PERIODIC SURFACES WHICH ARE EXTREMAL FOR ENERGY FUNCTIONALS CONTAINING CURVATURE FUNCTIONS

By

Johannes C.C. Nitsche

IMA Preprint Series # 785

March 1991

PERIODIC SURFACES WHICH ARE EXTREMAL FOR ENERGY FUNCTIONALS CONTAINING CURVATURE FUNCTIONS

JOHANNES C.C. NITSCHE*

Summary. The following investigation is an expanded version of the author's lecture at the IMA-Workshop "Statistical Thermodynamics and Differential Geometry of Micro-Structured Material", January 23, 1991[1].

After an expository introduction aimed at setting the subject into proper mathematical perspective, a proof is presented for the existence of triply periodic surfaces (interfaces) $\mathcal{S}$ embedded in R^3, subject to a volume constraint and stationary for a free energy functional $\mathcal{E} = \int\int_{\mathcal{S}} \Phi(H,K)dA$. Here H and K represent mean and Gaussian curvatures of $\mathcal{S}$, respectively, and dA denotes the area element. The details of the construction are carried out for the functional

$$\Phi(H,K) = \alpha + \beta H^2 - \gamma K\,. \tag{1}$$

More general integrands amenable to our method include

$$\Phi(H,K) = \Psi(H) - \gamma K\,, \tag{2}$$

given suitable structural conditions for the term $\Psi(H)$, and

$$\Phi(H,K) = \alpha + \beta H^2 - \gamma K + \delta K^2\,, \tag{3}$$

for small values of the elastic module δ. It turns out that minimal surfaces appear as members ('center surfaces' $\mathcal{S}_0$) of surface families governed by these energy functionals, very much in the same way in which they play this role vis-à-vis the surfaces of constant mean curvature in connection with the integrand $\Phi = 1$.

The surfaces of interest will be constructed from *surface patches* with the help of a general reflection principle applicable to the complemented fourth order Euler-Lagrange equations associated with (1), (2), (3). The existence proof for the latter requires methods from the theory of partial differential equations: derivation of a lower bound for the eigenvalue μ_2 of a specific second boundary value problem, Schauder estimates for the solutions of a Neumann problem in domains with corners, coupled with an alternating iteration technique.

It is hoped that the method of proof can be extended to include more general classes of free energy functionals. Also, it might be possible in a similar fashion to use periodic surfaces of constant mean curvature $H = H_0$ (value of the spontaneous curvature) as starting surfaces for the construction of periodic solutions for the Euler-Lagrange equations attached to W. Helfrich's energy functional with the integrand $\Phi(H,K) = \beta(H-H_0)^2+\gamma K$ and to other similar functionals.

---

*School of Mathematics, University of Minnesota, Minneapolis, MN 555455

[1]The original abstract appeared in the IMA Newsletter # 172, p. 6.

1. In the beginning, there were minimal surfaces — locally area minimizing structures for which, as J.B.M.C. Meusnier discovered in 1776, "the principal curvature radii are everywhere of opposite sign & equal."[2] The term mean curvature H for the arithmetic mean $1/R_1 + 1/R_2$ of the principal curvatures was subsequently suggested by Sophie Germain, 1831. The eminent physical significance of this quantity was recognized in 1805 and 1806 by T. Young and P.S. Laplace, respectively, in their studies of capillary phenomena which were later the subject of J.A.F. Plateau's celebrated experimental and theoretical investigations: The pressure difference at an interface is proportional to the mean curvature: $\Delta p \equiv 2\sigma H$, ($\sigma$ = surface tension). This has led to the simple macroscopic mathematical model — the limit case of a deeper interpretation considering all the masses as domain for interatomic and intermolecular cohesive forces — which assigns to an interface separating a liquid from the surrounding medium an energy $\mathcal{E}$ proportional to the surface area of the interface: $\mathcal{E} = \int\int dA$; see, e.g. [33]. The interface, considered as a lamina of negligible thickness, i.e. a sharp surface separating phases immune from the influence of bulk energy contributions and long-range interactions, appears thus as solution of the variational problem $\delta\mathcal{E} = 0$, that is, a minimal surface satisfying the equation $H = 0$: a partial differential equation of second order for the position vector of the surface in question. Experiments with soap films support the theory. If the surface is subject to a volume constraint, this differential equation becomes $H = c$. The constant c which is related, though not necessarily in a bijective way, to the volume fractions of the phases, leads to families of surfaces. In the presence of other forces, further terms may enter the expression for the free energy.

Much of the general interest in minimal surfaces stems from the aforementioned relations and, indeed, from their often striking similarity to the real interfaces, separating membranes and potential surfaces which, as has been pointed out, time and again, by many observers, are so abundant in nature, science and even (inspired by nature, but man-made) in architecture: labyrinthic structures found in botany and zoology, in equipotential and "Fermi"[3] surfaces, sandstone and other porous media, polymer blends, microemulsions and liquid crystals, to mention just a few.[4] Embedded infinite periodic minimal surfaces, simply periodic, doubly periodic and particularly triply periodic surfaces ("bicontinuous" structures, as defined by L.E. Scriven [49]; for further references, see [1],[2]) have recently attracted the attention of engineers and scientist as potential global models for the real interfaces and separating membranes in real materials and fluids, notwithstanding the basic shortcomings of the concept of periodicity for models in material science, compared with those incorporating more disordered structures.

Given that the sharp interfaces amenable to treatment with the tools of differential geometry are only approximations to the actual narrow zones separating phases (see, e.g.

[2] For an exposition of the history of minimal surfaces, see [37].

[3] i.e. surfaces of constant energy in the space of wave-vector components, developed in the mid-to-late twenties by Wolfgang Pauli and Arthur Sommerfeld; see e.g. [51], [18], [29], pp. 217-252, [37], A.8.12,[42].

[4] Regarding a descriptions of the "evidence" and further references, see e.g. [37], §279, A.8.12, A.8.32, [1],[2],[19],[20],[32],[39],[53].

[8],[22],[23],[45]), it must be asked to what extent minimal surfaces or surfaces of constant mean curvature — extremals for a variational problem based on the simple energy functional $\mathcal{E}$ above — are actually "implicated" in the various engineering applications. In many instances, a modification of the functional for the free surface energy incorporating curvature terms will prove to be appropriate. In other cases, e.g. for equipotential and Fermi surfaces which are often quite similar in appearance, topology and symmetry properties to periodic minimal surfaces, there seems to be at best a tenuous connection on physical grounds.[5] If the interfaces under scrutiny only "look like" minimal surfaces, they forfeit their right to all the special properties with which the latter are exclusively endowed: intrinsic geometric characteristics, analytic extendibility, with all the global constraints it entails, etc.

To illustrate the profound consequences of even extremely slight changes in appearance, consider the case of the so-called Schwarz P-surface[6] discussed below and depicted in figure 5. This surface intersects the planes of symmetry at right angles in nearly circular contours.[7] In fact, the maximum deviation of these contours from a circle, measured radially, is 0.479%. (The difference of curvatures amounts to 0.725%.) Despite this experimentally negligible difference of a mere $\frac{1}{2}\%$, its consequences are considerable: The only minimal surface with exact circles is the catenoid. Already H.A. Schwarz ([47], pp. 49, 116, 121-3) had commented on the agreement in shape between his solution of Plateau's problem through four sides of a regular tetrahedron and the parabolic hyperboloid with the same boundary. (The areas of the two surfaces differ by 0.12%; however, for a tetrahedron of side 1, the mean curvature of the hyperboloid becomes as large as $\sqrt{8/27} = 0.544$.) Similar examples of the extreme local closeness between periodic minimal surfaces and selected other surfaces abound. The scientific literature is replete with visualisations which are expressly identified as approximations (graphic, trigonometric, etc.) to proposed interfaces, justified or not, and even though the existence of the latter structures, as stationary global solutions to an appropriate complemented Euler-Lagrange equation, has often not yet been established. To cite two examples: one must consider the widely discussed trigonometric expressions

$$\cos(2\pi x) + \cos(2\pi y) + \cos(2\pi z) = 0 \tag{4}$$

[5] The Sommerfeld-Bethe article [51] in the *Handbuch der Physik* of 1933 contains drawings for simple crystal structures one of which, on p. 400, has conspicuous similarities with the so-called Schwarz P-surface. This picture has ben often reproduced, e.g. in [18], p. 7, [29], p. 228, [37], A.8.12. An interesting aside: These pictures were designed fifty years before the advent of computer graphics. Sommerfeld and Bethe had commissioned them from Rudolf Rühle, a foremost graphics expert who had earlier also computed and drawn the curves and function reliefs for the *Tables of Elementary Functions* by E. Jahnke and F. Emde.

[6] The qualifier "so-called" expresses the observation that a systematic nomenclature for all periodic surfaces of interest to scientists, and agreed on by all, has apparently not yet been developed.

[7] For explicit formulas, in terms of hyperelliptic integrals, see [36], p. 4.

and

$$\sin(2\pi x)\cos(2\pi y)+\sin(2\pi y)\cos(2\pi z)+\sin(2\pi z)\cos(2\pi x)=0 \tag{5}$$

as daring approximations — made under the spell of the beauty of minimal surfaces — to the so-called P-surface and A. Schoen's gyroid, respectively. The surface (4) has principal curvatures $\kappa_1 = 2\pi/\sqrt{6} = 2.565\ldots$ and $\kappa_2 = -4\pi/\sqrt{6} = 5.130\ldots$, and therefore mean curvature $H = -\pi/\sqrt{6} = 1.282\ldots$ in the points $(1/3, 1/3, 0)$, $(1/3, 0, 1, 3)$ and $(0, 1/3, 1/3)$[8]; see [5]. The surface (5) carries points in which $\mid H \mid$ exceeds the value 0.17. Thus the caveat incorporated in [37], p. xiii, is certainly in order :

> The material structure and the interaction of forces, on a *microscale,* which must provide the foundation leading to the various macroscopic energy expressions put forth in the literature are not fully understood today ... . Although it is enticing and exhilarating to recognize in electron microscope plates seemingly familiar shapes, e.g., periodic minimal surfaces, or singularity formations, there is often no firm theoretical basis at all to implicate such shapes and formations with certainty, particularly since specific terms appearing in the various macroscale energy expressions are frequently manipulated quite casually by their creators. Surely, the assertion that 'theoretical' (to boot, mostly numerical) data are in good agreement with specific experimental observations is, by itself, not sufficient justification for the validity of a comprehensive theory.

Does it matter whether the interfaces in real materials are merely close (and then, in what norm: $C^0, C^1, C^2, \ldots$?) to specific classes of differential geometric surfaces? The practitioner may judge[9]. But this point must be made: Without a justification based on fundamental principles of physics and chemistry, the predictive and modelling power derived from the theory of our extremal surfaces, as well as from the new shapes emerging in the geometric literature and appealing to the scientist, however convincing in many instances, will remain purely accidental.

Regarding useful forms for the energy functional fitting the specific requirements in practical applications, the mathematician depends on guidance from engineers, physicists

---

[8]All minimal surfaces with a representation of the form $f(x)+g(y)+h(z)=0$ have been determined in classical investigations by J. Weingarten and M. Fréchet; see e.g. [37], II.5.2. The surface (4) is not one of them. It is not clear whether the search for nonplanar minimal surfaces with a representation $f(x)g(y)+f(y)g(z)+f(z)g(x)=0$, or other similar representations suggested by the trigonometric approximations, would yield useful results. But who knows? Schwarz's minimal surface through four sides of a regular tetrahedron which is the surface patch for a triply periodic surface embedded in R^3 (historically, the very first) has a representation $f(x)f(y)+f(y)f(z)\ f(z)f(x)=-1$; see [37], p. 73. For specific surfaces given by an explicit representation in terms of concrete functions, it is a simple matter to compute the mean curvature with the help of the formulae of differential geometry ([37], pp. 52-3).

[9]The preface to [37] contains on p. xvi also the Platonic query: Are minimal surfaces constructs of the intellect, or do they exist in reality — creations or discoveries?

and chemists. Of course, there are also extensive theories relying on the stipulation of axioms, constitutive equations etc. . It is not clear so far to what extent these elaborate and more formal approaches have succeeded in applications to any concrete model problem[10].

2. While doubts may thus remain for the engineer concerning the appropriate form of the underlying energy functional for various concrete settings, it seems reasonable, from the mathematical point of view, to investigate specific classes of such functionals, in regard to the general properties of stationary solutions , to the form of vesicles, i.e. closed fluid membranes [17], and especially to the shape and the *existence* of periodic extremal surfaces. This is also the purpose of the following discussions. Interestingly, it turns out that minimal surfaces (but not necessarily surfaces of nonvanishing constant mean curvature) will retain their central role — however, so to speak, at a higher level — in the more general context considered here.

For a specific free energy functional with integrand (16) which is quadratic in the principal curvatures, we shall prove the existence of triply periodic extremal solution surfaces — sharp interfaces — embedded in R^3. In order to avoid overcharging the exposition, the technical details are carried out here for the case of surfaces of the topological type of the so-called Schwarz P-surface, undoubtedly among the most favored examples today. However, the method of proof applies equally to the cases of other topological types known for minimal surfaces. It is an extension of the method developed by the author for the existence proof of embedded triply periodic surfaces of constant mean curvature (short: H-surfaces). The specifics of this method have been described in his lecture [39][11]. Other recent existence proofs for periodic H-surfaces are those of H. Karcher [26] who utilizes a relation between periodic minimal surfaces in the sphere S^3 and surfaces of constant mean curvature in R^3 discovered by H.B. Lawson [31] and of N. Kapouleas [25] whose general construction is also applicable to certain disordered H-surfaces.

It is hoped that more general energy functionals involving the surface curvatures, particularly those which have shown physical promise[12] and others for which the integrand $\hat{\Phi}(\kappa_1, \kappa_2)$ is either a polynomial of degree greater than two or a general function subject to appropriate structural conditions, as for instance the conditions (10), (11) will be amenable to the methods developed here. Also our reflection principle (section 9 below) proves to be flexible and can be applied to more general constructions than those pursued here. In this sense, the present investigation should be considered only as a first step. Moreover,

[10] One is reminded of T. Wolfe's remarks concerning the development of U.S. architecture in the twentieth century ([56], pp. 27, 29): "... there came into being another unique phenomenon: the famous architect who did little or no building. ... Le Corbusier. ... He showed everybody how to become a famous architect without building buildings."

[11] Berlin, Sept. 1987. The author has also expounded his proof in lectures at several institutions, including the University of California, San Diego, Jan. 1986, and the University of Massachusetts, Nov. 1987.

[12] These include the Helfrich functional [17] with $\Phi(H, K) = c_1(H - H_0)^2 + c_2 K$ and the Hyde-Barnes-Ninham functional [20],[21], with $\Phi(H, K) = c_1(H - H_0)^2 + c_2(K - K_0)^2$, see further, e.g. [3].

it would seem highly desirable to complement the theoretical discussions presented here with numerical computations and graphic representations, in the same way in which D.M. Anderson's work [1], [2] has greatly helped the visualization of periodic H-surfaces. The author has not yet "seen" the extremal surfaces whose existence is proved in this note.

The general role possibly played by the curvatures in the present context has been paraphrased by H.G. v. Schnering and R. Nesper in this way ([53], p.1059):

> ... It appears that, in a very universal sense, the adaptation of structures to a collective order finds a natural solution through curvature.

This observation is of course more of a philosophic nature. It must be our goal to come to a quantitative understanding of the phenomena — from the physical-chemical foundations at the molecular level to the macroscopic mathematical existence proofs. There is no doubt that our subject transcends the confines of special disciplines and calls for collaborative efforts. The hope for such efforts and for new impulses had also been expressed by the author, coorganizer, with H.T. Davis, of the IMA-Workshop "Statistical Thermodynamics and Differential Geometry of Micro-Structured Material, January 21-25, 1991[13].

Playing with soap bubbles is probably as old as mankind itself and, existing or not — artists are not constrained by such considerations — the multifarious shapes of minimal surfaces and of related structures have captured the imagination of painters, sculptors and architects over the centuries and let them transform geometry into art ([35], §§6, 697, [37], §§6, 7). From the subject of periodic surfaces, only one of many chapters in the theory, figures 1 and 2 give two examples.

Figure 1: Man Ray Surface Réglée (Othello), 1936[14]

[13] His closing remarks contained this prediction: A whole new generation of chemical engineers is grappling now with the rudiments of differential geometry, while mathematicians make serious attempts at better comprehending the physical principles which govern interfacial phenomena. Soon, using the words of Professor Higgins, they will constitute two groups separated only by one common language.

[14] By courtesy of the Eugenia Cucalón Gallery, New York.

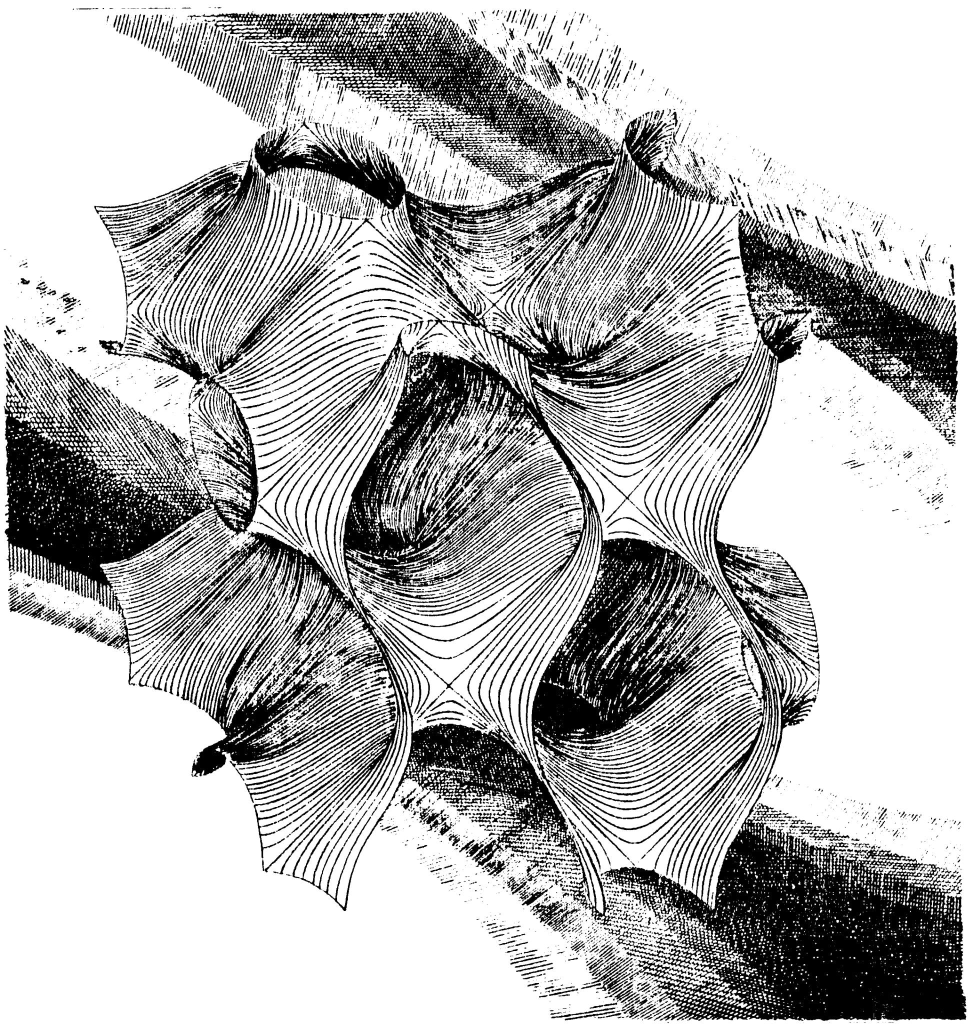

Figure 2: Patrice Jeener: Surface minimale à 3 periodes[15]

3. The discovery of triply periodic minimal surfaces is Hermann Amandus Schwarz's achievement (only one of many; see [37], §280, A.8.33) in the 19th century. From the very first — 1865, when he had just turned twenty-two — Schwarz visualized his solution surfaces of Plateau's problem for specific skew quadrilaterals also as fundamental domains (surface patches) for the creation of triply periodic minimal surfaces embedded in R^3: Already in [47], pp. 3–5, he describes clearly the labyrinthic nature with which these surfaces permeate space, dividing it into two highly intertwined regions of infinite connectivity.[16] Of course, for him, as for the mathematicians and crystallographers to

[15] From his work [24]. I am grateful to Mr. Jeener for this copy of his etching.

[16] K. Hattendorff's exposition of the work of Bernhard Riemann who had attacked Plateau's problem for certain quadrilaterals at about the same time (for specifics see [44], pp. 301, 334 and [37], §276, A.8.28) is confined to the following observation: "The minimal surface in question can be continuously extended across its original boundary .... If the construction is repeated for the new surface portions, the original surface can be continued arbitrarily far."

follow him[17], they were purely geometrical constructs; there is no hint at all about their possible role as physical interfaces (not even Joseph-Antoine-Ferdinand Plateau who entertained scientific contacts with Schwarz seemed to have had such notions). Subsequently, the associate fundamental domains, which meet the boundaries of certain tetrahedra at right angles, yielded further examples of periodic minimal surfaces, also already studied by Schwarz. Moreover, in view of a general reflection principle, the latter are suitable for the construction of triply periodic surfaces having constant, but nonvanishing, mean curvature.[18] Originally, most triply periodic minimal surfaces were generated from simply connected fundamental domains bounded by four (straight or curved) geodesics. Recently, however, far more elaborate fundamental domains have been introduced, notably by A.H. Schoen [46] and by W. Fischer & E. Koch [10]. They lead to an astounding wealth of new periodic minimal surfaces of distinct crystallographic types.

It is interesting to note that the existence of these new domains — not as physical models fashioned with some kind of elastic material or as approximations for numerical schemes, absent a convergence proof, but as *mathematical* minimal surfaces — is as yet unsecured in many cases and thus poses intriguing questions; see, e.g. [38]. A similar observation, even to a larger degree, applies to the surfaces satisfying more elaborate differential equations.

4. The creation of global periodic minimal surfaces with the help of the analytic extension of minimal surface patches across straight or plane boundary portions is made possible by virtue of two reflection principles also formulated by Schwarz ([47], pp. 111, 130, 181; see also [48], p.1237, [13], p. 300, [37], §150):

<u>1:</u> If a minimal surface is bounded by a straight line segment, then it can be analytically extended, as minimal surface, across this segment by reflection.

<u>2:</u> If a minimal surface meets a plane at right angle, then it can be analytically extended, as a minimal surface, across this plane by reflection.

Note that in the second case the "trace" of the surface on the plane is a geodesic of the surface.[19] Obviously, these reflection principles require a careful specification regarding the boundary regularity of the minimal surface.[20] In fact, Schwarz's own formulation was slightly different: (1) A straight line lying on the piece of a minimal surface is an axis of symmetry for the surface. (2) A plane intersecting the piece of a minimal surface orthogonally is a plane of symmetry for the surface.

It is an important observation that the second reflection principle, but not the first,

[17]E.R. Neovius, G. Tenius, A. Schoenflies, E. Stenius, O. Nicoletti, F. Marty, B. Steßmann, M. Wernick, T. Nagano, B. Smyth, among others.

[18]An existence proof, for a certain interval $| H | < H_0$, can be found in [39]; see further [2], [25], [26], also the remarks in section 2.

[19]For a discussion of early examples of such curves — astroid, Neil's parabola, cycloid etc., see e.g. [35], §766, [37], III.2.2.

[20]A complete discussion of this question can be found in [37], V.2.1, [35], VI.2.

is also valid for surfaces of constant, but nonzero, mean curvature: The reflection across a straight boundary segment causes the mean curvature, coherently oriented, to change sign, a fact which of course is of no consequence for minimal surfaces. Most other surface classes, for instance the related harmonic surfaces, i.e. surfaces for which the components of the position vector are harmonic functions, do *not* obey the reflection principles. On the other hand, both principles, in appropriate formulation, can be asserted for the solutions of the fourth order differential equations (17), (18), (19), (51) below.

5. One way of constructing periodic minimal surfaces is based on the Weierstrass representation formulae. The utilization of these is assisted by the fact that, for minimal surfaces, the implications of the two reflection principles mentioned above are closely related through O. Bonnet's concept of *adjoint minimal surfaces* ([37], III.2.7): If a minimal surface meets a plane at right angle, its adjoint surface contains a straight segment parallel to the normal of the plane. Therefore, the determination of a stationary minimal surface in a convex polyhedron is directly linked to the solution of Plateau's problem for the conjugate polygonal contour. (There remains, of course, the requirement, often quite difficult, to determine constants and to "kill periods".)

For example, this relation, applied to the surface patch representing the fundamental domain for the so-called Schwarz P-surface which will be of importance for us later on, leads to the following Weierstrass representation

$$\begin{aligned} x &= x(\sigma,\tau) = \Re \int_0^\omega (1-\omega^2)R(\omega)d\omega\,, \\ y &= y(\sigma,\tau) = \Re \int_0^\omega i(1+\omega^2)R(\omega)d\omega\,, \\ z &= z(\sigma,\tau) = \frac{1}{2} + \Re \int_0^\omega 2\omega R(\omega)d\omega\,. \end{aligned} \tag{6}$$

Here

$$R(\omega) = \frac{\kappa}{\sqrt{1+14\omega^4+\omega^8}}\,, \tag{7}$$

where the constant κ is determined by the condition

$$2\kappa \int_0^{\sqrt{2}-1} \frac{1+2t-t^2}{\sqrt{1+14t^4+t^8}}\,dt = 1\,,$$

so that $\kappa = .92742\ldots$ .The complex variable $\omega = \sigma + i\tau$ varies over the domain Π, hatched in figure 8 below, which is bounded by the σ- axis, the τ-axis and by the circles $(\sigma+1)^2+\tau^2 = 2$ and $\sigma^2+(\tau-1)^2 = 2$. The elementary tetrahedron has vertices $(0,0,0)$, $(0,0,1)$, $(1,0,0)$ and $(0,1,1)$. The angles between its faces are $\pi/2, \pi/2, \pi/2$ an $\pi/3$. The minimal surface patch (6) can therefore be continued analytically across all faces and appears as regular part of a larger surface piece.

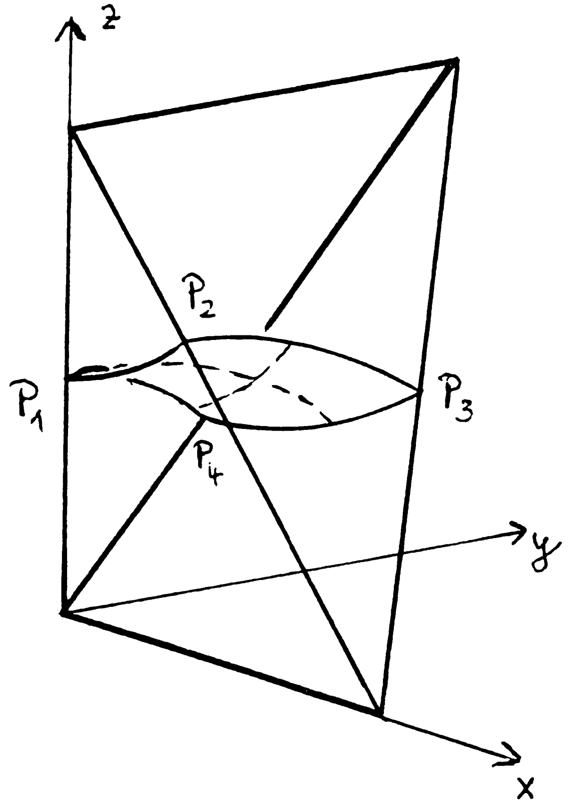

Figure 3: The elementary tetrahedron and the inscribed surface patch generating Schwarz's P-surface

Of course, the shape of the domain Π can be read off figure 3, without the necessity of determining the explicit representation (6): It is obtained from the spherical image of the surface patch by stereographic projection. With some computation, the representation (6) can be verified directly. For instance, along the circular arc $\omega = -1 + \sqrt{2}e^{i\vartheta}$,

$$1 + 14\omega^4 + \omega^8 = 16e^{4i\vartheta}\{16\cos^4\vartheta - 32\sqrt{2}cos^3\vartheta + 40cos^2\vartheta - 4\sqrt{2}cos\,\vartheta - 5\} \equiv 16e^{4i\vartheta}\Xi(\vartheta)\,,$$

so that

$$\frac{\partial}{\partial\vartheta}\frac{x+z}{\sqrt{2}} = 0\,,\ \frac{\partial}{\partial\vartheta}\frac{x-z}{\sqrt{2}} = \frac{\sin\vartheta}{\sqrt{\Xi(\vartheta)}}\,,\ \frac{\partial y}{\partial\vartheta} = \frac{\cos\vartheta}{\sqrt{\Xi(\vartheta)}}\,.$$

Six copies of the elementary tetrahedron fill the unit cube, and the concomitant analytic extension of the surface patch (6) leads to a larger surface piece of total curvature $-\pi$ which is sketched in figure 4.

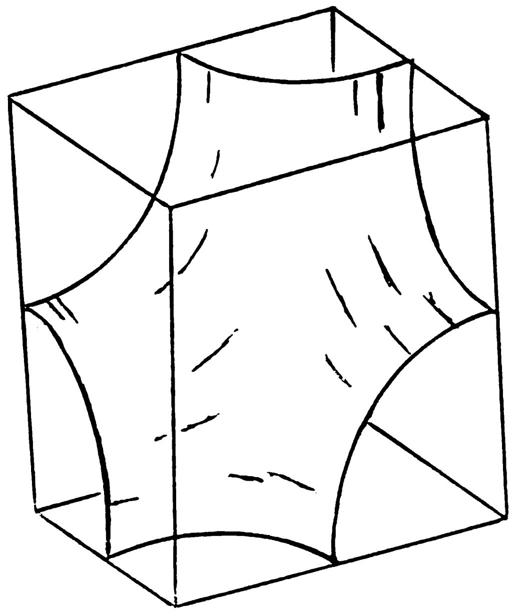

Figure 4: Six copies of the tetrahedron fill the unit cube and generate a piece of the P-surface

If eight of these cubes, i.e. 48 copies of the elementary tetrahedron are stacked, a portion of Schwarz's surface of topological type (see [37], §43) [1,6,-4] and total curvature -8π is obtained. Figure 5 depicts a sketch of this portion. (Photographs of models of this portion can be seen on the frontispiece of [37].)

For a later application in section 16 below, we mention that the Gaussian curvature of our surface satisfies everywhere the inequality $\mid K \mid \leq 4.65\ldots$.

6. Let $\mathcal{S}$ be a surface (interface) satisfying suitable regularity conditions to be specified later. We shall consider here energy functionals of the general form

$$\mathcal{E} \equiv \mathcal{E}(\mathcal{S}) = \int\int_{\mathcal{S}} \hat{\Phi}(\kappa_1, \kappa_2) dA\,. \tag{8}$$

In this expression, $\kappa_1 = 1/R_1$ and $\kappa_2 = 1/R_2$ denote the principal curvatures of $\mathcal{S}$. The integrand $\hat{\Phi}(\kappa_1, \kappa_2)$ is assumed to be a symmetric function of its argument. Under mild assumptions, $\hat{\Phi}$ can then be written in the form $\hat{\Phi}(\kappa_1, \kappa_2) = \Phi(H, K)$, where $H = (\kappa_1 + \kappa_2)/2$ and $K = \kappa_1 \kappa_2$ are the mean and Gaussian curvature of $\mathcal{S}$, respectively[21].

[21] For a symmetric, not necessarily homogeneous, polynomial $\hat{\Phi}$, this is a consequence of the fundamental theorem on symmetric polynomials.

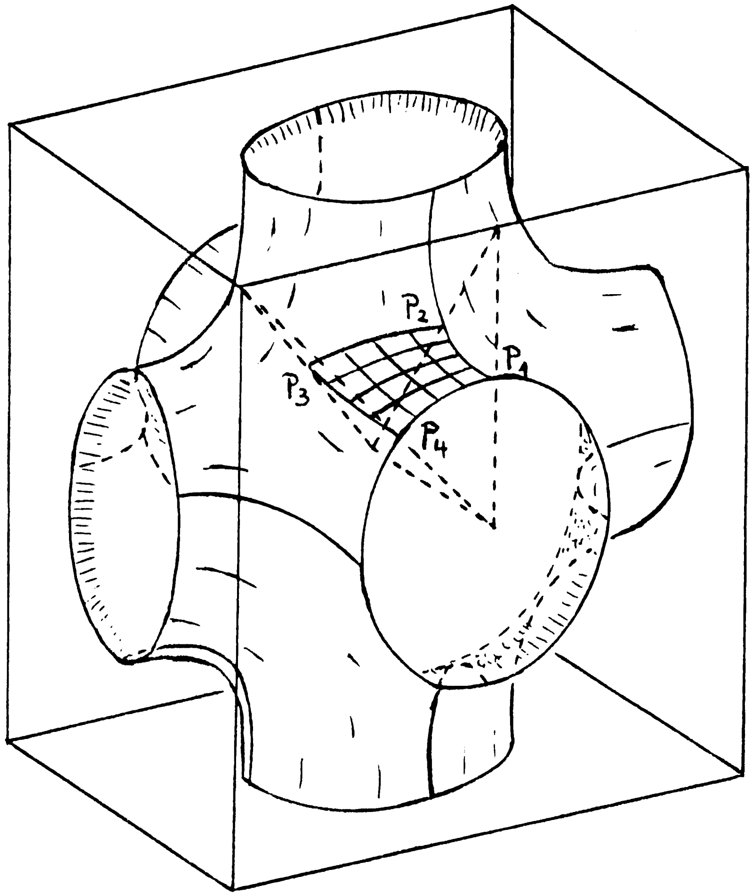

Figure 5: A section of Schwarz's P-surface made up by 48 copies of the fundamental surface patch (6). Its topological type is $[1, 6, -4]$

$\mathcal{E}$ is assumed to be *definite* in the following sense: There is a constant $c > -\infty$, possibly negative, such that $\mathcal{E}(\mathcal{T}) \geq c$ for every connected orientable surface $\mathcal{T}$ of regularity class C^2, *with* or *without* boundary. We allow the lower bound for $\mathcal{E}$ to be negative to reflect the fact that the free surface energy of an interface need not be positive; however, in the absence of information concerning the size and possible influence of specific physical constants (elastic moduli) entering the expression Φ, we must insist on the condition $c > -\infty$. The consequences of this condition regarding the structure of the integrand $\Phi(H, K)$ will be discussed below for the special example of the integrand (16).

The Euler-Lagrange equation associated with the variational problem $\delta\mathcal{E} = 0$ is complicated, and we will not write it down at this place for the general case that Φ depends in a nonlinear way on the variable K (see, however, section 16 below where the case

$\Phi(H,K) = \Psi(H) + \Psi^{(1)}(K)$ is discussed). If $\Phi(H,K)$ has the form $\Phi(H,K) = \Psi(H) - \gamma K$, then it will be ([37], p. 24)

$$\Delta\Psi_H + 2\{(2H^2 - K)\Psi_H - 2H\Psi\} = 0\,. \tag{9}$$

Of course, $\Delta\Psi_H = \Psi_{HH}\Delta H + \Psi_{HHH}\nabla(H,H)$. Here $\Psi_H = \partial\Psi/\partial H$ etc., and ∇, Δ represent the first and second Beltrami operator on $\mathcal{S}$. In view of this, the assumption

$$\Psi_{HH}(H) \geq m_0 > 0 \tag{10}$$

is natural. Here m_0 denotes a universal constant. We shall also stipulate that the following further structural conditions are satisfied, in addition to appropriate regularity assumptions:

$$\Psi(H) > 0\,,\ \Psi(-H) = \Psi(H)\,,\ \Psi'(0) = 0\,. \tag{11}$$

Note that the parameter γ does not appear in (9), a fact already observed by S.D. Poisson; see section 7. However, the presence of the term $-\gamma K$ influences the boundary conditions associated with (8) and the variational problem $\delta\mathcal{E} = 0$.

If the surface $\mathcal{S}$ is before us in a nonparametric representation $z = z(x,y)$ then the curvatures of $\mathcal{S}$ take the form ($p = z_x, q = z_y, r = z_{xx}, s = z_{xy}, t = z_{yy}, W = \sqrt{1+p^2+q^2}$)

$$H = \frac{(1+q^2)r - 2pqs + (1+p^2)t}{2W^3}\,, \tag{12}$$

$$K = \frac{rt - s^2}{W^4}\,. \tag{13}$$

Further, for a function $f(x,y)$ defined on $\mathcal{S}$, we have

$$\nabla(f,f) = \frac{1}{W^2}\{(1+q^2)f_x^2 - 2pqf_xf_y + (1+p^2)f_y^2\}\,, \tag{14}$$

$$\Delta f = \frac{1}{W^2}\{(1+q^2)f_{xx} - 2pqf_{xy} + (1+p^2)f_{yy}\} - \frac{2H}{W}(pf_x + qf_y)\,. \tag{15}$$

Equation (9) is a nonlinear partial differential equation of fourth order for the position vector of the surface under consideration. If $\mathcal{S}$ is not closed, then the vanishing of the first variation $\delta\mathcal{E}$ implies also certain boundary conditions. These boundary conditions are rather intricate; they also influence the behavior of the functional $\mathcal{E}$, particular in the case of "crumpled" boundaries. For the context of our present investigation, we shall come back to the subject of boundary conditions later. Again, in case that the surface $\mathcal{S}$ is subject to a volume constraint, the right hand side of (9) must be replaced by a (generally nonzero) constant.

It is clear from (15), but should be pointed out specifically here, that the Beltrami operator on the surface $\mathcal{S}$ depends on the coefficients of the first fundamental form of $\mathcal{S}$ and their first derivatives.

Equation (12) which defines the mean curvature can also be be interpreted as an elliptic differential equation for the a surface $z = z(x, y)$ of presribed mean curvature $H(x, y)$. If the function $z(x, y)$ is of regularity class $C^{m,\nu}$ in a region D of the (x, y)–plane, where $m \geq 2$ and $0 < \nu < 1$, then the mean curvature $H(x, y)$ of $\mathcal{S}$ is of regularity class $C^{m-2,\nu}$ in D. The theory of elliptic differential equations (see, e.g. [12]) guarantees the converse of this statement:

THEOREM. *Assume that $H(x, y) \in C^{m,\nu}(D)$, $m \geq 0$, $0 < \nu < 0$. If the function $z(x, y)$ is a C^2–solution (or a weak solution) of (H), then $z(x, y) \in C^{m+2,\nu}(D)$.*

This theorem is true also for the components of the position vector of the surface $\mathcal{S}$ if the latter is given in parametric representation.

For the time being, we shall work with an integrand of the special form

$$\Phi(H, K) = \alpha + \beta H^2 - \gamma K\,. \tag{16}$$

The requirement of definiteness implies the conditions $\alpha \geq 0, 0 \leq \gamma \leq \beta$. In fact, for the integrand (16), the value of $\mathcal{E}$ can be made arbitrarily negative as follows:

if $\alpha < 0$ — by choosing $\mathcal{T}$ as a large portion of a plane;

if $\beta < 0$ — by choosing $\mathcal{T}$ as a long portion of a thin cylinder;

if $\gamma < 0$ — by choosing $\mathcal{T}$ as a large portion, properly scaled, of a minimal surface with infinite total curvature;

if $\gamma > \beta$ — by choosing $\mathcal{T}$ as a collection of small semispheres connected by narrow bridges.[22] For an arrangement $\mathcal{T}_N$ of N such units (figure 6 depicts the case $N = 3$) we find that, approximately,

$$\mathcal{E}(\mathcal{T}_N) \leq 2\pi N^2\{(\alpha r^2 + \frac{\varepsilon}{8a}(4\alpha a^2 + \beta) - (\gamma - \beta)\}\,.$$

As a consequence, if $\gamma > \beta$ and if the parameters r, a, ε are suitably chosen, we see that $\mathcal{E}(\mathcal{T}_N) \to -\infty$ for $N \to \infty$.

The Euler-Lagrange equation, assuming the presence of a volume constraint, becomes

$$2\beta\{\mathbf{\Delta} H + 2H(H^2 - K)\} - 4\alpha H = c\,. \tag{17}$$

We shall call (17) the complemented equation (9). Equations (9) and (17) are in the same relation as the equations $H = 0$ for minimal surfaces and $H = c$ for H-surfaces. It had

[22] Of course, care must be taken in "fusing" the bridges to the semispheres so that all contributions to the integrals $\int\int H^2 dA$ and $\int\int K dA$ near the connecting regions remain under control.

been mentioned already earlier, that the constant c which is related, although generally not in a bijective way, to the volume fractions of the phases separated by $\mathcal{S}$, is the control parameter leading to a *family* of extremal surfaces.

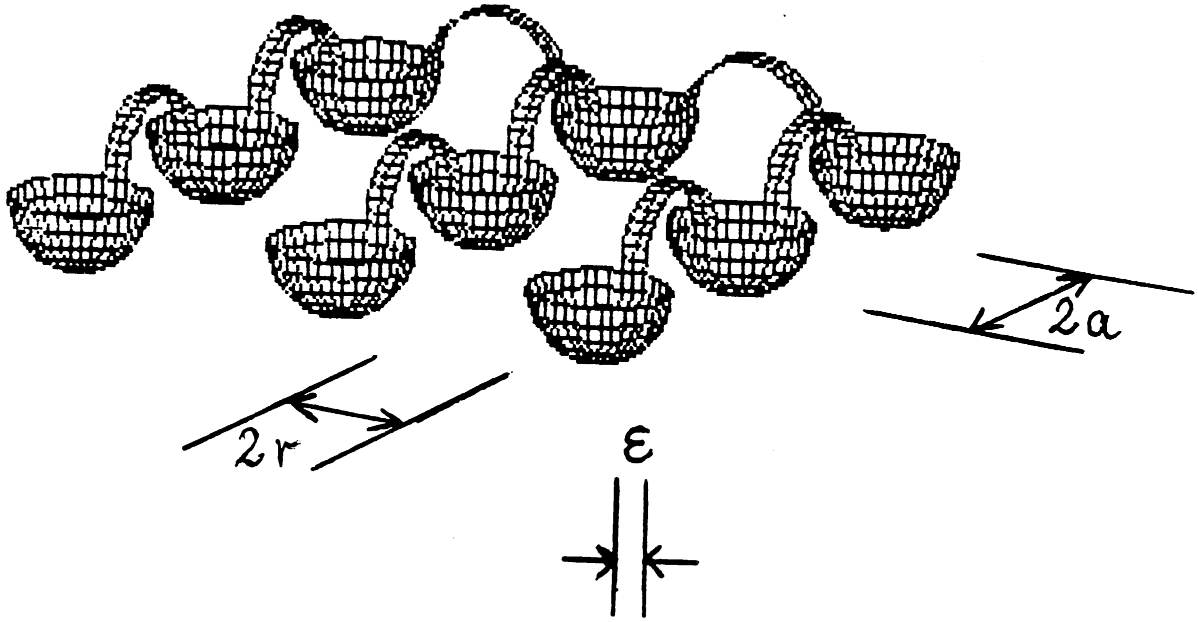

Figure 6: Sketch of the comparison surface $\mathcal{T}_3$

For the special case $\Phi(H,K) = H^2$, i.e. $\alpha = \gamma = 0$, $\beta = 1$, the differential equations (9) and its complemented equation (17) are reduced to the forms

(18) $$\Delta H + 2H(H^2 - K) = 0$$

and

(19) $$\Delta H + 2H(H^2 - K) = c\,,$$

respectively.

7. Given the recent renewed interest, in many quarters, for the consideration of functionals approximating the free energy of an interface which take into account also elastic properties, and therefore involve curvature functions, however imperfect at the present time, it seems appropriate to mention here that earlier mathematicians, notably Sophie Germain and Siméon Denis Poisson had come to consider expressions closely related to the functional $\int\int H^2 dA$ and to integrands involving the difference (in modern language) $H - H_0$ between the "elastic" and the "natural" curvatures (S. Germain [11], pp. 1,12) already long ago.

As it happened, in deference to the wishes of the emperor Napoléon, the *Institut de France* had put forth the following prize-question for October 1, 1811:

> To develop a mathematical theory for the vibrations of an elastic surface,
> and to compare its results with experimental observations [l'expérience].

The *Institut* posed the subject again for October 1, 1813, and once more for October 1, 1815. S. Germain competed each time, the second time receiving honorable mention, the third time gaining the prize. While these three entries are not in the published literature, Germain incorporated her main results in her investigation [11] and a later work, 1826, printed by the same publisher.

Poisson's contributions begin with his treatise [43]; this was read to the *Institut* on August 1, 1814, and is contained in the volume of the *Mémoirs* for 1812, published in 1814. The correctness of Germain's derivations has been widely criticized, as has been the physical relevance of her and Poisson's work[23](see e.g. [52], pp. 147-60, [27], further also [4]). However, from the point of view of the calculus of variations, important advances must be registered. On p. 224 of [43], Poisson formulates the differential equation (18) in lengthy nonparametric form (but this was long before E. Beltrami who was born in 1835). Highly interesting are also his observations on global curvature measures,

$$\Phi = (\frac{1}{R_1} + \frac{1}{R_2})^2 \,, \; \Phi = \frac{1}{R_1^2} + \frac{1}{R_2^2} \,, \; \Phi = \frac{1}{R_1 R_2} \,,$$

and on the relations between them (pp. 221–5): "I conclude this memoir by announcing a peculiar property of an elastic surface in equilibrium ..." In particular, Poisson observes expressly (pp. 224-5) that the addition of a term γK to the integrand $\Phi = H^2$ does *not* enter in the Euler-Lagrange equations (18); see (9), (17) — a prevenience of the global Gauss-Bonnet theorem. Poisson does not address the important question of boundary conditions. The curvature measure $C = (1/R_1^2 + 1/R_2^2)/2$ (the letter C presumably standing for Casorati), was later also favored by Casorati [7], albeit for purely geometric reasons.

Concerning compact (closed) solution surfaces of the variational problem $\delta\mathcal{E} = 0$ with the integrand $\Phi(H, K) = H^2$, see [55]. For such surfaces, due to the Gauss-Bonnet theorem, the total curvature $\int\int_{\mathcal{S}} K dA$ is a topological invariant (namely, 2π times the Euler characteristic of $\mathcal{S}$). If these surfaces have boundaries, particularly if the latter are crumpled, the addition of a term $-\gamma K$ to the integrand Φ cannot be ignored. Moreover, for an integrand of the form $\Phi(H, K) = \beta H^2 - \gamma K$, the functional $\mathcal{E}$ is invariant under similarities and conformal transformations of R^3, a somewhat unphysical property. The case

[23] G. Kirchhoff writes in [27], p. 52: "Notwithstanding the confirmations received by Sophie Germain's theory through experiments, this theory is not correct, because it allows conclusions which are in manifest contradiction to reality." Regarding [43], I. Todhunter says ([52], p. 212): "The Memoir considered as an exercise in mathematics is a fine specimen of Poisson's analytical skill, but it adds little to the discussion of the physical problem."

$\Phi(H,K) = c_1(H - H_0)^2 + c_2 K$ has been introduced by W. Helfrich [17] in his investigations of lipid bilayers. Helfrich's functional does not possess the invariance properties mentioned.

8. We consider a differential geometric surface $\mathcal{S} = \{\mathbf{x} = \mathbf{x}(u,v); (u,v) \in P\}$ of class $C^{4,\nu}(P)$, where the position vector $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ is defined over a domain P in the plane of the parameters u and v and $0 < \nu < 1$. The unit normal vector of $\mathcal{S}$ will be denoted by $\mathbf{X}(u,v) = (X(u,v), Y(u,v), Z(u,v))$. Without loss of generality, we may assume that $\mathcal{S}$ is before us in a representation based on isothermal parameters, so that the first fundamental form of $\mathcal{S}$ has the form $ds^2 = E(u,v)(du^2 + dv^2)$, i.e. $E(u,v) \equiv \mathbf{x}_u^2 = G(u,v) \equiv \mathbf{x}_v^2$ and $F(u,v) \equiv \mathbf{x}_u \cdot \mathbf{x}_v = 0$. The coefficients of the second fundamental form of $\mathcal{S}$ will be denoted by $L(u,v)$, $M(u,v)$, $N(u,v)$, as usual.

A *normal variation* $\mathbf{x}(u,v) \to \overline{\mathbf{x}}(u,v) = \mathbf{x}(u,v) + \zeta(u,v)\mathbf{X}(u,v)$ transforms the surface $\mathcal{S}$ into a new surface $\overline{\mathcal{S}}$. We shall restrict ourselves to small variations. For such variations, the function $\zeta(u,v) \in C^{4,\nu}(D)$ will be small in a suitable norm — for the case at hand the $C^{2,\nu}$–norm. It is important to note that u and v are generally not isothermal parameters on $\overline{\mathcal{S}}$. Therefore, denoting the standard Laplace operator by $\Delta \equiv \partial^2/\partial u^2 + \partial^2/\partial v^2$, the second Beltrami operators $\boldsymbol{\Delta}$ on $\mathcal{S}$ and $\overline{\boldsymbol{\Delta}}$ on $\overline{\mathcal{S}}$ applied to a function $f(u,v)$ become

$$\boldsymbol{\Delta} f = \frac{1}{E(u,v)}\,\Delta f\,, \tag{20}$$

$$\overline{\boldsymbol{\Delta}} f = \frac{1}{E(u,v)}\,\Delta f + \sum_{m=1}^{\infty} \Lambda^{(1)}_{m,1}(\zeta, f)\,. \tag{21}$$

Here the terms $\Lambda^{(1)}_{m,n}(\zeta, f)$ indicate polynomial expressions in $(\zeta, \zeta_u, \ldots, \zeta_{vv})$ and in $(f, f_u, \ldots, f_{vv})$ of degree m with respect to the first set of variables and of degree n with respect to the second set of variables, involving coefficients which depend on the vector $\mathbf{x}(u,v)$ and its first and second derivatives. (Since $\overline{\boldsymbol{\Delta}}$ is a linear operator, only terms of the form $\Lambda^{(1)}_{m,1}$ enter in (21).) The infinite sum converges for sufficiently small values of the ζ-dependent arguments, in fact, as long as the denominator $\overline{E}\,\overline{G} - \overline{F}^2$ in the expression of the Beltrami operator $\overline{\boldsymbol{\Delta}}$ does not vanish, i.e. as long as

$$\begin{aligned} \frac{1}{E^2}(\overline{E}\,\overline{G} - \overline{F}^2) &= 1 - 4H\zeta + \{2(2H^2 + K)\zeta^2 + \frac{\zeta_u^2 + \zeta_v^2}{E}\} \\ &\quad - 4HK\zeta^3 - \frac{2\zeta}{E^2}(N\zeta_u^2 - 2M\zeta_u\zeta_v + L\zeta_v^2) \\ &\quad + K^2\zeta^4 - K\zeta^2\frac{\zeta_u^2 + \zeta_v^2}{E} + 2H\zeta^2(N\zeta_u^2 - 2M\zeta_u\zeta_v + L\zeta_v^2) \end{aligned} \tag{22}$$

is positive. Let the bound $Z > 0$ be chosen such that the assumption $\| \zeta \|_{1,\nu} \leq Z$ guarantees this.

For the case that $\mathcal{S}$ is a minimal surface, equation (22) simplifies to (see [37]), §414)

$$\begin{aligned} \frac{1}{E^2}(\overline{E}\,\overline{G} - \overline{F}^2) &= 1 + 2K\zeta^2 + \frac{\zeta_u^2 + \zeta_v^2}{E} \\ &\quad + \frac{2\zeta}{E^2}(L\zeta_u^2 + 2M\zeta_u\zeta_v + N\zeta_v^2) \\ &\quad + K\zeta^2(K\zeta^2 - \frac{\zeta_u^2 + \zeta_v^2}{E}). \end{aligned} \tag{23}$$

A lengthy computation leads to the following expressions for the mean and Gaussian curvatures of $\overline{\mathcal{S}}$:

$$\overline{H} = H + (2H^2 - K)\zeta + \frac{1}{2E}\Delta\zeta + \sum_{m=2}^{\infty} \Omega_m^{(1)}(\zeta), \tag{24}$$

$$\begin{aligned} \overline{K} &= K + 2HK\zeta \\ &\quad + \frac{1}{2E^3}\{(L-N)E_u + 2ME_v\}\zeta_u + \frac{1}{2E^3}\{2ME_u - (L-N)E_v\}\zeta_v \\ &\quad + \frac{1}{E^2}\{N\zeta_{uu} - 2M\zeta_{uv} + L\zeta_{vv}\} + \sum_{m=2}^{\infty} \Omega_m^{(2)}(\zeta). \end{aligned} \tag{25}$$

Again, the terms $\Omega_m^{(1)}(\zeta)$ and $\Omega_m^{(2)}(\zeta)$ denote polynomial expressions of degree m in $(\zeta, \zeta_u, \dots, \zeta_{vv})$ with coefficients which depend on the vector $\mathbf{x}(u,v)$ and its first and second derivatives. These terms are well-defined but forbiddingly long; their explicit form is of secondary importance here, however. (For some of the terms, see [37], §414). The properties of the infinite series, here and later, are the same as mentioned in connection with (22) above.

At this point, we shall assume that our starting surface $\mathcal{S}$ is a minimal surface so that $H(u,v) = 0$. It will be our aim to determine the function $\zeta(u,v)$ in such a way that the surface $\overline{\mathcal{S}}$ is a solution for the differential equation (17). Since $\overline{\mathcal{S}}$ and $\mathcal{S}$ are close, the same will be true for the mean curvatures H and $\overline{H}$, i.e. $\overline{H}$ will be small, again in a suitable norm. It then follows from (21) that

$$\overline{\mathbf{\Delta}}\overline{H} = \frac{1}{E}\Delta\overline{H} + \sum_{m=1}^{\infty} \Lambda_{m,1}^{(1)}(\zeta, \overline{H}). \tag{26}$$

It is a fortunate circumstance that, in view of (24), (25), equation (17) — actually a partial differential equation of fourth order — can be written in the more manageable form of two simultaneous differential equations of second order for the quantities $\zeta(u,v)$ and $\overline{H}(u,v)$:

$$\Delta\zeta - 2EK\zeta = 2E\overline{H} + \sum_{m=2}^{\infty} \Omega_m^{(3)}(\zeta) \tag{27}$$

$$\Delta\overline{H} - 2EK\overline{H} - 2\tilde{\alpha}E\overline{H} = E\tilde{c} - 2E\overline{H}^3 + \sum_{m=1}^{\infty} \Lambda_{m,1}^{(2)}(\zeta, \overline{H}). \tag{28}$$

The inflected parameters are defined as follows: $\tilde{\alpha} = \alpha/\beta$, $\tilde{c} = c/2\beta$. This assumes that $\beta \neq 0$. The case in which $\beta = 0$ leads to the simpler problem

$$\overline{H} = -\frac{c}{4\alpha} = const\,, \tag{29}$$

$$\Delta\zeta - 2EK\zeta = -\frac{c}{2\alpha}E + \Omega_2 + \dots\,; \tag{30}$$

see [37], §414. This is the problem of the determination of surfaces of constant mean curvature, a special case; see the remarks in section 2.

We return to our main argument. The components of the position vector $\mathbf{x}(u,v)$ of $\mathcal{S}$ are real analytic functions of their arguments. As for the functions ζ and $\overline{H}$, we shall require them to be of the regularity class $C^{2,\nu}(P)$, and later $C^{2,\nu}(\overline{P})$. The following can be said, however: If $(\zeta, \overline{H})$ is a solution pair for the system of differential equations (27), (28) then, because of the meaning of ζ and $\overline{H}$ and in view of the theorem enunciated in section 6, the function $\zeta(u,v)$ actually belongs to the regularity class $C^{4,\nu}(P)$.

On the minimal surface $\mathcal{S}$, umbilic points are isolated, so that $K(u,v) \leq 0$ and that $K(u,v)$ vanishes at most at isolated points. We define the function

$$p(u,v) = -2E(u,v)K(u,v) \geq 0\,. \tag{31}$$

9. Here we shall prove our new reflection principle which plays an essential role for the construction, by analytic continuation, of periodic extremal surfaces associated with the variational problem under consideration.

Reflection Principle: *Let $\mathcal{T}$ be a solution surface of (17) part of whose boundary lies in a plane $\mathcal{P}$. It is assumed that $\mathcal{T}$ belongs to the regularity class $C^{4,\nu}$, up to the boundary part in $\mathcal{P}$.*[24] *If $\mathcal{T}$ meets the plane $\mathcal{P}$ at a right angle, and if the derivative of its mean curvature H normal to $\mathcal{P}$ is zero in the points of $\mathcal{P}$, then $\mathcal{T}$ can be extended across $\mathcal{P}$ as a $C^{4,\nu}$-solution of (17) by reflection.*

For the proof, we restrict ourselves to the general case $\beta \neq 0$. The special case $\beta = 0$ is simpler; see also the remarks in sections 2 and 8. In view of the local nature of the asserted property, we may assume that the surface $\mathcal{T}$ is given in nonparametric representation $z = z(x,y)$ and that $x = 0$ is the plane in question. Denote by D^+ the semidisc $\{x, y; x \geq 0, x^2+y^2 < \varepsilon^2\}$, by D^- the semidisc $\{x, y; x \leq 0, x^2+y^2 < \varepsilon^2\}$, and set $D = D^+ \cup D^-$. The function $z(x,y)$ is known to be of regularity class $C^{4,\nu}(D^+)$. For the purpose of clarity, we write $z(x,y) \equiv z^+(x,y)$ and denote our surface portion $\mathcal{T}$ by $\mathcal{T}^+$. The mean curvature

[24] These regularity assumptions will be satisfied in the applications to follow. A general theory regarding the boundary regularity for stationary solution surfaces of our variational problem has not yet been developed, however. For the case of stationary (not necessarily area minimizing) solution surfaces of the variational problem $\delta \int\int dA = 0$ subject to a volume constraint, see [16], and for the more restricted case of stationary minimal surfaces, [9], [15].

of $\mathcal{T}^+$, as defined by (12), is denoted by $H^+(x,y) \in C^{2,\nu}(D^+)$. Since $\mathcal{T}^+$ meets the plane $x = 0$ at a right angle, we have $z^+_x(0,y) = z^+_{xy}(0,y) = 0$.

The reflection of $\mathcal{T}^+$ on the plane $x = 0$ leads to the surface portion $\mathcal{T}^-$ represented by the function $z = z^-(x,y) = z^+(-x,y) \in C^{4,\nu}(D^-)$. The mean curvature $H^-(x,y)$ and the Gaussian curvature $K^-(x,y)$ of $\mathcal{T}^-$ are of class $C^{2,\nu}(D^-)$; moreover, we see from the properties of $z^+(x,y)$ and from (12), (13) that $H^-(x,y) = H^+(-x,y)$ and $K^-(x,y) = K^+(-x,y)$ for $(x,y) \in D^-$, so that we have $H^-_x(x,y) = -H^+_x(-x,y), H^-_{xx}(x,y) = H^+_{xx}(-x,y)$ etc. and, in particular, $H^-_x(0,y) = -H^+_x(0,y) = 0$. After an inspection of the form (15) of the second Beltrami operator, we thus can see that $\mathcal{T}^-$ is a solution surface of the differential equation in D^-.

From the above, it can be concluded that the function

$$z(x,y) = \begin{cases} z^+(x,y) \text{if } (x,y) \in D^+ \\ z^-(x,y) \text{if } (x,y) \in D^- \end{cases}$$

is (not of class $C^{4,\nu}(D)$, but certainly) of class $C^{2,\nu}(D)$. This is so, because the functions $z^+(x,y)$ and $z^-(x,y)$, as well as their first and second derivatives, have the same values along the segment $D^+ \cap D^-$ of the y-axis. Given the premise $H_x(0,y) = 0$, the same can be said for the function

$$H(x,y) = \begin{cases} H^+(x,y) \text{if } (x,y) \in D^+ \\ H^-(x,y) \text{if } (x,y) \in D^- \end{cases}.$$

We now have that $z(x,y)$ is a $C^{2,\nu}$–solution of (12) in all of D, where $H(x,y)$ itself is a $C^{2,\nu}$– function. By the theorem of section 6, $z(x,y)$ is in fact of regularity class $C^{4,\nu}(D)$.

The proof of our reflection principle is complete.

10. In the following discussions, we will be in the position to assume that $\mathcal{S}$ is the portion of a larger minimal surface $\{\mathbf{x} = \mathbf{x}(u,v); (u,v) \in Q\}$ whose parameter domain Q contains the closure of P: $\overline{P} \subset Q$. In fact, this larger minimal surface will be chosen as a piece of Schwarz's P-surface containing our surface patch (6). In the larger parameter domain Q, the set P defining this patch is a curved quadrilateral bounded by analytic arcs which meet at form corners where they form interior angles of $\pi/2, \pi/2, \pi/2$ and $\pi/3$. Along the boundary arcs, the minimal surface $\mathcal{S}$ intersects the faces of the fundamental tetrahedron at right angles. For a boundary arc of P which lies in a particular face of this tetrahedron, the vector $\mathbf{x}_u \wedge \mathbf{x}_v$ (the symbol $\wedge$ denotes the vector product) is parallel to this face. This condition can be written in the form $\partial \mathbf{x}/\partial n = 0$, where $\partial/\partial n$ denotes differentiation with respect to the normal direction at each point of ∂P. Of course, $H(u,v) \equiv 0$.

Our wish to apply the reflection principle of section 9 now guides us regarding the boundary conditions to be imposed on the functions ζ and $\overline{H}$. It can be concluded from

$$\overline{\mathbf{x}}_u \wedge \overline{\mathbf{x}}_v = -\zeta_u \mathbf{x}_u - \zeta_v \mathbf{x}_v + E\mathbf{X} + \dots$$

that the condition $\partial\zeta/\partial n = 0$ must be required along the boundary ∂P. As for $\overline{H}$, we have the same condition $\partial\overline{H}/\partial n = 0$. The boundary value problem to be solved is thus a Neumann problem, namely:

The differential equations (27), (28) in P, together with the boundary conditions

$$\frac{\partial\zeta}{\partial n} = 0 \, , \; \frac{\partial\overline{H}}{\partial n} = 0 \tag{32}$$

along ∂P.

11. Our boundary value problem for the differential equations (27), (28) has the following structure:

$$\mathcal{M}(\eta) \equiv \Delta\eta + r(u,v)\eta = f(u,v) \, , \; (u,v) \in P \, , \tag{33}$$

$$\frac{\partial\eta}{\partial n} = 0 \, , \; (u,v) \in \partial P \, . \tag{34}$$

Here $r(u,v) = p(u,v)$ for (27) and $r(u,v) = p(u,v) - 2\tilde{\alpha}E$ for (28).

The actual solution requires the use of apriori estimates for the solutions associated with the differential operator $\mathcal{M}$. For the time being, we consider the linear problem (33), (34) with a right hand side $f = f(u,v) \in C^{0,\nu}(\overline{P})$. We know that $r(u,v)$ is a real analytic function in $\overline{P}$. For the specific situation at hand, a recent theorem of N.M. Wigley [54], whose desirability for the present discussion, as well as for the earlier existence proof of periodic surfaces of constant mean curvature, had been suggested by the author, provides the required tools.[25]

THEOREM. *If the homogeneous problem associated with (33), (34) has no nontrivial solution, then this boundary value problem has a unique solution $\eta(u,v) \in C^{2,\nu}(\overline{P})$. Moreover, there is a constant C depending on $\nu, r(u,v)$ and P such that*

$$\| \eta \|_{2,\nu}(\overline{P}) \leq C \| f \|_{0,\nu}(\overline{P}) \, . \tag{35}$$

The norms in (35) are the usual Schauder norms. It is important to note that Wigley's theorem in this form is *not* true if P has interior angles larger than $\pi/2$. The theorem contains no numerical estimate for the constant C. Such an estimate would of course be helpful for the actual numerical solution of our differential equations.

12. Our first assignment now will be to study the solvability of the homogeneous boundary value problem. For this purpose, we consider the following eigenvalue problem:

$$\Delta\xi + \lambda p(u,v)\xi = 0 \, , \; (u,v) \in P \, , \tag{36}$$

$$\frac{\partial\xi}{\partial n} = 0 \, , \; (u,v) \in \partial P \, , \tag{37}$$

[25] For general information on boundary value problems in domains with corners, see [14] and [30].

where $p(u,v) = -2E(u,v)K(u,v) \geq 0$, see (31). The sequence of corresponding eigenvalues will be denoted by

$$\lambda_1 = 0 < \lambda_2 \leq \lambda_3 \leq \dots .$$

In connection with the case of equation (27), it is crucial for the application of Wigley's theorem to show that the value 1, coefficient of the linear term $-2EK\zeta$, is different from any of the eigenvalues λ_n. For the case at hand, we shall show in fact that $\lambda_2 > 1$. The discussion will benefit from a utilization of the knowledge that $\mathcal{S}$ is a minimal surface for which the Gauss mapping from $\mathcal{S}$ to the unit sphere and its equatorial plane, with the help of the normal vector $\mathbf{X} = (X, Y, Z)$,

$$u + iv \ \rightarrow \ \sigma + i\tau \equiv \frac{X + iY}{1 - Z},$$

is conformal. Therefore, if we define $\Delta_{(\sigma,\tau)} \equiv \partial^2/\partial\sigma^2 + \partial^2/\partial\tau^2$, our eigenvalue problem (36), (37) is transformed into the new eigenvalue problem

$$\Delta_{(\sigma,\tau)}\,\xi + \frac{8\lambda}{(1+\sigma^2+\tau^2)^2} = 0\ ,\ (\sigma,\tau) \in \Pi\,, \tag{38}$$

$$\frac{\partial\xi}{\partial n} = 0\ ,\ (\sigma,\tau) \in \partial\Pi\,. \tag{39}$$

The Gauss image of the surface patch (6) generating Schwarz's P-surface is shown in figure 7. It has area $\pi/6$.

The image Π in the σ,τ-plane of the parameter domain P of (6) in the (u,v)-plane is shown in figure 8. The fact that Π has interior angles $\pi/2, \pi/2, \pi/2$ and $2\pi/3$ is no contradiction to the conformality of the spherical mapping: The surface $\mathcal{S}$ has an umbilic point corresponding to the fourth corner.

The problem (38), (39) can be compared with the simpler eigenvalue problem

$$\Delta_{(\sigma,\tau)}\xi + 8\mu\xi = 0\ ,\ (\sigma,\tau) \in \Pi\,, \tag{40}$$

$$\frac{\partial\xi}{\partial n} = 0\ ,\ (\sigma,\tau) \in \partial\Pi \tag{41}$$

having the eigenvalues

$$\mu_1 < \mu_2 \leq \mu_3 \leq \dots .$$

It is well known (see, e.g. [6], p. 411) that $\mu_j \leq \lambda_j$. An estimate of [41], [40], p. 463, implies that $8\mu_2$ is not smaller than π^2/d^2, where d is the diameter of Π. It can be seen from figure 3 that $d = 2 - \sqrt{2}$ so that $8\mu_2 \geq \pi^2/(6 - 4\sqrt{2}) = 28.762\dots$, or $\lambda_2 \geq \mu_2 = 3.595\dots$.

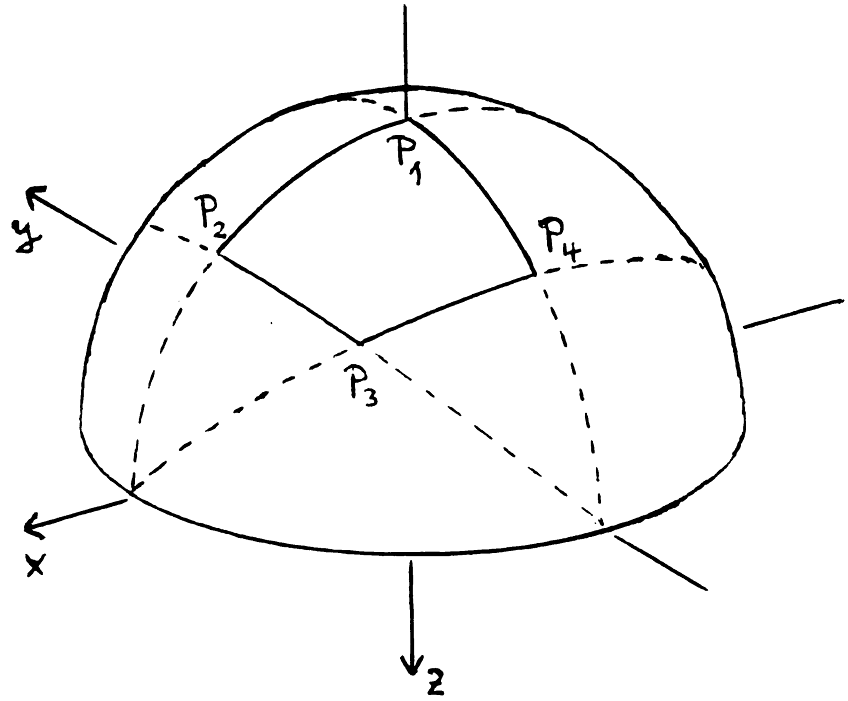

Figure 7: The spherical image of the minimal surface patch (6)

The inequality $\lambda_2 > 1$ guarantees that the assumptions of Wigley's theorem are satisfied regarding the differential operator on the left hand side of (27). Furthermore, since $\tilde{\gamma} \geq 0$ and $\tilde{\alpha}E \geq 0$, the same can be said regarding the other differential equation (28). The bigger of the two bounds established in (28) for the two differential equations will be denoted by $\mathcal{C}$.

13. A complete discussion of the eigenvalue problem (36), (37) for stationary minimal surfaces in tetrahedra, and in more general convex polyhedral domains of R^3, will be published elsewhere. Information concerning the eigenvalues λ_n is important in connection with the questions of stability, isolatedness and H-stability (in the sense of [34], [36], p. 8). It is interesting to note that while these questions have been treated extensively for the case of minimal surfaces with *fixed* boundaries, little is known for the case of *free* boundaries. In his investigations concerning the stability limit of portions of the helicoid ([47], pp. 151-67), H.A. Schwarz discusses a semifree problem (combination of Dirichlet and Neumann conditions). For the case of stationary surfaces in an ellipsoid, see [36].

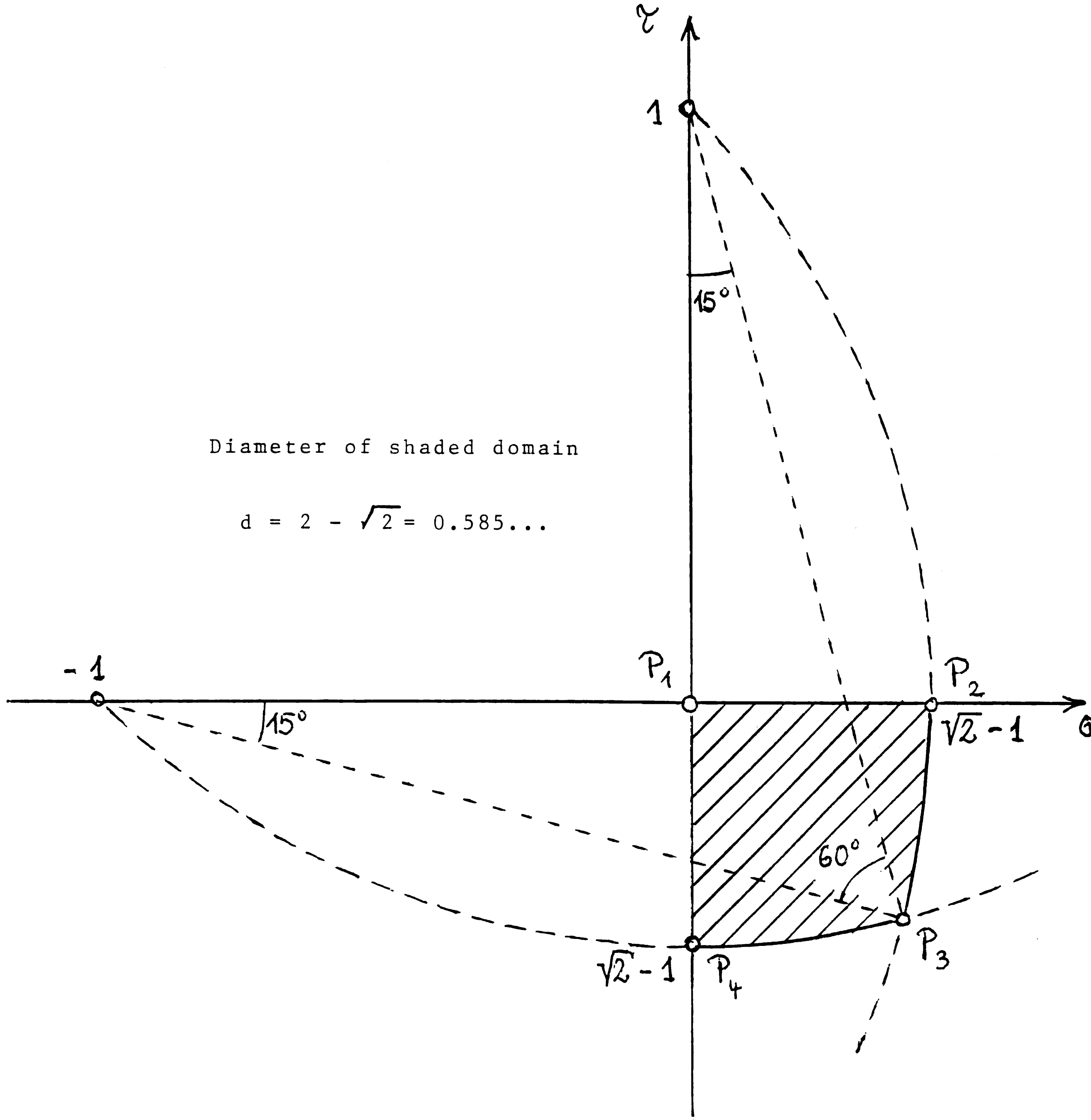

Figure 8: The spherical image in the (σ, τ)-plane
of the minimal surface patch (6)

14. We shall abbreviate the right hand sides in the two differential equations (27), (28) by writing

$$f = f(\zeta, \overline{H}) = f(\zeta(u,v), \overline{H}(u,v)) = 2E\overline{H} + \sum_{m=2}^{\infty} \Omega_m^{(3)}(\zeta)\,, \tag{42}$$

$$g = g(\zeta, \overline{H}) = g(\zeta(u,v), \overline{H}(u,v)) = E\tilde{c} - 2E\overline{H}^3 + \sum_{m=1}^{\infty} \Lambda_{m,1}^{(2)}(\zeta, \overline{H})\,. \tag{43}$$

In the following, we shall omit the subscripts in the $(2,\nu)$-norms, i.e. we shall write $\|\ \|_{2,\nu}(\overline{P}) \equiv \|\ \|$.

Let $\zeta \in C^{2,\nu}(\overline{P})$ and $\overline{H} \in C^{2,\nu}(\overline{P})$ be two functions subject to the preliminary inequalities $\|\zeta\| \le \min(1, Z)$ and $\|\overline{H}\| \le 1$. There is a constant $\mathcal{C}_1$ such that

$$\|f(\zeta, \overline{H})\|_{0,\nu}(\overline{P}) \le \mathcal{C}_1\{\|\overline{H}\| + \|\zeta\|^2\}\,, \tag{44}$$

$$\|g(\zeta, \overline{H})\|_{0,\nu}(\overline{P}) \le m + \mathcal{C}_1\{\|\overline{H}\|^3 + \|\zeta\| \cdot \|\overline{H}\|\}\,. \tag{45}$$

Here we have set $m = |\tilde{c}| \cdot \|E\|_{0,\nu}(\overline{P})$. For two function pairs $(\zeta^{(1)}, \overline{H}^{(1)})$ and $(\zeta^{(2)}, \overline{H}^{(2)})$ as above, and $f^{(k)} = f(\zeta^{(k)}, \overline{H}^{(k)}), g^{(k)} = g(\zeta^{(k)}, \overline{H}^{(k)})$, also

(46)

$$\|f^{(2)} - f^{(1)}\|_{0,\nu}(\overline{P}) \le \mathcal{C}_2\{\|\overline{H}^{(2)} - \overline{H}^{(1)}\| + (\|\zeta^{(1)}\| + \|\zeta^{(2)}\|) \cdot \|\zeta^{(2)} - \zeta^{(1)}\|\}\,,$$

$$\begin{aligned}\|g^{(2)} - g^{(1)}\|_{0,\nu}(\overline{P}) \le \mathcal{C}_2\{&(\|\zeta^{(1)}\| + \|\zeta^{(2)}\| + \|\overline{H}^{(1)}\| + \|\overline{H}^{(2)}\|) \cdot \|\overline{H}^{(2)} - \overline{H}^{(1)}\| \\ &+ (\|\overline{H}^{(1)}\| + \|\overline{H}^{(2)}\|) \cdot \|\zeta^{(2)} - \zeta^{(1)}\|\}\,,\end{aligned} \tag{47}$$

with a constant $\mathcal{C}_2$.

Our boundary value problem for the two equations (27) and (28) will be solved by an alternating technique comparable to the Gauss-Seidel iteration procedure in numerical analysis. Let $(\zeta, \overline{H})$ be a function pair. We shall associate with this pair an image pair of functions $(\zeta_t, \overline{H}_t)$ as follows. First we solve the Neumann problem for (28) with the right hand side $g(\zeta, \overline{H})$, to obtain a unique solution $\overline{H}_t$. Then we solve the Neumann problem for (27) with the right hand side $f(\zeta, \overline{H}_t)$, to obtain the unique solution ζ_t.

Assume that $\|\zeta\| \le a, \|\overline{H}\| \le a$. Then, as a consequence of (35) and (44), (45),

$$\begin{aligned}\|\overline{H}_t\| &\le \mathcal{C}\{m + \mathcal{C}_1(a^3 + a^2)\} \\ &\le m\mathcal{C} + 2\mathcal{C}\mathcal{C}_1 a^2\,,\end{aligned}$$

as well as

$$\begin{aligned}\|\zeta_t\| &\le \mathcal{C}\mathcal{C}_1\{\|\overline{H}_t\| + a^2\} \\ &\le m\mathcal{C}^2\mathcal{C}_1 + \mathcal{C}\mathcal{C}_1(1 + \mathcal{C}\mathcal{C}_1)a^2\,.\end{aligned}$$

Thus, if we choose

$$a \leq \min\,(1, Z, \frac{1}{4\mathcal{C}\mathcal{C}_1}, \frac{1}{2\mathcal{C}\mathcal{C}_1(1+2\mathcal{C}\mathcal{C}_1)})$$

and

$$m \leq \min\,(\frac{a}{2\mathcal{C}}, \frac{a}{2\mathcal{C}^2\mathcal{C}_1})\,,$$

then we can be sure that $\| \zeta_t \| \leq a, \| \overline{H}_t \| \leq a$. Furthermore, for two function pairs $(\zeta^{(1)}, \overline{H}^{(1)}), (\zeta^{(2)}, \overline{H}^{(2)})$ and their images, inequalities (46), (47) imply that

$$\| \overline{H}_t^{(2)} - \overline{H}_t^{(1)} \| \leq \mathcal{C}\mathcal{C}_2\{4a\| \overline{H}^{(2)} - \overline{H}^{(1))} \| + 2a \| \zeta^{(2)} - \zeta^{(1)} \|\}\,,$$

and

$$\begin{aligned} \| \zeta_t^{(2)} - \zeta_t^{(1)} \| &\leq \mathcal{C}\mathcal{C}_2\{\| \overline{H}_t^{(2)} - \overline{H}_t^{(1)} \| + 2a\| \zeta^{(2)} - \zeta^{(1)} \|\}\,, \\ &\leq \mathcal{C}\mathcal{C}_2\{\mathcal{C}\mathcal{C}_2(4a\| \overline{H}^{(2)} - \overline{H}^{(1)} \| + 2a(1+\mathcal{C}\mathcal{C}_2)\| \zeta^{(2)} - \zeta^{(1)} \|\}\,. \end{aligned}$$

It follows that

$$\| \zeta_t^{(2)} - \zeta_t^{(1)} \| + \| \overline{H}_t^{(2)} - \overline{H}_t^{(1)} \| \leq \mathcal{C}_3\{\| \zeta^{(2)} - \zeta^{(1)} \| + \| \overline{H}^{(2)} - \overline{H}^{(1)} \|\}\,, \tag{48}$$

with $\mathcal{C}_3 = 4a\mathcal{C}\mathcal{C}_2(1+\mathcal{C}\mathcal{C}_2)$. Then, if we restrict the bound a further by the condition

$$a < \frac{1}{4a\mathcal{C}\mathcal{C}_2(1+\mathcal{C}\mathcal{C}_2)}\,, \tag{49}$$

we see that the mapping $(\zeta, \overline{H}) \rightarrow (\zeta_t, \overline{H}_t)$ is a contracting mapping. Therefore, as a consequence of the well-known fixed point theorem in the Banach space $C^{2,\nu}(\overline{P})$, our boundary value problem (32) has a unique solution pair $(\zeta(u,v), \overline{H}(u,v))$ stisfying the inequalities $\| \zeta \| \leq a, \| \overline{H} \| \leq a$. By what was said at the end of section 6, these solutions belong to the regularity classes $C^{4,\nu}(\overline{P})$ and $C^{2,\nu}(\overline{P})$, respectively.

This completes the demonstration which established the existence of triply periodic extremal solution surfaces for the energy functional $\mathcal{E}$ with the integrand (16).

15. To summarize: Starting from a triply periodic minimal surface, here the Schwarz P-surface, the solutions of the Neumann problem for the complemented differential equations (27), (28) lead to a family of triply periodic surfaces which are extremal for the free energy functional (8) with integrand (16). These surfaces evolve from the P-surface as the operative (control) parameter c changes from the value 0 to (small) values $\neq 0$. The family depends on the physical parameters (elastic moduli) α and β. (As was mentioned earlier, the parameter γ does not enter the computations.) The case $\alpha \neq 0, \beta = 0$ corresponds to the case of constant mean curvature surfaces computed by D.M. Anderson and depicted in [1], [2]. The control parameter c is related, though generally not in a bijective way, to the volume fraction of the phases separated by our interfaces. Figure 9, taken from [2], p. 362, shows this relation for the case $\beta = 0$.

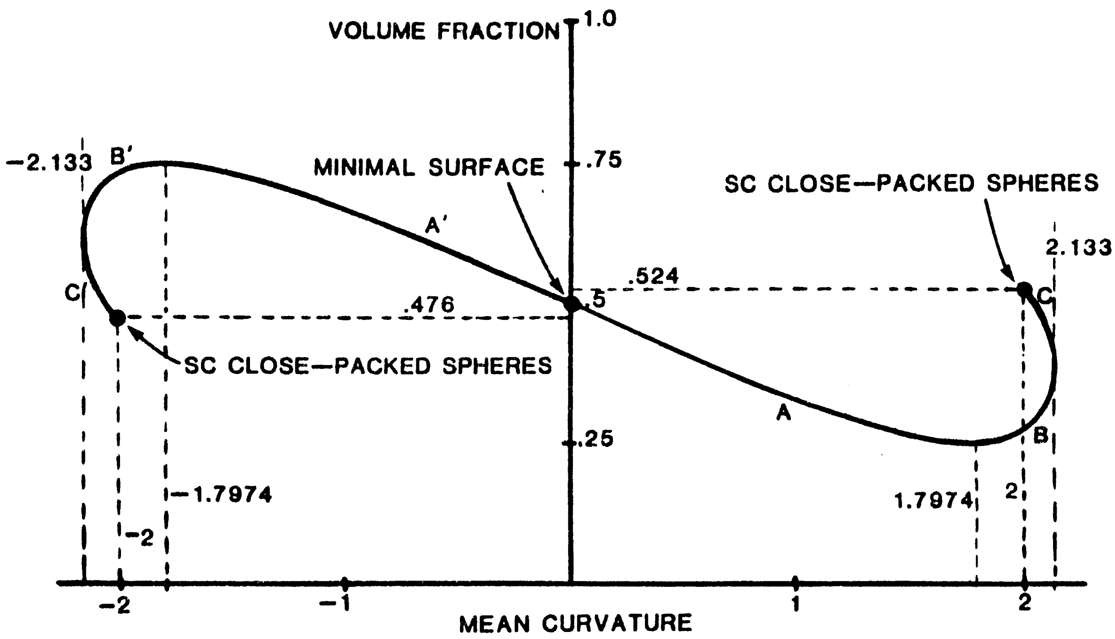

Figure 9: Plot of the volume fraction versus H

It is interesting to note that a volume fraction of approximately 1 : 3 cannot be exceeded by the family of H-surfaces emanating from the P-surface. A theoretical determination of the rate of change of this fraction at the value $c = 0$ in its dependence on the parameter α, and now also on the parameter β, has not yet been carried out.

16. Here we shall discuss briefly the case of energy functionals $\mathcal{E}$ with integrands $\Phi(H, K) = \Psi(H) + \Psi^{(1)}(K)$ which are somewhat more general than (16).

First, assume that $\Psi^{(1)}(K) = -\gamma K$ and that $\Psi(H)$ satisfies the structural conditions (10), (11). An inspection of the earlier existence proof shows that all arguments can be modified so as to cover the present case as well.

Next, consider the more general situation where the K-dependence of functional $\mathcal{E}$ is of the general form $\Psi^{(1)}(K)$, for instance $-\gamma K + \delta K^2$, with $\delta \geq 0$:

$$\Phi(H, K) = \Psi(H) - \gamma K + \delta K^2 \,. \tag{50}$$

The conditions of definiteness for the energy functional $\mathcal{E}$ with the integrand *(50) can be* derived in similar fashion as those for the integrand (16).

A lenghty computation shows that the complemented equation (9) must now be replaced by an equation of the form

$$\Delta\Psi_H + 2\{(2H^2 - K)\Psi_H - 2H\Psi\} + \mathcal{K}^{(1)}(K) + \mathcal{K}^{(2)}(K) = c\,. \tag{51}$$

Here

$$\mathcal{K}^{(1)}(K) = 4H\{K\Psi_K^{(1)}(K) - \Psi^{(1)}(K)\}\,, \tag{52}$$

where $\Psi_H^{(1)} = \partial\Psi^{(1)}/\partial H$ etc., as usual. For the special case of the functional (50), this term reduces to

$$\mathcal{K}^{(1)}(K) = 4\delta H K^2 \tag{53}$$

and leadsto a modification of the left hand side of (28) which becomes

$$\Delta\overline{H} + (1 + 2\delta \mid K \mid) p\overline{H} - 2\tilde{\alpha}E\overline{H}\,. \tag{54}$$

Recall that $K \leq 0$. As for the second term $\mathcal{K}^{(2)}(K)$ in (51), it seems to be difficult to set it in invariant form. Assuming that the surface in question is given by a representation with the help of isothermal parameters, this second term becomes

$$\begin{aligned} \mathcal{K}^{(2)}(K) = {} & \frac{2}{E^2}\Psi_{KK}^{(1)}\{NK_{uu} - 2MK_{uv} + LK_{vv}\} \\ & + \frac{1}{E^3}\Psi_{KK}^{(1)}\{(L-N)(E_uK_u - E_vK_v) + 2M(E_uK_v + E_vK_u)\} \\ & + \frac{2}{E^2}\Psi_{KKK}^{(1)}\{NK_u^2 - 2MK_uK_v + LK_v^2\}\,. \end{aligned} \tag{55}$$

It is necessary to convince oneself that the presence of the term $\mathcal{K}^{(2)}$ does not invalidate the refelection principle of section 9. In the present situation, the proof of the principle can be carried out utilizing the parametric representation for the surface. The surface $\mathcal{T} \equiv \mathcal{T}^+$ is assumed to be known for parameter values in the semidisk $D^+ = \{u, v; u \geq 0, u^2 + v^2 < \varepsilon^2\}$, so that $y_u(0,v) = z_u(0,v) = X(0,v) = 0$. As before, we set $\mathbf{x}(u,v) = \mathbf{x}^+(u,v)$ and define $x^-(u,v) = -x^+(u,v)$, $y^-(u,v) = y^+(u,v)$, $z^-(u,v) = z^+(u,v)$ for $(u,v) \in D^- = \{u, v; u \leq 0, u^2 + v^2 < \varepsilon^2\}$. Now we work with the vector

$$\mathbf{x}(u,v) = \begin{cases} \mathbf{x}^+(u,v)\text{if } (u,v) \in D^+ \\ \mathbf{x}^-(u,v)\text{if } (u,v) \in D^- \end{cases}.$$

It then follows that $X(u,v) = -X(-u,v), Y(u,v) = Y(-u,v), Z(u,v) = Z(-u,v)$, as well as $L(u,v) = L(-u,v), M(u,v) = -M(-u,v), N(u,v) = N(-u,v)$ and $K(u,v) = K(-u,v)$

etc. . Thus $\mathcal{K}^{(2)}(K(u,v)) = \mathcal{K}^{(2)}(K(-u,v))$. The reflection principle is a consequence of this.

Since the expression $\mathcal{K}^{(2)}$ does not generally vanish for minimal surfaces, the convergence proof of section 14 must be modified appropriately, namely,by treating δ as a second control parameter, along with c, that is, by solving our Neumann problems in the presence of the two operational parameters c and $\overline{\delta}$ which are zero at the beginning, and so that $\overline{\delta}$ increases ultimately to the value of the elastic module δ. For the functional (50), Wigley's existence theorem will be applicable as long as $1 + 2\delta \mid K \mid < \mu_2$. In view of the estimate for $\mid K \mid$ given at the end of section 5, this implies the restriction $\delta < .279\ldots$. Of course, the convergence proof based on apriori estimates similar to those developed in section 14 will require that δ be kept probably much smaller than this numerical bound.

The smallness of the expression $\mathcal{K}^{(2)}(K)$ will assure again that the mapping $(\zeta, \overline{H}) \to (\zeta_t, \overline{H}_t)$ is contracting. In view of the structure of (55), the convergence proof must be carried out under the regularity assumptions $\zeta(u,v) \in C^{4,\nu}, \overline{H}^{2,\nu} \in C^{2,\nu}$.

REFERENCES

[1] ANDERSON, D.M., *Studies in the microstructure of microemulsion*, Dissertation, Univ. of Minnesota (1986).

[2] ANDERSON, D.M., H.T. DAVIS, J.C.C. NITSCHE, & L.E. SCRIVEN, *Periodic surfaces of prescribed mean curvature*, Advances in Chem. Phys. **77**, Interscience, New York, 1990, pp. 337–96.

[3] ANDERSON, D.M., GRUNER, S.M., & S. LEIBLER, *Geometrical aspects of the frustration in the cubic phases of lyotropic liquid crystals*, Proc. Nat. Acad. Sci. USA **85** (1988), 5364–8.

[4] ARNOLD, D.H., *The Mécanique physique of Siméon Denis Poisson: The evolution and isolation in France of his approach to physical theory (1800-1840). VI*, Arch. Rat. Mech. Anal. **28** (1983), 343-67, X, Arch. Rat. Mech. Anal. **29** (1984), 287–307.

[5] BARNES, I.S., S.T. HYDE, & B.W. NINHAM, *The caesium chloride zero potential surface is not a minimal surface*, Colloque de Physique C7 **51**, suppl. au no. 23, 1990, 19–24.

[6] COURANT, R., & D. HILBERT, *Methods of mathematical physics. Vol. 1*, Interscience, New York, 1953.

[7] CASORATI, F., *Mesure de la courbure des surfaces suivant l'idée commune. Ses rapports avec les mesures de courbure Gaussienne et moyenne*, Acta Math. **14** (1890-1), 95–110.

[8] DAVIS, H.T., & L.E. SCRIVEN, *Stress and structure in fluid interfaces*, Adv. Chem. Phys. **49** (1982), 357–454.

[9] DZIUK, G., *Über die Stetigkeit teilwise freier Minimalflächen*, Manuscripta math. **36** (1981), 241–51.

[10] FISCHER, W., & E. KOCH, *New surface patches for minimal balance surfaces. I. Branched catenoids*, Acta Cryst. A **45** (1989), 166–9, II.*Multiple catenoids*, Acta Cryst. A **45** (1989), 169–74, III. *Infinite strips*, Acta Cryst. A **45** (1989), 485–90, IV. *Catenoids with spout-like attachments*, Acta Cryst. A **45** (1989), 558–63.

[11] GERMAIN, S., *Recherches sur la théorie des surfaces élastiques*, Huzard–Courcier, Paris, 1821.

[12] GILBARG, D., & N.S. TRUDINGER, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1983.

[13] GOURSAT, E., *Etude des surfaces qui admettant tout les plans de symétrie d'un polyèdre régulier*, Ann. Sci. Ec. Norm. Sup. (3) **4** (1887), 159–200, 241–312, 317–40.

[14] GRISVARD, P., *Elliptic problems in nonsmooth domains*, Pitman Advanced Pub. Program (Monographs and studies in math., 24), Boston, 1985.

[15] GRÜTER, M., S.HILDEBRANDT, & J.C.C. NITSCHE, *On the boundary behavior of minimal surfaces with a free boundary which are not minima of the area*, Manuscripta math. **35** (1981), 387–410..

[16] ———————, *Regularity for surfaces of constant mean curvature with free boundaries*, Acta Math. **156** (1986), 119–52.

[17] HELFRICH, W., *Elastic properties of lipid bilayers: Theory and possible experiments*, Z. Naturforsch. **28c** (1973), 693–703.

[18] HOCH, P.K., *A key concept from the electron theory of metals: History of the Fermi surface 1933–60*, Contemp. Phys. **24** (1983), 3–23.

[19] HOFFMAN, D.A., E.L. THOMAS, D.M. ANDERSON, & C.S. HENKEE, *Periodic area-minimizing surfaces in block-copolymers*, Nature **334** (1988), 599–601.

[20] HYDE, S.T., *Curvature and the global structure of interfaces in surfactant-water systems*, Colloque de Physique C7 **51**, suppl. au no. 23, 1990, 207–28.

[21] HYDE, S.T., I.S. BARNES, & B.W. NINHAM, *Curvature energy of surfactant interfaces confined to the plaquettes of a cubic lattice*, Langmuir, ACS J. Surf. Colloids **6** (1990),1055–62.

[22] ISRAELACHVILI, J.N., *Intermolecular and surface forces: with applications to colloidal and biological systems*, Academic Press, London–Orlando, 1985.

[23] ISRAELACHVILI, J.N., & P.M. MCGUIGGAN, *Forces between surfaces in liquids*, Science **241** (1988), 795–800.

[24] JEENER, P., *Surfaces minimales, objets à quatre dimensions, fractales, champs, arbres, cathédrales, ...*, Nathan, ACL-Editions, Paris, 1986.

[25] KAPOULEAS, N., *Complete constant mean curvature surfaces in euclidean three-space*, Ann. of Math. **131** (1990), 239–330.

[26] KARCHER, H., *The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions*, Manuscr. math. **64** (1989), 291–337.

[27] KIRCHHOFF, G.R., *Über das Gleichgewicht und die Bewegung einer elstischen Scheibe*, J. r. angew. Math. **40** (1850), 51–88.

[28] KIRK, K.L., S.M. GRUNER, & D.L. STEIN, *Thermodynamic model of the lamellar to inverse hexagonal phase transition of lipid membrane-water systems*, Biochem. **23** (1984), 1093–1102.

[29] KITTEL, C., *Introduction to solid state physics*, 6th ed, John Wiley & Sons, Inc., New York–Chichester–Brisbane–Toronto–Singapore, 1986.

[30] KONDRAT'EV, V.A., & O.A. OLEINIK, *Boundary value problems for partial differential equations in non-smooth domains*, Russian Math. Surv. **38.2** (1983), 1–86.

[31] LAWSON, H.B., *Complete minimal surfaces in S^3*, Ann. of Math. (2) **92** (1970), 335–74.

[32] MACKAY, A.L., *Equipotential surfaces in periodic charge distributions*, Angew. Chem., Int. Ed. Engl. **27** (1988), 849–50.

[33] MINKOWSKI, H., *Kapillarität.*, Enc. d. Math. Wissensch. 5.1.9, B.G. Teubner, Leipzig, 1903–21 (completed 1906), 558–613.

[34] NITSCHE, J.C.C., *Concerning the isolated character of solutions of Plateau's problem*, Math. Z. **109** (1969), 393–411.

[35] ———————, *Vorlesungen über Minimalflächen*, Springer-Verlag, Berlin–Heidelberg–New York, 1975.

[36] ———————, *Stationary partitioning of convex bodies*, Arch. Rat. Mech. Anal. **89** (1985), 1-19; corrigendum in Arch. Rat. Mech. Anal. **95** (1986), 389.

[37] ———————, *Lectures on minimal surfaces. Volume I*, Cambridge University Press, Cambridge–New York–
New Rochelle–Melbourne–Sydney, 1989.

[38] ———————, *The existence of surface patches for periodic minimal surfaces*, Colloque de Physique C7 **51**, suppl. au no. 23, 1990, 265–71.

[39] ———————, *Mathematik in Berlin, Born konkreter Geometrie über die Jahrhunderte*, In *Wissenschaft und Stadt*, ed. D. Heckelmann & O. Büsche. Colloquium Verlag, Berlin, to appear.

[40] PAYNE, L.E., *Isoperimetric inequalities and their applications.*., SIAM Review **9** (1967), 453–88.

[41] PAYNE, L.E., & H. WEINBERGER, *An optimal Poincaré inequality for convex domains*, Arch. Rat. Mech. Anal. **5** (1960), 286–92.

[42] PIPPARD, A.B., *Dynamics of conduction electrons*, Gordon and Breach, New York–London–Paris, 1965.

[43] POISSON, S.D., *Mémoire sur les surfaces élastiques*, Mém. Cl. Sci. Mathém. Phys. Inst. de France, 1812, 2nd pt., 167–225.

[44] RIEMANN, B., *Gesammelte mathematische Werke*, 2nd ed., B.G. Teubner, Leipzig, 1892.

[45] ROWLINSON, J.S., & B. WIDOM, *Molecular theory of capillarity*, Clarendon Press, Oxford, 1982.

[46] SCHOEN, A.H., *Infinite periodic minimal surfaces without self-intersections*, NASA Techn. Rep. D-5541, 1970.

[47] SCHWARZ, H.A., *Gesammelte Mathematische Abhandlungen*, Vol. 1, Springer, Berlin, 1890.

[48] ———————, *Zur Theorie der Minimalflächen, deren Begrenzung aus geradlinigen Strecken besteht*, Sitz.-Ber. Königl. Preuss. Akad. d. Wiss. Berlin, Phys.-Math. Cl. (1894), 1237–66.

[49] SCRIVEN, L.E., *Equilibrium bicontinuous structure*, Nature **263** (1976), 123–5.

[50] SMYTH, B., *Stationary minimal surfaces with boundary on a simplex*, Invent. math. **76** (1984), 411–20.

[51] SOMMERFELD, A., & H. BETHE, *Elektronentheorie der Metalle*, Handb.d. Physik, 2nd ed., 24.2, Verlag von Julius Springer, Berlin, 1933, 333–622.

[52] TODHUNTER, I., *A history of the theory of elasticity and of the strength of materials from Galilei to the present time. Vol. I. Galilei to Saint-Venant, 1639–1850*, Cambridge University Press, Cambridge,1886.

[53] VON SCHNERING, H.G., & R. NESPER, *How nature adapts chemical structures to curved surfaces*, Angew. Chem., Intern. Ed. Engl. **11** (1987), 1059–80.

[54] WIGLEY, N.M., *Schauder estimates in domains with corners*, Arch. Rat. Mech. Anal. **104** (1988), 271–6.

[55] WILLMORE, T.J., *Total curvature in Riemannian geometry*, Ellis Horwood Ltd., Chichester, 1982.

[56] WOLFE, T., *From Bauhaus to our house*, Washington Square Press, 1981.

Recent IMA Preprints

#	Author/s Title
699	**László Gerencsér**, Multiple integrals with respect to L-mixing processes
700	**David Kinderlehrer and Pablo Pedregal**, Weak convergence of integrands and the Young measure representation
701	**Bo Deng**, Symbolic dynamics for chaotic systems
702	**P. Galdi, D.D. Joseph, L. Preziosi, S. Rionero**, Mathematical problems for miscible, incompressible fluids with Korteweg stresses
703	**Charles Collins and Mitchell Luskin**, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem
704	**Peter Gritzmann and Victor Klee**, Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces
705	**A. Ronald Gallant and George Tauchen**, A nonparametric approach to nonlinear time series analysis: estimation and simulation
706	**H.S. Dumas, J.A. Ellison and A.W. Sáenz**, Axial channeling in perfect crystals, the continuum model and the method of averaging
707	**M.A. Kaashoek and S.M. Verduyn Lunel**, Characteristic matrices and spectral properties of evolutionary systems
708	**Xinfu Chen**, Generation and Propagation of interfaces in reaction diffusion systems
709	**Avner Friedman and Bei Hu**, Homogenization approach to light scattering from polymer-dispersed liquid crystal films
710	**Yoshihisa Morita and Shuichi Jimbo**, ODEs on inertial manifolds for reaction-diffusion systems in a singularly perturbed domain with several thin channels
711	**Wenxiong Liu**, Blow-up behavior for semilinear heat equations:multi-dimensional case
712	**Hi Jun Choe**, Hölder continuity for solutions of certain degenerate parabolic systems
713	**Hi Jun Choe**, Regularity for certain degenerate elliptic double obstacle problems
714	**Fernando Reitich**, On the slow motion of the interface of layered solutions to the scalar Ginzburg–Landau equation
715	**Xinfu Chen and Fernando Reitich**, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling
716	**C.C. Lim, J.M. Pimbley, C. Schmeiser and D.W. Schwendeman**, Rotating waves for semiconductor inverter rings
717	**W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya**, Multisummability of formal power series solutions of linear ordinary differential equations
718	**Peter J. Olver and Chehrzad Shakiban**, Dissipative decomposition of partial differential equations
719	**Clark Robinson**, Homoclinic bifurcation to a transitive attractor of Lorenz type, II
720	**Michelle Schatzman**, A simple proof of convergence of the QR algorithm for normal matrices without shifts
721	**Ian M. Anderson, Niky Kamran and Peter J. Olver**, Internal, external and generalized symmetries
722	**C. Foias and J.C. Saut**, Asymptotic integration of Navier–Stokes equations with potential forces. I
723	**Ling Ma**, The convergence of semidiscrete methods for a system of reaction-diffusion equations
724	**Adelina Georgescu**, Models of asymptotic approximation
725	**A. Makagon and H. Salehi**, On bounded and harmonizable solutions on infinite order arma systems
726	**San-Yih Lin and Yan-Shin Chin**, An upwind finite-volume scheme with a triangular mesh for conservation laws
727	**J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart**, On the dynamics of fine structure
728	**KangPing Chen and Daniel D. Joseph**, Lubrication theory and long waves
729	**J.L. Ericksen**, Local bifurcation theory for thermoelastic Bravais lattices
730	**Mario Taboada and Yuncheng You**, Some stability results for perturbed semilinear parabolic equations
731	**A.J. Lawrance**, Local and deletion influence
732	**Bogdan Vernescu**, Convergence results for the homogenization of flow in fractured porous media
733	**Xinfu Chen and Avner Friedman**, Mathematical modeling of semiconductor lasers
734	**Yongzhi Xu**, Scattering of acoustic wave by obstacle in stratified medium
735	**Songmu Zheng**, Global existence for a thermodynamically consistent model of phase field type
736	**Heinrich Freistühler and E. Bruce Pitman**, A numerical study of a rotationally degenerate hyperbolic system part I: the Riemann problem
737	**Epifanio G. Virga**, New variational problems in the statics of liquid crystals
738	**Yoshikazu Giga and Shun'ichi Goto**, Geometric evolution of phase-boundaries
739	**Ling Ma**, Large time study of finite element methods for 2D Navier–Stokes equations
740	**Mitchell Luskin and Ling Ma**, Analysis of the finite element approximation of microstructure in micromagnetics
741	**M. Chipot**, Numerical analysis of oscillations in nonconvex problems
742	**J. Carrillo and M. Chipot**, The dam problem with leaky boundary conditions
743	**Eduard Harabetian and Robert Pego**, Efficient hybrid shock capturing schemes
744	**B.L.J. Braaksma**, Multisummability and Stokes multipliers of linear meromorphic differential equations

745 **Tae Il Jeon and Tze-Chien Sun**, A central limit theorem for non-linear vector functionals of vector Gaussian processes
746 **Chris Grant**, Solutions to evolution equations with near-equilibrium initial values
747 **Mario Taboada and Yuncheng You**, Invariant manifolds for retarded semilinear wave equations
748 **Peter Rejto and Mario Taboada**, Unique solvability of nonlinear Volterra equations in weighted spaces
749 **Hi Jun Choe**, Holder regularity for the gradient of solutions of certain singular parabolic equations
750 **Jack D. Dockery**, Existence of standing pulse solutions for an excitable activator-inhibitory system
751 **Jack D. Dockery and Roger Lui**, Existence of travelling wave solutions for a bistable evolutionary ecology model
752 **Giovanni Alberti, Luigi Ambrosio and Giuseppe Buttazzo**, Singular perturbation problems with a compact support semilinear term
753 **Emad A. Fatemi**, Numerical schemes for constrained minimization problems
754 **Y. Kuang and H.L. Smith**, Slowly oscillating periodic solutions of autonomous state-dependent delay equations
755 **Emad A. Fatemi**, A new splitting method for scaler conservation laws with stiff source terms
756 **Hi Jun Choe**, A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities
757 **Haitao Fan**, A vanishing viscosity approach on the dynamics of phase transitions in Van Der Waals fluids
758 **T.A. Osborn and F.H. Molzahn**, The Wigner–Weyl transform on tori and connected graph propagator representations
759 **Avner Friedman and Bei Hu**, A free boundary problem arising in superconductor modeling
760 **Avner Friedman and Wenxiong Liu**, An augmented drift-diffusion model in semiconductor device
761 **Avner Friedman and Miguel A. Herrero**, Extinction and positivity for a system of semilinear parabolic variational inequalities
762 **David Dobson and Avner Friedman**, The time-harmonic Maxwell equations in a doubly periodic structure
763 **Hi Jun Choe**, Interior behaviour of minimizers for certain functionals with nonstandard growth
764 **Vincenzo M. Tortorelli and Epifanio G. Virga**, Axis-symmetric boundary-value problems for nematic liquid crystals with variable degree of orientation
765 **Nikan B. Firoozye and Robert V. Kohn**, Geometric parameters and the relaxation of multiwell energies
766 **Haitao Fan and Marshall Slemrod**, The Riemann problem for systems of conservation laws of mixed type
767 **Joseph D. Fehribach**, Analysis and application of a continuation method for a self-similar coupled Stefan system
768 **C. Foias, M.S. Jolly, I.G. Kevrekidis and E.S. Titi**, Dissipativity of numerical schemes
769 **D.D. Joseph, T.Y.J. Liao and J.-C. Saut**, Kelvin–Helmholtz mechanism for side branching in the displacement of light with heavy fluid under gravity
770 **Chris Grant**, Solutions to evolution equations with near-equilibrium initial values
771 **B. Cockburn, F. Coquel, Ph. LeFloch and C.W. Shu**, Convergence of finite volume methods
772 **N.G. Lloyd and J.M. Pearson**, Computing centre conditions for certain cubic systems
773 **João Palhoto Matos**, Young measures and the absence of fine microstructures in the $\alpha - \beta$ quartz phase transition
774 **L.A. Peletier & W.C. Troy**, Self-similar solutions for infiltration of dopant into semiconductors
775 **H. Scott Dumas and James A. Ellison**, Nekhoroshev's theorem, ergodicity, and the motion of energetic charged particles in crystals
776 **Stathis Filippas and Robert V. Kohn**, Refined asymptotics for the blowup of $u_t - \Delta u = u^p$.
777 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity
778 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximal smoothness of solutions to certain Euler–Lagrange equations from nonlinear elasticity
779 **Jack Carr and Robert Pego**, Self-similarity in a coarsening model in one dimension
780 **J.M. Greenberg**, The shock generation problem for a discrete gas with short range repulsive forces
781 **George R. Sell and Mario Taboada**, Local dissipativity and attractors for the Kuramoto–Sivashinsky equation in thin 2D domains
782 **T. Subba Rao**, Analysis of nonlinear time series (and chaos) by bispectral methods
783 **Nicholas Baumann, Daniel D. Joseph, Paul Mohr and Yuriko Renardy**, Vortex rings of one fluid in another free fall
784 **Oscar Bruno, Avner Friedman and Fernando Reitich**, Asymptotic behavior for a coalescence problem
785 **Johannes C.C. Nitsche**, Periodic surfaces which are extremal for energy functionals containing curvature functions
786 **F. Abergel and J.L. Bona**, A mathematical theory for viscous, free-surface flows over a perturbed plane
787 **Gunduz Caginalp and Xinfu Chen**, Phase field equations in the singular limit of sharp interface problems
788 **Robert P. Gilbert and Yongzhi Xu**, An inverse problem for harmonic acoustics in stratified oceans
789 **Roger Fosdick and Eric Volkmann**, Normality and convexity of the yield surface in nonlinear plasticity

Recent IMA Preprints

#	Author/s	Title
699	**László Gerencsér**, Multiple integrals with respect to L-mixing processes	
700	**David Kinderlehrer and Pablo Pedregal**, Weak convergence of integrands and the Young measure representation	
701	**Bo Deng**, Symbolic dynamics for chaotic systems	
702	**P. Galdi, D.D. Joseph, L. Preziosi, S. Rionero**, Mathematical problems for miscible, incompressible fluids with Korteweg stresses	
703	**Charles Collins and Mitchell Luskin**, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem	
704	**Peter Gritzmann and Victor Klee**, Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces	
705	**A. Ronald Gallant and George Tauchen**, A nonparametric approach to nonlinear time series analysis: estimation and simulation	
706	**H.S. Dumas, J.A. Ellison and A.W. Sáenz**, Axial channeling in perfect crystals, the continuum model and the method of averaging	
707	**M.A. Kaashoek and S.M. Verduyn Lunel**, Characteristic matrices and spectral properties of evolutionary systems	
708	**Xinfu Chen**, Generation and Propagation of interfaces in reaction diffusion systems	
709	**Avner Friedman and Bei Hu**, Homogenization approach to light scattering from polymer-dispersed liquid crystal films	
710	**Yoshihisa Morita and Shuichi Jimbo**, ODEs on inertial manifolds for reaction-diffusion systems in a singularly perturbed domain with several thin channels	
711	**Wenxiong Liu**, Blow-up behavior for semilinear heat equations:multi-dimensional case	
712	**Hi Jun Choe**, Hölder continuity for solutions of certain degenerate parabolic systems	
713	**Hi Jun Choe**, Regularity for certain degenerate elliptic double obstacle problems	
714	**Fernando Reitich**, On the slow motion of the interface of layered solutions to the scalar Ginzburg–Landau equation	
715	**Xinfu Chen and Fernando Reitich**, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling	
716	**C.C. Lim, J.M. Pimbley, C. Schmeiser and D.W. Schwendeman**, Rotating waves for semiconductor inverter rings	
717	**W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya**, Multisummability of formal power series solutions of linear ordinary differential equations	
718	**Peter J. Olver and Chehrzad Shakiban**, Dissipative decomposition of partial differential equations	
719	**Clark Robinson**, Homoclinic bifurcation to a transitive attractor of Lorenz type, II	
720	**Michelle Schatzman**, A simple proof of convergence of the QR algorithm for normal matrices without shifts	
721	**Ian M. Anderson, Niky Kamran and Peter J. Olver**, Internal, external and generalized symmetries	
722	**C. Foias and J.C. Saut**, Asymptotic integration of Navier–Stokes equations with potential forces. I	
723	**Ling Ma**, The convergence of semidiscrete methods for a system of reaction-diffusion equations	
724	**Adelina Georgescu**, Models of asymptotic approximation	
725	**A. Makagon and H. Salehi**, On bounded and harmonizable solutions on infinite order arma systems	
726	**San-Yih Lin and Yan-Shin Chin**, An upwind finite-volume scheme with a triangular mesh for conservation laws	
727	**J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart**, On the dynamics of fine structure	
728	**KangPing Chen and Daniel D. Joseph**, Lubrication theory and long waves	
729	**J.L. Ericksen**, Local bifurcation theory for thermoelastic Bravais lattices	
730	**Mario Taboada and Yuncheng You**, Some stability results for perturbed semilinear parabolic equations	
731	**A.J. Lawrance**, Local and deletion influence	
732	**Bogdan Vernescu**, Convergence results for the homogenization of flow in fractured porous media	
733	**Xinfu Chen and Avner Friedman**, Mathematical modeling of semiconductor lasers	
734	**Yongzhi Xu**, Scattering of acoustic wave by obstacle in stratified medium	
735	**Songmu Zheng**, Global existence for a thermodynamically consistent model of phase field type	
736	**Heinrich Freistühler and E. Bruce Pitman**, A numerical study of a rotationally degenerate hyperbolic system part I: the Riemann problem	
737	**Epifanio G. Virga**, New variational problems in the statics of liquid crystals	
738	**Yoshikazu Giga and Shun'ichi Goto**, Geometric evolution of phase-boundaries	
739	**Ling Ma**, Large time study of finite element methods for 2D Navier–Stokes equations	
740	**Mitchell Luskin and Ling Ma**, Analysis of the finite element approximation of microstructure in micromagnetics	
741	**M. Chipot**, Numerical analysis of oscillations in nonconvex problems	
742	**J. Carrillo and M. Chipot**, The dam problem with leaky boundary conditions	
743	**Eduard Harabetian and Robert Pego**, Efficient hybrid shock capturing schemes	
744	**B.L.J. Braaksma**, Multisummability and Stokes multipliers of linear meromorphic differential equations	

745 **Tae Il Jeon and Tze-Chien Sun**, A central limit theorem for non-linear vector functionals of vector Gaussian processes
746 **Chris Grant**, Solutions to evolution equations with near-equilibrium initial values
747 **Mario Taboada and Yuncheng You**, Invariant manifolds for retarded semilinear wave equations
748 **Peter Rejto and Mario Taboada**, Unique solvability of nonlinear Volterra equations in weighted spaces
749 **Hi Jun Choe**, Holder regularity for the gradient of solutions of certain singular parabolic equations
750 **Jack D. Dockery**, Existence of standing pulse solutions for an excitable activator-inhibitory system
751 **Jack D. Dockery and Roger Lui**, Existence of travelling wave solutions for a bistable evolutionary ecology model
752 **Giovanni Alberti, Luigi Ambrosio and Giuseppe Buttazzo**, Singular perturbation problems with a compact support semilinear term
753 **Emad A. Fatemi**, Numerical schemes for constrained minimization problems
754 **Y. Kuang and H.L. Smith**, Slowly oscillating periodic solutions of autonomous state-dependent delay equations
755 **Emad A. Fatemi**, A new splitting method for scaler conservation laws with stiff source terms
756 **Hi Jun Choe**, A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities
757 **Haitao Fan**, A vanishing viscosity approach on the dynamics of phase transitions in Van Der Waals fluids
758 **T.A. Osborn and F.H. Molzahn**, The Wigner–Weyl transform on tori and connected graph propagator representations
759 **Avner Friedman and Bei Hu**, A free boundary problem arising in superconductor modeling
760 **Avner Friedman and Wenxiong Liu**, An augmented drift-diffusion model in semiconductor device
761 **Avner Friedman and Miguel A. Herrero**, Extinction and positivity for a system of semilinear parabolic variational inequalities
762 **David Dobson and Avner Friedman**, The time-harmonic Maxwell equations in a doubly periodic structure
763 **Hi Jun Choe**, Interior behaviour of minimizers for certain functionals with nonstandard growth
764 **Vincenzo M. Tortorelli and Epifanio G. Virga**, Axis-symmetric boundary-value problems for nematic liquid crystals with variable degree of orientation
765 **Nikan B. Firoozye and Robert V. Kohn**, Geometric parameters and the relaxation of multiwell energies
766 **Haitao Fan and Marshall Slemrod**, The Riemann problem for systems of conservation laws of mixed type
767 **Joseph D. Fehribach**, Analysis and application of a continuation method for a self-similar coupled Stefan system
768 **C. Foias, M.S. Jolly, I.G. Kevrekidis and E.S. Titi**, Dissipativity of numerical schemes
769 **D.D. Joseph, T.Y.J. Liao and J.-C. Saut**, Kelvin–Helmholtz mechanism for side branching in the displacement of light with heavy fluid under gravity
770 **Chris Grant**, Solutions to evolution equations with near-equilibrium initial values
771 **B. Cockburn, F. Coquel, Ph. LeFloch and C.W. Shu**, Convergence of finite volume methods
772 **N.G. Lloyd and J.M. Pearson**, Computing centre conditions for certain cubic systems
773 **João Palhoto Matos**, Young measures and the absence of fine microstructures in the $\alpha - \beta$ quartz phase transition
774 **L.A. Peletier & W.C. Troy**, Self-similar solutions for infiltration of dopant into semiconductors
775 **H. Scott Dumas and James A. Ellison**, Nekhoroshev's theorem, ergodicity, and the motion of energetic charged particles in crystals
776 **Stathis Filippas and Robert V. Kohn**, Refined asymptotics for the blowup of $u_t - \Delta u = u^p$.
777 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity
778 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximal smoothness of solutions to certain Euler–Lagrange equations from nonlinear elasticity
779 **Jack Carr and Robert Pego**, Self-similarity in a coarsening model in one dimension
780 **J.M. Greenberg**, The shock generation problem for a discrete gas with short range repulsive forces
781 **George R. Sell and Mario Taboada**, Local dissipativity and attractors for the Kuramoto–Sivashinsky equation in thin 2D domains
782 **T. Subba Rao**, Analysis of nonlinear time series (and chaos) by bispectral methods
783 **Nicholas Baumann, Daniel D. Joseph, Paul Mohr and Yuriko Renardy**, Vortex rings of one fluid in another free fall
784 **Oscar Bruno, Avner Friedman and Fernando Reitich**, Asymptotic behavior for a coalescence problem
785 **Johannes C.C. Nitsche**, Periodic surfaces which are extremal for energy functionals containing curvature functions
786 **F. Abergel and J.L. Bona**, A mathematical theory for viscous, free-surface flows over a perturbed plane
787 **Gunduz Caginalp and Xinfu Chen**, Phase field equations in the singular limit of sharp interface problems
788 **Robert P. Gilbert and Yongzhi Xu**, An inverse problem for harmonic acoustics in stratified oceans
789 **Roger Fosdick and Eric Volkmann**, Normality and convexity of the yield surface in nonlinear plasticity