A TWO PARAMETER FAMILY OF PENSION CONTRIBUTION FUNCTIONS AND STOCHASTIC OPTIMIZATION

BY

THOMAS O'BRIEN

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455

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#	Author(s) Title
78	**Abstracts** for the Workshop on Bayesian Analysis in Economics and Game Theory
79	**G. Chichilnisky, G.M.Heal**, Existence of a Competitive Equilibrium in L_p and Sobolev Spaces
80	**Thomas Seidman**, Time-dependent Solutions of a Nonlinear System in Semiconductory Theory, II: Boundedness and Periodicity
81	**Yakar Kannai**, Engaging in R&D and the Emergence of Expected Non-convex Technologies
82	**Herve Moulin**, Choice Functions over a Finite Set: A Summary
83	**Herve Moulin**, Choosing from a Tournament
84	**David Schmeidler**, Subjective Probability and Expected Utility Without Additivity
85	**I.G. Kevrekidis, R. Aris, L.D. Schmidt, and S. Pelikan**, The Numerical Computation of Invariant Circles of Maps
86	**F. William Lawvere**, State Categories, Closed Categories, and the Existence Semi-Continuous Entropy Functions
87	**F. William Lawvere**, Functional Remarks on the General Concept of Chaos
88	**Steven R. Williams**, Necessary and Sufficient Conditions for the Existence of a Locally Stable Message Process
89	**Steven R. Williams**, Implementing a Generic Smooth Function
90	**Dilip Abreu**, Infinitely Repeated Games with Discounting: A General Theory
91	**J.S. Jordan**, Instability in the Implementation of Walrasian Allocations
92	**Myrna Holtz Wooders, William R. Zame**, Large Games: Fair and Stable Outcomes
93	**J.L. Noakes**, Critical Sets and Negative Bundles
94	**Graciela Chichilnisky**, Von Neumann-Morgenstern Utilities and Cardinal Preferences
95	**J.L. Ericksen**, Twinning of Crystals
96	**Anna Nagurney**, On Some Market Equilibrium Theory Paradoxes
97	**Anna Nagurney**, Sensitivity Analysis for Market Equilibrium
98	**Abstracts** for the Workshop on Equilibrium and Stability Questions in Continuum Physics and Partial Differential Equations
99	**Millard Beatty**, A Lecture on Some Topics in Nonlinear Elasticity and Elastic Stability
100	**Filomena Pacella**, Central Configurations of the N-Body Problem via the Equivariant Morse Theory
101	**D. Carlson and A. Hoger**, The Derivative of a Tensor-valued Function of a Tensor
102	**Kenneth Mount**, Privacy Preserving Correspondence
103	**Millard Beatty**, Finite Amplitude Vibrations of a Neo-hookean Oscillator
104	**D. Emmons and N. Yannelis**, On Perfectly Competitive Economies: Loeb Economies
105	**E. Mascolo and R. Schianchi**, Existence Theorems in the Calculus of Variations
106	**D. Kinderlehrer**, Twinning of Crystals (II)
107	**R. Chen**, Solutions of Minimax Problems Using Equivalent Differentiable Equations
108	**D. Abreu, D. Pearce, and E. Stacchetti**, Optimal Cartel Equilibria with Imperfect Monitoring
109	**R. Lauterbach**, Hopf Bifurcation from a Turning Point
110	**C. Kahn**, An Equilibrium Model of Quits under Optimal Contracting
111	**M. Kaneko and M. Wooders**, The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results
112	**Haim Brezis**, Remarks on Sublinear Equations
113	**D. Carlson and A. Hoger**, On the Derivatives of the Principal Invariants of a Second-order Tensor
114	**Raymond Deneckere and Steve Pelikan**, Competitive Chaos
115	**Abstacts** for the Workshop on Homogenization and Effective Moduli of Materials and Media
116	**Abstracts** for the Workshop on the Classifying Spaces of Groups
117	**Umberto Mosco**, Pointwise Potential Estimates for Elliptic Obstacle Problems
118	**J. Rodrigues**, An Evolutionary Continuous Casting Problem of Stefan Type
119	**C. Mueller and F. Weissler**, Single Point Blow-up for a General Semilinear Heat Equation

#	Author(s) Title
120	**D.R.J. Chillingworth**, Three Introductory Lectures on Differential Topology and its Applications
121	**Giorgio Vergara Caffarelli**, Green's Formulas for Linearized Problems with Live Loads
122	**F. Chiarenza and N. Garofalo**, Unique Continuation for Nonnegative Solutions of Schrodinger Operators
123	**J.L. Ericksen**, Constitutive Theory for some Constrained Elastic Crystals
124	**Minoru Murata**, Positive solutions of Schrodinger Equations
125	**John Maddocks and Gareth P. Parry**, A Model for Twinning
126	**M. Kaneko and M. Wooders**, The Core of a Game with a Continuum of Players and Finite Coalitions: Nonemptiness with Bounded Sizes of Coalitions
127	**William Zame**, Equilibria in Production Economies with an Infinite Dimensional Commodity Space
128	**Myrna Holtz Wooders**, A Tiebout Theorem
129	**Abstracts** for the Workshop on Theory and Applications of Liquid Crystals
130	**Yoshikazu Giga**, A Remark on A Priori Bounds for Global Solutions of Semi-linear Heat Equations
131	**M. Chipot and G. Vergara-Caffarelli**, The N-Membranes Problem
132	**P.L. Lions and P.E. Souganidis**, Differential Games and Directional Derivatives of Viscosity Solutions of Isaacs' Equations II
133	**G. Capriz and P. Giovine**, On Virtual Effects During Diffusion of a Dispersed Medium in a Suspension
134	**Fall Quarter Seminar Abstracts**
135	**Umberto Mosco**, Wiener Criterion and Potential Estimates for the Obstacle Problem
136	**Chi-Sing Man**, Dynamic Admissible States, Negative Absolute Temperature, and the Entropy Maximum Principle
137	**Abstracts** for the Workshop on Oscillation Theory, Computation, and Methods of Compensated Compactness
138	**Arie Leizarowitz**, Tracking Nonperiodic Trajectories with the Overtaking Criterion
139	**Arie Leizarowitz**, Convex Sets in R^n as Affine Images of some Fixed Set in R
140	**Arie Leizarowitz**, Stochastic Tracking with the Overtaking Criterion
141	**Abstracts** from the Workshop on Amorphous Polymers and Non-Newtonian Fluids
142	**Winter Quarter Seminar Abstracts**
143	**D.G. Aronson and J.L. Vazquez**, The Porous Medium Equation as a Finite-spee Approximation to a Hamilton-Jacobi Equation
144	**E. Sanchez-Palencia and H. Weinberger**, On the Edge Singularities of a Composite Conducting Medium
145	**Jon C. Luke**, Soliton Solutions in a Class of Fully Discrete Nonlinear Wave Equations
146	**Chi-Sing Man and H. Cohen**, A Coordinate-Free Approach to the Kinematics of Membranes
147	**J.L. Lions**, Asymptotic Problems in Distributed Systems
148	**Reiner Lauterbach**, An Example of Symmetry Breaking with Submaximal Isotrop Subgroup
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150	**B. Bozar-Karaklewicz and Jerry Bona**, Wave-dominated Shelves: A Model of Sand-Ridge Formation by Progressive, Infragravity Waves
151	**Abstracts** from the Workshop on Dynamical Problems in Continuum Physics
152	**V.I. Oliker**, The problem of Embedding S^n into R^{n+1} with Prescribed Gauss
153	**R. Batra**, The force on a Lattice Defect in an Elastic Body
154	**J. Fleckinger and Michael Lapidus**, Eigenvalues of Elliptic Boundary Value Problems with and Indefinite Weight Function
155	**R. Kohn and M. Vogelius**, Thin Plates with Rapidly Varying Thickness, and Their relation to Structural Optimization
156	**M. Gurtin**, Some Results and Conjectures in the Gradient Theory of Phase Transitions
157	**A. Novick-Cohen**, Energy Methods for the Cahn-Hilliard Equation

A TWO PARAMETER FAMILY OF PENSION CONTRIBUTION FUNCTIONS AND STOCHASTIC OPTIMIZATION

Thomas O'Brien

In a previous article the author has suggested a linear function of $A(t)$ (present value of future benefits) and $F(t)$ (fund) as pension contribution function in place of the form given in Trowbridge (1963) which is a one-parameter family of funding methods. Here we provide some theoretical justification for such a method by showing that, in the simplified model of this paper, the optimal solution of a stochastic control problem yields, as contribution function, an affine function of $A(t)$ and $F(t)$.

Keywords: Pension funding dynamics, Stochastic Control.

1. INTRODUCTION

In Trowbridge (1963), the unfunded present value family of pension funding methods is defined by the contribution formula

$$C_t = (k + d)(V_t - F_{t-1})$$

where V_t is the present value of benefits at time t, F_{t-1} is the fund amount at time $t - 1$, d is the discount rate, and k is a positive number less than 1. By suitable choice of k this method can be used to achieve any level of funding from class I through class V as described in Trowbridge (1952). However, the convergence to the ultimate funding level may be quite slow. This is especially true for small values of k. In Gasiewski (1985), computer simulations are carried out illustrating the use of a 2-parameter funding family. The contribution family is of the form $C(t) = c_1A(t) - c_2F(t)$ with c_1 and c_2 positive. By varying the parameters c_1 and c_2 both the ultimate level of funding and the rate of convergence to this level are controlled.

In the second section of this paper we outline the dynamic programming approach in controlled diffusion problems. In the final section we model the pension problem as a controlled diffusion process with the contribution function as control and then solve the problem obtaining as contribution function an affine function of $A(t)$ and $F(t)$.

2. CONTROLLED DIFFUSION PROCESSES

We give a brief sketch of the technique to be used in section 3. For more details see Arnold (1974) and Fleming and Rishel (1975).

We consider a system described by a stochastic differential equation

$$dX_t = f(t,X_t,u(t,X_t))dt + G(t,X_t,u(t,X_t))dW_t$$
$$X_{t_0} = c \qquad t \geq t_0 \tag{2.1}$$

where X_t and $f(t,X_u)$ assume values in R^d, $G(t,x,u)$ is $d \times m$ matrix valued and W_t is an m-dimensional Wiener process. If f and G are continuous and satisfy growth and Lipschitz conditions on $[t_0,T]$ then 2.1 has a unique solution and the resulting stochastic process, X_t, is a d-dimensional diffusion process. The control function, u, takes values in some R^p. The infinitesimal generator of this process is the differential operator

$$L^u = \frac{\partial}{\partial s} + \sum_{i=1}^{d} f_i(s,x,u) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} (G(s,x,u)G(s,x,u)')_{ij} \frac{\partial^2}{\partial x_i \partial x_j} . \tag{2.2}$$

The **costs** arising from the choice of control function u starting at time s in state X up to the instant T are given by

$$J(s,x,u) = E_{s,x}(\int_s^T k(r,X_t,u(r,X_r))dr + M(T,X_T)) \tag{2.3}$$

Here $E_{s,x}$ means expectation conditioned on being in state X at the time s. The function k represents running costs and M represents costs associated with being at X_T at the final time T. In our appliction, M will be zero. We seek an <u>optimal control function</u>, that is a control u^* from among a set U of admissible control functions which minimizes the function 2.3. We will write

$$W(s,x) = \min_{u \in U} J(s,x,u).$$

It follows from Bellman's optimality principal (a control function is optimal on $[t_0,T]$ iff it is optimal on every subinterval of the form $[s,T]$ where $t_0 \leq s \leq T$) that $W(s,x)$ satisfies Bellman's equation

$$0 = \min_u (L^u W(s,X) + k(s,x,u)) \qquad t_0 \leq s \leq T \tag{2.4}$$

with the end condition $W(T,X) = M(T,X)$. Thus the technique is to find the function $u^*(s,x,W)$ which minimizes $L^u W(s,x) + k(s,x,u)$ for each (s,x) in

$[t_0,T] \times R^d$. Substitution of this function u^* into the Bellman equation yields a partial differential equation

$$L^{u^*}W(s,x) + k(s,x,u^*) = 0 \qquad t_0 \leqslant s \leqslant T \tag{2.5}$$

with end condition $W(T,X) = M(T,X)$. The solution W is then inserted into the function u^* yielding the optimal control function which, slightly abusing notation, we write as

$$u^*(s,x) = u^*(s,x,W(s,x)).$$

3. OPTIMAL PENSION FUNDING

The genesis of this work may be found in the deterministic pension fund models of Bowers, Hickman, and Nesbitt (1976, 1979). In a previous paper the current author studied a stochastic pension fund model derived from the Bowers, Hickman, Nesbitt work. This effort also makes use of the basic framework of these earlier papers. The fund is for active plan members and the basic functions required for our model are:

$T(t)$	Annual rate of terminal funding normal cost for the plan at time t.
$A(t)$	Present value of future benefits at time t for the active members.
$F(t)$	The fund at time t.

In the above mentioned papers, population and salary were assumed to follow exponential growth paths. As a result, the terminal funding normal cost function had the form $T(t) = e^{\tau t}T_0$ where T_0 was the initial value. Here, in order to avoid later linearizations we will assume a linearly growing terminal funding normal cost. Thus $T(t) = (\tau t + 1)T_0 \quad t \geq 0$. It is assumed that all members enter the plan at age a and that all retirements occur at age r. Thus

$$A(t) = \int_a^r e^{-\delta_1(r-x)} T(t + r - x)dX = T_0\int_a^r e^{-\delta_1(r-x)} (\tau(t + r - x) + 1)dX$$

Here δ_1 is the fixed rate at which we discount for valuation purposes. Carrying out the integration and then differentiating with respect to t we obtain

$$\frac{dA(t)}{dt} = \frac{\tau}{\delta_1} T_0(1 - e^{-\delta_1(r-a)}) \tag{3.1}$$

Now let $X_t = (A(t),F(t))'$, the state vector of our system and let the contribution function (thought of as a control) be denoted $u(t,X_t)$. We then have

$$\frac{dF(t)}{dt} = u(t,X_t) + \delta F(t) - (\tau t + 1)T_0 \tag{3.2}$$

We next consider the growth rate τ and the fund earning rate δ as stochastic variables with $\tau = \tau_0 + B_1\xi_1$ and $\delta = \delta_0 + B_2\xi_2$ where ξ_1 and ξ_2 are independent white noise processes. Substituting these into 3.1 and 3.2 and combining the two into a single vector equation we obtain

$$dX_t = \begin{bmatrix} \alpha_1 \\ u(t,X_t) + \delta_0 F(t) - (\tau_0 t + 1)T_0 \end{bmatrix} dt + \begin{bmatrix} \beta_1 & 0 \\ -B_1 T_0 t & B_2 F(T) \end{bmatrix} dW_t$$

where

$$\alpha_1 = \frac{\tau_0}{\delta_1} T_0(1 - e^{-\delta_1(r-a)})$$

$$\beta_1 = \frac{B_1}{\delta_1} T_0(1 - e^{-\delta_1(r-a)})$$

and W_t is a 2-dimensional Wiener process.

We now consider the form of the cost function to be minimized. The control function u represents a real cost. Further, we are aiming for a certain level of funding as measured by the fund ratio $\frac{F(t)}{A(t)}$. Thus suppose we want a fund ratio $\frac{F(t)}{A(t)} = \eta$. Our cost function will penalize deviation from this goal. Consider a finite horizon from initial time 0 to time T and take

$$J(s,x,u) = E_{s,y} \int_s^T e^{-\rho t}(u^2 + \beta(\eta A - F)^2)dt$$

Here, as in much of the computation below we delete the arguments of some of the functions involved in order to avoid cumbersome expressions. With $W(s,x) = \min_{u \varepsilon U} J(s,x,u)$ the Bellman equation is

$$0 = W_s + \min_{u \varepsilon U} [\alpha_1 W_A + (u + \delta_0 F - (\tau_0 s + 1)T_0)W_F + \frac{1}{2} \beta_1^2 W_{AA}$$

$$- B_1 \beta_1 T_0 s W_{AF} + \frac{1}{2} (B_2^2 F^2 + B_1^2 T_0^2 s^2) W_{FF} + e^{-\rho s}[u^2 + \beta(\eta A - F)^2]] \tag{3.3}$$

Elementary calculus yields the minimizing u,

$$u^* = \frac{-W_F e^{\rho s}}{2} \tag{3.4}$$

Substituting u^* into 3.3 we get the partial differential equation

$$0 = W_s + \alpha_1 W_A + \left(\frac{-W_F e^{\rho s}}{2} + \delta_0 F - (\tau_0 s + 1)T_0\right)W_F + \frac{1}{2} \beta_1^2 W_{AA}$$

$$- \beta_1 B_1 T_0 s W_{AF} + \frac{1}{2} (B_2^2 F^2 + B_1^2 T_0^2 s^2) W_{FF} + e^{-\rho s} \left[\frac{W_F^2 e^{2\rho s}}{4} + \beta(\eta A - F)^2\right] \tag{3.5}$$

with boundary condition $W(T,X) \equiv 0$.

We try a solution of the form $X'QX + P'X + r(s)$ where

$$Q = \begin{bmatrix} q_1(s) & q_2(s) \\ q_2(s) & q_3(s) \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} p_1(s) \\ p_2(s) \end{bmatrix}$$

The function W and its needed partials are then:

$$W = q_1(s)A^2 + 2q_2(s)AF + q_3(s)F^2 + p_1(s)A + p_2(s)F + r(s)$$

$$W_s = \dot{q}_1(s)A^2 + 2\dot{q}(s)AF + \dot{q}_3(s)F^2 + \dot{p}_1(s)A + \dot{p}_2(s)F + \dot{r}(s)$$

$$W_A = 2(q_1(s)A + q_2(s)F) + p_1(s); \quad W_F = 2(q_2(s)A + q_3(s)F) + p_2(s)$$

$$W_{AA} = 2q_1(s) \; ; \quad W_{FF} = 2q_3(s) \; ; \quad W_{AF} = 2q_2(s)$$

Substituting these into 3.5 yields

$$\begin{aligned} 0 = {} & \dot{q}_1(s)A^2 + 2\dot{q}(s)AF + \dot{q}_3(s)F^2 + \dot{p}_1(s)A + \dot{p}_2(s)F + \dot{r}(s) \\ & + \alpha_1(2(q_1(s)A + q_2(s)F) + p_1(s)) - \frac{e^{\rho s}}{4}[2(q_2(s)A + q_3(s)F) + p_2(s)]^2 \\ & + (\delta_0 F - (\tau_0 s + 1)T_0)(2(q_2(s)A + q_3(s)F) + p_2(s)) + \beta_1^2 q_1(s) \\ & - 2B_1\beta_1 T_0 s q_2(s) + (B_2^2F^2 + B_1^2T_0^2s^2)q_3(s) + e^{-\rho s}\beta(\eta A - F)^2 \end{aligned}$$

Equating the coefficients of A^2, AF, F^2, A, F, and 1 to 0 yields the system of ordinary differential equations:

1) $\dot{q}_1(s) - e^{\rho s}q_2^2(s) + e^{-\rho s}\beta\eta^2 = 0$

2) $\dot{q}_2(s) - e^{\rho s}q_2(s)q_3(s) - e^{-\rho s}\beta\eta + \delta_0 q_2(s) = 0$

3) $\dot{q}_3(s) - e^{\rho s}q_3^2(s) + 2\delta_0 q_3(s) + B_2^2 q_3(s) + e^{-\rho s}\beta = 0$

4) $\dot{p}_1(s) + 2\alpha_1 q_1(s) - e^{\rho s}q_2(s)p_2(s) - 2(\tau_0 s + 1)T_0 q_2(s) = 0$

5) $\dot{p}_2(s) + 2\alpha_1 q_2(s) - e^{\rho s}q_3(s)p_2(s) + \delta_0 p_2(s) - 2(\tau_0 s + 1)T_0 q_3(s) = 0$

6) $\dot{r}(s) + \alpha_1 p_1(s) - \frac{e^{\rho s}}{4} p_2^2(s) - (\tau_0 s + 1)T_0 p_2(s) + \beta_1^2 q_1(s) -$

$$- 2B_1 T_0 \beta s q_2(s) + B_1^2 T_0^2 s^2 q_3(s) = 0$$

The boundary conditions are: $q_1(T) = q_2(T) = q_3(T) = p_1(T) = p_2(T) = r(T) = 0$.

Note that equation 3) involves only the function q_3. If we can solve it, then plugging the resulting function into equation 2) yields a first order linear equation which can be solved. We then can substitute the functions q_2 and q_3 into equation 5) obtaining a first order linear equation for p_2. Since $u^* = \frac{-W_F e^{\rho s}}{2}$ involves only the functions q_2, q_3 and p_2 we need not complete the solution of the system (though we observe that by the same method, q_1, p_1, and r may be obtained.

To solve 3), make the substitution $h(s) = [e^{\rho s} q_3(s)]^{-1}$. Equation 3) then becomes

$$-\dot{h}(s)\,\frac{e^{-\rho s}}{h^2(s)} - \frac{\rho e^{-\rho s}}{h(s)} - \frac{e^{-\rho s}}{h^2(s)} - \frac{2\delta_0 e^{-\rho s}}{h(s)} + \frac{B_2^2 e^{-\rho s}}{h(s)} + e^{-\rho s}\beta = 0.$$

Multiplying through by $-e^{\rho s}h^2(s)$ we obtain

$$\dot{h}(s) - \beta h^2(s) + (\rho - 2\delta_0 - B_2^2)h(s) + 1 = 0.$$

This is a separable equation and the quadratic polynomial $\beta h^2 - (\rho - 2\delta_0 - B_2^2)h - 1$ will have roots r_1 and r_2 with $r_1 < 0 < r_2$. This is so because β is positive and since ρ and δ_0 are both interest rates we will assume that $\rho < 2\delta_0$ and we then have the coefficient of h is positive. We carry out the integration and then substitute back to obtain q_3. The choice $-\beta T$ of constant of integration will satisfy the initial condition $q_3(T) = 0$. We finally have

$$q_3(s) = \frac{e^{-\rho s}(1 - e^{\beta(s-T)(r_2-r_1)})}{r_2 - r_1 e^{\beta(s-T)(r_2-r_1)}}$$

<u>Remark:</u> Since $r_2 > r_1$ and $0 < s < T$, $q_3(s)$ is positive.

We next solve for q_2 as described earlier subject to $q_2(T) = 0$. The solution may be written:

$$q_2(s) = e^{-\int_T^s(\delta_0 - e^{\rho t}q_3(t))dt}[\int_T^s e^{-\rho t} \beta n e^{\int_T^t(\delta_0 - e^{\rho u}q_3(u))du} dt]$$

Remark: Since the principal integrand in this expression is positive and we are integrating from right to left, q_2 is negative.

Continuing, we solve for p_2 in equation 5)

$$p_2(s)=e^{-\int_T^s(\delta_0-e^{\rho t}q_3(t))dt}[\int_T^s[2(\tau_0 t+1)T_0 q_3(t)-2\alpha_1 q_2(t)]e^{\int_T^t(\delta_0-e^{\rho u}q_3(u))du} dt].$$

Remark: For the same reason as for q_2, p_2 is negative.

We have now found the optimal control function.

$$u^* = \frac{-W_F e^{\rho s}}{2} = e^{\rho s}\{-q_2(s)A_s - q_3(s)F_s - \frac{1}{2} p_2(s)\}$$

It is given as an affine function of the state variables of the system, that is of the present value of future benefits and the fund at time s. The coefficients of A_s and F_s are positive and negative respectively in agreement with the form of the contribution function used by Gasiewski.

Conclusion

It is clear that a two parameter family of funding methods offers greater flexibility than the unfunded present value family. It is interesting that such a function should occur as the solution to an optimal control problem where the contribution function is the control. Though by no means is this result definitive it would seem that at the very least stochastic control theory offers a valuable way of thinking about pension funding and may in the future, with further development, become a practical method for its accomplishment.

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183 **J. Maddocks and J. Keller**, Mechanics of Robes
184 **F. Bernis**, Qualitative Properties for some nonlinear higher order
185 **F. Bernis**, Finite Speed of Propagation and Asymptotic Rates for some Nonlinear Higher Order Parabolic Equations with Absorption
186 **S. Reichelstein and S. Reiter**, Game Forms with Minimal Strategy Spaces
187 **T. Ding**, An Answer to Littlewood's Problem on Boundedness
188 **J. Rubinstein and R. Mauri**, Dispersion and Convection in Periodic Media
189 **W.H. Fleming and P.E. Souganidis**, Asymptotic Series and the Method of Vanishing Viscosity
190 **H. Beirao Da Veiga**, Existence and Asymptotic Behavior for Strong Solutions of Navier-Stokes Equations in the Whole Space
191 **L.A. Caffarelli, J.L. Vazquez, and N.I. Wolanski**, Lipschitz Continuity of Solutions and Interfaces of the N-Dimensional Porous Medium Equation
192 **R.Johnson**, m-Functions and Floquet Exponents for Linear Differential Systems
193 **F.V. Atkinson and L.A. Peletier**, Ground States and Dirichlet Problems for $-\Delta = F(U)$ in R^2
194 **G. Dal Maso, U. Mosco**, The Wiener Modulus of a Radial Measure
195 **H. A. Levine and H.F. Weinberger**, Inequalities between Dirichlet and Neumann Eigenvalues
196 **J. Rubinstein**, On the Macroscopic Description of Slow Viscous Flow Past a Random Array of Spheres
197 **G. Dal Maso and U. Mosco**, Wiener Criteria and Energy Decay for Relaxed Dirichlet Problems
198 **V. Oliker and P. Waltman**, On the Monge-Ampere Equation Arising in the Reflector Mapping Problem
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201 **P. Cannarsa and H. M. Soner**, On the Singularities of the Viscosity Solutions to Hamilton-Jacobi-Bellman Equations
202 **Andrew Majda**, Nonlinear Geometric Optics for Hyperbolic Systems of Conservation Laws
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211 **Arie Leizarowitz**, Control Problems with Random and Progressively Known Target
212 **R.W.R. Darling**, Ergodicity of a Measure-Valued Markov Chain Induced by Random Transformations
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214 **Arie Leizarowitz**, Controlling Diffusion Processes on Infinite Horizon with the Overtaking Criterion
215 **Millard Beatty**, The Poisson Function of Finite Elasticity
216 **David Terman**, Traveling Wave Solutions Arising From a Combustion Model
217 **Yuh-Jia Lee**, Sharp Inequalities and Regularity of Heat Semi-Group on Infinite Dimensional Spaces
218 **D. Stroock**, Lecture Notes
219 **Claudio Canuto**, Spectral Methods and Maximum Principle