NEW COARSE GRID OPERATORS FOR HIGHLY OSCILLATORY COEFFICIENT ELLIPTIC PROBLEMS

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NEW COARSE GRID OPERATORS FOR HIGHLY OSCILLATORY COEFFICIENT ELLIPTIC PROBLEMS

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Abstract

New coarse grid operators are developed for elliptic problems with highly oscillatory coefficients. These types of operators are necessary when the character of the differential equation on coarser grids (or longer wavelengths) is different from that on the fine grid. Elliptic problems for composite materials and different classes of hyperbolic problems are practical examples.

The new coarse grid operators can be constructed directly based on the homogenized differential operators or hierarchically computed from the finest grid. Numerical examples are presented showing that the homogenized form of the equations is very useful in the design of coarse grid operators for multigrid methods.

1 INTRODUCTION

Multigrid methods are usually not effective when applied to problems for which the standard coarse grid operators have significantly different properties from those of the fine grid operators [1,3,5,7,8]. In some of these problems the coarse grid operators should be constructed based on other

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principles than just simple restriction from the finest grid in order to restore the high efficiency of the multigrid method. Elliptic and parabolic equations with strongly variable coefficients and some hyperbolic equations are such problems. One feature in these problems is that the smallest eigenvalues in absolute value do not correspond to very smooth eigenfunctions. It is thus not easy to represent these eigenfunctions on the coarser grids.

We shall investigate elliptic equations with highly oscillatory coefficients,

$$\sum_j \frac{\partial}{\partial x_j} a_j(x, \frac{x}{\epsilon}) \frac{\partial}{\partial x_j} u_\epsilon(x) = f(x), \tag{1}$$

with $a_j(x, \eta)$ strictly positive, continuous and 1-periodic in each component of η. The parameter $\epsilon > 0$ should be regarded as small. This is one class of the problems discussed above for which there exists a fairly complete analytic theory such that a rigorous treatment is possible. This homogenization theory describes the dependence of the large scale features in the solutions on the smaller scales in the coefficients [2]. We shall consider model problems but there are also important practical applications of these equations in the study of elasticity and heat conduction for composite materials.

In this paper we analyze the multigrid methods for equation (1) by introducing new coarse grid operators, based on local or global homogenized operators of the equation. These operators can be numerically calculated from the finest grid operator, by solving a so-called cell problem [2].

The rest of the paper is organized as follows: we introduce our model problem and the theory of homogenization in section 2. In section 3, we briefly outline the multigrid method, then we derive our new coarse grid operators and discuss the convergence theory. In the last section, we present our numerical experiments. In a number of numerical tests we compare the convergence rate for different choices of parameter and coarse grid operators applied to a two dimensional elliptic model problem. We also consider a more realistic problem of heat conduction in a composite structure.

2 Model Problem

2.1 The Partial Differential Equation

Elliptic problems of the form

$$-\nabla \cdot a^{\epsilon}(x,y)\nabla u_{\epsilon} = f(x,y), \quad (x,y) \in \Omega = [0,1] \times [0,1], \tag{2}$$

will be considered, subject to Dirichlet boundary condition

$$u_{\epsilon}|_{\partial\Omega} = 0.$$

The function $a^{\epsilon}(x,y) = a(x/\epsilon, y/\epsilon)$ is strictly positive. And, $f(x,y)$ in the following is always assumed to be smooth. Since it contains a small parameter ϵ, the coefficient highly varies. The small parameter ϵ is the length of oscillation. From homogenization theory [2], it follows,

$$\max_{(x,y)\in\Omega} |u_{\epsilon} - u| \to 0, \quad as \quad \epsilon \to 0,$$

where u satisfies the following effective equation, which doesn't contain any oscillatory coefficients,

$$-A_{11}\frac{\partial^2 u}{\partial x^2} - (A_{12} + A_{21})\frac{\partial^2 u}{\partial y \partial x} - A_{22}\frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in \Omega, \tag{3}$$

subject to the same boundary condition. Here, the constant coefficients can be calculated from the following formula,

$$A_{ij} = \int_{\Omega} a(s_1,s_2)(\delta_{ij} - \frac{\partial \kappa_j}{\partial s_i})ds_1 ds_2, \quad i,j = 1,2,$$

and the auxiliary periodic functions κ_j are given by,

$$-\nabla_s \cdot a(s_1,s_2)\nabla_s \kappa_j = \frac{\partial a(s_1,s_2)}{\partial s_j}, \quad j = 1,2.$$

For simplicity, we consider a model problem. That is, the equation (2) with diagonal oscillatory coefficient,

$$a^{\epsilon}(x,y) = a(\frac{x-y}{\epsilon}). \tag{4}$$

From (3), we know that the corresponding homogenized equation is,

$$-\frac{(\mu+\bar{a})}{2}\frac{\partial^2 u}{\partial x^2}+(\mu-\bar{a})\frac{\partial^2 u}{\partial x \partial y}-\frac{(\mu+\bar{a})}{2}\frac{\partial^2 u}{\partial y^2}=f(x,y), \quad (x,y)\in\Omega, \quad (5)$$

where μ represents the harmonic average of coefficient $a^{\epsilon}(x,y)$, denoted by $\mu = m(1/a^{\epsilon})^{-1}$; and $\bar{a}$ represents the arithmetical average of coefficient $a^{\epsilon}(x,y)$, denoted by $\bar{a} = m(a^{\epsilon})$. Here, the mean value $m(f)$ of a $\epsilon-$periodic function is defined as,

$$m(f)=\frac{1}{\epsilon}\int_0^{\epsilon} f(x)dx.$$

2.2 Finite Difference Discretization

We discretize the unit square Ω into $N \times N$ equal cells with $(N-1)\times(N-1)$ grid points by taking grid step size h in both the x and y directions, where $h=\frac{1}{N}$. Particularly, we take such grid size h, which is smaller but has the same order as the wavelength ϵ. What's more, we require that the ratio of the wavelength to the grid size to be an irrational number, [4-6,8]. We call such a case the ergodic case, in which the difference operator would sample quite well the original differential operator. This is required in order to achieve the convergence of the different scheme to the solution of the differential equation, [4].

Denote $x_i = ih, y_j = jh$, and

$$\begin{aligned} a_{ij}^h &= a^{\epsilon}(x_i-\frac{h}{2},y_j)=a^{\epsilon}(x_i-\frac{h}{2}-y_j), & (6)\\ b_{ij}^h &= a^{\epsilon}(x_i,y_j-\frac{h}{2})=a^{\epsilon}(x_i-y_j+\frac{h}{2}), & (7)\\ f_{ij}^h &= f(x_i,y_j), \quad i,j=0,\cdots,N. & (8)\end{aligned}$$

The standard 5-point finite difference stencil of (2) and (4) at the h-grid level is

$$-D_+^i a_{ij}^h D_-^i u_{ij}^h - D_+^j b_{ij}^h D_-^j u_{ij}^h = f_{ij}^h. \quad (9)$$

D_+^i and D_-^i are the standard forward and backward divided differences in x direction, respectively; similarly, D_+^j and D_-^j are for y direction.

Using vector notation, we can rewrite (9) as

$$L_h U_h = F_h, \quad (10)$$

where L_h, A_j^h are tridiagonal matrices, written as

$$L_h = \frac{1}{h^2} Tridiag[B_{j-1}^h, A_j^h, B_j^h]_{j=1,\cdots,N-1}, \tag{11}$$

$$A_j^h = Tridiag[-a_{i-1j}^h, a_{i-1j}^h + a_{ij}^h + b_{ij}^h + b_{ij-1}^h, -a_{ij}^h]_{i=1,\cdots,N-1},$$

and B_j^h is a diagonal matrix, $B_j^h = Diag[-b_{i,j}^h]_{i=1,\cdots,N-1}$, $j = 1,\cdots,N-1$. Also,

$$U_h = (u_{11}^h, u_{21}^h, \cdots, u_{N-11}^h, \cdots, u_{1N-1}^h, u_{2N-1}^h, \cdots, u_{N-1N-1}^h)^T,$$

$$F_h = (f_{11}^h, f_{21}^h, \cdots, f_{N-11}^h, \cdots, f_{1N-1}^h, f_{2N-1}^h, \cdots, f_{N-1N-1}^h)^T.$$

3 The Multigrid Method

3.1 The Algorithm

The application of the two-level multigrid method to the equation (9) at the n-th iteration usually consists of the following three steps,

(1) Pre-smoothing : Compute $U_h^{n+\frac{1}{2}}$ by applying γ_1 steps of a given iteration method to (9) that starts with U_h^n at the finest grid. For convenience, we introduce the following notation:

$$U_h^{n+\frac{1}{2}} = S^{\gamma_1}(U_h^n, L_h, f_h);$$

(2) Coarser grid correction :

- Restrict the residual: $d_H = I_h^H(f_h - L_h U_h^{n+\frac{1}{2}})$;
- Solve the correction e_H: $L_H e_H = d_H$;
- Interpolate: $\tilde{U} = U_h^{n+\frac{1}{2}} + I_H^h e_H$;

(3) Post-smoothing:

$$U_h^{n+1} = S^{\gamma_2}(\tilde{U}, L_h, f_h).$$

From the above, one can immediately obtain the iteration operator M of the two-grid multigrid method:

$$M = S^{\gamma_2}(I - I_H^h L_H^{-1} I_h^H L_h) S^{\gamma_1}. \tag{12}$$

For the full multigrid method, instead of solving the correction exactly on the coarse H-grid level, we solve it by introducing another coarser grid, and applying the two-grid level multigrid method again. The same procedure can be repeated several times, until the coarsest level. In this paper, we always take the coarse grid step size to be as twice as the previous one. And, we consider the full multigrid cycle, denoted by $V(\gamma_1, \gamma_2)$, with γ_1 pre-smoothing steps and γ_2 post-smoothing steps at each level, till the coarsest grid level with grid step size $\frac{1}{2}$.

For a equation with smoothing coefficients, the standard coarse grid operator approximates quite well the fine grid operator, which, therefore, makes the spectral radius of M small. However, for a partial differential equation with non-smoothing coefficients, this is no longer true. In order to restore the high efficiency of multigrid methods, certain modifications are necessarily needed. For instance, appropriate smoothing iteration methods (such as line Gauss-Seidel method), combined with adapted interpolation operator are often suggested to interface problems. [1, 3]. Meanwhile, to ensure the approximation of the coarse grid operator to the fine grid operator, the variational coarse grid operator is recommended. Nevertheless, the coarse grid operator constructed in this way is awkward and costly. What we do in this paper is to introduce a new principle to define the coarse grid operator, so that a fast multigrid method can be obtained to solve the numerical solution of elliptic equations (1) with highly oscillatory coefficients.

3.2 Construction of Coarse Grid Operators

By considering the homogenized equation and the asymptotic behavior of eigenvalue problem [1,2,6,8,9], we realize that the small eigenvalues of the original oscillatory operator can be approximated by the corresponding homogenized eigenvalues. After a few steps of fine grid smoothing, the error is dominated by the low frequency modes, which can be approximated by the corresponding homogenized ones at the coarser grid level. Based on this idea, we construct the coarse grid operator directly from the homogenized operator. There are two ways to define it.

(1) Global Homogenized Coarse Grid Operator:

If we construct the coarse grid operator directly from the discretization of the corresponding homogenized operator (5), we get the *Global Homogenized Coarse Grid Operator* at H-grid level,

$$L_H = [-\frac{(\mu + \bar{a})}{2} D_+^i D_-^i - \frac{(\mu + \bar{a})}{2} D_+^j D_-^j + \theta(\mu - \bar{a}) D_0^i D_0^j]_{i,j=1,\cdots,\frac{1}{H}-1}, \quad (13)$$

where D_0 denotes the standard center divided difference. And, θ is a parameter.

However, as we may notice, there are many intermediate frequency modes which could still exist in the error in the process of the multigrid method after fine grid smoothing, unless a large number of smoothing iterations are used. These modes can not be approximated by the homogenized eigenmodes at the coarse grid level. Directly applying the global homogenized coarse grid operator is not very suitable in such a situation, and may delay the convergent rate. Based on this, a revised coarse grid operator is then introduced as follows.

(2) Local Homogenized Coarse Grid Operator:

In order to approximate the fine grid operator better, we construct another grid operator, called here the *Local Homogenized Coarse Grid Operator*, which still maintains the form of the homogenized operator as the global homogenized coarse grid operator does. But, the coefficients here are calculated locally. To show details of the derivation, we first divide the whole domain into many cells. For instance, at point (i,j) on the coarse $H-$grid level (see Figure 1 and 2), we define 4 cells, denoted by ***EH, WH, SH, NH***. We calculate the homogenized operator in each cell, and then use it to define the coarse grid operator. For our model problem, the coarse grid operator at H-grid level has the following form,

$$L_H = [-D_+^i a_{ij}^H D_-^i - D_+^j b_{ij}^H D_-^j + \theta D_0^i c_{ij}^H D_0^j + \theta D_0^j c_{ij}^H D_0^i]_{i,j=1,\cdots,\frac{1}{H}-1}, \quad (14)$$

where the coefficients are calculated as follows.

The coefficient a_{ij}^H is the coefficient of $\frac{\partial^2 u}{\partial x^2}$ in the homogenized equation (5) generated in ***WH***-cell. That is,

$$a_{ij}^H = \frac{1}{2}(\mu(WH) + \bar{a}(WH)),$$

where $\bar{a}(WH)$ and $\mu(WH)$ denote the arithmetical average and the harmonic average of all a_{ij}^h, b_{ij}^h in ***WH***-cell, respectively.

The coefficient b_{ij}^H is the coefficient of $\frac{\partial^2 u}{\partial y^2}$ in the homogenized equation (5) generated in ***SH***-cell. That is,

$$b_{ij}^H = \frac{1}{2}(\mu(SH) + \bar{a}(SH)),$$

where $\bar{a}(SH)$ and $\mu(SH)$ denote the arithmetical average and the harmonic average of all a_{ij}^h, b_{ij}^h in ***SH***-cell, respectively.

The coefficient c_{ij}^H is the coefficient of $\frac{\partial^2 u}{\partial x \partial y}$ in the homogenized equation (5) generated in cells ***EH,NH,SH,WH***. That is,

$$c_{ij}^H = \frac{1}{2}(\mu(EH, NH, SH, WH) - \bar{a}(EH, NH, SH, WH)),$$

where $\bar{a}(EH, NH, SH, WH)$ and $\mu(EH, NH, SH, WH)$ denote the arithmetical average and the harmonic average of all a_{ij}^h, b_{ij}^h in EH, NH, SH, WH-cells, respectively.

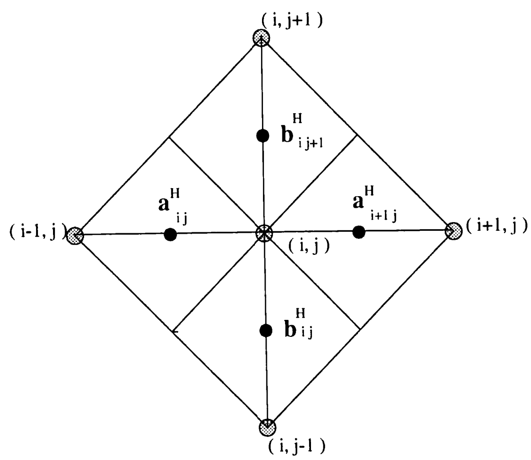

Figure 1: Coefficients for H-grid level at (i, j).

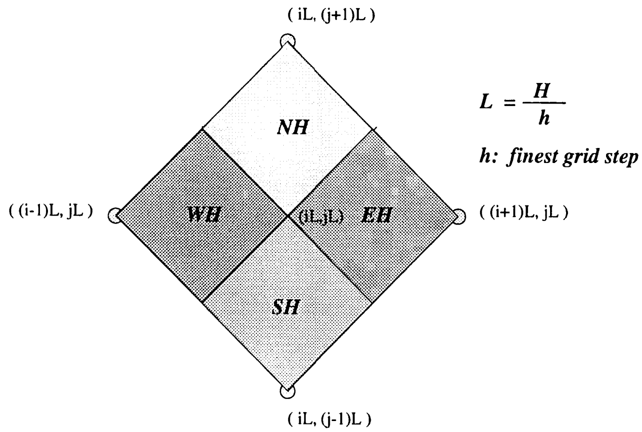

Figure 2: Construction of 4 cells on the coarse $H-$grid level at (i, j) with respect to the finest $h-$grid level at (iL, jL).

3.3 Construction of Interpolation

We apply the harmonic interpolation I_H^h, which is based on the continuity of $a^\epsilon(x,y)\frac{\partial u_\epsilon}{\partial x}$, and $a^\epsilon(x,y)\frac{\partial u_\epsilon}{\partial y}$, [1]. That is (See Figure 3), by setting

$$a^h_{2i-12j} D^i_- u^h_{2i-12j} = a^h_{2i2j} D^i_- u^H_{2i2j},$$

$$b^h_{2i2j-1} D^j_- u^h_{2i2j-1} = b^h_{2i2j} D^j_- u^H_{2i2j},$$

we solve

$$u^h_{2i-12j} = \frac{a^h_{2i-12j} u^H_{2i-22j} + a^h_{2i2j} u^H_{2i2j}}{a^h_{2i-12j} + a^h_{2i2j}};$$

$$u^h_{2i2j-1} = \frac{b^h_{2i2j-1} u^H_{2i2j-2} + b^h_{2i2j} u^H_{2i2j}}{b^h_{2i2j-1} + b^h_{2i2j}}.$$

At point $(2i-1, 2j-1)$, we use the following weighted interpolation,

$$u^h_{2i-12j-1} = \frac{a^h_{2i-12j-1} u^h_{2i-22j-1} + a^h_{2i2j-1} u^h_{2i2j-1} + b^h_{2i-12j-1} u^h_{2i-12j-2} + b^h_{2i-12j} u^h_{2i-12j}}{a^h_{2i-12j-1} + a^h_{2i2j-1} + b^h_{2i-12j-1} + b^h_{2i-12j}};$$

where $i, j = 1, \cdots, N/2-1$.

For the restriction operator I_h^H, we take the transpose of the prolongation, i.e.,

$$I_h^H = (I_H^h)^T.$$

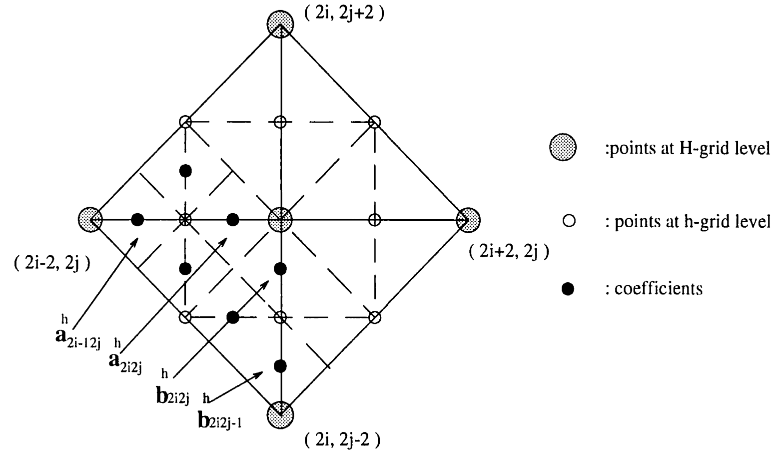

Figure 3: Interpolation from H-grid level to h-grid level.

3.4 Discussion of Theoretical Results

Using the Global Homogenized Coarse Grid Operator, the convergence of the two level method applied to elliptic equations with Dirichlet boundary conditions has been analyzed. [6]. In that paper, two classes of two dimensional elliptic equations are especially studied. The first case is the equation with coefficient oscillatory in x direction only; The second case is the equation with coefficient oscillatory diagonally. We prove that when both ϵ and h go to zeros, without requiring the ratio of h to ϵ to be small, as long as the numerical grids sample well the oscillating coefficients so that there are quite a lot cancellations among the high oscillations, one can improve the number of smoothing iteration in the two level method by using the global homogenized coarse grid operator. More precisely, in order that the two level method converges under the global homogenized coarse grid operator, the iteration number γ which is needed is as follows:

1. For the first case,
$$\gamma \geq Ch^{-1-1/3} \ln h;$$
2. For the second case,
$$\gamma \geq Ch^{-1-2/3} \ln h,$$

if the ratio of h to ϵ is a strict irrational number. For the detail of the proof, we refer to [6].

The theoretical results obtained seem to be a little bit pessimistic. However, the process of the theoretical proof indicates the role of homogenized operator in the convergence of multigrid methods. From a number of numerical experiments, [5], we can see that faster convergence rate in practice can be achieved than that required in the theoretical results. The theoretical convergence results are also better than what can be achieved if the coarse grid operator is constructed directly via averaging, [8]. If we replace the homogenized operator with the averaging coarse grid operator, by same analysis, all we can get is that the number of iteration number needed is $O(h^2)$, which is not even asymptotically better than just direct Jacobi method.

4 Numerical Results

4.1 Example 1

We study the mean rate [10] of convergence for the $V(\gamma_1, \gamma_2)-$cycle multigrid method by using the homogenized coarse grid operators and harmonic interpolation to (9). The mean rate of convergence for $V(\gamma_1, \gamma_2)$ is defined by

$$\rho = \left(\frac{\|L_h u^i - f_h\|_h}{\|L_h u^1 - f_h\|_h}\right)^{\frac{1}{i-1}} \tag{15}$$

where i is the smallest integer satisfying $\|L_h u^i - f_h\|_h \leq 1 \times 10^{-9}$.

In all the numerical experiments, unless noted otherwise, we consider the coefficient $a^\epsilon(x-y) = 2.1 + 2\sin(2\pi(x-y)/\epsilon)$, where the harmonic average μ and arithmetical average $\bar{a}$ are

$$\mu = 0.64,$$

$$\bar{a} = 2.1.$$

The smoothing iteration operator S is based on the following damped Jacobi iteration,

$$S = I - \omega h^2 L_h. \tag{16}$$

The finest grid points are on a 256×256 mesh, the step at the finest grid level h has an irrational relation with the wavelength ϵ, i.e., $\epsilon = \sqrt{2}h$. The step of the coarsest level equals to $1/2$, and ω in (16) is 0.095, which is tested numerically to be the best. In our numerical experiments, we always compare two cases by taking two different values for θ, where θ is a parameter introduced in (13) and (14). When $\theta = 1$, it means that the coarse grid operator used is the homogenized operator; When $\theta = 0$, it means that the coarse grid operator used is the operator in (13) and (14) without the cross terms. In the latter case, the coarse grid operator is no longer the homogenized operator.

In Figure 4, we plot spectral radius ρ for $V(\gamma, \gamma)$ as a function of smoothing step γ. Here, we notice that the rate of convergence for the multigrid method is faster for $\theta = 1$ than that of $\theta = 0$. It is clearer when the smoothing step is taken to be large.

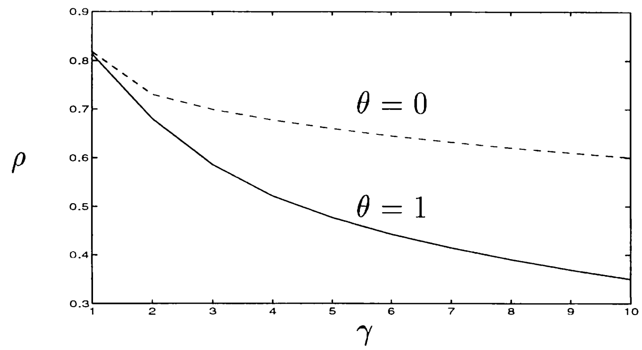

Figure 4: Spectral radius ρ as a function of smoothing step γ.

In Figure 5, we plot log(residual) with respect to the number of $V(3,3)$-cycles.

In Figure 4, 5, we applied the local homogenized coarse grid operator for all coarse grid levels. In fact, after a few coarse grid levels, the local homogenized coarse grid operator can be replaced by the global homogenized coarse grid operator. That's, we only need apply the local homogenized coarse grid operator at a few first coarse grid levels, then apply the global homogenized coarse grid operator at the rest of the coarse grid levels. Therefore, the computational number for the construction of the coefficients can be reduced. In Figure 6, we plot the spectral radius as a function of a level variable, after which we switch from the local homogenized coarse grid operator to the global homogenized coarse grid operator. This figure gives us some idea about how the process of eigenmodes of error in the multigrid method is reduced. Namely, after the finest grid level smoothing, there indeed exist

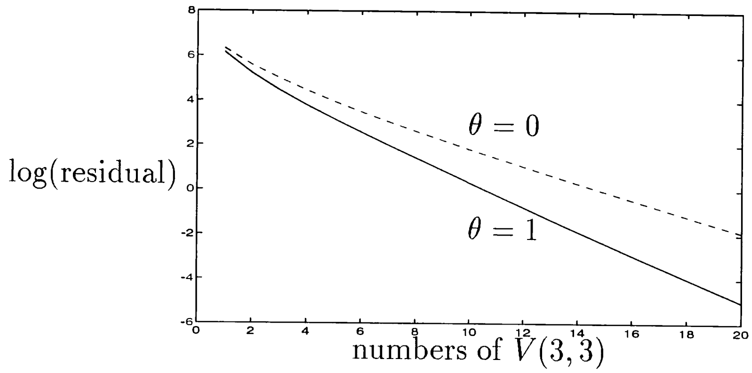

Figure 5: $\log(residual)$ as a function of number of $V(3,3)$.

many intermediate eigenmodes which are not quite close to the homogenized ones.

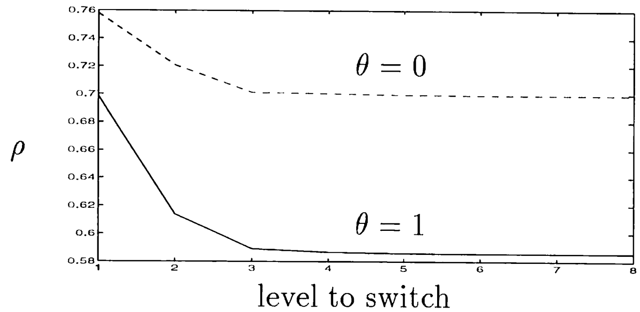

Figure 6: Spectral radius ρ as a function of a level variable for $V(3,3)$. And, x-axis represents the level number from finest to coarsest.

In Figure 7, we fix pre-smoothing to be 3, and calculate the spectral radius as a function of post-smoothing for $V(3,\gamma)$. From Figure 7, we can see that the more steps we provide for the post-smoothing, the more efficiency of the method with the cross terms. In fact this improvement occurs using either pre- or post-smoothing than that without the cross term's.

In Figure 8, we plot the spectral radius as a function of variable θ for $V(3,3)$. From this picture, we can see that with cross term, the convergence of the multigrid methods can be much improved.

In Figure 9, we plot the spectral radius as a function of $\frac{1}{h}$ for $V(3,3)$, where h is the grid step size of the finest level.

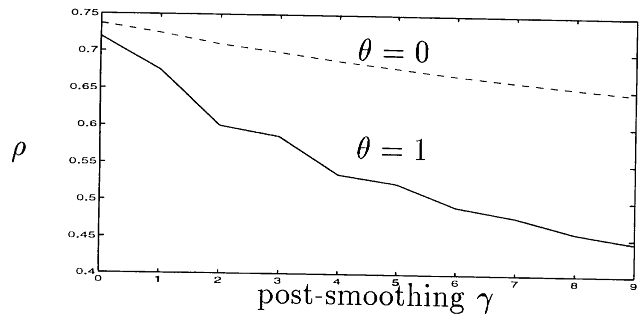

Figure 7: Spectral radius ρ as a function of post-smoothing γ for $V(3, \gamma)$.

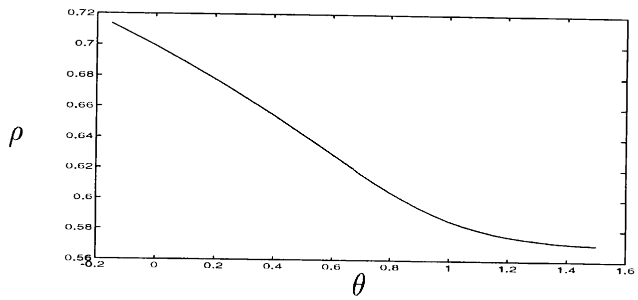

Figure 8: Spectral radius ρ as a function of variable θ.

We have showed by numerical experiments in vary different aspects that the homogenized operator is the correct one to define the coarse grid operator. With it, the rate of convergence of multigrid methods can be much improved. So far, in order to isolate the influence of the coarse grid approximation we have kept the smoothing operator fixed. It obviously also affect the performance. If we use SOR iteration method in (16), the convergence rate can be improved. We compare the convergence rate by choosing damped Jacobi iteration and SOR iteration in Figure 10.

4.2 Example 2

Consider a practical problem, described in Figure 11. This can be viewed as a wall with a composite material for insulation in the center. We are interested in the heat conduction in such a composite structure. The governing equation

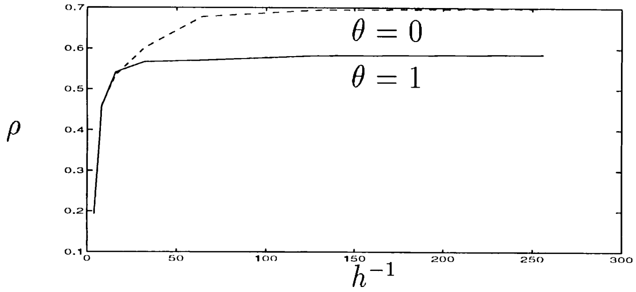

Figure 9: Spectral radius ρ as a function of $\frac{1}{h}$.

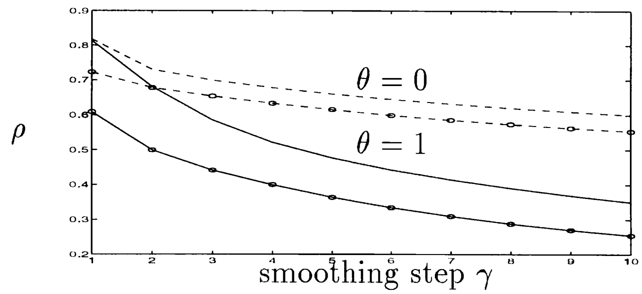

Figure 10: Spectral radius ρ as a function of smoothing steps γ for $V(\gamma, \gamma)$. Lines with circle for SOR iteration; otherwise, for Jacobi.

has the form as (2),

$$-\frac{\partial}{\partial x} C(x,y) \frac{\partial U}{\partial x} - \frac{\partial}{\partial y} C(x,y) \frac{\partial U}{\partial y} = 100,$$

in a rectangular domain $(x, y) \in \Omega = (0,1) \times (0,2)$. The boundary conditions and other parameters are given as indicated in Figure 11. Although the coefficient doesn't satisfy the periodic assumption, which is needed in homogenization theory, it still preserves some essential property as before from probablistic point view. Namely, it is highly oscillatory in the middle part of the domain. Standard discretization of the equation contains almost random distributed magnitude coefficients by taking $\epsilon = \sqrt{2}h$. In such a case, the coarse grid operators can still be generated by the similar idea as we introduce in the previous section. Since the conductivity here is strongly vary in x direction, and has two interfaces in y direction, the structure of the

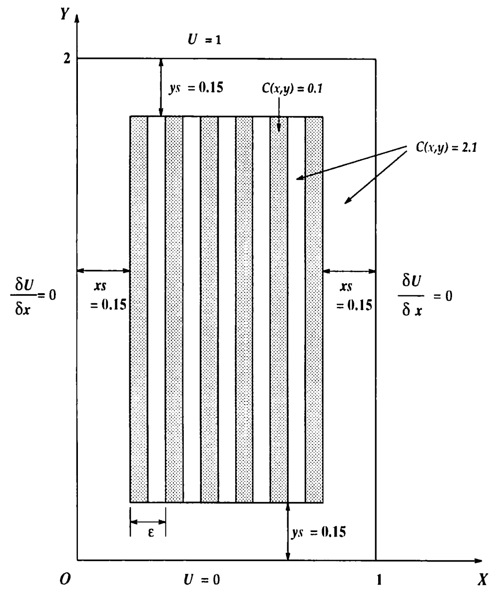

Figure 11: Model

homogenized coarse grid operator has approximately the following form,

$$-A_{11}\frac{\partial^2 U}{\partial x^2} - A_{22}\frac{\partial^2 U}{\partial y^2},$$

where A_{11} and A_{22} are defined by the following *MGH*. Different way of calculations of the coefficients for A_{11} and A_{22} leads us to different coarse grid operators. The following are specially what we are interested in. We name them in order:

MGH : A_{11} is generated by the local harmonic average of the coefficients, and A_{22} is generated by the local arithmetical average of the coefficients;

MGA : Both A_{11} and A_{22} are generated by the local arithmetical average of the coefficients;

SOR : Without any coarse grid correction, only apply SOR smoothing iteration directly.

In Table 1, we calculate the main rate ρ of the multigrid method $V(2L, 2L)$, with $\omega = 1.7$ in SOR. We compare the main rate ρ of convergence of the multigrid method, using the above three different methods. The number of grid points at finest level is 2^L. The grid step at coarsest level in the multigrid methods is $\frac{1}{2}$.

L	3	4	5	6	7	8
MGH	0.1514	0.3779	0.4005	0.3922	0.4752	0.4832
MGA	0.1416	0.3084	0.5168	0.5518	0.6358	0.7066
SOR	0.4596	0.7028	0.8843	0.9595		

Table 1: Spectral radius ρ.

All the examples indicate that the homogenized coarse grid operators are very useful to improve the rate of convergence of multigrid methods to elliptic equations with oscillatory coefficients. The effect is more pronounced in the asymptotic region. That is, the number of smoothing iteration is large.

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