SOME RANDOM PROCESSES RELATED TO AFFINE RANDOM WALKS

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ABSTRACT

This paper considers random processes of the form $X_{n+1} = a_n X_n + b_n \pmod{p}$ where (a_n, b_n) are independent random variables, p is an odd integer, and $P(a_n = (p+1)/2)$ is a positive constant. This paper searchs for the time it takes the sequence $X_0 = 0, X_1, X_2, ...$ to get close to uniformly distributed on $\mathbf{Z}/p\mathbf{Z}$. This paper shows that the order of this time will depend on the probabilities for a_n. In particular if a_n may take on values 1, 2, or $(p+1)/2$ and must take on at least 2 of these values and if b_n is independent of a_n, then this time depends on whether $P(a_n = 2) = P(a_n = (p+1)/2)$. This paper also considers some results when a_n and b_n are dependent.

INTRODUCTION

A pseudo-random number generator sometimes used on computers utilizes a recurrence equation of the form

$$X_{n+1} = aX_n + b \pmod{p}$$

where a and b are constants. Although the sequence $X_0 = 0, X_1, X_2, ...$ is deterministic, this sequence shares some properties of random sequences. See Knuth (1981) for more details.

Further work has examined random processes of the form

$$X_{n+1} = a_n X_n + b_n \pmod{p}$$

where the a_n's and b_n's are independent random variables with the a_n's identically distributed and the b_n's identically distributed. Cases where a_n has a fixed probability on $\mathbf{Z}^+$ and b_n has a fixed probability on $\mathbf{Z}$ have been explored in a number of previous works. See Chung Diaconis, and Graham (1987) and Hildebrand (1990, 1993a, 1993b). Questions where a_n has a distribution which depends on p (e.g. $P(a_n = (p+1)/2) = P(a_n = 1) = P(a_n = 2) = 1/3$ for odd values of p) appear in Diaconis (1988). The question of interest is how long does it take for X_n to get close to uniformly distributed on $\mathbf{Z}/p\mathbf{Z}$. Using random

walks on the affine group, Xu (1990) provides an upper bound to this time for certain values of p but wonders if the bound can be improved. This paper finds some lower bounds for this time and provides an upper bound which uses different techniques and a broader range of values of p than Xu.

Using the notation of the next section, this paper shows the following 3 theorems where a_n and b_n are assumed to be independent.

Theorem 1: If $P(a_n = (p+1)/2) = P(a_n = 2) = 1/2$ and $P(b_n = -1) = P(b_n = 1) = 1/2$ and $\epsilon > 0$ is given, then for some constant $c > 0$, if $n \geq c(\log p)^2$ then $||P_n - U|| < \epsilon$ for sufficiently large odd values of p.

Theorem 2: Suppose $P(a_n = (p+1)/2) = a$, $P(a_n = 1) = b$, $P(a_n = 2) = c$, at least 2 of a, b, and c are non-zero, $a + b + c = 1$, $P(b_n = 1) = d$, $P(b_n = 0) = e$, $P(b_n = -1) = f$, at least 2 of d, e, and f are non-zero, and $d + e + f = 1$. Let

$$g = \begin{cases} 2 & \text{if } a = c \\ 1 & \text{if } a \neq c \end{cases}.$$

Let $\epsilon > 0$ be given. Then for sufficiently large odd p, $||P_n - U|| < \epsilon$ if $n > c_1(\log p \log\log p)^g$ for some value $c_1 > 0$ (which may depend on a, b, c, d, e, and f but not p) but $||P_n - U|| < \epsilon$ for almost all odd p if $n > c_2(\log p)^g$ for some value $c_2 > 0$ (which also may depend on a, b, c, d, e, and f but not p). By almost all odd p, we mean that the proportion of odd p between 1 and p_0 satisfying this condition approaches 1 as $p_0 \to \infty$.

Theorem 3: With the notation of Theorem 2, there exists a value $c_3 > 0$ such that, given $\epsilon > 0$, $||P_n - U|| > 1 - \epsilon$ if $n < c_3(\log p)^g$ and p is odd.

Theorems 2 and 3 answer, up to a factor of $(\log\log p)^2$, a question posed on p. 35 of Diaconis (1988). This question has $a = b = c = 1/3$ and $d = e = f = 1/3$ and asks how long it takes for X_n to get close to uniformly distributed on $\mathbf{Z}/p\mathbf{Z}$.

This paper also shows the following results where a_n and b_n may be dependent.

Theorem 4: Let p be odd. Suppose (a_n, b_n) is defined so that $P(a_n = 2) = P(a_n = (p+1)/2) = (1/2)(1 - P(a_n = 1)) \neq 0$ and that b_n has a fixed distribution on $\mathbf{Z}$ and has finitely many possible values. Suppose the (a_n, b_n)'s are i.i.d. Let $\epsilon > 0$ be given. There exists a value $c > 0$ (not depending on p but depending on the values for the probabilities on (a_n, b_n)) such that if $n < c(\log p)^2$, then $||P_n - U|| > 1 - \epsilon$ for sufficiently large p.

Theorem 5: Suppose that p is odd and that (a_n, b_n) is chosen uniformly from $(2, 1)$, $(2, -1)$, $(((p+1)/2), ((p+1)/2))$, and $(((p+1)/2), -((p+1)/2))$. Then there exists a value $c > 0$ such that if $n > c(\log p)^3$, $||P_n - U|| \to 0$ as $p \to \infty$.

NOTATION AND BACKGROUND

Let P be a probability on $\mathbf{Z}/p\mathbf{Z}$. Define the variation distance of P from the uniform distribution U by

$$||P - U|| := \frac{1}{2} \sum_{s \in \mathbf{Z}/p\mathbf{Z}} |P(s) - \frac{1}{p}|.$$

One can readily show that

$$||P - U|| = \max_{A \subseteq \mathbf{Z}/p\mathbf{Z}} |P(A) - U(A)|.$$

This variation distance is the one defined in Diaconis (1988).

Suppose $X_0 = 0$ and

$$X_{n+1} = a_n X_n + b_n \pmod{p}$$

where the (a_n, b_n)'s are i.i.d. We shall define P_n to be the probability distribution of X_n (where X_n is viewed as a random variable on $\mathbf{Z}/p\mathbf{Z}$). By abuse of notation, we shall also call $||P_n - U||$ the distance of X_n from uniform.

Let X and Y be independent random variables on $\mathbf{Z}/p\mathbf{Z}$ with probability distributions P and Q. Let $P * Q$ be the probability distribution of $X + Y$. The following proposition will be useful:

Proposition 1:

$$||P * Q - U|| \leq ||P - U||$$

The proof is left as an exercise.

PROOF OF THEOREM 1

In this section, we shall prove Theorem 1. Throughout this section, we shall assume p is odd. The proof builds on the following lemmas.

Lemma 1: Suppose $Y_0 = 0$ and

$$Y_{n+1} = 2Y_n + b_n \pmod{p}$$

where b_n is as in Theorem 1. If $n > c_1 \log_2 p$ where $c_1 > 1$, then $||Q_n - U|| \to 0$ as $p \to \infty$ if Q_n is the probability distribution of Y_n.

Proof: If $n \geq 1$, then Y_n (viewed in $\mathbf{Z}$) is uniform on the odd integers from $-2^n + 1$ to $2^n - 1$. Since $2^n > p^{c_1}$ and $c_1 > 1$, the result follows by a straightforward consideration of these odd integers mod p. ∎

The next lemma is a property of random walks.

Lemma 2: Suppose $P(W_n = 1) = P(W_n = -1) = 1/2$. Let $V_n = \sum_{i=1}^{n} W_i$ (with $V_0 = 0$), let $M_n = \max_{i=0,...,n} V_i$, and let $m_n = \min_{i=0,...,n} V_0$. Given $c_1 > 0$, there exists a value c such that if $n > c(\log p)^2$ then $P(M_n - m_n \leq c_1 \log_2 p) < \epsilon/2$.

The proof of this lemma may be derived quickly from the Central Limit Theorem. ∎

Next observe that

$$X_1 = b_1$$

$$X_2 = a_2 b_1 + b_2$$

$$X_3 = a_3 a_2 b_1 + a_3 b_2 + b_3$$

...

Consider the sequence $a_{n+1}, a_{n+1}a_n, ..., a_{n+1}a_n...a_2$. Observe that this sequence can also be written as $2^{V_1}, 2^{V_2}, ..., 2^{V_n}$ where $V_1, ..., V_n$ are as in Lemma 2. If $j > 0$, 2^{-j} denotes $((p+1)/2)^j$ in the integers mod p since 2 is a unit in $\mathbf{Z}/p\mathbf{Z}$ and has multiplicative inverse $(p+1)/2$.

Suppose $a_1, ..., a_{n+1}$ are given such that $M_n - m_n > c_1 \log_2 p$ where the values M_n and m_n refer to the values $V_1, ..., V_n$ in the previous paragraph. Let $Z_{n+1} = a_{n+1}...a_2 b_1 + a_{n+1}...a_3 b_2 + ... + b_{n+1}$ be a random variable for these particular choices of $a_1, ..., a_{n+1}$ but with $b_1, ...b_{n+1}$ still being i.i.d. random variables with the same distribution as in Theorem 1. Let R_n be the distance of Z_n from uniform. Note that since p is odd, R_{n+1} is also the distance of $2^{-m_n} Z_{n+1}$ from the uniform. Note that

$$2^{-m_n} Z_{n+1} = \sum_{i=0}^{M_n - m_n} 2^i \tilde{b}_i + \sum_{i=M_n - m_n + 1}^{n} 2^{r(i)} \tilde{b}_i$$

where the values $r(i)$ are determined by $a_1, ..., a_{n+1}$ and $\tilde{b}_i$ are i.i.d. random variables with the same distribution of b_n. (The $\tilde{b}_i$'s are obtained from the b_i's by relabeling the subscripts.) Since $r(i)$ is determined and the $\tilde{b}_i$'s are i.i.d., Proposition 1 says that the distance of $2^{-m_n} Z_{n+1}$ from uniform is less than the distance of

$$\sum_{i=0}^{M_n - m_n} 2^i \tilde{b}_i$$

from uniform; the latter distance goes to to 0 as $p \to \infty$. Since $P(M_n - m_n \leq c_1 \log_2 p) < \epsilon/2$, we may thus conclude that $||P_n - U|| < \epsilon$ for large enough odd values of p. ∎

PROOF OF THEOREM 2

The technique illustrated by the proof of Theorem 1 is readily generalizable to other distributions for a_n and b_n. Theorem 2 proves a generalization, but the replacement for Lemma 1 is more complicated. Throughout this section, we assume p is odd.

Lemma 3: Suppose $Y_0 = 0$ and $Y_{n+1} = 2Y_n + b_n \pmod{p}$. Let Q_n be the probability distribution of Y_n. For some value $\tilde{c}_1 > 0$ if $n > \tilde{c}_1 \log p \log\log p$, then $||Q_n - U|| \to 0$ as $p \to \infty$. For some value $\tilde{c}_2 > 0$ if $n > \tilde{c}_2(\log p)$ then $||Q_n - U|| \to 0$ for almost all odd p. (Note $\tilde{c}_1$ and $\tilde{c}_2$ may depend on d, e, and f.)

This lemma is a straightforward generalization of Theorems 1 and 3 of Chung, Diaconis, and Graham (1987) and is left to the reader. ∎

Lemma 2 is replaced by the following lemma.

Lemma 4: Suppose $P(W_n = -1) = a$, $P(W_n = 0) = b$, and $P(W_n = 1) = c$ with a, b, and c as in Theorem 2. Let V_n, M_n, and m_n be obtained from $W_1, ..., W_n$ as in Lemma 2. Then given $\tilde{c}_1 > 0$ and $\tilde{c}_2 > 0$, there exist values $c_1 > 0$ and $c_2 > 0$ (which may depend on a, b, and c) such that $P(M_n - m_n \leq \tilde{c}_1 \log p \log\log p) < \epsilon/2$ if $n > c_1(\log p \log\log p)^g$ while $P(M_n - m_n \leq \tilde{c}_2 \log p) < \epsilon/2$ if $n > c_2(\log p)^g$ where g is as defined in Theorem 2.

This lemma can be shown from the Central Limit Theorem. ∎

The remainder of the proof of Theorem 2 is virtually identical to the last portion of the proof of Theorem 1 provided that one takes into account the two cases in Lemma 3. ∎

PROOF OF LOWER BOUNDS

The proof of Theorem 3 is straightforward in the case $g = 1$. Since there are no more than $9^{c_3 \log p} = p^{c_3 \log 9}$ possible values that X_n can have if $n = \lfloor c_3 \log p \rfloor$, then the set of also possible values of X_n will have probability under $p^{c_3 \log 9 - 1}$ under U. If $c_3 < 1/\log 9$, this implies that $||P_n - U|| \to 1$ as $p \to \infty$.

Next consider the case where $g = 2$. Define M_n and m_n as in the proof of Theorem 1. Let $\epsilon > 0$ be given. By elementary considerations from the Central Limit Theorem and a reflection principle, we can show that there exists a value $c_3 > 0$ such that if $n = \lfloor c_3(\log p)^2 \rfloor$, then $P(M_n - m_n > \lfloor (1/4)\log_2 p \rfloor) < \epsilon/2$. If $M_n - m_n \leq \lfloor (1/4)\log_2 p \rfloor$, consider

$$2^{\lfloor (1/4)\log_2 p \rfloor} X_{n+1} = (2^{\lfloor (1/4)\log_2 p \rfloor} a_n ... a_2)b_1 + (2^{\lfloor (1/4)\log_2 p \rfloor} a_n ... a_3)b_2 + ... + 2^{\lfloor (1/4)\log_2 p \rfloor} b_n.$$

Observe that since $((p+1)/2)2 \equiv 1 \pmod{p}$, then

$$2^{\lfloor (1/4)\log_2 p \rfloor} a_n ... a_2, 2^{\lfloor (1/4)\log_2 p \rfloor} a_n ... a_3, ..., 2^{\lfloor (1/4)\log_2 p \rfloor} \subseteq \left[1, 2^{2\lfloor (1/4)\log_2 p \rfloor}\right] \pmod{p}.$$

There are $\lfloor c_3(\log p)^2 \rfloor$ terms on the right. Mod p, each term is in the range

$$\left[-2^{2\lfloor (1/4)\log_2 p\rfloor}, 2^{2\lfloor (1/4)\log_2 p\rfloor}\right] \subseteq [-\sqrt{p}, \sqrt{p}].$$

Thus

$$2^{\lfloor (1/4)\log_2 p\rfloor} X_{n+1} \subseteq \left[-\sqrt{p}c_3(\log p)^2, \sqrt{p}c_3(\log p)^2\right] \pmod p.$$

Thus, for this choice of n,

$$\begin{aligned} ||P_n - U|| &> (1 - (\epsilon/2)) - \frac{1 + 2\sqrt{p}c_3(\log p)^2}{p} \\ &> 1 - \epsilon \end{aligned}$$

for sufficiently large p. ∎

Theorem 4 is a straightforward generalization of the previous theorem.

In some cases where b_n does not have a fixed distribution on $\mathbf{Z}$, similar claims may still be made:

Corollary: Let p be odd. If $P((a_n, b_n) = (2, 1)) = P((a_n, b_n) = ((p+1)/2, -(p+1)/2)) = 1/2$, then, given $\epsilon > 0$, there exists a values $c > 0$ such that if $n < c(\log p)^2$, then $||P_n - U|| > 1 - \epsilon$ for sufficiently large p.

Proof: X_n and $2X_n$ are the same distance from uniform. Since $2(-(p+1)/2) \equiv -1 \pmod p$, we have

$$2X_n = a_{n-1}...a_2\tilde{b}_1 + a_{n-1}...a_3\tilde{b}_2 + ... + \tilde{b}_{n-1} \pmod p$$

with $P((a_n, \tilde{b}_n) = (2, 2)) = P((a_n, \tilde{b}_n) = ((p+1)/2, -1)) = 0.5$. The previous theorem provides the lower bound for how long it takes for X_n to get close to uniform on $\mathbf{Z}/p\mathbf{Z}$.

PROOF OF THEOREM 5

The proof utilizes the upper bound lemma of Diaconis and Shahshahani. Let P be a probability on $\mathbf{Z}/p\mathbf{Z}$ and let

$$\hat{P}(k) = \sum_{j=0}^{p-1} P(j)q^{jk}$$

where $q := q(p) := e^{2\pi i/p}$. The expression $\hat{P}(k)$ is called the Fourier transform of P in $\mathbf{Z}/p\mathbf{Z}$. The upper bound lemma uses techniques from Fourier analysis to conclude

Lemma 5:

$$||P - U||^2 \leq \frac{1}{4}\sum_{k=1}^{p-1} |\hat{P}(k)|^2.$$

A generalization of this lemma is described and proved in Diaconis (1988).

The proof of the theorem shall use a recurrence relation among the Fourier transforms; a similar relation is used in Hildebrand (1990, 1993a, 1993b). The recurrence relation among the Fourier transforms will follow from the following lemma.

Lemma 6:

$$\begin{aligned} P(X_{n+1} = k) = &\frac{1}{4}P(X_n = ((p+1)/2)k - ((p+1)/2)) \\ &+ \frac{1}{4}P(X_n = ((p+1)/2)k + ((p+1)/2)) \\ &+ \frac{1}{4}P(X_n = 2k-1) + \frac{1}{4}P(X_n = 2k+1) \end{aligned}$$

The proof is straightforward and follows from the recurrence relation relating X_{n+1} to X_n.

The following lemma is similar to a recurrence in Hildebrand (1990,1993a):

Lemma 7:

$$\begin{aligned} \hat{P}_{n+1}(k) = &\frac{1}{4}\hat{P}_n(2k)q^k + \frac{1}{4}\hat{P}_n(2k)q^{-k} \\ &+ \frac{1}{4}\hat{P}_n(((p+1)/2)k)q^{((p+1)/2)k} + \frac{1}{4}\hat{P}_n(((p+1)/2)k)q^{-((p+1)/2)k} \end{aligned}$$

Proof: First observe that

$$\begin{aligned} \hat{P}_{n+1}(k) &= \sum_{j=0}^{p-1} P(X_{n+1} = j)q^{jk} \\ &= \sum_{j=0}^{p-1} \frac{1}{4}P(X_n = ((p+1)/2)j - ((p+1)/2))q^{jk} \\ &+ \sum_{j=0}^{p-1} \frac{1}{4}P(X_n = ((p+1)/2)j + ((p+1)/2))q^{jk} \\ &+ \sum_{j=0}^{p-1} \frac{1}{4}P(X_n = 2j-1)q^{jk} + \sum_{j=0}^{p-1} \frac{1}{4}P(X_n = 2j+1)q^{jk} \end{aligned}$$

Observe that the mapping from j to $((p+1)/2)j - ((p+1)/2)$ is a bijection on $\mathbf{Z}/p\mathbf{Z}$ since p is odd and since this mapping is the inverse of the bijection on $\mathbf{Z}/p\mathbf{Z}$ which sends j to

$2j+1$. Thus

$$\sum_{j=0}^{p-1} P(X_n = 2j+1)q^{jk} = \sum_{j=0}^{p-1} P(X_n = j)q^{(((p+1)/2)j-((p+1)/2))k}$$

$$= \sum_{j=0}^{p-1} P(X_n = j)q^{j(((p+1)/2)k)}q^{-((p+1)/2)k}$$

$$= \hat{P}_n(((p+1)/2)k)q^{-((p+1)/2)k}.$$

The other terms in the lemma follow similarly. ∎

The recurrence in Lemma 7 provides the key to the proof of Theorem 5. By Lemma 7, we conclude that

$$|\hat{P}_{n+1}(k)| \leq \frac{1}{2}|\cos 2\pi k/p|\ |\hat{P}_n(2k)| + \frac{1}{2}|\hat{P}_n(((p+1)/2)k)|.$$

Let $M_n = \max_{k\neq 0}|\hat{P}_n(k)|$. Observe that $M_0 = 1$ and that $M_{n+1} < M_n$ if $n \geq 0$. Also observe that if $k \in S := ((1/8)p, (3/8)p) \cup ((5/8)p, (7/8)p) \pmod p$ then $|\hat{P}_{n+a}(k)| \leq 0.8M_n$ for $a = 1, 2, 3, ...$ Thus we may claim that

$$|\hat{P}_{n+1}(k)| \leq \frac{1}{2}f(k)|\hat{P}_n(2k)| + \frac{1}{2}|\hat{P}_n(((p+1)/2)k)| \qquad (*)$$

where $f(k) = 0.8$ if $k \in S$ and $f(k) = 1$ of $k \notin S$.

Note that if $k \neq 0$ and $k \notin S$, then $2^b k \in S \pmod p$ for some value $b \leq (\log_2 p)$. Since p is odd, we may view $(p+1)/2$ as 2^{-1} in the multiplicative group of the units of $\mathbf{Z}/p\mathbf{Z}$. Let $d = \lfloor c_1(\log p)^2 \rfloor$. By $|\hat{P}_{n+d}(k)|$ by expanding $(*)$ recursively d levels. Define W_i and V_i as in Lemma 2. By the Central Limit Theorem, $V_n > \log_2 p$ with probability at least $c_2 > 0$. Thus at least the fraction c_2 of the terms will have a multiple of 0.8 coming from the $f(k)$ term in (*). Thus for all n and $k \neq 0$,

$$|\hat{P}_{n+d}(k)| \leq (0.8c_2 + 1(1-c_2))M_n.$$

Observe that that $c_3 := .8c_2 + (1-c_2) < 1$ and that $M_{n+d} \leq c_3 M_n$. Thus for some value $c_4 > 0$, $M_{d\lfloor c_4 \log p \rfloor} \leq c_3^{\lfloor c_4 \log p \rfloor} \leq 1/p^2$ for large enough p. By the upper bound lemma, if $n = d\lfloor c_4 \log p \rfloor$, then $||P_n - U||^2 < (1/4)((p-1)/p^2) \to 0$ as $p \to 0$. ∎

PROBLEMS FOR FURTHER STUDY

Theorems 2 and 3 do not provide sharp bounds on the time it takes for X_n to get close to uniform; the bounds differ by a factor of a constant times $(\log\log p)^g$. This time may vary

by such a factor; such variation appears in results proved in Chung, Diaconis, and Graham (1987) and Hildebrand (1993b). If $g = 2$, it is unclear whether such variation will hold, and this uncertainty provides a subject for further study.

Upper bounds for cases where a_n and b_n are dependent need improvement. Both generalizations of the method used in proving Theorem 5 and improvements of the upper bound in Theorem 5 form natural further problems worth studying.

The techniques in this paper can be extended to cases where a_n is either a, 1, or the multiplicative inverse of a if p and a are relatively prime. What happens if a_n takes on a broader range of values? For instance, what happens if a_n takes on the values 1, 2, 3, and the multiplicative inverses of 2 and 3 with certain probabilities?

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