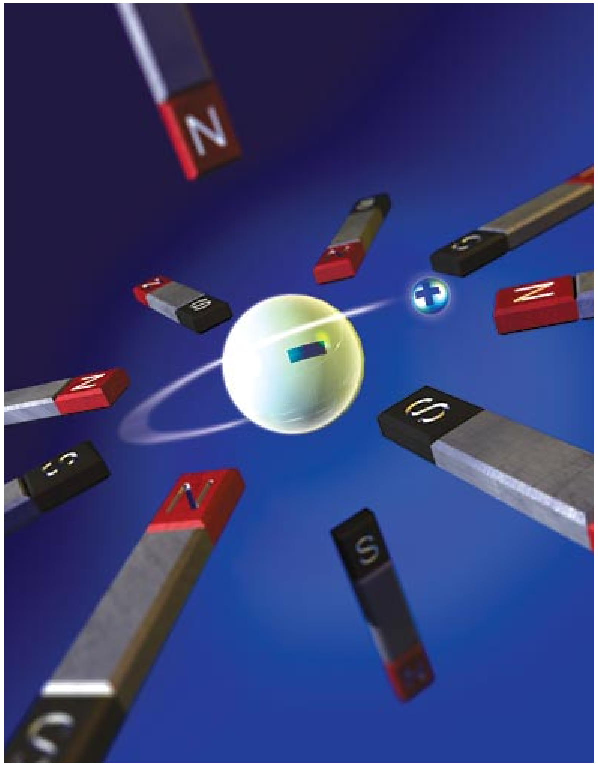
Anomalous magnetic moment of a bound electron



Picture: http://news.berkeley.edu/2010/11/17

Continuous Advances in QCD
May 12, 2016

Andrzej Czarnecki — University of Alberta

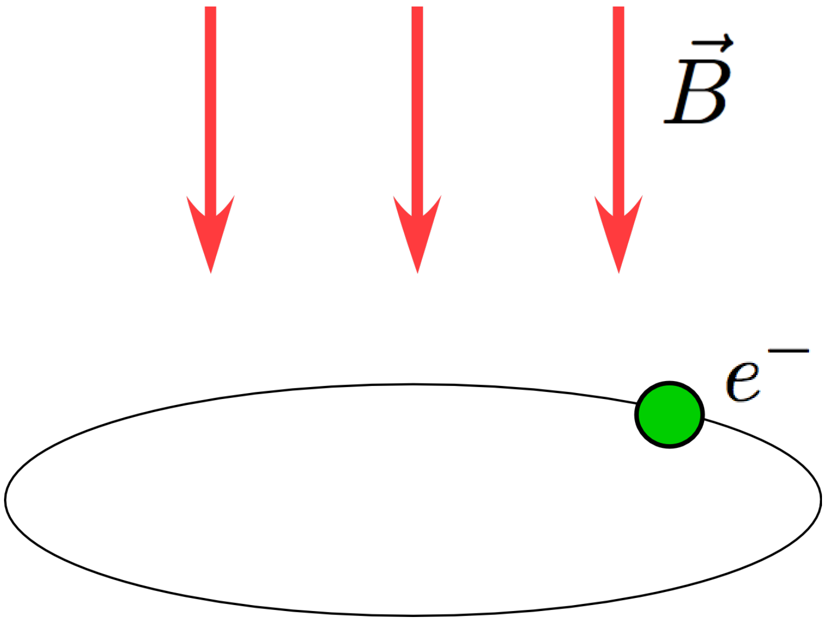
with M. Dowling, J. Piclum, R. Szafron

Outline: magnetic moment of a bound electron

Why useful?
- determination of the electron mass
- future determination of alpha
- indirectly related to muon g-2 (muonium)

Why interesting?
- quantum effects in external field
- simple system, model for more complex ones
- numerical estimates exist for large Z
- should be analytically feasible for small Z (many have tried)

Determination of the electron mass (20th century)



$$\omega_{\mathrm{cycl}} = \frac{eB}{m_e}$$

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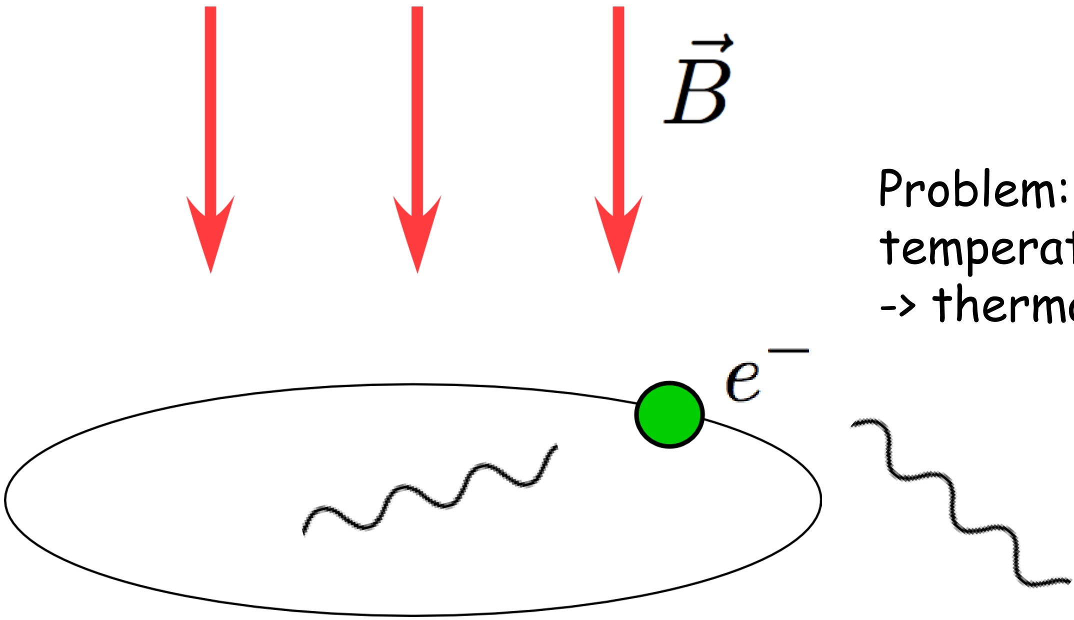

Problem: finite temperature (low, 4K) -> thermal photons

$$\omega_{\rm cycl} = \frac{eB}{m_e}$$

$$\frac{m_e}{\sqrt{1-v^2}}$$

$$\frac{\delta m_e}{m_e} \sim v^2 \sim \frac{kT}{m_e} \sim \frac{10^{-4}\frac{\rm eV}{\rm K}\cdot 4\ {\rm K}}{5\cdot 10^5\ {\rm eV}} \simeq 10^{-9}$$

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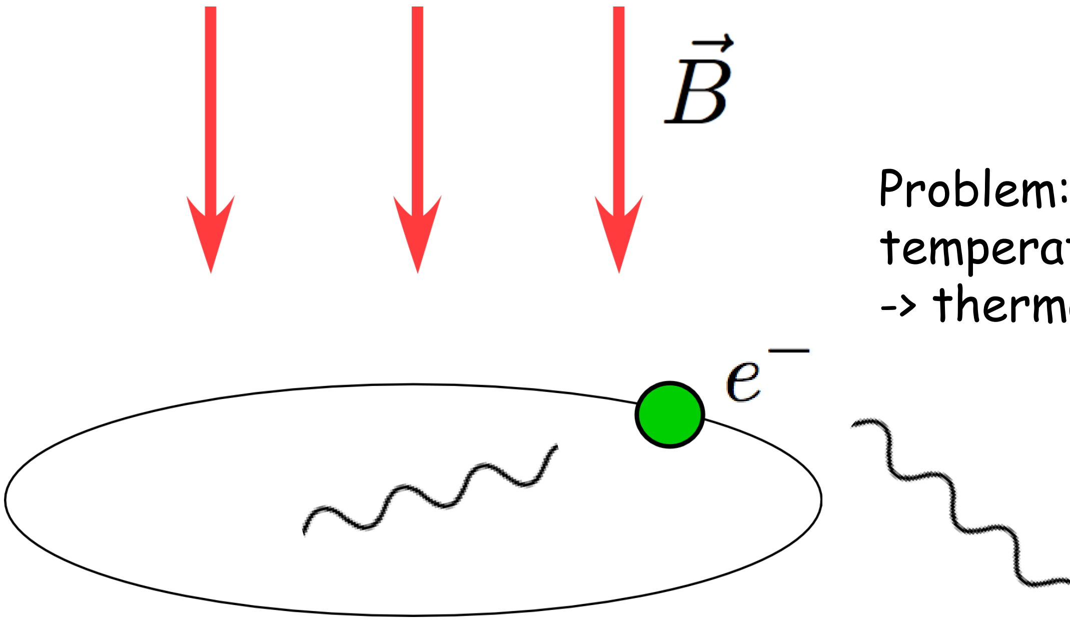

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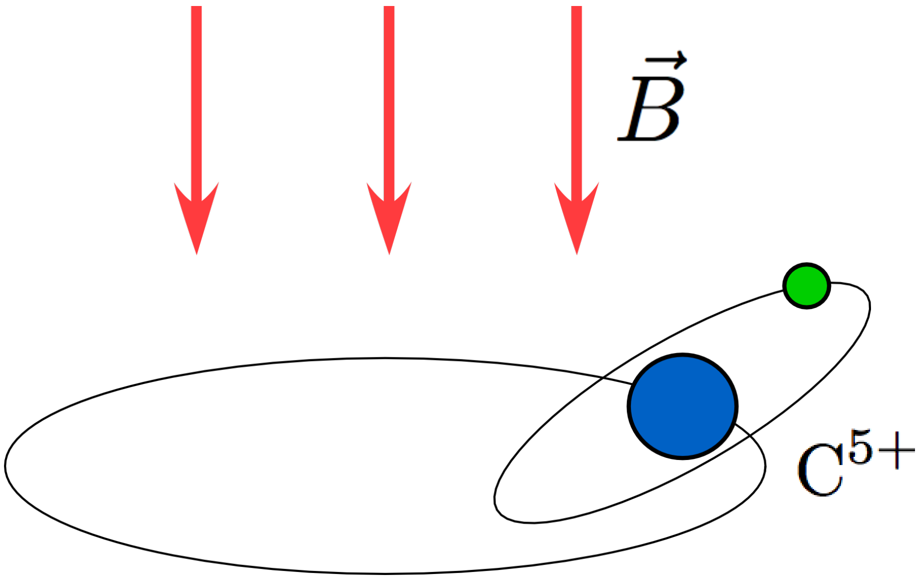
VOLUME 75, NUMBER 20 PHYSICAL REVIEW LETTERS 13 NOVEMBER 1995

Determination of the Electron's Atomic Mass and the Proton/Electron Mass Ratio via Penning Trap Mass Spectroscopy

$$\frac{m_e}{u} = 0.000\ 548\ 579\ 911\,(1)$$

2ppb relative error

21st century: anchor the electron in an ion

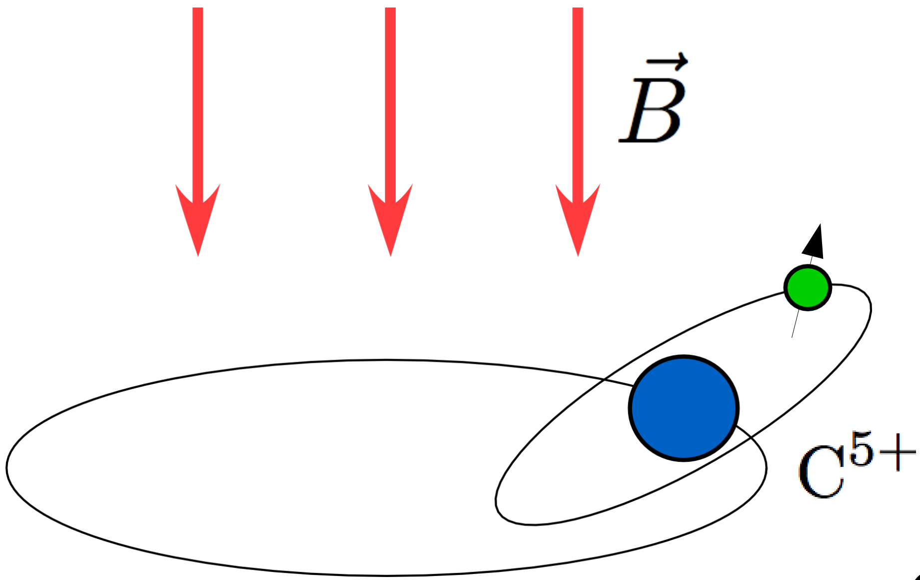


The speed of the electron becomes even higher,

$$v_e^2 \sim (Z\alpha)^2 \sim 10^{-3}$$

but is calculable in bound-state QED.

21^{st} century: anchor the electron in an ion



Larmor frequency $\omega_L = \frac{geB}{2m_e}$

Cyclotron frequency $\omega_{cycl} = \frac{ZeB}{M}$

$$m_e = \frac{g}{2Z}\frac{\omega_{cycl}}{\omega_L}M$$

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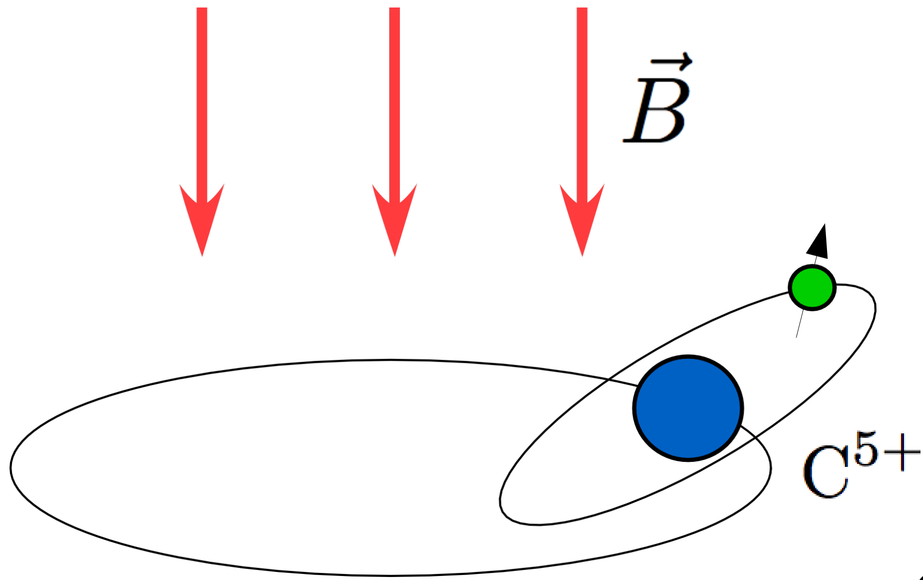

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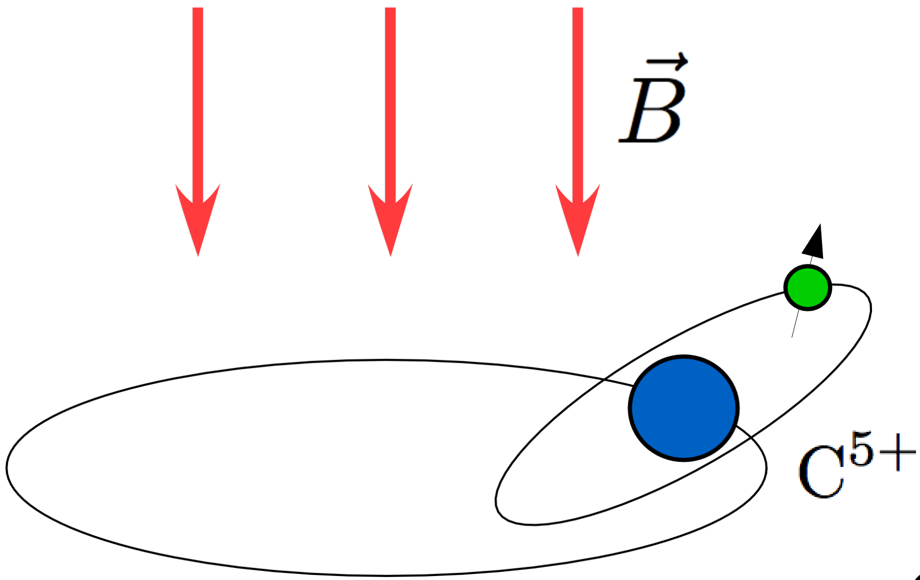
$$m_e = \frac{g}{2Z} \frac{\omega_{\rm cycl}}{\omega_L} M$$

Side remark: M and m_e have different origins! (QCD vs. Higgs)
Opportunity for searching for time variation.

21^{st} century: anchor the electron in an ion



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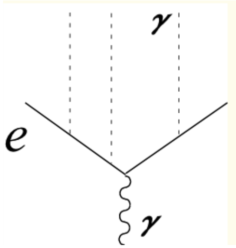
Interesting complication:
this g-factor is modified by the binding

Bound-electron g-2: the leading effect

Breit 1928: energy correction due to magnetic field in the hydrogen ground state.



$$\delta E = e \int d^3x f^2 v^* \left[1 - i\gamma \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{r}} \gamma^5\right] \gamma^5 \boldsymbol{A} \cdot \boldsymbol{\Sigma} \left[1 + i\gamma \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{r}} \gamma^5\right] v$$

$$g = 2 \cdot \frac{1}{3}\left(1 + 2\sqrt{1-(Z\alpha)^2}\right) \simeq 2\left(1 - \frac{(Z\alpha)^2}{3}\right)$$

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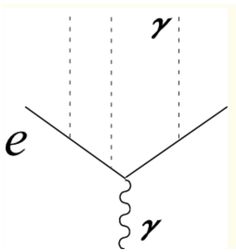

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Important: dependence on alpha; may be exploited to determine its value. (Use ions with various Z)

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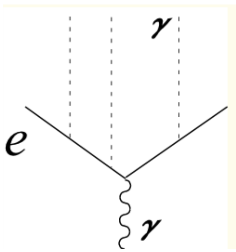

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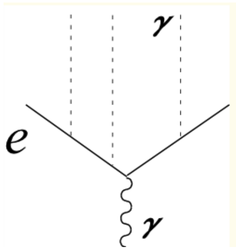
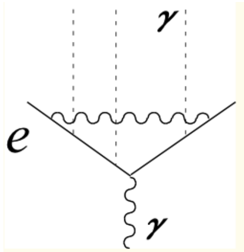
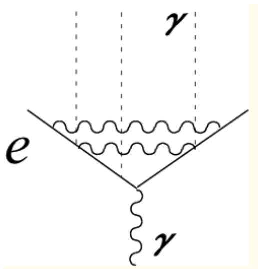
Valid to all orders in Zα

Harder to achieve when loops present.

Bound-electron g-2: binding and loops







$$g = 2 - \frac{2\,(Z\alpha)^2}{3} - \frac{(Z\alpha)^4}{6} + \ldots$$
$$+\frac{\alpha}{\pi}\left[1 + \frac{(Z\alpha)^2}{6} + (Z\alpha)^4\,(a_{41}\ln Z\alpha + a_{40}) + \ldots\right]$$
$$+\left(\frac{\alpha}{\pi}\right)^2\left[-0.65..\left(1 + \frac{(Z\alpha)^2}{6}\right) + (Z\alpha)^4\,(b_{41}\ln Z\alpha + b_{40}) + \ldots\right]$$

two-loop corrections

$$b_{41} = \frac{28}{9}$$
$$b_{40} = -16.4$$

Pachucki, AC Jentschura, Yerokhin (2005)

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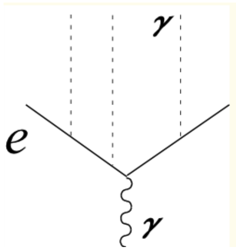

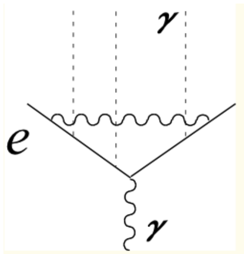

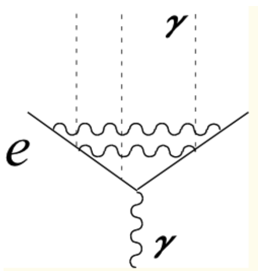

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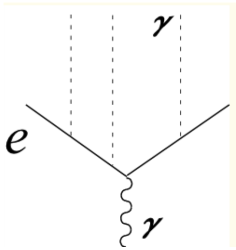
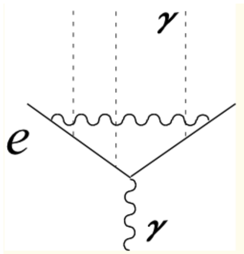
Together with experiments in Mainz, this improved the accuracy of m_e by about a factor 3,

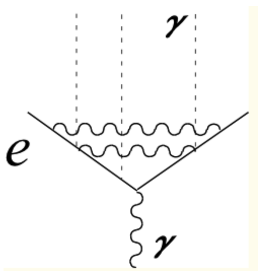
$$\frac{m_e}{u} = 0.000\;548\;579\;909\;32\,(29)\,(1)$$

theory error

Recent experimental improvement







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Nature 2014
Sturm et al

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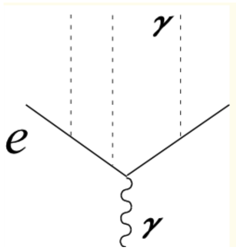

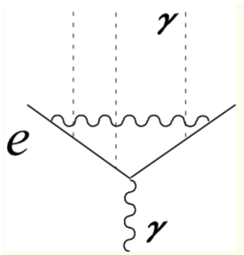

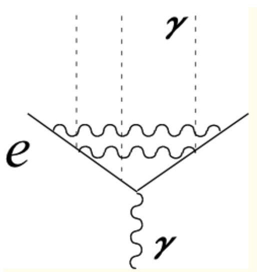

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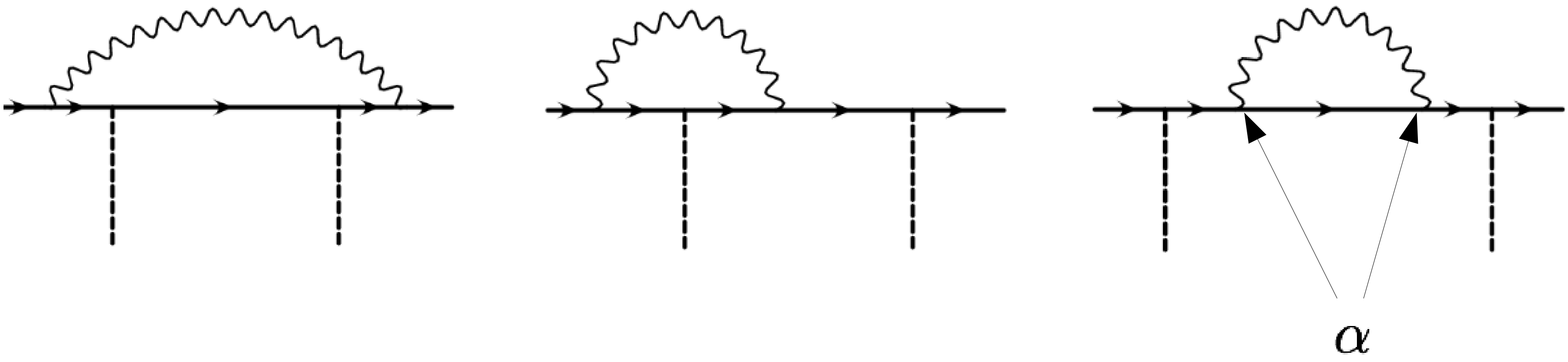
Nature 2014 Sturm et al

Next theory challenge: $(Z\alpha)^5$ effects.

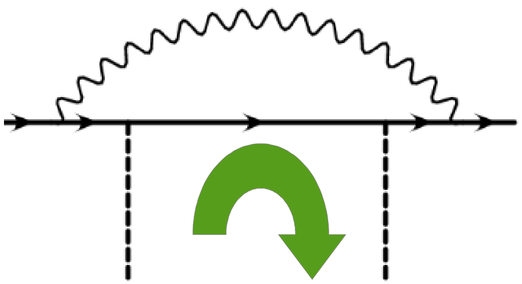
How to get higher-order corrections to g?

Warm-up: Lamb shift (no external magnetic field)

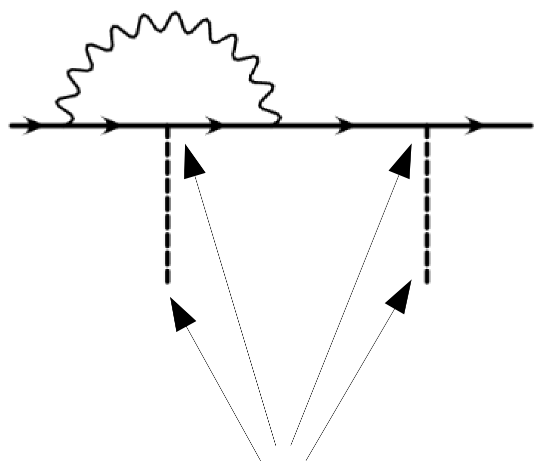


How are the higher-order corrections to g computed?

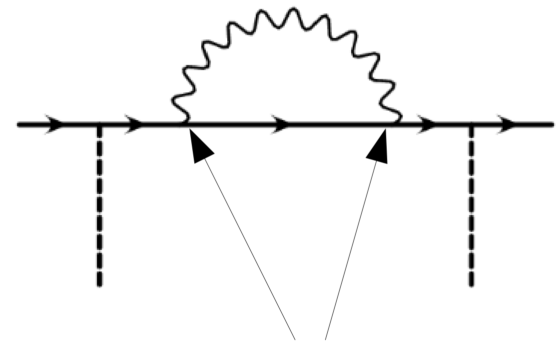
Warm-up: Lamb shift (no external magnetic field)



$$\left|\psi\left(r=0\right)\right|^2 \sim \left(Z\alpha\right)^3$$



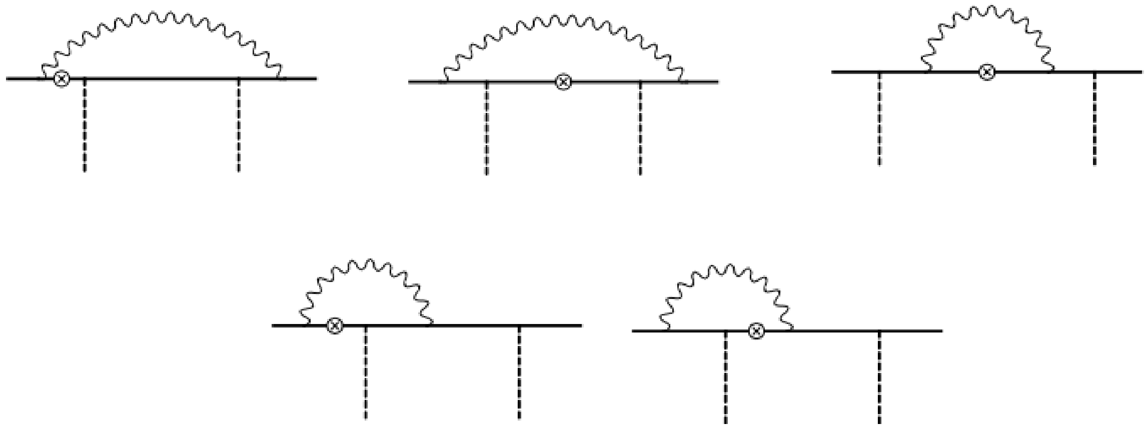
$$\left(Z\alpha\right)^2$$



$$\alpha$$

$$\Delta E \sim \alpha\left(Z\alpha\right)^5$$

To find Δg, consider the energy in a magnetic field



The result is *gauge*-invariant; but not yet complete.

What if the magnetic field couples to an external line?

Magnetic correction to the wave function

$$\Delta g = \frac{2|e|}{\mu_0 m B} \sum_n^{n \neq a} \frac{\langle a | \delta U | n \rangle \left\langle n \left| \vec{\alpha} \cdot \vec{A} \right| a \right\rangle}{E_a - E_n}$$

This sum can be done exactly
with help of virial identities

S. Karshenboim, V. Ivanov, and V. Shabaev

$$\Delta g = \frac{2\kappa m}{j(j+1)} \langle n\kappa | \delta U \frac{\kappa}{m^2} \left(I - |n\kappa\rangle \langle n\kappa|\right) \left[\left(E_{n\kappa} - \frac{m}{2\kappa} \right) r i \sigma_y + m r \sigma_x + \alpha Z i \sigma_y - \kappa \sigma_z \right] |n\kappa\rangle$$

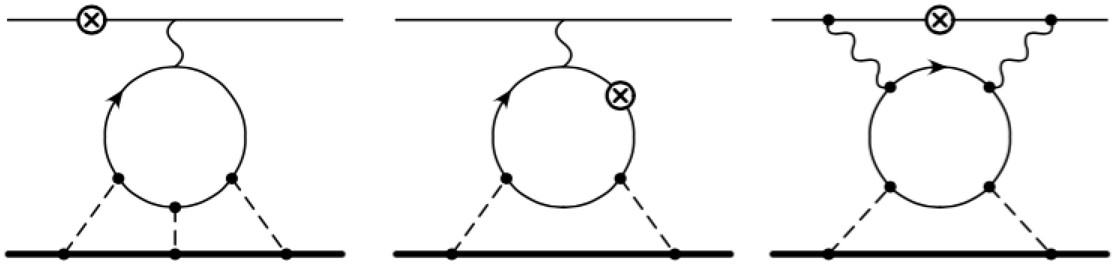
The result is especially simple if the short-distance perturbation is energy-independent (like the Uehling potential)

$$\Delta g = \frac{16}{3} \left(1 - \frac{1 - \sqrt{1-(Z\alpha)^2}}{2(Z\alpha)^2} \right) \left\langle \delta^3(r) \right\rangle_\psi \simeq 4 \left\langle \delta^3(r) \right\rangle_\psi$$

This is just Lamb

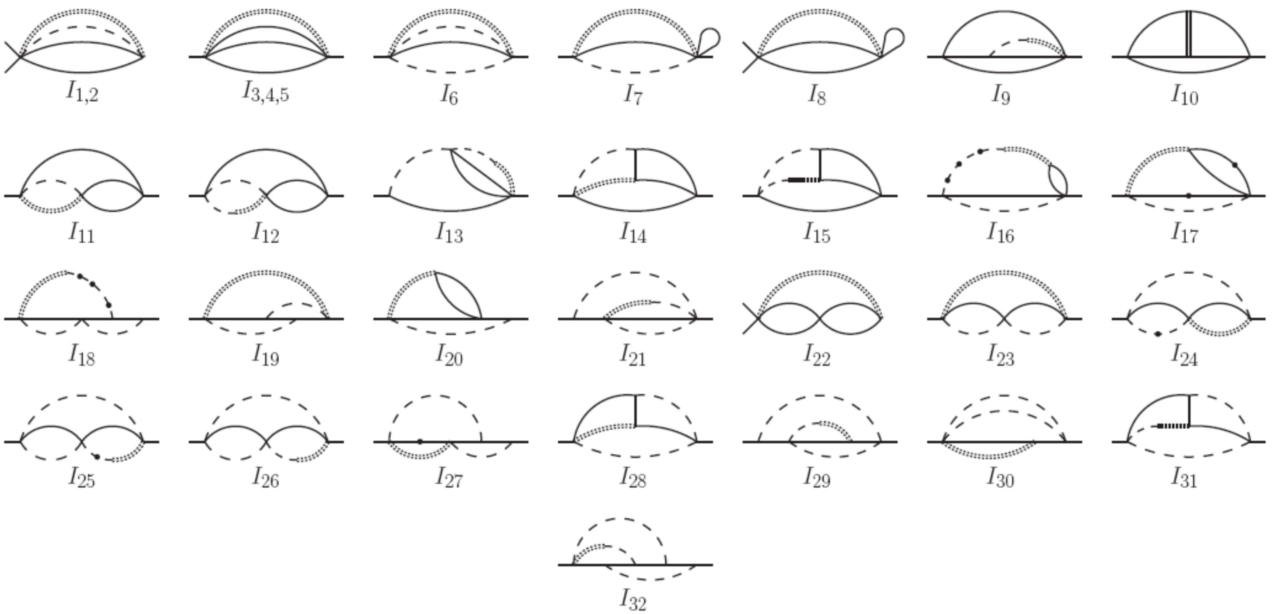
Next goal: $\alpha^2(Z\alpha)^5$ corrections to *g*

Examples:



More than 300 contributions.

A set of 32 master integrals



Typical expression

$$I_{24} = G(0,1,2,1,0,1,0) = \frac{2\pi^2}{\epsilon} - 162.745878930257(1) + 640.681562239(2)\epsilon$$
$$-9490.745115169417(3)\epsilon^2 + \mathcal{O}(\epsilon^3)\,,$$

Final results

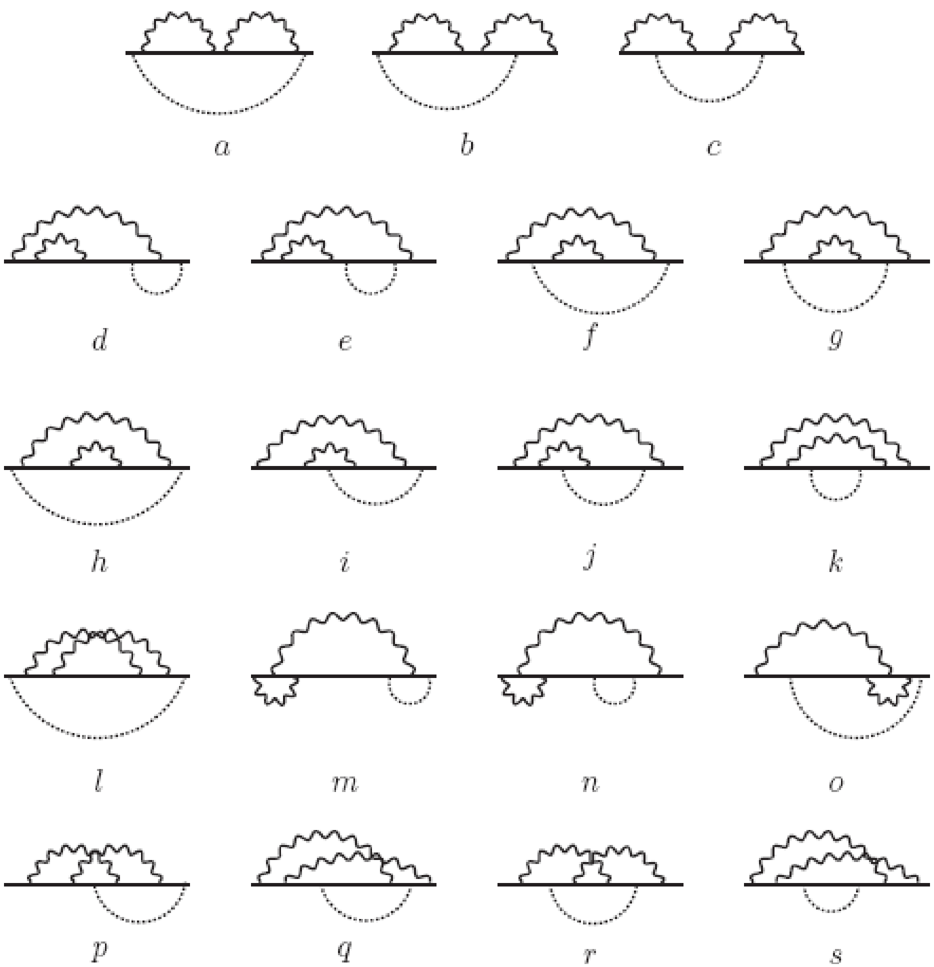
$$\delta E_{a-s} = \frac{\alpha^2 (Z\alpha)^5}{\pi n^3} \left(\frac{\mu}{m}\right)^3 m\,[-7.72381(4)]$$

Previous results

- 7.61(16) Pachucki 1994

- 7.724(1) Eides and Shelyuto, 1995

Reevaluation of the $\alpha^2(Z\alpha)^5$ Lamb shift



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Dowling, Mondejar, Piclum, AC, PRA 81, 022509

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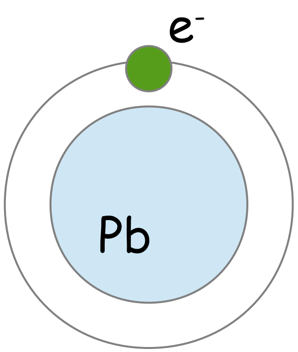
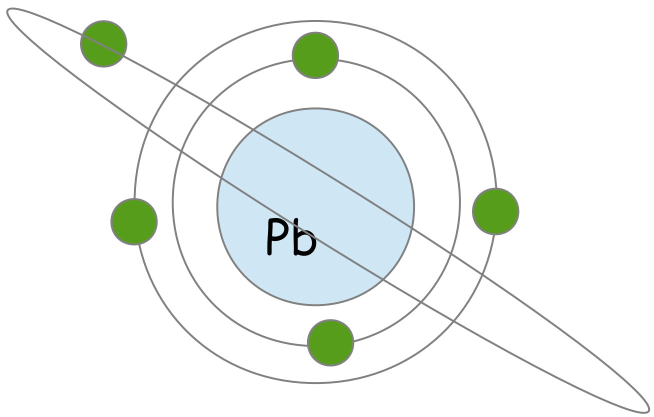
A new source of alpha: highly-charged ions

$$g \simeq 2 - \frac{2\,(Z\alpha)^2}{3} \longrightarrow \frac{\delta\alpha}{\alpha} \sim \frac{1}{(\alpha Z)^2}\sqrt{(\delta g_{\rm exp})^2 + (\delta g_{\rm th})^2}$$

large Z favorable



Hydrogen-like lead



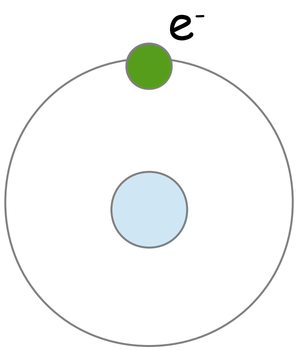
Boron-like lead

There is a combination of *g*-factors in both ions where the sensitivity to the nuclear structure largely cancels, but the sensitivity to alpha remains.

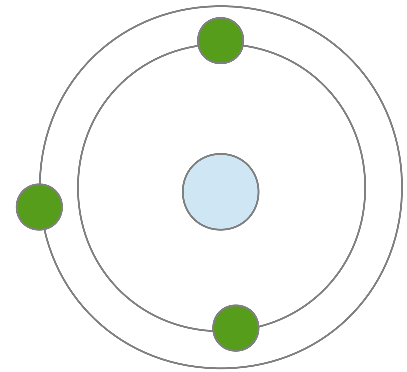
Shabaev, Glazov, Oreshkina, Volotka, Plunien, Kluge, Quint

New idea: medium-charged ions

$$g \simeq 2 - \frac{2\,(Z\alpha)^2}{3}$$



Hydrogen-like ion



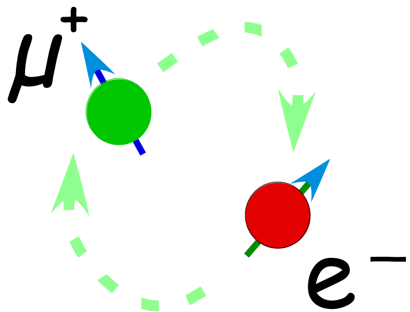
Lithium-like ion

Combine H-like and Li-like to remove nuclear dependence;
then combine with a different nucleus, to remove free-g dependence!

Much interesting theoretical work remains to be done!

Yerokhin, Berseneva, Harman, Tupitsyn, Keitel: PRL (2016)

New muonium HFS measurement in J-PARC

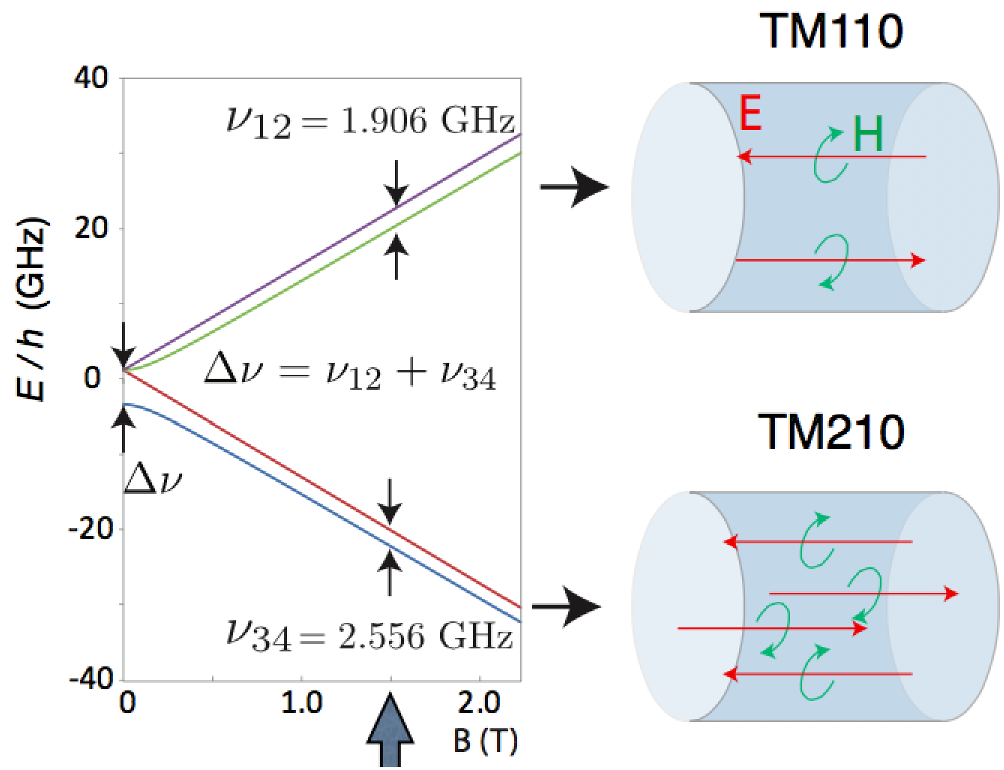


$$H = -(\vec{\mu}_e + \vec{\mu}_\mu)\cdot\vec{B} + c\vec{I}\cdot\vec{J}$$

Line width ~ 100 kHz (muon decay)
Aim at ~Hz accuracy

Z boson contribution -65 Hz

Caveat: the magnetic moment in muonium differs from that of a free muon (slightly). Theory input needed!



Hiroyuki A. Torii JPS Conf. Proc. 8, 025018 (2015)

Summary

* Binding modifies the electron g-factor

* Theory of this effect is more fun than for a free electron

* Synergy with beautiful experiments: mass of the electron and, in future, the fine structure constant.

* $\alpha(Z\alpha)^5$ effects almost finished; $\alpha^2(Z\alpha)^5$ hopefully soon.

* Opportunities for more theoretical improvement...

Can we use the electron to check muon g-2?

$$a_e = \frac{g_e - 2}{2}$$

Measured with relative error $25 \cdot 10^{-11}$

Phys. Rev. Lett. 100, 120801 (2008)

Provides the fine structure constant with the same precision,

$$\alpha^{-1}(a_e) = 137.035\,999\,1736\,(331)\,(86)$$

Phys. Rev. Lett. 109, 111807 (2012)

Experimental error dominates (for now)

Numerical errors from 4- and 5-loop diagrams