MAJORITY VOTE PROCESS ON GENERAL STRUCTURES

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

ZHENGYI GONG

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Dr. MAURY BRAMSON, ADVISOR

OCTOBER, 2024

Acknowledgments

This dissertation marks the culmination of my doctoral studies, and I would like to express my deepest gratitude to all those who have supported me throughout this challenging journey.

First and foremost, I am profoundly thankful to my advisor, Dr. Maury Bramson, whose guidance has not only sharpened my mathematical thinking and academic writing but also significantly improved my proficiency in English. Your mentorship has been vital to my growth as a young mathematician throughout, and I am deeply appreciative of your continued support and encouragement.

I extend my sincere appreciation to my committee members, Dr. Arnab Sen, Dr. Wei-Kuo Chen, and Dr. Kevin Leder, for their valuable time, feedback, and guidance during the preparation of this dissertation. Your support has been crucial in shaping its final form. I also wish to express my gratitude to Dr. Wai-Kit Lam, Dr. Tiefeng Jiang, and Dr. Peter Polacik. The knowledge and insights I gained from each of you have greatly enriched my academic journey.

I am deeply grateful to my parents for their unwavering mental support and encouragement, which have been instrumental in helping me reach this point in my academic career. Additionally, I would like to extend my heartfelt gratitude to my grandparents and other family members, whose love and support have been a constant source of strength throughout my life.

I would like to thank my girlfriend, Jingran Cao (who is about to receive her doctoral degree), for her constant companionship, providing both emotional support and joy during the more difficult phases of my doctoral studies.

My heartfelt thanks go to my fellow graduate students, Haoran Xue, Yun Li, Xuan Wu, and Chieh-Hsin Lai, for the many insightful discussions we have shared. Your collaboration has been key in shaping my research and understanding during the course of this dissertation.

Finally, I would like to acknowledge my friends, as well as those with whom I shared weightlifting and cycling activities. Your support and encouragement outside of academia provided a much-needed balance and helped sustain my motivation throughout my doctoral studies.

Abstract

The *majority vote process* (MVP) was one of the earliest *interacting particle systems* (IPS) to be explored, but little progress has been made except on $\mathbb{Z}$ and the d-ary regular trees $\mathbb{T}_d$, with $d \geq 5$. This thesis considers the majority vote process defined on more general *structures* $(S, \mathcal{N})$, where S denotes a set, and each $\mathcal{N}(x) \subseteq S$ denotes a *neighborhood of* x. Each *site* $x \in S$, at each time $t \geq 0$, is assumed to have one of two possible *opinions*, 0 or 1. At rate ε_0 (ε_1), the opinion of x is updated to 0 (1) irrespective of the opinions in $\mathcal{N}(x)$. At rate $1 - \varepsilon_0 - \varepsilon_1$, the opinion of x aligns with the absolute majority opinion in $\mathcal{N}(x)$, which is referred to as the *majority dynamics.* We investigate the ergodicity and equilibria behavior of the MVP, focusing on their dependence on $(S, \mathcal{N})$, ε_0 and ε_1.

A central concept will be that of *uniformly self-supporting structures.* A finite $A \subseteq S$ is a *self-supporting set* if A consisting of all 0s or all 1s implies no site from A can change opinion under the majority dynamics. A structure $(S, \mathcal{N})$ is a *uniformly self-supporting structure* if there exists a constant R and a family of self-supporting sets $\mathscr{E} = \{A_\alpha\}_{\alpha \in \Lambda}$ that covers S, each of which has a diameter bounded by R, such that, for every $A \in \mathscr{E}$ and $x \notin A$, there exists some $A' \in \mathscr{E}$, for which $x \in A'$ and $A \cap A' = \emptyset$.

Our main results are Theorems 1, 2, 3, and 4. Theorem 1 shows that the majority vote process on a uniformly self-supporting structure $(S, \mathcal{N})$ is ergodic if one also assumes that either ε_0 or ε_1 is significantly smaller than the other. Theorem 2 shows that the MVP converges to its unique equilibrium exponentially quickly when additional space homogeneity is assumed on $(S, \mathcal{N})$.

Theorems 3 and 4 consider examples of $(S, \mathcal{N})$ that are not uniformly self-supporting. Theorem 3 considers the *unoriented majority vote process with nearest-neighbor interactions (UMVP)* on $\mathbb{T}_d$, $d = 3, 4$. We demonstrate that when $d = 3$, the UMVP with sufficiently small noise admits at least two mutually singular equilibria; when $d = 4$, the UMVP with sufficiently small noise admits uncountably many mutually singular equilibria. Theorem 4 considers the *oriented majority vote process with nearest-neighbor interactions* (OMVP) on $\mathbb{Z}^d$, $d \geq 4$. We demonstrate such an OMVP with sufficiently small noise admits uncountably many mutually singular equilibria.

Contents

List of Figures

Chapter 1

Introduction

The field of *interacting particle systems* (IPS) has been extensively studied since its introduction in the late 1960s ([1], [2], [3], [4]). Initially motivated by statistical mechanics, biology, and social science, IPS models find natural formulations in various other domains, including social networks, tumor growth, the spread of infectious diseases, and behavioral systems. Prominent examples of IPS include the contact process, the voter model, the exclusion process, and the stochastic Ising process. Mathematically, interacting particle systems are continuous-time Markov processes on certain configuration spaces of particles.

In this thesis, we investigate the *majority vote process* (MVP), which can be described briefly as follows. At any given time $t \geq 0$, each *site* in the system is assumed to have one of two possible *opinions*, 0 or 1. The opinion at each site updates according to an exponentially distributed time, at the end of which the opinion is either set equal to the majority opinion of the neighboring sites, or it is randomly set equal to either 0 or 1.

To give a precise definition of the majority vote process, we introduce the following notation. Let $\mathfrak{P}(S)$ denote the power set of S. Consider a *structure* $(S, \mathcal{N})$, where S denotes an arbitrary set, and $\mathcal{N} : S \to \mathfrak{P}(S)$ denotes a *neighborhood function* such that there exists some constant K, satisfying $|\mathcal{N}(x)| < K$ for every $x \in S$. Three types of events – *vote events*, 0-*noise events*, and 1-*noise events* – can change the opinion at x, with the opinion at x remaining unchanged until the next such event occurs. At a vote event, the opinion at x becomes that of the majority opinion in $\mathcal{N}(x)$; in the case of a tie, the opinion at x remains unchanged. We will refer to this as the *majority dynamics*. At a 0-noise event (1-noise event), the opinion 0 (1) is assigned at x, irrespective of the current opinion of x. At each site $x \in S$, 0-noise events, 1-noise events, and vote events are assumed to occur at rates ε_0, ε_1 and $1 - \varepsilon_0 - \varepsilon_1$, respectively, with $\varepsilon_0, \varepsilon_1 \geq 0$ and $0 \leq \varepsilon_0 + \varepsilon_1 \leq 1$. We refer to ε_0, ε_1 and $\varepsilon = \varepsilon_0 + \varepsilon_1$ as the 0-*noise intensity*, 1-*noise intensity*, and *noise intensity*, respectively. Throughout this thesis, we assume that $\varepsilon_0 + \varepsilon_1 \leq 1$, for the majority vote process to be well-defined.

The majority vote process on $\mathbb{Z}^d$ was one of the earliest interacting particle systems to be investigated. The MVP has similar dynamics to those of the *stochastic Ising process*. The stochastic Ising process is the dynamic version of the famous Ising model, both of which are known to admit a phase transition on $\mathbb{Z}^d$ with $d \geq 2$, but not when $d = 1$. Due to this similarity, the majority vote process was believed to have a phase transition when $d \geq 2$, but not when $d = 1$. Despite its similar dynamics to the stochastic Ising process, the equilibria of the MVP do not admit an explicit expression, unlike the stochastic Ising process. (Subsection 1.3.1 will provide a detailed comparison between the majority vote process and the stochastic Ising process.)

Despite the relatively simple dynamics of the majority vote process, limited progress has been made since its introduction. Gray ([5]) showed that the majority vote process on $\mathbb{Z}$, with nearest-neighbor interactions, is ergodic whenever the noise event occurs at a strictly positive rate. Bramson and Gray ([6]) showed that the majority vote process on the d-ary regular tree, with $d \geq 5$, has uncountably many mutually singular equilibria when the noise event occurs at a sufficiently low rate. Little progress on the MVP has been made otherwise.

In this thesis, we investigate the ergodicity and equilibria behavior of the majority vote process, with a primary focus on understanding their dependence on $(S, \mathcal{N})$, ε_0 and ε_1. A basic result in the interacting particle system implies that, when the noise event occurs at a large enough rate compared with the vote event, the corresponding MVP is ergodic. (An IPS is said to be *ergodic* if it has a unique equilibrium and converges to the unique equilibrium starting from any initial configuration.) The MVP will therefore be ergodic when the noise terms $\varepsilon_0, \varepsilon_1$ are significantly greater than $1 - \varepsilon_0 - \varepsilon_1$. We will give a quick proof of this in Section 2.4. Consequently, research on the majority vote process mostly focuses on small noise cases.

A central concept of our research is that of the *uniformly self-supporting structure*. A finite $A \subseteq S$ is a *self-supporting set* if A consisting of all 0s or all 1s implies that no site from A can change opinion under the majority dynamics. A structure $(S, \mathcal{N})$ is a uniformly self-supporting structure if there exist a constant R and a family of self-supporting sets $\mathscr{E} = \{A_\alpha\}_{\alpha \in \Lambda}$, each of which, under the assigned metric, has a diameter bounded by R, and satisfies $\cup_{\alpha \in \Lambda} A_\alpha = S$, such that, for every $A \in \mathscr{E}$ and $x \notin A$, there exists some $A' \in \mathscr{E}$ satisfying $x \in A'$ and $A \cap A' = \emptyset$.

Examples of uniformly self-supporting structures $(S, \mathcal{N})$ include

$$S = \mathbb{Z}^d, \quad \mathcal{N}(x) = \{x \pm \mathbf{e}_i : 1 \leq i \leq d\} \quad \text{for each } x \in \mathbb{Z}^d, \tag{1.1}$$

where $\mathbf{e}_i$, $1 \leq i \leq d$, denotes the ith standard basis vector, and

$$S = \mathbb{Z}, \quad \mathcal{N}(x) = \{x, x \pm 1, x \pm 2\} \quad \text{for each } x \in \mathbb{Z}. \tag{1.2}$$

We will discuss these examples and prove they are uniformly self-supporting structures in Section 2.5. An Example of $(S, \mathcal{N})$ that is not a uniformly self-supporting structure is $S = \mathbb{Z}^2$, with each $\mathcal{N}(x) = \{x + \mathbf{e}_1, x + \mathbf{e}_2\}$.

Our main results are Theorems 1, 2, 3, and 4. Theorems 1 and 2 consider the majority vote process defined on a uniformly self-supporting structure $(S, \mathcal{N})$. Theorem 1 shows that such an MVP is ergodic if one also assumes that either ε_0 or ε_1 is significantly smaller than the other. Theorem 2 shows that, under similar conditions, the MVP on $(S, \mathcal{N})$ converges to its unique equilibrium exponentially quickly when $(S, \mathcal{N})$ is also assumed to be an $\mathcal{N}$-transitive structure. (An $\mathcal{N}$-transitive structure is, in essence, a uniformly self-supporting structure with some additional "space homogeneity". We will introduce this concept in Subsection 1.1.1.)

Theorems 3 and 4 consider the *unoriented majority vote process with nearest-neighbor interactions (UMVP)* on $\mathbb{T}_d$, the d-ary regular trees, with $d = 3, 4$, and the *oriented majority vote process with nearest-neighbor interactions (OMVP)* on $\mathbb{Z}^d$, $d \geq 4$, respectively. (We will define these two processes in Subsection 1.1.2.) We prove that, when $\varepsilon_0 + \varepsilon_1$ is sufficiently small, the corresponding majority vote process is non-ergodic. Under the same assumption, we also construct uncountably many mutually singular equilibria for the UMVP on $\mathbb{T}_4$, and for the OMVP on $\mathbb{Z}^d$ with $d \geq 4$.

The structure of this thesis is as follows. Chapter 1 serves as an introduction, primarily dealing with defining the majority vote process on general structures $(S, \mathcal{N})$, introducing our terminology and notation, presenting main results, and summarizing relevant background as well as previous related work. Chapter 2 introduces tools employed in the proofs of our main results and states some general results about the majority vote process. Among these tools, the most important ones are the graphical constructions (Section 2.1) and coupling (Section 2.3). The subsequent chapters, Chapters 3, 4, 5, and 6, provide proofs for Theorems 1, 2, 3, and 4, respectively. The proof of Theorem 2 (Chapter 4) partially relies on results from Chapter 3. Chapters 5 and 6 are independent and self-contained.

Chapter 1 itself is structured as follows. In Section 1.1, we introduce fundamental terminology and assumptions needed to articulate our main results. In Section 1.2, we present our main results – Theorems 1, 2, 3 and 4 – and discuss their interconnections. Section 1.3 introduces other related interacting particle systems, including the *stochastic Ising process* and the *voter model*, and compares them with the majority vote process. Section 1.4 gives a historical review of the majority vote process, and summarizes some of the most important results of the majority vote process and other related processes, including a discrete-time analog of the MVP studied by Toom ([7]). Lastly, we conclude this chapter with a more detailed summary of this thesis in Section 1.5.

1.1 Terminology for MVP and Uniformly Self-supporting Structures

This section introduces basic terminology for the majority vote process as well as assumptions and conditions needed to articulate our main theorems, Theorems 1, 2, 3, and 4. In Subsection 1.1.1, we introduce a metric $d(\cdot,\cdot)$ on S and revisit some crucial concepts, such as the self-supporting set and uniformly self-supporting structure, which have been introduced at the beginning of this chapter. These concepts are fundamental to Theorem 1. We conclude this subsection by introducing the $\mathcal{N}$*-transitive structure*, which is needed to state Theorem 2. Then, in Subsection 1.1.2, we introduce the OMVP and UMVP on graphs, whose behavior will be further investigated in Theorems 3 and 4.

1.1.1 Self-supporting Sets, Uniformly Self-supporting Structures and $\mathcal{N}$-transitive Structures

We introduce a metric $d(\cdot,\cdot)$ on S: Recall that $(S,\mathcal{N})$ is a pair where S denotes a set and $\mathcal{N}$ denotes a set-valued function on S such that each $\mathcal{N}(x)\subseteq S$, and there exists a constant K with $|\mathcal{N}(x)|<K$ for every $x\in S$. To introduce $d(\cdot,\cdot)$, we set $d(x,x)=0$, for every $x\in S$, and $d(x,y)=1$, for each $x,y\in S$ with either $y\in\mathcal{N}(x)$ or $x\in\mathcal{N}(y)$. We extend the definition of $d(\cdot,\cdot)$ to every $x,y\in S$ as follows. For $x\neq y\in S$ where $d(x,y)$ is not yet defined, we set

$$d(x,y)=\inf_{\gamma}\{r:\gamma=\{v_i\}_{i=0}^{r}\text{ with }v_0=x\text{ and }v_r=y\},\tag{1.3}$$

where the infimum in (1.3) is over all sequences $\gamma=\{v_i\}_{i=0}^{r}$ that satisfies $d(v_{i-1},v_i)=1$, for each $1\leq i\leq r$. We also set $d(x,y)=\infty$ if, for every $r\in\mathbb{N}$, such a sequence does not exist. It is straightforward to verify that $d(\cdot,\cdot)$ satisfies the positivity, symmetry, and triangle inequality conditions, and hence it is a metric on S. In addition, for each $x\in S$ and $R\in\mathbb{N}$, we set

$$B(x,R)=\{y\in S:d(x,y)\leq R\}.$$

For given $(S,\mathcal{N})$, recall a finite $A\subseteq S$ is a self-supporting set if A consisting of all 0s or all 1s implies that no site from A can change opinion under the majority dynamics. An equivalent condition for A self-supporting is

$$|A\cap\mathcal{N}(x)|\geq|\mathcal{N}(x)|/2,\quad\text{for every }x\in A.\tag{1.4}$$

It is easy to see A and B self-supporting implies that $A\cup B$ is self-supporting – for every $x\in A\cup B$, we have

$$\mathcal{N}(x)\cap(A\cup B)=(\mathcal{N}(x)\cap A)\cup(\mathcal{N}(x)\cap B),$$

with at least one of $|\mathcal{N}(x)\cap A|$ and $|\mathcal{N}(x)\cap B|$ no less than $|\mathcal{N}(x)|/2$. However, given A,B self-supporting, $A\cap B$ is not necessarily self-supporting. For example, in (1.2), $A=\{0,1,2\}$ and $B=\{2,3,4\}$ are both self-supporting sets, but $A\cap B=\{2\}$ is apparently not a self-supporting set.

Recall a structure $(S,\mathcal{N})$ is a uniformly self-supporting structure if there exists a constant R and a family of self-supporting sets $\mathscr{E}=\{A_\alpha\}_{\alpha\in\Lambda}$, satisfying

$$\begin{aligned}&\cup_{\alpha\in\Lambda} A_\alpha = S,\ \text{each } A_\alpha\subseteq B(x,R) \text{ for some } x\in S,\ \text{and}\\ &\text{for each } A\in\mathscr{E} \text{ and } x\notin A, \text{there exists some } A'\in\mathscr{E} \text{ satisfying}\\ &x\in A' \text{ and } A'\cap A=\emptyset.\end{aligned} \tag{1.5}$$

In addition, for each $A\in\mathscr{E}$, we say A is an *elementary self-supporting set*.

We next introduce the $\mathcal{N}$-transitive structure. We say $f:S\to S$ is an $\mathcal{N}$-*automorphism* if f is bijective and preserves neighborhood function $\mathcal{N}$, i.e., $f(\mathcal{N}(x))=\mathcal{N}(f(x))$ for every $x\in S$. (As usual, we write $f(A)=\{f(x):x\in A\}$ for each $A\subseteq S$.) A uniformly self-supporting structure $(S,\mathcal{N})$ is an $\mathcal{N}$*-transitive structure* if, for every $v_1,v_2\in S$, there exists an $\mathcal{N}$-automorphism $f:S\to S$ such that $f(v_1)=v_2$, and for every $A,A'\in\mathscr{E}$, there exists an $\mathcal{N}$-automorphism $f:S\to S$ such that $f(A)=A'$.

As a direct consequence of the $\mathcal{N}$-transitivity, every elementary self-supporting set $A\in\mathscr{E}$ has the same cardinality. Moreover, when $\mathcal{N}$-transitivity is assumed for a structure $(S,\mathcal{N})$, S has a strong space-homogeneity – every $x\in S$ is "locally identical". For example, $(\mathbb{Z}^d,\mathcal{N})$, where $\mathcal{N}(x)=\{x\pm\mathbf{e}_i:1\le i\le d\}$ for each $x\in S$, is $\mathcal{N}$-transitive.

In Section 2.5, we will introduce criteria to determine whether a structure $(S,\mathcal{N})$ is a uniformly self-supporting structure or an $\mathcal{N}$-transitive structure. We will also discuss important examples of uniformly self-supporting structures and $\mathcal{N}$-transitive structures in the same section.

1.1.2 MVP on Graphs: UMVP and OMVP

Recall we have defined the majority vote process on structures $(S,\mathcal{N})$ at the beginning of Chapter 1. To avoid trivialities, throughout this thesis, we assume S is infinite. A natural choice for S is the set of vertices of an infinite graph. This subsection explores cases where S is the set of vertices of a directed and undirected graph.

We first present some basic terminology of graph theory, which is consistent with definitions in [8]. An *undirected graph* is a pair $G=(V,E)$ of sets such that $E\subseteq V\times V$, i.e., elements of E, or *edges*, are 2-element subset of V. For each $e=(x,y)\in E$, we say x (y) is *incident* with e. Two vertices $x,y\in V$ are *adjacent* if $(x,y)\in E$.

A *path* in G is a non-empty graph $P = (V', E')$ of the form

$$V' = \{x_0, x_1, \cdots, x_k\} \subseteq V,$$

and

$$E' = \{(x_0, x_1), (x_1, x_2), \cdots, (x_{k-1}, x_k)\} \subseteq E,$$

where x_i, $0 \leq i \leq k$, are distinct vertices. A graph G is *connected* if it is non-empty and any two of its vertices are linked by a path in G. If $P = (V', E')$ is a path and $(x_k, x_0) \in E$, then $P' = (V', E' \cup \{(x_k, x_0)\})$ is a *cycle*. If a graph $G = (V, E)$ contains no cycle, then we say G is *acyclic*. An acyclic and connected graph G is a *tree*. A tree $G = (V, E)$ is a *d-ary regular tree* if each vertex $x \in V$ has exactly d adjacent vertices. (Or, equivalently, each $x \in V$ is incident with exactly d edges.) In this thesis, we let $\mathbb{T}_d$ denote the d-ary regular tree.

We next define the majority vote process on an undirected graph $G = (V, E)$ as follows. Consider the structure $(S, \mathcal{N})$, where $S = V$ and each $\mathcal{N}(x)$ is the set of all adjacent vertices of x, i.e.,

$$\mathcal{N}(x) = \{y \in V : (x, y) \in E\} \quad \text{for each } x \in V.$$

The majority vote process on such a structure $(S, \mathcal{N})$ is the *unoriented majority vote process with nearest-neighbor interactions* (UMVP) on G. In this thesis, we are interested in the behavior of the UMVP on $\mathbb{T}_d$, with $d = 3, 4$, and on hexagonal/triangular lattices, among others.

A *directed graph* is an *orientation* of an undirected graph $G = (V, E)$, where each $e = (x, y) \in E$ is oriented, with $x = init(e)$ and $y = ter(e)$. (Therefore $e_1 = (x, y)$ and $e_2 = (y, x)$ are considered as distinct edges.) Such a directed graph can arise from an undirected graph by directing every edge from one of its ends to the other. We say each edge $e = (init(e), ter(e)) \in E$ is *oriented from $init(e)$ to $ter(e)$*. If an edge is oriented from x to y, we refer to x as a *parent* of y and refer to y as an *offspring* of x. Let $O(x)$ denote the set of all offspring of x, i.e.

$$O(x) = \{y \in V : (x, y) \in E\} \quad \text{for each } x \in V.$$

We next define the majority vote process on a directed graph $G = (V, E)$. We consider the structure $(S, \mathcal{N})$, where $S = V$ and $\mathcal{N}(x) = O(x)$ for each $x \in V$. The majority vote process on such a structure $(S, \mathcal{N})$ is the *oriented majority vote process with nearest-neighbor interactions* (OMVP) on G. In this thesis, we are interested in the OMVP on $\mathbb{Z}^d$, with $\mathcal{N}(x) = \{x + \mathbf{e}_i : 1 \leq i \leq d\}$.

1.2 Main Results – Ergodicity and Nonergodicity; Convergence Rate

This section introduces our main results, Theorems 1, 2, 3, and 4. Theorems 1 and 2 consider the majority vote process on uniformly self-supporting structures and $\mathcal{N}$-transitive structures, respectively. As usual, we let ε_0 and ε_1 denote the 0-noise intensity and 1-noise intensity of the MVP, respectively.

Theorem 1 shows the MVP defined on a uniformly self-supporting structure $(S, \mathcal{N})$ is ergodic if one of ε_0 and ε_1 is significantly smaller than the other. Theorem 2 is a strengthened version of Theorem 1 when $(S, \mathcal{N})$ is assumed to be $\mathcal{N}$-transitive in addition, and shows the MVP converges to the unique equilibria exponentially quickly, under similar assumptions.

Theorems 3 and 4 consider examples of $(S, \mathcal{N})$ that are not uniformly self-supporting and the behavior of the corresponding MVP. Theorem 3 considers the UMVP on low-dimensional regular trees, namely on $\mathbb{T}_3$ and $\mathbb{T}_4$, and Theorem 4 considers the OMVP on d-dimensional lattices, with $d \geq 4$. In both cases, we show that the MVP is non-ergodic when $\varepsilon_1 + \varepsilon_2$ is sufficiently small. In addition, for the UMVP on $\mathbb{T}_4$ and the OMVP on $\mathbb{Z}^d$ with $d \geq 4$, we construct uncountably many mutually singular equilibria under the same assumptions.

We now state Theorem 1.

Theorem 1. *Consider a uniformly self-supporting structure $(S, \mathcal{N})$ and the majority vote process $\{\xi_t\}_{t\geq 0}$ on $(S, \mathcal{N})$. Let ε_0 and ε_1 denote the 0-noise and 1-noise intensities. Then, for every fixed $\varepsilon_0 \in (0, 1)$, there exists a constant $\varepsilon_1' > 0$ such that, for all $\varepsilon_1 \leq \varepsilon_1'$, ξ_t is ergodic.*

Due to symmetry, for every fixed $\varepsilon_1 \in (0, 1)$, there exists a constant $\varepsilon_0' > 0$ such that, for all $\varepsilon_0 \leq \varepsilon_0'$, ξ_t is ergodic. We conjecture that the majority vote process will be ergodic whenever $\varepsilon_0 \neq \varepsilon_1$, similar to the case of the stochastic Ising process with a nontrivial external field. However, different techniques will be necessary to show this.

It is elementary to show the majority vote process, regardless of the choice of $(S, \mathcal{N})$, ε_0, and ε_1, always has at least one equilibrium. (The same is true for very general interacting particle systems, see Chapter 1, Proposition 1.8 of [9].)

Recall ξ_t is determined by the noise events, vote events, and its initial configuration. Two majority vote processes on $(S, \mathcal{N})$ are *jointly defined* if they share the same sets of vote events and noise events. (We will revisit this definition in Section 2.3.)

Theorem 2 considers $\mathcal{N}$-transitive structures $(S, \mathcal{N})$ and the corresponding majority vote process $\{\xi_t\}_{t\geq 0}$ on $(S, \mathcal{N})$. We show that such a majority vote process converges to its unique equilibrium exponentially quickly, starting from any initial configuration, under the assumption that one of ε_0 and ε_1 is significantly smaller than the other.

Theorem 2. *Consider an $\mathcal{N}$-transitive structure $(S, \mathcal{N})$ and two jointly defined majority vote processes $\{\xi_t^{(1)}\}_{t\geq 0}$ and $\{\xi_t^{(2)}\}_{t\geq 0}$ on $(S, \mathcal{N})$, where the distribution of $\xi_0^{(1)}$ is an equilibrium. Let ε_0 and ε_1 denote the* 0*-noise and* 1*-noise intensities. Then, for every $\varepsilon_0 \in (0, 1)$, there exists some $\varepsilon_1'' > 0$, not depending on $\xi_0^{(2)}$, such that, for all $\varepsilon_1 \leq \varepsilon_1''$, there exist constants $C, \alpha > 0$ such that*

$$\mathbb{P}(\xi_t^{(1)}(x) \neq \xi_t^{(2)}(x)) \leq Ce^{-\alpha t}, \quad \text{for every } x \in S \text{ and } t \geq 0. \tag{1.6}$$

Since the initial distribution of $\xi_0^{(2)}$ is chosen arbitrarily, (1.6) implies the ergodicity of ξ_t. Similarly as in the case of Theorem 1, due to symmetry, for every $\varepsilon_1 \in (0, 1)$, there exists some $\varepsilon_0'' > 0$ such that, for all $\varepsilon_0 \leq \varepsilon_0''$, there exist some constants $C, \alpha > 0$ such that (1.6) holds. It is also worth noting that both Theorems 1 and 2 apply to majority vote processes featuring non-nearest-neighbor interactions, of which illustrative examples are provided in Section 2.5.

When $(S, \mathcal{N})$ is not uniformly self-supporting, the analysis of majority vote processes on $(S, \mathcal{N})$ can become considerably more intricate. Developing a comprehensive theory akin to Theorems 1 and 2 can become a formidable task in these cases. We instead focus on some specific cases of $(S, \mathcal{N})$ that are not uniformly self-supporting, and show the nonergodicity of the corresponding majority vote processes.

Our first example is the UMVP on $\mathbb{T}_3$ and $\mathbb{T}_4$. (Recall $\mathbb{T}_d$ denotes the d-ary regular tree, which is introduced in Subsection 1.1.2.) Bramson and Gray ([6]) showed UMVP on $\mathbb{T}_d$, with $d \geq 5$, are nonergodic and possess uncountably many mutually singular equilibria. We extend their nonergodicity result to $\mathbb{T}_3$ and $\mathbb{T}_4$, and construct uncountably many mutually singular equilibria for the UMVP on $\mathbb{T}_4$. We state our results as follows.

Theorem 3. *Consider the UMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{T}_3$ and $\mathbb{T}_4$. Let ε_0 and ε_1 denote the 0-noise and 1-noise intensities. When $\varepsilon_0 + \varepsilon_1$ is sufficiently small, ξ_t is nonergodic. Moreover, the UMVP on $\mathbb{T}_4$ has uncountably many mutually singular equilibria under the same assumptions.*

Our second example is the OMVP on $\mathbb{Z}^d$, with $d \geq 4$. Theorem 4 shows the nonergodicity of the OMVP when the noise intensity is sufficiently small.

Theorem 4. *Consider the OMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{Z}^d$ with $d \geq 4$. Let ε_0 and ε_1 denote the 0-noise intensity and 1-noise intensity, respectively. When $\varepsilon_0 + \varepsilon_1$ is sufficiently small, ξ_t is nonergodic and has uncountably many mutually singular equilibria.*

Comparing with Theorem 1, Theorems 3 and 4 reveal a contrasting behavior of the MVP on structures that are not uniformly self-supporting – when $\varepsilon_0 + \varepsilon_1$ is small enough, the MVP is non-ergodic. However, in the case of the MVP on a uniformly self-supporting structure, regardless of the value of $\varepsilon_0 + \varepsilon_1$, the MVP is always ergodic when one of $\varepsilon_0, \varepsilon_1$ is sufficiently smaller than the other.

1.3 Relationship Between MVP and Other IPS

This section discusses the relationship between the majority vote process and other interacting particle systems, namely the stochastic Ising process and the voter model. The next two subsections briefly introduce these two related processes and state their connections with the majority vote process.

1.3.1 Stochastic Ising Process

The stochastic Ising process was introduced as a dynamic version of the famous Ising model ([10]).

The Ising model was initially proposed in the 1920s by Wilhelm Lenz ([11]) and has served as a prominent model for ferromagnetism in statistical mechanics. The Ising model is one of the most well-studied statistical mechanics models due to its exact solvability (on $\mathbb{Z}^d$) and its exhibition of phase transitions for $d \geq 2$, but not when $d = 1$.

The Ising model considers an undirected graph $G = (V, E)$, where each vertex $x \in V$ has a state of either -1 or $+1$, representing a magnetic dipole moment. For this reason, we refer to the state of x as the *spin at* x. Let $\beta > 0$ and $h \in \mathbb{R}$ denote two constants. Gibbs measures on the state space $\mathscr{S}' = \{1, -1\}^V$ are introduced as the probability measures μ satisfying that, for each $x \in V$,

$$\mu(\eta : \eta(x) = \sigma(x) \mid \eta(u) = \sigma(u) \text{ for all } u \neq x) = \{1 + e^{-2\beta\left(\sum_{y\sim x}\sigma(y)+h\right)\sigma(x)}\}^{-1},$$

where $y \sim x$ exactly when $(x, y) \in E$. When $x \sim y$, we say y is a *neighboring site* of x. In statistical mechanics, β and h are often referred to as the inverse temperature and the external field, respectively. A fundamental question in statistical mechanics is whether a unique Gibbs measure exists for given β and h.

The *stochastic Ising process* is an interacting particle system that has the natural reversible dynamics associated with the Ising model. The *transition rate* $c(x, \sigma)$ is defined as the rate at which $\sigma(x)$ changes sign, and is given as

$$c(x, \sigma) = \{1 + e^{2\beta(\sum_{y\sim x}\sigma(y)+h)\sigma(x)}\}^{-1}. \tag{1.7}$$

A standard result from the interacting particle systems states that the equilibria (perhaps more than one) of the stochastic Ising process are the Gibbs measures of the corresponding Ising model.

The following connection between the stochastic Ising model and the majority vote process is worth pointing out. Consider the stochastic Ising process at zero temperature ($\beta = \infty$) and having no external field ($h = 0$), which is commonly referred to as *zero-temperature Glauber dynamics.* Then its flip rate (1.7) simplifies to

$$c(x, \sigma) = \begin{cases} 0 & \textit{if } (\sum_{y\sim x}\sigma(y))\sigma(x) > 0, \\ 1/2 & \textit{if } (\sum_{y\sim x}\sigma(y))\sigma(x) = 0, \\ 1 & \textit{if } (\sum_{y\sim x}\sigma(y))\sigma(x) < 0. \end{cases}$$

In other words, the spin at x flips at rate 1 when it disagrees with the majority of spins at its neighboring sites, at rate 0 when it agrees, and at rate 1/2 when there are an equal number of $+1$ and -1 spins among its neighboring sites. These dynamics closely resemble the majority dynamics of the majority vote process.

Well-known results [12] and [13] state that, when $G = \mathbb{Z}^d$ and $h \neq 0$, the Ising model has a unique Gibbs state, and the stochastic Ising process is ergodic. Our motivation for Theorem 1 stems partially from this result, as the unbalanced noise can be viewed to some extent as an external field. We note the stochastic Ising process is a reversible Markov process, and therefore its equilibria satisfy the detailed balance. Moreover, the equilibria of the stochastic Ising process can be given explicitly, up to a normalizing constant. In contrast, the majority vote process does not have this property, which makes it more difficult to investigate.

1.3.2 Voter model

The voter model was introduced independently by Clifford and Sudbury ([14]), and by Holley and Liggett ([15]). We briefly introduce its dynamics using our notation as follows: At each nonnegative time t, each site $x \in S$ is assumed to have an opinion of either 0 or 1. Each site waits an exponential amount of time with mean 1, immediately after which its opinion updates to the opinion it observes at a site $y \in \mathcal{N}(x)$, which is chosen according to some probability distribution $p(x, \cdot)$. The transition rate $c(x, \sigma)$ for each $x \in S$ and $\sigma \in \mathscr{S} = \{0, 1\}^S$ is therefore given by

$$c(x, \sigma) = \begin{cases} \sum_{y \in \mathcal{N}(x)} p(x, y)\sigma(y) & \text{if } \sigma(x) = 0, \\ \sum_{y \in \mathcal{N}(x)} p(x, y)[1 - \sigma(y)] & \text{if } \sigma(x) = 1. \end{cases}$$

A little thought reveals that the voter model has two obvious equilibria – the all-zero and all-one configurations. Consequently, it is never ergodic.

Two important differences exist between the dynamics of the voter model and the majority vote process. First, unlike the majority vote process, where a site aligns its opinion with the absolute majority of its neighbors at a vote event, the voter model changes its opinion by selecting a neighboring site at random. Second, there is no noise in the original voter model. (Granovsky and Madras ([16]) studied the voter model with the same type of noise as the MVP and showed the noisy voter model is ergodic whenever $\varepsilon_0 + \varepsilon_1 > 0$.)

The duality theory in the IPS is heavily used in the study of the voter model: It is well-known that the voter model has the coalescing random walk as its dual process. This often reduces problems involving the voter model to substantially simpler ones involving independent random walks – a property the MVP lacks.

1.4 Previous Work on MVP and Related Processes

The majority vote process was first introduced in the 1970s by Spitzer ([17]), where it was defined on $S = \mathbb{Z}^d$, with $\mathcal{N}(x) = \{x \pm \mathbf{e}_i : 1 \leq i \leq d\}$ for each $x \in S$, and $\varepsilon_0 = \varepsilon_1$. Progress on this model has since been limited. We summarize the scant literature on the majority vote process as follows.

A *spin system* is an IPS in which each site has two possible states. Gray ([5]) demonstrated that every attractive nearest-neighbor spin system on $\mathbb{Z}$, with strictly positive transition rates, is ergodic. It follows immediately that the majority vote process on $\mathbb{Z}$, with $\mathcal{N}(x) = \{x-1, x+1\}$ and $\varepsilon_0, \varepsilon_1 > 0$, is ergodic. (It is obvious that the majority vote process, with only one of ε_0 and ε_1 positive, is also ergodic.) We note Gray's contour method relies on the nearest-neighbor interactions. Unfortunately, very little is known about the majority vote processes featuring non-nearest-neighbor interactions, even on $\mathbb{Z}$, and we are not aware of any result (other than our Theorems 1 and 2) on the majority vote process on $\mathbb{Z}^d$, $d \geq 1$, with unbalanced noise $\varepsilon_0 \neq \varepsilon_1$.

Not much is known about the MVP defined on other graphs. Recall that $\mathbb{T}_d$ denotes the d-ary regular tree. Bramson and Gray ([6]) studied the UMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{T}_d$, with $d \geq 5$, and showed that ξ_t, with sufficiently small $\varepsilon_0 + \varepsilon_1$, has uncountably many mutually singular equilibria. They also established the existence of a non-trivial basin of attraction for these equilibria, on which they showed such convergence is exponentially fast. Our Theorem 3 extends some of their findings to the UMVP on $\mathbb{T}_3$ and $\mathbb{T}_4$.

The probabilistic cellular automata (PCA) are discrete-time analogs of the interacting particle system. The majority vote process can be modified to a PCA by updating the opinion of each site at each integer time following the same rules as the MVP. Toom ([7]) proved the PCA on $\mathbb{Z}^d$ is stable under the one-sided perturbation if and only if the corresponding Toom operator is an *eroder*. A consequence is that, the discrete-time analog of the MVP on $\mathbb{Z}^d$, with $d \geq 2$ and one-sided neighborhood

$\mathcal{N}(x) = \{x + \mathbf{e}_i : 1 \leq i \leq d\}$ for every $x \in \mathbb{Z}^d$, is nonergodic when $\varepsilon_0 + \varepsilon_1$ is sufficiently small. We will establish improved results for the (continuous-time) OMVP on $\mathbb{Z}^d$, $d \geq 4$, in Theorem 4.

The UMVP on finite graphs – for example, $[-N, N]^2 \cap \mathbb{Z}^2$ – are studied in [18] and [19], which provides some evidence for the non-ergodicity of the UMVP on $\mathbb{Z}^2$ with sufficiently small $\varepsilon_0 + \varepsilon_1$. Computer simulations of the majority vote process using Monte-Carlo methods can be found in [20], [21], and [22]. Nonetheless, given the fact that every finite state irreducible aperiodic Markov process is always ergodic, it is relatively difficult to gain intuition from computer simulations. Moreover, since the computational complexity grows exponentially as the dimension increases, simulations of the MVP on graphs other than $\mathbb{Z}$ are much harder to obtain.

1.5 Overview of the Thesis

We conclude Chapter 1 by outlining the subsequent chapters of this thesis.

In Chapter 2, we introduce some fundamental tools, including graphical constructions and couplings – this is done in Sections 2.1 and 2.3, respectively. These tools will be used frequently in the proof of our main theorems in later chapters. Using these tools, we demonstrate some general results about the majority vote process, including the monotonicity (see Section 2.2) and the large noise regime (see Section 2.4). We conclude Chapter 2 with important examples of uniformly self-supporting structures and $\mathcal{N}$-transitive structures in Section 2.5.

In Chapter 3, we prove Theorem 1: the majority vote process defined on a uniformly self-supporting structure $(S, \mathcal{N})$, with one of ε_0 and ε_1 significantly smaller than the other, is ergodic. We begin with an elementary proof, in the special case where $S = \mathbb{Z}$ and $\mathcal{N}(x) = \{x-2, x-1, x, x+1, x+2\}$, in Section 3.1. We subsequently extend this approach to prove Theorem 1 in the general setting. For this, we introduce concepts like the *d-path* and *s-path* in Section 3.2. By reducing the proof of Theorem 1 to its analog for the corresponding coupled process, and employing tools such as d-paths and s-paths to estimate the marginal density of the coupled process, we complete the proof of Theorem 1 in Sections 3.3, 3.4, and 3.5. Section 3.6 discusses consequences of Theorem 1, where we apply Theorem 1 to examples introduced in Section 2.5, and derive specific bounds for the MVP to be ergodic in different scenarios, including d-dimensional lattices, triangular lattices, hexagonal lattices, and defective lattices.

In Chapter 4, we prove Theorem 2, which establishes an exponential convergence rate for the MVP defined on an $\mathcal{N}$-transitive structure $(S, \mathcal{N})$. The proof of Theorem 2 is structured around a more delicate use of s-paths, and is done in three steps: First, in Proposition 4.1, we demonstrate that, under the $\mathcal{N}$-transitive assumption, s-paths can be constructed in a "more homogeneous" manner. We next introduce the *total lifetime* of an s-path in Subsection 4.1.2 and derive its conditional density in Proposition 4.5. Lastly, in Section 4.2, we establish a sharper upper bound on the marginal density of the coupled process, using the fact that there always exists an s-path with its total lifetime greater than t. This upper bound decays exponentially in time, which completes the proof of Theorem 2. In Section 4.4, we discuss the consequences of Theorem 2, and derive specific conditions for the MVP to converge exponentially quickly to its unique equilibrium in different scenarios, including d-dimensional square lattices, triangular lattices, and hexagonal lattices.

In Chapter 5, we prove Theorem 3, which shows that the UMVP on $\mathbb{T}_3$ and $\mathbb{T}_4$, with sufficiently small noise, is non-ergodic. The $\mathbb{T}_3$ and $\mathbb{T}_4$ cases are treated differently: For $S = \mathbb{T}_3$, we first show that, using Lemma 5.10, to prove Theorem 3, it suffices to prove its δ_t analog, where δ_t is the UMVP on $\mathbb{T}_3$ with "one-sided noise". We next introduce the *t-truss*, which is a variant of the *truss* introduced by Toom ([7]). We use the t-truss to estimate the marginal density of δ_t. Theorem 3 follows after proving the marginal density of δ_t is uniformly and strictly bounded by $1/2$. For $S = \mathbb{T}_4$, we introduce a *dominating process* $\{\delta_t\}_{t\geq 0}$, motivated by the work of Bramson and Gray ([6]). We first show, using Lemma 5.5, that δ_t bounds the deviation of the UMVP from certain initial configurations. (We refer to these special initial configurations as *compatible configurations.*) We then establish an upper bound on the marginal density of δ_t in Lemma 5.6. Combining Lemma 5.5 and Lemma 5.6, we show that distinct compatible configurations lead to mutually singular equilibria of the UMVP on $\mathbb{T}_4$. Theorem 3 then follows since there are uncountably many such compatible configurations.

In Chapter 6, we prove Theorem 4, which states that the OMVP on $\mathbb{Z}^d$, $d \geq 4$, with sufficiently small $\varepsilon_0 + \varepsilon_1$, is non-ergodic and has uncountably many mutually singular equilibria. In Section 6.3, we introduce a dominating process δ_t, and establish the analog of Lemma 5.5, Proposition 6.4, which then reduces the proof of Theorem 4 to establishing an upper bound on the marginal density of δ_t. Proposition 6.5 establishes this desired upper bound. The proof of Proposition 6.5 is given in Section 6.4: We first show that δ_t is a continuous-time spin system associated with a *shrinker*, which is introduced in Definition 6.9. We then use results from [23] and [24] to derive an upper bound for $\mathbb{P}(\delta_t(x) = 1)$. Lastly, in Section 6.5, we combine these results and prove Theorem 4.

Chapter 2

Basic Tools and General Results

In this chapter, we first provide a comprehensive overview of the fundamental tools needed in the proof of our main theorems, and then state some general results about the majority vote process using these standard methodologies. We outline this Chapter as follows.

In Section 2.1, we construct the majority vote process by employing a graphical construction. The approach of graphical construction serves as the foundation for our analysis of the MVP throughout this thesis.

We next discuss the monotonicity of the MVP in Section 2.2. We first introduce the monotonicity for a spin system and state a criterion for monotonicity in (2.8). We then prove the MVP's monotonicity by verifying its transition rates satisfy (2.8). We conclude this section with some important consequences of the MVP's monotonicity in Corollary 2.4.

In Section 2.3, we apply the graphical construction of the MVP to jointly define multiple copies of the MVP on the same probability space, and introduce the coupled process. After introducing the coupled process in Definition 2.5, we state some of its properties in Proposition 2.6, and derive an ergodicity criterion for the MVP in Proposition 2.7 using the coupled process. This criterion will be applied in the proofs of Theorems 1 and 2.

In Section 2.4, we introduce the large noise regime of the MVP – when the noise is "strong" enough, MVP is ergodic and converges exponentially quickly to the unique equilibrium starting from any initial distribution.

We conclude this chapter with Section 2.5, which provides examples of uniformly self-supporting structures and $\mathcal{N}$-transitive structures. We will revisit some of these examples in later chapters.

2.1 Graphical Construction

As usual, we let $\{\xi_t\}_{t\geq 0}$ denote the majority vote process on a structure $(S, \mathcal{N})$, where each $\xi_t(x)$ denotes the opinion of x at time t. The main idea of the MVP's graphical construction is to construct a timeline for each $x \in S$ that represents the evolution of its opinion $\xi_t(x)$. Since sample paths of the majority vote process can be obtained directly from these timelines, the graphical construction will serve as a useful tool for analyzing the process.

In this section, we first introduce the MVP's graphical construction on the space-time set $S \times [0, \infty)$. We next state the graphical induction principle, in Proposition 2.3, as a direct consequence of the MVP's graphical construction. This principle will be repeatedly applied in our proofs. Besides, another immediate advantage of employing such a graphical construction is to provide a simultaneous coupling of copies of the MVP with different initial configurations. In Section 2.3, we will discuss this idea carefully and use it to introduce the coupled process, on which the proofs of our main results, namely Theorems 1 and 2, rely heavily.

A *Poisson point process* on $(0, \infty)$ with intensity λ places points randomly such that the following are satisfied. The number of random points in $(a, b]$, denoted by $N(a, b)$, is Poisson distributed with mean $\lambda(b - a)$; and $N(a_0, b_0)$, $N(a_1, b_1)$, $\cdots$, $N(a_k, b_k)$ are independent given $a_{i-1} < b_{i-1} \leq a_i$, for all $1 \leq i \leq k$.

We next introduce 3 types of *event points* – 0-*noise points*, 1-*noise points*, and *vote points*, using a family of independent Poisson point processes.

Definition 2.1. *For given $\varepsilon_0, \varepsilon_1 \in [0, 1)$, with $\varepsilon_0 + \varepsilon_1 \leq 1$, and each $x \in S$, let $\{X_x^{(0)}\}_{x\in S}$, $\{X_x^{(1)}\}_{x\in S}$, and $\{X_x^{(2)}\}_{x\in S}$ denote families of independent Poisson point processes, placed on the timeline $\{x\} \times [0, \infty)$, with rates ε_0, ε_1 and $1 - \varepsilon_0 - \varepsilon_1$, respectively. Then, for each $x \in S$ and $t > 0$, we introduce the following terminology.*

(2.1.a) We say (x, t) is a 0-*noise point if and only if $(x, t) \in X_x^{(0)}$.*

(2.1.b) We say (x,t) is a 1*-noise point if and only if $(x,t) \in X_x^{(1)}$.*

(2.1.c) We say (x,t) is a vote point if and only if $(x,t) \in X_x^{(2)}$.

(2.1.d) We say (x,t) is an event point if and only if $(x,t) \in X_x^{(0)} \cup X_x^{(1)} \cup X_x^{(2)}$.

Since the Poisson point process is a continuous-time stochastic process, and that all $X_x^{(i)}$, for $x \in S$ and $i = 0,1,2$, are independent of each other, we can assume the following by ignoring a null event: For each $x, y \in S$ and $i, j \in 0,1,2$,

$$X_x^{(i)} \cap X_y^{(j)} = \emptyset, \quad \text{whenever } (x,i) \neq (y,j). \tag{2.1}$$

In other words, at each given time $t \geq 0$, there exists no more than one $y \in S$ with exactly one $i \in 0,1,2$ such that $(y,t) \in X_y^{(i)}$. For the same reason, we also assume that, for each $x \in S$,

$$\text{there is no accumulation point in } X_x^{(0)} \cup X_x^{(1)} \cup X_x^{(2)}, \tag{2.2}$$

i.e., for each $x \in S$ and any sequence $\{(x,t_i)\}_{i\in\mathbb{N}} \subseteq X_x^{(0)} \cup X_x^{(1)} \cup X_x^{(2)}$, the limit $\lim_{n\to\infty} t_i$ does not exist.

We will refer to $\{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x\in S}$ that satisfies (2.1) and (2.2) as a *realization of the Poisson clock*.

We assume that the majority vote process $\{\xi_t\}$ is RCLL (right continuous with left limits), and set

$$\xi_{t-} = \begin{cases} \lim_{s\nearrow t} \xi_s & \text{if } t > 0, \\ \xi_0 & \text{if } t = 0. \end{cases} \tag{2.3}$$

For every $x \in S$, we set $e(x,0) = 0$ and, for $t > 0$,

$$e(x,t) := \sup\{s < t : s = 0 \text{ or } (x,s) \in X_x^{(0)} \cup X_x^{(1)} \cup X_x^{(2)}\}. \tag{2.4}$$

For every spin system $\{\xi_t\}_{t\geq 0}$ that is defined using the graphical construction, we say ξ_t satisfies the *continuation rule* if the following is satisfied:

$$\xi_t(x) = \xi_{e(x,t)}(x), \quad \text{if } (x,t) \text{ is not an event point.} \tag{2.5}$$

We assume that the majority vote process satisfies the continuation rule.

For every $\xi \in \mathscr{S}$ and every $x \in S$, we define the *majority vote function* $M(\xi, x)$ as follows.

$$M(\xi, x) = \begin{cases} 1 & \text{if } \sum_{y \in \mathcal{N}(x)} \xi(y) > |\mathcal{N}(x)|/2, \\ 0 & \text{if } \sum_{y \in \mathcal{N}(x)} \xi(y) < |\mathcal{N}(x)|/2, \\ \xi(x) & \text{otherwise.} \end{cases} \tag{2.6}$$

In other words, $M(\xi, x)$ is the absolute majority opinion in $\mathcal{N}(x)$; if there are an equal number of sites having opinion 0 and 1 among $\mathcal{N}(x)$, then $M(\xi, x)$ is the current opinion of x.

We now construct a majority vote process using $\{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x \in S}$.

Definition 2.2 (Majority Vote Process)**.** *Consider a structure $(S, \mathcal{N})$ and constants $\varepsilon_0, \varepsilon_1 \in [0, 1)$, with $\varepsilon_0 + \varepsilon_1 \leq 1$. Let $\{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x \in S}$ denote a realization of the Poisson clock for this given choice of ε_0 and ε_1. A process $\{\xi_t\}_{t \geq 0}$ with the state space $\mathscr{S}$ is a majority vote process on $(S, \mathcal{N})$ if it satisfies the continuation rule (2.5), and the following three rules.*

(2.2.a) 0*-noise rule: For each $(x, t) \in X_x^{(0)}$, $\xi_t(x) = 0$.*

(2.2.b) 1*-noise rule: For each $(x, t) \in X_x^{(1)}$, $\xi_t(x) = 1$.*

(2.2.c) Majority vote rule: For each $(x, t) \in X_x^{(2)}$, $\xi_t(x) = M(\xi_{t-}, x)$.

Since the waiting time between each arrival of the Poisson point process is independently exponentially distributed, the majority vote process given by Definition 2.2 is consistent with the one introduced at the beginning of Chapter 1. Moreover, given an initial configuration and a realization of the Poisson clock, a sample path of the majority vote process can be constructed by Definition 2.2; therefore the majority vote process is almost surely well-defined.

Another merit of employing such a graphical construction for the MVP is that we can apply a graphical induction principle to prove that certain assertions $\mathcal{A}(x, t)$ hold almost surely for every $x \in S$ and $t \geq 0$. We state this graphical induction principle in the next proposition. Proposition 2.3 is a direct generalization of Proposition A.1 in [6]; hence we omit its proof here.

Proposition 2.3. *Let $\{\mathcal{A}(x, t)\}_{x \in S, t \geq 0}$ denote a collection of events satisfying the following conditions for all $x \in S$:*

1. *$\mathcal{A}(x,0)$ holds.*

2. *If $\mathcal{A}(y,e(y,t))$ holds for all $y \in \mathcal{N}(x) \cup \{x\}$, then $\mathcal{A}(x,t)$ holds.*

Then, off a set of probability 0 *that does not depend on $\{\mathcal{A}(x,t)\}_{x \in S, t \geq 0}$, $\mathcal{A}(x,t)$ holds for all $x \in S$ and $t \geq 0$.*

2.2 Monotonicity of MVP

This section explores the monotonicity of the majority vote process. We first introduce the monotonicity of general spin systems and state the criterion, in (2.8), for a spin system to be monotone. We then establish the monotonicity of the MVP by verifying its transition rates, denoted by $\{c(\xi,x)\}_{\xi \in \mathscr{S}, x \in S}$, satisfying (2.8). Lastly, we discuss some consequences of the MVP's monotonicity in Corollary 2.4. In particular, we use the MVP's monotonicity to derive an ergodicity criterion in (2.10)

Recall that a spin system is an interacting particle system on S, for which each site $x \in S$ has two possible values. Hence the majority vote process, defined in Definition 2.2, is a spin system. Other IPS introduced previously in Chapter 1 that are spin systems include the stochastic Ising process and the voter model (see Subsection 1.3).

We assume the state space for a spin system is $\mathscr{S} = \{0,1\}^S$. Let $c(\xi,x)$, the *transition rate*, denote the rate at which $\xi(x)$ changes from $\xi(x)$ to $1-\xi(x)$ when the spin system is in state ξ. It then follows from Definition 2.2 that the transition rates $c(x,\xi)$ of the majority vote process $\{\xi_t\}_{t \geq 0}$ can be formulated as

$$c(\xi,x) = \begin{cases} \varepsilon_1 & \text{if } \xi(x) = M(\xi,x) = 0, \\ 1-\varepsilon_0 & \text{if } \xi(x) = 1 - M(\xi,x) = 0, \\ \varepsilon_0 & \text{if } \xi(x) = M(\xi,x) = 1, \\ 1-\varepsilon_1 & \text{if } \xi(x) = 1 - M(\xi,x) = 1, \end{cases} \tag{2.7}$$

where ε_0 and ε_1 denote the 0-noise intensity and 1-noise intensity of the MVP, respectively, and $M(\xi,x)$ is introduced in (2.6). To see (2.7), for each $\xi \in \mathscr{S}$, on the one hand, if $\xi(x) = M(\xi,x) = 0(1)$, then only 1-noise events (0-noise events) can change $\xi(x)$, which contribute to the first and third cases on the right-hand side of (2.7). On the other hand, if $\xi(x) \neq M(\xi,x) = 0(1)$, then both vote events and 0-noise events

(1-noise events) can change $\xi(x)$, which contribute to the second and fourth cases on the right-hand side of (2.7).

For $\xi_1, \xi_2 \in \mathscr{S}$, we write $\xi_1 \leq \xi_2$ if $\xi_1(x) \leq \xi_2(x)$ for every $x \in S$. A spin system is *monotone* if, whenever $\xi_1 \leq \xi_2$, for each $x \in S$, its transition rates $c(\xi_1, x)$ and $c(\xi_2, x)$ satisfy

$$\begin{aligned} c(\xi_1, x) &\leq c(\xi_2, x) \quad \text{if } \xi_1(x) = \xi_2(x) = 0, \text{ and} \\ c(\xi_1, x) &\geq c(\xi_2, x) \quad \text{if } \xi_1(x) = \xi_2(x) = 1. \end{aligned} \tag{2.8}$$

We next demonstrate the monotonicity of the majority vote process by verifying that $\{c(\xi, x)\}_{\xi \in \mathscr{S}, x \in S}$ defined in (2.7) satisfies (2.8). Supposing $\xi_1 \leq \xi_2$, it follows immediately from (2.6) that

$$M(\xi_1, x) \leq M(\xi_2, x) \quad \text{for each } x \in S. \tag{2.9}$$

We first consider the case where $\xi_1(x) = \xi_2(x) = 0$. Since the transition rates in (2.7) are determined by $\xi(x)$ and $M(\xi, x)$, when $M(\xi_1, x) = M(\xi_2, x)$, we have $c(\xi_1, x) = c(\xi_2, x)$. Whereas when $M(\xi_2, x) = 1$ and $M(\xi_1, x) = 0$, since $\varepsilon_0 + \varepsilon_1 \leq 1$ is assumed in Definition 2.2, we have

$$c(\xi_2, x) = 1 - \varepsilon_0 \geq \varepsilon_1 = c(\xi_1, x).$$

Therefore, $c(\xi_2, x) \geq c(\xi_1, x)$ for each $x \in S$ with $\xi_1(x) = \xi_2(x) = 0$. Similarly, when $\xi_1(x) = \xi_2(x) = 1$, it also follows from (2.9) that $c(\xi_1, x) \geq c(\xi_2, x)$. Therefore the MVP is a monotone spin system.

We also note that (2.8) reveals a key feature of a monotone spin system: Its evolution tends to make a site agree with its neighboring sites. For instance, when ξ_2 has more sites with opinion 1 than ξ_1, ξ_2 has a higher transition rate at x where $\xi_2(x) = 0$, and vice versa.

Our next Corollary 2.4 states some consequences of the MVP's monotonicity. We note that Corollary 2.4 follows directly from Chapter III, Theorem 2.3 of [9]; hence we omit its proof here.

Corollary 2.4. *Consider two copies of the majority vote process on $(S, \mathcal{N})$, denoted by $\{\xi_t^{(0)}\}_{t \geq 0}$ and $\{\xi_t^{(1)}\}_{t \geq 0}$, such that*

$$\xi_0^{(0)}(x) \equiv 0, \quad \xi_0^{(1)}(x) \equiv 1, \quad \textit{for each } x \in S.$$

Setting $p_0(x, t) = \mathbb{P}(\xi_t^{(0)}(x) = 0)$, and $p_1(x, t) = \mathbb{P}(\xi_t^{(1)}(x) = 1)$, the following hold.

1. *For fixed $x \in S$, $p_0(x,t)$ and $p_1(x,t)$, as functions of t, are non-increasing:*

$$p_0(x,t) \leq p_0(x,s), \quad p_1(x,t) \leq p_1(x,s), \quad \text{for every } s \leq t.$$

2. *Let $\mu_t^{(0)}$ and $\mu_t^{(1)}$ denote the probability measures of $\xi_t^{(0)}$ and $\xi_t^{(1)}$, respectively. Then, there exist some $\nu^{(0)}, \nu^{(1)} \in \mathcal{P}$ such that $\mu_t^{(0)}$ and $\mu_t^{(1)}$ converge weakly to $\nu^{(0)}$ and $\nu^{(1)}$, respectively. Moreover, $\nu^{(0)}$ and $\nu^{(1)}$ are equilibria of the majority vote process.*

3. *The majority vote process ξ_t is ergodic if and only if*

$$\lim_{t\to\infty} p_0(x,t) = 1 - \lim_{t\to\infty} p_1(x,t) \quad \text{for every } x \in S. \tag{2.10}$$

2.3 Coupling

Coupling is one of the most important and most generally applicable techniques in interacting particle systems. Generally speaking, coupling is a construction of two or more stochastic processes on a common probability space. In this section, we introduce a natural coupling for the majority vote process by using its graphical construction, which was introduced in Section 2.1. We then define the *coupled majority vote process*, (or, for short, the *coupled process*), in Definition 2.5, such that any two majority vote processes with different initial configurations (in our application, often the all-zero and all-one configurations) can be realized on the same probability space. We then use this coupled majority vote process to derive an ergodicity criterion for the MVP (Proposition 2.7). We conclude this section by introducing terminology such as exponentially quick convergence and exponentially mixing.

As usual, we let $\{\xi_t\}_{t\geq 0}$ denote the majority vote process on a structure $(S, \mathcal{N})$. Recall that, by Definition 2.2, ξ_t is determined by its initial configuration and a realization of the Poisson clock. Let $\{\xi_t^{(1)}\}_{t\geq 0}$ and $\{\xi_t^{(2)}\}_{t\geq 0}$ denote two copies of the MVP, having different initial distributions μ_1, $\mu_2 \in \mathcal{P}$. (Recall $\mathcal{P}$ denotes the collection of all probability measures on $\mathscr{S}$.) We can construct $\xi_t^{(1)}$ and $\xi_t^{(2)}$ on the same probability space, by first introducing a probability measure μ on $\mathscr{S} \times \mathscr{S}$, with μ_1 and μ_2 as its two marginal distributions, and then let $\xi_t^{(1)}$ and $\xi_t^{(2)}$ share the same set of 0-noise points, 1-noise points and vote points in their graphical constructions. Then,

$\{(\xi_t^{(1)}, \xi_t^{(2)})\}_{t\geq 0}$ itself can be viewed as an interacting particle system, with its state space being $\mathscr{S}' = \{(0,0),(0,1),(1,0),(1,1)\}^S$. We refer to $\{(\xi_t^{(1)}, \xi_t^{(2)})\}_{t\geq 0}$, which is often denoted by $\{\eta_t\}_{t\geq 0}$ in our notation, as the *coupled majority vote process.* (See [25] for a more general version of coupling.) We also say $\xi_t^{(1)}$ and $\xi_t^{(2)}$ are *jointly defined on the same probability space.*

We next introduce the graphical construction of the *coupled majority vote process*, denoted by $\{\eta_t\}_{t\geq 0}$, in a similar way as Definition 2.2: The coupled majority vote process $\{\eta_t\}_{t\geq 0}$ is defined on the state space $\mathscr{S}'$ such that it satisfies the continuation rule (2.5), i.e.,

$$\eta_t(x) = \eta_{p(x,t)}(x) \text{ whenever } (x,t) \text{ is not an event point.} \tag{2.11}$$

Since the coupled majority vote process $\{\eta_t\}_{t\geq 0}$ is assumed to satisfy the continuation rule, we define

$$\eta_{t-} = \begin{cases} \lim_{s\nearrow t} \eta_s & \text{if } t > 0, \\ \eta_0 & \text{if } t = 0. \end{cases} \tag{2.12}$$

We define the coupled majority vote process counterpart of the majority vote function for every $\eta = (\xi_1, \xi_2) \in \mathscr{S} \times \mathscr{S}$ by setting

$$\overline{M}(\eta, x) = (M(\xi_1, x), M(\xi_2, x)) \text{ for every } x \in S,$$

where $M(\xi, x)$ is defined in (2.6). Then, the graphical construction of the coupled majority vote process can be given as follows.

Definition 2.5 (Graphical construction of coupled majority vote process)**.** *Given a structure $(S, \mathcal{N})$ and constants $\varepsilon_0, \varepsilon_1 \in [0,1]$, with $\varepsilon_0 + \varepsilon_1 \leq 1$, let $\{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x\in S}$ denote a realization of the Poisson clock, which was introduced in Definition 2.1. A process $\{\eta_t\}_{t\geq 0}$ with state space $\mathscr{S}'$ is a coupled majority vote process on $(S, \mathcal{N})$ if it satisfies the continuation rule (2.11) as well as the following rules.*

(2.5.a) 0*-noise rule: For each $(x,t) \in X_x^{(0)}$, $\eta_t(x) = (0,0)$.*

(2.5.b) 1*-noise rule: For each $(x,t) \in X_x^{(1)}$, $\eta_t(x) = (1,1)$.*

(2.5.c) Majority vote rule: For each $(x,t) \in X_x^{(2)}$, $\eta_t(x) = \overline{M}(\eta_{t-}, x)$.

For simplicity, we will refer to the coupled majority vote process as the *coupled process*. We next, in Propositions 2.6 and 2.7, state some of the most important properties and applications of the coupled process.

Proposition 2.6. *Consider the coupled process* $\eta_t = (\xi_t^{(1)}, \xi_t^{(2)})$ *satisfying* $\xi_0^{(1)}(x) \le \xi_0^{(2)}(x)$ *for every* $x \in S$*. Let* μ_t*,* $t \ge 0$*, denote the probability measure of each* η_t*. Then,*

$$\mu_t\{\xi_t^{(1)}(x) \le \xi_t^{(2)}(x) \textit{ for every } x \in S\} = 1. \tag{2.13}$$

Proof. We apply the graphical induction principle to prove (2.13). According to Proposition 2.3, since $\xi_0^{(1)}(x) \le \xi_0^{(2)}(x)$ for every $x \in S$, to prove (2.13), it suffices to show that $\xi_{e(y,t)}^{(1)}(y) \le \xi_{e(y,t)}^{(2)}(y)$ for every $y \in \mathcal{N}(x) \cup \{x\}$ implies $\xi_t^{(1)}(x) \le \xi_t^{(2)}(x)$.

If (x,t) is not an event point, it follows from the continuation rule (2.11) that $\xi_t^{(1)}(x) = \xi_{e(x,t)}^{(1)}(x)$ and $\xi_t^{(2)}(x) = \xi_{e(x,t)}^{(2)}(x)$. Then $\xi_t^{(1)}(x) \le \xi_t^{(2)}(x)$ follows from the assumption $\xi_{e(x,t)}^{(1)}(x) \le \xi_{e(x,t)}^{(2)}(x)$.

We now assume (x,t) is an event point and discuss different scenarios about the type of the event point (x,t).

If (x,t) is either a 0-noise point or a 1-noise point, we have $\xi_t^{(1)}(x) = \xi_t^{(2)}(x)$ by (2.5.a) and (2.5.b), implying $\xi_t^{(1)}(x) \le \xi_t^{(2)}(x)$.

We now assume (x,t) is a vote point. Under the assumption that $\xi_{e(y,t)}^{(1)}(y) \le \xi_{e(y,t)}^{(2)}(y)$ for all $y \in \mathcal{N}(x) \cup \{x\}$, it follows from the continuation rule (2.5) that $\xi_{t-}^{(1)}(y) \le \xi_{t-}^{(2)}(y)$ for all $y \in \mathcal{N}(x) \cup \{x\}$. According to (2.6), we have $M(\xi_{t-}^{(1)}, x) \le M(\xi_{t-}^{(2)}, x)$, which implies $\xi_t^{(1)}(x) \le \xi_t^{(2)}(x)$. This completes the proof of (2.13). $\square$

We next derive an ergodicity criterion for the majority vote process in terms of η_t. This criterion will be applied in the proof of Theorem 1 (Chapter 3), which reduces the proof of ergodicity of the majority vote process to that of (2.14). We also note that Proposition 2.7 is a consequence of Corollary 2.4 and Proposition 2.6, together with some extra work.

Proposition 2.7. *Consider a structure* $(S, \mathcal{N})$ *and the coupled process* $\{\eta_t\}_{t \ge 0}$*, with* $\eta_t = (\xi_t^{(1)}, \xi_t^{(2)})$ *for each* $t \ge 0$*, satisfying*

$$\eta_0(x) \equiv (0,1) \quad \textit{for every } x \in S.$$

Then, for each fixed $x \in S$, $\mathbb{P}(\eta_t(x) = (0,1))$, as a function of t, is non-increasing. Furthermore, letting $\{\xi_t\}_{t\geq 0}$ denote the majority vote process on $(S, \mathcal{N})$, whose 0-noise intensity and 1-noise intensity are the same as η_t, then, ξ_t is ergodic if and only if

$$\lim_{t\to\infty} \mathbb{P}(\eta_t(x) = (0,1)) = 0 \quad \text{for every } x \in S. \tag{2.14}$$

Proof. Since $\xi_t^{(1)}$, $\xi_t^{(2)}$ are jointly defined and satisfies $\xi_0^{(1)}(x) \leq \xi_0^{(2)}(x)$ for all $x \in S$, Proposition 2.6 implies

$$\mathbb{P}(\xi_t^{(1)}(x) \leq \xi_t^{(2)}(x)) = 1 \quad \text{for each } x \in S \text{ and } t \geq 0. \tag{2.15}$$

We therefore obtain

$$\mathbb{P}(\eta_t(x) = (0,1)) = 1 - \mathbb{P}(\eta_t(x) = (0,0)) - \mathbb{P}(\eta_t(x) = (1,1)). \tag{2.16}$$

We can rewrite $\mathbb{P}(\eta_t(x) = (0,0))$ in terms of $\xi_t^{(1)}(x)$ and $\xi_t^{(2)}(x)$ as follows.

$$\begin{aligned} \mathbb{P}(\eta_t(x) = (0,0)) &= \mathbb{P}(\xi_t^{(1)}(x) = 0 \mid \xi_t^{(2)}(x) = 0)\mathbb{P}(\xi_t^{(2)}(x) = 0) \\ &= \mathbb{P}(\xi_t^{(2)}(x) = 0), \end{aligned} \tag{2.17}$$

where the last equation follows from $\mathbb{P}(\xi_t^{(1)}(x) \leq \xi_t^{(2)}(x)) = 1$. Similarly, we also obtain

$$\mathbb{P}(\eta_t(x) = (1,1)) = \mathbb{P}(\xi_t^{(1)}(x) = 1). \tag{2.18}$$

On the other hand, $\xi_t^{(1)}$ and $\xi_t^{(2)}$ are copies of the MVP with the all-zero and all-one initial configurations, respectively. Therefore, for each $x \in S$ and $t \geq 0$, we have

$$\mathbb{P}(\xi_t^{(1)}(x) = 1) = 1 - p_0(x,t), \quad \mathbb{P}(\xi_t^{(2)}(x) = 0) = 1 - p_1(x,t), \tag{2.19}$$

where p_0 and p_1 were introduced in Corollary 2.4. Combining (2.17), (2.18), and (2.19), we obtain that

$$\mathbb{P}(\eta_t(x) = (0,1)) = p_0(x,t) + p_1(x,t) - 1. \tag{2.20}$$

According to Corollary 2.4, with x fixed, both $p_0(x,t)$ and $p_1(x,t)$ are non-increasing in t. Therefore $\mathbb{P}(\eta_t(x) = (0,1))$ is also non-increasing as a function of t.

Since $\lim_{t\to\infty} p_0(x,t)$ and $\lim_{t\to\infty} p_1(x,t)$ exist for each $x \in S$, by Corollary 2.6, it follows from (2.10) and (2.20) that ξ_t is ergodic if and only if (2.14) holds. This completes the proof of Proposition 2.7. $\square$

We will use Proposition 2.7 as the criterion to prove the ergodicity of the MVP under the assumptions of Theorem 1. For this reason, unless otherwise stated, we assume the coupled process η_t always has its initial configuration $\eta_0(x) \equiv (0,1)$ for all $x \in S$, throughout this thesis.

We next introduce exponentially quick convergence and exponential mixing and state a criterion using the coupled process.

Definition 2.8. *Let $\{\xi_t\}_{t\geq 0}$ and $\{\xi'_t\}_{t\geq 0}$ denote two jointly defined copies of the majority vote process on $(S,\mathcal{N})$ with initial distributions μ and μ', respectively. Suppose μ' is an equilibrium for the MVP. If there exist constants a and r such that*

$$\mathbb{P}(\xi_t(x) \neq \xi'_t(x)) \leq ae^{-rt} \text{ for all } x \in S \text{ and } t \geq 0, \tag{2.21}$$

then we say $\{\xi_t\}_{t\geq 0}$ converges exponentially quickly to μ' at rate r. Moreover, if (2.21) holds for $\mu = \mu'$ and independent initial configurations ξ_0 and ξ'_0, we say the majority vote process is exponentially mixing in μ' at rate r.

We conclude this section with Proposition 2.9, which states that, to show the majority vote process is ergodic and converges exponentially quickly to the unique equilibrium, it suffices to prove a coupled-process analog of (2.21) as follows.

Proposition 2.9. *Consider the coupled process $\{\eta_t\}_{t\geq 0}$ on $(S,\mathcal{N})$ with its initial distribution $\eta_0(x) \equiv (0,1)$ for every $x \in S$. Let $\{\xi_t\}_{t\geq 0}$ denote the majority vote process on $(S,\mathcal{N})$ with the same 0-noise intensity and 1-noise intensity. Suppose that there exist some constants $a, r > 0$ such that η_t satisfies*

$$\mathbb{P}(\eta_t(x) = (0,1)) \leq ae^{-rt} \quad \text{for every } x \in S \text{ and } t \geq 0. \tag{2.22}$$

Then, ξ_t is ergodic and converges exponentially quickly to the unique equilibrium ν starting from any initial distribution. In particular, ξ_t is exponentially mixing in ν.

We note Proposition 2.9 is a consequence of Proposition 2.6 and the triangle inequality.

Proof. For each $x \in S$, (2.22) implies that $\lim_{t\to\infty} \mathbb{P}(\eta_t(x) = (0,1)) = 0$. Then, it follows from Proposition 2.7 that ξ_t is ergodic. Let ν denote the unique equilibrium of ξ_t.

Let $\xi_t^{(0)}$, $\xi_t^{(1)}$, $\xi_t^{(2)}$, $\xi_t^{(3)}$ denote four jointly defined copies of the majority vote process, where $\xi_0^{(0)}(x) \equiv 0$ and $\xi_0^{(1)}(x) \equiv 1$ for all $x \in S$, and $\xi_0^{(2)}$ has ν as its initial distribution. We claim that, for each $x \in S$ and $t \geq 0$,

$$\mathbb{P}(\xi_t^{(0)}(x) \leq \xi_t^{(2)}(x), \xi_t^{(3)}(x) \leq \xi_t^{(1)}(x)) = 1. \tag{2.23}$$

We now prove (2.23): Since $\xi_t^{(0)}$ and $\xi_t^{(2)}$ are jointly defined copies of the MVP, with $\xi_0^{(0)}(x) = 0 \leq \xi_0^{(2)}(x)$ for each $x \in S$, it follows from Proposition 2.6 that, for each $x \in S$ and $t \geq 0$,

$$\mathbb{P}(\xi_t^{(0)}(x) \leq \xi_t^{(2)}(x)) = 1.$$

Similarly, we can obtain that, for each $x \in S$ and $t \geq 0$,

$$\mathbb{P}(\xi_t^{(0)}(x) \leq \xi_t^{(3)}(x)) = \mathbb{P}(\xi_t^{(2)}(x) \leq \xi_t^{(1)}(x)) = \mathbb{P}(\xi_t^{(3)}(x) \leq \xi_t^{(1)}(x)) = 1.$$

This completes the proof of (2.23).

As a direct consequence of (2.23), we obtain that, for each $x \in S$ and $t \geq 0$,

$$\mathbb{P}(\xi_t^{(2)}(x) \neq \xi_t^{(3)}(x)) \leq \mathbb{P}(\xi_t^{(0)}(x) \neq \xi_t^{(1)}(x)) = \mathbb{P}(\eta_t(x) = (0,1)). \tag{2.24}$$

Since the initial distribution of $\xi_0^{(3)}$ can be chosen arbitrarily, it follows from (2.22) and (2.24) that, ξ_t converges exponentially quickly to ν, starting from any initial distribution. Moreover, setting the distribution of $\xi_0^{(3)}$ as ν and assuming that $\xi_0^{(2)}$ and $\xi_0^{(3)}$ are independent, then it follows from (2.22) and (2.24) that ξ_t is exponentially mixing in ν. This completes the proof of Proposition 2.9. $\square$

2.4 Large Noise Regime

In this section, we will apply a general result from the interacting particle system literature to show that, when the 0-noise and 1-noise intensities, denoted by ε_0 and ε_1, satisfy the lower bounds in (2.28), with minimal assumptions on $(S, \mathcal{N})$, the majority vote process on $(S, \mathcal{N})$ is ergodic and converges exponentially quickly to the unique equilibrium. We will refer to values of ε_0 and ε_1 with this behavior as the *large noise regime* of the majority vote process. The large noise regime can be viewed as the MVP analog of the paramagnetic phase of various statistical mechanics models, when the system is in high temperatures (above the Curie temperature). We note that a similar large noise regime exists for a wide class of interacting particle systems: In particular, any IPS of finite range, when perturbed with a strong enough independent noise, is ergodic([3], [26], [27], [28]).

The structure of this section is as follows. We first introduce the *minimal transition rate*, and state an ergodicity criterion that applies to a wide range of spin systems. We then verify that when $\varepsilon_0, \varepsilon_1$ satisfy (2.28), the MVP meets this criterion, therefore is ergodic, and converges exponentially quickly to the unique equilibrium.

We will consider a metric space S, equipped with a metric $d(\cdot,\cdot) : S \times S \to [0,\infty)$. Recall that $\{c(\xi,x)\}_{\xi\in\mathscr{S},x\in S}$ denotes the transition rate of a spin system on S, where $\mathscr{S} = \{0,1\}^S$ is the state space. A spin system on S is of *finite range* if there exists a constant M such that $c(\xi_1,x) = c(\xi_2,x)$ holds for each $x \in S$ and every $\xi_1,\xi_2 \in \mathscr{S}$ satisfying $\xi_1(y) = \xi_2(y)$ whenever $d(x,y) \leq M$. A little thought reveals that the majority vote process, as well as the stochastic Ising process and the voter model, are all spin systems of finite range.

For a spin system of a finite range, we define its *minimal transition rate* $\tilde{\varepsilon}$ as follows.

$$\tilde{\varepsilon} = \inf_{x\in S}\Big(\inf_{\xi_1,\xi_2}[c(\xi_1,x) + c(\xi_2,x)]\Big), \tag{2.25}$$

where the inner fold infimum is taken over all $\xi_1,\xi_2 \in \mathscr{S}$ satisfying $\xi_1(y) = \xi_2(y)$ for all $y \neq x$, and $\xi_1(x) = 1 - \xi_2(x)$, i.e., ξ_1 and ξ_2 only differ at x. From its definition, it is obvious that $\tilde{\varepsilon}$ characterizes a minimal transition rate and is independent of the configuration.

Well-known results ([3], [26], [27], [28]) in the interacting particle systems literature assert that the IPS is ergodic when the dependence of $c(\xi,x)$ on ξ and x, which will be characterized by $\tilde{M}$ in (2.26), is weak relative to an underlying minimal transition rate, which is measured by $\tilde{\varepsilon}$. We reformulate this ergodicity criterion in terms of spin systems as follows.

Proposition 2.10. *Consider a spin system of finite range on S, with transition rates $\{c(\xi,x)\}_{\xi\in\mathscr{S},x\in S}$. Let $\tilde{\varepsilon}$ denote the minimal transition rate, defined in (2.25). We set*

$$\tilde{M} = \sup_{x\in S}\sum_{u\neq x}\sup_{\xi_1,\xi_2}|c(\xi_1,x) - c(\xi_2,x)|, \tag{2.26}$$

where the supremum is taken over all $\xi_1,\xi_2 \in \mathscr{S}$ satisfying $\xi_1(x) = \xi_2(x)$ for all $x \neq u$, and $\xi_1(u) \neq \xi_2(u)$. Then, if $\tilde{M} < \tilde{\varepsilon}$, the spin system is ergodic. Under the same *assumptions, let ν denote the unique equilibrium and $\mu \in \mathcal{P}$ denote an arbitrary probability measure. We consider two jointly defined copies of this spin system, denoted by $\{\xi_t^{(0)}\}_{t\geq 0}$ and $\{\xi_t^{(1)}\}_{t\geq 0}$, whose initial distributions are ν and μ, respectively. Then, for each $x \in S$ and $t \geq 0$,*

$$\mathbb{P}(\xi_t^{(0)}(x) \neq \xi_t^{(1)}(x)) \leq \left(\sup_{x\in S, \xi\in\mathscr{S}} c(\xi, x) \right) \frac{e^{-(\tilde{\varepsilon}-\tilde{M})t}}{\tilde{\varepsilon} - \tilde{M}}. \tag{2.27}$$

We note that, since the spin system is assumed to be of finite range, only finitely many terms in the summation of (2.26) are non-zero, therefore $\tilde{M}$ is always finite. We next apply Proposition 2.10 to the MVP and obtain the following ergodicity theorem.

Proposition 2.11. *Consider the majority vote process $\{\xi_t\}_{t\geq 0}$ on $(S, \mathcal{N})$, where $(S, \mathcal{N})$ satisfies $\mathcal{N}(x) \leq K$ for every $x \in S$. Let ε_0 and ε_1 denote its 0-noise intensity and 1-noise intensity, respectively. Assuming ε_0 and ε_1 satisfy*

$$\varepsilon_0, \varepsilon_1 > \frac{K+1}{2K+4}, \tag{2.28}$$

then $\{\xi_t\}_{t\geq 0}$ is ergodic and converges exponentially quickly to the unique equilibrium, starting from any initial distribution.

Proof. We first derive $\tilde{\varepsilon}$ and $\tilde{M}$ for the majority vote process. Recall the transition rate $\{c(\xi, x)\}_{\xi\in\mathscr{S}, x\in S}$ of the MVP is given in (2.7).

We begin with $\tilde{\varepsilon}$. Fix any arbitrary $x \in S$. For every ξ_1, ξ_2 with $\xi_1(y) = \xi_2(y)$ for all $y \neq x$ and $\xi_1(x) \neq \xi_2(x)$, a little thought reveals that

$$c(\xi_1, x) + c(\xi_2, x) \geq \varepsilon_0 + \varepsilon_1, \tag{2.29}$$

where the equation holds only when $|\mathcal{N}(x)|$ is even and there are an equal number of sites with opinions 0 and 1. (In this case, a vote event does not change the opinion of x in ξ_1 or ξ_2.) Since the lower bound in (2.29) is uniform in x, we obtain that

$$\tilde{\varepsilon} \geq \varepsilon_0 + \varepsilon_1. \tag{2.30}$$

We next estimate $\tilde{M}$. For every fixed $x \in S$, $c(\xi_1, x) \neq c(\xi_2, x)$ only when there exists at least one $y \in \mathcal{N}(x) \cup \{x\}$ such that $\xi_1(y) \neq \xi_2(y)$. Recall we assume $|\mathcal{N}(x)| \leq K$ for every $x \in S$. Since the inner supremum in (2.26) is taken over $\xi_1, \xi_2 \in S$ that only differ at one site u, the number of u such that $\sup|c(\xi_1, x) - c(\xi_2, x)| \neq 0$ is at most $K+1$. For such a pair $\xi_1, \xi_2 \in \mathscr{S}$,

$$|c(\xi_1, x) - c(\xi_2, x)| \leq \max(1 - 2\varepsilon_0, 1 - 2\varepsilon_1).$$

Hence we obtain

$$\tilde{M} \leq (K+1)\max(1 - 2\varepsilon_0, 1 - 2\varepsilon_1) = (K+1)(1 - 2\min(\varepsilon_0, \varepsilon_1)). \tag{2.31}$$

Assuming ε_0 and ε_1 satisfy (2.28), it follows from (2.30) and (2.31) that

$$\tilde{M} - \tilde{\varepsilon} < (K+1) - 2(K+1)\min(\varepsilon_0, \varepsilon_1) - 2\min(\varepsilon_0, \varepsilon_1) < 0.$$

Therefore Lemma 2.10 implies the ergodicity of the MVP and its exponentially quick convergence to the unique equilibrium, which completes the proof of Proposition 2.11. □

2.5 Examples of Uniformly Self-supporting and $\mathcal{N}$-transitive Structures

In this section, we discuss important examples of uniformly self-supporting structures and $\mathcal{N}$-transitive structures, including the nearest-neighbor interactions on $\mathbb{Z}^d$ (Example 2.12), defective lattices (Examples 2.14, 2.15), and an example featuring non-nearest-neighbor interactions on $\mathbb{Z}$ (Example 2.13). We conclude this section with examples of $(S, \mathcal{N})$ that are not uniformly self-supporting structures (see Examples 2.16, 2.17).

Let $\mathfrak{P}(S)$ denote the power set of S. Recall a structure $(S, \mathcal{N})$ consists of a set S and a neighborhood function $\mathcal{N} : S \to \mathfrak{P}(S)$, such that there exists a constant $K > 0$, with $|\mathcal{N}(x)| < K$ for every $x \in S$. Recall that the uniformly self-supporting structures and $\mathcal{N}$-transitive structures have been introduced at the beginning of Chapter 1. In a nutshell, to show a structure $(S, \mathcal{N})$ is a uniformly self-supporting structure, it suffices to find a constant R and a collection of self-supporting sets $\mathscr{E}$ satisfying (1.5). To show a structure $(S, \mathcal{N})$ is an $\mathcal{N}$-transitive structure, in addition to showing $(S, \mathcal{N})$ is a uniformly self-supporting structure, we need to show that, for each given $x, y \in S$, there exists an $\mathcal{N}$-automorphism f such that $f(x) = y$; and for each given $A, A' \in \mathscr{E}$, there exists an $\mathcal{N}$-automorphism g such that $g(A) = A'$.

In the remainder of this section, we discuss important examples of structures $(S, \mathcal{N})$, beginning with the nearest-neighbor interactions on $\mathbb{Z}^d$.

Example 2.12 (Nearest-Neighbor Interactions on $\mathbb{Z}^d$)**.** *Consider $S = \mathbb{Z}^d$, and $\mathcal{N}(x) = \{x \pm \mathbf{e}_j : 1 \le j \le d\}$, for each $x \in \mathbb{Z}^d$, where $\mathbf{e}_j$, $1 \le j \le d$, denotes the jth standard unit vector. We claim that $(S, \mathcal{N})$ is an $\mathcal{N}$-transitive structure.*

Proof. We first show that $(\mathbb{Z}^d, \mathcal{N})$ is a uniformly self-supporting structure. Consider the lattice hyper-rectangle as follows. Let $x_i, x_i' \in \mathbb{Z}$, where $x_i < x_i'$ for each $1 \le i \le d$. We set

$$R(x_1, x_1', \cdots, x_d, x_d') = \{(y_i)_{i=1}^d \in \mathbb{Z}^d : x_i \le y_i \le x_i' \text{ for every } 1 \le i \le d\}.$$

In other words, $R(x_1, x_1', \cdots, x_d, x_d')$ is the d-dimensional hyper-rectangle, with $(x_i)_{i=1}^d$ and $(x_i')_{i=1}^d$ as the vertices on its main diagonal. A little thought shows that the d-dimensional hyper-rectangle $R(x_1, x_1', \cdots, x_d, x_d')$ is self-supporting. To see this, for every $x \in R(x_1, x_1', \cdots, x_d, x_d')$, we have $R(x_1, x_1', \cdots, x_d, x_d') \cap \mathcal{N}(x) \ge d$. For each $x \in \mathbb{Z}^d$, letting $x = (x_i)_{i=1}^d$, we set

$$Q(x) = R(x_1 - 1, x_1, x_2 - 1, x_2, \cdots, x_d - 1, x_d). \tag{2.32}$$

Then each $Q(x)$ is a lattice d-hypercube of side length 1, with x being one of its vertices. Letting

$$R = 2, \quad \mathscr{E} = \{Q(x) : x \in \mathbb{Z}^d\}, \tag{2.33}$$

we claim that such an R and $\mathscr{E}$ satisfy (1.5).

Since each $x \in Q(x)$, we have $\cup_{x \in \mathbb{Z}^d} Q(x) = \mathbb{Z}^d$. Given $(\mathbb{Z}^d, \mathcal{N})$, the metric $d(\cdot, \cdot)$ introduced in (1.3) is equivalent to the L^1 norm on $\mathbb{Z}^d$. Therefore $Q(x) \subseteq B(x, 2)$ for every $x \in \mathbb{Z}^d$. To show R and $\mathscr{E}$ satisfy (1.5), it suffices to show that, for every $x, y \in \mathbb{Z}^d$, if $y \notin Q(x)$, there exists some $z \in \mathbb{Z}^d$, with $y \in Q(z)$ and $Q(x) \cap Q(z) = \emptyset$. Letting $z = (z_i)_{i=1}^d$, we set

$$z_i = \begin{cases} y_i & \text{if } y_i < x_i, \\ y_i + 1 & \text{otherwise.} \end{cases} \tag{2.34}$$

A little thought reveals that $y \in Q(z)$ and $Q(z) \cap Q(x) = \emptyset$. This completes the proof of $(\mathbb{Z}^d, \mathcal{N})$ a uniformly self-supporting structure.

For each $a = (a_i)_{i=1}^d \in \mathbb{Z}^d$, let T_a denote the usual translation, i.e., $T_a : \mathbb{Z}^d \to \mathbb{Z}^d$ with $T_a(x) = (x_1 + a_1, x_2 + a_2, \cdots, x_d + a_d)$ for each $x = (x_i)_{i=1}^d \in \mathbb{Z}^d$. A little thought reveals that T_a is an $\mathcal{N}$-automorphism. For every $x = (x_i)_{i=1}^d, y = (y_i)_{i=1}^d \in \mathbb{Z}^d$, setting $a = (y_i - x_i)_{i=1}^d$, then we have $T_a(x) = y$. Since all elementary self-supporting sets given by (2.33) are d-hypercubes of length 1, for every $A, A' \in \mathscr{E}$, there exists some translation T_a such that $T_a(A) = A'$. This completes the proof of $(S, \mathcal{N})$ an $\mathcal{N}$-transitive structure. $\square$

For similar reasons, many other lattice systems (e.g., triangular lattice, hexagonal lattice), with the nearest-neighbor interaction, are $\mathcal{N}$-transitive structures.

Our next example of $\mathcal{N}$-transitive structures features non-nearest-neighbor interactions.

Example 2.13 (A non-nearest-neighbor interaction on $\mathbb{Z}$)**.** *Consider $S = \mathbb{Z}$ and $\mathcal{N}(x) = \{x-2, x-1, x, x+1, x+2\}$ for each $x \in \mathbb{Z}$. We claim that $(S, \mathcal{N})$ is an $\mathcal{N}$-transitive structure.*

Proof. We first show that such a structure $(\mathbb{Z}, \mathcal{N})$ is a uniformly self-supporting structure. For each $x \in \mathbb{Z}$, we set $C(x) = \{x-1, x, x+1\}$. Each $C(x)$ itself is a self-supporting set since

$$|C(x) \cap \mathcal{N}(y)| = 3, \quad \text{for every } x \in \mathbb{Z} \text{ and } y \in C(x).$$

Setting

$$R = 1, \quad \mathscr{E} = \{C(x) : x \in Z\}, \tag{2.35}$$

we claim that R and $\mathscr{E}$ satisfy (1.5).

First, since $x \in C(x)$, we have $\cup_{x \in \mathbb{Z}} C(x) = \mathbb{Z}$. Furthermore, since the metric $d(\cdot, \cdot)$, introduced in (1.3), satisfies

$$d(x, y) \le |x - y| \quad \text{for every } x, y \in \mathbb{Z},$$

for each $x \in \mathbb{Z}$, $C(x) \subseteq B(x, 1)$. Lastly, for every $x, y \in \mathbb{Z}$ with $y \notin C(x)$, assuming $y > x$, then we have $y \in C(y+1)$ and $C(y+1) \cap C(x) = \emptyset$. (If $y < x$, then we have

$y \in C(y-1)$ and $C(y-1) \cap C(x) = \emptyset$.) Therefore, $(\mathbb{Z}, \mathcal{N})$ is a uniformly self-supporting structure.

We next show that $(\mathbb{Z}, \mathcal{N})$ is an $\mathcal{N}$-transitive structure. For every $a \in \mathbb{Z}$, let T_a denote the translation $T_a : \mathbb{Z} \to \mathbb{Z}$ with $T_a(y) = y + a$ for every $y \in \mathbb{Z}$. Then T_a is an $\mathcal{N}$-automorphism. For every $x, y \in \mathbb{Z}$, let $a = y - x$, then we have $T_a(x) = y$. Since each $C(x)$ is a set of 3 consecutive integers, for every $x, y \in \mathbb{Z}$, there exists some translation T such that $T(C(x)) = C(y)$, which completes the proof that $(S, \mathcal{N})$ is an $\mathcal{N}$-transitive structure. $\square$

We will revisit Example 2.13 in Chapter 3. Our next example is a "defective lattice": Example 2.14 is an analog of Example 2.12, with $d = 2$ and the site $(0,0)$ removed from $\mathbb{Z}^2$.

Example 2.14 (A defective lattice)**.** *Set* $S = \mathbb{Z}^2_* = \mathbb{Z}^2 - \{(0,0)\}$ *and*

$$\mathcal{N}(x) = \begin{cases} \{x \pm \mathbf{e}_1, x \pm \mathbf{e}_2\} - \{(0,0)\} & \textit{for } x = (0,1), (0,-1), (1,0), (-1,0), \\ \{x \pm \mathbf{e}_1, x \pm \mathbf{e}_2\} & \textit{otherwise.} \end{cases} \tag{2.36}$$

We claim that $(S, \mathcal{N})$ *is a uniformly self-supporting structure.*

Proof. Recall that $Q(x)$ is introduced in (2.32). In particular, when $d = 2$, for each $x = (x_1, x_2) \in \mathbb{Z}^2$,

$$Q(x) = \{(x_1 - 1, x_2 - 1), (x_1 - 1, x_2), (x_1, x_2 - 1), (x_1, x_2)\}.$$

Setting

$$C = \{(-1,0), (0,1), (0,-1), (1,0), (-1,1), (1,1), (1,-1), (-1,-1)\},$$

and

$$R = 2, \quad \mathscr{E} = \{Q(y) : y \in \mathbb{Z}^2 - Q(1,1)\} \cup \{C\}, \tag{2.37}$$

we claim that R and $\mathscr{E}$ satisfy (1.5).

First, a little thought reveals that

$$\bigcup_{A \in \mathscr{E}} A = \left(\cup_{y \in \mathbb{Z}^2 - Q(1,1)} Q(y) \right) \cup C = \mathbb{Z}^2 - \{(0,0)\}.$$

Since the metric $d(\cdot,\cdot)$, introduced in (1.3), satisfies

$$d(x,y) \leq ||x-y||,$$

where $||\cdot||$ denotes the L_1 norm on $\mathbb{Z}^2$, we have

$$Q(y) \subseteq B(y,2) \quad \text{for every } y \in \mathbb{Z}^2 - \{(0,0)\},$$

and

$$C \subseteq B(x,4), \quad \text{where } x = (1,1).$$

Therefore, to show R and $\mathscr{E}$, defined in (2.37), satisfy (1.5), it suffices to show that,

$$\begin{aligned} &\text{for each } x \in S \text{ and } A \in \mathscr{E} \text{ such that } x \notin A, \text{there exists} \\ &\text{some } A' \in \mathscr{E},\ \text{satisfying } x \in A' \text{ and } A' \cap A = \emptyset. \end{aligned} \tag{2.38}$$

There are two possibilities about A: $A \neq C$ and $A = C$. If $A \neq C$, then $A = Q(y)$ for some $y \in S$. Consider

$$\max\{d(w,y) : w \in S \text{ and } x \in Q(w)\}, \tag{2.39}$$

and let z denote the maximizer in (2.39): Since there are only finitely many w with $x \in Q(w)$, such a maximizer always exists. A little thought shows that $x \in Q(z)$ and $Q(z) \cap Q(y) = \emptyset$.

We next assume $A = C$ and $x \notin C$. For each $y \in S$, we write

$$d(y,C) = \inf\{d(y,w) : w \in C\}.$$

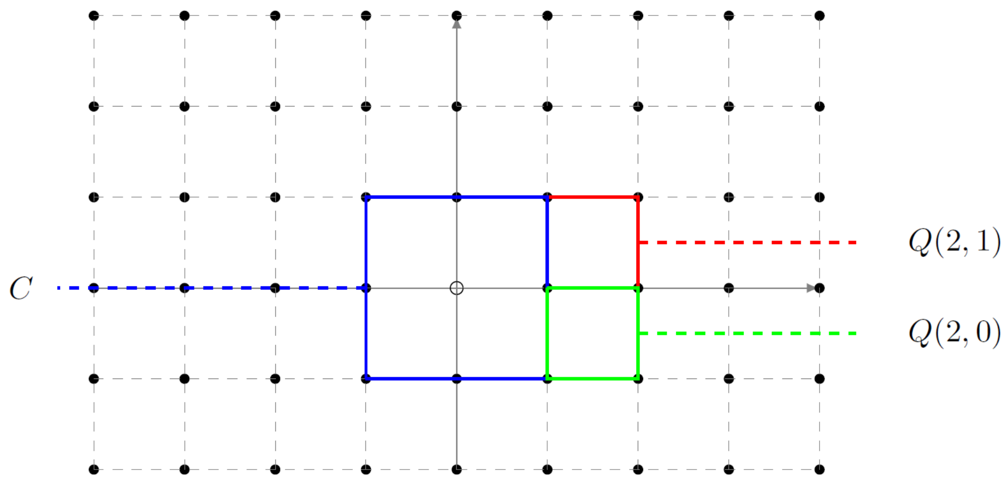
Consider

$$\max\{d(w,C) : w \in S \text{ and } x \in Q(w)\}, \tag{2.40}$$

and let z denote the maximizer in (2.40). A little thought shows that $x \in Q(z)$ and $Q(z) \cap C = \emptyset$. This completes the proof of (2.38), hence implies that $(S,\mathcal{N})$ is a uniformly self-supporting structure. $\square$

Next, we will provide more examples of defective lattices. Their proofs are similar to that of Example 2.14, and hence are omitted here.

Figure 2.1: Examples of elementary self-supporting sets in Example 2.13



Example 2.15 (More examples of defective lattices)**.** *Given $G \subseteq \mathbb{Z}^d$, consider the defective lattice where $S = \mathbb{Z}^d - G$ and*

$$\mathcal{N}(x) = S \cap \{x \pm \mathbf{e}_i : 1 \leq i \leq d\}, \quad \text{for each } x \in S.$$

We claim $(S, \mathcal{N})$ is uniformly self-supporting in the following three cases.

1. *$G = \{0, 2\mathbf{e}_1\}$.*

2. *$G = (n\mathbb{Z})^d = \{(nx_i)_{i=1}^d : x_i \in \mathbb{N}$ for every $1 \leq i \leq d\}$, where $n \in \mathbb{Z}$ satisfies $n \geq 4$.*

3. *For every $x \neq y \in G$, $||x - y|| > 3$, where $|| \cdot ||$ denotes the L_1 norm on $\mathbb{Z}^d$.*

We conclude this section with examples of $(S, \mathcal{N})$ that are not uniformly self-supporting structures. The structures in Examples 2.16 and 2.17 are not uniformly self-supporting structures due to the lack of self-supporting sets. In each case, we briefly illustrate why.

Example 2.16. *Consider $S = \mathbb{Z}^d$ and $\mathcal{N}(x) = \{x + \mathbf{e}_i : 1 \leq i \leq d\}$ for each $x \in \mathbb{Z}^d$. We claim that $(S, \mathcal{N})$ is not a uniformly self-supporting structure.*

Example 2.16 is the "one-sided" version of Example 2.12. A little thought shows that every finite set $A \subseteq S$ does not satisfy (1.4), implying no finite set $A \subseteq S$ is self-supporting. Therefore, $(S, \mathcal{N})$ is not a uniformly self-supporting structure. For example, consider $d = 2$ and $A = \{(0,0), (1,0), (0,1), (1,1)\}$. It is easy to see that A is self-supporting in Example 2.12, but not here, since $\mathcal{N}(1,1) \cap A = \emptyset$, which does not satisfy (1.4).

Example 2.17. *Consider $S = \mathbb{T}_d$, the d-ary tree, (which was introduced in Subsection 1.1.2), with $d \geq 3$, and let each $\mathcal{N}(x)$ be the set of all adjacent sites of x. We claim that $(S, \mathcal{N})$ is not a uniformly self-supporting structure.*

We claim that every finite $A \subseteq \mathbb{T}_d$, with $d \geq 3$, is not self-supporting. For example, consider $d = 3$ and $A = \{x, y, z\} \subseteq \mathbb{T}_3$, where x denotes an arbitrary site and $y \neq z \in \mathcal{N}(x)$. Since y and z are not adjacent,

$$|A \cap \mathcal{N}(y)| = |A \cap \mathcal{N}(z)| = 1,$$

implying that A is not self-supporting.

Fix an arbitrary $x \in S$, for every finite $A \subseteq S$, consider

$$D = \max\{d(x, y), y \in A\} \tag{2.41}$$

where $d(\cdot, \cdot)$ denotes the metric introduced in (1.3), which coincides with the graph distance in Example 2.17. Since A is finite, the maximum (2.41) exists, and there is some $z \in A$ such that $d(x, z) = D$. Let $z_1, z_2, \cdots, z_d$ denote d sites in $\mathcal{N}(z)$. A little thought shows that there exists exactly one $1 \leq j \leq d$ such that $d(z_j, x) = D - 1$, and

$$d(z_i, x) = D + 1 \quad \text{for every } i \neq j.$$

It therefore follows from (2.41) that $|\mathcal{N}(z) \cap A| \leq 1$, and hence A is not self-supporting.

Chapter 3

Proof of Theorem 1: Ergodicity

Chapter 3 establishes the ergodicity of the majority vote process, defined on a uniformly self-supporting structure $(S, \mathcal{N})$, under the assumption that either its 0-noise intensity or 1-noise intensity is significantly smaller than the other. We first outline this chapter as follows.

To better illustrate our ideas, we prove, in Section 3.1, a one-dimensional case of Theorem 1 – $S = \mathbb{Z}$ and $\mathcal{N}(x) = \{x-2, x-1, x, x+1, x+2\}$. We first introduce *good sets*, s-paths, and successful s-paths in Subsection 3.1.1. In the proof of the general case, the notion of good sets will be replaced with elementary self-supporting sets, which was introduced in Section 1.1. Subsequently, in Subsection 3.1.2, we state, in Propositions 3.6 and 3.7, important properties of s-paths, upon which the proof of this special case heavily relies. Lastly, we combine previous results and prove this special case of Theorem 1 in Subsection 3.1.3. The proof of this simple case encapsulates all essential ideas for proving Theorem 1 in the general case, and has the merit of avoiding some combinatorial intricacies. We also note skipping Section 3.1 does not compromise the completeness of the proof for the general case.

In Section 3.2, we extend this approach to the general case, introducing d-paths, s-paths, and successful s-paths. We also discuss some of their important properties in the same section, namely Propositions 3.10, 3.15, and 3.16, which are crucial to our proof of Theorem 1. The proof of Propositions 3.15 and 3.16 will be delayed until the end of this chapter.

Section 3.3 presents the proof of Theorem 1, assuming Propositions 3.15 and 3.16.

We first apply Proposition 2.6 and relate the ergodicity of the majority vote process to (2.14) for the coupled process $\{\eta_t\}_{t\geq 0}$. We next apply Propositions 3.10 and 3.15, which reduces the proof of (2.14) to that of proving (3.64) and (3.68). The proof of (3.64) and (3.68) follows from Propositions 3.16, 3.14, Lemma 3.13, and a proper order of choosing ε_1, r and T. Sections 3.4 and 3.5 are dedicated to proving Propositions 3.15 and 3.16, respectively.

Section 3.6 presents important applications of Theorem 1. We apply Theorem 1 to examples introduced in Section 2.5, and derive specific bounds for the corresponding MVP to be ergodic in different scenarios, including d-dimensional lattices, triangular lattices, hexagonal lattices, and defective lattices.

3.1 An Illustrative 1-dimensional Example

We set $S = \mathbb{Z}$ and $\mathcal{N}(x) = \{x-2, x-1, x, x+1, x+2\}$ throughout this section and consider the MVP $\{\xi_t\}_{t\geq 0}$ defined on such a structure $(S, \mathcal{N})$. As usual, we let ε_0 and ε_1 denote the 0-noise intensity and 1-noise intensity, respectively, where we have assumed that $\varepsilon_0 + \varepsilon_1 \leq 1$. (Otherwise, the majority vote process is not well-defined.) We claim the following.

Theorem 5. *Consider the structure $(S, \mathcal{N})$, where $S = \mathbb{Z}$ and $\mathcal{N}(x) = \{x-2, x-1, x, x+1, x+2\}$ for each $x \in \mathbb{Z}$. Let $\{\xi_t\}_{t\geq 0}$ denote the majority vote process on $(S, \mathcal{N})$, with ε_0 and ε_1 denoting its 0-noise intensity and 1-noise intensity, respectively. For every fixed $\varepsilon_0 \in (0,1)$, ξ_t is ergodic if ε_1 satisfies*

$$\varepsilon_1 < 2\varepsilon_0/69. \tag{3.1}$$

We first provide the intuition for our proof of Theorem 5, which the reader should feel free to skip. Let $\{\eta_t\}_{t\geq 0}$ denote the coupled process on $\mathbb{Z}$ with the initial configuration $\eta_0(x) \equiv (0,1)$ for all $x \in \mathbb{Z}$. (Recall the coupled process was introduced in Definition 2.5.) We will use Proposition 2.7 as the criterion to show the ergodicity of ξ_t. For this, it suffices to show that

$$\lim_{t\to\infty} \mathbb{P}(\eta_t(x) = (0,1)) = 0 \quad \text{for all } x \in S. \tag{3.2}$$

To estimate $\mathbb{P}(\eta_t(x) = (0,1))$, we focus on the rates of "death" and "birth" of $(0,1)$ sites in η_t. By Definition 2.5, if (x,t) is a noise point, then $\eta_t(x) = (0,0)$ or $(1,1)$.

In other words, a noise point (x,t) always "clears" a $(0,1)$ site at x. On the other hand, vote points can "reinstate" a $(0,1)$ site only under certain scenarios (a necessary condition for such reinstatement is characterized in Proposition 3.7). Our strategy for proving Theorem 5 is to show that, when the noise intensities ε_0 and ε_1 satisfy (3.1), the mechanism of reinstating $(0,1)$ sites is significantly weaker than the one that clears $(0,1)$ sites, eventually leading to (3.2).

The remainder of Section 3.1 is dedicated to the proof of Theorem 5 and is structured as follows: Subsection 3.1.1 introduces terminology and essential tools for the proof, among which the most important ones are good sets, s-paths, and successful s-paths. Two important properties of successful s-paths are stated in Subsection 3.1.2, Propositions 3.6 and 3.7. These propositions are crucial to our proof. Lastly, Subsection 3.1.3 combines all previous results and proves Theorem 5.

3.1.1 Terminology: Good Sets, s-paths and Successful s-paths

Recall that a metric $d(\cdot,\cdot)$ has been introduced in (1.3). A little thought shows that this metric is equivalent to the Euclidean norm on $\mathbb{Z}$, since

$$|x-y| \le d(x,y) \le 2|x-y|.$$

For simplicity, let $d(\cdot,\cdot)$ denote the Euclidean norm throughout this section. Then, for every $x \in \mathbb{Z}$ and $r \in \mathbb{N}$, we have $B(x,r) = [x-r, x+r] \cap \mathbb{Z}$. For each finite $A \subseteq \mathbb{Z}$, consider the event

$$\mathcal{G}(A,t) := \{\eta_s(y) = (0,0) \textit{ for some } s \in (0,t) \text{ and every } y \in A\}. \tag{3.3}$$

In particular, when $A = B(x,r)$, we let $\mathcal{G}(x,r,t)$ denote the event $\mathcal{G}(A,t)$. Then, we claim the following.

Proposition 3.1. *For an arbitrary $x \in \mathbb{Z}$, $r \in \mathbb{N}$ and every $p \in (0,1)$, there exists some $T > 0$, which is determined by ε_0 and r, such that*

$$\mathbb{P}(\mathcal{G}^c(x,r,t)) < p \quad \textit{for every } t > T. \tag{3.4}$$

Proof. Recall that $e(x,t)$, introduced in (2.4), is the time of the latest event point occurring at x before time t; and $X_x^{(0)}$, introduced in Definition 2.1, denotes the collection of all 0-noise points at x. For every finite $A \subseteq \mathbb{Z}$ and $0 < s < t$, consider the event

$$\mathscr{B}(A, s, r) := \{e(y,t) > s, \text{ and } (y, e(y,t)) \in X_y^{(0)}, \text{ for every } y \in A\}. \tag{3.5}$$

Due to the independence of 0-noise points at distinct sites, we have

$$\mathbb{P}(\mathscr{B}(A, s, t)) = (\varepsilon_0(1 - e^{s-t}))^{|A|}. \tag{3.6}$$

In particular,

$$\mathbb{P}(\mathscr{B}(A, i, i+1)) = (\varepsilon_0(1 - e^{-1}))^{|A|} > (\varepsilon_0/2)^{|A|}. \tag{3.7}$$

We also note that, since $\mathscr{B}(A, i, i+1)$ is determined by the event points occurring in A during time $(i, i+1)$, for every $i \neq j \in \mathbb{N}$, $\mathscr{B}(A, i, i+1)$ and $\mathscr{B}(A, j, j+1)$ are independent.

By Definition 2.5, conditioning on $\mathscr{B}(A, s, r)$, we have $\eta_r(y) = (0, 0)$ for all $y \in A$. Letting $A = B(x, r)$, for every $m \in \mathbb{N}$, we obtain

$$\mathbb{P}(\mathcal{G}^c(x, r, m)) \leq \mathbb{P}(\bigcap_{i=0}^{m-1} \mathscr{B}^c(A, i, i+1)) < (1 - (\varepsilon_0/2)^{2r+1})^m. \tag{3.8}$$

Let m denote the smallest integer satisfying

$$m > \frac{\ln p}{\ln(1 - (\varepsilon_0/2)^{2r+1})}. \tag{3.9}$$

Then, for every fixed choice of $x \in \mathbb{Z}$ and $r \in \mathbb{N}$, when $t > m$, it follows from (3.8) and (3.9) that

$$\mathbb{P}(\mathcal{G}^c(x, r, t)) < \mathbb{P}(\mathcal{G}^c(x, r, m)) < (1 - (\varepsilon_0/2)^{2r+1})^m < p. \tag{3.10}$$

We set $T = 1 + \ln p / \ln(1 - (\varepsilon_0/2)^{2r+1})$, which satisfies (3.4), and hence completes the proof of Proposition 3.1. $\square$

We next state a necessary condition for $\eta_t(x) = (0, 1)$.

Proposition 3.2. *Let $\{\eta_t\}_{t \geq 0}$ denote the coupled process on $(\mathbb{Z}, \mathcal{N})$, where $\mathcal{N}(x) = \{x-2, x-1, x, x+1, x+2\}$ for all $x \in \mathbb{Z}$. Suppose that $\eta_t(x) = (0, 1)$ for some $x \in \mathbb{Z}$ and $t \geq 0$, and that (x, t) is not an event point. Conditioning on $\{e(x, t) > 0\}$, the following holds.*

1. *$(x, e(x, t))$ is a vote point.*

2. There exists some $y \in \mathcal{N}(x)$ with $\eta_{e(x,t)-}(y) = (0,1)$.

Proof. Since (x,t) is not an event point, it follows from the continuation rule (2.11) that $\eta_{e(x,t)}(x) = \eta_t(x) = (0,1)$. We first show $(x, e(x,t))$ is a vote point: Supposing otherwise, then $(x, e(x,t))$ is a noise point. Regardless of which type of noise point $(x, e(x,t))$ is, we always have $\eta_t(x) \neq (0,1)$ by Definition 2.5, which leads to contradictions. Hence $(x, e(x,t))$ is a vote point.

We next prove, by contradiction, that there exists some $y \in \mathcal{N}(x)$ with $\eta_{e(x,t)-}(y) = (0,1)$. Suppose that $\eta_{e(x,t)-}(y) \neq (0,1)$ for all $y \in \mathcal{N}(x)$. Then $\eta_{e(x,t)-}(y) \in \{(0,0),(1,1)\}$ for all $y \in \mathcal{N}(x)$. Since $(x, e(x,t))$ is a vote point, it then follows from (2.5.c) that $\eta_{e(x,t)}(x) \in \{(0,0),(1,1)\}$, contradicting the assumption that $\eta_t(x) = (0,1)$. This completes the proof of Proposition 3.2. $\square$

In a nutshell, Proposition 3.2 shows that the event $\{\eta_t(x) = (0,1)\}$ leads to a vote point $(x, e(x,t))$ and a site $y \in \mathcal{N}(x)$ with $\eta_{e(x,t)-}(y) = (0,1)$. Recall that we have assumed that $\{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x \in S}$ are pairwise disjoint. Letting $s = e(x,t)$, then (y,s) can not be an event point. Therefore Proposition 3.2 applies to $\{\eta_s(y) = (0,1)\}$, which leads to another vote point $(y, e(y,s))$ and a site $z \in \mathcal{N}(y)$ with $\eta_{e(y,s)-}(z) = (0,1)$. Let $(y_0, s_0) = (x, e(x,t))$ and $(y_1, s_1) = (y, e(y,s))$. We can repeatedly apply Proposition 3.2 to each $\eta_{s_i}(y_i)$, and obtain a sequence of event points $\{(y_i, s_i)\}_{i=0}^k$ with $\eta_{s_i}(y_i) = (0,1)$. We formulate this observation in Proposition 3.3.

Proposition 3.3. *Let $\{\eta_t\}_{t \geq 0}$ denote the coupled process on $(\mathbb{Z}, \mathcal{N})$, where $\mathcal{N}(x) = \{x-2, x-1, x, x+1, x+2\}$ for all $x \in \mathbb{Z}$. Suppose that $\eta_T(x) = (0,1)$ for some $x \in \mathbb{Z}$ and $T > 0$, then, there exists, almost surely, a finite sequence of vote points $\{(x_i, t_i)\}_{i=0}^m$, with $(x_0, t_0) = (x, e(x,T))$, $x_i \in \mathcal{N}(x_{i-1})$, $t_i = e(x_i, t_{i-1})$ for $1 \leq i \leq m$, $t_m = 0$, and satisfying the following:*

1. *For every $0 \leq i \leq m$, $\eta_{t_i}(x_i) = (0,1)$.*

2. *If $x_i \neq x_{i-1}$, then $\eta_{t_i-}(x_{i-1}) \neq (0,1)$.*

Proof. We use an inductive argument to prove Proposition 3.3. Suppose that (x,t) is not an event point, since $\eta_t(x) = (0,1)$, (x_0, t_0) is a vote point and $\eta_{t_0}(x_0) = (0,1)$. According to Proposition 3.2, there are two possibilities for $\eta_{t_0-}(x_0)$: If $\eta_{t_0-}(x_0) = (0,1)$, then we set $x_1 = x_0$. Otherwise, there exists some $y \in \mathcal{N}(x_0)$ with $y \neq x_0$ and $\eta_{t_0-}(y) = (0,1)$. In this case, we set $x_1 = y$ and $t_1 = e(x_1, t_0)$. In either case, it is straightforward to verify that (x_0, t_0) and (x_1, t_1) satisfy all conditions in Proposition 3.3.

Suppose that $\{(x_i, t_i)\}_{i=0}^{k}$ has been constructed such that it satisfies all conditions in Proposition 3.3, and has $t_k > 0$ (if $t_k = 0$, $\{(x_i, t_i)\}_{i=0}^{k}$ is the desired sequence), there are again two possibilities for $\eta_{t_k-}(x_k)$: If $\eta_{t_k-}(x_k) = (0,1)$, then we set $x_{k+1} = x_k$. Otherwise, there exists some $y \in \mathcal{N}(x_k)$ such that $y \neq x_k$ and $\eta_{t_k-}(y) = (0,1)$. In this case, we set $x_{k+1} = y$ and $t_{k+1} = e(x_{k+1}, t_k)$. It is easy to verify that the sequence $\{(x_i, t_i)\}_{i=0}^{k+1}$ still satisfies all conditions in Proposition 3.3.

Since the sequence $\{t_i\}_{i=0}^{k}$ is strictly decreasing, and each $(t_{i+1} - t_i)$, $1 \leq i \leq k$, is independently exponentially distributed with mean 1, such a construction (until reaching $t_m = 0$) ends in finite steps almost surely. $\square$

We refer to such a sequence satisfying conditions in Proposition 3.3 as a *d-path*. We will revisit this concept in our proof of the general case.

Our next observation pertains to the connection between a $(0,1)$ site and the occurrence of some "nearby" 1-noise points. Suppose that $\eta_T(x) = (0,1)$, and that there exists some elementary self-supporting set $A \ni x$ such that $\eta_t(y) = (0,0)$ for all $y \in A$ and some $t < T$. Then there exists some $z \in A$ and $t' \in (t, T)$ such that (z, t') is an 1-noise point. To see this, if we assume that there is no 1-noise point occurs in A between t and T, then all sites from A will remain in state $(0,0)$, contradicting the assumption that $\eta_T(x) = (0,1)$.

For every $x \in \mathbb{Z}$, we define $C(x) = \{x-1, x, x+1\}$. We say a set A is a *good set* if $A = C(y)$ for some $y \in \mathbb{Z}$, i.e., A consists of 3 consecutive integers.

Recall that $\mathscr{B}(A, s, t)$ has been introduced in (3.5). Letting $\mathscr{B}(A, s)$ denote the event $\mathscr{B}(A, 0, s)$, for each good set A, we introduce

$$e_B(A, t) = \sup\{s < t : \mathscr{B}(A, s) \text{ occurs}\}, \tag{3.11}$$

and set $e_B(A, t) = -1$ if the supremum in (3.11) is taken over an empty set. In other words, $e_B(A, t)$ traces the latest time $s < t$ when $\mathscr{B}(A, t)$ occurs.

We also introduce

$$e_{\varepsilon 1}(A, t) = \sup\{s < t : (y, s) \in X_y^{(1)}, \text{ for some } y \in A\}, \tag{3.12}$$

and define $e_{\varepsilon 1}(A,t) = 0$ if the supremum in (3.12) is taken over an empty set.

For every good set A, we introduce a stochastic process $\{\chi_t^{(A)}\}_{t\geq 0}$ as follows.

$$\chi_t^{(A)} = \begin{cases} 1 & \text{if } e_B(A,t) \leq e_{\varepsilon 1}(A,t), \\ 0 & \text{otherwise.} \end{cases}$$

We say A is *activated* at time t if $\chi_t^{(A)} = 1$. In particular, every good set A is activated at $t = 0$.

For each $A_1, A_2 \subseteq \mathbb{Z}$, we set

$$d(A_1, A_2) = \inf\{|x-y| : x \in A_1, y \in A_2\}.$$

We next introduce s-paths and successful s-paths.

Definition 3.4 (s-path)**.** *Let $l \in \mathcal{N}$ denote any positive integer and A_i, $0 \leq i \leq l$, denote any good set. The sequence $\{A_i\}_{i=0}^{l}$ is an s-path if it satisfies $0 < d(A_{i-1}, A_i) \leq 3$ for each $1 \leq i \leq l$.*

Definition 3.5 (Successful s-path)**.** *An s-path $\{A_i\}_{i=0}^{l}$ is successful at time t if there exists a strictly decreasing sequence $\{s_i\}_{i=0}^{l}$, with $s_0 = t$, and satisfying the following conditions.*

(3.5.a) For each $0 \leq i \leq l$, $\chi_{s_i}^{(A_i)} = 1$.

(3.5.b) For each $0 \leq i \leq l$, set

$$r_i = \inf\{r \leq s_i : \chi_u^{(A_i)} = 1 \textit{ for all } u \in (r, s_i)\}. \tag{3.13}$$

Then $s_i > r_{i-1}$ for every $1 \leq i \leq l$.

(3.5.c) If $A_i \cap A_j \neq \emptyset$ for some $i < j$, then $r_i > s_j$.

We conclude this subsection with the intuition for introducing successful s-paths, which the reader should feel free to skip. In essence, we want the reader to envision a successful path as a union of cylinders $A_i \times [r_i, s+i]$ in spacetime $\mathbb{Z} \times [0,T]$, whose existence is ensured by the event $\{\eta_t(x) = (0,1)\}$. Our proof of (3.1) will be structured around the use of successful s-paths: In Proposition 3.6, we first show that the probability of a given s-path $\{A_i\}_{i=1}^{l}$ being successful decays exponentially as $l \to \infty$. Then, in Proposition 3.7, we show that, conditioning on $\mathcal{G}(0, R, t)$, there exists a successful s-path, with a length no shorter than $(R-1)/6$. The proof of (3.1) then follows from a union bound (3.26) and choosing R, ε_1, T in an appropriate order.

3.1.2 Properties of Successful s-paths

In this subsection, we state two important properties of successful s-paths. We begin with the probability of a given s-path being successful. Recall that $\mathcal{G}(A, t)$ denotes the event that there exists some $s < t$ such that $\eta_s(y) = (0, 0)$ for all $y \in A$. Also, conditioning on $\mathscr{B}(A, s)$, $\eta_s(y) = (0, 0)$ for all $y \in A$.

Proposition 3.6. *Let ℓ denote an arbitrary positive integer and consider an s-path $\{A_i\}_{i=0}^{\ell}$. Let $\mathcal{E}(\{A_i\}_{i=0}^{\ell}, t)$ denote the event that $\{A_i\}_{i=0}^{\ell}$ is successful at time t. Let A denote the smallest set of consecutive integers that contains all A_i for $0 \le i \le \ell$. Then,*

$$\mathbb{P}(\mathcal{E}(\{A_i\}_{i=0}^{\ell}, t) | \mathcal{G}(A, t)) \le \left(\frac{6\varepsilon_1}{3\varepsilon_1 + 2\varepsilon_0} \right)^{\ell+1}. \tag{3.14}$$

Proof. If $\{A_i\}_{i=0}^{\ell}$ is successful at time t, then $\{A_i\}_{i=0}^{h}$ is also successful at time t for every $h \le \ell$, as we can take the exact same choice of s_i for every $0 \le i \le h$. We write the probability on the left side of (3.14) as

$$\begin{aligned} \text{LHS of (3.14)} = & \mathbb{P}(\mathcal{E}(\{A_0\}, t) | \mathcal{G}(A, t)) \\ & \prod_{h=1}^{\ell} \mathbb{P}(\mathcal{E}(\{A_i\}_{i=0}^{h}, t) | \mathcal{E}(\{A_i\}_{i=0}^{h-1}, t), \mathcal{G}(A, t)) \end{aligned} \tag{3.15}$$

We begin by estimating the first term – $\{A_0\}$ is successful if and only if A_0 is activated at time t. Given the occurrence of $\mathscr{B}(A_0, s)$ for some $s < t$, A_0 is successful if and only if at least one 1-noise point occurs before the latest occurrence of $\mathscr{B}(A_0, s)$. A little thought reveals that

$$\mathbb{P}(\mathcal{E}(A_0, t) | \mathcal{G}(A, t)) < \frac{3\varepsilon_1}{3\varepsilon_1 + 2\varepsilon_0}. \tag{3.16}$$

We estimate the second term in (3.15). A little thought shows that, conditioning on $\mathcal{E}(\{A_i\}_{i=0}^{h-1}, t)$, $\{A_i\}_{i=0}^{h}$ is successful at time t if at least one of the following conditions holds:

1. A_h is activated at time r_{h-1}.

2. There exists at least one 1-noise point (y, t') for some $y \in A_h$ and $t' \in (r_{h-1}, s_{h-1})$.

Recall that whether A_h is activated at time r_{h-1} or not is determined by $e_B(A_h, r_{h-1})$ and $e_{\varepsilon 1}(A_h, r_{h-1})$. It then follows from $A_h \cap A_{h-1} = \emptyset$ (which implies that the event points in A_h and A_{h-1} are independent) and (3.5.c) that

$$\mathbb{P}(A_h \text{ is activated at time } r_{h-1} | \mathcal{G}(A, t), \mathcal{E}(\{A_i\}_{i=0}^{h-1}, t)) \leq \frac{3\varepsilon_1}{3\varepsilon_1 + 2\varepsilon_0}. \tag{3.17}$$

We estimate the probability that there exists at least one 1-noise point (y, t') for some $y \in A_h$ and $t' \in (r_{h-1}, s_{h-1})$. Note that, since $r_{h-2} - r_{h-1}$ is exponentially distributed and $s_{h-1} > r_{h-2}$. Therefore

$$\mathbb{P}(\text{There is no 1-noise point in } A_h \text{ during } (r_{h-1}, s_{h-1})) \geq \frac{1}{1 + 3\varepsilon_1^2}. \tag{3.18}$$

Combining (3.17) and (3.18), we obtain that

$$\mathbb{P}(\mathcal{E}(\{A_i\}_{i=0}^{h}, t) | \mathcal{E}(\{A_i\}_{i=0}^{h-1}, t), \mathcal{G}(A, t)) \leq \frac{6\varepsilon_1}{3\varepsilon_1 + 2\varepsilon_0}. \tag{3.19}$$

Applying the upper bounds in (3.16) and (3.19) to (3.15) completes the proof of Proposition 3.6. □

Our next proposition shows the existence of at least one successful s-path conditioning on $\{\eta_t(x) = (0, 1)\}$ and $\mathcal{G}(0, r, t)$.

Proposition 3.7. *Let $\{\eta_t\}_{t \geq 0}$ denote the coupled process on $(\mathbb{Z}, \mathcal{N})$, where $\mathcal{N}(x) = \{x-2, x-1, x, x+1, x+2\}$ for all $x \in \mathbb{Z}$. Conditioning on $\{\eta_t(0) = (0, 1)\}$ and $\mathcal{G}(0, r, t)$ for some $t > 0$ and $r \geq 2$, there exists an s-path $\{A_i\}_{i=0}^{l}$ satisfying the following:*

1. *$0 \in A_0$ and $l \geq r/6$.*

2. *$A_i \subseteq B(0, r)$ for each $0 \leq i \leq l$.*

3. *$\{A_i\}_{i=0}^{l}$ is successful at time t.*

Proof. Conditioning on $\mathcal{G}(0, r, t)$, there exists some $s < t$ such that $\eta_s(y) = (0, 0)$ for all $y \in B(0, r)$. We first choose a d-path $\sigma = \{(x_i, t_i)\}_{i=0}^{m'}$, satisfying Proposition 3.3.

Letting $m = \min\{j : t_j < s\}$, Proposition 3.3 implies that $|x_m| > r$. (Otherwise, it contradicts the assumption that $\eta_s(y) = (0,0)$ for all $y \in B(0,r)$.)

We next construct a sequence of good sets $\{A_i\}_{i=0}^{l}$ recursively, beginning with $A_0 = \{-1, 0, 1\}$.

Setting

$$k_{-1} = 0, \quad k_0 = min\{k \in \mathbb{N} : x_k \notin A_0\},$$

We note such a k_0 always exists since $x_m \notin A_0$. We let A_1 to be any good set satisfying $x_{k_0} \in A_1$ and $A_1 \cap A_0 = \emptyset$, noting that such an A_1 always exists since $x_{k_0} \notin A_0$. There are two possibilities for A_1: $A_1 \subseteq B(0,r)$ or $A_1 \not\subseteq B(0,r)$. If $A_1 \not\subseteq B(0,r)$, we will take the s-path as $\{A_0\}$, which has already been proven to be successful at time t. If $A_1 \subseteq B(0,r)$, we continue our construction. Since $x_{k_0-1} \in A_0$ and $x_{k_0} \in \mathcal{N}(x_{k_0-1})$, we have $d(A_0, A_1) \leq d(x_{k_0-1}, x_{k_0}) = 1$. Let $s_1 = t_{k_0}$. Since $\eta_{t_{k_0}}(x_{k_0}) = (0,1)$, A_1 is activated at time s_1. To continue our construction, we set

$$k_1 = \min\{k > k_0 : x_k \notin A_1\}.$$

Again, this minimum is ensured not to be taken over an empty set since $A_1 \subseteq B(0,r)$ and $x_m \notin B(0,r)$.

Suppose that $\{A_i\}_{i=0}^{h-1}$ is constructed such that $\{A_i\}_{i=0}^{h-1}$ is successful at time t. At each step when introducing A_i, $1 \leq i \leq h-1$, we define

$$k_i = \min\{k > k_{i-1} : x_k \notin A_{i-1}\}. \tag{3.20}$$

We next describe how to choose A_h. We first take A_h, for the time being, to be any good set satisfying $x_{k_{h-1}} \in A_h$ such that $A_h \cap A_{h-1} = \emptyset$. We also set $s_h = t_{k_{h-1}}$. If multiple possible A_h exist, we will prioritize the one(s) that satisfies (3.5.c), if any exists. Since

$$d(A_{h-1}, A_h) \leq d(x_{k_{h-1}, x_{k_{h-1}-1}}) = 1,$$

if follows from Definition 3.4 that $\{A_i\}_{i=0}^{h}$ is an s-path.

Next, we determine whether this newly obtained s-path is successful at time t. For simplicity of notation, we set $y = x_{k_{h-1}}$ and $s = t_{k_{h-1}}$. Since $\eta_s(y) = (0,1)$ and $y \in A_h$, A_h is activated at $s_h(= s)$. It follows from Proposition 3.3 that $A_{h-1} \ni x_{k_{h-1}-1}$ is also activated at time s. Hence we have $s_h > r_{h-1}$, implying (3.5.a) and (3.5.b).

Since we have assumed that $\{A_i\}_{i=0}^{h-1}$ is successful at time t, and adding an extra set A_h does not affect $\{s_i\}_{i=0}^{h-1}$ and $\{r_i\}_{i=0}^{h-1}$, hence (3.5.c) still holds for $1 \le j \le h-1$. Therefore, if (3.5.c) holds for $j = h$, the s-path $\{A_i\}_{i=0}^{h}$ is successful at time t, which completes the inductive step.

If (3.5.c) is not satisfied for $j = h$, we need to "trim" the sequence $\{A_i\}_{i=0}^{h-1}$ in the following way. Let Γ denote the set of all $i < h$ such that (3.5.c) does not hold for A_i and A_h, i.e., $A_i \cap A_h \neq \emptyset$ and $r_i < s_h$. Letting

$$u = \min\{i : i \in \Gamma\},$$

then we have

$$u \le h - 1. \tag{3.21}$$

We now introduce a new s-path $\{C_i\}_{i=0}^{u}$ as follows. We set $C_i = A_i$ for all $0 \le i \le u$, in other words, we set $\{C_i\}_{i=0}^{u}$ to be the first $u+1$ terms of the original s-path $\{A_i\}_{i=0}^{h-1}$. (Hence $\{C_i\}_{i=0}^{u}$ is an s-path.) Let $\{s_i'\}_{i=0}^{u}$ denote the s-sequence of the new s-path $\{C_i\}_{i=0}^{u}$ and set $s_i' = s_i$ for all $0 \le i \le u$. Setting

$$k_u' = \min\{k \ge k_{h-1} : x_k \notin C_u\}, \tag{3.22}$$

since $x_m \notin B(0, r)$, the minimum in (3.22) is not taken over an empty set. If $k_u' \ge m$, we complete our construction and $\tau = \{C_i\}_{i=0}^{u}$ is the desired s-path. We now assume that $k_u' < m$. For simplicity of notation, we let $y = x_{k_u'}$ and $s = t_{k_u'}$. Since $y \notin C_u$, there exists a good set $A \ni y$ such that $A \cap C_u = \emptyset$. We set $C_{u+1} = A$. In addition, we set $s_{u+1}' = s$ and

$$k_{u+1}' = \min\{k > k_u' : x_k \notin C_{u+1}\}. \tag{3.23}$$

We first show that $\{C_i\}_{i=0}^{u+1}$ is an s-path, which, in essence, follows from the triangle inequality. Since $(y, s) \in \sigma$, it follows form Proposition 3.3 that at either $d(y, A_h) = 1$ or $d(y, C_u) = 1$. If $d(y, C_u) = 1$, then we have

$$d(C_{u+1}, C_u) \le d(y, C_u) = 1.$$

On the other hand, if $d(y, A_h) = 1$, since $A_h \cap C_u \neq \emptyset$, we have

$$d(C_{u+1}, C_u) \le d(y, A_h) + |A_h| - 1 \le 3.$$

Therefore $\{C_i\}_{i=0}^{u}$ is still an s-path.

Since our approach of choosing C_{u+1} is exactly the same as that of choosing A_r, $\{C_i\}_{i=0}^{u+1}$ satisfies (3.5.a) and (3.5.b). It is worth noting that such a choice of C_{u+1} does not necessarily imply (3.5.c) yet, but ensures $k'_u > k_{h-1}$. (This follows from (3.22) and $x_{k_{h-1}} \in A_h$.) This is a crucial improvement as we have "gone down further" along the d-path.

If (3.5.c) is satisfied, we will use $\{C_i\}_{i=0}^{u+1}$ as the new $\{A_i\}_{i=1}^{r-1}$, which completes our inductive step. If (3.5.c) is not satisfied, we will make the same modification to $\{C_i\}_{i=0}^{u}$ as was done to $\{A_i\}_{i=0}^{r-1}$. After each modification, we will obtain a strictly shorter s-path $\{C'_i\}_{i=0}^{w+1}$, which follows from (3.21), with a greater or equal k'_w, which follows from (3.23). Since both the old s-path $\{A_i\}_{i=1}^{h-1}$ and the d-path σ are of finite lengths, (3.5.c) will be satisfied after a finite number of such modifications.

Repeating the inductive step until having $d(A_l, x_m) \leq 3$. This ensures that each A_k, $0 \leq k \leq l$, satisfies $A_k \subseteq B(0,r)$. Then, we obtain an s-path $\tau = \{A_i\}_{i=0}^{l}$, which is successful at time t, with $d(x_m, A_l) \leq 3$. Since each A_i is a set of three consecutive integers, we have

$$r \leq |x_m - 0| \leq 3 + \sum_{i=1}^{l} (2 + d(A_i, A_{i-1})) < 6l, \tag{3.24}$$

which completes the proof of Proposition 3.7. □

3.1.3 Proof of Theorem 5

This subsection proves Theorem 5. Let $x \in \mathbb{Z}$ denote an arbitrary site. According to Proposition 2.6, with x fixed, $\mathbb{P}(\eta_t(x) = (0,1))$ is non-increasing as a function of t. To prove (3.2), it suffices to show that, for every fixed x and every $p > 0$, there exists some T such that

$$\mathbb{P}(\eta_T(x) = (0,1)) < p. \tag{3.25}$$

Since $(\mathbb{Z}, \mathcal{N})$ is $\mathcal{N}$-transitive, (which was proved in Example 2.13,) $\mathbb{P}(\eta_T(x))$ does not depend on the choice of x. To demonstrate the existence of some T satisfying (3.25) for each $x \in \mathbb{Z}$, it suffices to prove (3.25) for some $T > 0$ and $x = 0$.

Let $A = B(0, R)$, where the value of R will be specified later. We bound $\mathbb{P}(\eta_T(0) =$

$(0,1))$ using the union bound:

$$\mathbb{P}(\eta_T(0) = (0,1)) \leq \mathbb{P}(\mathcal{G}^c(A,T)) + \mathbb{P}(\eta_T(x) = (0,1)|\mathcal{G}(A,T)), \tag{3.26}$$

where $\mathcal{G}(A,t)$ was introduced at the beginning of Subsection 3.1.1. To bound $\mathbb{P}(\eta_T(0) = (0,1))$ by p, it suffices to bound both terms on the right-hand side of (3.26) by $p/2$.

We first bound the first term on the right-hand side of (3.26). Proposition 3.1 implies that there exists a sufficiently large T_0, depending on ε_0 and R, such that

$$\mathbb{P}(\mathcal{G}^c(A, T_0(\varepsilon_0, R))) < p/2. \tag{3.27}$$

For fixed $\varepsilon_0 \in (0,1)$, $p > 0$ and R, $\mathbb{P}(\mathcal{G}^c(A,T))$ can be bounded by $p/2$ by taking R sufficiently large.

To obtain an upper bound on the second term on the right-hand side of (3.26), we apply Propositions 3.6 and 3.7. Conditioning on $\mathcal{G}(A,T)$, Proposition 3.7 implies the existence of at least one successful s-path $\{A_i\}_{i=0}^{l}$ with $A_0 = \{-1,0,1\}$ and $l \geq R/6$. Therefore,

$$\mathbb{P}(\eta_T(0) = (0,1)|\mathcal{G}(A,T)) \leq \sum_{j \geq R/6}^{\infty} 6^j \left(\frac{6\varepsilon_1}{3\varepsilon_1 + 2\varepsilon_0}\right)^{j+1}, \tag{3.28}$$

where the total number of the s-paths of length j, with $A_0 = \{-1,0,1\}$, produces the factor 6^j, and the second factor follows directly from Proposition 3.6. Therefore,

$$\mathbb{P}(\eta_T(0) = (0,1) \mid \mathcal{G}(A,T)) \leq \frac{q^{(R+1)/6}}{6(1-q)}, \tag{3.29}$$

where $q = 36\varepsilon_1/(3\varepsilon_1 + 2\varepsilon_0)$.

We note that, when ε_0 and ε_1 are given, the upper bound in (3.29) does not depend on T and is determined solely by the choice of R. A little thought reveals that (3.1) implies $q < 1/2$. For any fixed values of $p, T > 0$ and $\varepsilon_0 \in (0,1)$, and ε_1 taken sufficiently small so as to satisfy (3.1), it follows that

$$\mathbb{P}(\eta_T(0) = (0,1) \mid \mathcal{G}(A,T)) < p/2,$$

for R large enough such that

$$\frac{q^{(R+1)/6}}{6(1-q)} \leq \frac{q^{(R+1)/6}}{3} < \frac{p}{2}. \tag{3.30}$$

We now demonstrate that, for every $\varepsilon_0 \in (0,1)$, if ε_1 is small enough to satisfy (3.1), then, for every $p > 0$, there exists some T such that $\mathbb{P}(\eta_T(0) = (0,1)) < p$, and hence (3.25) holds, which will complete the proof of Theorem 5. With fixed ε_0 and $p > 0$, we first choose R large enough such that (3.30) holds. It then follows from (3.29) and (3.30) that, for every $T > 0$,

$$\mathbb{P}(\eta_T(0) = (0,1) \mid \mathcal{G}(A,T)) < p/2. \tag{3.31}$$

We next choose $T = T_0$ such that (3.27) holds. Substituting both terms on the right-hand side of (3.26) using upper bounds in (3.27) and (3.31), (3.25) follows from (3.26). This completes the proof Theorem 5.

3.2 d-paths and s-paths

The remainder of this chapter is dedicated to the proof of Theorem 1 for the general case. Unless otherwise stated, we assume that $(S, \mathcal{N})$ denotes any uniformly self-supporting structure. As usual, we let $\{\xi_t\}_{t\geq 0}$ denote the majority vote process on $(S, \mathcal{N})$, and ε_0 and ε_1 denote its 0-noise and 1-noise intensity. We let $\{\eta_t\}_{t\geq 0}$ denote the coupled process, which was introduced in Definition 2.5, with its initial configuration $\eta_0(x) \equiv (0,1)$ for all $x \in S$.

The proof for the general case follows essentially the same idea as that of Section 3.1: We will use Proposition 2.7 as the criterion for proving the ergodicity of ξ_t. Since Proposition 2.7 reduces the proof of the ergodicity of ξ_t to that of (2.14) for the coupled process η_t, our proof will also be constructed around the use of η_t. In this section, we introduce some crucial concepts, such as the d-path and successful s-path, both of which are defined in terms of η_t. The d-path will be used to construct a successful s-path, conditioned on $\{\eta_t(x) = (0,1)\}$. Since $\{\eta_t(x) = (0,1)\}$ implies the existence of at least one successful s-path (see Proposition 3.15), we can bound $\mathbb{P}(\eta_t(x) = (0,1))$ by summing the probability of a given s-path being successful over all possible s-paths.

We outline this section as follows. We first introduce the d-path in Subsection 3.2.1. In a nutshell, a d-path is a finite sequence of "consecutive" vote points (x_i, t_i), $1 \leq i \leq r$, with $\eta_{t_i}(x_i) = (0,1)$ for all $1 \leq i \leq r$. Lemma 3.8 and Proposition 3.10 ensure the existence of such a sequence, conditioning on $\{\eta_t(x) = (0,1)\}$. Moving forward, we introduce s-paths in Subsection 3.2.2 – an s-path is a sequence of elementary self-supporting sets such that every two consecutive ones are non-intersecting, with their distance bounded by a constant. We next introduce successful s-paths in the same subsection: whether a given s-path is successful or not is determined by the Poisson clock $\{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x\in S}$. We conclude this section with some important properties of the successful s-paths in Subsection 3.2.3, with the most crucial ones being Propositions 3.15 and 3.16. Proposition 3.15 gives conditions under which the coupled process will have a successful s-path of at least a given length. Proposition 3.16 gives an upper bound on the probability of any given s-path being successful. For readability, the proofs of Propositions 3.15 and 3.16 are given at the end of this chapter.

3.2.1 Definition and Existence of d-paths

We first state a simple observation about the consequences of $\eta_t(x) = (0,1)$, which helps to motivate the introduction of the d-path. Recall that $e(x,t)$, which has been introduced in (2.4), denotes the time of the latest event point at x before time t, and $X_x^{(0)}$, $X_x^{(1)}$, and $X_x^{(2)}$ denote the set of 0-noise points, 1-noise points, and vote points, occurring at x.

Lemma 3.8. *Consider the coupled process $\{\eta_t\}_{t\geq 0}$ on $(S,\mathcal{N})$. Suppose that $\eta_t(x) = (0,1)$ for some $x \in S$ and $t > 0$, with $e(x,t) > 0$, and assume that (x,t) is not an event point. Then the following holds.*

(3.8.a) $(x, e(x,t))$ *is a vote point.*

(3.8.b) There exists some $y \in \mathcal{N}(x)\cup\{x\}$ such that $\eta_{e(x,t)-}(y) = (0,1)$.

Proof. For simplicity of notation, let $s = e(x,t)$. Since it is assumed that (x,t) is not an event point, then $e(x,t) < t$. We first prove (3.8.a) by contradiction. Supposing otherwise, then $(x,s) \in X_x^{(0)} \cup X_x^{(1)}$. If $(x,s) \in X_x^{(0)}$, then $\eta_s(x) = (0,0)$; whereas if $(x,s) \in X_x^{(1)}$, then $\eta_s(x) = (1,1)$. In either case, the continuation rule (2.5) implies that $\eta_t(x) \neq (0,1)$, contradicting the assumption.

We next prove (3.8.b) by contradiction. Setting $n_0 = |\mathcal{N}(x)|$, and letting n_{00}, n_{01} and n_{11} denote the numbers of $y \in \mathcal{N}(x)\cup\{x\}$ such that $\eta_{s-}(y) = (0,0)$, $\eta_{s-}(y) = (0,1)$,

$\eta_{s-}(y) = (1,1)$, a little thought reveals that $\eta_s(x) = (0,1)$ implies that

$$n_{00} + n_{01} \geq n_0/2, \quad n_{01} + n_{11} \geq n_0/2. \tag{3.32}$$

Since $n_{00} + n_{01} + n_{11} = n_0$, for both inequalities in (3.32) to hold, either $n_{01} > 0$, or $n_{00} = n_{11} = n_0/2$. Since the former already implies that there exists some $y \in \mathcal{N}(x)$ such that $\eta_{e(x,t)-}(y) = (0,1)$, to prove (3.8.b), it suffices to show $\eta_{e(x,t)-}(x) = (0,1)$ if $n_{00} = n_{11} = n_0/2$. Supposing otherwise, then $\eta_{e(x,t)-}(x) = (0,0)$ or $(1,1)$. In either case, it follows from $n_{00} = n_{11} = n_0/2$ and (2.5.c) that $\eta_t(x) \neq (0,1)$. This completes the proof of Lemma 3.8. □

In summary, for each $\eta_t(x) = (0,1)$, (3.8.a) ensures the latest event point at x is a vote point, and (3.8.b) implies that there exists some $y \in \mathcal{N}(x) \cup \{x\}$ with $\eta_{e(x,t)-}(y) = (0,1)$. This leads to the introduction of the d-path, similar to that in Section 3.1.

Definition 3.9 (d-path)**.** *For a given $x \in S$ and $t > 0$, a d-path starting at (x,t) is a finite sequence of event points $\{(x_i,t_i)\}_{i=0}^m$, with $(x_0,t_0) = (x, e(x,t))$, such that the following hold:*

(3.9.a) For every $1 \leq i \leq m$, $x_i \in \mathcal{N}(x_{i-1}) \cup \{x_{i-1}\}$, and $t_i = e(x_i, t_{i-1})$.

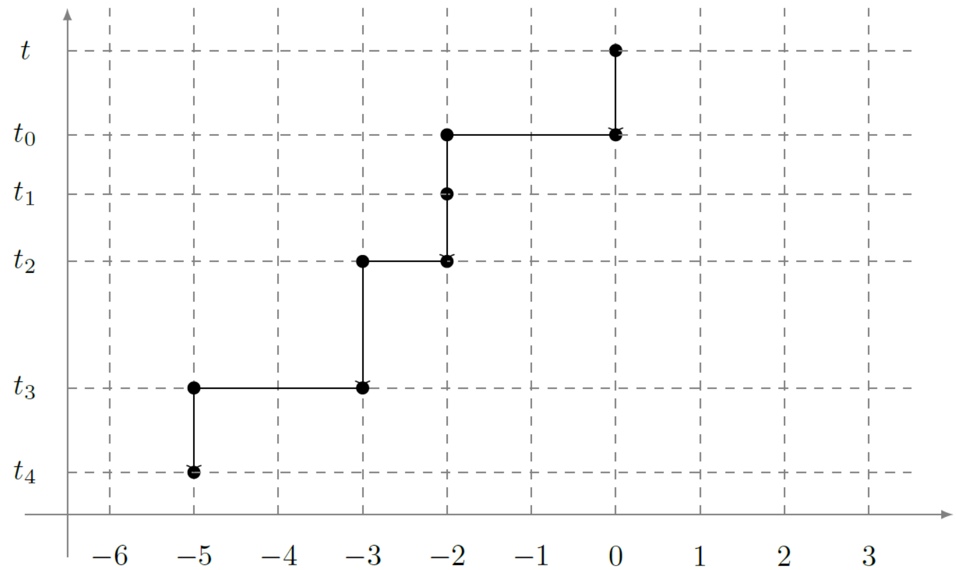
(3.9.b) For every $0 \leq i \leq m-1$, $(x_i,t_i) \in X_x^{(2)}$ and $\eta_{t_i}(x_i) = (0,1)$.

(3.9.c) For every $1 \leq i \leq m$, if $x_i \neq x_{i-1}$, then $\eta_{t_{i-1}-}(x_{i-1}) \neq (0,1)$.

In Figure 3.1, we present a diagram for an example of the d-path, where $S = \mathbb{Z}$ and $\mathcal{N}(x) = \{x-2, x-1, x+1, x+2\}$, for each $x \in \mathbb{Z}$. We briefly explain the existence of such a sequence conditioned on $\{\eta_t(x) = (0,1)\}$ – the first two conditions (3.9.a) and (3.9.b) are natural consequences of Lemma 3.8; whereas the last condition (3.9.c) can be achieved by "not diverting the d-path" as long as (3.9.a) and (3.9.b) hold for the same x. Our next Proposition formulates this idea and proves the almost sure existence of such a finite d-path, conditioned on $\{\eta_t(x) = (0,1)\}$.

Proposition 3.10. *Consider the coupled process $\{\eta_t\}_{t\geq 0}$ on $(S, \mathcal{N})$. Suppose that $\eta_t(x) = (0,1)$ for some $x \in S$ and $t > 0$. Then there exist almost surely some $m \in \mathbb{N}$ and a d-path $\{(x_i,t_i)\}_{i=0}^m$, with $(x_0,t_0) = (x, e(x,t))$ and $t_m = 0$.*

Figure 3.1: An example of a d-path



A (segment of) d-path $\sigma = \{(x_i, t_i)\}_{i=0}^{4}$ is displayed: For each $1 \le i \le 4$, $x_i \in \mathcal{N}(x_{i-1})$ and $t_i = e(x_i, t_{i-1})$. Each event point $(x_i, t_i) \in \sigma$, for $0 \le i \le 4$, is a vote point.

Proof. We will construct $\{(x_i, t_i)\}_{i=0}^{m}$ recursively to prove Proposition 3.10. Since event points are determined by Poisson point processes, (x, t) is not an event point with probability 1. Assuming that (x, t) is not an event point, Lemma 3.8 then applies. Letting $t_0 = e(x, t)$, it follows from the continuation rule (2.11) and (3.8.a) that $\eta_{t_0}(x_0) = (0, 1)$ and (x_0, t_0) is a vote point. Now there are two possibilities for $\eta_{t_0-}(x_0)$: If $\eta_{t_0-}(x_0) = (0, 1)$, we set $x_1 = x_0$. Otherwise, it follows from (3.8.b) that there exists some $y \in \mathcal{N}(x_0)$ with $\eta_{t_0-}(y) = (0, 1)$. We then set $x_1 = y$ and $t_1 = e(x_1, t_0)$. It is easy to verify that (x_0, t_0) and (x_1, t_1) satisfy (3.9.a), (3.9.b), and (3.9.c).

Suppose that $\{(x_i, t_i)\}_{i=0}^{k}$ are constructed to satisfy (3.9.a), (3.9.b), (3.9.c), and $t_k > 0$ (If $t_k = 0$, $\{(x_i, t_i)\}_{i=0}^{k}$ is the desired d-path). There are again two possibilities for $\eta_{t_k-}(x_k)$: If $\eta_{t_k-}(x_k) = (0, 1)$, we set $x_{k+1} = x_k$; Otherwise, Lemma 3.8 applies and there exists some $y \in \mathcal{N}(x_k)$ with $\eta_{t_k-}(y) = (0, 1)$. We then set $x_{k+1} = y$ and $t_{k+1} = e(x_{k+1}, t_k)$. It is easy to verify that the sequence $\{(x_i, t_i)\}_{i=0}^{k+1}$ still satisfies (3.9.a), (3.9.b), and (3.9.c).

We next show that m is almost surely finite. Recall that $\{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x\in S}$ is determined by an independent family of Poisson point processes. Therefore, $t_i - t_{i+1}$, $0 \leq i \leq m-2$, is a family of independent random variables, with each $t_i - t_{i+1}$, $0 \leq i \leq m-2$, exponentially distributed, with $\mathbb{E}(t_i - t_{i+1}) = 1$. When m is fixed, $\sum_{i=0}^{m-2}(t_i - t_{i+1})$ is Gamma distributed, with its shape parameter $m-1$ and scale parameter 1. Since

$$t = \sum_{i=0}^{m-1}(t_i - t_{i+1}) > \sum_{i=0}^{m-2}(t_i - t_{i+1}),$$

for every $t > 0$ and $M \in \mathbb{N}$,

$$\mathbb{P}(m \geq M) \leq \mathbb{P}(\sum_{i=0}^{M-2}(t_i - t_{i+1}) < t) = \frac{\gamma(M-1,t)}{(M-1)!}, \tag{3.33}$$

where $\gamma(s,x)$ is the lower incomplete Gamma function

$$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt.$$

With t fixed, it follows from (3.33) that $\mathbb{P}(m \geq M) \to 0$ as $M \to \infty$, hence m is almost surely finite. $\square$

3.2.2 Terminology: s-paths and Successful s-paths

This subsection introduces two crucial concepts for the proof of Theorems 1 and 2, namely the s-path and successful s-path.

Recall that, for a given uniformly self-supporting structure $(S, \mathcal{N})$, there exist a constant $R > 0$ and a set $\mathscr{E}$ of self-supporting sets, satisfying (1.5), and also that every $A \in \mathscr{E}$ is an elementary self-supporting set. In the remainder of this chapter, we fix a choice of R and $\mathscr{E}$ satisfying (1.5).

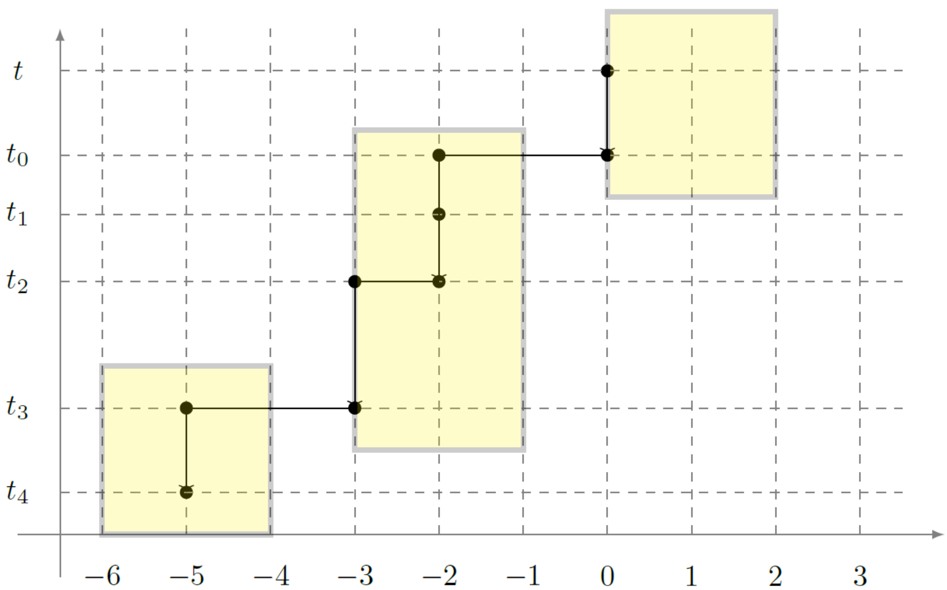
Recall a metric $d(\cdot,\cdot)$ on elements $x, y \in S$ was introduced at the beginning of Subsection 1.1.1. We extend the definition of $d(\cdot,\cdot)$ to all sets $A_1, A_2 \subseteq S$ as follows.

$$d(A_1, A_2) = \inf\{d(x_1, x_2) : x_1 \in A_1, x_2 \in A_2\}. \tag{3.34}$$

We note that, since $d(\cdot,\cdot)$ only takes integer-values, and $d(x_1, x_2) \geq 0$ for every $x_1, x_2 \in S$, for each $A_1, A_2 \subseteq S$, the infimum in (3.34) can be attained by some $x_1 \in A_1$ and $x_2 \in A_2$.

We next introduce the s-path.

Figure 3.2: An example of an s-path



A (segment of) an s-path $\tau = \{A_i\}_{i=0}^{3}$ around the d-path σ, which was presented in Figure 3.1, is displayed: Here, $A_0 = \{0, 1, 2\}$, $A_1 = \{-3, -2, -1\}$, and $A_2 = \{-6, -5, -4\}$.

Definition 3.11 (s-path)**.** *A finite sequence of elementary self-supporting sets $\tau = \{A_i\}_{i=0}^{m}$ is an s-path if, for every $1 \leq i \leq m$, $A_{i-1} \cap A_i = \emptyset$, and $d(A_i, A_{i-1}) \leq 2R$, where R is the same constant as in (1.5).*

We note that an s-path $\tau = \{A_i\}_{i=0}^{m}$ is a deterministic sequence of elementary self-supporting sets and does not rely on the realization of the Poisson clock, unlike d-paths. In Figure 3.2, we present an example of an s-path τ, structured around the d-path σ given in Figure 3.1.

To introduce the successful s-paths, we first introduce the following notation. For a finite $A \subseteq S$ and $0 < s < r$, consider the event

$$\mathscr{B}(A, s, r) := \{e(y, r) > s, \text{ and } (y, e(y, r)) \in X_y^{(0)} \text{ for every } y \in A\}. \tag{3.35}$$

In particular, for every finite $A \subseteq S$ and $t > 0$, we let $\mathscr{B}(A, t)$ denote the event $\mathscr{B}(A, 0, t)$. In other words, $\mathscr{B}(A, t)$ denotes the event that $(y, e(y, t))$ is a 0-noise point for every

$y \in A$. It follows from Definition 2.5 that, conditioning on $\mathscr{B}(A,t)$, $\eta_t(y) = (0,0)$ for every $y \in A$.

We next define

$$e_B(A,t) = \sup\{s < t : \mathscr{B}(A,s) \text{ occurs}\}, \tag{3.36}$$

and set $e_B(A,t) = -1$ if the supremum in (3.36) is taken over an empty set. We also define

$$e_{\varepsilon 1} = \sup\{s < t : (y,s) \in X_y^{(1)} \text{ for some } y \in A\}, \tag{3.37}$$

and set $e_{\varepsilon 1} = 0$ if the supremum in (3.37) is taken over an empty set. We also set

$$e_\varepsilon(A,t) = \max\{e_B(A,t), e_{\varepsilon 1}(A,t)\}. \tag{3.38}$$

For every elementary self-supporting set A, we introduce a stochastic process $\{\chi_t^{(A)}\}_{t\geq 0}$ as follows.

$$\chi_t^{(A)} = \begin{cases} 1 & \text{if } e_{\varepsilon 1}(A,t) > e_B(A,t), \\ 0 & \textit{otherwise}. \end{cases} \tag{3.39}$$

For each elementary self-supporting set $A \subseteq S$, we say A is *activated* at time t if $\chi_t^{(A)} = 1$. We also say an elementary self-supporting set A is *activated* at time $t-$ if $\chi_{t-}^{(A)} = 1$, where $\chi_{t-}^{(A)} = \lim_{r\to t-} \chi_r^{(A)}$.

We next introduce the *successful s-path*.

Definition 3.12. *Let $\tau = \{A_i\}_{i=0}^m$ denote an s-path. We say τ is successful at time t if there exists a non-increasing sequence $\{s_i\}_{i=0}^m$, with $s_0 = t$, such that the following hold:*

(3.12.a) For every $0 \leq i \leq m$, A_i is activated at time s_i.

(3.12.b) Setting

$$r_i = \inf\{t \leq s_i : A_i \textit{ is activated during } (t, s_i)\}, \tag{3.40}$$

for every $0 \leq i \leq m$, then $s_i > r_{i-1}$ for every $1 \leq i \leq m$.

(3.12.c) Whenever $A_i \cap A_j \neq \emptyset$ for some $i < j$, we have $r_i > s_j$.

We will refer to $\{s_i\}_{i=0}^m$ and $\{r_i\}_{i=0}^m$ as the *s-sequence* and *r-sequence* of τ, respectively.

3.2.3 Properties of s-paths

In this subsection, we state some important properties of the s-paths and successful s-paths. As usual, $(S, \mathcal{N})$ denotes a uniformly self-supporting structure, and $x \in S$ denotes an arbitrary site.

We begin with a combinatorial estimate for the number of s-paths $\{A_i\}_{i=1}^m$ with a fixed length m and satisfying $x \in A_1$.

Lemma 3.13. *Let $x \in S$ denote an arbitrary site and $N(x, j)$ denote the number of s-paths $\{A_i\}_{i=1}^j$ with $x \in A_1$. Then there exists some constant C_0, not depending on x, such that*

$$N(x, j) \leq C_0^j, \quad \text{for every } x \in S. \tag{3.41}$$

Proof. We first estimate the number of all possible elementary self-supporting sets $A_1 \ni x$. Setting

$$\mathscr{E}_x = \{A \in \mathscr{E} : x \in A\}, \tag{3.42}$$

we claim that

$$|\mathscr{E}_x| \leq 2^{3(K-1)^{2R}}, \tag{3.43}$$

where K is the constant such that $|\mathcal{N}(x)| \leq K$ for every $x \in S$. To see (3.43), we first note that, for every $A \in \mathscr{E}_x$, there exists some $y \in S$ such that $A \subseteq B(y, R)$. Then it follows from the triangle inequality that $A \subseteq B(x, 2R)$. Therefore, (3.43) follows from the simple inequality that

$$|B(x, 2R)| \leq 1 + \sum_{j=1}^{2R} (K-1)^j < 3(K-1)^{2R}. \tag{3.44}$$

Hence, there are no more than $2^{3(K-1)^{2R}}$ different ways of choosing A_1.

Suppose that $\{A_i\}_{i=1}^{k-1}$ has already been chosen and satisfies $0 < d(A_i, A_{i-1}) \leq 2R$ for every $2 \leq i \leq k-1$. We first estimate the number of all possible A_k, such that $0 < d(A_{k-1}, A_k) \leq 2R$.

By (1.5), every elementary self-supporting set $A \subseteq B(y, R)$ for some $y \in S$. It follows from the triangle inequality that, for every $A \in \mathscr{E}$,

$$A \subseteq B(z, 2R) \quad \text{for every } z \in A. \tag{3.45}$$

Since $d(A_{k-1}, A_k) \leq 2R$, there exist some $w \in A_{k-1}$ and $u \in A_k$ such that

$$d(w, u) = d(A_{k-1}, A_k) \leq 2R. \tag{3.46}$$

We claim the number of all possible A_k does not exceed

$$C_0 = 3(K-1)^R \cdot 3(K-1)^{2R} \cdot 2^{3(K-1)^{2R}}, \tag{3.47}$$

where K is a constant such that $|\mathcal{N}(x)| \leq K$ for every $x \in S$. To see this, we first note that, since A_{k-1} has already been chosen, there are no more than $|A_{k-1}|$ different ways of choosing $w \in A_{k-1}$ to satisfy (3.46). Since $A_{k-1} \subseteq B(y, R)$ for some $y \in S$, $|A_{k-1}| \leq |B(y, R)|$. Moreover, we have

$$|B(y, R)| < 1 + K \sum_{i=0}^{R-1} (K-1)^i \leq 3(K-1)^R, \tag{3.48}$$

which gives the first factor in (3.47).

After choosing w, we choose u to satisfy (3.46). Since $d(w, u) \leq 2R$, we have $u \in B(w, 2R)$. Hence, there is no more than $|B(w, 2R)|$ different ways of choosing u, which contributes to the second factor in (3.47). After choosing u, (3.45) and the triangle inequality implies that $A_k \subseteq B(u, 2R)$. Hence, there are at most $2^{3(K-1)^{2R}}$ different ways of choosing A_k, which is the third factor in (3.47). This completes the proof of Lemma 3.13. $\square$

For each $x \in S$, we define its *self-supporting support* as

$$\Psi(x) = \bigcup_{A \in \mathscr{E}_x} A, \tag{3.49}$$

where $\mathscr{E}_x$ was introduced in (3.42). For each finite $A \subseteq S$, we define its *self-supporting closure* as

$$\tilde{A} = \bigcup_{x \in A} \Psi(x). \tag{3.50}$$

It is easy to verify that, for every finite $A \subseteq S$, $\tilde{A}$ is finite, satisfies

$$\tilde{A} \subseteq \bigcup_{y \in A} B(y, 2R),$$

and is self-supporting.

Let $\tilde{B}(x,r)$ denote the self-supporting closure of $B(x,r)$, and let $\mathcal{G}(x,r,t)$ denote the event that there exists some $0 < s < t$ such that $\eta_s(y) = (0,0)$, for all $y \in \tilde{B}(x,r)$. Our next proposition shows that, for a fixed $r \in \mathbb{N}$, $\{\mathbb{P}(\mathcal{G}(x,r,t))\}_{x\in S}$ converges uniformly to 1 as $t \to \infty$.

Proposition 3.14. *For an arbitrary $r \in \mathbb{N}$ and $\delta \in (0,1)$, there exists some $T = T(\delta, r, \varepsilon_0) > 0$ such that*

$$\mathbb{P}(\mathcal{G}(x,r,t)) \geq \delta, \quad \textit{for every } x \in S \textit{ and } t \geq T. \tag{3.51}$$

Proof. Since $\mathcal{G}(x,r,s)$ implies $\mathcal{G}(x,r,t)$ for every $s < t$, for fixed x and r, $\mathbb{P}(\mathcal{G}(x,r,t))$ is increasing as a function of t. To prove (3.51), it suffices to prove

$$\mathbb{P}(\mathcal{G}^c(x,r,T)) \leq 1 - \delta, \quad \text{for every } x \in S. \tag{3.52}$$

Since $\tilde{B}(x,r)$ is the self-supporting closure of $B(x,r)$, we have $\tilde{B}(x,r) \subseteq B(x,r+2R)$, and

$$|\tilde{B}(x,r)| < 3(K-1)^{r+2R}. \tag{3.53}$$

To simplify notation, we set $A = \tilde{B}(x,r)$, and consider the event

$$\mathscr{B}(A,s,r) := \{e(y,r) > s \text{ and } (y, e(y,t)) \in X_y^{(0)} \text{ for all } y \in A\}. \tag{3.54}$$

Due to the independence of 0-noise points at distinct sites,

$$\mathbb{P}(\mathscr{B}(A,s,r)) = (\varepsilon_0(1 - e^{s-r}))^{|A|}. \tag{3.55}$$

In particular, it follows from (3.53) that

$$\mathbb{P}(\mathscr{B}(A,i,i+1)) = (\varepsilon_0(1-e^{-1}))^{|A|} > (\varepsilon_0/2)^{u(r)}, \tag{3.56}$$

where $u(r) = 3(K-1)^{r+2R}$.

Since $\mathscr{B}(A,i,i+1)$ implies $\mathcal{G}(x,r,i+1)$ for every $i \in \mathbb{N}$,

$$\mathbb{P}(\mathcal{G}^c(x,r,m)) \leq \mathbb{P}\Big(\bigcap_{i=0}^{m-1} \mathscr{B}^c(A,i,i+1)\Big) < (1 - (\varepsilon_0/2)^{u(r)})^m. \tag{3.57}$$

Let m denote the smallest integer satisfying

$$m > \frac{\ln(1-\delta)}{\ln(1-(\varepsilon_0/2)^{u(r)})}. \tag{3.58}$$

Then, for every fixed $r \in \mathbb{N}$, it follows from (3.57) and (3.58) that

$$\mathbb{P}(\mathcal{G}^c(x,r,m)) < 1-\delta, \quad \text{for every } x \in S. \tag{3.59}$$

This completes the proof of Proposition 3.14. $\square$

We conclude this section with the most crucial properties of successful s-path, namely Propositions 3.15 and 3.16. In a nutshell, Proposition 3.15 shows that, conditioning on $\{\eta_t(x) = (0,1)\} \cap \mathcal{G}(x,r,t)$, there exists at least one s-path $\{A_i\}_{i=0}^m$, with $A_0 \ni x$, which is successful at time t and has a length that is bounded from below by a constant fraction of r. For better readability, we delay the proof of Proposition 3.15 until Section 3.4.

Proposition 3.15. *Let $x \in S$ denote an arbitrary site. For every $t > 0$ and $r \in \mathbb{N}$ with $r > 5R$, conditioning on $\{\eta_t(x) = (0,1)\} \cap \mathcal{G}(x,r,t)$, there exists an s-path $\{A_i\}_{i=0}^m$ that is successful at time t, with $x \in A_0$, $A_i \subseteq \tilde{B}(x,r)$ for each $0 \le i \le m$, and $m \ge r/(4R)$.*

We next state Proposition 3.16, whose proof will be delayed until Section 3.5. Proposition 3.16 gives an upper bound on the probability of a given s-path of length l being successful at time t. Moreover, the upper bound given in Proposition 3.16 does not depend on t.

Proposition 3.16. *Let $x \in S$ denote an arbitrary site and $r, m \in \mathbb{N}$ denote two positive integers. Consider an s-path $\tau = \{A_i\}_{i=0}^m$, with $A_0 \ni x$ and $A_i \subseteq B(x,r)$ for each $0 \le i \le m$. Let $\mathcal{E}(\tau,t)$ denote the event that τ is successful at time t. Then*

$$\mathbb{P}(\mathcal{E}(\tau,t) \mid \mathcal{G}(x,r,t)) < \left(\frac{2}{1+q}\right)^{m+1}, \tag{3.60}$$

where $q = 2\varepsilon_0/3(K-1)^R\varepsilon_1$.

3.3 Proof of Theorem 1 Assuming Propositions 3.15, 3.16

This section states the proof of Theorem 1 assuming Propositions 3.15 and 3.16. We consider the majority vote process $\{\xi_t\}_{t\ge 0}$ on a uniformly self-supporting structure $(S,\mathcal{N})$. Recall that there exists a constant K such that $|\mathcal{N}(x)| < K$ for every $x \in S$. As usual, we let ε_0 and ε_1 denote its 0-noise and 1-noise intensities, and assume that ε_0 and ε_1 satisfy

$$\varepsilon_1 \le \frac{2\varepsilon_0}{3(K-1)^R(4C_0-1)}, \tag{3.61}$$

where R is the constant satisfying (1.5) and C_0 is the constant introduced in Lemma 3.13, which does not depend on ε_0 and ε_1. We note that, for given $\varepsilon_0 \in (0,1)$, ε_1 can always be taken small enough to satisfy (3.61). In the remainder of this subsection, we prove that ξ_t is ergodic whenever (3.61) is satisfied.

We will use Proposition 2.7 to prove the ergodicity of ξ_t, which asserts that, for ξ_t to be ergodic, it suffices for (2.14) to hold for the coupled process $\{\eta_t\}_{t\geq 0}$.

For a fixed $x \in S$, the probability $\mathbb{P}(\eta_t(x) = (0,1))$ is non-increasing as a function of t. To show (2.14), it suffices to show that, for each given $x \in S$ and $p \in (0,1)$, there exists some $T > 0$ such that,

$$\mathbb{P}(\eta_T(x) = (0,1)) < p. \tag{3.62}$$

For this, we will employ the union bound

$$\mathbb{P}(\eta_T(x) = (0,1)) \leq \mathbb{P}(\mathcal{G}^c(x,r,T)) + \mathbb{P}(\eta_T(x) = (0,1) \mid \mathcal{G}(x,r,T)), \tag{3.63}$$

where $\mathcal{G}(x,r,T)$ was introduced in Proposition 3.15, and $r \in \mathbb{N}$ will be determined later.

To bound the first term on the right-hand side of (3.63), we apply Proposition 3.14: For any given ε_0, ε_1 and r, we can choose $T_0 > 0$ large enough such that

$$\mathbb{P}(\mathcal{G}^c(x,r,T)) < \frac{p}{2} \quad \text{for each } x \in S \text{ and } T > T_0. \tag{3.64}$$

In particular, we can choose T_0 large enough such that

$$T_0 > \frac{\ln(1-p/2)}{\ln(1-(\varepsilon_0/2)^{3(K-1)^{r+R}}} + 1. \tag{3.65}$$

Then (3.64) follows directly from (3.57).

To obtain an upper bound on the second term on the right-hand side of (3.63), we apply Propositions 3.15, 3.16 and Lemma 3.13. First, conditioning on $\mathcal{G}(x,r,T)$, Proposition 3.15 implies that there exists at least one s-path $\tau = \{A_i\}_{i=0}^{l}$, with $x \in A_0$ and $l \geq r/4R$, such that τ is successful at time T. The following holds after applying a simple union-bound:

$$\mathbb{P}(\eta_T(x) = (0,1) \mid \mathcal{G}(x,r,T)) \leq \sum_{\sigma} \mathbb{P}(\mathcal{E}(\tau,T) \mid \mathcal{G}(x,r,T)), \tag{3.66}$$

where the summation in (3.66) is taken over all s-paths σ satisfying the conditions given in Proposition 3.15, and $\mathcal{E}(\tau,t)$ denotes the event that the s-path τ is successful at time t.

We next rewrite the summation in (3.66) by first summing over all path τ of the same length, then summing over all possible lengths l, satisfying $l \geq r/4R$. Letting $\Sigma(l)$ denotes the set of all s-path $\{A_i\}_{i=0}^{l}$ with $x \in A_0$, then, it follows from (3.66) that

$$\mathbb{P}(\eta_T(x) = (0,1) \mid \mathcal{G}(x,r,T)) \leq \sum_{l=\lfloor r/4R \rfloor}^{\infty} \sum_{\tau \in \Sigma(l)} \mathbb{P}(\mathcal{E}(\tau,T) \mid \mathcal{G}(x,r,T)), \tag{3.67}$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Furthermore, it follows from Lemma 3.13 that

$$|\Sigma(l)| < C_0^{l+1},$$

where C_0 was introduced in (3.47). On the other hand, Proposition 3.16 implies that, for an s-path τ whose length is $l+1$,

$$\mathbb{P}(\mathcal{E}(\tau,t) \mid \mathcal{G}(x,r,T) \leq \left(\frac{2}{1+q}\right)^{l+1},$$

where $q = 2\varepsilon_0/3(K-1)^R\varepsilon_1$. Combining these two estimates, it follows from (3.67) that

$$\mathbb{P}(\eta_T(x) = (0,1)|\mathcal{G}(x,r,T)) \leq \sum_{j=l_0}^{\infty} C_0^j (\frac{2}{1+q})^j, \tag{3.68}$$

where $l_0 = \lfloor r/4R \rfloor$. We note that, when ε_0 and ε_1 are given, the upper bound in (3.68) does not depend on the choice of T and is determined by r. When ε_0 and ε_1 satisfy (3.61), $2C_0/(1+q) < 1/2$. We can choose r large enough such that

$$\sum_{j=l_0}^{\infty} C_0^j \left(\frac{2}{1+q}\right)^j \leq 2\left(\frac{2C_0}{1+q}\right)^{r/4R} \leq \frac{p}{2}. \tag{3.69}$$

We next demonstrate that, for every $\varepsilon_0 \in (0,1)$ and $p \in (0,1)$, if ε_1 satisfies (4.21), then, for every $p > 0$, there exists some T, such that (3.62) holds for every $x \in S$. For this, we need to choose r, T as follows. We first choose r large enough such that (3.69) holds. We next choose $T > T_0$, which is given in (3.65) and depends on p, r, ε_0 and ε_1. This implies (3.64). Then, it follows from the union bound (3.63) that

$$\mathbb{P}(\eta_T(x) = (0,1)) \leq \mathbb{P}(\mathcal{G}^c(x,r,T)) + \mathbb{P}(\eta_T(x) = (0,1) \mid |\mathcal{G}(x,r,T)) < p.$$

This completes the proof of Theorem 1.

3.4 Proof of Proposition 3.15

This section gives the long proof of Proposition 3.15. As usual, throughout this section, $(S, \mathcal{N})$ denotes a uniformly self-supporting structure. Recall that there is a fixed constant $R > 0$ and a collection of self-supporting sets $\mathcal{E}$, such that R and $\mathcal{E}$ satisfy (1.5). Also, recall every $A \in \mathcal{E}$ is an elementary self-supporting set, and each $\mathcal{E}_x$ denotes the collection of elementary self-supporting sets $A \ni x$.

We let $\{\eta_t\}_{t \geq 0}$ denote the coupled process on $(S, \mathcal{N})$, with its initial configuration $\eta_0(x) \equiv (0,1)$ for every $x \in S$. Recall that $\mathcal{G}(x,r,t)$ denotes the event that there exists some $0 < s < t$ such that $\eta_s(y) = (0,0)$ for all $y \in \tilde{B}(x,r)$, where $\tilde{B}(x,r)$ is the self-supporting closure of $B(x,r)$, and $B(x,r)$ denotes the r-radius ball, centered at x, with respect to the distance $d(\cdot,\cdot)$, which was introduced in (1.3).

Proposition 3.15 states that, for each $r > 5R$, if one conditions on $\{\eta_t(x) = (0,1)\} \cap \mathcal{G}(x,r,t)$, then there exists an s-path $\tau = \{A_i\}_{i=0}^m$, with $m \geq r/4R$, $x \in A_0$, and with

$$A_i \subseteq \tilde{B}(x,r) \quad \text{for each } 0 \leq i \leq m, \tag{3.70}$$

that is successful at time t.

To prove Proposition 3.15, we first choose any d-path $\sigma = \{(x_i, t_i)\}_{i=0}^M$ satisfying Proposition 3.10, and then construct a successful s-path "around" σ. The proof is divided into the following three parts – we first prove that, for each $x \in S$ and $t \geq 0$ with $\eta_t(x) = (0,1)$, every elementary self-supporting set $A \ni x$ is activated at time t. This is done in Lemma 3.17. Since every event point $(y,s) \in \sigma$ has $\eta_s(y) = (0,1)$, each d-path σ leads to a collection of elementary self-supporting sets, each of which is activated at a time t_i. Next, among these elementary self-supporting sets, we choose appropriate ones to form an s-path $\tau = \{A_i\}_{i=0}^m$ that is successful at time t. Lastly, we show this successful s-path also satisfies (3.70) and $m \geq r/4R$.

We begin with Lemma 3.17.

Lemma 3.17. *Let A denote an elementary self-supporting set and x denote a site $x \in A$. Conditioning on $\{\eta_t(x) = (0,1)\}$, A is activated at time t.*

Proof. We prove Lemma 3.17 by contradictions. Supposing otherwise, then we have $u = e_B(A,t) > e_{\varepsilon_1}(A,t)$. It then follows from (3.35) and (3.36) that

$$(y, e(y,u)) \in X_y^{(0)} \quad \text{for each } y \in A. \tag{3.71}$$

Since η_t satisfies the continuation rule (2.11), (3.71) implies that $\eta_u(y) = (0,0)$ for every $y \in S$.

Since $e_{\varepsilon 1}(A,t) < u$, every event point (y,s) with $y \in A$ and $u < s \leq t$ is either a 0-noise point or a vote point. Then, a little thought shows that $\eta_t(y) = (0,0)$ for every $y \in A$. This contradicts the assumption that $\eta_t(x) = (0,1)$, hence completing the proof of Lemma 3.17. $\square$

We next fix a d-path $\sigma = \{(x_i,t_i)\}_{i=0}^M$ satisfying Proposition 3.10, and find the smallest m' such that $x_{m'} \notin \tilde{B}(x,r)$.

Construction of σ and m'

Conditioning on $\{\eta_t(x) = (0,1)\}$, Proposition 3.10 implies that there exists a d-path $\sigma = \{(x_i,t_i)\}_{i=0}^M$, with $(x_0,t_0) = (x, e(x,t))$ and $t_M = 0$, satisfying (3.9.a), (3.9.b), and (3.9.c). Conditioning on $\mathcal{G}(x,r,t)$, there exists some $s \in (0,t)$ such that $\eta_s(y) = (0,0)$ for all $y \in \tilde{B}(x,r)$. Since $\{t_i\}_{i=0}^{m'}$ is strictly decreasing, there exists some $w \in \mathbb{N}$ such that $t_{w+1} < s \leq t_w$. A little thought shows that (3.9.b) implies that

$$\eta_s(x_{w+1}) = \eta_{t_{w+1}}(x_{w+1}) = (0,1).$$

Since $\eta_s(y) = (0,0)$ for every $y \in \tilde{B}(x,r)$, $x_{w+1} \notin \tilde{B}(x,r)$. Setting

$$m' = \min\{j : x_j \notin \tilde{B}(x,r)\}, \tag{3.72}$$

then, the minimum in (3.72) is not taken over an empty set. Moreover, the minimality of m' ensures that

$$x_i \in \tilde{B}(x,r) \quad \text{for every } 0 \leq i \leq m' - 1. \tag{3.73}$$

With such a d-path σ and m' chosen, in the remainder of this section, we will first construct an s-path $\tau = \{A_i\}_{i=0}^m$, around this σ, with the value of m to be determined later. After choosing τ, we prove that τ satisfies Proposition 3.15. The s-path τ will be constructed recursively, beginning with A_0 and A_1. We will introduce an auxiliary sequence $\{k_i\}_{i=-1}$, with $k_{-1} = 0$ – the value of k_i is determined after choosing each A_i.

Construction of A_0 and A_1

We begin with the construction of A_0. Recall that $\mathcal{E}_x$ was introduced in (3.42), with $\mathcal{E}_x \neq \emptyset$ for every $x \in S$. According to the triangle inequality, for each $x \in S$ and every $A \in \mathcal{E}_x$,

$$A \subseteq B(x, 2R) \subseteq \tilde{B}(x, 2R). \tag{3.74}$$

Conditioning on $\{\eta_t(x) = (0,1)\}$, it follows from Lemma 3.17 that every elementary self-supporting set $A \ni x$ is activated at time t. Setting A_0 to be any $A \in \mathcal{E}_x$ and $s_0 = t$, then A_0 is activated at time s_0. With $s_0 = t$, r_0 is defined, by (3.40), as

$$r_0 = \inf\{u < t : A_0 \text{ is activated during } (u,t)\}. \tag{3.75}$$

Since $x \in B(x,r)$ and $A_0 \in \mathcal{E}_x$, $A_0 \subseteq \tilde{B}(x,r)$.

In addition to A_0 and s_0, we also define

$$k_0 = \min\{k \in \mathbb{N} : x_k \notin A_0\}.$$

Such a k_0 always exists since $x_{m'} \notin \tilde{B}(x,r)$ and $A_0 \subseteq \tilde{B}(x,R)$. Since

$$A_0 \subseteq B(x, 2R) \subsetneq \tilde{B}(x,r),$$

and

$$d(x_{k_0-1}, x_{k_0}) = 1,$$

a little thought shows that $k_0 < m'$. (We note that the sequence $\{k_i\}_{i=-1}^{m-1}$ will be constructed along with the construction of $\{A_i\}_{i=0}^{m}$ and $\{s_i\}_{i=0}^{m}$.)

We proceed to choose A_1. To simplify notation, we let

$$y = x_{k_0}, \quad s = t_{k_0}.$$

Since $y \notin A_0$, by (1.5), there exists some $A' \in \mathcal{E}_y$, such that $A_0 \cap A' = \emptyset$. We set $A_1 = A'$. Since $y \notin A_0$ and $x_{k_0-1} \in A_0$,

$$0 < d(A_0, A_1) \leq d(x_{k_0}, x_{k_0-1}) = 1.$$

Therefore the sequence $\{A_i\}_{i=0}^{1}$ forms an s-path. Lastly, we set $s_1 = s$.

We next verify this s-path $\{A_0, A_1\}$ is successful at time t. For this, it suffices to verify that (3.12.a), (3.12.b) and (3.12.c) hold for $i = 0, 1$.

Since $(x_{k_0}, t_{k_0}) \in \sigma$, (3.9.b) implies that $\eta_{t_{k_0}}(x_{k_0}) = (0,1)$. Hence, it follows from Lemma 3.17 that A_1 is activated at time $s_1 = t_{k_0}$. Therefore, (3.12.a) is satisfied for $i = 1$. Since we have already demonstrated that A_0 is activated at $s_0 = t$, (3.12.a) is satisfied for $i = 0, 1$.

For every $(x_i, t_i) \in \sigma$, (3.9.b) implies that $\eta_{t_i}(x_i) = (0,1)$. For every $0 \leq j \leq k_0$, since $\eta_{t_j}(x_j) = (0,1)$, and $x_j \in A_0$, it follows from Lemma 3.17 that A_0 is activated during $[t_1, t]$ and hence $r_0 < s_1$, which implies (3.12.b).

Since $A_0 \cap A_1 = \emptyset$, (3.12.c) is automatically satisfied. This completes the proof that $\{A_i\}_{i=0}^{1}$ is successful at time t.

After choosing A_1, we set

$$k_1 = \min\{k > k_0 : x_k \notin A_1\}.$$

This completes the initial step of our construction.

Suppose that an s-path $\{A_i\}_{i=0}^{l-1}$ is constructed to satisfy (3.12.a), (3.12.b), and (3.12.c). Also, after choosing each A_i, we set

$$k_i = \min\{k > k_{i-1} : x_k \notin A_{i-1}\} \quad \text{for } 0 \leq i \leq l-1. \tag{3.76}$$

Then, $\{k_i\}_{i=-1}^{l-1} \subseteq \mathbb{N}$ is a strictly increasing sequence. If $k'_h \geq m'$ for some $h \in \mathbb{N}$, where m' was defined in (3.72), then $\{A_i\}_{i=0}^{h}$ is our desired s-path. Hence we can assume that $k_i < m'$ for each $0 \leq i \leq l-1$.

We summarize the procedure of selecting A_l as follows. We first choose a temporary A_l, using a similar approach when choosing A_1. After choosing this temporary A_l, we verify if the sequence $\{A_i\}_{i=0}^{l}$ is successful at time t. Since the way we choose this temporary A_l ensures (3.12.a) and (3.12.b), we focus on whether or not (3.12.c) is satisfied for $j = l$. If $\{A_i\}_{i=0}^{l}$ is successful at time t, then this inductive step is completed and we proceed to choose A_{l+1}. Whereas, if $\{A_i\}_{i=0}^{l}$ is not successful at time t due to the violation of (3.12.c) for $j = l$, then we will "trim" $\{A_i\}_{i=0}^{l-1}$ into a shorter s-path. Moreover, the way we trim $\{A_i\}_{i=0}^{l-1}$ ensures the shortened sequence "utilizes" more sites in σ. Since $\{A_i\}_{i=0}^{l-1}$ and σ are of finite lengths, such a modification of $\{A_i\}_{i=0}^{l}$ can be done only finitely many times. We then show that, after finitely many times of such modifications, we will obtain a successful s-path.

Construction of "temporary" A_l given $\{A_i\}_{i=0}^{l-1}$

We next demonstrate how to choose A_l. Again, for simplicity of notation, we set $y = x_{k_{l-1}}$ and $s = t_{k_{l-1}}$. Since $y \notin A_{l-1}$, according to (1.5), there exists some $A' \in \mathcal{E}_y$ such that $A' \cap A_{l-1} = \emptyset$. We set $A_l = A'$, $s_l = s$, and

$$k_l = \min\{k > k_{l-1} : x_k \notin A_l\}.$$

We first verify that $\{A_i\}_{i=0}^{l}$ is an s-path. Since $y \in \mathcal{N}(x_{k_{l-1}-1})$, according to (3.9.a),

$$d(y, x_{k_{l-1}-1}) = 1.$$

Since $x_{k_{l-1}-1} \in A_{l-1}$, and $A' \cap A_{l-1} = \emptyset$,

$$0 < d(A', A_{l-1}) \leq d(y, x_{k_{l-1}-1}) = 1.$$

Therefore $\{A_i\}_{i=0}^{l}$ is an s-path.

Since we did not make any changes to the choice of s_i and A_i for $0 \leq i \leq l-1$, (3.12.a) and (3.12.b) still hold for $0 \leq i \leq l-1$. We next verify that (3.12.a) and (3.12.b) hold for $i = l$.

Since $\eta_s(y) = (0, 1)$, and $y \in A_l$, Lemma 3.17 implies that A_l is activated at time $s_l = s$. Therefore (3.12.a) holds for $i = l$.

It follows from the minimal property (3.76) that

$$x_j \in A_{l-1} \quad \text{for every } k_{l-2} \leq j \leq k_{l-1} - 1. \tag{3.77}$$

For every $k_{l-2} \leq j \leq k_{l-1} - 1$, since $\eta_{t_j}(x_j) = (0, 1)$, it follows from Lemma 3.17 that A_{l-1} is activated during time $[t_{l-2}, t_{l-1}]$. Therefore,

$$t_{k_{l-1}} = s_l > r_{l-1},$$

which implies (3.12.b) for $i = l$.

Now, if A_l and s_l satisfy (3.12.c), then we can conclude that $\{A_i\}_{i=0}^{l}$ is successful at time t, which completes this step of choosing A_l. But, if (3.12.c) is violated, we need to "trim" the sequence $\{A_i\}_{i=0}^{l-1}$ as follows.

Modification of $\{A_i\}_{i=0}^{l-1}$

Since $\{A_i\}_{i=0}^{l-1}$ has been constructed to satisfy (3.12.c), the only possible value of j for which (3.12.c) might not hold is $j = l$. Let Γ denote the set of all $i < l$ such that (3.12.c) does not hold for A_i and A_l, i.e.,

$$\Gamma = \{i : 0 \leq i < l,\, A_i \cap A_l \neq \emptyset,\, r_i < s_l\}, \tag{3.78}$$

and set

$$u = \min\{i : i \in \Gamma\}.$$

We introduce another s-path $\{C_i\}_{i=0}^{u+1}$ by setting $C_i = A_i$ for every $0 \leq i \leq u$. Let $\{s_i'\}_{i=0}^{u}$ denote the s-sequence of the new s-path $\{C_i\}_{i=0}^{u+1}$, and set $s_i' = s_i$ for every $0 \leq i \leq u$. Since $\{C_i\}_{i=0}^{u}$ is the first $u+1$ terms of the original s-path $\{A_i\}_{i=0}^{l-1}$, a little thought shows that $\{C_i\}_{i=0}^{u}$ is a successful s-path at time t.

We proceed to choose C_{u+1}, using a similar approach when choosing A_l: We set

$$k_u' = \min\{k \geq k_{l-1} : x_k \notin C_u\}, \tag{3.79}$$

where k_{l-1} is defined by (3.76). Since we have assumed that $x_{m'} \notin \tilde{B}(x, r)$ and $m' > k_{l-1}$, the minimum in (3.79) is not taken over an empty set. If $k_u' \geq m'$, then we have completed our construction and $\tau = \{C_i\}_{i=0}^{u}$ is the desired s-path. Assuming $k_u' < m'$, for simplicity of notation, we write

$$z = x_{k_u'}, \quad h = t_{k_u'}.$$

Since $z \notin C_u$, there exists some $A'' \in \mathcal{E}_z$, such that $A'' \cap C_u = \emptyset$. We set $C_{u+1} = A''$, $s_{u+1}' = h$, and

$$k_{u+1}' = \min\{k > k_u' : x_k \notin C_{u+1}\}. \tag{3.80}$$

We first show that $\{C_i\}_{i=0}^{u+1}$ is an s-path. To see this, we need to discuss the following two possibilities for k_u' and k_{l-1}: If $y = x_{k_{l-1}} \notin C_u$ (this is possible since C_u was selected before y), then $k_u' = k_{l-1}$, and hence $y = z$. In this case, since $y \in A_l$ and $A_l \cap C_u \neq \emptyset$, it follows from the triangle inequality that

$$d(C_{u+1}, C_u) \leq d(y, z') \leq 2R, \tag{3.81}$$

where $z' \in S$ denotes a site satisfying $z' \in A_l \cap C_u$. On the other hand, if $y \in C_u$, then $z \neq y$, and hence $x_{k'_u-1} \in C_u$. Therefore,

$$d(C_{u+1}, C_u) \leq d(z, x_{k'_u-1}) = 1. \tag{3.82}$$

In either case, (3.81) and (3.82) imply that $\{C_i\}_{i=0}^{u+1}$ is an s-path.

We next show that $\{C_i\}_{i=0}^{u+1}$ satisfies (3.12.a) and (3.12.b). For this, it suffices to show that (3.12.a) and (3.12.b) hold for $i = u+1$, since $\{C_i\}_{i=0}^{u}$ is successful at time t.

Since $(z, h) \in \sigma$, Proposition 3.10 implies that $\eta_h(z) = (0, 1)$. Then, according to Lemma 3.17, C_{u+1} is activated at time $h = s'_{u+1}$, which implies (3.12.a) for $i = u+1$.

To show (3.12.b) holds for $i = u+1$, we discuss the same two possibilities as before: If $y \notin C_u$, then

$$y = z, \quad h = s = s_l.$$

Since we have assumed that $r_u < s_l$ in (3.78),

$$s_{u+1} = h = s > r_u,$$

which implies (3.12.b) for $i = u+1$.

On the other hand, if $y \in C_u$, then

$$z \neq y, \quad k'_u > k_{l-1}.$$

It therefore follows from the minimal property (3.79) that $z \neq x_{k'_u-1}$, $z \in C_{u+1}$ and $x_{k'_u-1} \in C_u$. According to Proposition 3.10, for every $k_{l-2} \leq j \leq k_{l-1} - 1$, $\eta_{t_j}(x_j) = (0, 1)$. Hence, it follows from Lemma 3.71 that $r_u < h$, which implies that (3.12.b) holds for $i = u+1$.

Such a choice of C_{u+1} does not necessarily imply (3.12.c) yet, but ensures $k'_u \geq k_{l-1}$, and $u < l-1$. If (3.12.c) is satisfied for $j = u+1$, then $\{C_i\}_{i=0}^{u+1}$ is successful at time t, which completes our inductive step. Whereas if (3.12.c) is not satisfied, we will make the same modification to $\{C_i\}_{i=0}^{u}$ as was done to $\{A_i\}_{i=0}^{l-1}$ in this subsection. After each modification, we will obtain a strictly shorter s-path $\{C'_i\}_{i=0}^{w+1}$, with a greater or equal k'_w. Since both the s-path and σ are of finite length, and (3.12.c) always holds when the s-path has length 2, (which is proved in the previous subsection, after choosing A_1), (3.12.c) will be satisfied after a finite number of such modifications.

Proof of (3.70) and $m \geq r/4R$

We repeat this inductive step until $k_m \geq m'$. Then, we obtain an s-path $\tau = \{A_i\}_{i=0}^m$, which is successful at time t, with $d(x_{m'}, A_l) \leq R$.

We verify that τ satisfies (3.70), which follows directly from the construction itself: Since each k_i, $0 \leq i \leq m-1$, satisfies $k_i < m'$; and $A_i \in \mathcal{E}_{x_{k_i}}$ for every $0 \leq i \leq m$, it follows from (3.49) and (3.50) that each $A_i \subseteq \tilde{B}(x, r)$.

We next prove $m \geq r/4R$. Since $d(x_{m'}, A_l) \leq R$, we have

$$r + R \leq d(x_{m'}, x) \leq m(2R + 2R), \tag{3.83}$$

where the first inequality in (3.83) follows from (3.9.a), and the second inequality follows from the fact that $d(A_{i-1}, A_i) \leq 2R$ for each $1 \leq i \leq m$. Then the inequality $m \geq r/4R$ follows immediately from (3.83). This completes the proof of Proposition 3.15.

3.5 Proof of Proposition 3.16

This section proves Proposition 3.16. Recall that an s-path $\{A_i\}_{i=0}^m$ is successful at time t if there exists a non-increasing s-sequence $\{s_i\}_{i=0}^m$, with $s_0 = t$, satisfying (3.12.a), (3.12.b) and (3.12.c).

We outline the proof of Proposition 3.16 as follows. First, in Proposition 3.18, we develop some crucial properties of $\chi_t^{(A)}$. In particular, we derive, as a direct result of Proposition 3.18, an upper bound (3.92) on the probability of an elementary self-supporting set being activated at any given time t. We then, in Subsection 3.5.1, reduce the proof of Proposition 3.16 to that of (3.96). For this, we rewrite the probability for a given s-path to be successful as a product (3.94), and estimate each factor of that product separately. Lastly, we prove (3.96) in Subsection 3.5.2.

Recall that an elementary self-supporting set A is activated at time t if $\chi_t^{(A)} = 1$, or equivalently, if $e_{\varepsilon 1}(A, t) > e_B(A, t)$, where $e_{\varepsilon_1}(A, t)$ and $e_B(A, t)$ were introduced in (3.37) and (3.36). Proposition 3.18 states some important properties of $\chi_t^{(A)}$.

Proposition 3.18. *Let $A, A_1, A_2 \in \mathcal{E}$ denote any elementary self-supporting sets, and $\{\chi_t^{(A)}\}_{t\geq 0}$, $\{\chi_t^{(A_1)}\}_{t\geq 0}$, and $\{\chi_t^{(A_2)}\}_{t\geq 0}$ be defined in (3.39). Then the following hold.*

(1) For each elementary self-supporting set $A \subseteq S$ and $t > 0$,

$$\mathbb{P}(\chi_t^{(A)} = 1 |\, e_B(A,t) > 0) \leq \frac{|A|\varepsilon_1}{|A|\varepsilon_1 + 2\varepsilon_0}. \tag{3.84}$$

(2) If $A_1 \cap A_2 = \emptyset$, then $\chi_s^{(A_1)}$ and $\chi_t^{(A_2)}$ are independent for every $s, t > 0$.

Proof. Recall that $e_{\varepsilon 1}(x,t)$ denotes the last 1-noise point occurring at x before time t. We set

$$Z = \max\{e_{\varepsilon 1}(y,t) : y \in A\}. \tag{3.85}$$

Conditioning on $\cup_{y\in A}\{e_{\varepsilon 1}(y,t) \neq 0\}$, $t - Z$ is exponentially distributed with rate $|A|\varepsilon_1$. Let $\mathscr{B}(A,s,t)$ be defined in (3.54). Then, for every fixed $s \in (0,t)$, conditioning on $\{e_B(A,t) > 0,\, Z = s\}$, $\{\chi_t^{(A)} = 1\}$ occurs if and only if $\mathscr{B}(A,s,t)$ does not occur. Therefore,

$$\begin{aligned}\mathbb{P}(\chi_t^{(A)} = 1 \mid e_B(A,t) > 0) = &\int_0^t \mathbb{P}(\mathscr{B}^c(A,z,t) \mid e_B(A,t) > 0,\, Z = z) \\ &n\varepsilon_1 e^{-n\varepsilon_1(t-z)}\, dz,\end{aligned} \tag{3.86}$$

where $n = |A|$. Conditioning on $\{Z = s\}$, $\mathscr{B}(A,s,t)$ is determined by 0-noise points in A between during time (s,t), as there is no 1-noise point in A during (s,t). Let $e_{\varepsilon 0}(x,t)$ denote the latest 0-noise point occurring at x before time t, i.e.,

$$e_{\varepsilon 0}(x,t) = \sup\{s < t : (x,s) \in X_x^{(0)}\}, \tag{3.87}$$

where the supremum in (3.87) is set to be 0 if taken over an empty set. Then

$$\mathbb{P}(\mathscr{B}^c(A,z,t) \mid e_B(A,t) > 0, Z = z) = 1 - (1 - e^{-\varepsilon_0(t-z)})^n. \tag{3.88}$$

Combining (3.86) and (3.88), we obtain

$$\mathbb{P}(\chi_t^{(A)} = 1 | e_B(A,t) > 0) = \int_0^t n\varepsilon_1 e^{-n\varepsilon_1(t-z)}(1 - (1 - e^{-\varepsilon_0(t-z)})^n)\, dz. \tag{3.89}$$

Setting $\beta = n\varepsilon_1/\varepsilon_0$, we integrate the right-hand side of (3.89) and obtain

$$\begin{aligned}\mathbb{P}(\chi_t^{(A)} = 1 | e_B(A,t) > 0) &\leq n\varepsilon_0\varepsilon_1 \int_0^1 (1 - u^n)(1-u)^{1+\beta}\, du \\ &\leq \frac{\beta}{2+\beta} - n\varepsilon_0\varepsilon_1 B(n+1, 2+\beta),\end{aligned} \tag{3.90}$$

where $B(u,v) = \int_0^1 x^{u-1}(1-x)^{v-1}\,dx$ denotes the real-valued Beta function of the Euler integral. To see (3.90), the first inequality follows from the substitution $u = 1-e^{-\varepsilon_0(t-z)}$, and the second inequality follows directly from the definition of Beta function and $\varepsilon_0, \varepsilon_1 \le 1$.

Since the Beta function $B(u,v)$ is positive, the desired bound (3.84) follows directly from (3.90).

We next prove (2). Set

$$\mathscr{F}(A,t) = \mathscr{F}(\{(y,s) \in X_y^{(0)} \cap X_y^{(1)} \cup X_y^{(2)} : y \in A \text{ and } s < t\}), \tag{3.91}$$

where $\mathscr{F}(X)$ denotes the σ-field generated by random points in X. Then it follows from the proof of (1) that

$$\{\chi_t^{(A_1)} = 1\} \in \mathscr{F}(A_1,t), \quad \{\chi_t^{(A_2)} = 1\} \in \mathscr{F}(A_2,t).$$

Since $A_1 \cap A_2 = \emptyset$, $\{\chi_t^{(A_1)}\}_{t\ge 0}$ and $\{\chi_t^{(A_2)}\}_{t\ge 0}$ are independent. □

3.5.1 Demonstration of Proposition 3.16

We will prove (3.60) with an inductive argument.

Since $(S,\mathcal{N})$ is a uniformly self-supporting structure, according to (1.5), each elementary self-supporting set $A \in \mathcal{E}$ satisfies $A \subseteq B(y,R)$ for some $y \in S$. Then, as a direct consequence of Proposition 3.18, it follows from (3.84) that, for each elementary self-supporting set $A \subseteq S$,

$$\mathbb{P}(\chi_t^{(A)} = 1 \mid e_B(A,t) > 0) \le \frac{\alpha}{2+\alpha}, \tag{3.92}$$

where $\alpha = 3(K-1)^R \varepsilon_1/\varepsilon_0$.

When $k = 0$, the s-path τ consists of only one elementary self-supporting set A_0, (3.12.b) and (3.12.c) are therefore automatically satisfied. Letting $\tau = \{A_0\}$, it follows from (3.92) that

$$\mathbb{P}(\mathcal{E}(\{A_0\},t) \mid \mathcal{G}(x,r,t)) = \mathbb{P}(\chi_t^{(A_0)} = 1 \mid \mathcal{G}(x,r,t)) \le \frac{\alpha}{2+\alpha}. \tag{3.93}$$

Let $\tau_j = \{A_i\}_{i=0}^j$ for $1 \le j \le k$. We claim that if $\mathcal{E}(\tau_l,t)$ holds, then $\mathcal{E}(\tau_j,t)$ holds for every $l \ge j$. To see this, suppose that τ_i is successful at time t. By Definition 3.12, there exists an s-sequence $\{s_i\}_{i=0}^{l}$, satisfying (3.12.a), (3.12.b) and (3.12.c). Using the same choice of s_i for each $0 \leq i \leq j$, in the case of τ_j, (3.12.a), (3.12.b), and (3.12.c) will still hold for τ_j, implying that τ_j is also successful at time t. Hence, we may write $\mathbb{P}(\mathcal{E}(\tau, t))$ as a product as follows.

$$\mathbb{P}(\mathcal{E}(\tau, t)) = \mathbb{P}(\mathcal{E}(\{A_0\}, t) \mid \mathcal{G}(x, r, t)) \prod_{h=1}^{k} \mathbb{P}(\mathcal{E}(\tau_h, t) \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t)). \tag{3.94}$$

Since (3.93) already gives an upper bound on $\mathbb{P}(\mathcal{E}(\{A_0\}, t) \mid \mathcal{G}(x, r, t))$, we only need to derive an upper bound on the second term in (3.94).

Conditioning on $\mathcal{E}(\tau_{h-1}, t)$, there exists a sequence $\{s_i\}_{i=0}^{h-1}$ satisfying (3.12.a), (3.12.b) and (3.12.c). (The r-sequence $\{r_i\}_{i=0}^{h-1}$ is also given by (3.40).) For τ_h to be successful at time t, it suffices that, for some $s_h \in (r_{h-1}, s_{h-1})$, the following hold.

1. A_h is activated at s_h.

2. $s_h < r_i$ for every $0 \leq i \leq h-1$ with $A_i \cap A_h \neq \emptyset$.

For simplicity, we let $\mathcal{H}(s)$ denote the event that $s < r_j$ for every $j \leq h-1$ with $A_l \cap A_j \neq \emptyset$.

There are two possibilities for such an s_h to exist – either there exists some 1-noise point (y, u) with $y \in A_h$ and $u \in (r_{h-1}, s_{h-1})$, or $\chi_{r_{h-1}}^{(A_h)} = 1$. We let $\mathcal{I}(r_{h-1}, s_{h-1})$ denote the event that there exists some 1-noise point (y, u), with $y \in A_h$ and $u \in (r_{h-1}, s_{h-1})$. Then,

$$\begin{aligned} \mathbb{P}(\mathcal{E}(\tau_h, t) \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t)) \leq \mathbb{P}(\mathcal{I}(r_{h-1}, s_{h-1}) \cap \mathcal{H}(s) \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t)) \\ + \mathbb{P}(\{\chi^{(A_h)}(r_{h-1}) = 1\} \cap \mathcal{H}(r_{h-1}) \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t)). \end{aligned} \tag{3.95}$$

We claim that the first term on the right-hand side of (3.95) satisfies the following:

$$\mathbb{P}(\mathcal{I}(r_{h-1}, s_{h-1}) \cap \mathcal{H}(s) \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t)) \leq \frac{\alpha}{\alpha + 2}. \tag{3.96}$$

We will leave the proof of (3.96) until the end of this section.

We next estimate the second term on the right-hand side of (3.95). Since $r_{h-1} < r_j$ for all j such that $A_l \cap A_j \neq \emptyset$, $\chi_{r_{h-1}}^{(A_h)}$ is independent of $\mathcal{E}(\tau_{h-1}, t)$. It therefore follows from (3.92) that

$$\mathbb{P}(\{\chi^{(A_h)}(r_{h-1}) = 1\} \cap \mathcal{H}(r_{h-1}) \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t)) \leq \frac{\alpha}{\alpha + 2}. \tag{3.97}$$

Combining (3.96) and (3.97), we obtain

$$\mathbb{P}(\mathcal{E}(\tau_h, t) \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t)) \leq \frac{2\alpha}{\alpha + 2}. \tag{3.98}$$

With (3.98), the inequality (3.60) then follows from (3.93) and (3.94), which completes the proof of Proposition 3.16. □

3.5.2 Proof of (3.96)

Since $u < r_j$ for all j such that $A_l \cap A_j \neq \emptyset$, the 1-noise point (y, u) is independent of $\mathcal{E}(\tau_{h-1}, t)$. Since $A_{h-1} \cap A_h = \emptyset$, $(r_{h-1} - s_{h-1})$ is independent of $\mathscr{F}(y, u)$. Therefore,

$$\mathbb{P}(\mathcal{I}(r_{h-1}, s_{h-1}) \cap \mathcal{H}(s) \mid s_{h-1} - r_{h-1} = v) = |A_h|\varepsilon_1 e^{-|A_h|\varepsilon_1 v}, \tag{3.99}$$

since the 1-noise point in A_h is Poisson distributed with rate $|A_h|\varepsilon_1$.

From the definition of r_{h-1} (3.40), r_{h-1} is the first time when a 1-noise point occurs in A_{h-1} after $e_B(A_{h-1}, s_{h-1})$. Let K denote the number of 1-noise points in A_{h-1} during time $(e_B(A_{h-1}, s_{h-1}), s_{h-1})$, it follows from Proposition 3.18 that K is geometrically distributed, with

$$\mathbb{P}(K = i \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t)) = q_1^{i-1}(1 - q_1), \tag{3.100}$$

where $q_1 \leq \frac{|A_{h-1}|\varepsilon_1}{|A_{h-1}|\varepsilon_1 + 2\varepsilon_0}$. We note that the power $i - 1$ in (3.100) follows from the fact that A_{h-1} is activated at time s_{h-1}, which is implied by $\mathcal{E}(\tau_{h-1}, t)$.

Conditioning on $K = k$, $(s_{h-1} - r_{h-1})$ is the sum of k independent and exponentially distributed random variables, each of which has a mean $|A_{h-1}|\varepsilon_1$. Therefore, $(s_{h-1} - r_{h-1})$ is Gamma distributed, with the shape parameter k and the rate parameter $|A_{h-1}|\varepsilon_1$. In other words, the conditional density of $(s_{h-1} - r_{h-1})$ can be written as

$$f_{sr}(v \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t)) = \sum_{k=1}^{\infty} q_1^{k-1}(1 - q_1) f_{\Gamma(k, |A_{h-1}|\varepsilon_1)}(v), \tag{3.101}$$

where $f_{\Gamma(k, |A_{h-1}|\varepsilon_1)}(v)$ is the probability density function of a Gamma distribution with the shape parameter k and rate parameter $|A_{h-1}|\varepsilon_1$, i.e.,

$$f_{\Gamma(\alpha, \beta)}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\beta x}. \tag{3.102}$$

Combining (3.99) and (3.101), we obtain

$$\begin{aligned}&\mathbb{P}(\mathcal{I}(r_{h-1}, s_{h-1}) \cap \mathcal{H}(s) \mid \mathcal{E}(\tau_{h-1}, t), \mathcal{G}(x, r, t))\\ &\leq \int_{v=0}^{\infty} |A_h|\varepsilon_1 e^{-|A_h|\varepsilon_1 v} \sum_{k=1}^{\infty} q_1^{i-1}(1-q_1) f_{\Gamma(k,|A_{h-1|\varepsilon_1})}(v)\, dv \leq \frac{1}{1+2/\alpha},\end{aligned}$$

where the last inequality follows $|A_{h-1}| \leq 3(K-1)^R$. This completes the proof of (3.96), and hence completes the proof of Proposition 3.16.

3.6 Applications of Theorem 1 – Lattices and Defective Lattices

In this section, we apply Theorem 1 to different examples of $(S, \mathcal{N})$ introduced in Section 2.5, and derive corresponding conditions under which the majority vote process on these $(S, \mathcal{N})$ is ergodic. These examples include d-dimensional lattices $\mathbb{Z}^d$ (Corollary 3.19), 2-dimensional hexagonal lattices (Corollary 3.20), 2-dimensional triangular lattices (Corollary 3.21) and defective lattices (Corollary 3.22).

We begin with the UMVP on $S = \mathbb{Z}^d$ and $\mathcal{N}(x) = \{x \pm \mathbf{e}_i : 1 \leq i \leq d\}$, where $\mathbf{e}_i$ denotes the ith standard basis vector. We claim the following.

Corollary 3.19. *Consider the UMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{Z}^d$, with 0-noise intensity ε_0 and 1-noise intensity ε_1. For every $\varepsilon_0 \in (0,1)$, ξ_t is ergodic if $\varepsilon_1 \in (0,1)$ satisfies*

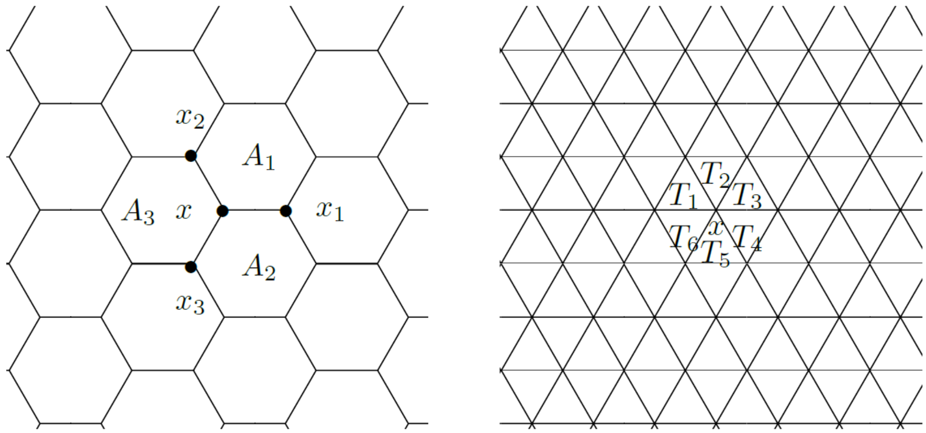
$$\varepsilon_1 \leq \frac{\varepsilon_0}{2^{d-1}(4 \cdot 13^d - 1)}. \tag{3.103}$$

Proof. Recall that we demonstrated, in Example 2.12, that $(S, \mathcal{N})$ is uniformly self-supporting with $R = 2$. Since all elementary self-supporting sets are d-hypercubes of side length 1, a little thought shows that $C_0 = 13^d$ satisfies (3.41), and $q = \varepsilon_0/(2^{d-1}\varepsilon_1)$ satisfies (3.60). It then follows from (3.64), (3.68), and (3.69) that ξ_t is ergodic under the assumption (3.103). This completes the proof of Corollary 3.19. □

In particular, when $S = \mathbb{Z}^2$, the bound in (3.103) follows from

$$\varepsilon_1 \leq \frac{\varepsilon_0}{1350}. \tag{3.104}$$

Figure 3.3: Hexagonal and triangular lattices



A hexagonal lattice, where each $y \in S$ has 3 sites in $\mathcal{N}(y)$, (for example, $\mathcal{N}(x) = \{x_1, x_2, x_3\}$), and a triangular lattice, where each $y \in S$ has 6 sites in $\mathcal{N}(y)$ are displayed.

Also, when $S = \mathbb{Z}^3$, the bound in (3.103) follows from

$$\varepsilon_1 \leq \frac{\varepsilon_0}{35148}. \tag{3.105}$$

Our next corollary pertains to the UMVP defined on the 2-dimensional hexagonal lattice.

Corollary 3.20. *Let $(S, \mathcal{N})$ denote the hexagonal lattice with nearest-neighbor interactions; here, $|\mathcal{N}(x)| = 3$ (see Figure 3.3). Consider the corresponding UMVP on $(S, \mathcal{N})$ $\{\xi_t\}_{t\geq 0}$, with 0-noise intensity ε_0 and 1-noise intensity ε_1. For every $\varepsilon_0 \in (0,1)$, ξ_t is ergodic if $\varepsilon_1 \in (0,1)$ satisfies*

$$\varepsilon_1 \leq \frac{2\varepsilon_0}{6(4 \cdot 3^9 - 1)}. \tag{3.106}$$

Proof. To prove Corollary 3.20, we first show the structure $(S, \mathcal{N})$ is uniformly self-supporting. For every $y \in S$, there are 3 "unit hexagons" that have y as a vertex. For example, in Figure 3.3, let A_1 denote the hexagon that has x, x_1, x_2 as its vertices, let A_2 denote the hexagon that has x, x_1, x_3 as its vertices and let A_3 denote the hexagon that has x, x_2, x_3 as its vertices. Then A_1, A_2 and A_3 are self-supporting. A little thought shows that, if we take $\mathscr{E}$ to be the collection of all such hexagons, $\mathscr{E}$ and $R = 3$ satisfy (1.5). Hence, $(S, \mathcal{N})$ is a uniformly self-supporting structure.

We next prove the UMVP on $(S, \mathcal{N})$ is ergodic when ε_0 and ε_1 satisfy (3.106). Since all elementary self-supporting sets are hexagons of side length 1, a little thought shows that $C_0 = 3^9$ satisfies (3.41) in Lemma 3.13, and $q = \varepsilon_0/(3\varepsilon_1)$ satisfies (3.60) in Proposition 3.16. It then follows from (3.64), (3.68), and (3.69) that ξ_t is ergodic under the assumption (3.106). This completes the proof of Corollary 3.20. □

Our next corollary discusses the UMVP defined on a 2-dimensional triangular lattice.

Corollary 3.21. *Let $(S, \mathcal{N})$ denote the triangular lattice with the nearest-neighbor interactions; here, $|\mathcal{N}(x)| = 6$ for each $x \in S$ (see Figure 3.3). Consider the corresponding majority vote process $\{\xi_t\}_{t \geq 0}$, with 0-noise intensity ε_0 and 1-noise intensity ε_1. For every $\varepsilon_0 \in (0, 1)$, ξ_t is ergodic if $\varepsilon_1 \in (0, 1)$ satisfies*

$$\varepsilon_1 \leq \frac{2\varepsilon_0}{5(4 \cdot 6^3 - 1)}. \tag{3.107}$$

The following proof of Corollary 3.21 adopts the same strategy as the proof of Corollary 3.20, which the reader should feel free to skip.

Proof. To prove Corollary 3.21, we first show such a $(S, \mathcal{N})$ is a uniformly self-supporting structure. We then verify that (3.61) is satisfied under the assumption (3.107).

We first show the structure $(S, \mathcal{N})$ is uniformly self-supporting. For each $y \in S$, there are 6 "unit triangles" that have y as a vertex. For example, in Figure 3.3, 6 unit triangles containing x are T_i, $1 \leq i \leq 6$. Every three adjacent unit triangles, (for example, T_1, T_2, and T_3), form a "unit trapezoid". A little thought shows that we can take $\mathscr{E}$ as the collection of all such trapezoids. Then $(S, \mathcal{N})$ is uniformly self-supporting with $R = 3$.

We next prove the UMVP on $(S, \mathcal{N})$ is ergodic when ε_0 and ε_1 satisfy (3.107). Since all elementary self-supporting sets are trapezoids, each of which consists of 5 sites, a little thought shows that $C_0 = 6^3$ satisfies (3.41) in Lemma 3.13, and $q = 2\varepsilon_0/(5\varepsilon_1)$ satisfies (3.60) in Proposition 3.16. It then follows from (3.64), (3.68), and (3.69) that ξ_t is ergodic under the assumption (3.107). This completes the proof of Corollary 3.21. □

Our next Corollary concerns the UMVP defined on 2-dimensional defective lattice $S = \mathbb{Z}^2_* = \mathbb{Z}^2 - \{(0, 0)\}$, which is introduce in Example 2.14.

Corollary 3.22. *Let* $S = \mathbb{Z}_*^2 = \mathbb{Z}^2 - \{(0,0)\}$ *and*

$$\mathcal{N}(x) = \begin{cases} \{x \pm \mathbf{e}_1, x \pm \mathbf{e}_2\} - \{(0,0)\} & \text{for } x = (0,1), (0,-1), (1,0), (-1,0), \\ \{x \pm \mathbf{e}_1, x \pm \mathbf{e}_2\} & \text{otherwise.} \end{cases} \tag{3.108}$$

Consider the corresponding UMVP $\{\xi_t\}_{t \geq 0}$*, with* 0*-noise intensity* ε_0 *and* 1*-noise intensity* ε_1*. For every* $\varepsilon_0 \in (0,1)$*,* $\{\xi_t\}$ *is ergodic if* $\varepsilon_1 \in (0,1)$ *satisfies*

$$\varepsilon_1 \leq \frac{\varepsilon_0}{4(4 \cdot 15^2 - 1)} = \frac{\varepsilon_0}{3596}. \tag{3.109}$$

Proof. Recall that, in Example 2.14, we have proved this $(S, \mathcal{N})$ is uniformly self-supporting, with $R = 4$. Moreover, we have demonstrated that each x has at most 4 elementary self-supporting sets $A \ni x$. Therefore, $C_0 = 15^2$ satisfies (3.41) in Lemma 3.13, and $q = \varepsilon_0/(4\varepsilon_1)$ satisfies (3.60) in Proposition 3.16. It then follows from (3.64), (3.68), and (3.69) that ξ_t is ergodic under the assumption (3.109). This completes the proof of Corollary 3.22. $\square$

Chapter 4

Proof of Theorem 2: Rate of Convergence

This chapter considers $\mathcal{N}$-transitive structures $(S, \mathcal{N})$ and their corresponding majority vote processes $\{\xi_t\}_{t\geq 0}$. As usual, we let ε_0 and ε_1 denote the 0-noise and 1-noise intensities. In addition, we assume that ε_0 and ε_1 satisfy (3.61), which we assumed to prove Theorem 1, i.e., ξ_t converges to its unique equilibrium ν from any initial distribution. This chapter proves that, under further assumptions on ε_0, ε_1, and $(S, \mathcal{N})$, ξ_t converges exponentially quickly to ν, starting from any initial distribution.

The structure of this chapter is as follows. As usual, we let $\{\eta_t\}_{t\geq 0}$ denote the coupled process, introduced in Definition 2.5, with initial configuration $\eta_t(x) \equiv (0, 1)$, for all $x \in S$. By Proposition 2.9, to prove (1.6), it suffices to prove its η_t analog, namely (2.22).

Proving (2.22) requires a more intricate use of the s-path under the $\mathcal{N}$-transitive assumption. To do so, in Section 4.1, we investigate additional properties of $\mathcal{N}$-transitive structures $(S, \mathcal{N})$, and introduce the *total lifetime* of a successful s-path. We next prove that, for $\eta_t(x) = (0, 1)$, there exists an s-path τ that is successful at time t, with a total lifetime $U(\tau, t) \geq t$. This is done in Proposition 4.1 and Corollary 4.4. Proposition 4.1 is the $\mathcal{N}$-transitive version of Proposition 3.15. The proof of Proposition 4.1 will be delayed until Section 4.3. We next derive an upper bound on the conditional density of $U(\tau, t)$, conditioned on the event that τ is successful at time t, which is done in

Proposition 4.5.

By Applying Proposition 4.1, Corollary 4.4, and Proposition 4.5, we prove Theorem 2 in Section 4.2. The proof of Proposition 4.1 is given in Section 4.3, which completes the proof of Theorem 2.

This chapter concludes with some applications of Theorem 2: In Section 4.4, we study examples discussed in Section 2.5, and derive the corresponding conditions under which ξ_t converges exponentially quickly to its unique equilibrium.

4.1 $\mathcal{N}$-Transitivity and Total Lifetime

This section presents the preparatory work for the proof of Theorem 2. Subsection 4.1.1 discusses key properties of $\mathcal{N}$-transitive structures $(S, \mathcal{N})$ that are used in the proof. We develop the stronger "$\mathcal{N}$-transitive versions" of Propositions 3.15 and 3.16, which are presented as Propositions 4.1 and 4.2, respectively.

We first highlight some key differences between Propositions 3.15 and 4.2 and their $\mathcal{N}$-transitive versions. In Proposition 4.1, we will construct an s-path, which consists of "congruent" elementary self-supporting sets, around a d-path $\{(x_i, t_i)\}_{i=0}^{m'}$ with $t_{m'} = 0$. Unlike in Proposition 3.15, where the construction terminates once the newly added elementary self-supporting set is close enough to the boundary of $\tilde{B}(x, r)$, the construction in Proposition 4.1 terminates when a newly added elementary self-supporting set has been activated since the initial time. Using this, Proposition 4.2, which is the $\mathcal{N}$-transitive version of Proposition 3.16, develops the probability for such an s-path being successful.

Subsection 4.1.2 introduces the total lifetime of a successful s-path and explores its essential properties. The simple Corollary 4.4 points out that each successful s-path satisfying Proposition 4.1 has its total lifetime strictly greater than t. We conclude this section with Proposition 4.5, which provides an upper bound on the conditional density for the total lifetime. We point out that, although the definition of the total lifetime does not rely on the $\mathcal{N}$-transitive assumption, Proposition 4.5 is only valid for $\mathcal{N}$-transitive structure $(S, \mathcal{N})$, and plays a crucial role in the proof of Theorem 2.

4.1.1 Properties of $\mathcal{N}$-transitive Structures

Recall that, for an $\mathcal{N}$-transitive structure $(S, \mathcal{N})$, there exist a constant R and a family of self-supporting sets $\mathscr{E}$, satisfying (1.5), with each $A, A' \in \mathscr{E}$ satisfying $|A| = |A'|$. Throughout this chapter, we fix such a family $\mathscr{E}$ and let $k = |A|$ for every $A \in \mathscr{E}$.

We begin with the $\mathcal{N}$-transitive version of Proposition 3.15. Recall that each successful s-path $\tau = \{A_i\}_{i=0}^m$ is associated with an s-sequence $\{s_i\}_{i=0}^m$ and an r-sequence $\{r_i\}_{i=0}^m$, given in Definition 3.12, and that r_i denotes the last time, before s_i, when A_i is not activated.

Proposition 4.1. *Consider an $\mathcal{N}$-transitive structure $(S, \mathcal{N})$ and the majority vote process $\{\xi_t\}_{t\geq 0}$ defined on $(S, \mathcal{N})$. Let $\{\eta_t\}_{t\geq 0}$ denote the corresponding coupled process with initial configuration $\eta_0(x) \equiv (0, 1)$, for every $x \in S$. Suppose that $\eta_t(x) = (0, 1)$ for some $x \in S$ and $t > 0$. Then, for some $m \in \mathbb{N}$, there exists an s-path $\tau = \{A_i\}_{i=0}^m$, with $x \in A_0$, that is successful at time t, and for which $r_m = 0$, and $r_i > 0$, for each $0 \leq i \leq m - 1$.*

Proposition 4.1 is the $\mathcal{N}$-transitive analog of Proposition 3.15. Hence, our strategy for proving Proposition 4.1 is analogous to that of Proposition 3.15. Recall that, in the proof of Proposition 3.15, we first identified a d-path $\sigma = \{x_i, t_i\}_{i=0}^{m'}$, satisfying Proposition 3.10, with $(x_0, t_0) = (x, e(x, t))$ and $t_{m'} = 0$. We next constructed an s-path recursively "around" this d-path σ. To adapt this strategy to the proof of Proposition 4.1, noting that the only extra requirement for this s-path in Proposition 4.1 is $r_i > 0$ for every $0 \leq i < m$ and $r_m = 0$, it will suffice for us to make a slight modification regarding the terminating condition for our construction.

Recall that, in the proof of Proposition 3.15, the construction of the s-path $\{A_i\}_{i=0}^m$ terminates when we find an event point $(x_l, t_l) \in \sigma$, with $x_l \notin \tilde{B}(x, r)$. In contrast, in the proof of Proposition 4.1, since we now no longer require the s-path to be contained in $\tilde{B}(x, r)$, it will suffice for us to terminate our construction when finding an elementary self-supporting set A_m, with $r_m = 0$. Other than this, the proof of Proposition 4.1 is similar to that of Proposition 3.15. For readability, we defer the proof of Proposition 4.1 until the end of this chapter.

We next develop the $\mathcal{N}$-transitive version of Proposition 3.16. Recall that, for every self-supporting set $A \subseteq S$, $e_B(A, t)$ was introduced in (3.36). For an s-path $\tau = \{A_i\}_{i=0}^m$, with $m \geq 1$, and arbitrary $t > 0$, we set

$$\mathcal{R}'(\tau, t) = \{r_i > 0 \text{ for every } 0 \leq i \leq m\}. \tag{4.1}$$

We note that, since r_i, $0 \leq i \leq m$, is only defined for an s-path that is successful at time t, $\mathcal{R}'(\tau, t)$ is only well-defined for these s-paths. For this reason, we introduce

$$\mathcal{R}(\tau, t) = \mathcal{R}'(\tau, t) \cap \mathcal{E}(\tau, t), \tag{4.2}$$

where $\mathcal{E}(\tau, t)$ denotes the event that τ is successful at time t. Then $\mathcal{R}(\tau, t)$ is well-defined for every s-path τ and every $t > 0$.

Proposition 4.2. *Consider an $\mathcal{N}$-transitive structure $(S, \mathcal{N})$ and an s-path $\tau = \{A_i\}_{i=0}^m$, with $m \geq 1$. Then*

$$\mathbb{P}(\mathcal{R}(\tau, t)) < \left(\frac{2}{1+q}\right)^{m+1}, \tag{4.3}$$

where $q = 2\varepsilon_0/(k\varepsilon_1)$, and k denotes the cardinality of every elementary self-supporting set A_i.

Our proof of Proposition 4.2 employs an inductive argument that is similar to that of Proposition 3.16. The differences between the proofs of Propositions 3.16 and 4.2 are as follows. In Proposition 3.16, the probability in (3.60) is conditioned on $\mathcal{G}(x, r, t)$, which enables us to apply Proposition 3.71. In contrast, in the proof of Proposition 4.2, to apply Proposition 3.71, we need to show that $\mathcal{R}(\tau, t)$ implies $\{e_B(A_i, s_i) > 0\}$ for every $0 \leq i \leq m$.

Proof. We will prove (4.3) by an inductive argument: We first write the probability $\mathbb{P}(\mathcal{R}(\tau, t))$ as a product (4.4), and next estimate each factor of this product by applying Proposition 3.17.

Let $\tau_j = \{A_i\}_{i=0}^j$ for each $0 \leq j \leq m$. As in the proof of Proposition 3.16, if $\mathcal{R}(\tau_l, t)$ holds, then $\mathcal{R}(\tau_j, t)$ holds for every $1 \leq j \leq l$. We can therefore write $\mathbb{P}(\mathcal{R}(\tau, t))$ as the following product.

$$\mathbb{P}(\mathcal{R}(\tau, t)) = \mathbb{P}(\mathcal{R}(\tau_0, t)) \cdot \prod_{h=1}^{m} \mathbb{P}(\mathcal{R}(\tau_h, t) \mid \mathcal{R}(\tau_{h-1}, t)). \tag{4.4}$$

We next estimate each factor separately.

We begin with $\mathbb{P}(\mathcal{R}(\tau_0,t))$: Since $r_0 > 0$ implies $e_B(A_0,t) > 0$, Proposition 3.17 follows, and so we obtain

$$\mathbb{P}(\mathcal{R}(\tau_0,t)) \leq \mathbb{P}(\chi_t^{(A_0)} = 1, e_B(A_0,t) > 0) \leq \frac{\alpha}{\alpha+2} = \frac{1}{1+q}, \tag{4.5}$$

where the last equation follows from the definition of q.

We next derive an upper bound on each factor of the product on the right-hand side of (4.4). Conditioning on $\mathcal{R}(\tau_{h-1},t)$, τ_{h-1} is successful at time t, and hence there exists a sequence $\{s_i\}_{i=0}^{h-1}$ satisfying (3.12.a), (3.12.b) and (3.12.c). Moreover, the r-sequence $\{r_i\}_{i=0}^{h-1}$ are also given by (3.40). Therefore, $\mathcal{R}(\tau_h,t)$ holds if and only if there exists some $s_h \in (r_{h-1}, s_{h-1})$, such that the following hold.

1. A_h is activated at s_h.

2. $s_h < r_i$, for every $0 \leq i \leq h-1$ with $A_i \cap A_h \neq \emptyset$.

3. $e_B(A_h, s_h) > 0$. (This is equivalent to $r_h > 0$.)

For simplicity, we let $\mathcal{H}(s)$ denote the event that $s < r_j$ for every $0 \leq j \leq h-1$ such that $A_h \cap A_j \neq \emptyset$. Recall that $\mathcal{I}(r_{h-1}, s_{h-1})$ denotes the event that there exists some 1-noise point (y,u) with $y \in A_h$ and $u \in (r_{h-1}, s_{h-1})$. A little thought shows

$$\begin{aligned} \mathbb{P}(\mathcal{R}(\tau_h,t) \mid \mathcal{R}(\tau_{h-1},t)) &\leq \mathbb{P}(\mathcal{I}(r_{h-1},s_{h-1}) \cap \mathcal{H}(s) \mid \mathcal{R}(\tau_{h-1},t)) \\ &+ \mathbb{P}(\{\chi^{(A_h)}(r_{h-1}) = 1, e_B(A_h, r_{h-1}) > 0\} \cap \mathcal{H}(r_{h-1}) \mid \mathcal{R}(\tau_{h-1},t)). \end{aligned} \tag{4.6}$$

(Since (4.6) follows from the same reasoning as (3.95), we omit its proof here.)

We next estimate the two terms on the right-hand side of (4.6). We can bound the first term using the same approach as that of (3.96), and obtain

$$\mathbb{P}(\mathcal{I}(r_{h-1},s_{h-1}) \cap \mathcal{H}(s) \mid \mathcal{R}(\tau_{h-1},t)) \leq \frac{1}{1+q}. \tag{4.7}$$

We next bound the second term by applying Proposition 3.18. Since $r_{h-1} < r_j$ for all j such that $A_l \cap A_j \neq \emptyset$, $\chi_{r_{h-1}}^{(A_h)}$ is independent of $\mathcal{R}(\tau_{h-1},t)$. It therefore follows from (3.92) that

$$\mathbb{P}(\{\chi^{(A_h)}(r_{h-1}) = 1, e_B(A_h, r_{h-1}) > 0\} \cap \mathcal{H}(r_{h-1}) \mid \mathcal{R}(\tau_{h-1},t)) \leq \frac{1}{1+q}. \tag{4.8}$$

Combining (4.7) and (4.8), we obtain

$$\mathbb{P}(\mathcal{R}(\tau_h, t) \mid \mathcal{R}(\tau_{h-1}, t)) \leq \frac{2}{1+q}, \quad \text{for every } 1 \leq h \leq m. \tag{4.9}$$

Since (4.9) provides an upper bound on each factor of the product on the right-hand side of (4.4), (4.3) then follows from (4.5) and (4.9). This completes the proof of Proposition 4.2. □

4.1.2 Total Lifetime and its Properties

This subsection introduces the *total lifetime* for a successful s-path $\tau = \{A_i\}_{i=0}^m$, which will serve as an important tool for the proof of Theorem 2. We develop an upper bound on the conditional density of the total lifetime, in Proposition 4.5, which will play a significant role in establishing the exponential rate of convergence of ξ_t.

Definition 4.3 (Total Lifetime)**.** *Consider an s-path $\tau = \{A_i\}_{i=0}^m$ that is successful at time t. Let $\{s_i\}_{i=0}^m$ and $\{r_i\}_{i=0}^m$ denote its s-sequence and r-sequence. We define the total lifetime of τ at time t by*

$$U(\tau, t) = \sum_{i=0}^{m} (s_i - r_i). \tag{4.10}$$

Our next corollary is a direct consequence of the definition of the total lifetime.

Corollary 4.4. *Let $\tau = \{A_i\}_{i=0}^m$ denote an s-path that is successful at time t, with $r_m = 0$. Then*

$$U(\tau, t) \geq t. \tag{4.11}$$

Proof. Recall that, for an s-path τ that is successful at time t, (3.12.b) implies that $s_i > r_{i-1}$ for $1 \leq i \leq m$. Therefore,

$$U(\tau, t) = s_0 - r_0 + \sum_{i=0}^{m} (s_i - r_i) > t - r_0 + \sum_{i=1}^{m} (r_{i-1} - r_i) = t. \tag{4.12}$$

This completes the proof of Corollary 4.4. □

Before stating Proposition 4.5, we will first recall some notation and previous results. For a given $\mathcal{N}$-transitive structure $(S, \mathcal{N})$, there exists a collection of self-supporting sets

$\mathscr{E}$, satisfying (1.5), such that every $A \in \mathscr{E}$ has the same cardinality. (We refer to each $A \in \mathscr{E}$ as an elementary self-supporting set.) We let k denote the cardinality of A.

For given $m \geq 1$ and an s-path $\tau = \{A_i\}_{i=0}^m$, recall that $\mathcal{E}(\tau, t)$, (which depends on m, t and the sequence $\{A_i\}_{i=0}^m$), denotes the event that τ is successful at time t, and that $\mathcal{R}(\tau, t)$, (which depends on m, t and the r-sequence $\{r_i\}_{i=0}^m$), denotes the event that $\tau = \{A_i\}_{i=0}^m$ is successful at time t, and satisfies that $r_i > 0$, for each $0 \leq i \leq m$. For every s-path $\tau = \{A_i\}_{i=0}^m$, we set $T(\tau) = \{A_i\}_{i=0}^{m-1}$, denoting the s-sequence obtained by removing A_0 from τ.

Proposition 4.1 states that, conditioning on $\{\eta_t(x) = (0,1)\}$, there exists an s-path $\tau = \{A_i\}_{i=0}^m$ that is successful at time t, and satisfies $r_i > 0$, for every $0 \leq i < m$, and $r_m = 0$. Conditioning on $\mathcal{E}(\tau, t)$, the total lifetime $U(\tau, t)$ is the sum of K independent and exponentially distributed random variables, with K a random integer independent of these exponentially distributed random variables, and whose (conditional) distribution does not depend on t. Therefore, $U(\tau, t)$ admits a conditional density. Our next proposition, Proposition 4.5, gives an upper bound on the conditional density of $U(\tau, t)$, conditioned on $\mathcal{E}(\tau, t) \cap \mathcal{R}(T(\tau), t)$.

Proposition 4.5. *Consider an $\mathcal{N}$-transitive structure $(S, \mathcal{N})$ and the corresponding majority vote process $\{\xi_t\}_{t\geq 0}$. Let ε_0 and ε_1 denote the 0-noise and 1-noise intensities. Let $\tau = \{A_i\}_{i=0}^m$ denote an s-path, with $m \geq 1$, and $|A_i| = k$, for every $0 \leq i \leq m$. Then, for every $u > 0$, the following holds almost surely:*

$$\frac{d\mathbb{P}(U(\tau, t) \in (0, u) \mid \mathcal{E}(\tau, t) \cap \mathcal{R}(T(\tau), t))}{du} < \varepsilon_1 k \frac{(\varepsilon_1 k u)^{m+1}}{(m+1)!} e^{-3\varepsilon_1 k u/4}. \tag{4.13}$$

To prove the exponential convergence of ξ_t, we require a sharper estimate of $\mathbb{P}(\eta_t(x) = (0,1))$ than what was established in the proof of Theorem 1 (for example, (3.64) and (3.68)). For this, we will use Proposition 4.5, which relies on the assumption that each elementary self-supporting set has the same cardinality. It enables us to establish a sharper estimate for the event $\{U(\tau, t) > t\}$ and hence the exponential convergence rate of ξ_t. This is where the proof of Theorem 2 deviates from that of Theorem 1. We conclude this section with the proof of Proposition 4.5.

4.1.3 Proof of Proposition 4.5

Let $\tau = \{A_i\}_{i=0}^m$ denote an s-path that is successful at time t, which satisfies $r_i > 0$, for every $0 \le i \le m-1$, and $r_m = 0$. Recall that, in the proof of Proposition 3.16, we have already demonstrated that each $s_i - r_i$ is the sum of at least one independent and exponentially distributed random variables, with mean $\varepsilon_1 d$. Suppose that each $s_i - r_i$ is the sum of $K_i,\ 0 \le i \le m$, independent and exponentially distributed random variables. Then, conditioning on $\mathcal{E}(\tau, t) \cap \mathcal{R}(T(\tau), t)$, $\{K_i\}_{i=0}^m$ is a family of i.i.d. random variables, each having a probability mass function

$$\mathbb{P}(K_i = k_i \mid \mathcal{R}(T(\tau), t)) = p^{k_i - 1}(1-p), \quad \text{for } k_i = 1, 2, \cdots, \tag{4.14}$$

where $p = k\varepsilon_1/(k\varepsilon_1 + 2\varepsilon_0)$. Our next lemma gives the probability mass function of $\sum_{i=0}^m K_i$.

Lemma 4.6. *Consider i.i.d. geometric distributed random variables X_i, $1 \le i \le n$, each of which has the probability mass function*

$$\mathbb{P}(X_i = j) = q^{j-1}(1-q) \textit{ for } j = 1, 2, \cdots, \tag{4.15}$$

where $q \in (0,1)$ is a fixed constant. Letting $Y = \sum_{i=1}^n X_i$, then Y has probability mass function

$$\mathbb{P}(Y = k) = \binom{k-1}{n-1}(1-q)^n q^{k-n}, \textit{ for } k = n, n+1, \cdots. \tag{4.16}$$

Proof. Since we can interpret the geometric distributed random variable as the number of independent Bernoulli trials required to get one failure (with probability $1-p$), the sum of n independent geometric distributed random variable can be interpreted as the number of independent Bernoulli trials it takes to get the nth failure. When $Y = k$, except for the k-th trial being a failure, there are $n-1$ failures among the first $k-1$ trails, which produces the factor $\binom{k-1}{n-1}$. $\square$

Recall that, by Definition 4.3, $U(\tau, t) = \sum_{i=0}^m (s_i - r_i)$ is the sum of $(\sum_{i=0}^m k_i)$ number of independent exponentially distributed random variables, each of which has mean $\varepsilon_1 d$. Conditioned on $\{K = k_i : 0 \le i \le m\}$, $U(\tau, t)$ is Gamma distributed, with shape parameter $(\sum_{i=0}^m k_i)$ and rate $\varepsilon_1 d$. Let $f_U(s)$ denote the left-hand side of (4.13), the conditional density of $U(\tau, t)$, then Lemma 4.6 implies that, for every $u > 0$,

$$f_U(u) = \sum_{K=m+1}^{\infty} \binom{K-1}{m} p^{K-m-1}(1-p)^{m+1} f_{\Gamma(K,\varepsilon_1 d)}(u), \tag{4.17}$$

where $f_{\Gamma(\alpha,\beta)}$ is the probability density function of the Gamma distribution with the shape parameter α and the rate parameter β (see (3.102)), and the constant p on the right-hand side of (4.17) is defined by $p = \varepsilon_1 d/(\varepsilon_1 d + \varepsilon_0^d) = q/(1+q)$.

We first substitute $f_{\Gamma(K,\varepsilon_1 d)}(u)$ with (3.102), and p with $p = q/(1+q)$, and then rewrite the summation in (4.17) as follows.

$$\begin{aligned} f_U(u) =& \varepsilon_1 d e^{-\varepsilon_1 du} (\frac{q}{1+q})^{m+1} \frac{(\varepsilon_1 du)^m}{m!} \\ & \cdot \sum_{K=m+1}^{\infty} \frac{(\varepsilon_1 du)^{K-(m+1)}}{(1+q)^{K-(m+1)}(K-(m+1))!}. \end{aligned} \tag{4.18}$$

By (4.21), $q = 2\varepsilon_0/k\varepsilon_1 > 3$, and hence the last factor on the right-hand side of (4.18) satisfies

$$\sum_{K=m+1}^{\infty} \frac{(\varepsilon_1 du)^{K-(m+1)}}{(1+q)^{K-(m+1)}(K-(m+1))!} < e^{-\varepsilon_1 du/4}, \quad \text{for every } u > 0. \tag{4.19}$$

Since $q/(1+q) < 1$, (4.18) and (4.19) imply (4.13). This completes the proof of Proposition 4.5. $\square$

4.2 Proof of Theorem 2 Assuming Proposition 4.1

This section presents the proof of Theorem 2 assuming Proposition 4.1. Recall that $\{\eta_t\}_{t\geq 0}$ denotes the coupled majority vote process, with initial distribution $\eta_0(x) \equiv (0,1)$ for all $x \in S$. Consider two jointly defined majority vote processes $\{\xi_t^{(1)}\}_{t\geq 0}$ and $\{\xi_t^{(2)}\}_{t\geq 0}$ with arbitrary initial distributions. As a direct consequence of Proposition 2.9, for every $x \in S$ and $t \geq 0$,

$$\mathbb{P}(\xi_t^{(1)}(x) \neq \xi_t^{(2)}(x)) \leq \mathbb{P}(\eta_t(x) = (0,1)).$$

To prove (1.6), it suffices to prove its η_t analog – that there exist some constants $C, \alpha > 0$ such that

$$\mathbb{P}(\eta_t(x) = (0,1)) \leq Ce^{-\alpha t}, \quad \text{for every } x \in S \text{ and } t \geq 0. \tag{4.20}$$

We assume that ε_0 and ε_1 satisfy

$$v := \frac{k\varepsilon_1}{k\varepsilon_1 + 2\varepsilon_0} < \frac{1}{24C_0}, \tag{4.21}$$

where k is the cardinality of every elementary self-supporting set, and C_0 is the constant introduced in Lemma 3.13. We also note that both k and C_0 are determined solely by $(S, \mathcal{N})$, and do not rely on the choice of ε_0 and ε_1. For any arbitrary $\varepsilon_0 \in (0, 1)$, one can always take ε_1 small enough such that (4.21) holds.

In the remainder of this section, we prove that, under the assumption (4.21), there exist constants $C, \alpha > 0$ such that (4.20) holds. Since $(S, \mathcal{N})$ is $\mathcal{N}$-transitive, $\mathbb{P}(\eta_t(x) = (0, 1))$ does not depend on the choice of x, and so it suffices to show (4.20) for any given x.

In the remainder of this section, we will fix an arbitrary choice of x. By Proposition 4.1, $\{\eta_t(x) = (0, 1)\}$ implies that there exists an s-path $\tau = \{A_i\}_{i=0}^m$ that is successful at time t, with $x \in A_0$, $r_m = 0$, and $r_i > 0$ for every $0 \le i \le m - 1$. Furthermore, Corollary 4.4 implies that such an s-path τ satisfies $U(\tau, t) > t$. Recall that, for an s-path $\tau = \{A_i\}_{i=0}^m$, $\mathcal{E}(\tau, t)$ denotes the event that τ is successful at time t and satisfies $r_i > 0$ for every $0 \le i \le m$. Recall that, for every s-path $\tau = \{A_i\}_{i=0}^m$, $T(\tau) = \{A_i\}_{i=0}^{m-1}$. It follows from Proposition 4.1 and Corollary 4.4 that

$$\mathbb{P}(\eta_t(x) = (0, 1)) \le \sum_{\tau} \mathbb{P}(\mathcal{E}(\tau, t) \cap \{U(\tau, t) > t\} \cap \mathcal{R}(T(\tau), t)), \tag{4.22}$$

where the summation in (4.22) is over all s-paths $\tau = \{A_i\}_{i=0}^m$, with $x \in A_0$.

We set

$$\Lambda_1 = \sum_{|\tau|=1} \mathbb{P}(\mathcal{E}(\tau, t) \cap \{U(\tau, t) > t\} \cap \mathcal{R}(T(\tau), t)),$$

where the summation is over all s-paths $\tau = \{A_0\}$, with $x \in A_0$; and set

$$\Lambda_2 = \sum_{|\tau|>1} \mathbb{P}(\mathcal{E}(\tau, t) \cap \{U(\tau, t) > t\} \cap \mathcal{R}(T(\tau), t)),$$

where the summation is over all s-paths $\tau = \{A_i\}_{i=0}^m$, with $x \in A_0$ and $m \ge 1$. We estimate Λ_1 and Λ_2 separately.

We first consider the case where τ consists of only one elementary self-supporting set. Suppose that $\tau = \{A_0\}$. Then

$$\mathbb{P}(\mathcal{E}(\tau, t) \cap \{U(\tau, t) \ge t\} \cap \mathcal{R}(T(\tau), t)) \le \mathbb{P}(\chi_s^{A_0} = 1 \text{ for every } 0 \le s \le t). \tag{4.23}$$

Recall that, for every $0 < s < r$, $\mathscr{B}(A_0, s, r)$ denotes the event that $e(y, r) > s$ and $(y, e(y, t)) \in X_y^{(0)}$ for every $y \in A_0$. A little thought shows that $\mathscr{B}(A_0, s, r)$ implies $\{\chi_r^{A_0} = 0\}$. Furthermore, since $\mathscr{B}(A_0, i, i+1)$ and $\mathscr{B}(A_0, j, j+1)$ are independent for every $i \neq j$,

$$\begin{aligned} \mathbb{P}(\chi_s^{A_0} = 1 \text{ for every } 0 \le s \le t) &\le \prod_{i=0}^{\lfloor t \rfloor - 1} \mathbb{P}(\mathscr{B}^c(A_0, i, i+1)) \\ &\le (1 - (\varepsilon_0(1 - e^{-1}))^k)^{t-1}, \end{aligned} \tag{4.24}$$

where the second inequality in (4.24) follows from (3.56). Setting

$$C_1 = C_0(1 - (\varepsilon_0^k(1 - e^{-1})^k)^{-1}, \quad \alpha_1 = -k \ln(1 - \varepsilon_0^k(1 - e^{-1})^k),$$

it follows from (4.24) that

$$\Lambda_1 = \sum_{|\tau|=1} \mathcal{E}(\tau, t) \cap \{U(\tau, t) > t\} \cap \mathcal{R}(T(\tau), t) \le C_1 e^{-\alpha_1 t}, \tag{4.25}$$

where the summation is taken over all $\tau = \{A_0\}$ such that $x \in A_0$.

We next bound each term in the summation in Λ_2 in terms of the conditional density of $U(\tau, t)$ as follows:

$$\begin{aligned} &\mathbb{P}(\mathcal{E}(\tau, t) \cap \{U(\tau, t) > t\} \cap \mathcal{R}(T(\tau), t)) \le \mathbb{P}(\mathcal{E}(T(\tau), t) \cap \mathcal{R}(T(\tau), t)) \\ &\cdot \int_t^\infty \frac{d\mathbb{P}(U(\tau, t) \in (0, s) \mid \mathcal{E}(\tau, t) \cap \mathcal{R}(T(\tau), t)}{ds} \, ds, \end{aligned} \tag{4.26}$$

where $\tau = \{A_i\}_{i=0}^m$ denotes an s-path satisfying $x \in A_0$ and $m \ge 1$.

Since the upper bound on the conditional density of the total lifetime $U(\tau, t)$, which was given in Proposition 4.5, is determined by the length of τ, we first sum over all s-paths of the same length, then sum over all possible lengths. We claim that it follows from (4.13) and (4.26) that

$$\Lambda_2 \le \sum_{m=1}^\infty C_0^{m+1} \left(\frac{2}{1+q} \right)^m \int_t^\infty \varepsilon_1 k \frac{(k\varepsilon_1 s)^m}{m!} e^{-3k\varepsilon_1 s/4} \, ds, \tag{4.27}$$

where C_0 is defined in Lemma 3.13 and q is defined in Proposition 4.2. To see (4.27), we first note that C_0^{m+1} is an upper bound on the number of all possible s-paths, with $A_0 \ni x$ and having length $m+1$; and the factor $(2/(1+q))^m$ follows from Proposition 3.16.

We next estimate the right-hand side of (4.27) by applying the Fubini Theorem to exchange the order of integration over s and summation over m. We then obtain

$$\Lambda_2 < \int_0^t 2k\varepsilon_1 C_1 e^{-3\varepsilon_1 ks/4} \sum_{m=1}^{\infty} \frac{(2vC_1 k\varepsilon_1 s)^m}{m!}\, ds, \tag{4.28}$$

where $v = 2/(1+q)$ was introduced in (4.21). Since

$$\sum_{m=1}^{\infty} \frac{(2vC_1\varepsilon_1 ks)^m}{m!} < e^{2vC_1 k\varepsilon_1 s}, \text{ for all } s > 0, \tag{4.29}$$

substituting the upper bound given in (4.29) into (4.28), we obtain that

$$\Lambda_2 < 2k\varepsilon_1 C_1 \int_t^{\infty} e^{(-3/4+2vC_1)k\varepsilon_1 s}\, ds. \tag{4.30}$$

Since ε_0 and ε_1 are assumed to satisfy (4.21), $2vC_0 < 1/4$. Therefore,

$$\Lambda_2 < 4e^{-k\varepsilon_1 t/2}. \tag{4.31}$$

Combining (4.25) and (4.31), (4.20) follows from (4.22), which completes the proof of Theorem 2. □

4.3 Proof of Proposition 4.1

In this section, we prove Proposition 4.1. We will adopt a similar strategy to that of proving Proposition 3.15: We will first find a d-path $\sigma = \{(x_i, t_i)\}_{i=0}^{M}$ satisfying Proposition 3.10, and then construct an s-path around this d-path $\tau = \{A_i\}_{i=0}^{m}$. The s-path τ will be constructed similarly to what was done in Section 3.4: A sequence $\{k_i\}_{i=-1}^{m-1}$ will be constructed, along with $\{A_i\}_{i=0}^{m}$. When choosing A_l given $\{A_i\}_{i=0}^{l-1}$, we will first choose a temporary A_l, satisfying (3.12.a) and (3.12.b), then modify it if A_l does not satisfy (3.12.c).

We point out a key difference between Propositions 3.15 and 4.1. In Proposition 4.1, we require that $r_i > 0$, for every $0 \le i \le m-1$, and $r_m = 0$. This results in a different terminating condition than that of Proposition 3.15. (When determining A_l given $\{A_i\}_{i=0}^{l-1}$, if A_l satisfies $r_l = 0$, we will terminate the construction.)

Construction of σ

We first determine a d-path σ. By Proposition 3.10, conditioning on $\{\eta_t(x) = (0, 1)\}$, there exists a d-path $\sigma = \{(x_i, t_i)\}_{i=0}^{M}$ that satisfies (3.9.a), (3.9.b), and (3.9.c), with $(x_0, t_0) = (x, e(x, t))$ and $t_M = 0$. We next construct an s-path around this d-path σ, beginning with A_0.

Construction of A_0

We first take $A_0 \ni x$ to be any elementary self-supporting set, and set $s_0 = t$. We claim that the s-path $\tau_0 = \{A_0\}$ is successful at time t. Since τ_0 being successful at time t follows the same reasoning as in the proof of Proposition 3.15 (Section 3.5), we omit its proof here.

Since τ_0 is successful at time t, r_0 is defined, by (3.40), as

$$r_0 = \inf\{u < t : A_0 \text{ is activated during } (u, t)\}. \tag{4.32}$$

If $r_0 = 0$, we take $\tau = \tau_0$ as the intended s-path. (This reflects a different terminating condition, compared with that in the proof of Proposition 3.15.) It is straightforward to verify that τ satisfies Proposition 4.1.

If $r_0 \neq 0$, we set $k_{-1} = 0$ and

$$k_0 = \min\{k \in \mathbb{N} : x_k \notin A_0\}. \tag{4.33}$$

A little thought shows that such a k_0 always exists: If not, then $x_k \in A_0$, for every $0 \leq k \leq M$. Since $t_M = 0$, it follows from Lemma 3.17 that A_0 is activated during time $(0, t)$, contradicting the assumption that $r_0 > 0$.

Construction of A_1

We next determine A_1. For simplicity, we set

$$y = x_{k_0}, \quad s = t_{k_0}.$$

Since $y \notin A_0$, by (1.5), there exists some $A' \in \mathcal{E}_y$ such that $A_0 \cap A' = \emptyset$. Setting $A_1 = A'$ and $s_1 = s$, we claim that $\tau_1 = \{A_i\}_{i=0}^{1}$ is an s-path that is successful at time t. Since this follows from the same reasoning as that in the proof of Proposition 3.15, we omit its proof here.

Since A_1 and s_1 are determined, r_1 is defined, by (3.40). We next determine if r_1 satisfies the terminating condition: If $r_1 = 0$, we take $\tau = \tau_1$ as the desired s-path. In this case, since τ_1 is successful at time t and satisfies $r_1 = 0$ and $r_0 > 0$, τ_1 satisfies Proposition 4.1.

If $r_1 > 0$, we set

$$k_1 = \min\{k > k_0 : x_k \notin A_1\}. \tag{4.34}$$

This completes the construction of A_1.

Construction of A_l

Suppose that $\{A_i\}_{i=0}^{l-1}$ is an s-path that is successful at time t, and assume that $r_{l-1} > 0$: If $r_{l-1} = 0$, then r_{l-1} satisfies the terminating condition, and hence $\tau_{l-1} = \{A_i\}_{i=0}^{l-1}$ is the intended s-path. Suppose that a sequence $\{k_i\}_{i=-1}^{l-1}$ is constructed, along with $\{A_i\}_{i=0}^{l-1}$, and satisfies $k_{-1} = 0$, and

$$k_i = \min\{k > k_{i-1} : x_k \notin A_{i-1}\}, \text{ for } 0 \le i \le l-1. \tag{4.35}$$

We summarize our strategy for choosing A_l as follows. This approach is essentially a variant of that in Section 3.5: We first select a temporary A_l, in a manner similar to how we choose A_1. We then determine whether the s-path $\{A_i\}_{i=0}^{l}$ is successful at time t, in particular, whether (3.12.c) is satisfied for $j = l$. If the s-path $\{A_i\}_{i=0}^{l}$ is successful at time t, we confirm A_l as our choice. However, if the s-path $\{A_i\}_{i=0}^{l}$ is not successful at time t due to the violation of (3.12.c) (the way we choose A_l ensures (3.12.a) and (3.12.b)), we will "trim" the s-path $\{A_i\}_{i=0}^{l-1}$ (possibly multiple times) until the shortened sequence is successful at time t, and "uses" more sites in σ.

We begin by describing how to select a temporary A_l. Again, for simplicity, we set $y = x_{k_{l-1}}$ and $s = t_{k_{l-1}}$. Since $y \notin A_{l-1}$, there exists some $A' \in \mathcal{E}_y$ such that $A' \cap A_{l-1} = \emptyset$. We then set $A_l = A'$, $s_l = s = t_{k_{l-1}}$, and

$$k_l = \min\{k > k_{l-1} : x_k \notin A_l\}.$$

It is straightforward to verify that $\tau_l = \{A_i\}_{i=0}^{l}$ is an s-path and satisfies conditions

(3.12.a) and (3.12.b). Since this follows from the same reasoning as in Section 3.5, we omit its proof here.

We next determine whether (3.12.c) holds for $j = l$. If A_l and s_l satisfy (3.12.c), then we conclude that $\{A_i\}_{i=0}^{l}$ is successful at time t, which completes this inductive step of choosing A_l.

However, if (3.12.c) does not hold for $j = l$, we will "trim" the sequence $\{A_i\}_{i=0}^{l-1}$ into a shorter s-path. Since this process of trimming is the same as that in Section 3.4, we omit it here.

We will repeat this recursive construction of τ until $r_m = 0$. Since $(x_M, t_M) \in \sigma$ satisfies $t_M = 0$ for some finite M, such a construction ends after a finite number of steps. This completes the proof of Proposition 4.1.

4.4 Applications of Theorem 2

In this section, we apply Theorem 2 to some examples that were discussed in Section 2.5. Our first corollary, Corollary 4.7, pertains to the MVP on $\mathbb{Z}$, with $\mathcal{N}(x) = \{x - 2, x-1, x, x+1, x+2\}$. Recall that, in Section 3.1, we proved the ergodicity of such an MVP when ε_0 and ε_1 satisfy $\varepsilon_1 \le 2\varepsilon_0/69$.

Corollary 4.7. *Let $S = \mathbb{Z}$ and $\mathcal{N}(x) = \{x-2, x-1, x, x+1, x+2\}$, for each $x \in \mathbb{Z}$. Consider the majority vote process $\{\xi_t\}_{t\ge 0}$ on $(S, \mathcal{N})$, and let ε_0, ε_1 denote the 0-noise and 1-noise intensities. For any given $\varepsilon_0 \in (0,1)$, if ε_1 satisfies*

$$\varepsilon_1 \le \frac{2\varepsilon_0}{789}\,, \tag{4.36}$$

then $\{\xi_t\}_{t\ge 0}$ is ergodic and converges exponentially quickly to its unique equilibrium ν, starting from any initial distribution. In particular, ξ_t is exponentially mixing in ν.

Proof. Recall that, in Section 4.2, we proved that (4.21) implies that ξ_t is ergodic and converges exponentially quickly to its unique equilibrium. To prove Corollary 4.7, it suffices to verify (4.21) under the assumption (4.36).

Recall that, in Example 2.13, we proved that $(S, \mathcal{N})$ is an $\mathcal{N}$-transitive structure, with $R = 2$, $k = 3$, and

$$\mathscr{E} = \{C(x) : x \in \mathbb{Z}\},$$

where $C(x) = \{x-1, x, x+1\}$ for each $x \in \mathbb{Z}$. Then a little thought shows that $C_0 = 11$ satisfies (3.41). (4.21) follows directly from (4.36), which completes the proof of Corollary 4.7. □

Our next corollary, Corollary 4.8, demonstrates the exponential convergence of the UMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{Z}^d$. Recall that, in Corollary 3.19, we proved that ξ_t is ergodic when ε_0 and ε_1 satisfy (3.103).

Corollary 4.8. *Consider the UMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{Z}^d$, and let ε_0 and ε_1 denote its 0-noise and 1-noise intensities. For any given $\varepsilon_0 \in (0,1)$, if ε_1 satisfies*

$$\varepsilon_1 \leq \frac{\varepsilon_0}{2^{d-1}(24 \cdot 13^d - 1)}, \tag{4.37}$$

then $\{\xi_t\}_{t\geq 0}$ is ergodic, and converges exponentially quickly to its unique equilibrium ν, starting from any initial distribution. In particular, ξ_t is exponentially mixing in ν.

Proof. We already proved, in Example 2.12, that $(S, \mathcal{N})$ is $\mathcal{N}$-transitive, with $k = 2^d$. To prove Corollary 4.8, it suffices to verify (4.21) under the assumption (4.37).

Noting that in Corollary 3.19, we demonstrated that $C_0 = 13^d$ satisfies (3.41). (4.21) follows directly from (4.37), which completes the proof of Corollary 4.8. □

Our next two corollaries show the exponential convergence of the UMVP on the hexagonal lattices and triangular lattices. Recall the UMVP $\{\xi_t\}_{t\geq 0}$ on the hexagonal lattice was introduced in Corollary 3.20, where S denotes the 2-dimensional hexagonal lattice and each site $x \in S$ satisfies $|\mathcal{N}(x)| = 3$ (see Figure 3.3). In Corollary 3.20, we proved that ξ_t is ergodic if ε_0 and ε_1 satisfy (3.106). In the next corollary, Corollary 4.9, we will establish its exponentially quick convergence to the unique equilibrium.

Corollary 4.9. *Consider the UMVP $\{\xi_t\}_{t\geq 0}$ on the hexagonal lattice, and let ε_0 and ε_1 denote the 0-noise and 1-noise intensities. For any given $\varepsilon_0 \in (0,1)$, if ε_1 satisfies*

$$\varepsilon_1 \leq \frac{\varepsilon_0}{3(24 \cdot 3^9 - 1)}, \tag{4.38}$$

then $\{\xi_t\}_{t\geq 0}$ is ergodic and converges exponentially quickly to the unique equilibrium ν starting from any initial distribution. In particular, ξ_t is exponentially mixing in ν.

Proof. Recall that, in Corollary 3.20, we proved that $(S, \mathcal{N})$ is a uniformly self-supporting structure by taking $\mathscr{E}$ to be the collection of all "unit hexagons" and $R = 3$. A little thought shows that, with the same choice of $\mathscr{E}$ and R, $(S, \mathcal{N})$ is $\mathcal{N}$-transitive. Moreover, since $C_0 = 3^9$ satisfies (3.41), (4.21) follows directly from (4.38). This completes the proof of Corollary 4.9. $\square$

Recall the UMVP $\{\xi_t\}_{t\geq 0}$ on the triangular lattice was introduced in Corollary 3.21, where S denotes the triangular lattice and each site $x \in S$ satisfies $|\mathcal{N}(x)| = 6$ (see Figure 3.3). In Corollary 3.21, we proved that ξ_t is ergodic if ε_0 and ε_1 satisfy (3.107). In the next corollary, Corollary 4.10, we will establish its exponentially quick convergence to the unique equilibrium.

Corollary 4.10. *Consider the UMVP $\{\xi_t\}_{t\geq 0}$ on the triangular lattice. Let ε_0 and ε_1 denote the* 0*-noise and* 1*-noise intensities. For any given $\varepsilon_0 \in (0, 1)$, if ε_1 satisfies*

$$\varepsilon_1 \leq \frac{2\varepsilon_0}{3(24 \cdot 6^3 - 1)}, \tag{4.39}$$

then ξ_t is ergodic and converges exponentially quickly to its unique equilibrium ν, starting from any initial distribution. In particular, ξ_t is exponentially mixing in ν.

Proof. Recall that, in Corollary 3.21, we already proved that $(S, \mathcal{N})$ is a uniformly self-supporting structure by taking $R = 3$ and $\mathscr{E}$ to be the collection of all "unit trapezoids" as in Figure 3.3. A little thought shows that $(S, \mathcal{N})$ is an $\mathcal{N}$-transitive structure with the same choice of R and $\mathscr{E}$. Since we demonstrated, in the proof of Corollary 3.21, that $C_0 = 6^3$ satisfies (3.41), (4.21) follows directly from (4.39). This completes the proof of Corollary 4.10. $\square$

Chapter 5

Unoriented Majority Vote Process on Low Dimensional Trees

This chapter discusses the behavior of the unoriented majority vote process (UMVP) defined on low-dimensional trees. Let $\mathbb{T}_d$ denote the d-ary regular tree and recall that the UMVP defined on $\mathbb{T}_d$, denoted by $\{\xi_t\}_{t\geq 0}$, was introduced in Subsection 1.1.2. As usual, let $\varepsilon_0, \varepsilon_1 \in [0, 1]$ denote the 0-noise and 1-noise intensities, and assume that $\varepsilon_0 + \varepsilon_1 \leq 1$. Bramson and Gray ([6]) studied the UMVP defined on $\mathbb{T}_d$, with $d \geq 5$, and proved that ξ_t is non-ergodic and possesses uncountably many mutually singular equilibria when $\varepsilon_0 + \varepsilon_1$ is sufficiently small. In this chapter, we will address the non-ergodicity of the remaining cases, namely the non-ergodicity of the UMVP on $\mathbb{T}_3$ and $\mathbb{T}_4$, under the assumption that $\varepsilon_0 + \varepsilon_1$ is sufficiently small.

The chapter is structured as follows: In Section 5.1, we will show contrasting behavior of the UMVP on $\mathbb{T}_4$, under the assumptions that $\varepsilon_0 + \varepsilon_1$ is sufficiently large and small. In Subsection 5.1.1, we will show that when $\varepsilon_0 + \varepsilon_1 > 4/5$, ξ_t is ergodic and converges exponentially quickly to the unique equilibrium, starting from any initial distribution.

The main part of Section 5.1 is dedicated to proving that ξ_t is non-ergodic and possesses uncountably many mutually singular equilibria when $\varepsilon_0 + \varepsilon_1$ is sufficiently small. In Subsection 5.1.2, we will state this non-ergodicity result (Theorem 6) and provide an overview of its proof.

In Subsection 5.1.3, we will introduce the *compatible configurations*, which, when used as the initial configuration of ξ_t, will lead to uncountably many mutually singular equilibria.

In Subsection 5.1.4, we will introduce the *dominating process on* $\mathbb{T}_4$, denoted by $\{\delta_t\}_{t\geq 0}$, in Definition 5.4. We will establish the "domination property" of δ_t: Letting ξ_t denote a copy of the UMVP whose initial configuration is compatible, then δ_t controls the deviation of ξ_t from the initial configuration ξ_0. This will be done in Lemma 5.5. We will also show that the marginal density of $\delta_t(x)$ is uniformly (in both t and x) bounded – this will be done in Lemma 5.6. Subsection 5.1.4 concludes with a concentration inequality, stated in Lemma 5.7. The subsequent Subsection 5.1.5 proves Lemma 5.6.

In Subsection 5.1.6, we will apply Lemmas 5.5, 5.6, and 5.7 to construct a family of uncountably many equilibria of ξ_t, and prove that these equilibria are mutually singular with each other. This approach is similar to the proof of Theorem 2 in [6].

In Section 5.2, we will discuss the behavior of the UMVP on $\mathbb{T}_3$. We will begin with its behavior when $\varepsilon_0 + \varepsilon_1$ is large: When $\varepsilon_0 + \varepsilon_1 > 3/4$, ξ_t is ergodic and converges exponentially quickly to the unique equilibrium, starting from any initial distribution.

The main part of Section 5.2 will prove the non-ergodicity of the UMVP on $\mathbb{T}_3$, when $\varepsilon_0 + \varepsilon_1$ is sufficiently small. In Subsection 5.2.1, we will state this non-ergodicity result and provide an overview of the proof.

In Subsection 5.2.2, we will introduce the *dominating process on* $\mathbb{T}_3$, denoted by $\{\delta_t\}_{t\geq 0}$. This will be done in Definition 5.9. In Lemma 5.10, we will show that, for a copy of the UMVP ξ_t, with all-zero or all-one initial configuration, $\mathbb{P}(\xi_t(x) \neq \xi_0(x)) \leq \mathbb{P}(\delta_t(x) = 1)$, for every $x \in \mathbb{T}_3$ and $t \geq 0$.

In Subsection 5.2.3, we will introduce the *t-truss* (Definition 5.11) and *selection* (Definition 5.13), the former of which is a variant of the *truss* introduced by Toom ([7]). We will state some important properties of the selection and t-truss in Propositions 5.14 and 5.19.

In Subsection 5.2.4, we will prove the non-ergodicity of ξ_t, using Propositions 5.14 and 5.19. Section 5.2 concludes with the proof of Propositions 5.14 and 5.19: We will prove Proposition 5.14 in Subsection 5.2.5, and Proposition 5.19 in Subsection 5.2.6.

5.1 UMVP on $\mathbb{T}_4$

This section investigates the ergodic behavior of the UMVP on $\mathbb{T}_4$, which was introduced in Subsection 1.1.2. Throughout this section, we let $\{\xi_t\}_{t\geq 0}$ denote the UMVP on $\mathbb{T}_4$, and let $\varepsilon_0, \varepsilon_1 \in [0,1]$ denote the 0-noise and 1-noise intensities. Let $(S, \mathcal{N})$ denote the structure where $S = \mathbb{T}_4$, and, for each $x \in S$, $\mathcal{N}(x) = \{y \in \mathbb{T}_4 : y \text{ is adjacent to } x\}$. Recall that, in Example 2.17, we proved that such a structure $(S, \mathcal{N})$ is not uniformly self-supporting.

We will discuss the ergodic behavior of ξ_t under two assumptions – when $\varepsilon_0 + \varepsilon_1$ is sufficiently small, and when it is sufficiently large (but does not exceed 1). We begin with the large noise regime of ξ_t.

5.1.1 Ergodicity for Large $\varepsilon_0 + \varepsilon_1$

As discussed in Section 2.4, when the noise event occurs at a significantly higher rate compared with the vote event, the corresponding majority vote process is ergodic. Hence, for both $\varepsilon_0, \varepsilon_1$ large, the ergodicity of ξ_t follows directly from Proposition 2.11.

Our next proposition, Proposition 5.1, moderately increases the region where we can show the large noise regime holds for the UMVP on $\mathbb{T}_4$, which is stated in (5.1). We claim that, for sufficiently large $\varepsilon_0 + \varepsilon_1$, ξ_t is ergodic and converges exponentially quickly to its unique equilibrium. This behavior contrasts with that of ξ_t for sufficiently small $\varepsilon_0 + \varepsilon_1$, as detailed in Theorem 6. The proof of Proposition 5.1 adopts an idea similar to that of Proposition 2.11, and is included here for completeness. The primary difference between the proofs of Propositions 5.1 and 2.11 results from the symmetry of $\mathbb{T}_4$, which reduces the proof of Proposition 5.1 to the computation of a differential inequality.

Proposition 5.1. *Consider the UMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{T}_4$, and let ε_0 and ε_1 denote the 0-noise and 1-noise intensities. When $\varepsilon_0, \varepsilon_1 \in (0,1)$ satisfy*

$$\varepsilon_0 + \varepsilon_1 > 4/5, \tag{5.1}$$

ξ_t is ergodic and converges exponentially quickly to its unique equilibrium, starting from any initial configuration.

Proof. Let $\{\eta_t\}_{t\geq 0}$ denote the coupled process on $(S, \mathcal{N})$, and assume that η_t has the same 0-noise and 1-noise intensities as ξ_t, and also that $\eta_0(x) \equiv (0,1)$ for every $x \in S$. By Proposition 2.9, to show ξ_t is ergodic and converges to the unique equilibrium exponentially quickly, it suffices to prove that, under the assumption (5.42), there exist some $a, r > 0$ such that

$$\mathbb{P}(\eta_t(x) = (0,1)) \leq ae^{-rt} \quad \text{for every } x \in \mathbb{T}_4 \text{ and } t \geq 0. \tag{5.2}$$

Due to the space homogeneity of $(S, \mathcal{N})$, $\mathbb{P}(\eta_t(x) = (0,1))$ does not depend on the choice of x. Set $h(t) = \mathbb{P}(\eta_t(x) = (0,1))$. Since $\{\eta_t\}_{t\geq 0}$ is a Feller process, $h(t)$ is differentiable in t. Furthermore, we claim that

$$h'(t) \leq -(5\varepsilon - 4)h(t), \tag{5.3}$$

where $\varepsilon = \varepsilon_0 + \varepsilon_1$. Supposing (5.3) holds, and setting $r = 5\varepsilon - 4$ and $a = 1$, (5.2) then follows from Gronwall's inequality. To show (5.2), it suffices to prove (5.3) for every $t \geq 0$.

We prove that (5.3) holds for every $t \geq 0$: According to Proposition 2.6, for each $x \in \mathbb{T}_4$ and $t \geq 0$,

$$\mathbb{P}(\eta_t(x) = (1,0)) = 0,$$

We therefore can assume that $\eta_t(x) \in \{(0,0), (1,1), (0,1)\}$.

By (2.5.a) and (2.5.b), $\eta_t(x)$ changes from $(0,1)$ to $(0,0)$ or $(1,1)$ at least at rate ε, which produces the term $-\varepsilon h(t)$ on the right-hand side of (5.3).

On the other hand, $\eta_t(x)$ changes from $(0,0)$ or $(1,1)$ to $(0,1)$ only when (x,t) is a vote point and there exists some $y \in \mathcal{N}(x)$ with $\eta_{t-}(y) = (0,1)$. Let $\mathscr{Q}(x,t)$ denote the event

$$\mathscr{Q}(x,t) = \{\exists y \in \mathcal{N}(x) : \eta_{t-}(y) = (0,1)\}.$$

Then

$$\mathbb{P}(\mathscr{Q}(x,t) \cap \{\eta_t(x) \neq (0,1)\}) \leq \sum_{y\in\mathcal{N}(x)} \mathbb{P}(\eta_{t-}(y) = (0,1)) = 4h(t), \tag{5.4}$$

where the last equation follows from $|\mathcal{N}(x)| = 4$, for each $x \in \mathbb{T}_4$. $\eta_t(x)$ changes from $(0,0)$ or $(1,1)$ to $(0,1)$ at most at rate $4h(t)$. This leads to the term $4(1-\varepsilon)h(t)$ to the right-hand side of (5.3), which completes the proof of (5.3), and hence completes the proof of Proposition 5.1. $\square$

5.1.2 Non-ergodicity for Small $\varepsilon_0 + \varepsilon_1$

Bramson and Gray ([6]) demonstrated that when $\varepsilon_1 + \varepsilon_2$ is sufficiently small, the majority vote process ξ_t on $\mathbb{T}_d$, with $d \geq 5$, is non-ergodic and has uncountably many mutually singular equilibria. Their proof is structured around the use of "tilings" and a collection of auxiliary processes $\{\delta_t\}_{t\geq 0}$. Building on their approach but with a different "tiling", our next theorem extends this result to $\mathbb{T}_4$. This non-ergodicity contrasts with the behavior of the majority vote process defined on a uniformly self-supporting structure, demonstrated in Theorems 1 and 2 (see Chapters 3 and 4): On $\mathbb{T}_4$, regardless of the value of $\varepsilon_0 + \varepsilon_1$, as long as one of ε_0 and ε_1 is significantly smaller than the other, the majority vote process will be ergodic.

Theorem 6. *Consider the UMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{T}_4$, and let ε_0 and ε_1 denote the 0-noise and 1-noise intensities. When $\varepsilon_0, \varepsilon_1 \in [0, 1]$ satisfy*

$$\varepsilon_0 + \varepsilon_1 < \frac{1}{12}, \tag{5.5}$$

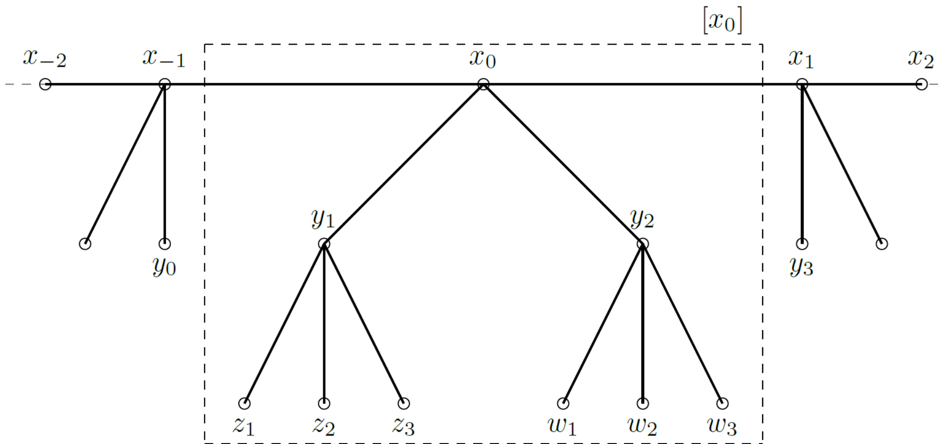
ξ_t is non-ergodic and has uncountably many mutually singular equilibria.

We outline the proof of Theorem 6 as follows: In Subsection 5.1.3, we introduce a family of uncountably many *compatible configurations*. To do so, we first introduce the *s-line*, which is a preassigned doubly infinite sequence of adjacent vertices on $\mathbb{T}_4$. We next partition $\mathbb{T}_4$ into *s-tiles* according to this s-line. A configuration is defined to be compatible if all sites from the same s-tile have the same opinion. When these compatible configurations are used as initial configurations for ξ_t, they lead to a family of uncountably many equilibria for ξ_t. To complete the proof of Theorem 6, it suffices to demonstrate that these equilibria are mutually singular with respect to each other.

Subsection 5.1.4 introduces the *dominating process on* $\mathbb{T}_4$, denoted by $\{\delta_t\}_{t\geq 0}$. We state some properties of δ_t in the same section: Let ξ_t denote a copy of the UMVP, with a compatible configuration. In Lemma 5.5, we show that, when jointly defined with ξ_t, δ_t controls the amount ξ_t deviates from the initial (compatible) configuration ξ_0. In Lemma 5.6, we establish a uniform upper bound on the marginal distribution of δ_t.

Subsection 5.1.5 proves Lemma 5.6, which completes the proof of Theorem 6. The proof of Lemma 5.6 is structured around the differential inequality (5.25). We first show

Figure 5.1: An example of an s-tiling with a given s-line $\{x_i\}_{i\in\mathbb{Z}}$



Here, $\mathcal{U}(x_0) = \{y_1, y_2\}$, $\mathcal{U}(y_1) = \{z_1, z_2, z_3\}$, and $\mathcal{U}(y_2) = \{w_1, w_2, w_3\}$. Therefore, y_0 and y_1 are not in the same s-tile; but y_1 and y_2 are in the same s-tile.

the marginal density of δ_t satisfies (5.25), and next prove (5.25), which implies (5.16) by contradiction.

In Subsection 5.1.6, we apply Lemmas 5.5, 5.6, and a concentration inequality (Lemma 5.7) to prove Theorem 6. By Lemma 5.5, to show the equilibria introduced by different compatible configurations are mutually singular, it suffices to show (5.41) for δ_t. Given the independent structure of δ_t, and the uniform upper bound on its marginal as in Lemma 5.6, Lemma 5.7 can be applied, which implies (5.41).

5.1.3 Compatible Configurations

This subsection introduces a family of uncountably many compatible configurations on $\mathbb{T}_4$, which will be used in Subsection 5.1.6 to introduce uncountably many mutually singular equilibria. We first define the *s-line*, then use the s-line to introduce the compatible configurations.

Let $\sigma = \{x_i\}_{i\in\mathbb{Z}} \subseteq \mathbb{T}_4$ denote a fixed doubly infinite sequence of sites, such that x_i and x_{i+1} are adjacent, for every $i \in \mathbb{Z}$. We refer to $\sigma = \{x_i\}_{i\in\mathbb{Z}}$ as an *s-line.* Sites on the s-line are *s-sites*, and the other sites are *l-sites*.

Two l-sites $x_1, x_2 \in \mathbb{T}_4$ are *in the same s-tile* if there exists a finite path $\{y_i\}_{i=1}^m$, with $y_1 = x_1$, $y_m = x_2$, such that no more than one y_i, for $1 \leq i \leq m$, is an s-site. We write $x_1 \sim x_2$ if x_1, x_2 are in the same s-tile (see Figure 5.1). We claim that

$$\text{“} \sim \text{” is an equivalence relation.} \tag{5.6}$$

Proof of (5.6) . To show that "$\sim$" is an equivalence relation, it suffices to show that "$\sim$" is reflexive, symmetric, and transitive.

First, "$\sim$" is reflexive since $x \sim x$, for each $x \in \mathbb{T}_4$. If $\{y_i\}_{i=1}^m$ is a path, so is $\{y_{m+1-i}\}_{i=1}^m$. $x \sim y$ implies $y \sim x$, which implies that "$\sim$" is symmetric.

We will next show "$\sim$" is transitive. Suppose that $x_1 \sim x_2$, and $x_2 \sim x_3$, for some $x_1, x_2, x_3 \in \mathbb{T}_4$. Let $p_1 = \{v_i\}_{i=1}^r$ and $p_2 = \{w_i\}_{i=1}^s$ denote the paths connecting x_1 and x_2, x_2 and x_3, respectively. Then p_1 and p_2 both contain at most one s-site.

We join p_1 and p_2 to form a new path. Since p_1 and p_2 are paths, $v_i \neq v_j$, ($w_i \neq w_j$), for every $i \neq j$. We claim that, for every $y_1, y_2 \in p_1 \cap p_2$, and any path p' connecting y_1 and y_2, $p' \subseteq p_1 \cap p_2$. To see this, note that T_4 is a tree and hence has no cycle, the path connecting y_1, y_2 is unique, and hence $p' \subseteq p_1$. Similarly, $p' \subseteq p_2$. Let $p_0 = p_1 \cap p_2$. A little thought shows that p_0 is a path, with x_2 as one end. Let y' denote the other end, and set

$$p = (p_1 - p_0) \sqcup \{y'\} \sqcup (p_2 - p_0), \tag{5.7}$$

where "$\sqcup$" denotes the non-intersecting union, and $p_i - p_0$, $i = 1, 2$, denotes the path obtained by removing p_0 from p_i, for $i = 1, 2$. (It is straightforward to verify that p is a path.)

We next show that p, introduced in (5.7), contains no more than one s-site. Since $p \subseteq (p_1 \cup p_2)$, if at least one of such p_1 and p_2 contains no s-site, p contains no more than one s-site. Supposing that both p_1 and p_2 contain exactly one s-site, we let s_1 and s_2 denote the s-site in p_1 and p_2, respectively. To show p contains at most one s-site, it suffices to show that $s_1 = s_2$.

Letting $p_i' \subseteq p_i$, $i = 1, 2$, denote the path connecting s_i and x_2, then p_i' contains no other s-site besides s_i. Since s_1 and s_2 are s-sites, there exists a path $p'' \subseteq \{x_i\}_{i \in \mathbb{Z}}$ connecting s_1 and s_2. If $s_1 \neq s_2$, then p'', p_1', p_2' form a cycle, contradicting the fact that $\mathbb{T}_4$ has no cycle. $s_1 = s_2$, implying that p contains at most one s-site. $x_1 \sim x_3$, and hence "$\sim$" is transitive. This completes the proof of (5.6). $\square$

We will refer to the equivalence classes introduced by "$\sim$" as *s-tiles*. It follows from the definition of "$x \sim y$" that every tile contains exactly one s-site x. Unless stated otherwise, we will use x as the representative of its equivalence class, and let $[x]$ denote the s-tile containing x.

We introduce the compatible configuration on $\mathbb{T}_4$, which is the $\mathbb{T}_4$ analog of the compatible configurations in [6].

Definition 5.2. *For a given s-line $\{a_i\}_{i\in\mathbb{Z}}$, let $[x]$ denote the equivalence class determined by "$\sim$". A configuration $\zeta \in \Xi$ is compatible if, for each $x \in \mathbb{T}_4$, and every $y_1, y_2 \in [x]$, $\zeta(y_1) = \zeta(y_2)$.*

In other words, ζ is compatible if opinions in the same s-tile are unanimous. When the noise parameter $\varepsilon = 0$, for every compatible configuration ζ, the majority vote process ξ_t with the initial configuration $\xi_0(x) \equiv \zeta(x)$ is "stable", in the sense that $\xi_t(x) = \zeta(x)$, for all $t \geq 0$ and $x \in \mathbb{T}_4$. (This is due to the fact that every site in ζ has at least two neighbors with the same opinion.) For every doubly infinite sequence $a = \{a_i\}_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, we let $\zeta^{(a)}$ denote the compatible configuration, where $\zeta(x_i) = a_i$, for every $i \in \mathbb{Z}$. In the remainder of this section, we will fix an s-line $\{x_i\}_{i\in\mathbb{Z}}$ and hence determine the s-tiles.

5.1.4 Dominating Process

We introduce the dominating process on $\mathbb{T}_4$, denoted by $\{\delta_t\}_{t\geq 0}$, by its graphical construction.

We introduce a set-valued function $\mathcal{U}$ to "orient" $\mathbb{T}_4$. Recall that $d(\cdot,\cdot)$ denotes the graph distance on $\mathbb{T}_4$.

Definition 5.3. *For every $x \in \mathbb{T}_4$, we define a set-valued function $\mathcal{U} : S \to 2^S$ as follows.*

(5.3.a) For every s-site x, we set $\mathcal{U}(x) = \{y \in \mathcal{N}(x) : y \text{ is an l-site}\}$.

(5.3.b) For every l-site $y \in [x]$, we set $\mathcal{U}(y) = \{z \in \mathcal{N}(y) : d(z,x) > d(y,x)\}$.

It follows from Definition 5.3 that $|\mathcal{U}(x)| = 2$, for every s-site $x \in \mathbb{T}_4$; and $|\mathcal{U}(y)| = 3$, for every l-site $y \in \mathbb{T}_4$. For future convenience, for every $x \in \mathbb{T}_4$, we write

$$\mathcal{U}^1(x) = \mathcal{U}(x), \quad \mathcal{U}^i(x) = \bigcup_{y \in \mathcal{U}^{i-1}(x)} \mathcal{U}(y), \quad \text{for } i \geq 2. \tag{5.8}$$

For every $x \in \mathbb{T}_4$, we define $\mathcal{D}(x)$, the *descendent* of x, by

$$\mathcal{D}(x) = \bigcup_{i=1}^{\infty} \mathcal{U}^i(x). \tag{5.9}$$

We next define the dominating process $\{\delta_t\}_{t \geq 0}$, by using its graphical construction. The dominating process was initially introduced by Bramson and Gray ([6]), and our dominating process is a variant of that in [6].

Recall that event points were introduced in Definition 2.1, and were assumed to satisfy (2.1) and (2.2). As before, for each $x \in S$, $X_x^{(0)}$, $X_x^{(1)}$ and $X_x^{(2)}$ denote the 0-noise, 1-noise and vote points at x.

Definition 5.4 (Dominating Process on $\mathbb{T}_4$)**.** *A spin system $\{\delta_t\}_{t \geq 0}$ is a dominating process on $\mathbb{T}_4$ if it satisfies the continuation rule (2.5), and the following three rules:*

(5.4.a) All-zero initial configuration: $\delta_0(x) = 0$ for every $x \in S$.

(5.4.b) δ-vote rule: If x is an l-site and $(x,t) \in X_x^{(2)}$, then

$$\delta_t(x) = \begin{cases} 1, & \text{if } \sum_{y \in \mathcal{U}(x)} \delta_{t-}(y) \geq 2, \\ 0, & \text{if } \sum_{y \in \mathcal{U}(x)} \delta_{t-}(y) = 0, \\ \delta_{t-}(x), & \text{otherwise.} \end{cases} \tag{5.10}$$

If x is an s-site, and $(x,t) \in X_x^{(2)}$, then

$$\delta_t(x) = \begin{cases} 1, & \text{if } \sum_{y \in \mathcal{U}(x) \cup \{x\}} \delta_{t-}(y) \geq 1, \\ 0, & \text{if } \sum_{y \in \mathcal{U}(x) \cup \{x\}} \delta_{t-}(y) = 0. \end{cases} \tag{5.11}$$

(5.4.c) Noise rule: If $(x,t) \in X_x^{(0)} \cup X_x^{(1)}$, then $\delta_t(x) = 1$.

We will refer to $\varepsilon = \varepsilon_0 + \varepsilon_1$ as the *noise parameter* of the dominating process. An immediate consequence of Definition 5.4 is that the transition rate of $\{\delta_t\}_{t \geq 0}$ satisfies

(2.8). Therefore, δ_t is a monotone spin system. Since (2.8) can be verified directly by (5.4.b) and (5.4.c), we omit its demonstration here.

Since the dominating process $\{\delta_t\}_{t\geq 0}$ was defined by its graphical construction, δ_t can be jointly defined with the UMVP on $\mathbb{T}_4$, as long as the noise parameter of δ_t, denoted by ε, and the 0-noise and 1-noise intensities of the UMVP, denoted by ε_0 and ε_1, satisfy $\varepsilon = \varepsilon_0 + \varepsilon_1$. Our next lemma, Lemma 5.5, illustrates the "dominating" property of δ_t, when δ_t is jointly defined along with a copy of the UMVP.

Lemma 5.5. *Let $\{\xi_t\}_{t\geq 0}$ denote the UMVP on $\mathbb{T}_4$, with $\xi_0 = \zeta = \zeta^{(a)}$ for some $a \in \{0,1\}^{\mathbb{Z}}$. Let ε_0 and ε_1 denote its 0-noise and 1-noise intensities and set $\varepsilon = \varepsilon_0 + \varepsilon_1$. Let $\{\delta_t\}_{t\geq 0}$ denote the dominating process, with noise intensity ε. Supposing that δ_t is jointly defined with ξ_t, then*

$$|\xi_t(x) - \zeta^{(a)}(x)| \leq \delta_t(x), \quad \textit{for every } x \in T_4 \textit{ and } t \geq 0. \tag{5.12}$$

Proof. We will prove (5.12) by applying Proposition 2.3, the graphical induction principle. Let $\mathcal{A}(x,t)$ denote the event that $|\xi_t(x) - \zeta^{(a)}(x)| \leq \delta_t(x)$ holds for a given $x \in \mathbb{T}_4$ and $t \geq 0$. Let $\mathcal{A}(x,t-)$ denote the event that $|\xi_{t-}(x) - \zeta^{(a)}(x)| \leq \delta_{t-}(x)$, where $\xi_{t-}(x)$ and $\delta_{t-}(x)$ were introduced by (2.3). It follows from Proposition 2.3 that, to prove (5.12), it suffices to show the following.

(5.5.a) $\mathcal{A}(x,0)$ holds for all $x \in \mathbb{T}_4$.

(5.5.b) For each $x \in \mathbb{T}_4$, if $\mathcal{A}(y,e(y,t))$ holds for all $y \in \mathcal{N}(x) \cup \{x\}$, then $\mathcal{A}(x,t)$ holds.

Since $\delta_t(x) \equiv 0$ for all $x \in \mathbb{T}_4$, (5.5.a) follows. To prove (5.12), it suffices to prove (5.5.b).

In the remainder of this proof, we fix any $x \in \mathbb{T}_4$ and suppose that, for each $y \in \mathcal{N}(x) \cup \{x\}$, $\mathcal{A}(y,e(y,t))$ holds. If (x,t) is not an event point, then $\mathcal{A}(x,t)$ follows from $\mathcal{A}(x,e(x,t))$ and the continuation rule (2.5).

We assume that (x,t) is an event point. Since $\mathcal{A}(y,e(y,t))$ holds for all $y \in \mathcal{N}(x) \cup \{x\}$, $\mathcal{A}(y,t-)$ holds for all such y due to the continuation rule. We next discuss different scenarios of the event point (x,t).

If (x,t) is a noise point, then $\delta_t(x) = 1$ by (5.4.c). Since the left-hand side of (5.12) is no larger than 1, $\mathcal{A}(x,t)$ holds regardless of the value of $\xi_t(x)$.

We assume that (x, t) is a vote point and discuss the following two cases – when x is an s-site and when x is an l-site. We note that if $\delta_t(x) = 1$, (5.12) always holds. Tt suffices to prove (5.12) when $\delta_t(x) = 0$.

Suppose that x is an s-site and $\delta_t(x) = 0$. Let $\mathcal{U}(x) = \{y_1, y_2\}$, then (5.11) implies that $\delta_{t-}(y_1) = \delta_{t-}(y_2) = \delta_{t-}(x) = 0$. Since $\mathcal{A}(y_i, t-)$ holds, for $i = 1, 2$, and $\mathcal{A}(x, t-)$ holds,

$$\xi_{t-}(y_1) = \xi_{t-}(y_2) = \xi_{t-}(x) = \zeta(x).$$

It then follows from (2.2.c) that $\xi_t(x) = \zeta(x)$, hence (5.12) holds.

We now assume that x is an l-site and $\delta_t(x) = 0$. Considering $\mathcal{U}(x) = \{y_1, y_2, y_3\}$, there are only two possibilities according to (5.10):

1. $\delta_{t-}(y_1) = \delta_{t-}(y_2) = \delta_{t-}(y_3) = 0$.

2. At least two $y \in \{y_1, y_2, y_3\}$ have $\delta_{t-}(y) = 0$, and $\delta_{t-}(x) = 0$

In the first case, since $\mathcal{A}(y_i, t-)$ holds for $i = 1, 2, 3$, $\xi_{t-}(y_i) = \zeta(y_i)$ for $i = 1, 2, 3$. Since y_1, y_2, y_3, and x are in the same s-tile, $\zeta(y_i) = \zeta_(x)$ holds for $i = 1, 2, 3$. Therefore,

$$\xi_{t-}(y_1) = \xi_{t-}(y_2) = \xi_{t-}(y_3) = \zeta(x). \tag{5.13}$$

Since (x, t) is a vote point and (5.13), we obtain that $\xi_t(x) = \zeta(x)$, and hence (5.12) holds.

In the second case, without loss of generality, we assume that $\delta_{t-}(y_1) = \delta_{t-}(y_2) = \delta_{t-}(x) = 0$. Since $\mathcal{A}(y_i, t-)$ holds for $i = 1, 2$, and $\mathcal{A}(x, t-)$ holds,

$$\xi_{t-}(y_1) = \zeta(y_1), \quad \xi_{t-}(y_2) = \zeta(y_2), \quad \xi_{t-}(x) = \zeta(x). \tag{5.14}$$

Since y_1, y_2 and x are in the same s-tile, (5.14) implies that

$$\xi_{t-}(y_1) = \xi_{t-}(y_2) = \xi_{t-}(x) = \zeta(x). \tag{5.15}$$

$\mathcal{A}(x, t)$ follows from (x, t) being a vote point and (5.15). This completes the proof of (5.5.b), and hence the proof of Lemma 5.5. $\square$

Not only does δ_t control the deviation of ξ_t from its initial compatible configuration $\zeta^{(a)}$, but it also exhibits an independence structure, which ξ_t does not have: For every

$x, y \in \mathbb{T}_4$, with $\mathcal{D}(x) \cap \mathcal{D}(y) = \emptyset$, $\delta_t(x)$ and $\delta_t(y)$ are independent. Consequently, for each $x \in S$ and every $n \in \mathbb{N}$, $\{\delta_t(y)\}_{y \in \mathcal{D}^n(x)}$ is a family of independent Bernoulli random variables.

Our next lemma states an upper bound on the marginal density of $\delta_t(x)$, for every l-site $x \in \mathbb{T}_4$ and $t \geq 0$.

Lemma 5.6. *Let $\delta_t(x)$ denote the dominating process, and $\varepsilon \in (0, 1)$ denote its noise intensity. If $\varepsilon < 1/12$, then, for every l-site $x \in \mathbb{T}_4$,*

$$\mathbb{P}(\delta_t(x) = 1) < 2\varepsilon. \tag{5.16}$$

We delay the proof of Lemma 5.6 until Subsection 5.1.5, and conclude this subsection with a concentration inequality.

Lemma 5.7. *Let X_i, $1 \leq i \leq n$, denote i.i.d. Bernoulli random variables. Assuming $\mathbb{E}X_i = \mu$ for every $1 \leq i \leq n$, then, for every $\alpha > 0$, the following holds:*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq (1+\alpha)\mu\right) \leq \left(\frac{e^{\alpha}}{(1+\alpha)^{1+\alpha}}\right)^{n\mu}. \tag{5.17}$$

Proof. Since X_i is a Bernoulli random variable, it has the moment generating function

$$\mathbb{E}e^{\lambda X_i} = \mu e^{\lambda} + 1 - \mu.$$

Since X_i, $1 \leq i \leq n$, are independent of each other, the moment generating function for $\sum_{i=1}^{n} X_i$ can be written as the product

$$\mathbb{E}e^{\lambda \sum X_i} = (\mu e^{\lambda} + 1 - \mu)^n. \tag{5.18}$$

Applying Chebyshev's inequality on the left-hand side of (5.17), we obtain

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq (1+\alpha)\mu\right) \leq \frac{(\mathbb{E}e^{\lambda X_1})^n}{e^{\lambda(1+\alpha)n\mu}} = \left(\frac{\mu e^{\lambda} + 1 - \mu}{e^{\lambda(1+\alpha)\mu}}\right)^n. \tag{5.19}$$

Taking $\lambda = \log(1+\alpha)$ in (5.19), we obtain the desired bound in (5.17). This completes the proof of Lemma 5.7. □

5.1.5 Proof of Lemma 5.6

This subsection proves Lemma 5.6. A little thought shows that it follows from Definition 5.4 and the symmetry of $\mathbb{T}_4$ that each $\delta_t(x)$, where x is an s-site, is identically distributed. Therefore, $\mathbb{P}(\delta_t(x) = 1)$ does not depend on the choice of the l-site x. We set $p(t) = \mathbb{P}(\delta_t(x) = 1)$. In this subsection, we will derive a differential equation that $p(t)$ satisfies, and apply a standard topological bootstrapping argument to prove (5.16).

We fix any l-site $x \in \mathbb{T}_4$, and let $\mathcal{U}(x) = \{y_1, y_2, y_3\}$. On the one hand, it follows from Definition 5.4 that $\delta_t(x)$ can change from 0 to 1 in two scenarios – either when (x, t) is a noise point, by (5.4.c); or when (x, t) is a vote point and at least two $y \in \{y_1, y_2, y_3\}$ have $\delta_{t-}(y) = 1$, by (5.4.b).

Let $H(x, t)$ denote the event that $\delta_t(y_i) = \delta_t(y_j) = 1$, for some $i \neq j \in \{1, 2, 3\}$. Therefore, $\delta_t(x)$ changes from 0 to 1 at rate

$$\varepsilon \mathbb{P}(\delta_t(x) = 0) + (1 - \varepsilon)\mathbb{P}(\{\delta_t(x) = 0\} \cap H(x, t)). \tag{5.20}$$

On the other hand, by (5.4.b), $\delta_t(x)$ changes from 1 to 0 only when (x, t) is a vote point, and $\delta_t(y_i) = 0$ for $i = 1, 2, 3$. Letting $K(x, t)$ denote the event that $\delta_t(y_1) = \delta_t(y_2) = \delta_t(y_3) = 0$, then $\delta_t(x)$ changes from 1 to 0 at rate

$$(1 - \varepsilon)\mathbb{P}(\{\delta_t(x) = 1\} \cap K(x, t)). \tag{5.21}$$

Combining (5.20) and (5.21), we obtain

$$\begin{aligned} \frac{d}{dt} p(t) = \varepsilon \mathbb{P}(\delta_t(x) = 0) &+ (1 - \varepsilon)\mathbb{P}(\{\delta_t(x) = 0\} \cap H(x, t)) \\ &- (1 - \varepsilon)\mathbb{P}(\{\delta_t(x) = 1\} \cap K(x, t)). \end{aligned} \tag{5.22}$$

Since $K(x, t) \cap H(x, t) = \emptyset$,

$$\begin{aligned} \mathbb{P}(\{\delta_t(x) = 0\} \cap H(x, t)) &- \mathbb{P}(\{\delta_t(x) = 1\} \cap K(x, t)) \\ &\leq \mathbb{P}(H(x, t)) - \mathbb{P}(\delta_t(x) = 1). \end{aligned} \tag{5.23}$$

Since $\delta_t(y_i)$, $i = 1, 2, 3$, are independent and identically distributed,

$$\mathbb{P}(H(x, t)) = 3(1 - p(t))p^2(t) + p^3(t). \tag{5.24}$$

Therefore, $p(t)$ satisfies

$$\begin{aligned} p'(t) &\le \varepsilon(1-p(t)) + (1-\varepsilon)(3(1-p(t))p^2(t) + p^3(t) - p(t)) \\ &= \varepsilon - p(t) + 3(1-\varepsilon)p^2(t) - (1-\varepsilon)p^3(t). \end{aligned} \tag{5.25}$$

We rewrite (5.25) as follows.

$$p'(t) + p(t) \le \varepsilon + 3(1-\varepsilon)p^2(t) - (1-\varepsilon)p^3(t). \tag{5.26}$$

We claim that $p(t) < 2\varepsilon$, for all $t \ge 0$: Since $p(0) = 0 < 2\varepsilon$ and $p(t)$ is differentiable, there exists some $u > 0$ such that $p(t) \le 2\varepsilon$ holds for every $t \in [0, u]$. To show (5.16), it suffices to show that, for each given $u > 0$ such that $p(t) \le 2\varepsilon$ for every $t \in [0, u]$, the following holds:

$$p(t) < 2\varepsilon \text{ for } t \in [0, u]. \tag{5.27}$$

Since $p(t) \le 2\varepsilon$ for $t \in [0, u]$, it follows from (5.26) that

$$p'(t) + p(t) \le \varepsilon + 3(2\varepsilon)^2 < 2\varepsilon, \tag{5.28}$$

where the last inequality follows from the assumption that $\varepsilon < 1/12$.

Suppose that (5.27) does not hold. Since $p(t)$ is differentiable, there exists some $u' \in [0, u]$ such that $p(u') = 2\varepsilon$. Setting

$$v = \inf\{u' \in [0, u] : p(u) = 2\varepsilon\}, \tag{5.29}$$

it follows from (5.28) that $p'(v) < 0$. This contradicts the monotonicity of $\{\delta_t\}_{t \ge 0}$. (Recall we proved that δ_t is a monotone spin system. It follows from Proposition 2.4 that $p'(t) \ge 0$.) This completes the proof of (5.27). $\square$

5.1.6 Proof of Theorem 6

In this subsection, we prove Theorem 6 by applying Lemmas 5.5, 5.6, and 5.7. This proof is structured similarly to that of Theorem 2 in [6]: We first construct an equilibrium $\mu^{(a)}$, for each given doubly infinite sequence $a \in \{0, 1\}^{\mathbb{Z}}$. We next show that, for every distinct a_1, $a_2 \in \{0, 1\}^{\mathbb{Z}}$, $\mu^{(a_1)}$ and $\mu^{(a_2)}$ are mutually singular. Since $\{0, 1\}^{\mathbb{Z}}$ is uncountable, ξ_t has uncountably many mutually singular equilibria.

As usual, we let $\{x_i\}_{i \in \mathbb{Z}}$ denote the s-line, with $x_i \in \mathcal{N}(x_{i+1}) \cap \mathcal{N}(x_{i-1})$ for every $i \in \mathbb{Z}$. For every $a \in \{0, 1\}^{\mathbb{Z}}$, let $\zeta^{(a)}$ denote the compatible configuration, where

$\zeta(x_i) = a_i$, for every $i \in \mathbb{Z}$. Let $\{\xi_t^{(a)}\}_{t\geq 0}$ denote the UMVP on $\mathbb{T}_4$, with its initial configuration $\xi_0^{(a)} = \zeta^{(a)}$.

For each $t \geq 0$, letting μ_t^a denote the probability distribution of $\xi_t^{(a)}$, we consider the Cesaro averages:

$$\bar{\mu}_t^{(a)} := \frac{1}{t}\int_0^t \mu_s^{(a)} ds,\ t \geq 0. \tag{5.30}$$

Since the state space Ξ is compact when equipped with the product topology, the space of probability measures on Ξ is sequentially compact when equipped with the weak$*$ topology. Hence, there exists a subsequence $\{t_i\}_{i=1}^{\infty}$, with $\lim_{i\to\infty} t_i = \infty$, such that

$$\mu^{(a)} = \lim_{i\to\infty} \bar{\mu}_{t_i}^{(a)}, \tag{5.31}$$

where the limit in (5.31) is taken with respect to the weak$*$ topology. By definition, $\mu^{(a)}$ is an equilibrium.

We next show that the equilibria μ^{ζ} obtained from distinct compatible configurations are mutually singular. Recall that $\mathcal{U}^n$, introduced in (5.8), satisfies

$$|\mathcal{U}^n(x)| = \begin{cases} 2\cdot 3^{n-1}, & \text{if } x \text{ is an s-site,} \\ 3^n, & \text{if } x \text{ is an l-site.} \end{cases} \tag{5.32}$$

For each s-site x, $\{\delta_t(y)\}_{y\in\mathcal{D}^n(x)}$ is a family of independent random variables. Since $\{\delta_t(y)\}_{y\in\mathcal{D}^n(x)}$ is a family of independent Bernoulli random variables, we apply Lemma 5.7, with $\alpha = 1$, and obtain that

$$\mathbb{P}\left(\frac{1}{2\cdot 3^{n-1}}\sum_{s\in\mathcal{U}^n(x)} \delta_t(x) \geq 2\mathbb{E}\delta_t(x)\right) \leq \left(\frac{e}{4}\right)^{n\mathbb{E}\delta_t(x)}. \tag{5.33}$$

Moreover, Lemma 5.6 implies that $E\delta_t(x) \leq 2\varepsilon$. It therefore follows from (5.33) that, for each s-site x

$$\mathbb{P}\left(\frac{1}{2\cdot 3^{n-1}}\sum_{s\in\mathcal{U}^n(x)} \delta_t(x) \geq 4\varepsilon\right) \leq \left(\frac{e}{4}\right)^{2^n C}, \tag{5.34}$$

where $C := \mathbb{E}\delta_t(x) \leq 2\varepsilon$.

For any given s-site $x \in \mathbb{T}_4$ and every $a \in \{0,1\}^{\mathbb{Z}}$, we consider the event

$$F(x,a) := \left\{\xi \in \mathscr{S} : \limsup_{n\to\infty} \frac{1}{2\cdot 3^{n-1}}\sum_{s\in\mathcal{U}^n(x)} |\zeta^{(a)}(y) - \xi(y)| \leq 8\varepsilon\right\}. \tag{5.35}$$

We claim that

$$\mu^{(a)}(F(x,a)) = 1, \quad \text{for every s-site } x, \text{ and every } a \in \{0,1\}^{\mathbb{Z}}. \tag{5.36}$$

To see (5.36), noting that for every $A \subseteq \mathcal{D}^n(x)$, it follows from Lemma 5.5 that

$$\mu_t^{(a)}(\xi : \xi(y) \neq \zeta^{(a)}(y) \text{ for every } y \in A) \leq \mathbb{P}(\delta_t(y) = 1 : y \in A). \tag{5.37}$$

It hence follows from (5.34) that

$$\mu_t^{(a)}\left(\xi \in \mathscr{S} : \frac{1}{2 \cdot 3^{n-1}} \sum_{s \in \mathcal{U}^n(x)} |\zeta^{(a)}(y) - \xi(y)| > 8\varepsilon\right) \leq (\frac{e}{4})^{2^n C}. \tag{5.38}$$

Since $C = \mathbb{E}\delta_t(y)$ does not depend on the choice of y (as long as y is an s-site), the right-hand side of (5.38) is summable, for every $t \geq 0$. The Borel-Cantelli Lemma implies that

$$\mu_t^{(a)}(F(x,a)) = 1, \quad \text{for every } t \geq 0. \tag{5.39}$$

Since $\mu^{(a)}$, which was defined by (5.31), is obtained by taking a subsequential limit of the Cesaro averages of $\mu_t^{(a)}$,

$$\mu^{(a)}((F(x,a)) = 1, \quad \text{for each s-site } x \in \mathbb{T}_4. \tag{5.40}$$

For distinct $a, a' \in \{0,1\}^{\mathbb{Z}}$, there exists some $i \in \mathbb{Z}$, with $a_i \neq a'_i$. Letting $\mu^{(a)}$ and $\mu^{(a')}$ denote the equilibria obtained by taking a subsequential limit of $\bar{\mu}_t^{(a)}$ and $\mu_t^{(a')}$, then

$$\mu^{(a)}(F(x_i,a)) = \mu^{(a')}(F(x_i,a')) = 1. \tag{5.41}$$

If $\varepsilon < 1/12$, then $F(x_i,a) \cap F(x_i,a') = \emptyset$, which implies that $\mu^{(a)}$ and $\mu^{(a')}$ are mutually singular. Since the sequence space $\{0,1\}^{\mathbb{Z}}$ is uncountable, ξ_t has uncountably many mutually singular equilibria. This completes the proof of Theorem 6. □

5.2 UMVP on $\mathbb{T}_3$

This section investigates the ergodic behavior of the UMVP on $\mathbb{T}_3$. Throughout this section, let $(S, \mathcal{N})$ denote the structure where $S = \mathbb{T}_3$, and, for each $x \in S$, $\mathcal{N}(x) = \{y \in \mathbb{T}_3 : y \text{ is adjacent to } x\}$. Recall that, in Example 2.17, we proved that such a structure $(S, \mathcal{N})$ is not uniformly self-supporting.

First, we briefly discuss the behavior of ξ_t when $\varepsilon_0 + \varepsilon_1$ is large. Recall that, in Section 2.4, we demonstrated that when the noise event occurs at a significantly higher rate compared with the vote event, the corresponding majority vote process is ergodic. In Proposition 5.1, we proved that, when $\varepsilon_0 + \varepsilon_1$ is large enough, the UMVP on $\mathbb{T}_4$ is ergodic, and converges to its unique equilibrium, starting from any initial distribution. Our next result, Proposition 5.8, shows that ξ_t also has this behavior when $\varepsilon_0 + \varepsilon_1$ is large. As in Section 5.1, Proposition 5.8 is primarily included to contrast with that of the behavior of ξ_t when $\varepsilon_0 + \varepsilon_1$ is sufficiently small. Since the proof of Proposition 5.8 is similar to that of Proposition 5.1, we omit its proof here.

Proposition 5.8. *Consider the UMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{T}_3$, and let ε_0 and ε_1 denote the* 0*-noise and* 1*-noise intensities. When $\varepsilon_0, \varepsilon_1 \in (0,1)$ satisfy*

$$\varepsilon_0 + \varepsilon_1 > 3/4, \tag{5.42}$$

ξ_t is ergodic and converges exponentially quickly to the unique equilibrium, starting from any initial configuration.

We note that different bounds in (5.1) and (5.42) arise from the different number of sites in each $\mathcal{N}(x)$, for $x \in \mathbb{T}_3$ and $x \in \mathbb{T}_4$: Recall that, in the proof of Proposition 5.1, the size $|\mathcal{N}(x)|$ is used to obtain the crucial inequality (5.4). Substituting $4h(x)$ with $3h(x)$ on the right-hand side of (5.4), the proof of Proposition 5.8 can be obtained from that of Proposition 5.1.

5.2.1 Non-ergodicity for Small $\varepsilon_0 + \varepsilon_1$

Our next theorem establishes the non-ergodicity of ξ_t when $\varepsilon_0 + \varepsilon_1$ is sufficiently small. As on $\mathbb{T}_4$, this behavior differs from that of the majority vote process on a uniformly self-supporting structure.

Theorem 7. *Consider the UMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{T}_3$, and let ε_0 and ε_1 denote its* 0*-noise and* 1*-noise intensities. When ε_0 and ε_1 satisfy*

$$\varepsilon_0 + \varepsilon_1 < \frac{1}{12 \cdot 10^7}, \tag{5.43}$$

ξ_t is non-ergodic and admits at least two mutually singular equilibria.

The non-ergodicity result in Theorem 7 is weaker than those of $\mathbb{T}_d$, with $d \geq 4$, namely, Theorem 6 and Theorem 1 in [6]. In the proofs of the latter two cases, one disregards the influence of one or two sites from each $\mathcal{N}(x)$ to "orient" $\mathbb{T}_d$. However, in $\mathbb{T}_3$, each $\mathcal{N}(x)$ consists of only three sites, making it impossible to ignore any site's influence. Proving Theorem 7 requires a more delicate "contour method". To achieve this, we will construct the *t-truss*, which is a variant of the *truss* introduced in [7].

The remainder of Section 5.2 is devoted to the proof of Theorem 7. We outline this proof as follows: Since the UMVP on $\mathbb{T}_3$ is a monotone spin system, it follows from Proposition 2.10 that, to prove Theorem 7, it suffices to show that, for every $x \in \mathbb{T}_3$, $\lim_{t\to\infty} \mathbb{P}(\xi_t(x) = 1) < 1/2$, where ξ_t is a copy of the UMVP on $\mathbb{T}_3$, with the initial configuration $\xi_0(y) \equiv 0$, for every $y \in \mathbb{T}_3$.

In Subsection 5.2.2, we introduce the *dominating process on* $\mathbb{T}_3$, denoted by $\{\delta_t\}_{t\geq 0}$. This process serves the same purpose on $\mathbb{T}_3$ as the dominating process of $\mathbb{T}_4$: δ_t controls the deviation of ξ_t (with certain initial configurations) from its initial configuration ξ_0. The dominating process reduces the proof of Theorem 7 to proving an inequality for the marginal density of δ_t. However, the dynamics of the dominating process on $\mathbb{T}_3$ differs from those of $\mathbb{T}_4$: Here, δ_t is a copy of the UMVP, jointly defined with ξ_t, but has different 0-noise and 1-noise intensities. The use of δ_t remains the same as that of $\mathbb{T}_4$: In Lemma 5.10, we prove that δ_t, when jointly defined with a copy of the UMVP ξ_t that has all-zero or all-one initial configuration, bounds the deviation of ξ_t from its initial configuration ξ_0. As a direct consequence of Lemma 5.10 and Corollary 2.4, to prove the non-ergodicity of ξ_t, it suffices to prove that $\lim_{t\to\infty} \mathbb{P}(\delta_t(x)) < 1/2$, for every $x \in \mathbb{T}_3$, or (5.61).

Subsection 5.2.3 is where the proof of Theorem 7 differs most significantly from that of Theorem 6. In Subsection 5.2.3, we introduce the t-truss and discuss some of its properties. Unlike in the proof of Theorem 6, where the technique of s-tiling was employed, the concept of the t-truss is unique to the proof of Theorem 7. In Proposition 5.19, we prove that, supposing $\delta_t(x) = 1$ for some $t \geq 0$ and $x \in \mathbb{T}_3$, then there exists a successful t-truss, rooted at x. To prove an upper bound on the marginal density $\mathbb{P}(\delta_t(x) = 1)$, it suffices to derive an upper bound on the probability for each t-truss to be successful, and then to sum over all possible t-truss. This approach is distinct from the method used in proving Theorem 6.

The proof of Theorem 7 will be structured around the use of the t-truss: In Subsection 5.2.4, we prove (5.61) assuming Propositions 5.14 and 5.19. Subsections 5.2.5 and 5.2.6 prove Proposition 5.14 and Proposition 5.19, respectively. This will complete the proof of Theorem 7.

5.2.2 Dominating Process

This subsection introduces the dominating process on $\mathbb{T}_3$, denoted by $\{\delta_t\}_{t\geq 0}$, and state its "dominating" property in Lemma 5.10. Lemma 5.10 states that, for a copy ξ_t of the UMVP on $\mathbb{T}_3$ that has all-zero or all-one initial configuration, δ_t, when jointly defined with ξ_t, controls the deviation of ξ_t from its initial configuration.

In the remainder of this section, we fix the choice of ε_0 and ε_1 that satisfies (5.43). Recall that a realization of the Poisson clock $X = \{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x\in S}$ is a family of event points, sampled from independent Poisson point processes (see Definition 2.1). As before, a realization of the Poisson clock is assumed to satisfy (2.1) and (2.2).

We define the *dominating process on* $\mathbb{T}_3$ by its graphical construction.

Definition 5.9 (Dominating Process on $\mathbb{T}_3$). *Let $\{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x\in S}$ denote a realization of the Poisson clock. A process $\{\delta_t\}_{t\geq 0}$ is a dominating process on $\mathbb{T}_3$ if it satisfies the continuation rule (2.5) and the following three rules.*

(5.9.a) δ-noise rule: For each $(x,t) \in X_x^{(0)} \cup X_x^{(1)}$, $\delta_t(x) = 1$.

(5.9.b) Majority vote rule: For each $(x,t) \in X_x^{(2)}$, $\delta_t(x) = M(\xi_{t-}, x)$.

(5.9.c) For each $x \in \mathbb{T}_3$, $\delta_0(x) \equiv 0$.

Since the introduction of δ_t uses the same collection of event points of ξ_t, δ_t can be jointly defined with ξ_t. Comparing Definition 5.9 with Definition 2.2 shows that δ_t is a copy of the UMVP on $\mathbb{T}_3$, whose 0-noise intensity and 1-noise intensity are 0 and $\varepsilon_0 + \varepsilon_1$, respectively, and having all-zero initial configuration.

Let $\{\xi_t^{(0)}\}_{t\geq 0}$ and $\{\xi_t^{(1)}\}_{t\geq 0}$ denote the UMVP on $\mathbb{T}_3$, with all-zero and all-one initial configurations, respectively, i.e., $\xi_0^{(0)}(x) = 1 - \xi_0^{(1)}(x) \equiv 0$, for every $x \in \mathbb{T}_3$. Our next lemma, Lemma 5.10, illustrates the "domination property" of δ_t: When jointly defined with $\xi_t^{(0)}$ or $\xi_t^{(1)}$, δ_t controls the deviation of $\xi_t^{(0)}$ ($\xi_t^{(1)}$) from its initial all-zero (all-one) configuration. The proof of Lemma 5.10 uses the same approach as that of Lemma 5.5, and is included for completeness. The reader should feel free to skip it.

Lemma 5.10. *Let δ_t denote the dominating process that is jointly defined with $\xi_t^{(0)}$ and $\xi_t^{(1)}$. Then, for each $x \in \mathbb{T}_3$ and $t \geq 0$,*

$$\mathbb{P}(\xi_t^{(0)}(x) = 1) \leq \mathbb{P}(\delta_t(x) = 1), \quad \mathbb{P}(\xi_t^{(1)}(x) = 0) \leq \mathbb{P}(\delta_t(x) = 1). \tag{5.44}$$

Lemma 5.10 is a natural result of the monotonicity of the majority vote process: Compared with $\xi_t^{(0)}$, δ_t follows the same majority vote rule (2.2.c), but has "one-sided" noise. When jointly defined with $\xi_t^{(0)}$, δ_t only differs at noise points, in the sense that the 0-noise points for $\xi_t^{(0)}$ are the 1-noise points for δ_t. Cumulatively, we expect δ_t to have more sites of opinion 1 than $\xi_t^{(0)}$ has.

Proof. We note that, due to symmetry, to prove Lemma 5.10, it suffices to show (5.44) for one of $\xi_t^{(0)}$ and $\xi_t^{(1)}$. In the remainder of this subsection, our goal is to prove (5.44) for $\xi_t^{(0)}$.

Since δ_t is jointly defined with $\xi_t^{(0)}$ and $\xi_t^{(1)}$, to prove (5.44), it suffices to show

$$\xi_t^{(0)}(x) \leq \delta_t(x), \quad \text{for every } x \in \mathbb{T}_3 \text{ and } t \geq 0. \tag{5.45}$$

We will prove (5.45) by applying Proposition 2.3, the graphical induction principle.

Let $\mathcal{A}(x,t)$ denote the event that $\xi_t^{(0)}(x) \leq \delta_t(x)$ holds, for a given $x \in \mathbb{T}_3$ and $t \geq 0$. Let $\mathcal{A}(x,t-)$ denote the event that $\xi_{t-}^{(0)}(x) \leq \delta_{t-}(x)$ holds, for a given $x \in \mathbb{T}_3$ and $t \geq 0$, where $\xi_{t-}^{(0)}(x)$ and $\delta_{t-}(x)$ were introduced by (2.3). It follows from Proposition 2.3 that, to prove (5.45), it suffices to show the following:

(5.10.a) $\mathcal{A}(x,0)$ holds for all $x \in \mathbb{T}_3$.

(5.10.b) For each $x \in \mathbb{T}_3$, if $\mathcal{A}(y,e(y,t))$ holds for all $y \in \mathcal{N}(x) \cup \{x\}$, then $\mathcal{A}(x,t)$ holds.

Since $\delta_t(x) \equiv 0$ for all $x \in \mathbb{T}_3$, (5.10.a) follows immediately. To prove (5.45) for all $x \in \mathbb{T}_3$ and $t \geq 0$, it suffices to prove (5.10.b).

In the remainder of this proof, we fix any $x \in \mathbb{T}_3$. Supposing $\mathcal{A}(y,e(y,t))$ holds for all $y \in \mathcal{N}(x) \cup \{x\}$, then $\mathcal{A}(x,e(x,t))$ holds. If (x,t) is not an event point, then the continuation rule (2.5) implies (5.10.b) directly.

We assume that (x,t) is an event point. Since $\mathcal{A}(y, e(y,t))$ holds for every $y \in \mathcal{N}(x) \cup \{x\}$, $\mathcal{A}(y, t-)$ holds for all such y by the continuation rule (2.5) and the assumption (2.1).

If (x,t) is a noise point, then $\delta_t(x) = 1$ by (5.9.a). Since the left-hand side of (5.45) is no larger than 1, $\mathcal{A}(x,t)$ holds regardless of the value of $\xi_t^{(0)}(x)$. To prove (5.45), it suffices to prove (5.45) when (x,t) is a vote point.

Assuming that (x,t) is a vote point, since (5.45) always holds when $\delta_t(x) = 1$, to prove (5.45), it suffices to prove (5.45) when $\delta_t(x) = 0$. Let $\mathcal{N}(x) = \{y_1, y_2, y_3\}$. Since we have assumed that $\delta_t(x) = 0$, at least two $y \in \{y_1, y_2, y_3\}$ satisfy $\delta_{t-}(y) = 0$. Without loss of generality, we suppose that $\delta_{t-}(y_1) = \delta_{t-}(y_2) = 0$.

Since we have assumed that $\mathcal{A}(y_1, t-)$ and $\mathcal{A}(y_2, t-)$ hold, it follows from $\delta_{t-}(y_1) = \delta_{t-}(y_2) = 0$ that $\xi_{t-}^{(0)}(y_1) = \xi_{t-}^{(0)}(y_2) = 0$. Therefore, when (x,t) is a vote point, $\xi_t^{(0)}(x) = 0$, and hence $\mathcal{A}(x,t)$ holds. This proves (5.10.b), and hence completes the proof of Lemma 5.10. $\square$

5.2.3 Terminology: t-truss and Successful t-truss

This subsection introduces the *t-truss*, which will serve as the most important tool to control the marginal density $\mathbb{P}(\delta_t(x) = 1)$. Our t-truss is a variant of the *truss*, which was introduced by Toom ([7]). Recall that we have fixed a realization of the Poisson clock $X = \{X_x^{(0)}, X_x^{(1)}, X_x^{(2)}\}_{x \in S}$, and X is assumed to satisfy (2.1) and (2.2). We introduce the t-truss as follows.

Definition and properties of t-truss

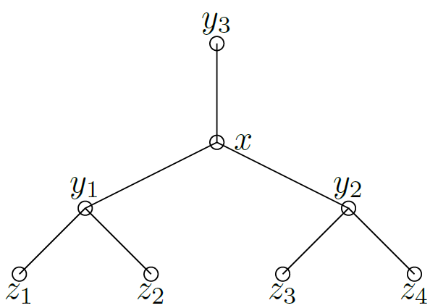
For each $(x,t) \in X$, we set

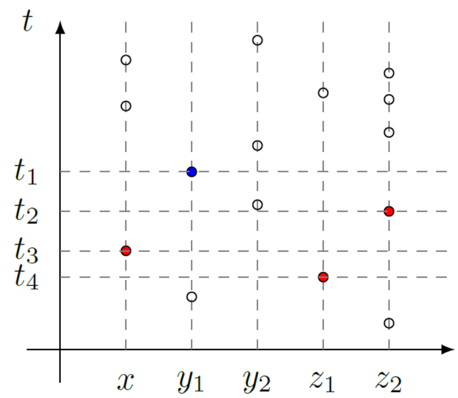
$$\mathcal{W}(x,t) = \{(y, e(y,t)) : y \in \mathcal{N}(x)\}, \tag{5.46}$$

where $e(x,t)$ was introduced in (2.4). In other words, $\mathcal{W}(x,t)$ is the collection of the latest event points at each $y \in \mathcal{N}(x)$ before time t. For each $(x,t) \in X$, we set

$$\begin{aligned} \mathcal{Y}(x,t) = \{(y,s) \in X : (x,t) \in \mathcal{W}(y,s), s > t, \text{ and} \\ (X_y^{(0)} \cup X_y^{(1)} \cup X_y^{(2)}) \cap (\{y\} \times (t,s)) = \emptyset\}, \end{aligned} \tag{5.47}$$

Figure 5.2: Examples of $\mathcal{W}(x,t)$, $\mathcal{Y}(x,t)$ and $\mathcal{Z}(x,t)$





On the left, $x, y_1, y_2, y_3, z_1, z_2, z_3, z_4 \in \mathbb{T}_3$ are displayed. On the right, a realization of the Poisson clock X on $\mathbb{T}_3$ is shown, where each circle represents an event point. By definition (5.46), $\mathcal{W}(y_1,t_1) = \{(x,t_3),(z_1,t_4),(z_2,t_2)\}$. Hence, by definition (5.47), $(y_1,t_1) \in \mathcal{Y}(x,t_3) \cap \mathcal{Y}(z_1,t_4) \cap \mathcal{Y}(z_2,t_2)$. By definition (5.48), $\mathcal{Z}(x,t_3) = \{(z_1,t_4),(z_2,t_2)\}$, $\mathcal{Z}(z_1,t_4) = \{(x,t_3),(z_2,t_2)\}$, and $\mathcal{Z}(z_2,t_2) = \{(x,t_3),(z_1,t_4)\}$.

and

$$\begin{aligned}\mathcal{Z}(x,t) = \{(z,w) \in X : z \neq x, \text{ and} \\ (z,w) \in \mathcal{W}(y,s) \text{ for some } (y,s) \in \mathcal{Y}(x,t)\}.\end{aligned} \tag{5.48}$$

It follows from (5.47) and (5.48) that $(z,w) \in \mathcal{Z}(x,t)$ implies that $(x,t) \in \mathcal{Z}(z,w)$. Figure 5.2 presents an example of $\mathcal{W}(x,t)$, $\mathcal{Y}(x,t)$, and $\mathcal{Z}(x,t)$, for a given realization of the Poisson clock.

We introduce the t-truss. Recall that a tree is a connected (undirected) graph with no cycle.

Definition 5.11 (t-truss)**.** *For a realization of the Poisson clock X satisfying 2.1 and 2.2, let $T = (V(T), E(T))$ denote a finite graph on X, with $V(T) \subseteq X$. Let $(x,t) \in V(T)$ denote a vertex of T such that $t \geq s$, for every $(y,s) \in V(T)$. (Such an (x,t) always exists by 2.1.) We say T is a t-truss if it satisfies the following.*

(5.11.a) T is a tree.

(5.11.b) *Each* $(y,s) \in V(T)$ *is incident with at most* 4 *edges.*

(5.11.c) *For every* $(y,s) \in V(T)$*, letting* $\sigma = \{(y_i,s_i)\}_{i=0}^{r}$ *denote the path connecting* (x,t) *and* (y,s)*, with* $(y_0,s_0) = (x,t)$*, and* $(y_r,s_r) = (y,s)$*, then, for every* $1 \le i \le r$*,*

$$(y_i,s_i) \in \mathcal{W}(y_{i-1},s_{i-1}) \cup \mathcal{Z}(y_{i-1},s_{i-1}). \tag{5.49}$$

Using the realization of the Poisson clock X shown in Figure 5.2, we present examples of graphs on X that are t-trusses and those that are not. These examples are presented in Figure 5.3.

Our next lemma estimates the number of all possible t-trusses with l edges.

Lemma 5.12. *For a fixed choice of the Poisson clock* X *satisfying 2.1 and 2.2, let* $N(l)$ *denote the number of all possible t-trusses with* l *edges. Then*

$$N(l) \le 10^{2l-1}. \tag{5.50}$$

Proof. Since T is a tree, there exists a closed path $\tau = \{(y_i,s_i)\}_{i=0}^{2l}$, with $(y_0,s_0) = (y_{2r},s_{2r}) = (x,t)$, such that each edge $e \in E(T)$ appears exactly twice in τ. Therefore, the number of possible t-trusses is no more than the number of such paths.

A little thought shows it follows form (5.49) that, for every (y_{i-1},s_{i-1}) and (y_i,s_i), with $(y_{i-1},s_{i-1}) \in W(y_i,s_i)$, there exists some $j < i$ such that $(y_j,s_j) \in W(y_{j-1},s_{j-1})$.

We estimate the number of such paths: Supposing that $\{(y_i,s_i)\}_{i=0}^{k-1}$ is determined for some $k \ge 1$, then, there are only three possibilities when choosing (y_k,s_k):

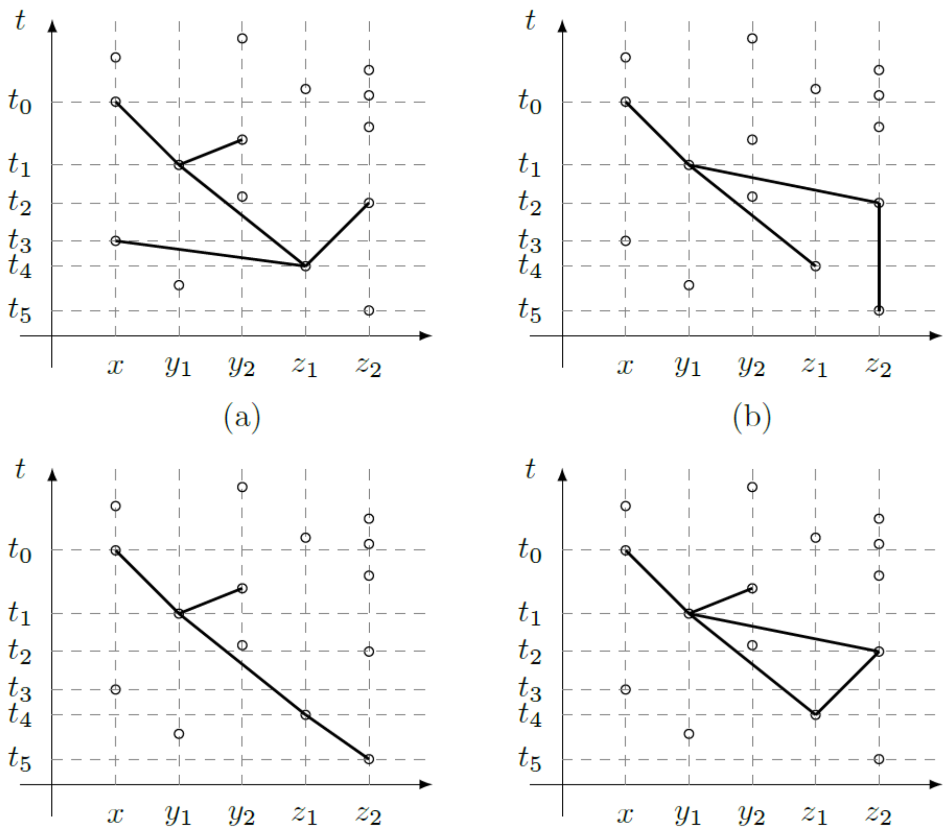
1. $(y_k,s_k) \in \mathcal{W}(y_{k-1},s_{k-1})$: There are at most 3 such event points.

2. $(y_k,s_k) \in \mathcal{Z}(y_{k-1},s_{k-1})$: There are at most 6 such event points.

3. (y_k,s_k) such that $(y_{k-1},s_{k-1}) \in \mathcal{W}(y_k,s_k)$: A little thought shows that (y_k,s_k) has been traversed by the path $\{(y_i,s_i)\}_{i=0}^{k-1}$. It therefore follows from (5.49) that such a (y_k,s_k) is unique.

Since the last vertex of the path σ, (y_{2l},s_{2l}), coincides with (x,t), the number of all possible paths is bounded by 10^{2l-1}. This completes the proof of Lemma 5.12. $\square$

Figure 5.3: Examples of graphs that are not t-trusses.



Let $x, y_1, y_2, y_3, z_1, z_2, z_3, z_4 \in \mathbb{T}_3$ denote the vertices that were displayed in Figure 5.2. The graph displayed in (a) is a t-truss, while those displayed in (b), (c), and (d) are not: In (b), the edge connecting (z_2, t_2) and (z_2, t_5) violates (5.11.c). In (c), the edge connecting (z_1, t_4) and (z_2, t_5) violates (5.11.c). The graph displayed in (d) (bottom-right) is not a tree.

Selections and successful t-truss

We introduce the *selection* on X: Recall that, for each $x \in \mathbb{T}_3$ and an event point $(x,t) \in X_x^{(0)} \cup X_x^{(1)} \cup X_x^{(2)}$, $\mathcal{W}(x,t)$ was introduced in (5.46).

Definition 5.13 (selection)**.** *We fix a realization of the Poisson clock X satisfying (2.1) and (2.2). A selection $\mathcal{V}$ is a set-valued function $\mathcal{V} : X \to \mathfrak{P}(X)$, satisfying $\mathcal{V}(y,s) \subseteq \mathcal{W}(x,t)$, such that, for each $(x,t) \in X$, $|\mathcal{V}(x,t)| = 0$ or 2.*

In practice, when a Poisson clock X is fixed, for a given $(x,t) \in X$ with $\delta_t(x) = 1$, $\mathcal{V}(x,t)$ will be set as a subset of $\mathcal{W}(x,t)$, such that each $(y,s) \in \mathcal{V}(x,t)$ satisfies $\delta_s(y) = 1$.

We introduce the following notation: For each $A \subseteq X$, we set

$$\mathcal{V}(A) = \bigcup_{(y,s)\in A} \mathcal{V}(y,s), \tag{5.51}$$

We also set $\mathcal{V}^0(x,t) = \{(x,t)\}$, $\mathcal{V}^1(x,t) = \mathcal{V}(x,t)$, and

$$\mathcal{V}^j(x,t) = \mathcal{V}(\mathcal{V}^{j-1}(x,t)), \quad \text{for } j = 2,3,\cdots. \tag{5.52}$$

Let $\mathcal{V}^\infty(x,t)$ denote the union

$$\mathcal{V}^\infty(x,t) = \bigcup_{j=0}^{\infty} \mathcal{V}^j(x,t). \tag{5.53}$$

Recall $\{\delta_t\}_{t\geq 0}$ denotes the dominating process on $\mathbb{T}_3$. Our next proposition, Proposition 5.14, shows that, for each event point (x,t) satisfying $\delta_t(x) = 1$, there exists a selection $\mathcal{V}$, such that every $(y,s) \in \mathcal{V}^\infty$ satisfies $\delta_s(y) = 1$.

Proposition 5.14. *Let X denote any realization of the Poisson clock satisfying (2.1) and (2.2). Supposing that $\delta_t(x) = 1$ for some $(x,t) \in X$, then there exists a selection $\mathcal{V}_{(x,t)}$, satisfying the following:*

(5.14.a) $|\mathcal{V}^\infty_{(x,t)}(x,t)| < \infty$.

(5.14.b) For each $y \in \mathbb{T}_3$ and every noise point $(y,s) \in \mathcal{V}^\infty_{(x,t)}(x,t)$, $\mathcal{V}_{(x,t)}(y,s) = \emptyset$.

(5.14.c) For every $(y,s) \in \mathcal{V}^\infty_{(x,t)}$, $\delta_s(y) = 1$.

For readability, we leave the proof of Proposition 5.14 until the end of this section.

In the remainder of this subsection, we fix any $(x,t) \in X$ with $\delta_t(x) = 1$, and let $\mathcal{V}_{(x,t)}$ denote a selection satisfying (5.14.a), (5.14.b) and (5.14.c). (Proposition 5.14 ensures the existence of such a selection $\mathcal{V}$.) For simplicity, we abbreviate $\mathcal{V}_{(x,t)}$ and $\mathcal{V}^\infty(x,t)$ as $\mathcal{V}$ and $\mathcal{V}^\infty$, respectively.

We introduce a binary relation "$\sim$" on $\mathcal{V}^\infty$ as follows. For every $(y,s) \in \mathcal{V}^\infty$, we set $(y,s) \sim (y,s)$. For every (y_1,s_1), $(y_2,s_2) \in \mathcal{V}^\infty$ satisfying

$$\{(y_1,s_1),(y_2,s_2)\} \subseteq \mathcal{V}(y,s) \text{ for some } (y,s) \in \mathcal{V}^\infty, \tag{5.54}$$

and

$$\mathcal{V}^\infty(y_1,s_1) \cap \mathcal{V}^\infty(y_2,s_2) \neq \emptyset, \tag{5.55}$$

we set $(y_1,s_1) \sim (y_2,s_2)$. For every (y_1,s_1), $(y_2,s_2) \in \mathcal{V}^\infty$ where $(y_1,s_1) \sim (y_2,s_2)$ is not defined, we set $(y_1,s_1) \sim (y_2,s_2)$ if there exists a sequence $\{(z_i,r_i)\}_{i=1}^u \subseteq \mathcal{V}^\infty$ such that $(y_1,s_1) \sim (z_1,r_1)$, $(y_2,s_2) \sim (z_u,r_u)$, and

$$(z_i,r_i) \sim (z_{i+1},r_{i+1}) \text{ for every } i \text{ satisfying } i \geq 1 \text{ and } i+1 \leq r. \tag{5.56}$$

We claim that "$\sim$" is an equivalent relation on $\mathcal{V}^\infty$.

To see this, it follows from its definition that "$\sim$" is reflexive and symmetric. Moreover, (5.56) implies that "$\sim$" is transitive. Therefore, "$\sim$" is an equivalence relation. We next introduce the equivalence class introduced by "$\sim$".

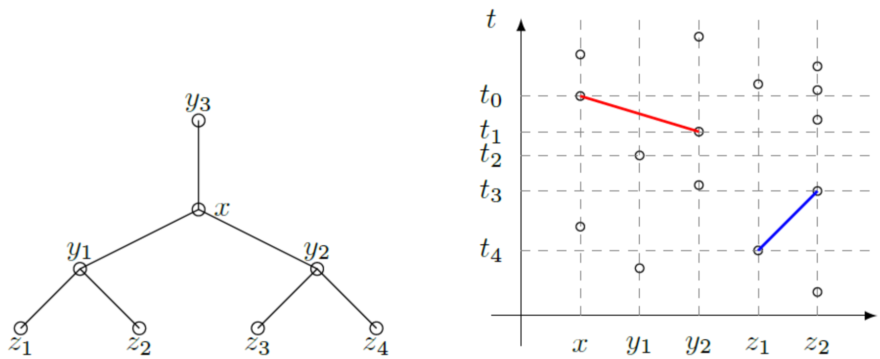
Definition 5.15. *Let $\mathcal{V}$ denote a selection satisfying Proposition 5.14, and $\mathcal{V}^\infty = \mathcal{V}^\infty(x,t)$, which was defined in (5.53). For each $(y,s) \in \mathcal{V}^\infty$, let $[(y,s)]$ denote the equivalence class of (y,s).*

We note that the class $[(x,t)]$ contains only one event point (x,t). If $(y,s) \in \mathcal{V}^\infty$ is a noise point, it follows from (5.14.b) that $[(y,s)]$ consists of exactly one event point (y,s).

We next introduce two types of edges on $\mathcal{V}^\infty$: *arrows* and *forks*. Recall that $\mathcal{Z}(y,s)$ was defined in (5.48).

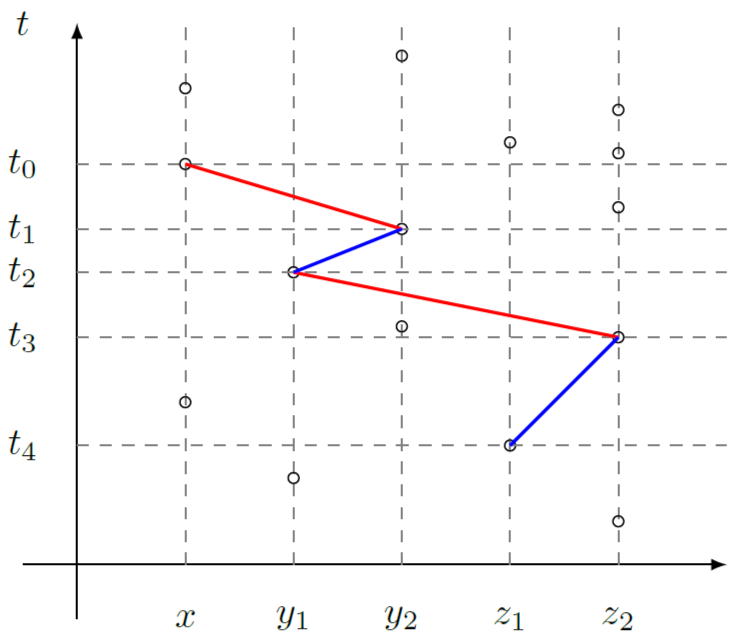
Definition 5.16 (arrows)**.** *Given a selection $\mathcal{V}$, we consider an edge $e = ((y_1,s_1),(y_2,s_2))$ with $(y_i,s_i) \in \mathcal{V}^\infty$ and $s_1 < s_2$. We say e is an arrow if $(y_1,s_1) \in \mathcal{V}(y_2,s_2)$.*

Figure 5.4: Examples of arrows and forks.



On the left, $x, y_1, y_2, y_3, z_1, z_2, z_3, z_4 \in \mathbb{T}_3$ are displayed. On the right, a realization of the Poisson clock X on $\mathbb{T}_3$ is shown, where each circle represents an event point. We define a selection on X as follows: $\mathcal{V}(x, t_0) = \{(y_1, t_2), (y_2, t_1)\}$, $\mathcal{V}(y_1, t_2) = \{(z_1, t_4), (z_2, t_3)\}$, and, for every event point $(y, s) \in X$ that is neither (x, t_0) nor (y_1, t_2), $\mathcal{V}(y, s) = \emptyset$. It follows from Definition 5.13 that $\mathcal{V}$ is a selection on X. The edge connecting (x, t_0) and (y_2, t_1) is an arrow since $y_2 \in \mathcal{N}(x)$. The edge connecting (z_1, t_4) and (z_2, t_3) is a fork.

Figure 5.5: An example of a successful t-truss



Using the same selection $\mathcal{V}$ defined in Figure 5.4, we introduce a t-truss $T = (V(T), E(T))$ as follows: $V(T) = \{(x,t_0),(y_2,t_1),(y_1,t_2),(z_2,t_3),(z_1,t_4)\}$, and $E(T) = \{e_1,e_2,e_3,e_4\}$, where $e_1 = ((x,t_0),(y_2,t_1))$, $e_2 = ((y_1,t_2),(z_2,t_3))$ are two arrows, and $e_3 = ((y_2,t_1),(y_1,t_2))$, $e_4 = ((z_1,t_4),(z_2,t_3))$ are two forks. A little thought shows that T is successful if event points (y_2,t_1), (z_2,t_3) and (z_1,t_4) are noise points.

Definition 5.17 (forks)**.** *Given a selection $\mathcal{V}$, an edge $e = ((y_1,s_1),(y_2,s_2))$ with $(y_i,s_i) \in \mathcal{V}^\infty : i = 1,2$ is a fork if $(y_1,s_1) \not\to (y_2,s_2)$ and $(y_1,s_1) \in \mathcal{Z}(y_2,s_2)$.*

Figure 5.4 presents an example of an arrow and fork.

We next introduce the *successful t-truss*. Recall the t-truss was introduced in Definition 5.11. Given a selection $\mathcal{V}$ satisfying Proposition 5.14, an event point $(y,s) \in \mathcal{V}(T)$ is a *leaf* if $\mathcal{V}(y,s) = \emptyset$.

Definition 5.18. *Given a selection $\mathcal{V}$ and a t-truss $T = (V(T),E(T))$ on $\mathcal{V}^\infty$. We say T is a successful t-truss with $\mathcal{V}$ if T satisfies the following.*

1. *$E(T)$ consists of arrows and forks only.*

2. Every leaf $(y,s) \in V(T)$ is a noise point, i.e.,

$$(y,s) \in X_y^{(0)} \cap X_y^{(1)}, \quad \textit{for each } (y,s) \in V(T) \textit{ with } \mathcal{V}(y,s) = \emptyset. \tag{5.57}$$

Using the same realization of the Poisson clock, we draw the diagram of a successful t-truss in Figure 5.5.

We conclude this subsection with Proposition 5.19, which demonstrates that the event $\{\delta_t(x) = 1\}$ implies the existence of at least one successful t-truss.

Proposition 5.19. *Consider a selection $\mathcal{V}$ satisfying (5.14.a), (5.14.b) and (5.14.c). Supposing $\delta_t(x) = 1$, for some $x \in \mathbb{T}_3$ and $t > 0$, then there exists a successful t-truss T such that*

$$a(T) + f(T) < 4n(T), \tag{5.58}$$

where $a(T)$, $f(T)$ denote the number of arrows and forks in $E(T)$, and $n(T)$ denotes the number of leaves in $V(T)$.

For readability, the proof of Proposition 5.19 is delayed until Subsection 5.2.6.

5.2.4 Proof of Theorem 7 Assuming Propositions 5.14 and 5.19

This subsection proves Theorem 7: the UMVP on $\mathbb{T}_3$ is non-ergodic when $\varepsilon_0 + \varepsilon_1$ is sufficiently small, assuming Propositions 5.14 and 5.19.

Let $\{\xi_t^{(0)}\}_{t\geq 0}$ and $\{\xi_t^{(1)}\}_{t\geq 0}$ denote the UMVP on $\mathbb{T}_3$, with all-zero and all-one initial configurations. Since the majority vote process is a monotone spin system, by Corollary 2.4, to show that ξ_t is non-ergodic, it suffices to show that $\xi_t^{(0)}$ and $\xi_t^{(1)}$ have distinct limiting distributions. For this, it suffices to show

$$\lim_{t\to\infty} \mathbb{P}(\xi_t^{(0)}(x) = 1) < 1/2, \quad \text{for every } x \in \mathbb{T}_3; \tag{5.59}$$

and

$$\lim_{t\to\infty} \mathbb{P}(\xi_t^{(1)}(x) = 0) < 1/2, \quad \text{for every } x \in \mathbb{T}_3. \tag{5.60}$$

To prove (5.59) and (5.60), by (5.44), it suffices to show

$$\lim_{t\to\infty} \mathbb{P}(\delta_t(x) = 1) < \frac{1}{2}, \quad \text{for every } x \in \mathbb{T}_3. \tag{5.61}$$

We use the t-truss, which was introduced in Subsection 5.2.3, to prove (5.61). Supposing $\delta_t(x) = 1$ for some $x \in \mathbb{T}_3$ and $t \geq 0$, Proposition 5.14 implies the existence of a selection satisfying (5.14.a), (5.14.b) and (5.14.c).

Let $\mathcal{V}$ denote such a selection, and Θ_n denote the collection of t-trusses on $\mathcal{V}^\infty(x,t)$ with n leaves. It follows from Proposition 5.19 that there exist some $N \in \mathbb{N}$, and a successful t-truss $T \in \Theta_N$, satisfying (5.58). Therefore,

$$\mathbb{P}(\delta_t(x) = 1) \leq \sum_{n=1}^{\infty} \sum_{T \in \Theta_n} \mathbb{P}(T \textit{ is successful with } \mathcal{V}) \leq \sum_{n=1}^{\infty} |\Theta_n| \varepsilon^n. \tag{5.62}$$

Since $E(T)$ consists of arrows and forks only, and satisfies (5.58), if follows from Lemma 5.12 that

$$|\Theta_n| \leq 10^{8n} - 1. \tag{5.63}$$

Since $\varepsilon = \varepsilon_0 + \varepsilon_1 < (12 \cdot 10^7)^{-1}$, the desired bound (5.61) follows from (5.62) and (5.63). This completes the proof of Theorem 7. $\square$

5.2.5 Proof of Proposition 5.14

This subsection proves Proposition 5.14. We will construct a selection $\mathcal{V}$ satisfying (5.14.b) and (5.14.c), and show that $|\mathcal{V}^\infty(x,t)|$ is almost surely finite. We begin with a simple observation.

Lemma 5.20. *For a given $y \in \mathbb{T}_3$ and $(y,s) \in X_y^{(2)}$, with $\delta_s(y) = 1$, there exist at least two $y_1, y_2 \in \mathcal{N}(x)$, such that $\delta_{e(y_1,s)}(y_1) = \delta_{e(y_2,s)}(y_2) = 1$.*

Proof. Since $(y,s) \in X_y^{(2)}$ satisfies $\delta_s(y) = 1$, it follows from (2.2.c) that there are at least two distinct $y_1, y_2 \in \mathcal{N}(y)$, with $\delta_{s-}(y_1) = \delta_{s-}(y_2) = 1$. If follows from the continuation rule (2.5) that $\delta_{e(y_1,s)}(y_1) = \delta_{e(y_2,s)}(y_2) = 1$. This demonstrates 5.20. $\square$

We will construct $\mathcal{V}$ by repeatedly applying Lemma 5.20. We begin with (x,t). We set

$$\mathcal{V}(x,t) = \begin{cases} \emptyset & \text{if } (x,t) \in X_x^{(0)} \cup X_x^{(1)}, \\ \{(y_1, e(y_1,t)), (y_2, e(y_2,t))\} & \text{otherwise,} \end{cases} \tag{5.64}$$

where $y_1, y_2 \in \mathcal{N}(x)$ satisfy $\delta_{e(y_1,t)}(y_1) = \delta_{e(y_2,t)}(y_2) = 1$. (The existence of such y_1 and y_2 is implied by Lemma 5.20.)

If (x, t) is a noise point, then $\mathcal{V}^\infty(x,t) = \{(x,t)\}$, which satisfies (5.14.a), (5.14.b), and (5.14.c).

Suppose that $\mathcal{V}(y,s)$ is defined for every $(y,s) \in \cup_{i=0}^{j-1}\mathcal{V}^i(x,t)$ and satisfies (5.14.b) and (5.14.c). For every $(y,s) \in \mathcal{V}^j(x,t)$, we set

$$\mathcal{V}(y,s) = \begin{cases} \emptyset & \text{if } (y,s) \in X_y^{(0)} \cup X_y^{(1)}, \\ \{(y_1', e(y_1', s)), (y_2', e(y_2', s))\} & \text{otherwise,} \end{cases} \tag{5.65}$$

where $y_1', y_2' \in \mathcal{N}(y)$ and satisfy $\delta_{e(y_1',s)}(y_1') = \delta_{e(y_2',s)}(y_2') = 1$. (The existence of such y_1, y_2 is implied by Lemma 5.20.) It is straightforward to verify that (5.14.b) and (5.14.c) hold for $(y_i', e(y_i', s))$, $i = 1, 2$.

In the remainder of the proof, we show that $|\mathcal{V}^\infty(x,t)|$ is almost surely finite. Assuming (x,t) is a vote point (otherwise, $\mathcal{V}^\infty(x,t) = \{(x,t)\}$), for every vote point $(y,s) \in \mathcal{V}^\infty(x,t)$, it follows from (5.65) that there exists some $(y_1, s_1) \in \mathcal{V}^\infty(x,t)$, with $y \in \mathcal{N}(y_1)$. Consider a sequence of event points $\{(y_i, s_i)\}_{i=0}^m$, with $(y_0, s_0) = (x,t)$, and for each $1 \le i \le m$, $y_i \in \mathcal{N}(y_{i-1})$, $s_i = e(y_i, s_{i-1})$. Since $s_{i-1} - s_i = s_{i-1} - e(y_i, s_{i-1})$, $1 \le i \le m$, are independently and exponentially distributed, their sum is Gamma distributed. It therefore follows from

$$t = s_0 \ge \sum_{i=1}^m (s_{i-1} - s_i) \ge \sum_{i=1}^m (s_{i-1} - s_i), \tag{5.66}$$

that, for any given $t > 0$,

$$\mathbb{P}(\sum_{i=1}^m (s_{i-1} - s_i) \le t) \le \frac{\gamma(m,t)}{\Gamma(m)} = e^{-t} \sum_{j=m}^\infty \frac{t^j}{j!}. \tag{5.67}$$

Supposing that $|\mathcal{V}^\infty(x,t)| = \infty$, we claim that, for each $L > 0$, there exists a sequence of event points $\{(y_i, s_i)\}_{i=0}^m$, with $(y_0, s_0) = (x,t)$, $m > L$, and for each $1 \le i \le m$, $y_i \in \mathcal{N}(y_{i-1})$, $s_i = e(y_i, s_{i-1})$. To see this, note that if the maximum length of such a sequence is $m' < \infty$, then $|\mathcal{V}^\infty(x,t)| < 2^{m'+1}$, which contradicts the assumption that $|\mathcal{V}^\infty(x,t)| = \infty$.

For each $L > 0$, it follows from (5.66) and (5.67) that

$$\mathbb{P}(|\mathcal{V}^\infty(x,t)| = \infty) \le \mathbb{P}(\sum_{i=1}^m (s_{i-1} - s_i) \le t) \le e^{-t} \sum_{j=m}^\infty \frac{t^j}{j!}, \tag{5.68}$$

where $m = \lfloor L \rfloor$ denoting the largest integer that does not exceed L. Taking the limit $L \to \infty$ on both sides of (5.68), we obtain that $\mathbb{P}(|\mathcal{V}^\infty(x,t)| = \infty) = 0$. This completes the proof of Proposition 5.14. □

5.2.6 Demonstration of Proposition 5.19

In this section, we prove Proposition 5.19. Proposition 5.14 implies that, conditioning on $\{\delta_t(x) = 1\}$, there exists some selection $\mathcal{V}_{(x,t)}$ satisfying (5.14.a), (5.14.b) and (5.14.c). In this proof, we fix an $(x,t) \in X$ such that $\delta_t(x) = 1$, and also a selection $\mathcal{V}_{(x,t)}$, satisfying (5.14.a), (5.14.b) and (5.14.c). For readability, we abbreviate $\mathcal{V}_{(x,t)}$ and $\mathcal{V}^\infty(x,t)$ as $\mathcal{V}$ and $\mathcal{V}^\infty$, respectively.

For every $(y_1,s_1),(y_2,s_2) \in \mathcal{V}^\infty$, it follows from (2.1) that $s_1 \neq s_2$. We therefore write $\mathcal{V}^\infty = \{(y_i,s_i)\}_{i=0}^M$ such that $(y_0,s_0) = (x,t)$, and, for every $1 \le i \le M$, $s_{i-1} > s_i$.

We construct a successful t-truss $T = (V(T), E(T))$ on $\mathcal{V}^\infty$ recursively, such that T satisfies Proposition 5.19. Recall that arrows and forks were introduced in Definitions 5.16 and 5.17. The construction of T will be done recursively: Throughout the construction, S_i and V_i, $1 \le i \le M$, will denote sets of event points, A_i, $1 \le i \le M$, will denote sets of arrows, F_i, $1 \le i \le M$, will denote sets of forks, and Λ_i, $1 \le i \le M$, will denote sets of integers (indexes). We will construct each of S_i, V_i, A_i, F_i, and Λ_i, for $1 \le i \le M$, then use S_M, V_M, A_M, and F_M to construct a t-truss T, and prove that T is successful.

We let $e(i,j)$ denote the following edge:

$$e(i,j) = ((y_i,s_i),(y_j,s_j)), \quad \text{for } 0 \le i < j \le k.$$

Construction of V_1, A_1, F_1, S_1, Λ_1

Recall that $[(y_1,s_1)]$ denotes the equivalence class of (y_1,s_1), induced by the binary relation "$\sim$". Let $|[(y_1,s_1)]|$ denote the cardinality of the equivalence class $[(y_1,s_1)]$. (Since $\mathcal{V}^\infty$ is almost surely finite, so is $[(y_1,s_1)]$.) There are two possibilities: $|[(y_1,s_1)]| = 1$, or $|[(y_1,s_1)]| > 1$.

If $|[(y_1,s_1)]| = 1$, then we set

$$V_1 = \{(y_0,s_0),(y_1,s_1)\}, \quad A_1 = \{e(0,1)\}. \tag{5.69}$$

Since $(y_1, s_1) \in \mathcal{V}(y_0, s_0)$, it follows from Definition 5.16 that the edge $e(0,1)$ is an arrow. We also set $S_1 = F_1 = \emptyset$, and $\Lambda_1 = \{0,1\}$. This completes the construction of of V_1, A_1, F_1, S_1, and Λ_1.

If $[(y_1, s_1)] > 1$, we set $V_1 = V_0$, $A_1 = F_1 = \emptyset$, $S_1 = \{(y_1, s_1)\}$, and $\Lambda_1 = \{0,1\}$.

In both cases, every $(y, s) \in V_1$ is incident with at most one arrow from A_1. Considering a graph $T_1 = (V(T_1), E(T_1))$, where $V(T_1) = S_1 \cup V_1$, and $E(T_1) = A_1 \cup F_1$, then T_1 admits no cycle.

We will construct V_k, A_k, F_k, S_k, given $\{V_i\}_{i=0}^{k-1}$, $\{S_i\}_{i=0}^{k-1}$, $\{A_i\}_{i=0}^{k-1}$ and $\{F_i\}_{i=0}^{k-1}$ in the following way:

Recursive construction of V_k, A_k, F_k, S_k, Λ_k given $\{V_i\}_{i=0}^{k-1}$, $\{A_i\}_{i=0}^{k-1}$, $\{F_i\}_{i=0}^{k-1}$, $\{S_i\}_{i=0}^{k-1}$, $\{\Lambda_i\}_{i=0}^{k-1}$

Suppose that $\{V_i\}_{i=0}^{k-1}$, $\{S_i\}_{i=0}^{k-1}$, and $\{A_i\}_{i=0}^{k-1}$ are constructed to satisfy that, for each $1 \le i \le k-1$,

$$\text{each } (y, s) \in V_i \text{ is incident with at most two arrows } a_1, a_2 \in A_i \tag{5.70}$$

and also that

$$\text{each } (y, s) \in S_i \text{ is incident with at most one arrow from } A_i. \tag{5.71}$$

We set

$$\Lambda_{k-1} = \{0 \le i \le k-1 : (y_i, s_i) \in V_{k-1} \cup S_{k-1}\}.$$

We consider the graph $T_{k-1} = (V(T_{k-1}), E(T_{k-1}))$, where $V(T_{k-1}) = V_{k-1} \cup S_{k-1}$, and $E(T_{k-1}) = A_{k-1} \cup F_{k-1}$, and we assume that T_{k-1} admits no cycle. We will construct V_k, A_k, F_k, S_k in a way such that (5.70) and (5.71) are satisfied for $i = k$, and the graph $T_k = (V_k \cup S_k, A_k \cup F_k)$ admits no cycle.

Setting $W = [(y_k, s_k)]$, then there are two possibilities of W – $|W| > 1$ or $|W| = 1$. If $|W| > 1$, then we set $A_k = A_{k-1}$, $F_k = F_{k-1}$, $V_k = V_{k-1}$ and $S_k = S_{k-1}$. (5.70) and (5.71) are still satisfied for $i = k$. Considering the graph $T_k = (V(T_k), E(T_k))$, where $V(T_k) = V_k \cup S_k = V_{k-1} \cup S_{k-1} = V(T_{k-1})$, and $E(T_k) = A_k \cup F_k = A_{k-1} \cup F_{k-1} = E(T_{k-1})$, then T_k admits no cycle.

We now assume $|W| = 1$ and set $V_k = V_{k-1} \cup \{(y_k, s_k)\}$. We will “add” an arrow or a fork in T according to the following steps.

Step 1 If there exists some $i \in \Lambda_{k-1}$ such that $e(i,k)$ is a fork, then we set

$$F_k = F_{k-1} \cup \{e(i,k)\}, \tag{5.72}$$

and skip the next two steps.

Step 2 If there exists some $i \in \Lambda_{k-1}$ such that $e(i,k)$ is an arrow, then we set

$$A_k = A_{k-1} \cup \{e(i,k)\}. \tag{5.73}$$

If multiple such i exist, we set i equal to the largest and skip the next step.

Step 3 If, for every $i \in \Lambda_{k-1}$, the edge $e(i,k)$ is not a fork or arrow, then we set $V_k = V_{k-1}$.

Let $j \leq k-1$ denote the largest integer such that

$$(y_k, s_k) \in \mathcal{V}(y_j, s_j). \tag{5.74}$$

We note that such a j always exists by the definition of $\mathcal{V}^\infty$. Letting (y_h, s_h) denote the other event point in $\mathcal{V}(y_j, s_j)$, then there are two possibilities of h and k: If $h < k$, then $|[(y_h, s_h)]| > 1$. (Otherwise, $(y_h, s_h) \in V_{k-1}$, contradicting the assumption that every $e(i,k)$ is not a fork or an arrow.) We set $S_k = S_{k-1} \cup \{(y_h, s_h)\}$. The maximality of j ensures that $e(h,k)$ is a fork. We also set

$$A_k = A_{k-1}, \quad F_k = F_{k-1} \cup \{e(h,k)\}. \tag{5.75}$$

If $h > k$, we set

$$S_k = S_{k-1} \cup \{(y_j, h_j)\}, \quad A_k = A_{k-1} \cup \{e(j,k)\}, \tag{5.76}$$

and $F_k = F_{k-1}$. Since $(y_k, s_k) \in \mathcal{V}(y_j, s_j)$, $e(j,k)$ is an arrow. This completes Step 3.

We have defined $\mathcal{V}_k$, A_k, F_k and S_k through these three steps. Let $T_k = (V(T_k), E(T_k))$ denote the graph where $V(T_k) = V_k \cup S_k$, and $E(T_k) = A_k \cup F_k$. We will demonstrate that T_k has no cycle, and that V_k, S_k and A_k satisfy (5.70) and (5.71).

Compared with T_{k-1}, T_k has at most two extra event points in $V(T)$ and one extra edge that is incident with the extra event point(s). Therefore, T_k admits no cycle.

We verify that V_k, S_k and A_k satisfy (5.70) and (5.71). Since exactly one of A_k and F_k differs from A_{k-1} and F_{k-1}, we discuss these two possibilities: If $F_k \neq F_{k-1}$, then (5.70) and (5.71) hold for $i = k$.

If $A_{k-1} \neq A_k$, then

$$A_k = A_{k-1} \cup \{e(i,k)\},$$

for some $0 \leq i < k$. If $(y_i, s_i) \in S_{k-1}$, then we claim (y_i, s_i) is incident to exactly one arrow $e(i,k)$: If not, it follows from (5.73) that there is some $j < k$, with

$$(y_j, s_j) \in \mathcal{V}(y_i, s_i), \quad e(i,j) \in A_{k-1}. \tag{5.77}$$

In this case, the maximality of (5.73) implies that $e(j,k)$ is a fork, satisfying the condition in Step 1. This contradicts the assumption $F_{k-1} \neq F_k$.

If $(y_i, s_i) \in V_{k-1}$, a little thought shows that (y_i, s_i) is incident to exactly one arrow $e(i,k)$. In both cases, V_k, S_k and A_k satisfy (5.70) and (5.71). This completes the construction of V_k, A_k, F_k and S_k.

Construction of T

We repeat the construction of V_k, A_k, F_k, S_k, for $1 \leq k \leq M$. Using V_M, A_M, F_M, and S_M, we construct a t-truss T as follows.

We set

$$\tilde{V} = V_M \cup S_M, \quad \tilde{E} = A_M \cup F_M.$$

If the graph $\tilde{T} = (\tilde{V}, \tilde{E})$ is connected, then we take $T = \tilde{T}$ as the desired t-truss.

We consider the following sets if $\tilde{T}$ is not connected.

$$A(\mathcal{V}^\infty) := \{((y,s),(y',s')) : (y',s') \in \mathcal{V}(y,s)\},$$

and

$$F(\mathcal{V}^\infty) := \{(y,s),(y',s') \in \mathcal{V}^\infty : ((y,s),(y',s')) \text{ is a fork}\}.$$

In other words, $A(\mathcal{V}^\infty)$ and $F(\mathcal{V}^\infty)$ are the set of all possible arrows and forks on $\mathcal{V}^\infty$.

We set

$$E_0 = \{((y,s),(y',s')) \in A(\mathcal{V}^\infty) : ((y',s'),(y'',s'')) \in F(\mathcal{V}^\infty) \\ \text{with } (y'',s'') \in \mathcal{V}(y,s) \text{ and } s' < s''\}. \tag{5.78}$$

In other words, if there exist event points (y,s), (y',s') and (y'',s''), such that $\mathcal{V}(y,s) = \{(y',s'),(y'',s'')\}$ (with $s' < s''$), and the edge $((y',s'),(y'',s''))$ is a fork, then $e = ((y,s),(y',s')) \in E_0$.

Setting

$$V' = \mathcal{V}^\infty, \quad E' = \Big(A(\mathcal{V}^\infty) \cup F(\mathcal{V}^\infty)\Big) \cap E_0^c,$$

then a little thought shows that (V',E') is connected, and $\tilde{E} \subseteq E'$, as a direct result of the maximality of (5.73).

Let $\bar{T} = (V(\bar{T}), E(\bar{T}))$ denote the minimal connected subgraph of (V',E'), such that $\tilde{V} \subseteq V(\bar{T})$ and $\tilde{E} \subseteq E(\bar{T})$. A little thought shows that every leave of $\bar{T}$ is a noise point.

We claim that $\bar{T}$ is a t-truss: Since $\bar{T}$ is the minimal connected subgraph of (V',E'), and (V',E') does not admit any cycle, $\bar{T}$ is a tree. Since $E' \subseteq (A(\mathcal{V}^\infty) \cup F(\mathcal{V}^\infty))$, $\bar{T}$ satisfies (5.11.c). To show $\bar{T}$ is a t-truss, it suffices to show that $\bar{T}$ satisfies (5.11.b). To see this, since each $(y,s) \in V_M$ is incident with at most 2 arrows, and each $(y,s) \in S_M$ is incident with at most 1 arrows, then each $(y,s) \in V(\bar{T})$ is incident with at most 2 arrows. On the other hand, it follows from Definition 5.17 that each event point is incident with at most 2 forks. Therefore, $\bar{T}$ satisfies (5.11.b).

Setting $T = \bar{T}$, since we already proved that every leave of T is a noise point, it follows that T is a successful t-truss. To show T satisfies Proposition 5.19, it suffices to prove (5.58) for T.

Proof of (5.58)

In the remainder of this subsection, we prove that T satisfies (5.58). It follows from the construction of each V_k, $1 \le k \le M$, that V_M includes all event points $(y,s) \in \mathcal{V}^\infty$ with $|[y,s]| = 1$. Since all noise points $(y,s) \in \mathcal{V}^\infty$ satisfy $|[y,s]| = 1$, V_M, and hence $V(\bar{T})$ contains all noise points in $\mathcal{V}^\infty$.

Consider the graph $G_0 = (V(G_0), E(G_0))$ where $V(G_0) = \mathcal{V}^\infty$, and $E(G_0) = A(\mathcal{V}^\infty)$. Letting $d_0(\cdot,\cdot)$ denote the graph distance on G_0, we define

$$u(y,s) = d_0((y,s),(y_0,s_0)), \quad \text{for each } (y,s) \in \mathcal{V}^\infty, \tag{5.79}$$

and

$$B(r) = \{(y,s) \in \mathcal{V}^\infty \cap T : u(y,s) = r\}, \quad \text{for each } r \in \mathbb{N}. \tag{5.80}$$

In addition, we set $B(0) = \{(y_0, s_0)\}$. By the minimality of T, each arrow $a = ((y,s),(y',s')) \in E(T)$, with $s > s'$, satisfies $(y,s) \in B(n)$ and $(y',s') \in B(n+1)$ for some n. Letting a_n denote the number of such arrows, and l_n denote the number of noise points in $B(n)$, we claim

$$|B(n+1)| \geq 2(a_{n-1} - l_n), \quad \text{for each } n \in \mathbb{N}. \tag{5.81}$$

To see (5.81), consider $a_1, a_2 \in E(T) \cap A(\mathcal{V}^\infty)$, with $a_1 = ((z_1, r_1), (z_1', r_1'))$ and $a_2 = ((z_2, r_2), (z_2', r_2'))$, such that (z_1, r_1), $(z_2, r_2) \in B(n-1)$ and (z_1', r_1'), $(z_2', r_2') \in B(n)$. Since T is a minimal connected subgraph of (V', E'), $(z_1', r_1') \neq (z_2', r_2')$. (Otherwise, the path connecting (y_0, s_0) and (z_1', r_1'), and the path connecting (y_0, s_0) and (z_2', r_2') form a cycle.) (5.81) follows from the observation that all leaves of T are noise points.

Since $\mathcal{V}^\infty$ is almost surely finite, only finitely many $B(n)$ are non-empty. Summing over both sides of (5.81) from $n = 2$ to infinity, we obtain that

$$\sum_{i=2}^{\infty} |B(i)| \geq 2(a - l). \tag{5.82}$$

On the other hand, since T is a tree,

$$\sum_{i=2}^{\infty} |B(i)| = |V(T)| - |B(0)| - |B(1)| \leq a + f. \tag{5.83}$$

Hence, it follows from (5.82) and (5.83) that

$$a \leq 2l + f. \tag{5.84}$$

Definition 5.17 states that, if a fork exists between $(y', s'), (y'', s'') \in V(T)$, then $\mathcal{V}^\infty(y', s') \cap \mathcal{V}^\infty(y'', s'') = \emptyset$. Since T is a tree, and all its leaves are noise points, $f \leq l$. (5.58) follows from (5.84) and $f \leq l$. This completes the proof of Proposition 5.19. $\square$

Chapter 6

Oriented Majority Vote Process on $\mathbb{Z}^d$

Recall that the oriented majority vote process with nearest-neighbor interactions (OMVP) was introduced in Subsection 1.1.2. In this chapter, we will discuss the behavior of the OMVP on $\mathbb{Z}^d$, with $d \geq 4$. As usual, we let $\{\xi_t\}_{t\geq 0}$ denote the OMVP, and $\varepsilon_0, \varepsilon_1 \in [0,1]$ denote the 0-noise and 1-noise intensities. The goal of this chapter is to demonstrate the contrasting behavior of ξ_t, when the noise intensity $\varepsilon_0 + \varepsilon_1$ is small and when it is large: When $\varepsilon_0 + \varepsilon_1$ is large, ξ_t will be ergodic and converge exponentially fast to its unique equilibrium, starting from any initial distribution. Conversely, when $\varepsilon_0 + \varepsilon_1$ is sufficiently small, ξ_t is non-ergodic and has uncountably many mutually singular equilibria.

This chapter is structured as follows. In Section 6.1, we will review the basic setting of the OMVP on $\mathbb{Z}^d$, and examine its behavior when $\varepsilon_0 + \varepsilon_1$ is large. As before, the ergodicity of ξ_t under large $\varepsilon_0 + \varepsilon_1$ is established through a standard argument and is included primarily to contrast with the behavior when $\varepsilon_0 + \varepsilon_1$ is sufficiently small.

The main part of Chapter 6 is dedicated to proving that ξ_t is non-ergodic and possesses uncountably many mutually singular equilibria when $\varepsilon_0 + \varepsilon_1$ is sufficiently small. In Section 6.2, we will state Theorem 8: When $\varepsilon_0 + \varepsilon_1$ is small enough to satisfy (6.3), ξ_t is non-ergodic and has uncountably many mutually singular equilibria. In the same section, we will provide an overview of the proof of Theorem 8.

In Section 6.3, we will introduce the *dominating process on* $\mathbb{Z}^d$, denoted by $\{\delta_t\}_{t\geq 0}$, which serves as the $\mathbb{Z}^d$ analog of the dominating process on $\mathbb{T}_d$. We will relate δ_t with ξ_t through Propositions 6.4 and 6.5. In the same section, we will introduce a family of uncountably many *compatible configurations*, which will lead to mutually singular equilibria of the OMVP. (This will be proved in Section 6.5.) We will prove Proposition 6.4 in Subsection 6.3.2, and Proposition 6.5 in Section 6.4. We will conclude this chapter with the proof of Theorem 8 in Section 6.5.

6.1 Definitions and Large Noise Regime

Recall the OMVP on a directed graph $G = (V, E)$ was introduced in Section 1.3. Let $V = \mathbb{Z}^d$ and $E = \{e = (x, x + \mathbf{e}_i) : x \in \mathbb{Z}^d, 1 \leq i \leq d\}$, where $\mathbf{e}_i$ denotes the ith vector in the standard basis. Each edge $e = (x, x + \mathbf{e}_i) \in E$ is assumed to be directed from x to $x + \mathbf{e}_i$. For every $x \in \mathbb{Z}^d$, we set

$$O(x) = \{x + \mathbf{e}_i : 1 \leq i \leq d\}. \tag{6.1}$$

In other words, $O(x)$ consists of all vertices that are incident with an edge directed from x. The OMVP on $\mathbb{Z}^d$, denoted by $\{\xi_t\}_{t\geq 0}$, is defined as the majority vote process on $(S, \mathcal{N})$, where $S = \mathbb{Z}^d$, and $\mathcal{N}(x) = O(x)$, for each $x \in \mathbb{Z}^d$.

Our next Proposition, Proposition 6.1, shows that when $\varepsilon_0 + \varepsilon_1$ is large enough, the OMVP on $\mathbb{Z}^d$ will be ergodic, and converge to the unique equilibrium exponentially fast, starting from any initial distribution. Since the proof of Proposition 6.1 uses the same technique as that of Propositions 5.1 and 5.8, we omit its proof here. Recall we assumed that $\varepsilon_0 + \varepsilon_1 \leq 1$, so that the rates of all events are nonnegative.

Proposition 6.1. *Consider the OMVP* $\{\xi_t\}_{t\geq 0}$ *on* $\mathbb{Z}^d$*, with* $d \geq 2$*. Let* ε_0 *and* ε_1 *denote the* 0*-noise and* 1*-noise intensities. When* $\varepsilon_0, \varepsilon_1 \in [0, 1]$ *satisfy*

$$\varepsilon_0 + \varepsilon_1 > 1 - \frac{1}{d+1}, \tag{6.2}$$

ξ_t *is ergodic and converges exponentially quickly to the unique equilibrium, starting from any initial distribution.*

The bound in (6.2) is consistent with the bounds given in (5.1) and (5.42), noting that $|\mathcal{N}(x)| = d$ for every $x \in \mathbb{Z}^d$: Recall that, in the proof of Proposition 5.1, $|\mathcal{N}(x)|$

is used to obtain the crucial inequality (5.4). Substituting $4h(x)$ by $dh(x)$ on the right-hand side of (5.4), the proof of Proposition 6.1 can be obtained directly from that of Proposition 5.1.

6.2 Non-ergodicity for Small $\varepsilon_0 + \varepsilon_1$

In the remainder of this chapter, we examine the behavior of the OMVP when $\varepsilon_0+\varepsilon_1$ is small. Gray ([23]) demonstrated that the OMVP on $\mathbb{Z}^2$ is non-ergodic when $\varepsilon_0+\varepsilon_1$ is sufficiently small. Using a similar approach, the non-ergodicity of the OMVP on $\mathbb{Z}^3$ can be obtained under the same assumptions. Our next theorem, Theorem 8, establishes the non-ergodicity of ξ_t when $\varepsilon_0+\varepsilon_1$ is sufficiently small. As on $\mathbb{T}_d$, with $d \geq 3$, this behavior contrasts with that of the majority vote process on a uniformly self-supporting structure.

Theorem 8. *Consider the OMVP $\{\xi_t\}_{t\geq 0}$ on $\mathbb{Z}^d$, with $d \geq 4$. Let ε_0 and ε_1 denote the 0-noise and 1-noise intensities. Supposing $\varepsilon_0, \varepsilon_1$ satisfy*

$$\varepsilon_0 + \varepsilon_1 < (10^{(250+2d)/4}(d-1)^{7/4})^{-250d}, \tag{6.3}$$

then ξ_t is non-ergodic and has uncountably many mutually singular equilibria.

We summarize the proof of Theorem 8 as follows. Our strategy is similar to that for proving Theorem 6: We will introduce a family of uncountably many *compatible configurations*, with each compatible configuration leading to an equilibrium for ξ_t. A family of *dominating processes* will be introduced to show that these equilibria are mutually singular with respect to each other.

In Section 6.3, we will introduce a family of uncountably many *compatible configurations*, which is the $\mathbb{Z}^d$ analog of Definition 5.2. We will partition $\mathbb{Z}^d$ into $d-1$ dimensional hyperplane $\{A_i\}_{i\in\mathbb{Z}}$. A configuration is defined to be *compatible* if, for each $i \in \mathbb{Z}$, all sites from A_i have the same opinion. When these compatible configurations are used as initial configurations for ξ_t, they lead to a family of uncountably many equilibria. To complete the proof of Theorem 8, it suffices to demonstrate that these equilibria are mutually singular with respect to each other.

In Subsection 6.3.1, we will introduce the *dominating process on* $\mathbb{Z}^d$, the $\mathbb{Z}^d$ analog of Definition 5.4. We let $\{\delta_t\}_{t\geq 0}$ denote this process and state some of its properties.

In Corollary 6.3, we will prove that, for given $s, t > 0$, $\delta_t(x)$ and $\delta_s(y)$ are independent if x, y are not on the same hyperplane A_i. In Proposition 6.4, we will prove that, when jointly defined with ξ_t, δ_t controls the amount ξ_t deviates from the initial compatible configuration ξ_0. Proposition 6.4 is the $\mathbb{Z}^d$ analog of Lemma 5.5, and will be proved in Subsection 6.3.2. In Proposition 6.5, we will establish a uniform upper bound on the marginal density of δ_t. Although Proposition 6.5 is the $\mathbb{Z}^d$ analog of Lemma 5.6, proving it requires significantly different techniques, which will be detailed in Section 6.4.

Section 6.4 is where the proof of Theorem 8 differs most significantly from that of Theorem 6. We will project the dominating process δ_t onto distinct hyperplanes $\{A_i\}_{i\in\mathbb{Z}}$. For each projected process $\tilde{\delta}_t$, we will prove that $\tilde{\delta}_t$ is a *continuous-time spin systems associated with a shrinker operator*. This enables us to apply estimates from [23] and [24], which will imply Proposition 6.5, after some work.

Section 6.5 will prove Theorem 8. We will introduce an uncountable subset $\Lambda \subseteq \{0,1\}^{\mathbb{Z}}$ and use each $a \in \Lambda$ to introduce a compatible configuration ζ^a. Each ζ^a will lead to an equilibrium $\mu^{(a)}$, and we will prove that these equilibria $\mu^{(a)}$, for $a \in \Lambda$, are mutually singular with respect to each other. To achieve this, we will apply Propositions 6.7 and 6.5. This will complete the proof of Theorem 8.

6.3 Compatible Configuration and Dominating Process

This section introduces the *compatible configurations* and *dominating process*. We also discuss some important properties of the dominating process. We outline this section as follows: In Subsection 6.3.1, we introduce the dominating process $\{\delta_t\}_{t\geq 0}$ on $\mathbb{Z}^d$. We next state some of its properties in Corollary 6.3, Proposition 6.4, and Proposition 6.5. Subsection 6.3.2 proves Proposition 6.4. (The proof of Proposition 6.5 will be given in Section 6.4.)

For each $1 \leq j \leq d$, we define a projection $\pi_j : \mathbb{Z}^d \to \mathbb{Z}$ by setting

$$\pi_j(x) = x_j, \quad \text{for each } x = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d.$$

For each $m \in \mathbb{Z}$, we let A_m denote the $d-1$ dimensional hyperplane

$$A_m = \{x \in \mathbb{Z}^d : \pi_d(x) = m\}.$$

For each doubly infinite sequence $a \in \{0,1\}^{\mathbb{Z}}$, where $a = \{a_m\}_{m\in\mathbb{Z}}$, we introduce a *compatible configuration* $\zeta^{(a)} \in \mathscr{S}$ by setting

$$\zeta^{(a)}(x) = a_{\pi_d(x)}, \quad \text{for each } x \in \mathbb{Z}^d.$$

In other words, we partition $\mathbb{Z}^d$ into infinitely many $d-1$ dimensional hyperplanes and assign a unanimous opinion to each hyperplane according to the sequence $a \in \{0,1\}^{\mathbb{Z}}$.

We next introduce the dominating process on $\mathbb{Z}^d$.

6.3.1 Dominating Process and its Properties

As on $\mathbb{T}_4$, the dominating process on $\mathbb{Z}^d$ is defined so as to create some independence by ignoring the potential influence from $\xi_t(x+\mathbf{e}_d)$ to $\xi_t(x)$. To compensate for this change, at each vote point (x,t), $\delta_t(x)$ slightly favors the opinion 1 over 0.

We will define the dominating process $\{\delta_t^{(\varepsilon)}\}_{t\geq 0}$ by its graphical construction. Recall that event points $\cup_{x\in\mathbb{Z}^d}(X_x^{(0)} \cup X_x^{(1)} \cup X_x^{(2)})$ are sampled from a family of independent Poisson point processes, given in Definition 2.1. As usual, we will refer to both a 0-noise point and 1-noise point as a noise point. For every $x \in \mathbb{Z}^d$, we set

$$\mathcal{U}(x) = \{x + \mathbf{e}_i : 1 \leq i \leq d-1\}. \tag{6.4}$$

In other words, for each $x = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d$, $\mathcal{U}(x) = O(x) - \{x + \mathbf{e}_d\}$.

Definition 6.2 (Dominating Process on $\mathbb{Z}^d$)**.** *Consider $S = \mathbb{Z}^d$ and a constant $\varepsilon \in (0,1)$. A process $\{\delta_t\}_{t\geq 0}$, with the state space $\mathscr{S} = \{0,1\}^S$, is a dominating process on $\mathbb{Z}^d$ if it satisfies the continuation rule (2.5) and the following three rules.*

(6.2.a) All-zero initial configuration: $\delta_0(x) = 0$ for every $x \in \mathbb{Z}^d$.

(6.2.b) δ-vote rule: If (x,t) is a vote point, then

$$\delta_t(x) = \begin{cases} 1 & \text{if } \sum_{y\in\mathcal{U}(x)\cup\{x\}} \delta_{t-}(y) \geq (d-1)/2, \\ 0 & \text{if } \sum_{y\in\mathcal{U}(x)\cup\{x\}} \delta_{t-}(y) < (d-1)/2. \end{cases} \tag{6.5}$$

(6.2.c) Noise rule: If (x,t) is a noise point, then $\delta_t(x) = 1$.

We will refer to $\varepsilon = \varepsilon_0 + \varepsilon_1$ as the *noise parameter* of δ_t. Let ξ_t denote a copy of the UMVP on $\mathbb{Z}^d$, with the same 0-noise and 1-noise intensities as δ_t. Since the graphical construction of δ_t is determined by the collection of event points $\cup_{x\in S}(X_x^{(0)} \cup X_x^{(1)} \cup X_x^{(2)})$, δ_t can be jointly defined with ξ_t.

Since we assumed that δ_t satisfies the continuation rule (2.5), $\{\delta_t\}_{t\geq 0}$ is an rcll process. Let $\delta_{t-} \in \mathscr{S}$ denote the configuration where, for every $x \in \mathbb{Z}^d$,

$$\delta_{t-}(x) = \begin{cases} \lim_{s\nearrow t} \delta_s(x) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases} \tag{6.6}$$

An immediate result of Definition 6.2 is the independence of the dominating process between different hyperplanes A_m and $A_{m'}$. We write this result as our next corollary.

Corollary 6.3. *Let $\{\delta_t\}_{t\geq 0}$ denote the dominating process on $\mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$ with $\pi_d(x) \neq \pi_d(y)$, and $s, t > 0$, $\delta_t(x)$ and $\delta_s(y)$ are independent.*

Proof. It follows from (6.2.b) and (6.2.c) that $\delta_t(x)$ is determined by the initial configuration δ_0 and event points $\cup_{z\in A_{\pi_d(x)}}(X_z^{(0)} \cup X_z^{(1)} \cup X_z^{(2)})$. Since $\delta_0(x) \equiv 0$, and

$$\left(\bigcup_{z\in A_{\pi_d(x)}} (X_z^{(0)} \cup X_z^{(1)} \cup X_z^{(2)})\right) \cap \left(\bigcup_{z\in A_{\pi_d(y)}} (X_z^{(0)} \cup X_z^{(1)} \cup X_z^{(2)})\right) = \emptyset,$$

$\delta_t(x)$ and $\delta_s(y)$ are independent, for every $t, s > 0$. This completes the proof of Corollary 6.3. $\square$

For each sequence $a \in \{0,1\}^{\mathbb{Z}}$, let $\{\xi_t^{(a)}\}_{t\geq 0}$ denote the OMVP on $\mathbb{Z}^d$, with the initial configuration $\xi_0^{(a)} = \zeta^{(a)}$. In Proposition 6.4, we will show that, when jointly defined with $\{\xi_t^{(a)}\}_{t\geq 0}$, the dominating process $\{\delta_t\}_{t\geq 0}$ bounds the deviation of $\xi_t^{(a)}$ from its compatible initial configuration $\zeta^{(a)}$.

Proposition 6.4. *For a given sequence $a \in \{0,1\}^{\mathbb{Z}}$, we let $\{\xi_t^{(a)}\}_{x\geq 0}$ denote a copy of the OMVP on $\mathbb{Z}^d$, with $d \geq 4$, and assume that $\xi_t^{(a)}$ has the initial configuration $\xi_0^{(a)}(x) \equiv \zeta^a(x)$, for all $x \in \mathbb{Z}^d$. Consider the dominating process $\{\delta_t\}_{t\geq 0}$ that is jointly defined with $\xi_t^{(a)}$. We claim the following holds almost surely:*

$$|\xi_t^{(a)}(x) - \zeta^a(x)| \leq \delta_t(x), \quad \text{for each } x \in \mathbb{Z}^d, \text{and } t \geq 0. \tag{6.7}$$

Since the graphical construction of δ_t and ξ_t are defined using the same set of event points $\cup_{x\in\mathbb{Z}^d}(X_x^{(0)} \cup X_x^{(1)} \cup X_x^{(2)})$, Proposition 2.3 applies. Therefore, we can prove Proposition 6.4 using the graphical induction principle. For readability, the proof of Proposition 6.4 is delayed until Subsection 6.3.2. The proof of Proposition 6.4 uses the same approach as that of Lemma 5.5, and is only included for completeness.

We conclude this subsection with a uniform upper bound on the marginal density of $\delta_t(x)$, stated in Proposition 6.5. Although Proposition 6.5 is the $\mathbb{Z}^d$ analog of Lemma 5.6, proving it requires significantly different techniques.

Proposition 6.5. *Let $\{\delta_t\}_{t\geq 0}$ denote the dominating process on $\mathbb{Z}^d$, and let ε denote the noise intensity. If ε satisfies*

$$\varepsilon < (10^{(250+2d)/4}(d-1)^{7/4})^{-250d}, \tag{6.8}$$

then, for each $x \in \mathbb{Z}^d$ and $t \geq 0$,

$$\mathbb{P}(\delta_t(x) = 1) < \frac{1}{10^{7500}}. \tag{6.9}$$

The proof of Proposition 6.5 is divided into three parts: Recall that, for each $m \in \mathbb{Z}$, $A_m = \{x = (x_i)_{i=1}^d \in \mathbb{Z}^d : x_d = m\}$ denotes the $d-1$ dimensional hyperplane. To prove Proposition 6.5, we will first project δ_t onto hyperplanes A_m, for each $m \in \mathbb{Z}$. By Corollary 6.3, for each $x \in A_m$, the projected process $\tilde{\delta}_t(x)$ has the same marginal density as that of $\delta_t(x)$. We will show that $\tilde{\delta}_t$ is a continuous-time spin system associated with a *Toom operator* T (see (18.3.1) of [23]). We will prove that the corresponding Toom operator T is a *shrinker*. Lastly, we will show that (6.9) is a consequence of Theorem 18.3.1 of [23] and Theorem 1 of [24]. The proof of Proposition 6.5 will be delayed until Section 6.4.

6.3.2 Proof of Proposition 6.4

This subsection proves Proposition 6.4 by applying the graphical induction principle (Proposition 2.3). The proof of Proposition 6.4 uses the same technique as that of Lemma 5.5 and is included only for completeness. Hence, the reader should feel free to skip it.

To prove Proposition 6.4, it suffices to prove (6.7) for a copy of the OMVP $\{\xi_t^{(a)}\}_{t\geq 0}$, with the initial configuration $\xi_0^{(a)} = \zeta^{(a)}$. Due to symmetry, we can assume that $a_i = 0$

for all $i \in \mathbb{Z}$. (Otherwise one can swap the roles of 0 and 1 in this proof.) Since a has been fixed, we abbreviate $\xi_t^{(a)}$ as ξ_t, in the remainder of this subsection.

We let $\mathcal{A}(x,t)$ denote the event that (6.7) holds for a given $x \in \mathbb{Z}^d$ and $t \geq 0$, and let $\mathcal{A}(x,t-)$ denote the event that (6.7) holds for $\xi_{t-}(x)$ and $\delta_{t-}(x)$. Proposition 2.3 states that to prove (6.7) for all $x \in \mathbb{Z}^d$ and $t \geq 0$, it suffices to prove the following:

(6.4.a) $\mathcal{A}(x,0)$ holds for all $x \in \mathbb{Z}^d$.

(6.4.b) For each $x \in \mathbb{Z}^d$, if $\mathcal{A}(y,e(y,t))$ holds for all $y \in \mathcal{N}(x) \cup \{x\}$, then $\mathcal{A}(x,t)$ holds.

Since (6.4.a) follows directly from the assumption that $\xi_0(x) \equiv 0$, to prove (6.7), it suffices to prove (6.4.b).

In the remainder of this proof, we fix any $x \in \mathbb{Z}^d$, and assume that $\mathcal{A}(y,e(y,t))$ holds, for all $y \in \mathcal{N}(x) \cup \{x\}$. Since $\mathcal{A}(y,e(y,t))$ holds for all $y \in \mathcal{N}(x) \cup \{x\}$, $\mathcal{A}(x,e(x,t))$ holds. If (x,t) is not an event point, (6.7) follows from the continuation rule (2.5).

We assume that (x,t) is an event point. Since $\mathcal{A}(y,e(y,t))$ holds for every $y \in \mathcal{N}(x) \cup \{x\}$, $\mathcal{A}(y,t-)$ holds for every such y due to the continuation rule (2.5). We next discuss three possibilities of (x,t): when (x,t) is a 0-noise point, 1-noise point, or vote point.

If (x,t) is either a 0-noise point or 1-noise point, $\delta_t(x) = 1$ by (6.2.c). Since the left-hand side of (6.7) is no greater than 1, $\mathcal{A}(x,t)$ holds, regardless of the value of $\xi_t(x)$.

We now consider the possibility that (x,t) is a vote point. Since $\delta_t(x) = 1$ ensures (6.7), we also assume $\delta_t(x) = 0$.

Setting

$$y_i = x + \mathbf{e}_i, \quad \text{for } 1 \leq i \leq d,$$

then $\mathcal{U}(x) = \{y_1, y_2, \cdots, y_{d-1}\}$ by (6.4). Since $\delta_t(x) = 0$, (6.5) implies that at most $d/2 - 1$ number of $y \in \mathcal{U}(x) \cup \{x\}$ satisfies $\delta_{t-}(y) = 1$. We further divide this into the following 2 cases, according to the parity of d.

When $d = 2k$, for some $k \geq 2$

Since d is even, by (6.5), there are at most $k-1$ number of $y \in \mathcal{U}(x) \cup \{x\}$ satisfying $\delta_{t-}(y) = 1$. Then there are two possibilities, depending on whether $\delta_{t-}(x) = 1$: If

$\delta_{t-}(x) = 1$, then at most $k-2$ number of $y \in \mathcal{U}(x)$ has $\delta_{t-}(y) = 1$. Therefore,

$$\sum_{y \in \mathcal{N}(x)} \delta_{t-}(y) \leq 1 + \sum_{y \in \mathcal{U}(x)} \delta_{t-}(y) \leq k-1. \tag{6.10}$$

Since $\mathcal{A}(y_i, t-)$ holds, for $1 \leq i \leq d$, at most $k-1$ number of $y \in \mathcal{N}(x)$ has $\xi_{t-}(y) = 1$. It therefore follows from (6.10) that

$$\sum_{y \in \mathcal{N}(x)} \xi_{t-}(y) \leq \sum_{y \in \mathcal{N}(x)} \delta_{t-}(y) \leq k-1.$$

Then (2.6) and (2.2.c) implies that $\xi_t(x) = 0$, and hence (6.7) holds for (x, t).

If $\delta_{t-}(x) = 0$, then at most $k-1$ values of $y \in \mathcal{U}(x)$ have $\delta_{t-}(y) = 1$, and hence

$$\sum_{y \in \mathcal{N}(x)} \delta_{t-}(y) \leq 1 + \sum_{y \in \mathcal{U}(x)} \delta_{t-}(y) \leq k. \tag{6.11}$$

Since it was assumed that $\mathcal{A}(x, t-)$ holds and $\delta_t(x) = 0$, one must have $\delta_{t-}(x) = 0$. Since $\mathcal{A}(y_i, t-)$ holds for $1 \leq i \leq d$, at most $k-1$ number of $y \in \mathcal{N}(x)$ satisfy $\xi_{t-}(y) = 1$. Therefore, (6.11) implies

$$\sum_{y \in \mathcal{N}(x)} \xi_{t-}(y) \leq \sum_{y \in \mathcal{N}(x)} \delta_{t-}(y) \leq k.$$

Since $\xi_{t-}(x) = 0$, it follows from (2.6) and (2.2.c) that $\xi_t(x) = 0$, which implies (6.7). This completes the proof of (6.7) for d even.

When $d = 2k+1$ for some $k \geq 2$

Assuming d is odd, then at most $k-1$ number of $y \in \mathcal{U}(x) \cup \{x\}$ satisfy $\delta_{t-}(y) = 1$. Since $\mathcal{A}(y, t-)$ holds for every $y \in \mathcal{N}(x) \cup \{x\}$,

$$\sum_{y \in \mathcal{N}(x) \cup \{x\}} \xi_{t-}(y) \leq \sum_{y \in \mathcal{N}(x) \cup \{x\}} \delta_{t-}(y) \leq k.$$

Therefore, regardless of the value of $\xi_{t-}(x)$, the following always holds:

$$\sum_{y \in \mathcal{N}(x)} \xi_{t-}(y) \leq k.$$

It therefore follows from (2.6) and (2.2.c) that $\xi_t(x) = 0$, which implies that (6.7) holds for (x, t). This completes the proof of (6.7) when d is odd, and hence the proof of Proposition 6.4.

6.4 Proof of Proposition 6.5

This section proves Proposition 6.5. We outline our proof as follows: We will project the dominating process δ_t onto hyperplanes $A_m = \{x = (x_i)_{i=1}^d \in \mathbb{Z}^d : x_d = m\}$, for each $m \in \mathbb{Z}$. It follows from Corollary 6.3 that these projected processes are independent of each other. Letting $\tilde{\delta}_t$ denote a given projected process, we will show that $\tilde{\delta}_t$ is a continuous-time spin system associated with a shrinker operator (defined in Definitions 6.7 and 6.9). This enables us to apply estimates in [23] and establish an upper bound on $\mathbb{P}(\tilde{\delta}_t(x) = 1)$ in terms of the generic population process, which was introduced by Bramson and Gray ([24]). This will be done in Lemma 6.12. (The generic population process, for our purposes, can be considered as a generalization of the majority vote process that has one-sided noise.) We will employ it to establish a uniform upper bound on $\mathbb{P}(\tilde{\delta}_t(x) = 1)$: In Proposition 6.13, we will apply estimates in [24] to establish an upper bound on the marginal density of the associated generic population process. Lastly, we will apply Lemma 6.12 and Proposition 6.13 to complete the proof of Proposition 6.5.

6.4.1 Terminology: Projected Dominating Process, Spin System Associated with a Toom operator

This subsection is dedicated to introducing terminology. We will introduce the *projected dominating process*, denoted by $\tilde{\delta}_t$, which has the same marginal distribution as of δ_t (Corollary 6.3). We will also review the concept of the *(continuous-time) spin system associated with a Toom operator*, initially introduced by Gray ([23]).

Projected Dominating Process

By Corollary 6.3, the probability measure of δ_t is a product measure on the produce space $\prod_{m\in\mathbb{Z}}\{0,1\}^{A_m}$, where $A_m = \{x = (x_i)_{i=1}^d : x_d = m\}$, for each $m \in \mathbb{Z}$. For us, it will be more convenient to view $\{\delta_t\}$ as a composition of (infinitely many) independent particle systems on A_m, for $m \in \mathbb{Z}$. For each $m \in \mathbb{Z}$, we introduce

$$P_m : \mathbb{Z}^{d-1} \to \mathbb{Z}^d : (x_1, x_2, \cdots, x_{d-1}) \mapsto (x_1, x_2, \cdots, x_{d-1}, m). \tag{6.12}$$

Let $\delta_t^{(m)}(x) = \delta_t(P_m(x))$, for each $x \in \mathbb{Z}^{d-1}$, and $\nu_t^{(m)}$ denote the probability measure (on $\mathbb{Z}^{d-1}$) of $\delta_t^{(m)}$. Letting ν_t denote the probability measure of δ_t, then Corollary 6.3

implies that ν_t is the product measure of $\nu_t^{(m)}$, for $m \in \mathbb{Z}$:

$$\nu_t = \prod_{m \in \mathbb{Z}} \nu_t^{(m)}. \tag{6.13}$$

To prove (6.9), it suffices to prove the following $\delta_t^{(m)}$ analog:

$$\mathbb{P}(\delta_t^{(m)}(x) = 1) \leq 10^{-7500}, \quad \text{for every } m \in \mathbb{Z}. \tag{6.14}$$

Due to symmetry, $\delta_t^{(m)}$ is identically distributed, for each $m \in \mathbb{Z}$. To prove (6.14), it suffices to prove (6.14) for $m = 0$. In the remainder of this proof, we abbreviate $\delta_t^{(0)}$ as $\tilde{\delta}_t$, for every $t \geq 0$.

We next show that $\tilde{\delta}_t$ is a *spin system associated with a Toom operator* T, for which the concept was initially introduced in Section 18.3 of [23]. To align with their notations, we introduce the following terminology.

Spin Systems Associated with a Toom Operator

We introduce the *Toom operator*. For simplicity, for given $A \subseteq \mathbb{Z}^{d-1}$ and $x \in \mathbb{Z}^{d-1}$, we let $A + x$ denote the translation of A by x, i.e.,

$$A + x = \{y + x : y \in A\}.$$

Definition 6.6 (Toom operator)**.** *An operator $T : \mathfrak{P}(\mathbb{Z}^{d-1}) \to \mathfrak{P}(\mathbb{Z}^{d-1})$ is a Toom operator if it satisfies the following:*

(6.6.a) Monotonicity: if $A \subseteq B \subseteq \mathbb{Z}^{d-1}$, then $T(A) \subseteq T(B)$.

(6.6.b) Translation-invariance: $T(A + x) = T(A) + x$, for every $x \in \mathbb{Z}^{d-1}$.

(6.6.c) Neighborhood N: There exists some finite $N \subseteq \mathbb{Z}^{d-1}$ such that, for every $x \in \mathbb{Z}^{d-1}$ and $A \subseteq \mathbb{Z}^{d-1}$, $x \in T(A)$ holds if and only if $x \in T(A \cap (N + x))$ holds.

(6.6.d) Two traps: $T(\emptyset) = \emptyset$ and $T(\mathbb{Z}^{d-1}) = \mathbb{Z}^{d-1}$.

Conditions (6.6.a), (6.6.b), and (6.6.d) are relatively straightforward to understand, while (6.6.c) can be viewed as an analog of the finite range condition discussed in Section 2.4.

Recall that a spin system is an interacting particle system whose state space $\mathscr{S} = \{0,1\}^S$. For a given spin system $\{\xi_t\}_{t\geq 0}$, its *birth rates* $\{b_x(\xi)\}_{x\in S,\xi\in\mathscr{S}}$ and *death rates* $\{d_x(\xi)\}_{x\in S,\xi\in\mathscr{S}}$ are introduced to satisfy the following: For every $t, h \geq 0$ and $x \in S$,

$$\mathbb{P}(\xi_{t+h}(x) = 1 \mid \xi_t(x) = 0) = b_x(\xi_t)h + o(h),$$

and

$$\mathbb{P}(\xi_{t+h}(x) = 0 \mid \xi_t(x) = 1) = d_x(\xi_t)h + o(h).$$

For each $\xi \in \{0,1\}^S$, let $A(\xi) \subseteq S$ denote the "occupied sites" in ξ, i.e.,

$$A(\xi) = \{y \in S : \xi(y) = 1\}. \tag{6.15}$$

We introduce the following:

Definition 6.7 (Spin system associated with a Toom operator)**.** *Consider a spin system $\{\delta_t\}_{t\geq 0}$ and a Toom operator T. Letting $\{b_x(\delta)\}_{x\in S,\delta\in\mathscr{S}}$ and $\{d_x(\delta)\}_{x\in S,\delta\in\mathscr{S}}$ denote the birth and death rates of δ_t, we say δ_t is a spin system associated with a Toom operator T if there exists a constant $\varepsilon \in [0,1]$ such that, for each $x \in S$ and $\delta \in \{0,1\}^S$, $b_x(\delta)$ and $d_x(\delta)$ satisfy*

$$b_x(\delta) = \begin{cases} 1 & \text{if } x \in T(A(\delta)), \\ \varepsilon & \text{otherwise;} \end{cases} \tag{6.16}$$

and

$$d_x(\delta) = \begin{cases} 0 & \text{if } x \in T(A(\delta)), \\ 1-\varepsilon & \text{otherwise.} \end{cases} \tag{6.17}$$

Intuitively, the dynamics of a spin system δ_t associated with a Toom operator T consist of two parts: At rate ε, $\delta_t(x)$ is updated to 1. At rate $1-\varepsilon$, $\delta_t(x)$ is updated to 1 if $x \in T(A(\delta_{t-}))$, and to 0 if $x \notin T(A(\delta_{t-}))$. These dynamics resemble those of the dominating process (and hence the projected dominating process): the δ-noise rule and δ-vote rule in 6.2.

For each $x \in \mathbb{Z}^{d-1}$, we define $\tilde{\mathcal{U}}(x)$ by

$$\tilde{\mathcal{U}}(x) = \{x + \mathbf{e}_i : 1 \leq i \leq d-1\}.$$

We claim the following:

Lemma 6.8. *For each $m \in \mathbb{Z}$, we let $\delta_t^{(m)}$ denote the projected dominating process on $\mathbb{Z}^{d-1}$. Then $\delta_t^{(m)}$ is a spin system associated with the Toom operator T, where*

$$T(A) = \left\{ x \in \mathbb{Z}^d : |\tilde{\mathcal{U}}(x) \cap A| \geq \frac{d-1}{2} \right\}, \quad \textit{for each } A \subset \mathbb{Z}^{d-1}. \tag{6.18}$$

Proof. Due to the symmetry of $\mathbb{Z}^{d-1}$, to prove Lemma 6.8, it suffices to prove it for $m = 0$. For simplicity, we set $\delta_t = \tilde{\delta}_t^{(m)}$, for each $t \geq 0$. Let $\{b_x(\delta)\}_{x \in S, \delta \in \mathscr{S}}$ and $\{d_x(\delta)\}_{x \in S, \delta \in \mathscr{S}}$ denote the birth and death rates of δ_t. To show δ_t is a spin system associated with a Toom operator T, we will first show that T is a Toom operator, next show the birth and death rates of δ_t satisfy (6.16) and (6.17).

By Definition 6.6, to show that T is a Toom operator, it suffices to show that T satisfies (6.6.a), (6.6.b), (6.6.c) and (6.6.d).

We verify that T satisfies (6.6.a). For each $x \in \mathbb{Z}^{d-1}$ and any given $A \subset B \subset \mathbb{Z}^{d-1}$, if $|\tilde{\mathcal{U}}(x) \cap A| \geq (d-1)/2$, then $|\tilde{\mathcal{U}}(x) \cap B| \geq (d-1)/2$. It follows from (6.18) that $T(A) \subset T(B)$, which shows that T satisfies (6.6.a).

Since $\tilde{\mathcal{U}}(x) = \tilde{\mathcal{U}}(0) + x$, for each $x \in \mathbb{Z}^{d-1}$, T satisfies (6.6.b). Taking $N = \tilde{\mathcal{U}}(0)$, it is straightforward to verify that T satisfies (6.6.c). If $A = \emptyset$, then $\tilde{\mathcal{U}}(x) \cap A = \emptyset$, for each $x \in \mathbb{Z}^{d-1}$. If $A = \mathbb{Z}^{d-1}$, $\tilde{\mathcal{U}}(x) \cap A = \mathbb{Z}^{d-1}$, for each $x \in \mathbb{Z}^{d-1}$. By (6.18), T satisfies (6.6.d). Therefore, T defined in (6.18) is a Toom operator.

We show that for each $x \in \mathbb{Z}^{d-1}$ and $\delta \in \mathscr{S} = \{0,1\}^{\mathbb{Z}^{d-1}}$, $b_x(\delta)$ and $d_x(\delta)$ satisfy (6.16) and (6.17). We consider the following two possibilities: $x \in T(A(\delta))$ and $x \notin T(A(\delta))$. If $x \in T(A(\delta))$ and $\delta(x) = 0$, it follows from (6.2.b) and (6.2.c) that either a vote event or a noise event will change the value of $\delta(x)$. Therefore, $b_x(\delta) = 1$. If $x \in T(A(\delta))$ and $\delta(x) = 1$, it follows from (6.2.b) and (6.2.c) that neither a vote event nor a noise event can change the value of $\delta(x)$. And hence $d_x(\delta) = 0$.

If $x \notin T(A(\delta))$ and $\delta(x) = 0$, it follows from (6.2.b) and (6.2.c) that only a noise event can change the value of $\delta(x)$, and hence $b_x(\delta) = \varepsilon$. If $x \notin T(A(\delta))$ and $\delta(x) = 1$, it follows from (6.2.b) and (6.2.c) that only a vote event can change the value of $\delta(x)$, and hence $d_x(\delta) = 1 - \varepsilon$. This completes the proof that $b_x(\delta)$, $d_x(\delta)$, and T satisfy (6.16) and (6.17), and hence the proof of Lemma 6.8. □

6.4.2 Shrinker

In this subsection, we will introduce the *shrinker*, a special class of Toom operators, and show that the Toom operator T introduced by (6.18) is a shrinker.

For vectors $v, w \in \mathbb{R}^{d-1}$, we define the half-space $\mathcal{H}(v,w)$ by

$$\mathcal{H}(v,w) = \{x \in \mathbb{R}^{d-1} :< x - w, v >< 0\},$$

where $< \cdot, \cdot >$ denotes the usual Euclidean inner product. We let $H(v,w)$ denote the "lattice half-space" by

$$H(v,w) = \mathcal{H}(v,w) \cap \mathbb{Z}^{d-1}.$$

For each $v \in \mathbb{R}^{d-1} - \{\mathbf{0}\}$ and a given Toom operator T, we introduce the *speed coefficient* $\alpha(v)$ (which depends on T and v), by

$$\alpha(v) = \sup\{a \in \mathbb{R} : 0 \in T(H(v, -av))\}. \tag{6.19}$$

In Figure 6.1, we consider the Toom operator corresponding to the discrete-time OMVP on $\mathbb{Z}^2$: Let $T' : \mathbb{Z}^2 \to \mathbb{Z}^2$ denote the operator where, for each $A \subseteq \mathbb{Z}^2$,

$$\begin{aligned} T'(A) =& \{y \in \mathbb{Z}^2 : y + \mathbf{e}_i \in A, \text{ for } i = 1,2\} \\ & \cup \{y \in A : \{y + \mathbf{e}_1, y + \mathbf{e}_2\} \cap A \neq \emptyset\}. \end{aligned} \tag{6.20}$$

As one can see from diagrams (a) and (b), T' has speed coefficients $\alpha(\mathbf{e}_1) = 0$ and $\alpha(-\mathbf{e}_1 - \mathbf{e}_2) = 1/2$.

We next introduce the *shrinker*, initially introduced by Gray ([23]).

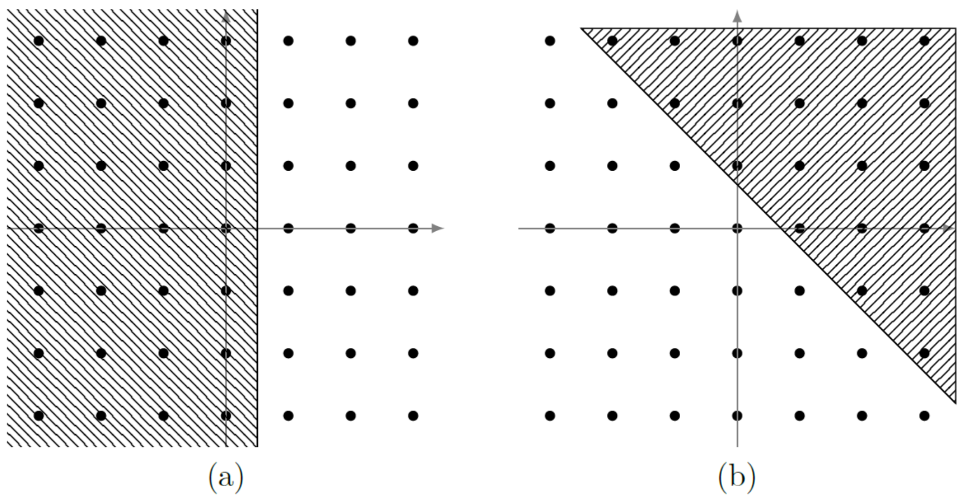
Definition 6.9 (shrinker)**.** *A Toom operator is a shrinker if there is some $V \subset \mathbb{R}^{d-1}$ such that α is nonnegative on V, and $\mathcal{H}(V, \alpha I) = \mathbb{R}^{d-1}$, where*

$$\mathcal{H}(V, \alpha I) := \bigcup_{v \in V} \mathcal{H}(v, \alpha(v)v). \tag{6.21}$$

The terminology "shrinker" is intended to suggest how T affects a set of vacant sites: Supposing that only a finite number of vacant sites exist in a configuration $\delta \in \{0,1\}^{\mathbb{Z}^{d-1}}$ and letting $A(\delta)$ denote the set of occupied sites in δ, then (6.21) implies that there exist some $n \in \mathbb{N}$ and $A \subset \mathbb{Z}^{d-1}$, such that $A \subseteq A(\delta)$,

$$T^{i-1}(A) \subseteq T^i(A), \quad \text{for each } 1 \leq i \leq n,$$

Figure 6.1: An example of a shrinker



In (a): The half-space $H(\mathbf{e}_1, -a\mathbf{e}_1)$ and lattice half-space $\mathcal{H}(\mathbf{e}_1, -a\mathbf{e}_1)$ are displayed, assuming $a < 0$. As one can see, if $a < 0$, then $0 \in \mathcal{H}(\mathbf{e}_1, -a\mathbf{e}_1)$, $|\mathcal{H}(\mathbf{e}_1, -a\mathbf{e}_1) \cap \{\mathbf{e}_1, \mathbf{e}_2\}| = 1$, and hence $0 \in T'(H(\mathbf{e}_1, -a\mathbf{e}_1)$. On the contrary, if $a > 0$, then $0 \notin \mathcal{H}(\mathbf{e}_1, -a\mathbf{e}_1)$, $|\mathcal{H}(\mathbf{e}_1, -a\mathbf{e}_1) \cap \{\mathbf{e}_1, \mathbf{e}_2\}| \leq 1$, and hence $0 \notin T'(H(\mathbf{e}_1, -a\mathbf{e}_1))$. By (6.19), $\alpha(\mathbf{e}_1) = 0$. In (b): Letting $\mathbf{v} = -(\mathbf{e}_1 + \mathbf{e}_2)$, the half-space $H(\mathbf{v}, -a\mathbf{v})$ and lattice half-space $\mathcal{H}(\mathbf{v}, -a\mathbf{v})$ are displayed, assuming $a < 1/2$. As in (a), one can show that $0 \in T'(H(\mathbf{v}, -a\mathbf{v}))$ if $a < 1/2$, and that $0 \notin T'(H(\mathbf{v}, -a\mathbf{v}))$ otherwise. By (6.19), $\alpha(\mathbf{v}) = 1/2$.

and $T^n(A) = \mathbb{Z}^{d-1}$. In other words, the set of vacant sites "shrinks" every time after applying T. For example, the Toom operator introduced by (6.20) is a shrinker, since $V = \{\mathbf{e}_1, \mathbf{e}_2, -(\mathbf{e}_1 + \mathbf{e}_2)\}$ satisfies Definition 6.9.

We claim that the operator T defined by (6.18) is a shrinker.

Proposition 6.10. *The Toom operator T introduced by (6.18) is a shrinker.*

Proof. We will prove Proposition 6.10 by constructing some $V \subseteq \mathbb{R}^{d-1}$ such that $\mathcal{H}(V, \alpha I) = \mathbb{R}^{d-1}$. Letting $\mathbf{e}_i$ denote the ith standard vector of $\mathbb{R}^{d-1}$, for $1 \le i \le d-1$, we set

$$V = \{\mathbf{e}_i : 1 \le i \le d-1\} \cup \{\mathbf{v} = -\sum_{i=1}^{d-1} \mathbf{e}_i\}. \tag{6.22}$$

By Definition 6.9, to show Claim 6.10, it suffices to show $\mathcal{H}(V, \alpha I) = \mathbb{R}^{d-1}$.

We claim the following.

Lemma 6.11. *Consider the Toom operator T introduced in (6.18) and, for each $v \in \mathbb{R}^{d-1}$, let $\alpha(v)$ denote the speed coefficient as in (6.19). We claim the following:*

$$\alpha(\mathbf{e}_i) = \alpha(-\mathbf{e}_i) = 0, \quad \textit{for every } 1 \le i \le d-1. \tag{6.23}$$

Proof. Due to the symmetry of the operator T, to show (6.23), it suffices to show (6.23) for $i = 1$.

For every $a < 0$ and $2 \le i \le d-1$, since $< \mathbf{e}_i + a\mathbf{e}_1, \mathbf{e}_1 >= a < 0$, one must have $e_i \in H(\mathbf{e}_1, -a\mathbf{e}_1)$. Therefore,

$$|\tilde{\mathcal{U}}(0) \cap H(\mathbf{e}_1, -a\mathbf{e}_1)| \ge d-2.$$

It hence follows from (6.18) that $0 \in T(H(\mathbf{e}_1, -a\mathbf{e}_1))$.

On the other hand, for every $a \ge 0$ and $2 \le i \le d-1$, since $< \mathbf{e}_i + a\mathbf{e}_1, \mathbf{e}_1 >= a \ge 0$, one must have $\mathbf{e}_i \notin H(\mathbf{e}_1, -a\mathbf{e}_1)$, and hence

$$|\tilde{\mathcal{U}}(0) \cap H(\mathbf{e}_1, -a\mathbf{e}_1)| \le 1.$$

By definition (6.18), $0 \notin T(H(\mathbf{e}_1, -a\mathbf{e}_1))$. It therefore follows from (6.19) that $\alpha(\mathbf{e}_1) = \alpha(-\mathbf{e}_1) = 0$. This completes the proof of (6.23). $\square$

We next evaluate $\alpha(\mathbf{v})$ using the same approach as the proof of Lemma 6.11. For every $a < 1/(d-1)$, a little thought shows that

$$|\tilde{\mathcal{U}}(0) \cap H(\mathbf{v}, -a\mathbf{v})| = d - 1.$$

It follows from (6.18) that $0 \in T(H(\mathbf{v}, -a\mathbf{v}))$. On the other hand, for every $a \geq 1/(d-1)$, a little thought shows that

$$|\tilde{\mathcal{U}}(0) \cap H(\mathbf{v}, -a\mathbf{v})| = 0.$$

It therefore follows from (6.18) that $0 \notin T(H(\mathbf{v}, -a\mathbf{v}))$. Consequently, $\alpha(\mathbf{v}) = 1/(d-1)$. Combining this with (6.23), it is straight forward to verify that

$$\mathcal{H}(V, \alpha I) = \mathbb{R}^{d-1}. \tag{6.24}$$

This completes the proof of Proposition 6.10. □

6.4.3 Demonstration of (6.14)

This subsection is dedicated to demonstrating (6.14). Recall we proved that $\tilde{\delta}_t$ is a continuous-time spin system associated with a shrinker T, defined by (6.18). Applying estimates in the proof of Theorem 18.3.1 of [23], namely (18.4.1), (18.4.5), and (18.4.6), one can demonstrate Lemma 6.12. Recall the projected dominating process $\tilde{\delta}_t$ was introduced by (6.12).

Lemma 6.12. *Let $\{\tilde{\delta}_t\}_{t\geq 0}$ denote the projected dominating process on $\mathbb{Z}^{d-1}$, and let ε denote the noise parameter of $\tilde{\delta}_t$. Letting J denote a constant satisfying $J < (1/\varepsilon)^{1/250^{d-1}}$, then, for each $x \in \mathbb{Z}^{d-1}$ and $t \geq 0$,*

$$\mathbb{P}(\tilde{\delta}_{Jt}(x) = 1) \leq J^{-240(d-1)} + \mathbb{P}(x \in \zeta_{Jt}), \tag{6.25}$$

where $\zeta_{Jt} \subseteq \mathbb{R}^{d-1}$ denotes the generic population process introduced in [24], with the following parameters: The occupation rate $\alpha = 1/(d-1)$, which is set equal to the maximum speed coefficient on V, where V was defined by (6.22). The interaction rate $\beta(J) = -500(d-1)^2 J^2$, the constant $\theta(J) = -1/(3\beta(J))$, and the error rate $\varepsilon(J) = J^{-240(d-1)}$.

We summarize here the proof of Lemma 6.12: In [23], Gray compared a spin system (in our application, $\tilde{\delta}_t$) with a discrete-time auxiliary process that was denoted by η_t. (This η_t is unrelated to the coupled process introduced in Definition 2.5.) Using the graphical inductive principle, Gray showed that

$$\tilde{\delta}_t(x) \leq \eta_t(x), \quad \text{for each } x \in \mathbb{Z}^{d-1} \text{ and } t \in \mathbb{N}.$$

The discrete-time auxiliary process η_t consists of two parts, $\zeta_t \subseteq \mathbb{R}^{d-1}$ and $H_t \subseteq \mathbb{Z}^{d-1}$, where ζ_t is precisely the generic population process introduced in [24], and H_t is an "error process" that controls the deviation of η_t from ζ_t. For each $x \in \mathbb{Z}^{d-1}$ and $t \geq 0$, since

$$\mathbb{P}(\tilde{\delta}_t(x) = 1) \leq \mathbb{P}(\eta_t(x) = 1) \leq \mathbb{P}(x \in H_t) + \mathbb{P}(x \in \zeta_t),$$

to bound $\mathbb{P}(\tilde{\delta}_{Jt}(x) = 1)$, it suffices to develop upper bounds on $\mathbb{P}(x \in H_t)$ and $\mathbb{P}(x \in \zeta_t)$. Gray proved that, when $J < (1/\varepsilon)^{1/250^{d-1}}$,

$$\mathbb{P}(x \in H_{Jt}) < J^{-240(d-1)}, \quad \text{for each } x \in \mathbb{Z}^{d-1} \text{ and } t \geq 0. \tag{6.26}$$

This implies (6.25).

To derive an upper bound on $\mathbb{P}(x \in \zeta_{Jt})$, we apply estimates from [24], namely (2-10), (2-21), (2-24) and (2-26). We combine these estimates into the next proposition.

Proposition 6.13. *Let ζ_t denote the generic population process satisfying Lemma 6.12. For each $x \in \mathbb{Z}^{d-1}$, $t \geq 0$, and $J > 0$,*

$$\mathbb{P}(x \in \zeta_{Jt}) \leq \limsup_{\ell \to \infty} z_1(\ell, J) + \sum_{\ell=1}^{\infty} z_2(\ell, J), \tag{6.27}$$

where

$$z_1(\ell, J) = C_1^{(\ell^2 + d\ell)} J^{10\ell^2 + 4d - 2^\ell d}, \quad \text{for each } \ell \in \mathbb{N} \text{ and } J > 0, \tag{6.28}$$

and

$$z_2(\ell, J) = C_2^{\ell^2} J^{-2^\ell}, \quad \text{for each } \ell \in \mathbb{N} \text{ and } J > 0, \tag{6.29}$$

with constants C_1, C_2 determined by d solely, satisfying $C_2 < 10^{25} d^d$.

We summarize the proof of Proposition 6.13: In [24], Bramson and Gray used a "renormalization argument" to derive an upper bound on $\mathbb{P}(x \in \zeta_t)$. Let $\zeta_t^{(\ell)}$ denote the generic population process, with parameters

$$\begin{aligned} \alpha^{(\ell)} = \alpha/2^l = 1/(2^l(d-1)), \quad \varepsilon^{(\ell)} = (\varepsilon(J))^{2^\ell} \\ \beta^{(\ell)} = -500(d-1)J^2 2^\ell, \quad \theta^{(\ell)} = 1/\left(3 \cdot 10^{4\ell} 2^{2\ell^2+2\ell}(d-1)^{3\ell/2}\right). \end{aligned} \tag{6.30}$$

They proved that, for every $k \in \mathbb{N}$ and $t \geq 0$,

$$\zeta_t \subseteq \cup_{\ell=1}^k N_t^{(\ell)} \bigcup \zeta_t^{(k)}, \tag{6.31}$$

where, for every $\ell \in \mathbb{N}$, $N_t^{(\ell)}$ is an "error set", satisfying

$$\mathbb{P}(x \in N_{Jt}^{(\ell)}) \leq \frac{\varepsilon^{(\ell)} 2^{2d+1} C^{(l)} (d+1)!}{(d-1)^{(d+1)/2}}, \quad \text{for each } x \in \mathbb{Z}^{d-1} \text{ and } t \geq 0. \tag{6.32}$$

They also proved that, for every $\ell \in \mathbb{N}$, $x \in \mathbb{Z}^{d-1}$ and $t \geq 0$,

$$\mathbb{P}(x \in \zeta_{Jt}^{(\ell)}) \leq \frac{\varepsilon^{(\ell)} 2^{(l+3)(d+1)} |\beta^{(\ell)}|^{d+1} C^{(l)} (d+1)!}{(\theta^{(\ell)})^{d+1}}, \tag{6.33}$$

where $C^{(\ell)}$ is a constant defined by

$$C^{(\ell)} = (2^{47} \cdot 5^{18} \cdot 10^{2d} (d-1)^7 J^4)^{2\ell^2}, \quad \text{for each } \ell \in \mathbb{N}. \tag{6.34}$$

It follows from (6.31) that, for each $x \in \mathbb{Z}^{d-1}$ and $t \geq 0$,

$$\mathbb{P}(x \in \zeta_{Jt}) \leq \limsup_{l \to \infty} \mathbb{P}(x \in \zeta_{Jt}^{(\ell)}) + \sum_{l=0}^{\infty} \mathbb{P}(x \in N_{Jt}^{(\ell)}). \tag{6.35}$$

Substituting $\mathbb{P}(x \in \zeta_{Jt}^{(\ell)})$ and $\mathbb{P}(x \in N_{Jt}^{(\ell)})$ by (6.33) and (6.32), (6.27) follows directly from (6.30) and (6.34). This completes the proof of Proposition 6.13.

We now prove (6.14) by applying Lemma 6.12 and Proposition 6.13.

Proof of (6.14). Taking

$$J = 10^{(250+2d)/4} (d-1)^{7/4}, \tag{6.36}$$

since ε satisfies (6.8), $\varepsilon J^{250(d-1)} < 1$. It therefore follows from Lemma 6.12 and Proposition 6.13 that, for each $x \in \mathbb{Z}^{d-1}$ and $t \geq 0$,

$$\begin{aligned} \mathbb{P}(\tilde{\delta}_{Jt}(x) = 1) \leq & J^{-240(d-1)} + \limsup_{\ell \to \infty} C_1^{(\ell^2+d\ell)} J^{10\ell^2+4d-2^\ell d} \\ & + \sum_{\ell=1}^{\infty} C_2^{\ell^2} J^{-2^\ell}. \end{aligned} \tag{6.37}$$

Substituting J by (6.36), a further simple estimate of (6.37) gives that, for each $x \in \mathbb{Z}^{d-1}$ and $t \geq 0$,

$$\mathbb{P}(\tilde{\delta}_t(x) = 1) < J^{-120(d-1)} < 10^{-7500}. \tag{6.38}$$

This completes the proof of (6.14). $\square$

6.5 Demonstration of Theorem 8

This section proves Theorem 8. The structure of this proof is similar to that of Theorem 6 (Section 5.1) and Theorem 2 of [6]: We will introduce an appropriate uncountable subset $\Lambda \subseteq \{0,1\}^{\mathbb{Z}}$. Recall that, for each $a \in \{0,1\}^{\mathbb{Z}}$, the compatible configuration $\zeta^{(a)}$ was introduced in Subsection 6.3.1. Let $\xi_t^{(a)}$ denote the OMVP on $\mathbb{Z}^d$, with $\zeta^{(a)}$ as the initial configuration. For each $a \in \{0,1\}^{\mathbb{Z}}$, we will use $\xi_t^{(a)}$ to introduce an equilibrium $\mu^{(a)}$. We will show that these equilibria $\mu^{(a)}$, for $a \in \Lambda$, are mutually singular with respect to each other. For this, we will apply Propositions 6.7 and 6.5. This will complete the proof of Theorem 8.

We introduce an equivalence relation "$\sim$" on the sequence space $L = \{0,1\}^{\mathbb{Z}}$ as follows: For $a, b \in L$, we set $a \sim b$ if a, b differ by at most finitely many terms. It is easy to see that "$\sim$" defines an equivalence relation on L. We claim the following:

Claim 6.14. *Letting $\bar{\Lambda}$ denote the collection of equivalence class induced by "$\sim$", $\bar{\Lambda}$ is uncountable.*

Proof. We introduce a map: $f : L = \{0,1\}^{\mathbb{Z}} \to [0,1]$ as follows.

$$f(a) = \sum_{i=1}^{\infty} 2^{-a_i}, \quad \text{for each } a \in L.$$

For every $a, b \in L$ with $a \sim b$, a little thought shows $f(a) - f(b) \in \mathbb{Q}$.

We introduce another relation "$\sim_{\mathbb{Q}}$" on $[0,1]$ as follows: For every $x, y \in [0,1]$, $x \sim_{\mathbb{Q}} y$ if $x - y \in \mathbb{Q}$. It is well known that "$\sim_{\mathbb{Q}}$" is an equivalence relation since $\mathcal{Q}$ is a group. Moreover, the equivalence relation "$\sim_{\mathbb{Q}}$" has uncountably many equivalence classes. (Each class contains at most countably many real numbers, and $[0,1]$ is uncountable.) Since f is subjective, and, for each $a, b \in L$, $a \sim b$ implies $f(a) \sim_{\mathbb{Q}} f(b)$, $\bar{\Lambda}$ is uncountable. This completes the proof of the claim. $\square$

In the remainder of the proof of Theorem 8, we will select a representative a from each equivalence class in $\bar{\Lambda}$, use each $\zeta^{(a)}$ to introduce an equilibrium for ξ_t and show these equilibria are mutually singular with respect to each other.

For each equivalence class in $\bar{\Lambda}$, we choose an arbitrary element to be its representative. Let Λ denote the collection of such representatives. Recall that $\{\xi_t^{(a)}\}_{t\geq 0}$ denotes the OMVP, with the initial configuration $\xi_0^{(a)}(x) = \zeta^{(a)}(x)$, for all $x \in \mathbb{Z}^d$. For each $a \in \Lambda$, we let μ_t^a denote the probability measure of $\xi_t^{(a)}$, and consider the following Cesaro averages:

$$\bar{\mu}_t^{(a)} := \frac{1}{t}\int_0^t \mu_s^{(a)}\, ds.$$

Letting $\mathcal{P}$ denote the space of probability measures on $\mathscr{S} = \{0,1\}^S$, since $\mathscr{S}$ is compact under the product topology, $\mathcal{P}$ is sequentially compact, under the weak$*$ topology. Therefore, a subsequential limit $\mu^{(a)}$ exists, such that

$$\mu^{(a)} = \lim_{m\to\infty} \bar{\mu}_{t_m}^{(a)}, \quad \text{for some subsequence } t_m \uparrow \infty.$$

It is well-known that $\mu^{(a)}$ is an equilibrium for ξ_t.

We show that subsequential limits $\mu^{(a)}, \mu^{(b)}$, obtained from distinct $a, b \in \Lambda$, are mutually singular. Consider distinct $a, b \in \Lambda$, where $a = \{a_n\}_{n\in\mathbb{Z}}$ and $b = \{b_n\}_{n\in\mathbb{Z}}$. Since a, b differ at infinitely many terms, there exists a subsequence $\{k_i\}_{i=1}^\infty \subseteq \mathbb{Z}$, such that $a_{k_i} \neq b_{k_i}$ for every $i \in \mathbb{N}$. Setting $y_i = k_i \mathbf{e}_d$, for $i \in \mathbb{N}$, we consider the event

$$F(a) = \left\{\xi \in \mathscr{S} : \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |\zeta^{(a)}(y_i) - \xi(y_i)| \leq 2\cdot 10^{-7500}\right\}. \tag{6.39}$$

We claim that $\mu^{(a)}(F(a)) = 1$.

Proposition 6.4 implies that, for every $t \geq 0$,

$$\mu_t^{(a)}(\xi : |\zeta^{(a)}(y_i) - \xi_t(y_i)| = 1) \leq \mathbb{P}(\delta_t(y_i) = 1). \tag{6.40}$$

Applying (6.40) for each $1 \leq i \leq n$, we obtain

$$\mu_t(F(a)) \leq \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n \delta_t(y_i) \leq 2\cdot 10^{-7500}\right). \tag{6.41}$$

Corollary 6.3 and Proposition 6.5 imply that $\{\delta_t(y_i)\}_{i=1}^\infty$ is a family of independent Bernoulli random variables satisfying

$$\mathbb{E}\delta_t(y_i) = \mathbb{P}(\delta_t(y_i) = 1) < 10^{-7500}.$$

It therefore follows from Lemma 5.7 that

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}\delta_t(y_i) > 4 \cdot 10^{-7500}\right) < \left(\frac{e}{4}\right)^{nC}, \tag{6.42}$$

where $C = \mathbb{E}\delta_t(y_i)$ is a constant that does not depend on y_i. Since the right-hand side of (6.42) is summable for every fixed t, the Borel-Cantelli Lemma applies, and hence

$$\mu_t^{(a)}(F(a)) = 1, \quad \text{for every } t \geq 0. \tag{6.43}$$

Since $\mu^{(a)}$ is obtained by taking a subsequential limit of the Cesaro averages of $\mu_t^{(a)}$, $\mu^{(a)}(F(a)) = 1$. Similarly, $\mu^{(b)}(F(b)) = 1$. Since a and b differ at infinitely many terms, $F(a) \cap F(b) = \emptyset$, and hence $\mu^{(a)}$ and $\mu^{(b)}$ are mutually singular.

Setting

$$\Theta = \{\mu^{(a)} : a \in \Lambda\}, \tag{6.44}$$

Θ is therefore a set of uncountably many mutually singular equilibria. This completes the proof of Theorem 8.

References

[1] Frank Spitzer. Random processes defined through the interaction of an infinite particle system. In *Probability and Information Theory: Proceedings of the International Symposium at McMaster University, Canada, April, 1968*, pages 201–223. Springer, 2006.

[2] Frank Spitzer. Interaction of markov processes. In *Random Walks, Brownian Motion, and Interacting Particle Systems: A Festschrift in Honor of Frank Spitzer*, pages 66–110. Springer, 1991.

[3] Roland L'vovich Dobrushin. Markov processes with a large number of locally interacting components: existence of a limit process and its ergodicity. *Problemy peredachi informatsii*, 7(2):70–87, 1971.

[4] Roland L'vovich Dobrushin. Markov processes with many locally interacting components–the reversible case and some generalizations. *Problemy peredachi informatsii*, 7(3):57–66, 1971.

[5] Lawrence F Gray. The positive rates problem for attractive nearest neighbor spin systems on $\mathbb{Z}$. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 61(3):389–404, 1982.

[6] Maury Bramson and Lawrence F. Gray. The majority vote process and other consensus processes on trees. *Ann. Appl. Probab.*, 31(1):169–198, 2021. ISSN 1050-5164.

[7] A. L. Toom. Stable and attractive trajectories in multicomponent systems. In

Multicomponent random systems, volume 6 of *Adv. Probab. Related Topics*, pages 549–575. Dekker, New York, 1980.

[8] Reinhard Diestel. Graph theory 3rd ed. *Graduate texts in mathematics*, 173(33): 12, 2005.

[9] Thomas M. Liggett. *Interacting particle systems.* Classics in Mathematics. Springer-Verlag, Berlin, 2005. ISBN 3-540-22617-6. Reprint of the 1985 original.

[10] Roy J Glauber. Time-dependent statistics of the ising model. *Journal of mathematical physics*, 4(2):294–307, 1963.

[11] Ernst Ising. *Beitrag zur theorie des ferro-und paramagnetismus.* PhD thesis, Grefe & Tiedemann Hamburg, Germany, 1924.

[12] David Ruelle. On the use of "small external fields" in the problem of symmetry breakdown in statistical mechanics. *Annals of Physics*, 69(2):364–374, 1972.

[13] Nick Crawford and Wojciech De Roeck. Stability of the uniqueness regime for ferromagnetic glauber dynamics under non-reversible perturbations. In *Annales Henri Poincaré*, volume 19, pages 2651–2671. Springer, 2018.

[14] Peter Clifford and Aidan Sudbury. A model for spatial conflict. *Biometrika*, 60(3): 581–588, 1973.

[15] Richard A Holley and Thomas M Liggett. Ergodic theorems for weakly interacting infinite systems and the voter model. *The annals of probability*, pages 643–663, 1975.

[16] Boris L Granovsky and Neal Madras. The noisy voter model. *Stochastic Processes and their applications*, 55(1):23–43, 1995.

[17] Frank Spitzer. Interaction of markov processes. *Advances in Mathematics*, 5(2):246–290, 1970. ISSN 0001-8708. doi: https://doi.org/10.1016/0001-8708(70)90034-4. URL `https://www.sciencedirect.com/science/article/pii/0001870870900344`.

[18] P Balister, B Bollobás, and R Kozma. Mean field models of probabilistic cellular automata, 2004.

[19] Paul Balister, Béla Bollobás, J Robert Johnson, and Mark Walters. Random majority percolation. *Random Structures & Algorithms*, 36(3):315–340, 2010.

[20] Mario J de Oliveira. Isotropic majority-vote model on a square lattice. *Journal of Statistical Physics*, 66(1):273–281, 1992.

[21] Francisco WS Lima, Umberto L Fulco, and Raimundo N Costa Filho. Majority-vote model on a random lattice. *Physical Review E*, 71(3):036105, 2005.

[22] Jane Molofsky, Richard Durrett, Jonathan Dushoff, David Griffeath, and Simon Levin. Local frequency dependence and global coexistence. *Theoretical population biology*, 55(3):270–282, 1999.

[23] Lawrence F Gray. Toom's stability theorem in continuous time. In *Perplexing problems in probability*, pages 331–353. Springer, 1999.

[24] Maury Bramson and Lawrence Gray. A useful renormalization argument. In *Random walks, Brownian motion, and interacting particle systems*, pages 113–152. Springer, 1991.

[25] David Griffeath. *Additive and cancellative interacting particle systems*, volume 724 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979. ISBN 3-540-09508-X.

[26] Wayne G Sullivan. A unified existence and ergodic theorem for markov evolution of random fields. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 31(1):47–56, 1974.

[27] Richard A Holley and Daniel W Stroock. A martingale approach to infinite systems of interacting processes. *The Annals of Probability*, pages 195–228, 1976.

[28] Lawrence Gray and David Griffeath. On the uniqueness of certain interacting particle systems. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 35(1):75–86, 1976.