. The proof of (3.6) will be split into two lemmas.

Lemma 3.1. *There exist constants* ε_0, $C > 0$ *such that*

(3.7) $$\|\nabla\underset{\sim}{e}_u\|^2_{\sigma^{-2}} \le \frac{C}{\varepsilon} E_h + \langle \operatorname{div} \underset{\sim}{e}_u, \sigma^{-2} e_p\rangle + \frac{C}{\varepsilon}\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + C(\varepsilon + \frac{h}{\theta})\|e_p\|^2_{\sigma^{-2}} ,$$

for all $0 < \varepsilon \leqslant \varepsilon_0$.

<u>Proof</u>: Note first that the following equality holds

$$\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} = \langle \nabla \underset{\sim}{e}_u,\ \sigma^{-2}\nabla(\underset{\sim}{u}-\hat{\underset{\sim}{u}})\rangle - \langle \nabla \underset{\sim}{e}_u,\ \hat{\underset{\sim}{e}}_u \cdot \nabla\sigma^{-2}\rangle + \langle \nabla \underset{\sim}{e}_u,\ \nabla(\sigma^{-2}\hat{\underset{\sim}{e}}_u)\rangle$$

$$=: \mathrm{I} + \mathrm{II} + \mathrm{III}\ .$$

The first two terms are easily bounded on account of property (2.3). Indeed, we can write

$$\mathrm{I} + \mathrm{II} \le \varepsilon\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + \frac{C}{\varepsilon}\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + \frac{C}{\varepsilon} E_h\ .$$

Now, set $\underset{\sim}{\psi} := \sigma^{-2}\hat{\underset{\sim}{e}}_u$ and decompose the remaining term as follows

$$\mathrm{III} = \langle \nabla \underset{\sim}{e}_u,\ \nabla(\underset{\sim}{\psi}-\hat{\underset{\sim}{\psi}})\rangle + \langle \nabla \underset{\sim}{e}_u,\ \nabla\hat{\underset{\sim}{\psi}}\rangle =: \mathrm{IV} + \mathrm{V}\ .$$

A straightforward application of the superapproximation property (2.6) yields

$$\mathrm{IV} \le C\frac{h}{\theta}\|\nabla \underset{\sim}{e}_u\|_{\sigma^{-2}}\left(\|\hat{\underset{\sim}{e}}_u\|_{\sigma^{-4}} + \|\nabla\hat{\underset{\sim}{e}}_u\|_{\sigma^{-2}}\right)$$

$$\le C\frac{h}{\theta}\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + C\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + C\,E_h\ .$$

Using the first error equation (3.1), we can rewrite term V; namely

$$\mathrm{V} = \langle \mathrm{div}(\hat{\underset{\sim}{\psi}}-\underset{\sim}{\psi}),\ e_p\rangle + \langle \mathrm{div}\ \underset{\sim}{\psi},\ e_p\rangle =: \mathrm{VI} + \mathrm{VII}.$$

Applying property (2.6) again, we get

$$\mathrm{VI} \le C\frac{h}{\theta}\|e_p\|_{\sigma^{-2}}\left(\|\hat{\underset{\sim}{e}}_u\|_{\sigma^{-4}} + \|\nabla\hat{\underset{\sim}{e}}_u\|_{\sigma^{-2}}\right)$$

$$\le \frac{h}{\theta}\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + C\frac{h}{\theta}\|e_p\|^2_{\sigma^{-2}} + C\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + C\,E_h\ .$$

Finally, in view of the definition of $\underset{\sim}{\psi}$, term VII can be bounded as follows

$$\text{VII} \leq \varepsilon \|e_p\|^2_{\sigma^{-2}} + \frac{C}{\varepsilon}\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + \frac{C}{\varepsilon} E_h + \langle \operatorname{div} \underset{\sim}{e}_u, \sigma^{-2} e_p \rangle .$$

Taking ε and h/θ small enough, term $\|\nabla \underset{\sim}{e}_u\|_{\sigma^{-2}}$ on the right hand side can be absorbed into the left, thus concluding the proof. ◇

In order to bound the second term on the right hand side of (3.7), we shall make use of the coerciveness provided by the modified divergence equation.

<u>**Lemma 3.2**</u> <u>There exist constants</u> $C, \eta_0 > 0$ <u>such that</u>

$$\langle \operatorname{div} \underset{\sim}{e}_u, \sigma^{-2} e_p \rangle + \alpha\, h^2 \|\nabla e_p\|^2_{\sigma^{-2}}$$

(3.8)

$$\leq C\left[\frac{h}{\theta}\|e_p\|^2_{\sigma^{-2}} + \left(\eta + \frac{\alpha}{\eta} + \frac{h}{\theta}\right)\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + \frac{1}{\eta} E_h\right],$$

<u>for all</u> $0 < \eta \leq \eta_0$.

<u>Proof:</u> Set $\omega := \sigma^{-2} e_p$ and use the second error equation (3.2) combined with the definition of ω to arrive at

$$\langle \operatorname{div} \underset{\sim}{e}_u, \omega \rangle + \alpha h^2 \|\nabla e_p\|^2_{\sigma^{-2}}$$

$$= -\alpha h^2 \langle \nabla e_p, e_p \nabla \sigma^{-2} \rangle + \alpha h^2 \sum_T \langle \Delta \underset{\sim}{e}_u, \nabla \omega \rangle_T$$

$$+ \langle \operatorname{div} \underset{\sim}{e}_u, \omega - \hat{\omega} \rangle + \alpha h^2 \langle \nabla e_p, \nabla(\omega - \hat{\omega}) \rangle$$

$$- \alpha h^2 \sum_T \langle \Delta \underset{\sim}{e}_u, \nabla(\omega - \hat{\omega}) \rangle$$

$$=: \text{I} + \ldots + \text{V} .$$

We now proceed to estimate the previous terms separately. Owing to property (2.3), term I becomes

$$I \le C \frac{h}{\theta} \alpha h^2 \|\nabla e_p\|^2_{\sigma^{-2}} + \frac{h}{\theta} \|e_p\|^2_{\sigma^{-2}} .$$

Based again on the definition of ω, we can rewrite term II as follows

$$II = \alpha h^2 \sum_T \langle \Delta \underset{\sim}{e}_u, \sigma^{-2} \nabla e_p \rangle_T + \alpha h^2 \sum_T \langle \Delta \underset{\sim}{e}_u, e_p \nabla \sigma^{-2} \rangle =: II_1 + II_2 .$$

Then, property (2.3) and an inverse inequality lead to

$$II_1 \le \alpha h^2 \|\nabla e_p\|_{\sigma^{-2}} \left(\sum_T \|\Delta \underset{\sim}{e}_u\|^2_{\sigma^{-2},T}\right)^{1/2}$$
$$\le \eta \alpha h^2 \|\nabla e_p\|^2_{\sigma^{-2}} + \frac{C}{\eta} \alpha \|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + \frac{C}{\eta} E_h ,$$

as well as

$$II_2 \le \frac{h}{\theta} \|e_p\|^2_{\sigma^{-2}} + C \frac{h}{\theta} \|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + C E_h .$$

We now evaluate term III. Let us first decompose ω,

$$\omega = \sigma^{-2}(p-\hat{p}) + \sigma^{-2}\hat{e}_p =: \omega_1 + \omega_2 .$$

Then, by virtue of (2.5) and the local boundedness of the interpolant operator I_h (which implies that $\|\hat{\omega}_1\|_{\sigma^2} \le C\|\omega_1\|_{\sigma^2}$), we easily find out that

$$\mathrm{III} \le C\|\nabla \underset{\sim}{e}_u\|_{\sigma^{-2}}\left(\|p-\hat{p}\|_{\sigma^{-2}} + \frac{h}{\theta}\|\hat{e}_p\|_{\sigma^{-2}}\right)$$

$$\le \frac{h}{\theta}\|e_p\|^2_{\sigma^{-2}} + C\left(\frac{h}{\theta} + \eta\right)\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + \frac{C}{\eta}E_h .$$

An analogous argument, now invoking (2.6) rather than (2.5), provides the following bound for term IV; namely

$$\mathrm{IV} \le C\,\alpha h^2\|\nabla e_p\|_{\sigma^{-2}}\left(\|\nabla\omega_1\|_{\sigma^2} + \|\nabla(\omega_2-\hat{\omega}_2)\|_{\sigma^2}\right) =: \mathrm{IV}_1 + \mathrm{IV}_2 .$$

Using the definition of ω_1 in conjunction with (2.3) yields

$$\|\nabla\omega_1\|_{\sigma^2} \le C\left(\|\nabla(p-\hat{p})\|_{\sigma^{-2}} + \frac{1}{\theta}\|p-\hat{p}\|_{\sigma^{-2}}\right) .$$

At the same time, property (2.6) gives

$$\|\nabla(\omega_2-\hat{\omega}_2)\|_{\sigma^2} \le C\,\frac{h}{\theta}\left(\frac{1}{\theta}\|\hat{e}_p\|_{\sigma^{-2}} + \|\nabla\hat{e}_p\|_{\sigma^{-2}}\right) .$$

Combining these estimates, we get

$$\mathrm{IV} \le C\alpha\left(\eta + \frac{h}{\theta}\right)h^2\|\nabla e_p\|^2_{\sigma^{-2}} + \frac{h}{\theta}\|e_p\|^2_{\sigma^{-2}} + \frac{C}{\eta}E_h .$$

To bound the remaining term V we make use of an inverse inequality which allows a control of the Laplacian, and split ω as before. After some simple calculations, we obtain

$$\mathrm{V} \le \frac{h}{\theta}\|e_p\|^2_{\sigma^{-2}} + C\,\frac{h}{\theta}\,\alpha h^2\|\nabla e_p\|^2_{\sigma^{-2}} + C\left(\frac{h}{\theta} + \eta\right)\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + \frac{C}{\eta}E_h .$$

Finally, a proper choice of η and θ allows the term $\alpha h^2 \|\nabla e_p\|^2_{\sigma^{-2}}$ to be hidden into the left hand side so yielding the asserted estimate. ◇

The next step consists of combining estimates (3.7) and (3.8). Once we do so, we realize that parameters η and h/θ might be taken even smaller than before so as to have control of the norm $\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}}$. At this stage, since we already have a bound from below for η, we can take $\alpha > 0$ small enough and absorbe $\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}}$ into the left hand side. This provides the desired estimate (3.6).

3.2. Error estimates for velocity.

In this subsection, the velocity error is evaluated in terms of errors for gradients of velocity and pressure. In fact, we have

Lemma 3.3 There exist constants $K_0, C > 0$ such that if $\theta = K\, h|\log h|^{1/2}$, then

$$\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} \leq CK^{-1}\left(\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + h^2\|\nabla e_p\|^2_{\sigma^{-2}}\right) + C\,E_h \,, \tag{3.9}$$

for all $K \geq K_0$.

Proof: As usual, we employ a duality argument. Set $\underset{\sim}{b} := \sigma^{-4}\underset{\sim}{e}_u$ and denote by $(\underset{\sim}{v}, q) \in \underset{\sim}{X} \times M$ the solution of the adjoint problem

$$\begin{cases} -\Delta \underset{\sim}{v} + \nabla q = \underset{\sim}{b}\,, & \text{in } \Omega \\ \operatorname{div} \underset{\sim}{v} = 0\,, & \text{in } \Omega \\ \underset{\sim}{v} = \underset{\sim}{0}\,, & \text{on } \partial\Omega\,. \end{cases}$$

Moreover, due to (2.11) we have

$$\|D^2\underset{\sim}{v}\|_{\sigma^2} + \|\nabla q\|_{\sigma^2} \le C \frac{|\log\theta|^{1/2}}{\theta} \|\underset{\sim}{e}_u\|_{\sigma^{-4}} . \tag{3.10}$$

Then, we can decompose the norm $\|\underset{\sim}{e}_u\|_{\sigma^{-4}}$ as follows

$$\begin{aligned}\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} &= \langle\nabla\underset{\sim}{e}_u, \nabla(\underset{\sim}{v}-\hat{\underset{\sim}{v}})\rangle - \langle \operatorname{div} \underset{\sim}{e}_u, q-\hat{q}\rangle \\ &\quad + \langle\nabla\underset{\sim}{e}_u, \nabla\hat{\underset{\sim}{v}}\rangle - \langle \operatorname{div} \underset{\sim}{e}_u, \hat{q}\rangle \\ &=: I + II + III + IV .\end{aligned}$$

By virtue of a priori estimate (3.10), terms I and II can be bounded together by

$$\begin{aligned} I + II &\le C \frac{h}{\theta} |\log\theta|^{1/2} \|\nabla\underset{\sim}{e}_u\|_{\sigma^{-2}}\|\underset{\sim}{e}_u\|_{\sigma^{-4}} \\ &\le \frac{1}{K}\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + \frac{C}{K}\|\nabla\underset{\sim}{e}_u\|^2_{\sigma^{-2}} .\end{aligned}$$

To estimate the last two terms we make use of the error equations. Indeed, by (3.1) we get

$$\begin{aligned} III &= \langle e_p, \operatorname{div}(\hat{\underset{\sim}{v}}-\underset{\sim}{v})\rangle = \langle\nabla e_p, \underset{\sim}{v}-\hat{\underset{\sim}{v}}\rangle \\ &\le C\frac{h^2}{\theta} |\log\theta|^{1/2}\|\nabla e_p\|_{\sigma^{-2}}\|\underset{\sim}{e}_u\|_{\sigma^{-4}} \\ &\le \frac{1}{K}\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + \frac{C}{K} h^2\|\nabla e_p\|^2_{\sigma^{-2}} ,\end{aligned}$$

where we have used (3.10) again. On the other hand, using the second error equation (3.2), an inverse inequality, the local boundedness of I_h (to replace $\|\nabla\hat{q}\|_{\sigma^2}$ by $\|\nabla q\|_{\sigma^2}$) together with (3.10), it is not difficult to conclude that

$$IV \le C\alpha h\Big[h\|\nabla e_p\|_{\sigma^{-2}} + \|\nabla \hat{\underset{\sim}{e}}_u\|_{\sigma^{-2}} + h\Big(\sum_T \|\Delta(\underset{\sim}{u}-\hat{\underset{\sim}{u}})\|^2_{\sigma^{-2},T}\Big)^{1/2}\Big]\|\nabla q\|_{\sigma^2}$$

$$\le \frac{1}{K}\|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + \frac{C}{K}\|\nabla\underset{\sim}{e}_u\|^2_{\sigma^{-2}} + \frac{C}{K}\alpha h^2\|\nabla e_p\|^2_{\sigma^{-2}} + C\,E_h \,.$$

Collecting all these estimates and taking K big enough, we reach the desired result. ◇

3.3. Error estimates for pressure.

As in §3.2, our present object is to bound the pressure error in terms of gradient errors.

Lemma 3.4. _There exist constants_ $K_0, C > 0$ _such that if_ $\theta \ge Kh$, _then_

(3.11) $$\|e_p\|^2_{\sigma^{-2}} \le C\Big(h^2\|\nabla e_p\|^2_{\sigma^{-2}} + \|\nabla\underset{\sim}{e}_u\|^2_{\sigma^{-2}} + E_h\Big) ,$$

for all $K \ge K_0$.

Proof: The key idea is again to use a duality argument. To this end, set $g := \sigma^{-2}e_p$ and denote by $(\underset{\sim}{v},q) \in \underset{\sim}{X} \times M$ the solution of the auxiliary problem

$$\begin{cases} -\Delta \underset{\sim}{v} + \nabla q = \underset{\sim}{0} & , \quad \text{in } \Omega \\ \operatorname{div} \underset{\sim}{v} = g - m & , \quad \text{in } \Omega \\ \underset{\sim}{v} = \underset{\sim}{0} & , \quad \text{on } \partial\Omega , \end{cases}$$

where $m := |\Omega|^{-1}\int_\Omega g$. The pair $(\underset{\sim}{v},q)$ clearly exists because the compatibility condition (2.8) holds. Moreover, it follows from (2.11) and (2.3) that

$$(3.12) \qquad \|D^2\underset{\sim}{v}\|_{\sigma^2} + \|\nabla q\|_{\sigma^2} \le C\big(\|\nabla e_p\|_{\sigma^{-2}} + \frac{1}{\theta}\|e_p\|_{\sigma^{-2}}\big) .$$

Now, since both p and p_h have mean value zero, we can write

$$\|e_p\|^2_{\sigma^{-2}} = \langle e_p, g-m\rangle = \langle e_p, \operatorname{div} \underset{\sim}{v}\rangle$$
$$= \langle e_p, \operatorname{div}(\underset{\sim}{v}-\hat{\underset{\sim}{v}})\rangle + \langle e_p, \operatorname{div} \hat{\underset{\sim}{v}}\rangle =: I + II .$$

Term I can be easily bounded on account of (3.12). In fact, we have

$$I = \langle \nabla e_p, \hat{\underset{\sim}{v}}-\underset{\sim}{v}\rangle \le C\, h^2\|\nabla e_p\|_{\sigma^{-2}}\big(\|\nabla e_p\|_{\sigma^{-2}} + \frac{1}{\theta}\|e_p\|_{\sigma^{-2}}\big)$$
$$\le C\, h^2\|\nabla e_p\|^2_{\sigma^{-2}} + \frac{1}{K}\|e_p\|^2_{\sigma^{-2}} .$$

On the other hand, we make use of the error equation (3.1) to express II as follows

$$II = \langle \nabla \underset{\sim}{e}_u, \nabla\hat{\underset{\sim}{v}}\rangle = \langle \nabla \underset{\sim}{e}_u, \nabla(\hat{\underset{\sim}{v}}-\underset{\sim}{v})\rangle + \langle \nabla \underset{\sim}{e}_u, \nabla \underset{\sim}{v}\rangle =: III + IV$$

Applying again (3.12) results in

$$\mathrm{III} \leq C\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + C\, h^2 \|\nabla e_p\|^2_{\sigma^{-2}} + \frac{1}{K}\|e_p\|^2_{\sigma^{-2}} .$$

The analysis of the remaining term requires a special care. Indeed, since $-\Delta \underset{\sim}{v} + \nabla q = \underset{\sim}{0}$ we get

$$\mathrm{IV} = \langle \mathrm{div}\, \underset{\sim}{e}_u, q\rangle = \langle \mathrm{div}\, \underset{\sim}{e}_u, q-\hat{q}\rangle + \langle \mathrm{div}\, \underset{\sim}{e}_u, \hat{q}\rangle ,$$

and observe that the first term can be bounded in a similar fashion to term III. So, it only remains to estimate the last term, which is decomposed by virtue of error equation (3.2) and the local boundedness of I_h as follows

$$\langle \mathrm{div}\, \underset{\sim}{e}_u, \hat{q}\rangle \leq C\alpha h\Big[h\|\nabla e_p\|_{\sigma^{-2}} + \|\nabla \hat{\underset{\sim}{e}}_u\|_{\sigma^{-2}} + h\Big(\sum_T \|\Delta(\underset{\sim}{u}-\hat{\underset{\sim}{u}})\|^2_{\sigma^{-2},T}\Big)^{1/2}\Big]\|\nabla q\|_{\sigma^{-2}} .$$

Here we have also used an inverse inequality. A further use of a priori estimate (3.12) leads to the final bound

$$\mathrm{IV} \leq C\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + C\, h^2 \|\nabla e_p\|_{\sigma^{-2}} + \frac{1}{K}\|e_p\|^2_{\sigma^{-2}} + C\, E_h .$$

Consequently, collecting the previous bounds and choosing K large enough allows the term $\|e_p\|^2_{\sigma^{-2}}$ to be hidden into the left hand side. The lemma is thus complete. ◇

3.4. Final rates of convergence

Inserting the estimates (3.9) and (3.11) into (3.6) and taking $\theta = Kh|\log h|^{1/2}$, we get

$$\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + \alpha h^2 \|\nabla e_p\|^2_{\sigma^{-2}} \leq C[(\varepsilon K)^{-1} + \varepsilon + K^{-1}](\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + h^2 \|\nabla e_p\|^2_{\sigma^{-2}}) + \frac{C}{\varepsilon} E_h .$$

Then, with a suitable choice of ε and K we can further control the first two terms on the right hand side and obtain the following error estimate in weighted norms

$$\|\nabla \underset{\sim}{e}_u\|^2_{\sigma^{-2}} + \alpha h^2 \|\nabla e_p\|^2_{\sigma^{-2}} + \|\underset{\sim}{e}_u\|^2_{\sigma^{-4}} + \|e_p\|^2_{\sigma^{-2}} \leq C\, E_h .$$

The desired maximum norm error estimates are a trivial consequence of them and property (2.4). For details we refer to Ciarlet [5, p. 163].

Let us now conclude by establishing the final rates of convergence for the proposed Petrov-Galerkin approximation.

<u>Corollary 3.1</u> <u>Assume that</u> $(\underset{\sim}{u},p) \in [W^{k+1,\infty}(\Omega)]^2 \times W^{k,\infty}(\Omega)$. <u>Then, there exists a constant</u> $C > 0$ <u>such that</u>

$$(3.13) \qquad |\log h|^{-1/2}\|\underset{\sim}{e}_u\|_{L^\infty} + h\|\nabla \underset{\sim}{e}_u\|_{L^\infty} + h\|e_p\|_{L^\infty} \leq Ch^{k+1}|\log h|(\|\underset{\sim}{u}\|_{W^{k+1,\infty}} + \|p\|_{W^{k,\infty}}) .$$

For this error estimate to hold the continuous solution $(\underset{\sim}{u},p)$ must satisfy an a priori regularity which is hard to check. The following result is an attempt to weaken this constraint.

Corollary 3.2 Assume that $\underset{\sim}{f} \in [W^{k-1,\infty}(\Omega)]^2$ and $\partial\Omega \in C^{k+1}$. Then there exists a constant $C > 0$ such that

$$(3.14) \qquad |\log h|^{-1/2} \|\underset{\sim}{e}_u\|_{L^\infty} + h\|\nabla \underset{\sim}{e}_u\|_{L^\infty} + h\|e_p\|_{L^\infty} \leq C\, h^{k+1} |\log h|^2 \, \|\underset{\sim}{f}\|_{W^{k-1,\infty}} .$$

Proof: The argument is based on a priori estimate (2.10). Indeed, the interpolation error on the right hand side of (3.3) can be easily bounded by

$$E = C\, h^{k+1-2/s} |\log h| \left(\|\underset{\sim}{u}\|_{W^{k+1,s}(\Omega)} + \|p\|_{W^{k,s}(\Omega)} \right) ,$$

which in view of (2.10) becomes

$$E \leq C\, h^{k+1} |\log h| \; s h^{-2/s} \, \|\underset{\sim}{f}\|_{W^{k-1,\infty}(\Omega)} .$$

The assertion then follows from taking $s = |\log h|$. ◇

Let us finally conclude by making some comments. The maximum norm error estimates (3.3), (3.13) and (3.14) can be interpreted as the natural L^∞-analogues of the energy bounds recently proved by Brezzi-Douglas [3]. Moreover, they proposed a modification of the Hughes-Balestra-Franca method which is possibly a bit cheaper computationally and maintains the same accuracy [3, §4]. Our analysis still applies to this case providing quasi-optimal L^∞-error estimates similar to those proved here.

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