A PROBABILITY INEQUALITY FOR LINEAR COMBINATIONS OF BOUNDED RANDOM VARIABLES*

by

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ABSTRACT

Let $Y_1, \ldots, Y_n$ be independent random variables with mean zero such that $|Y_i| \leq i$, $i = 1, \ldots, n$, and let $\theta_1, \ldots, \theta_n$ be real numbers satisfying $\sum_1^n \theta_i^2 = 1$. Set $T_n(\theta) = \sum_1^n \theta_i Y_i$ and let $\varphi(x) = (2\pi)^{-\frac{1}{2}}\exp[-\frac{1}{2}x^2]$.

Theorem.

For $\alpha > 0$, and for all $\theta_1, \ldots, \theta_n$,

$$P\{|T_n(\theta)| \geq \alpha\} \leq 2 \inf_{0 \leq u \leq \alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3} \varphi(x)dx \leq 12 \frac{\varphi(\alpha)}{\alpha} \inf_{0 \leq \delta \leq \alpha^2} \frac{\exp[\delta/2(2-\delta/\alpha^2)]}{\delta^3(1-\delta/\alpha^2)^4}.$$

1. Introduction.

Let $U_1, \ldots, U_n$ be independent random variables with $P\{U_i = 1\} = P\{U_i = -1\} = \frac{1}{2}$, $i = 1, \ldots, n$. Further, let $\mathfrak{F}_1$ be the class of functions $f: R \to R$ such that (i) f is symmetric and has a derivative f' and (ii) $\frac{1}{t}[f'(t+\Delta) - f'(-t+\Delta)]$ is non-decreasing in $t > 0$ for each $\Delta \geq 0$. As in Eaton (1970), set $T_n(\theta) = \sum_1^n \theta_i U_i$ where $\theta_1, \ldots, \theta_n$ are real numbers and $\sum_1^n \theta_i^2 = 1$. With $T_n \equiv \frac{1}{\sqrt{n}} \sum_1^n U_i$, we have

Proposition 1.

For each $f \in \mathfrak{F}_1$,

$$\mathcal{E}f(T_n(\theta)) \leq \mathcal{E}f(T_n) \leq \mathcal{E}f(T_{n+1}) \tag{1.1}$$

for $n = 1, 2, \ldots$.

Proof:

See Eaton (1970).

Proposition 2.

If $f \in \mathfrak{F}_1$ and if there exists a $\delta > 0$ and a constant M such that $\mathcal{E}|f(T_n)|^{1+\delta} \leq M$ for all n, then

$$\mathcal{E}f(T_n) \leq \mathcal{E}f(Z) \tag{1.2}$$

where Z has a unit normal distribution.

Proof:

See Eaton (1970).

The purpose of this paper is to use (1.1) and (1.2) to obtain an upper bound for $P\{|T_n(\theta)| \geq \alpha\}$ for $\alpha > 0$. Consider an $f \in \mathfrak{F}_1$ so that (1.2) holds, and so that $f \geq 0$ and $f(x) \geq 1$ if $|x| \geq \alpha$. It follows immediately, using (1.1) and (1.2), that

$$P\{|T_n(\theta)| \geq \alpha\} \leq \mathcal{E}f(T_n(\theta)) \leq \mathcal{E}f(T_n) \leq \mathcal{E}f(Z). \tag{1.3}$$

Now, to derive a probability bound, we would like to minimize the right hand side of (1.3) for all functions f for which (1.3) is valid. However, the class $\mathfrak{F}_1$ is rather difficult to describe in a manner which allows the minimization of $\mathcal{E}f(Z)$. The following lemma gives a useful sufficient condition for a symmetric function f to be in $\mathfrak{F}_1$.

Lemma 1.

Suppose $f: R \to R$ is symmetric, f''' exists and $f'''(x)$ is non-decreasing for $x \geq 0$. Then $f \in \mathfrak{F}_1$.

Proof:

For $t > 0$ and $\Delta \geq 0$

$$f'''(t+\Delta) - f'''(-t+\Delta) \geq 0$$

so

$$t[f'''(t+\Delta) - f'''(-t+\Delta)] + f''(t+\Delta) + f''(-t+\Delta) \geq f''(t+\Delta) + f''(-t+\Delta).$$

Hence

$$\frac{d}{dt}[t(f''(t+\Delta) + f''(-t+\Delta))] \geq \frac{d}{dt}[f'(t+\Delta) - f'(-t+\Delta)] .$$

Therefore

$$t[f''(t+\Delta) + f''(-t+\Delta)] \geq f'(t+\Delta) - f'(-t+\Delta) .$$

But

$$\frac{d}{dt}\left[\frac{f'(t+\Delta) - f'(-t+\Delta)}{t}\right] = \frac{t[f''(t+\Delta) + f''(-t+\Delta)] - [f'(t+\Delta) - f'(-t+\Delta)]}{t^2} \geq 0.$$

Thus $f \in \mathfrak{F}_1$ and the proof is complete.

2. The Basic Inequality.

To obtain a probability inequality for $P\{|T_n(\theta)| \geq \alpha\}$, fix $\alpha > 0$ and let $\mathfrak{F}_\alpha$ denote the class of functions f which are symmetric and satisfy

$$(2.1)\quad \begin{cases} f(x) = \frac{1}{3!}\int_0^x (x-u)^3 dF(u)\ , \quad x \geq 0 \\ f(\alpha) = \frac{1}{3!}\int_0^\alpha (\alpha-u)^3 dF(u) = 1 \end{cases}$$

Here, F is a non-decreasing function on $[0, \infty)$ with $F(0) = 0$ and $F(+\infty) < +\infty$. Define $(\cdot)_+$ by $(v)_+ = v$ if $v \geq 0$ and $(v)_+ = 0$ if $v < 0$. Then, $f \in \mathfrak{F}_\alpha$ iff

$$(2.2) \qquad \begin{cases} f(x) = \frac{1}{3!}\int_0^\infty [(|x| - u)_+]^3 dF(u) \; ; \; x \in R \\ f(\alpha) = 1 \end{cases}$$

<u>Proposition 3</u>.

If $f \in \mathfrak{F}_\alpha$, then

$$(2.3) \qquad P\{|T_n(\theta)| \geq \alpha\} \leq \mathcal{E}f(T_n(\theta)) \leq \mathcal{E}f(Z)$$

where Z is $N(0, 1)$.

<u>Proof:</u>

Since $f'''(x) = F(x)$, $x \geq 0$, $f'''(x)$ is non-decreasing for $x > 0$. By Lemma 1, $f \in \mathfrak{F}_1$. Further, $f'(x) = \frac{1}{2}\int_0^x (x-u)^2 dF(x) \geq 0$ for $x \geq 0$ so $f(x)$ is increasing for $x \geq 0$. Since $f(\alpha) = 1$, $f(x) \geq 1$ if $|x| \geq \alpha$. Combining the above and applying Prop. 1, we have

$$(2.4) \qquad P\{|T_n(\theta)| \geq \alpha\} \leq \mathcal{E}f(T_n(\theta)) \leq \mathcal{E}f(T_n).$$

But,

$$(2.5) \qquad \mathcal{E}|f(T_n)|^2 = |\frac{1}{3!}\int_0^\infty [(|T_n| - u)_+]^3 dF(u)|^2 \leq \mathcal{E}[\frac{1}{3!}|T_n|^3 F(+\infty)]^2$$

$$= \left(\frac{F(+\infty)}{6}\right)^2 \mathcal{E}T_n^{\,6} \leq M$$

for some constant M and for all n. By Prop. 2, $\mathcal{E}f(T_n) \leq \mathcal{E}f(Z)$. This completes the proof.

From the above proposition, we have

$$(2.6) \qquad P\{|T_n(\theta)| \geq \alpha\} \leq \inf_{f \in \mathfrak{F}_\alpha} \mathcal{E}f(Z).$$

Proposition 4.

For $\alpha > 0$,

$$(2.7)\qquad \inf_{f\in\mathfrak{F}_\alpha} \mathcal{E}f(Z) = 2 \inf_{0\le u\le\alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3}\varphi(x)dx.$$

Proof:

For $\alpha > 0$,

$$(2.8)\qquad \inf_{f\in\mathfrak{F}_\alpha} \mathcal{E}f(Z) = 2 \inf_{f\in\mathfrak{F}_\alpha} \frac{1}{3!}\int_0^\infty\int_0^\infty [(x-u)_+]^3 dF(u)\varphi(x)dx$$

$$= 2 \inf_F \frac{1}{3!}\int_0^\infty w(u)dF(u)$$

where F is non-decreasing, $F(+\infty) < +\infty$, $\frac{1}{3!}\int_0^\alpha (\alpha-u)^3 dF(u) = 1$ and $w(u) \equiv \int_0^\infty [(x-u)_+]^3\varphi(x)dx$. But

$$(2.9)\qquad 2 \inf_F \frac{1}{3!}\int_0^\infty w(u)dF(u) \ge 2 \inf_F \int_0^\alpha \frac{w(u)}{(\alpha-u)^3}\,\frac{(\alpha-u)^3}{3!}\,dF(u) \ge 2 \inf_{0\le u\le\alpha} \frac{w(u)}{(\alpha-u)^3}.$$

However, it is easy to see that one has equality in both of the inequalities in (2.9) since a choice of F can be made which gives equality. Since $w(u) = \int_u^\infty (x-u)^3\varphi(x)dx$, (2.7) holds.

Theorem 1.

For $\alpha > 0$,

$$(2.10)\qquad P\{|T_n(\theta)| \ge \alpha\} \le 2 \inf_{0\le \hat{u}\le\alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3}\varphi(x)dx.$$

Proof:

This follows immediately from (2.6) and Prop. 4.

The explicit minimization of the right hand side of (2.10) has not been accomplished. The following gives some upper bounds for this minimum.

$$(2.11)\qquad H(\alpha,u) \equiv \int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3}\varphi(x)dx = \int_0^\infty \frac{x^3}{(\alpha-u)^3}\ \varphi(x+u)dx$$

$$= \frac{\varphi(\alpha)}{\alpha}\ \frac{\alpha}{(\alpha-u)^3}\ e^{-\frac{1}{2}(u^2-\alpha^2)} \int_0^\infty x^3 e^{-ux} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{\varphi(\alpha)}{\alpha}\ \frac{\alpha}{u^4(\alpha-u)^3}\ e^{-\frac{1}{2}(u^2-\alpha^2)} \int_0^\infty x^3 e^{-x} e^{-\frac{1}{2}(x^2/u^2)} dx\ .$$

Set $u = \alpha - \frac{\delta}{\alpha}$ for $0 \leq \delta \leq \alpha^2$ so

$$(2.12)\qquad H(\alpha,u) = \frac{\varphi(\alpha)}{\alpha}\ \frac{e^\delta}{\delta^3}\ \frac{e^{-\frac{1}{2}(\delta^2/\alpha^2)}}{(1-\frac{\delta}{\alpha^2})^4} \int_0^\infty x^3 e^{-x} e^{-\frac{1}{2}(x^2/u^2)} dx\ .$$

Now, $\frac{e^\delta}{\delta^3}$ is minimized by setting $\delta = 3$ and $\int_0^\infty x^3 e^{-x} e^{-\frac{1}{2}(x^2/u^2)} dx \leq \int_0^\infty x^3 e^{-x} dx = 6.$ Thus, for $\alpha > \sqrt{3}$

$$(2.13)\qquad \inf_{0\leq u\leq\alpha} H(\alpha,\ u) \leq \frac{6e^3}{27}\ \frac{\varphi(\alpha)}{\alpha}\ \frac{e^{-\frac{1}{2}(9/\alpha^2)}}{(1-\frac{3}{\alpha^2})^4}\ .$$

Corollary 1.

For $\alpha > \sqrt{3}$,

$$(2.14)\qquad P\{|T_n(\theta)| \geq \alpha\} \leq \frac{12e^3}{27}\ \frac{\varphi(\alpha)}{\alpha}\ \frac{e^{-\frac{1}{2}(9/\alpha^2)}}{(1-\frac{3}{\alpha^2})^4}$$

for all $\theta_1,\ldots,\theta_n$ and $n = 1,2,\ldots.$

It is easy to show that $\frac{e^{-\frac{1}{2}(9/\alpha^2)}}{(1-\frac{3}{\alpha^2})^4}$ is a decreasing function of α for $\alpha > \sqrt{3}$. Thus, we have

Corollary 2.

For $\alpha \geq \alpha_0 > \sqrt{3}$, let $K = K(\alpha_0) = \frac{12e^3}{27}\ \frac{e^{-\frac{1}{2}(9/\alpha_0^2)}}{(1-\frac{3}{\alpha_0^2})^4}$. Then

$$(2.15)\qquad P\{|T_n(\theta)| \geq \alpha\} \leq K\,\frac{\varphi(\alpha)}{\alpha}\ .$$

The estimates used to derive (2.14) and (2.15) are quite crude. Some numerical work indicates that for all $\alpha > \sqrt{2}$, $\inf_{0\leq u\leq\alpha} H(\alpha,\ u) \leq \frac{6e^3}{27}\ \frac{\varphi(\alpha)}{\alpha}$. However, a proof of this inequality has not yet been constructed.

3. <u>An Extension to Bounded Random Variables</u>.

It was shown by the author (Eaton (1972)) that the inequality of Theorem 1 was valid for any independent <u>symmetric</u> random variables $X_1,\dots, X_n$ such that $|X_i| \leq 1$, $i = 1,\dots, n$ and $\tilde{T}_n(\theta) \equiv \sum_1^n \theta_i X_i$, $\Sigma\theta_i^2 = 1$. After the appearance of this result, W. Hoeffding informed the author that an alternative argument could be used to establish the validity of Theorem 1 for independent random variables $Y_1,\dots, Y_n$ such that $\mathcal{E}Y_i = 0$, $|Y_i| \leq 1$ for $i = 1,\dots, n$. It is this elegant argument which is presented in this section.

As above, let $Y_1,\dots, Y_n$ be independent random variables with $\mathcal{E}Y_i = 0$ and $|Y_i| \leq 1$, $i = 1,\dots, n$. The following lemma due to G. A. Hunt (1955) is needed.

<u>Lemma 2</u>.

Suppose $g: \prod_{i=1}^n [-1,1] \to R$ where g is convex in each argument when the remaining $n - 1$ arguments are held fixed. Then

$$(3.1) \qquad \mathcal{E}g(Y_1,\dots, Y_n) \leq \mathcal{E}g(U_1,\dots, U_n).$$

Now, let $\theta_1,\dots, \theta_n$ be real numbers such that $\Sigma\theta_i^2 = 1$ and set $S_n(\theta) = \sum_i^n \theta_i Y_i$ and $T_n(\theta) = \sum_1^n \theta_i U_i$. For $u \geq 0$, define $f_u: R \to [0, \infty)$ by

$$(3.2) \qquad f_u(x) = [(|x| - u)_+]^3 .$$

<u>Theorem 2</u>.

For each $\alpha > 0$,

$$(3.3) \qquad P\{|S_n(\theta)| \geq \alpha\} \leq 2 \inf_{0 \leq u \leq \alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3}\, \varphi(x)dx .$$

<u>Proof</u>:

For $0 \leq u < \alpha$, it is clear that

$$(3.4) \qquad P\{|S_n(\theta)| \geq \alpha\} \leq \frac{\mathcal{E}f_u(S_n(\theta))}{(\alpha-u)^3}$$

since $f_u \geq 0$ and $\frac{f_u(x)}{(\alpha-u)^3} \geq 1$ if $|x| \geq \alpha$. But $g(Y_1,\ldots, Y_n) \equiv \mathcal{E}f_u(\sum_1^n \theta_i Y_i)$ satisfies the assumption of Lemma 2. Thus $\mathcal{E}f_u(S_n(\theta)) = \mathcal{E}f_u(\Sigma\theta_i Y_i) \leq \mathcal{E}f_u(\Sigma\theta_i U_i) = \mathcal{E}f_u(T_n(\theta))$. Using Propositions 1 and 2 on f_u, we have

$$(3.5) \qquad \mathcal{E}f_u(S_n(\theta)) \leq \mathcal{E}f_u(T_n(\theta)) \leq \mathcal{E}f_u(T_n) \leq \mathcal{E}f_u(Z).$$

Combining (3.4) and (3.5) yields

$$(3.6) \qquad P\{|S_n(\theta)| \geq \alpha\} \leq \frac{\mathcal{E}f_u(Z)}{(\alpha-u)^3}$$

for $0 \leq u < \alpha$. Thus,

$$(3.7) \qquad P\{|S_n(\theta)| \geq \alpha\} \leq \inf_{0\leq u\leq\alpha} \frac{\mathcal{E}f_u(Z)}{(\alpha-u)^3} = 2 \inf_{0\leq u\leq\alpha} \int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3}\varphi(x)dx .$$

This completes the proof.

Corollary 3.

Corollaries 1 and 2 are valid with $T_n(\theta)$ replaced by $S_n(\theta)$.

Proof:

This is clear from the discussion in Section 2.

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