

MIXING PROPERTIES FOR RANDOM WALK IN RANDOM SCENERY

BY

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MIXING PROPERTIES FOR RANDOM WALK IN RANDOM SCENERY

W. Th. F. den Hollander

Abstract

Consider the lattice Z^d , $d \geq 1$, together with its points and on it a random walk that is scenery perceived at a given time is a pattern in a finite box around his current position. Under a sequence of probability distributions governing walk and coloring, we prove asymptotic independence of local sceneries perceived at times 0 and T_k , in the limit as $k \rightarrow \infty$, where T_k is the (random) k -th hitting time of a black point. An immediate corollary of this result is the convergence in distribution of the interarrival times between successive black hits, i.e. of $T_{k+1} - T_k$ as $k \rightarrow \infty$. The limit distribution is expressed in terms of the distribution of the first hitting time T_1 . The proof uses coupling arguments.

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1. Statement of result

Consider the lattice of d -dimensional integers Z^d , $d \geq 1$, together with two independent probabilistic structures:

- a stochastic coloring $C := (C(z))_{z \in Z^d}$, assigning either of the colors black or white to each point of the lattice,
- a random walk $W := (W_n)_{n \geq 0}$ on the points of the lattice, starting at the origin ($W_0 = 0$).

The formal set-up is as follows. Let C be the set of all possible colorings, F_C the σ -algebra generated by the cylinder sets and P_C a probability measure on (C, F_C) having the properties:

- (A1) P_C is stationary and ergodic (w.r.t. lattice translations).
- (A2) $0 < q := P_C(C(0) = \text{black}) < 1$.

Let W be the set of all possible walks (starting at 0), F_W the σ -algebra generated by the cylinder sets and P_W a probability measure on (W, F_W) such that:

- (A3) The increments $W_{n+1} - W_n$, $n \geq 0$, are i.i.d. with probability distribution function $p: Z^d \rightarrow [0, 1]$ which is aperiodic, i.e. there is no proper sublattice containing the support of p .

Then the combination of walk and coloring is described by the product probability space (Ω, F, P) given by $\Omega = C \times W$, $F = F_C \times F_W$ and $P = P_C \times P_W$.

We shall be concerned with the local color patterns the walker sees around himself while stepping through the lattice.

Definitions:

- (i) A local scenery s consists of a finite set $F_s \subset Z^d$ and a black-white coloring of the points of F_s . The color in s of $z \in F_s$ is denoted by $s(z)$.
- (ii) The local scenery s is perceived at time n , an event which will be denoted by $[s]_n$, if $C(W_n + z) = s(z)$ for each $z \in F_s$.

The following proposition may be inferred from Meilijson (1974).

Proposition. Let L_p be the smallest sublattice containing the set $\{z-z': p(z)p(z')>0\}$.

Suppose that:

(P1) P_C is ergodic w.r.t. translations in L_p if $L_p \neq \{0\}$, and P_C is strongly mixing if $L_p = \{0\}$.

Then for any local sceneries s and t ,

$$(\star) \quad \lim_{n \rightarrow \infty} P([s]_0 \cap [t]_n) = P([s]_0) P([t]_0).$$

The converse is true also: (\star) holds iff (P1) does. □

Assumption (P1) is satisfied e.g. when p is strongly aperiodic (in which case $L_p = \mathbb{Z}^d$) or when P_C is strongly mixing, meaning that for all (cylinder) sets A and B in F_C

$$\lim_{|z| \rightarrow \infty} P_C(A \cap T^z B) = P_C(A) P_C(B),$$

where $T^z B$ is the translate of B over the vector z . (Note that the latter implies ergodicity w.r.t. translations in any sublattice.) The proof of the above proposition is given in section 2. This is essentially a restatement and refinement of the ideas of Meilijson in the present context and is included here for reasons of exposition. Counterexamples to (\star) are easily constructed within the class of periodic P_C , i.e. for color distributions obtained from a given infinite periodic coloring by assigning equal probability to all distinct translates.

Our main result involves a version of the preceding proposition with a different time scale, viz. one in which time is counted according to the number of visits to

black points. Let

$T_k :=$ (random) time at which the walker hits a black point
for the k -th time ($k \geq 1$).

Our basic assumptions (A1-3) imply that the sequence of colors of the points visited by the walker, which we shall henceforth refer to as the color sequence, is stationary and ergodic (Kasteleyn (1985); Kakutani (1951) theorem 3). Therefore, in particular, $T_k < \infty$ P -a.s. for all $k \geq 1$.

Theorem. Suppose that:

(T1) $\sum_{z, z'} p(z)p(z')P_C(C(z) \neq C(z')) > 0$ (see corollary 2 below).

(T2) P_C has trivial σ -algebra at infinity.

Then for any local sceneries s and t such that $0 \notin F_t$,

(**) $\lim_{k \rightarrow \infty} P([s]_0 \cap [t]_{T_k}) = P([s]_0) P([t]_0 | C(0) = \text{black}).$ □

Let $K_n = \{z \in \mathbb{Z}^d : |z^i| \leq n, 1 \leq i \leq d\}$, let F_C^n be the σ -algebra generated by the cylinder sets in $\mathbb{Z}^d \setminus K_n$, and $F_C^\infty = \bigcap_{n \geq 0} F_C^n$. Then (T2) requires that all elements of F_C^∞ have probability 0 or 1, or equivalently, that for any (cylinder) set A in F_C

$$P_C(A | F_C^\infty) = P_C(A) \text{ a.s.}$$

Obviously this is stronger than the strong mixing property mentioned above, for the latter only requires that A is asymptotically independent of any far away cylinder set and not necessarily of the whole infinite coloring on the outside of a large cube.

An important class of probability measures for which (T2) is satisfied is the

class of Gibbs states which satisfy (A1) (Ruelle (1978) Theorem 1.11). Here (A1) implies (T2) because of the assumption, inherent in the definition of Gibbs states, that for each coloring each lattice site contributes only a finite amount to the interaction potential, which puts a restriction on the correlations.

It is important to note that (***) is a much deeper result than (*). The point to appreciate is that T_k is a random variable that depends both on the walk and on the coloring, and a change of the colors anywhere in the lattice may (and will in general) change the distribution of the position the walker occupies at time T_k . Therefore, (***) is in no way directly related to (*). To illustrate this, consider the example with $d=1$, $p(1)=1$ and P_C the distribution obtained by randomly coloring points black, replacing each black point by a pair of neighboring black points and then coloring the remaining points white. In this case (*) holds but (***) does not. The reverse situation is also possible. Again take $d=1$ and $p(1)=1$, and now let P_C be the strictly periodic distribution where the black points occupy a sublattice. Then (***) holds but (*) does not.

Keane and Den Hollander (1986) recently proved (***) for the case where P_C is the Bernoulli measure and p is transient random walk. The proof of the above extension will be given in section 3 and is based on coupling arguments.

As will become clear from the proof later on, (T1) alone implies that

$$(***) \quad \lim_{k \rightarrow \infty} \{ P([s]_0 \cap [t]_{T_k}) - P([s]_0 \cap [t]_{T_{k+1}}) \} = 0.$$

Since (A1-3) are easily shown to imply that (***) holds at least in the weaker sense of Cesaro (see the remarks at the end of section 3), it thus is not unlikely that (***) is true without (T2), but more information is needed to settle this question. (A small step in this direction can be achieved though by showing that (***) holds under (T1) when P_C is periodic. This follows from the easily established fact that when

P_C is periodic $P([s]_0 \cap [t]_{T_k})$ is asymptotically periodic in k .)

An immediate corollary of our theorem is the convergence in distribution of the interarrival times between successive black hits. Indeed, let $n_0 := T_1$ and $n_k := T_{k+1} - T_k$, $k \geq 1$, then we have the following.

Corollary 1. When $(\star\star)$ holds, then for any nonnegative integer m ,

$$\lim_{k \rightarrow \infty} P([n_k > m]) = q^{-1} P([n_0 = m]).$$

□

To deduce this, first note that n_k depends only on the local sceneries perceived at time T_k . Then use that $P([n_k > m] | C(0) = \text{black})$ is independent of k and is equal to $q^{-1} P([n_0 = m])$, the latter being a consequence of the stationarity and ergodicity of the color sequence (Kasteleyn (1985)).

Assumption (T1) requires that with positive probability the support of p contains points of both colors. The role of this assumption becomes clear from the following lemmas, which we prove in section 4.

Lemma 1. Suppose that $P_W(W_n = 0 \text{ for some } n > 0) > 0$. If (T1) fails, then the color sequence is a.s. periodic and so is P_C . □

Lemma 2. If (T1) fails then so does (T2), unless $d=1$ and $p(1)=1$ or $p(-1)=1$. In the latter case $(\star\star)$ holds under (T2). □

Thus, by lemma 2, our most general result is:

Corollary 2. $(\star\star)$ holds under (T2) alone. □

Finally, in the context of ergodic theory (*) and (**) are equivalent to strong mixing of, respectively, the dynamical system associated with the local sceneries as perceived by the walker in the course of time and the so-called "induced" dynamical system obtained by conditioning on the sceneries which have a black origin (see Keane and Den Hollander (1986); in this paper the latter strong mixing property was given the name "kasteleyn mixing").

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2. Proof of proposition

For each $z \in \mathbb{Z}^d$, let $t+z$ be the translate over z of the local scenery t , i.e. $F_{t+z} = F_t + z$ and $(t+z)(z'+z) = t(z')$ for $z' \in F_t$. By the independence of walk and coloring

$$P([s]_0 \cap [t]_n) = \sum_z P_W(W_n = z) P([s]_0 \cap [t+z]_0).$$

First assume that p is strongly aperiodic. Then it is known that for each positive integer m ,

$$\lim_{n \rightarrow \infty} \sum_z |P_W(W_n = z) - |K_m|^{-1} \sum_{z' \in K_m} P_W(W_n = z+z')| = 0,$$

which says that the random walk spreads locally uniformly over Z^d (see Meilijson (1974); the results in this paper relate to $d=1$ but are easily carried over to $d>1$; the proof depends on a coupling argument of Ornstein (1969)). Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} P([s]_0 \cap [t]_n) &= \lim_{n \rightarrow \infty} \sum_z \{ |K_m|^{-1} \sum_{z' \in K_m} P_W(W_n = z+z') \} P([s]_0 \cap [t+z]_0) \\ &= \lim_{n \rightarrow \infty} \sum_y P_W(W_n = y) \{ |K_m|^{-1} \sum_{y' \in K_m} P([s]_0 \cap [t+y+y']_0) \}. \end{aligned}$$

Since, by (A1), for every y

$$\lim_{m \rightarrow \infty} |K_m|^{-1} \sum_{y' \in K_m} P([s]_0 \cap [t+y+y']_0) = P([s]_0) P([t]_0),$$

this yields (*) because of bounded convergence and the fact that $\sum_y P_W(W_n = y) = 1$ for all n .

If p is not strongly aperiodic, then the random walk spreads locally uniformly over the sublattice L_p and its translates, and hence it is enough to assume ergodicity w.r.t. translations in L_p . More precisely, L_p may have dimension d or $d-1$; in the first case the walker moves cyclically between the translates of L_p , while in the second it glides from one parallel translate to another, spreading in both cases in the direction of L_p . There is one exceptional case, viz. $L_p = \{0\}$ when p is deterministic. By (A3), p must then be $d=1$ with $p(1)=1$ or $p(-1)=1$. It is clear that in this case (*) needs strong mixing of P_C .

The above argument also shows that (P1) is necessary for (*). This completes the proof.

Note: A more commonly adopted definition of strong aperiodicity is that for all $z \in Z^d$ there is no proper sublattice containing the set $\{z+z': p(z') > 0\}$.

3. Proof of theorem

Abbreviate $S := \mathbb{Z}^d$, fix an arbitrary positive integer m , let

$$K_m(z) := \{z' \in S : |(z'-z)^i| \leq m, 1 \leq i \leq d\}$$

and define random variables

$$\begin{aligned} B_n &:= \#\{k \in [0, n] : C(W_k) = \text{black}\} \\ H_n(z) &:= \#\{k \in [0, n] : W_k = z\} \\ \tau_n(z) &:= \inf\{k > n : W_k = z\} \\ u_n &:= \text{local scenery perceived at time } n \text{ in } K_m(W_n) \end{aligned} \quad (z \in S, n \geq 0).$$

Consider two copies of S , denoted by S^1 and S^2 , each of which accommodates a stochastic coloring and a random walk, and which are coupled in a way as is described below. (Upper indices 1 and 2 will always refer to S^1 and S^2 .)

For the coupling of the colorings C^1 and C^2 we shall need the following lemma, the proof of which will be carried out in part 3B below.

Coupling lemma. Suppose that P_C satisfies (T2). Let C_m^1 and C_m^2 be arbitrary colorings of $K_m = K_m(0)$, each with positive probability. Then there exists a coupling of the colorings C^1 and C^2 , described by a probability measure on $(C^1 \times C^2, F_C^1 \times F_C^2)$ to be denoted by $P_C^{1,2}(\cdot | C_m^1, C_m^2)$, with the following properties:

- C^1 and C^2 coincide with C_m^1 and C_m^2 inside K_m and are distributed according to the conditional probability measure obtained from P_C by conditioning on C_m^1 and C_m^2 , respectively.
- With probability 1 there exists a random integer $\rho \in (m, \infty)$ such that $C^1(z) = C^2(z)$ for all $z \in S \setminus K_\rho$ (ρ will be the smallest such integer).

Moreover, if $d=1$ then there exists a coupling which has the additional property:

(c) With probability 1, C^1 and C^2 contain equal numbers of black points both in $[-\rho,0]$ and in $(0,\rho]$. \square

For the coupling of C^1 and C^2 we choose the one described by the lemma.

Next, choose $\epsilon > 0$ and integer N , and let $M = [\epsilon N]$ ($[.]$ denotes the integer part).

The random walks W^1 and W^2 are coupled as follows:

- (1) W^1 and W^2 trace the same path according to the rule p , until time N .
- (2) At time N , W^1 and W^2 are "uncoupled" and they proceed by making a succession steps, independently and according to the rule p , but conditioned on having to end up at the same site after each pair of steps that are taken.
- (3) This "uncoupling" continues until either time $N+2M$ is reached or $B_n^1 = B_n^2$ for some random time n in $[N, N+2M]$. After that, W^1 and W^2 are "recoupled" and they again continue in unison, but now forever.

As a result of the uncoupling at time N , W^1 and W^2 can visit different lattice points (and thus hit different colors) at times $N+1$, $N+3$, etc., until they are recoupled. The net effect of this uncoupling will be, and this will be seen to be the crux of the proof, that the difference between B_n^1 and B_n^2 accumulated at time $n=N$ gets pushed to zero after time N and remains fixed at zero for some time afterwards (see (3.1) below).

We shall denote by $P^{1,2}(\cdot | C_m^1, C_m^2)$, $P(\cdot | C_m^1)$ and $P(\cdot | C_m^2)$ the probability measures describing the coupled system and its marginals. One easily checks that each of the walks separately is controlled by the same rule p , while each of the colorings separately is controlled by the same probability measure P_C , but with the colorings conditioned on having to coincide with C_m^1 and C_m^2 , respectively, inside the box K_m . Thus the marginals are just the probability measures obtained from P by conditioning on these colorings inside the box.

Now we are ready to lay out the scheme of the proof. Let

$$I = I(\epsilon, N) := [N, N+2M],$$

$$J = J(\epsilon, N) := (N+2M, N+4M].$$

The main part of the proof consists in showing that

$$(3.1) \quad \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} P^{1,2}(B_n^1 = B_n^2 \text{ for all } n \in J \mid C_m^1, C_m^2) = 1.$$

The proof of (3.1) will be carried out in part 3A below and will require the use of (T1) and of properties (b) and (c) in the coupling lemma.

Continuing from (3.1), we may use that $W_n^1 = W_n^2$ for all $n \geq N+2M$ to obtain

$$(3.2) \quad \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} P^{1,2}(B_n^1 = B_n^2 \text{ and } u_n^1 = u_n^2 \text{ for all } n \in J \mid C_m^1, C_m^2) = 1,$$

provided we show that

$$(3.3) \quad \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} P^{1,2}(K_\rho \cap (\cup_{n \in J} K_m(W_n^1)) = \emptyset \mid C_m^1, C_m^2) = 1,$$

so that the box K_ρ at which S^1 and S^2 differ in coloring falls outside each of the local (box) sceneries perceived by W^1 and W^2 in J . But for any positive integer R ,

$$\begin{aligned} & P^{1,2}(K_\rho \cap (\cup_{n \in J} K_m(W_n^1)) \neq \emptyset \mid C_m^1, C_m^2) \\ & \leq P_C^{1,2}(\rho > R \mid C_m^1, C_m^2) + \sum_{z \in K_{R+m}} P_{W^1}(\tau_{N+2M}(z) \leq N+4M) \end{aligned}$$

and as part of the proof of (3.1) we shall show that for each $z \in S$

$$(3.4) \quad \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} P_W(\tau_N(z) \leq N+4M) = 0,$$

so that (3.3) will follow by letting $R \rightarrow \infty$ afterwards.

Next, using that

$$(3.5) \quad P(\lim_{k \rightarrow \infty} k^{-1}T_k = q^{-1}) = 1,$$

which is a consequence of the stationarity and ergodicity of the color sequence (see e.g. Breiman (1968) Chapter 6), we know that

$$P^{1,2}(\lim_{k \rightarrow \infty} k^{-1}T_k^1 = \lim_{k \rightarrow \infty} k^{-1}T_k^2 = q^{-1} \mid C_m^1, C_m^2) = 1.$$

Thus, for any $\epsilon > 0$

$$\lim_{N \rightarrow \infty} P^{1,2}(T_j^1, T_j^2 \in J \mid C_m^1, C_m^2) = 1$$

with $j = [(1+3\epsilon)Nq]$. Together with (3.2) this gives

$$(3.6) \quad \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} P^{1,2}(u_{T_j^1}^1 = u_{T_j^2}^2 \mid C_m^1, C_m^2) = 1.$$

From (3.6) we proceed as follows. Returning to the notation used in sections 1 and 2, we let $[t]_n$ be the event that $u_n = t$, where we consider only local sceneries t with $F_t \cap C_m \setminus \{0\}$. Then,

$$\begin{aligned}
& |P([t]_{T_j} | C_m^1) - P([t]_{T_j} | C_m^2)| \\
&= |P^{1,2}(u_{T_j}^1 = t | C_m^1, C_m^2) - P^{1,2}(u_{T_j}^2 = t | C_m^1, C_m^2)| \\
&\leq P^{1,2}(u_{T_j}^1 \neq u_{T_j}^2 | C_m^1, C_m^2).
\end{aligned}$$

Since the marginals are independent of N and ϵ , and since $(1+3\epsilon)q < 1$ for ϵ sufficiently small, this combines with (3.6) to give

$$(3.7) \quad \lim_{k \rightarrow \infty} \{ P([t]_{T_k} | C_m^1) - P([t]_{T_k} | C_m^2) \} = 0.$$

We may now complete the proof by the observation that for any t the sequence of indicators $1\{[t]_{T_k}\}$, $k \geq 1$, conditioned on the origin being black is stationary and ergodic (Keane and Den Hollander (1986)), so that in particular for all $k \geq 1$,

$$P([t]_{T_k} | C(0)=\text{black}) = P([t]_0 | C(0)=\text{black}).$$

For we can choose m large enough so that $F_s, F_t C K_m$, average in (3.7) over those C_m^1 which realize $[s]_0$ and those C_m^2 which realize $C(0)=\text{black}$, and then we arrive at (**):

$$\lim_{k \rightarrow \infty} P([t]_{T_k} | [s]_0) = P([t]_0 | C(0)=\text{black}).$$

3A. Proof of (3.1) and (3.4)

Let $\Delta B_n^{1,2} := B_n^1 - B_n^2$, $n \geq 0$. Since the uncoupling of the walks in I stops as

soon as $\Delta B_n^{1,2} = 0$ for some $n \in I$, we have for any positive integer R ,

$$\begin{aligned}
 (3.8) \quad & P^{1,2}(\Delta B_n^{1,2} \neq 0 \text{ for some } n \in J \mid C_m^1, C_m^2) \\
 & \leq P^{1,2}(\rho > R \mid C_m^1, C_m^2) \\
 & \quad + P^{1,2}(\Delta B_n^{1,2} \neq 0 \text{ for all } n \in I; \rho \leq R \mid C_m^1, C_m^2) \\
 & \quad + \sum_{z \in K_R} P^{1,2}(\tau_N^1(z) \leq N+4M \mid C_m^1, C_m^2).
 \end{aligned}$$

We shall show that each of the three terms in the r.h.s. of (3.8) tends to zero when we take limits in the order $N \rightarrow \infty$, $\epsilon \rightarrow 0$, $R \rightarrow \infty$. This will yield (3.1) and (3.4).

The first term is independent of N and ϵ and tends to zero as $R \rightarrow \infty$ because $\rho < \infty$ a.s. The summand in the third term equals $P_W(\tau_N(z) \leq N+4M)$. Fix R . Since for transient random walk $\lim_{N \rightarrow \infty} P_W(\tau_N(z) < \infty) = 0$ for each $z \in S$, we need only to worry about the recurrent case. Now clearly,

$$\begin{aligned}
 E_W(H_{N+8M}(z) - H_N(z)) & \geq E_W(H_{N+8M}(z) - H_N(z) ; \tau_N(z) \leq N+4M) \\
 & \geq P_W(\tau_N(z) \leq N+4M) E_W H_{4M}(0)
 \end{aligned}$$

and since $E_W H_n(z) \leq E_W H_n(0)$ for all z and n , it is enough to show that for each $z \in S$

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \{E_W H_{N+8M}(z) - E_W H_N(z)\} / E_W H_{4M}(z) = 0.$$

But for recurrent random walk it is known that $E_W H_n(z) \sim n^\gamma L(n)$, $n \rightarrow \infty$, for some $0 \leq \gamma \leq 1/2$ and some slowly varying function L (Spitzer (1976) section 7), and so

the first limit gives $\{(1+8\epsilon)^Y - 1\}/(4\epsilon)^Y$ and the second limit gives zero.

Thus we are left to deal with the second term in the r.h.s. of (3.8). For any positive integer K ,

$$(3.9) \quad \begin{aligned} & P^{1,2}(\Delta B_n^{1,2} \neq 0 \text{ for all } n \in I; \rho \leq R \mid C_m^1, C_m^2) \\ & \leq P^{1,2}(|\Delta B_N^{1,2}| > K; \rho \leq R \mid C_m^1, C_m^2) \\ & \quad + P^{1,2}(W_n^1 \in K_R \text{ or } W_n^2 \in K_R \text{ for some } n \in I \mid C_m^1, C_m^2) \\ & \quad + 2P^{1,2}(\Delta B_{n;N}^{1,2} > -K \text{ and } W_n^1, W_n^2 \notin K_\rho \text{ for all } n \in I \mid C_m^1, C_m^2). \end{aligned}$$

Here $\Delta B_{n;N}^{1,2} := \Delta B_n^{1,2} - \Delta B_N^{1,2}$, $n \geq N$, and in the last term the symmetry between W^1 and W^2 is used. We shall show that each of the three terms in the r.h.s. of (3.9) tends to zero in the appropriate limit.

The first term is independent of ϵ and equals

$$(3.10) \quad \begin{aligned} & \sum_{U, V \subset K_R} P_C^{1,2}(U^{1,2} = U; V^{1,2} = V; \rho \leq R \mid C_m^1, C_m^2) \\ & \quad \times P_W(|\sum_{z \in U} H_N(z) - \sum_{z' \in V} H_N(z')| > K), \end{aligned}$$

where we introduce the random sets

$$\begin{aligned} U^{1,2} &= \{ z \in K_R : C^1(z) = \text{black}, C^2(z) = \text{white} \} \\ V^{1,2} &= \{ z \in K_R : C^1(z) = \text{white}, C^2(z) = \text{black} \}. \end{aligned}$$

We claim that (3.10) tends to zero as $N \rightarrow \infty$, for R fixed, if we let $K \rightarrow \infty$ depending on N such that

(I) $K=o(N^{\frac{1}{2}})$, K sufficiently close to $N^{\frac{1}{2}}$.

To see this, note that (3.10) is bounded above by $K^{-1} |K_R| E_W H_N(0)$. Indeed this tends to zero under (I), provided $P_W(W_n=0)=o(n^{-\frac{1}{2}})$, $n \rightarrow \infty$. The latter property is shared by all random walks except those that fall in the class $d=1$, $\sum_z |z|^2 p(z) < \infty$ and $\sum_z z p(z)=0$ (Spitzer (1976) section 7). For this class, however, we can use property (c) in the coupling lemma, which says that when $d=1$ the coupling can be chosen in such a way that $|U^{1,2}|=|V^{1,2}|$ with probability 1 given $\rho \leq R$. Using the latter we can then bound (3.10) above by

$$P_W(\frac{1}{2} |K_R| \sup_{z, z' \in K_R} |H_N(z) - H_N(z')| > K)$$

and since it is known that for all random walks in this class

$$\lim_{n \rightarrow \infty} n^{-\frac{1}{2}-\delta} \sup_{z \in S} |H_n(z) - H_n(z+1)| = 0 \text{ a.s. for any } \delta > 0$$

(Csáki and Révész (1983) lemma 5), it again follows that the limit of (3.10) is zero under (I).

The second term in the r.h.s. of (3.9) is bounded above by $2P_W(W_n \in K_R)$ for some $n \in I$ which tends to zero in the appropriate limit by the argument we just gave for the third term in the r.h.s. of (3.8).

Thus, to finish the proof, we are left to deal with the third term in the r.h.s. of (3.9) and it is here that the uncoupling of W^1 and W^2 in I comes into full play. Because of this uncoupling, $\Delta B_{N+2i; N}^{1,2} =: X_i$, $0 \leq i \leq M$, performs a random walk on the integers Z (starting at 0) with the following properties:

(a) The absolute increments $|X_{i+1} - X_i|$ take values 0 or 1 and are dependent.

(b) The sign of $X_{i+1} - X_i$ is random (i.e. + or -, each with probability 1/2 and independently of previous increments).

Property (b) is a consequence of the symmetry between W^1 and W^2 at each pair of steps taken.

We proceed as follows. Fix $\epsilon > 0$ and let

$$Y_M := \sum_{i=0}^{M-1} |X_{i+1} - X_i|$$

be the random variable denoting the total number of "actual transitions" by the random walk X_i in the interval I . By property (b) above it will be clear that, given any value of Y_M , the total displacement X_M is distributed as the position of a simple random walk on Z , with independent increments, after Y_M steps and starting at 0. Thus, when we let W_n^{srw} denote the position of simple random walk on Z at time n and P_W^{srw} the corresponding probability measure, then we have, for any positive integer L ,

$$\begin{aligned} (3.11) \quad & P^{1,2}(\Delta B_{n;N}^{1,2} > -K \text{ and } W_n^1, W_n^2 \notin K_\rho \text{ for all } n \in I \mid C_m^1, C_m^2) \\ &= \sum_{j \geq 0} P^{1,2}(Y_M = j; W_n^1, W_n^2 \notin K_\rho \text{ for all } n \in I \mid C_m^1, C_m^2) \\ & \quad \times P_W^{\text{srw}}(W_n^{\text{srw}} > -K \text{ for all } n \in [0, j]) \\ & \leq P^{1,2}(Y_M < L; W_n^1, W_n^2 \notin K_\rho \text{ for all } n \in I \mid C_m^1, C_m^2) \\ & \quad + P_W^{\text{srw}}(W_n^{\text{srw}} > -K \text{ for all } n \in [0, L]). \end{aligned}$$

Now, because the walkers cannot see the difference between C^1 and C^2 when they stay outside the box K_ρ , the first term in the r.h.s. of (3.11) is bounded above by

$$(3.12) \quad \bar{P}^{1,2}(Y_M < L \mid C_m^1),$$

where $\bar{P}^{1,2}$ is the probability measure for the coupled system that is obtained by letting C^1 and C^2 be identical over the whole lattice and distributed according to P_C and by uncoupling W^1 and W^2 over the whole interval I . In (3.12) we condition on $C^1=C^2$ having to coincide with C_m^1 inside K_m , but of course the same bound is true for C_m^2 . Now, under $\bar{P}^{1,2}$

(c) $|X_{i+1}-X_i|$, $i \geq 0$, is a stationary and ergodic 0-1 sequence.

This is a consequence of (A1) and (A3) in section 1 (Kakutani (1951) theorem 3). Hence, by the ergodic theorem,

$$\bar{P}^{1,2}(\lim_{M \rightarrow \infty} M^{-1} Y_M = \bar{E}^{1,2}|X_1| \mid C_m^1) = 1.$$

By assumption (T1), $\bar{E}^{1,2}|X_1| = \bar{P}^{1,2}(X_1 \neq 0) > 0$, and therefore the first term in the r.h.s. of (3.11) tends to zero if we let $L \rightarrow \infty$ depending on N such that

$$(II) \quad L = o(N).$$

(recall that $M = \lfloor \epsilon N \rfloor$). The second term also has limit zero if

$$(III) \quad K = o(L^{1/2}),$$

because of the well-known space/time scaling property of simple random walk. Now, collecting conditions (I-III), we see that K and L can be chosen in such a way that all three conditions are met. This completes the proof of (3.1) and (3.4).

3B. Proof of coupling lemma

To prove the coupling lemma we shall need the following theorem.

Maximal coupling theorem (Goldstein (1979)). Let $(X_n^1)_{n \geq 0}$ and $(X_n^2)_{n \geq 0}$ be arbitrary sequences of random variables taking values in the same Borel space, and let P^1 and P^2 denote their respective probability measures. There exists a successful coupling $P^{1,2}$, meaning that $P^{1,2}(X_n^1 = X_n^2 \text{ for all } n \text{ sufficiently large}) = 1$, iff P^1 and P^2 agree on the tail σ -algebra (i.e. their restrictions to the tail σ -algebra are identical). \square

From this theorem it follows that there exists a coupling of the colorings with properties (a) and (b) in the coupling lemma iff for every choice of C_m^1 and C_m^2 , $P_C(\cdot | C_m^1)$ and $P_C(\cdot | C_m^2)$ agree on F_C^∞ . But this is equivalent to (T2), which also shows that the coupling lemma cannot be true without (T2).

The existence of a coupling in $d=1$ with the additional property (c) is less immediate. If we set

$$\begin{aligned} X_n &= 1 \{ C(n)=\text{black} \}, \\ S_n &= 0, S_n - S_{n-1} = X_n, \quad n \in \mathbb{Z}, \end{aligned}$$

then Goldstein's theorem shows that such a coupling exists iff the process $(S_n)_{n \in \mathbb{Z}}$ has trivial σ -algebra at infinity. Now, because the random variables S_n are increasing sums over the random variables X_n , it is far from obvious that this triviality carries over from (T2). The fact that it does may be deduced from techniques developed by Berbee (1979) and relies heavily on the stationarity of the X -process. The proof is rather lengthy (and somewhat tedious) and will therefore be omitted. The interested reader should study Chapters 5 and 6 of Berbee. In the following we shall

attempt a heuristic argument that is more straightforward.

Consider two independent realizations S_n^1 and S_n^2 of the S -process, each under P_C . Then $S_n^1 - S_n^2$ is a "random walk" on Z with stationary and ergodic increments which have zero expectation and bounded first absolute moment. Dekking (1982) has shown that such a random walk is recurrent, i.e. it returns to 0 infinitely often with probability 1. Now suppose that we again consider two independent realizations but this time under $P_C(\cdot | C_m^1)$ and $P_C(\cdot | C_m^2)$, respectively. Then the above is still true in that with probability 1 there exists a doubly infinite sequence of random integers

$$\dots < m_{-2} < m_{-1} < m_0 (=0) < m_1 < m_2 < \dots$$

along which $S_n^1 = S_n^2$. Given such a sequence, let

$$Y_k^i = (X_n^i)_{n \in (m_{k-1}, m_k]}, \quad k \in Z, \quad i=1,2,$$

be the sequence of random vectors describing the X^1 and X^2 -process inbetween the intersection points m_k . Since, by (T2), $P_C(A|C_m^1) = P_C(A|C_m^2) = 0$ or 1 for all $A \in F_C^\infty$, the same must be true on the σ -algebra at infinity of the Y -processes, so that in particular these measures coincide on this σ -algebra. Goldstein's theorem then tells us that there exists a successful coupling (at both ends) of $(Y_k^1)_{k \in Z}$ and $(Y_k^2)_{k \in Z}$. But, by construction, such a coupling is also a successful coupling of $(S_n^1)_{n \in Z}$ and $(S_n^2)_{n \in Z}$.

The above argument is flawed in that it does not take into account the fact that the m_k are random and depend on the realization of the Y^1 and Y^2 -process. The reader is invited to do the patching up.

Remarks:

(1) As was mentioned in section 1, assumption (T1) alone is enough to obtain (***) . The proof is easy. Suppose that we couple S^1 and S^2 by letting C^1 and C^2 be identical and distributed according to P_C and by uncoupling W^1 and W^2 at time 0. In this case the marginals both equal P . Then, because of (T1), the random walk performed by $\Delta B_n^{1,2}$ will a.s. hit any point of Z in the course of time. When it hits $+1$ we recouple, and $\Delta B_n^{1,2}$ is fixed at $+1$ for all subsequent n . Since $u_n^1 = u_n^2$ after the recoupling, we immediately obtain (***) .

(2) To prove that (**) holds in the weaker sense of Cesaro, as was claimed in section 1, first note that for any local scenery t the sequence of indicators $1\{[t]_n\}$, $n \geq 0$, is stationary and ergodic (Keane and Den Hollander (1986)). Hence

$$\lim_{k \rightarrow \infty} k^{-1} \sum_{n=0}^{k-1} 1\{[t]_n\} = P([t]_0) \quad P\text{-a.s.}$$

Next, let t be such that $0 \notin F_t$ and let t' be the local scenery with $F_{t'} = F_t \cup \{0\}$, $t'(z) = t(z)$ for $z \in F_t$ and $t'(0) = \text{black}$. Then clearly, for any k ,

$$\sum_{n=0}^{T_k} 1\{[t']_n\} = \sum_{n=0}^k 1\{[t]_{T_n}\},$$

and if we now use (3.5) we get

$$\lim_{k \rightarrow \infty} k^{-1} \sum_{n=0}^{k-1} 1\{[t]_{T_n}\} = q^{-1} P([t']_0) = P([t]_0 | C(0) = \text{black}) \quad P\text{-a.s.}$$

Integration over those colorings which realize $[s]_0$ yields the result. (Incidentally, (3.5) follows from the same argument, by taking for t the local scenery with $F_t = \emptyset$.)

4. Proof of lemma 1 and 2

For each $n \geq 0$, let $V_n = \{z \in \mathbb{Z}^d : P_W(W_n = z) > 0\}$ be the set of points the walker can reach at time n . Suppose that (T1) does not hold. Then the points in V_1 are P_C -a.s. all identically colored. From stationarity and induction on n it follows that for each n the points in V_n are P_C -a.s. identically colored. If p is non-deterministic, then for n large the set V_n will contain points arbitrarily far apart. Hence P_C cannot satisfy (T2) (it is not even strongly mixing). Since the only aperiodic deterministic p is the one with $d=1$ and $p(1)=1$ or $p(-1)=1$, this proves the first part of lemma 2.

Now suppose that $P_W(W_n = 0 \text{ for some } n > 0) > 0$. Then there exists an integer $1 \leq j < \infty$ such that $P_W(W_{kj} = 0) > 0$ for all $k \geq 0$. This in turn implies that for each $i=0, 1, \dots, j-1$ all the points in the union $\bigcup_{k \geq 0} V_{kj+i}$ are P_C -a.s. identically colored, which is the same as saying that the color sequence is P -a.s. periodic. But the color sequence is ergodic and therefore it must P -a.s. consist of a single periodic color sequence or one of its translates. But this, in turn, can happen only when P_C is periodic, as is easily seen from (A1) and (A3). This proves lemma 1.

The second part of lemma 2 can be read off from property (c) in the coupling lemma.

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