The Supermoduli Space of Genus Zero SUSY Curves with Ramond Punctures

A Dissertation

SUBMITTED TO THE FACULTY OF THE

UNIVERSITY OF MINNESOTA

BY

Nadia Ott

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF

DOCTOR OF PHILOSOPHY

Advisor Alexander Voronov

August 2020

Copyright © Nadia Ott. All rights reserved.

Acknowledgments

I am grateful to Ugo Bruzzo, Ionuţ Ciocan-Fontanine, Giulio Codogni, Daniel J. Diroff, Ron Donagi, Vivek Shende, Rita Fioresi, and Daniel Hernández Ruipérez for helpful discussions. I would also like to thank Alexander Voronov for being a wonderful advisor and introducing me to the subject.

Abstract

A construction of the supermoduli space $\mathfrak{M}_{0,n_R}$ of super Riemann surfaces of genus zero with n_R Ramond punctures as a quotient Deligne-Mumford superstack of dimension $(n_R - 3 \,|\, n_R/2 - 2)$ is presented.

Contents

0.1 Introduction

Super Riemann surfaces (SUSY curves) arise in the formulation of superstring theory, and their moduli spaces, called *supermoduli space*, are the integration spaces for superstring scattering amplitudes. A SUSY curve is described by the data $(X, \mathcal{D})$, where X is a a compact complex supermanifold of dimension $1|1$, and $\mathcal{D}$ is a rank $(0|1)$ maximally non-integrable sub-bundle of the tangent bundle called the *SUSY structure*.

Despite this slightly exotic definition, SUSY curves and their moduli have mathematical properties quite similar to those of Riemann surfaces. In fact, if one studies families parameterized only by commuting (as opposed to anticommuting) variables, then SUSY curves are nothing other than spin curves. In full generality, SUSY curves have unobstructed deformation theory and hence may be expected to have smooth moduli. Our interest here is in the moduli problem, which we will study from the point of view of algebraic geometry. The algebro-geometric approach to supermoduli theory was initiated by Deligne in a famous letter [9] to Yu. I. Manin, where the existence of a compactified moduli of stable SUSY curves was sketched, appealing to the (unwritten) generalization of Schlessinger's conditions for pro-representability and Artin's existence theorems to supergeometry.[1] Other historical works on moduli from other perspectives include [17, 10, 6].

We are interested specifically in genus zero SUSY curves, and in giving a construction of the moduli space by an explicit quotient presentation (rather than by an abstract existence argument).The desire to have a concrete presentation was ex-

[1] After the original appearance of the present article, a detailed argument for existence of moduli of stable SUSY curves was given in [21], along the lines of Deligne's letter but without explicit use of the Artin theorems.

pressed by E. Witten in a letter to A. S. Schwarz [26]. As with ordinary curves, genus zero SUSY curves present a certain challenge, as they have an infinitesimal group of automorphisms, and so in order for the moduli problem to be representable by a Deligne-Mumford stack, we must introduce punctures. (These punctures also appear naturally in the superstring theory.) In fact, there are two kinds of punctures on a SUSY curve.

One kind, the Neveu-Schwarz punctures, are entirely analogous to marked points on ordinary Riemann surfaces, and their moduli are straightforward to understand. In particular, such a puncture is equivalent to marking a point on the curve, and the moduli space $\mathfrak{M}_{0,n_{\mathrm{NS}}}$ of SUSY curves of genus $g = 0$ and $n_{\mathrm{NS}} \geq 3$ Neveu-Schwarz punctures may be described just like in the classical case:

$$\mathfrak{M}_{0,n_{\mathrm{NS}}} \cong \mathrm{Conf}(\mathbb{P}^{1|1}, n_{\mathrm{NS}})/\operatorname{OSp}(1|2,\mathbb{C}),$$

where $\mathrm{Conf}(\mathbb{P}^{1|1}, n_{\mathrm{NS}})$ denotes the configuration space of n_{NS} distinct, labeled points on the complex projective superspace $\mathbb{P}^{1|1}$ and $\operatorname{OSp}(1|2,\mathbb{C})$ stands for the orthosymplectic supergroup, which is the group of automorphisms of $\mathbb{P}^{1|1}$ preserving the standard SUSY structure on it.

By contrast, the Ramond punctures are more subtle. To explain what these are (following [27]), first recall that étale locally on a SUSY curve $(X, \mathcal{D})$ there are coordinates z, θ such that $\mathcal{D}$ is generated by the vector field

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$$

from which we compute that $\frac{1}{2}[\mathcal{D}, \mathcal{D}] = \frac{\partial}{\partial z}$.

Now consider a subbundle $\mathcal{D}$ of the tangent bundle locally generated by the vector field

$$D_\theta = \frac{\partial}{\partial \theta} + h(z)\theta \frac{\partial}{\partial z}$$

where $h(z)$ is some degree n_R polynomial. Then $\frac{1}{2}[\mathcal{D},\mathcal{D}] = h(z)\frac{\partial}{\partial z}$. Thus, D_θ is integrable along $h(z)=0$.

A SUSY curve with n_R Ramond punctures is described by three pieces of data $(X,\mathcal{D},R)$; where X and $\mathcal{D}$ are as before with the exception that we now ask $\mathcal{D}$ to be integrable exactly along a codimension $(1|0)$-subsupermanifold, *i.e.* a *divisor* R of degree n_R. We call $\mathcal{D}$ the SUSY structure, though note $(X,\mathcal{D})$ is **not** a SUSY curve by our previous definition, as $\mathcal{D}$ is not everywhere non-integrable. We say that the SUSY structure on X degenerates along R, whose components we term *Ramond punctures*. Beware: a codimension $(1|0)$ divisor on a supercurve is not determined by merely specifying marked points.

The distribution generated by the vector field D_θ above is an example of a local description of a SUSY structure on a SUSY curve with Ramond punctures.

Let us now describe the basic idea behind the construction of the moduli space of genus zero curves with n_R Ramond punctures, $\mathfrak{M}_{0,n_R}$.

It is useful to take the "dualized" definition of a SUSY structure: a rank $(1|0)$ subbundle $\mathcal{L}$ of the cotangent bundle meeting certain non-integrability conditions. We will[2] call $\mathcal{L}$ the *SUSY line bundle*. In [27], Witten gives a complete characterization of a genus zero SUSY curve with Ramond punctures over a point. Over $\operatorname{Spec} k$, any supercurve supporting a SUSY structure with n_R Ramond punctures is isomorphic to the *weighted superprojective space* $\mathbb{WP}^{1|1}(1,1\,|\,1-n_R/2)$ (henceforth $\mathbb{WP}$) and that the SUSY line bundle is isomorphic to $\mathcal{O}_{\mathbb{WP}}(-2)$. A SUSY structure on $\mathbb{WP}$ can then be specified by giving a global section of $\mathcal{H}om(\mathcal{O}_{\mathbb{WP}}(-2),\Omega) \cong \Omega^1_{\mathbb{WP}}(2)$. This assignment is unique up to an invertible factor.

[2]not following any standard convention

We first extend the characterization in [27] to families of SUSY curves over bosonic schemes and then to families of SUSY curves over arbitrary superschemes. For the latter we use tools from super deformation theory. The general results from super deformation theory relevant to our moduli problem are found Section 0.3.1.

Globalizing the facts that (1) a genus zero SUSY curve over a point is $\mathbb{WP}$ and (2) $\mathbb{WP}$ has no even deformations, we show:

Theorem 0.1.1 (Theorem 0.5.6)**.** *Given a family of genus zero SUSY curves over a bosonic base, the underlying family of supercurves is étale locally trivial.*

We explicitly compute the global sections of $\mathrm{H}^0(\mathbb{WP}, \Omega^1_{\mathbb{WP}}(2)$ and prove that the SUSY structures on $\mathbb{WP}$ are the quotient of an open subscheme Y_b (described in Section 0.5.5) of $\mathrm{H}^0(\mathbb{WP}, \Omega^1_{\mathbb{WP}}(2)$ by the invertible functions $\mathrm{H}^0(\mathcal{O}^*_{\mathbb{WP}})$. We show:

Theorem 0.1.2 (Theorem 0.5.17)**.** *The moduli space of SUSY structure* (Definition 0.5.16) *on* $\mathbb{WP}$ *is represented by the algebraic space*

$$\mathrm{Y}_b/\mathbb{G}_m$$

of dimension $n_R + 1$*, where* $\mathbb{G}_m$ *acts on* Y_b *by its identification with* $\mathrm{H}^0(\mathcal{O}^*_{\mathbb{WP}})$*.*

Any family of supercurves X/T underlying a genus zero SUSY curve Σ with $n_R \geq 4$ Ramond punctures is étale locally isomorphic to $\mathbb{WP} \times T_b$. We add to Σ the data of such an isomorphism, and prove that:

Theorem 0.1.3 (Theorem 0.5.18)**.** *The bosonic moduli space* $(\mathfrak{M}_{0,n_R})_b$ *underlying* $\mathfrak{M}_{0,n_R}$ *is represented by the Deligne-Mumford stack*

$$[(\mathrm{Y}_b/\mathbb{G}_m)/\operatorname{Aut}(\mathbb{WP})_b]$$

of dimension $n_R - 3$*, where* $\mathrm{Aut}(\mathbb{WP})_b$ *denotes the bosonic reduction of the supergroup scheme* $\mathrm{Aut}(\mathbb{WP})$

The supergroup scheme $\mathrm{Aut}(\mathbb{WP})$ is described in Section 0.4.4.

In Section 0.4.1, we study the first order deformation theory of genus zero supercurves, showing in particular that they are *not* (in general) rigid. This is quite in contrast to their bosonic counterpart $\mathbb{P}^1$ as well as to the projective superspace $\mathbb{P}^{1|1}$ (the supercurve underlying a SUSY curve when $n_R = 0$). Later in Section 0.5.4, we prove that $\mathbb{WP}$ has a universal deformation space (Theorem 0.5.20), denoted by S, and give an explicit construction of the universal deformation, denoted by Z, of $\mathbb{WP}$, in two ways. The first way: Z is glued from two copies of $\mathbb{A}_S^{1|1}$, using the (constant in S) $z \to 1/w$ gluing for the even coordinate and a (varying in S) gluing for the odd coordinate (53). The second way: Z is a hypersurface given by an explicit equation inside $\mathbb{P}^1 \times \mathbb{A}^{0|n_R/2}$ (Lemma 0.5.21).

The space S is an affine space of dimension $(0|n_R/2-2)$, matching the odd (at this point, expected) dimension of $\mathfrak{M}_{0,n_R}$. This suggests to construct $\mathfrak{M}_{0,n_R}$ using a bundle with purely even fibers over S. We have already parameterized the supercurves; it remains to parameterize the SUSY structures on them.

We show (Corollary 0.5.8) that any SUSY line bundle on $\mathbb{WP} \times T_b$ admits a unique deformation to a line bundle over the original family over T. From the gluing description of Z, we see that the usual description of the line bundle $\mathcal{O}(n)$ (transition function z^n over the overlap of charts) can still be used to define a line bundle over Z, which we denote $\mathcal{O}_Z(n)$. From the second description, we see that Z is projective, *i.e.*, it is equal to Proj of a superalgebra. In particular, it follows from the first description that a SUSY line bundle for Z must be isomorphic to our aforementioned $\mathcal{O}_Z(-2)$. Thus the SUSY structures on Z/S are the quotient of an open subscheme Y (described in Section 0.5.5) of $\mathrm{H}^0(Z, \Omega_{Z/S} \otimes \mathcal{O}_Z(2))$ by the invertible functions $\mathrm{H}^0(Z, \mathcal{O}_Z^*)$. We explicitly compute the above space of global sections in Section 0.5.5, using our defining Čech cover of Z, and summarize this result in Theorem 0.5.27 below.

Theorem 0.1.4 (Theorem 0.5.27)**.** *The moduli space (Definition 0.5.26) of SUSY structures on the universal deformation Z of $\mathbb{WP}$ is represented by the algebraic superspace over S,*

$$[Y/\mathbb{G}_m \times (\mathbb{G}_a^{0|1})^{n_R/2} \times S]$$

of relative dimension $(n_R + 1|n_R/2 + 2)$ over S and where $\mathbb{G}_m \times (\mathbb{G}_a^{0|1})^{n_R/2} \times S$ acts on Y by its identification with $\mathrm{H}^0(z, \mathcal{O}_Z^*)$

The supergroup $\mathbb{G}_m \times (\mathbb{G}_a^{0|1})^{n_R/2}$ is described in Section 0.4.4.

We denote by $\mathbb{E}$ the base change to Y_b of the universal object (some stacky SUSY curve) over $(\mathfrak{M}_{0,n_R})_b$. In Section 0.5.7 we give an explicit description of the deformation space of $\mathbb{E}$ (Theorem 0.5.29) as an algebraic superspace of dimension $(n_R - 3|n_R/2 - 2)$.

We also give (after base change to Y) an explicit description of the universal deformation for $\mathbb{E}$ as the supercurve $Z \times_S Y$ with SUSY structure generated by the global section ϖ_z (see (62)) of $\mathrm{H}^0(\Omega^1_{Z \times_S Y/Y}(2))$.

To any given family of SUSY curves Σ over T the data we add (after étale base change) the data of the following isomorphisms: The first is an isomorphism between $\Sigma \times_T T_b$ and $\mathbb{E}$. The second is an isomorphism (of deformations) between X/T and

$Z \times_{S,f} T$, where $f : T \to S$ is the unique morphism described in Theorem 0.5.20. We then prove the main result of the paper:

Theorem 0.1.5 (Theorem 0.5.30)**.** *The Deligne-Mumford superstack* $\mathfrak{M}_{0,n_R}$ *may be expressed as the quotient superstack*

$$[\mathcal{D}ef_{\mathbb{E}}/\mathbb{Z}/2\mathbb{Z}]$$

of dimension $(n_R - 3|n_R/2 - 2)$*, where* $\mathbb{Z}/2\mathbb{Z}$ *is the subgroup of* $\mathrm{Aut}(\mathbb{WP})$ *generated by the canonical automorphism* Γ*.*

Acknowledgments

I am grateful to Ugo Bruzzo, Ionuţ Ciocan-Fontanine, Giulio Codogni, Daniel J. Diroff, Ron Donagi, Rita Fioresi, and Daniel Hernández Ruipérez for helpful discussions.

0.2 Notation and Conventions

All superalgebras will be over an algebraically closed field $\mathbf{k}$ of characteristic zero. All superschemes will be assumed to be over $\operatorname{Spec}\mathbf{k}$, unless otherwise specified. If X is a super "object", we will use X_{bos} or, X_b to denote its underlying bosonic space. We will also assume that all superschemes are Noetherian, and locally of finite type.

0.3 Algebraic Supergeometry

In this section we will give an account of algebraic supergeometry with a focus on those theorems and definitions that are relevant to the moduli problem of genus zero super Riemann surfaces (SUSY curves) with Ramond punctures. Our definitions regarding superschemes and their morphisms follow those given in [4]. For the various definitions involving superstacks we follow [2], as well as [3]. Our definitions regarding Krull superdimension and regularity follow [20].

0.3.1 Superschemes

A superscheme may be thought of as a generalization of a scheme to include anti-commuting coordinates by defining a topological space X to have a structure sheaf of $\mathbb{Z}_2$-graded algebras. The majority of the classical definitions from scheme theory carry over to the super setting without any added difficulty. We, therefore, focus primarily on the concepts in superscheme theory that are unique to supergeometry. Virtually all new concepts in supergeometry are concerned with connecting their geometry to that of their underlying bosonic schemes. We assume that the reader is familiar with the basic definitions regarding superalgebras. We recommend [25] and [5] for an introduction to the subject.

Definition 0.3.1. *(Superspace). A* superspace *is a* locally superringed space $(X, \mathcal{O}_X)$*, i,e there is a $\mathbb{Z}_2$-grading on the structure sheaf $\mathcal{O}_X$*

$$\mathcal{O}_X = \mathcal{O}_X^+ \oplus \mathcal{O}_X^-,$$

where $\mathcal{O}_X^+$ (resp. $\mathcal{O}_X^+-$) is called the sheaf of even (resp. odd) functions*, making the structure sheaf a sheaf of supercommutative rings such that the stalks are local rings.*

We will refer to the ideal generated by $\mathcal{O}_X^-$ in $\mathcal{O}_X$ as the ideal sheaf of odd nilpotents and denote it by $\mathcal{J}_X$, or simply $\mathcal{J}$. The space $(X, \mathcal{O}_X/\mathcal{J})$ is an ordinary space denoted by X_{bos}, or sometimes X_b, and is referred to as the *bosonic truncation* of X. The surjection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}$ induced a closed immersion of $i : X_b \hookrightarrow X$. If the closed immersion i has a section $j : X \to X_b$, then we say that X is *projected*. To any superspace X corresponds a superspace $\operatorname{Gr} X$, called the *associated graded of* X having the same topological space as X but with a $\mathbb{Z} \times \mathbb{Z}_2$ graded structure sheaf

$$\mathcal{O}_{\operatorname{Gr} X} = \bigoplus_{i \geq 0} \mathcal{O}_X[i]$$

where $\mathcal{O}_X[0] = \mathcal{O}_X/\mathcal{J}$ and $\mathcal{O}_X[i] = \mathcal{J}^i/\mathcal{J}^{i+1}$ and such that $\mathcal{O}_{\operatorname{Gr} X}$ is generated by $\mathcal{O}_X[1]$ over $\mathcal{O}_X[0] = \mathcal{O}_{X_b}$. A superspace X isomorphic to its associated graded $\operatorname{Gr} X$ is called *split*.

Definition 0.3.2 (Superscheme)**.** *A* superscheme $(X, \mathcal{O}_X)$ *is a superspace* $(X, \mathcal{O}_X)$ *such that* $\mathcal{O}_X^-$ *is a quasi-coherent sheaf of* $\mathcal{O}_X^+$*-modules. A* morphism of superschemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a continuous map* $f : X \to Y$ *of topological spaces such that the induced map* $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ *is a morphism of sheaves of* $\mathbb{Z}_2$*-graded rings,* i.e.*, it preserves the* $\mathbb{Z}_2$*-grading, and it induces a local morphism between the stalks.*

Remark 0.3.3. The quasi-coherence condition comes from the characterization of complex superspaces (see [24], Proposition 1.1.3) as superspaces for which $\mathcal{O}_X^-$ is a coherent sheaf of $\mathcal{O}_X^+$-modules. Note, that we only require $\mathcal{O}_X^-$ to be quasi-coherent. We will call a superscheme *Noetherian* if the scheme $(X, \mathcal{O}_X^+)$ is Noetherian and $\mathcal{O}_X^-$ is a coherent sheaf of $\mathcal{O}_X^+$-modules.

Example 0.3.4 (Affine Superscheme)**.** *For a supercommutative ring* R*, the standard construction of an affine scheme generalizes easily to give an affine superscheme* $\operatorname{Spec} R$*, see [18, 19, 5].*

The basic examples of superschemes are super affine space and super projective space are defined below. In addition, we will define weighted super projective space, as this space will play a central role in the moduli problem.

Example 0.3.5 (Affine Superspace)**.** Given a super vector space V of dimension $m|n$ over the ground field k, we define the corresponding *affine superspace* as

$$\mathbb{V} = \operatorname{Spec} S(V^*),$$

where V^* stands for the dual super vector space and S denotes the supersymmetric algebra. The standard *affine* $(m|n)$*-superspace* is the affine superspace associated with the vector superspace $V = k^{m|n}$. If we choose coordinates $x_1, \ldots, x_m, \theta_1, \ldots, \theta_n \in V^*$ on V, then

$$\mathbb{A}^{m|n} := \operatorname{Spec} \mathbf{k}[x_1, \cdots, x_m, \theta_1, \cdots, \theta_n].$$

The bosonic truncation of $\mathbb{A}^{m|n}$ is the affine space $\mathbb{A}^m$. The super affine space is split and may be regarded as a super ringification of its bosonic truncation, the affine space $\mathbb{A}^m$, with a structure sheaf

$$\mathcal{O}_{\mathbb{A}^{m|n}} = S(\Pi\mathcal{O}^n_{\mathbb{A}^m}),$$

where Π denotes the parity change operation, which shifts the grading by 1 modulo 2. The supersymmetric algebra of a purely odd linear object, in this case the sheaf $\Pi\mathcal{O}^n_{\mathbb{A}^m}$ of $\mathcal{O}_{\mathbb{A}^m}$-modules, is well-known as the exterior (Grassmann) algebra of the corresponding purely even object:

$$S(\Pi\mathcal{O}^n_{\mathbb{A}^m}) = \bigwedge(\mathcal{O}^n_{\mathbb{A}^m}).$$

We will avoid using the exterior algebra, because its natural extension to the super world is super anticommutative rather than supercommutative.

Example 0.3.6 (Super Projective Space)**.** Given a super vector space V of dimension $m + 1|nq$, the $(m|n)$-dimensional super projective space $\mathbb{P}(V)$ may be defined as the superspace of lines, *i.e.*, (1—0)-dimensional vector subspaces of V. More technically,

$\mathbb{P}(V)$ is the superscheme representing the functor of points that assigns a superscheme T the set of *supplemented* (1—0)-line subbundles of the trivial super vector bundle $T \times V$ over T. One may also think of $\mathbb{P}(V)$ as the quotient

$$\mathbb{P}(V) = (\mathbb{V} \setminus \{0\})/\mathbb{G}_m$$

of the super affine space $\mathbb{V}$ with deleted origin by the multiplicative group $\mathbb{G}_m = \mathrm{GL}(1)$ acting on $\mathbb{V} \setminus \{0\}$ by dilations. Finally, the construction of a projective spectrum generalizes to the super case, and one may identify

$$\mathbb{P}(V) := \operatorname{Proj} S(V^*),$$

where the algebra $S(V^*)$ is $\mathbb{Z} \oplus \mathbb{Z}_2$-graded with the $\mathbb{Z}$-grading coming from the symmetric power and the $\mathbb{Z}_2$-grading coming from the $\mathbb{Z}_2$-grading in V^*. It turns out the super projective space over k is isomorphic to a split superscheme defined by the ordinary projective space $\mathbb{P}(V^+)$ and structure sheaf

$$\mathcal{O}_{\mathbb{P}(V)} \cong S((V^-)^* \otimes \mathcal{O}_{\mathbb{P}(V^+)}(-1)),$$

see [19, Proposition 4.3.5]. If $V = k^{m+1|n}$ with coordinates $x_0, \ldots, x_m, \theta_1, \ldots, \theta_n$, then

$$\mathbb{P}^{m|n} := \operatorname{Proj} \mathbf{k}[x_0, \cdots, x_m, \theta_1, \cdots, \theta_n],$$

where all the generators have degree one in the $\mathbb{Z}$-grading.

Example 0.3.7 (Weighted Projective Superspace)**.** Given a weight vector $(a_0, \ldots, a_m \,|\, \alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^{m+1|n}$ with $a_i > 0$, $i = 0, \ldots, m$, the *weighted super projective space* $\mathbb{WP}^{m|n}(a_0, \ldots, a_m \,|\, \alpha_1, \ldots, \alpha_n)$ is defined as

$$\mathbb{WP}^{m|n}(a_0, \ldots, a_m \,|\, \alpha_1, \ldots, \alpha_n) := \operatorname{Proj} \mathbf{k}[x_0, \cdots, x_m, \theta_1, \cdots, \theta_n],$$

where the $\mathbb{Z}$-grading on the polynomial superalgebra $\mathbf{k}[x_0, \cdots, x_m, \theta_1, \cdots, \theta_n]$ is defined by declaring the degree of x_i to be a_i and the degree of θ_j to be α_j for all i and j. One may also identify the weighted super projective space as the quotient

$$(\mathbb{A}^{m+1|n} \setminus \{0\})/\mathbb{G}_m,$$

where the multiplicative group acts on the super affine space according to the given weights:

$$g(x_0, \ldots, x_m \,|\, \theta_1, \ldots, \theta_n) = (g^{a_0}x_0, \ldots, g^{a_m}x_m \,|\, g^{\alpha_1}\theta_1, \ldots, g^{\alpha_n}\theta_n).$$

We will be interested in particular in the weighted super projective space $\mathbb{WP}^{1|1}(1, 1 \,|\, m)$ for $m \in \mathbb{Z}$, in which case it is also split and isomorphic to the ordinary projective line $\mathbb{P}^1$ equipped with the structure sheaf

$$\mathcal{O}_{\mathbb{WP}^{1|1}(1,1\,|\,m)} = S\left(\Pi\mathcal{O}_{\mathbb{P}^1}(-m)\right).$$

Krull Superdimension and Regularity

There is some subtlety in defining the Krull dimension of a superalgebra. Ordinarily, the Krull dimension of a ring R is defined to be the maximum length of all chains of prime ideals in R. For a superalgebra $R = R_0 \oplus R_1$, the ideal of odd nilpotent $\mathcal{J}$ is contained in every prime ideal and thus

$$\operatorname{Kdim} R = \operatorname{Kdim} R_0.$$

In [20], the notion of Krull dimension is extended to superalgebras as follows: First, take any generators $y_1, \ldots, y_s$ of R_1 and define $y_1, \ldots, y_s$ to form a *sytem of odd parameters for R* if there exists a longest chain of prime ideals

$$\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_n$$

in R_0 such that $\mathrm{Ann}_{R_0}(y^{\underline{s}}) \subseteq \mathfrak{p}_0 \subseteq \mathfrak{p}_0$, where $y^{\underline{s}}$ denotes the sum of all products of the y_i.

Definition 0.3.8 (Krull Superdimension)**.** *The Krull superdimension of a superring R is defined as*

$$\operatorname{Kdim} R = r|s$$

where $r = \operatorname{Kdim} R_0$ and s is the cardinality of a system of odd parameters for R.

Remark 0.3.9. *If $X = \operatorname{Spec} A$ is an affine superscheme, then $\dim X := \operatorname{Kdim} A$.*

The definition of a Noetherian, regular local, superring can now be defined exactly as in the classical setting.

Definition 0.3.10 (Regular Superring)**.** *A local Noetherian superring $(A, \mathfrak{m})$ is regular if $\operatorname{Kdim} A = \dim_{\boldsymbol{k}} \mathfrak{m}/\mathfrak{m}^2$. Here*

$$\mathfrak{m}/\mathfrak{m}^2 = \left(\mathfrak{m}_0/\mathfrak{m}_0^2 + A_1^2\right) \oplus (A_1/\mathfrak{m}_0 A_1)$$

is seen to be a finite-dimensional super vector space over $\boldsymbol{k}$.

Definition 0.3.11. *A locally Noetherian superscheme X is regular at a point $x \in X$ if $\mathcal{O}_{X,x}$ is a regular local rings, i.e $\operatorname{Kdim} \mathcal{O}_{X,x} = \dim_{\boldsymbol{k}} (\mathfrak{m}_x/\mathfrak{m}_x^2))^*$. Otherwise, we say that X is* singular *at $x \in X$.*

Smooth and Étale Morphisms

Definition 0.3.12 (Smooth Morphism)**.** *A morphism $f : X \to Y$ of superschemes is* smooth *at $x \in X$ if the following hold:*

(a) f is of finite type at x.

(b) *f is flat at x.*

(c) *If $y = f(x)$, then $X_y = X \times_Y \operatorname{Spec} k(y)$ is regular at x.*

We say that f is smooth of relative dimension $m|n$ *if f is smooth and for each $y = f(x)$, $\dim_{k(y)} X_y = (m, n)$.*

Definition 0.3.13 (Etale Morphism)**.** *A smooth morphism $f : X \to Y$ of superschemes is étale if for every $y = f(x)$, $\dim_{k(y)} X_y = (0, 0)$. We say that $f : X \to Y$ is an étale covering of Y if f is also surjective.*

Definition 0.3.14 (Etale topology)**.** *The étale topology on* SupSch *is the Grothendieck topology on* SupSch *whose coverings are surjective étale morphisms.*

Deformation Theory for Superschemes

In this section, we generalize classical results from deformation theory to smooth superschemes and vector bundles. The theorems in this section are all straightforward generalizations to supergeometry of classical theorems in deformation theory, and we, do not provide any proofs. We recommend [24, 21] for further reading on deformation theory for complex superanalytic space.

Let A be a Noetherian superalgebra over $\mathbf{k}$. A *square zero extension* of A is a surjection $A' \to A$ with $A' \in (\mathrm{Noeth}_{\mathbf{k}})$ and with a square-zero kernel $I \subseteq A'$. We can picture this as the exact sequence of $\mathbf{k}$-superalgebras

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0.$$

Remark 0.3.15. *Here I is naturally a finite A-module: if $a \in A$ and $i \in I$, we take an element $a' \in A'$ in the pre-image of a and define $a \cdot i = a'i \in I$. Suppose $a'' \in A'$*

is another element in the pre-image of a so that $a' - a'' \in I$, then

$$(a' - a'')i = 0$$

because $I^2 = (0)$, and so $a'i = a''i$. and therefore the A-module structure on I is well-defined.

Infinitesimal Deformations of Smooth Superschemes. Let X be a superscheme over $\operatorname{Spec} A$ with and let A' be a square-zero extension of A. We call a triple $(\mathcal{X}, \pi, i)$ represented by the cartesian diagram

$$\eta : \begin{array}{ccc} X & \xrightarrow{i} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec} \mathbf{k} & \longrightarrow & \operatorname{Spec} A \end{array}$$

with π flat and surjective and a i a closed immersion induced by an isomorphism $\phi : \mathcal{X} \times_A \operatorname{Spec} \mathbf{k} \xrightarrow{\sim} X$, an *infinitesimal deformation* of X over $\operatorname{Spec} A'$. An *isomorphism*

$$(\mathcal{X}, \pi, i) \xrightarrow{\sim} (\mathcal{X}', \pi', i')$$

of infinitesimal deformations of X over $\operatorname{Spec} A'$ is an isomorphism $f : \mathcal{X} \to \mathcal{X}'$ of superschemes over $\operatorname{Spec} A'$ such that $\phi' \circ f|_{\operatorname{Spec} \mathbf{k}} \circ \phi^{-1} = \mathrm{id}_X$. We say that a deformation $\mathcal{X}$ of X is *trivial* if $\mathcal{X}$ is isomorphic to the *trivial deformation* $X \times_k \operatorname{Spec} A$.

Remark 0.3.16. The A-supermodule $I = \ker(A' \to A)$ can be factored into a composition of square-zero extensions $A' = A_0 \to \cdots \to A_n = A$ such that $I_j = ker(A_j \to A_{j+1})$ is a A_{j+1}-supermodule of rank $(1|0)$ or rank $(0|1)$. We call a square-zero extension with kernel of rank $(1|0)$ (resp. rank $(0|1)$) an *even tiny extension* (resp. an *odd tiny extension*).

Theorem 0.3.17. *Any deformation of a smooth affine superscheme is isomorphic to the trivial deformation.*

We omit the proof of Theorem 0.3.17 because it follows exactly as in the classical case (e.g Theorem 1.2.4 in [23]).

Theorem 0.3.18 (Deformations of Smooth Superschemes)**.** *Let* $i : A' \to A$ *be a square-zero extension with* $I = \ker(A' \to A)$ *and let* X_0 *be a smooth superscheme over* $\operatorname{Spec} A$*. Then, there is an obstruction*

$$o(X_0, i) \in H^2(X_0, f_0^* I \otimes \mathcal{T}_{X_0/A})$$

(where $f_0 : X_0 \to \operatorname{Spec} A$ *and where* $\mathcal{T}_{X_0/A}$ *denotes the relative tangent sheaf of* X_0 *over* $\operatorname{Spec} A$*) whose vanishing is necessary and sufficient for the existence of a deformation* X *of* X_0 *over* $\operatorname{Spec} A'$*. When the obstruction vanishes, the set of isomorphism classes of such deformations is a torsor for* $H^1(X_0, f_0^* I \otimes \mathcal{T}_{X_0/A})$*. The automorphism group of any fixed deformation* X *of* X_0 *over* $\operatorname{Spec} A'$ *is isomorphic to* $H^0(X_0, f_0^* I \otimes \mathcal{T}_{X_0/A})$*.*

We omit the proof of Theorem 0.3.18, which goes exactly as in the classical case (e.g., Theorem 8.5.9(b) in [11]).

Deformations of Vector Bundles. We will also need the notion of a deformation of a vector bundle on X. Let X_0 be a smooth superscheme over $\operatorname{Spec} A$, E_0 be a vector bundle on X_0, and let (X, π, i) be a deformation of X_0 over $\operatorname{Spec} A'$. A *deformation of a vector bundle* E_0 *over* X is is a pair (E, j) where E is a vector bundle on X and j is an $\mathcal{O}_X$-linear map $E \to i_* E_0$ inducing an isomorphism $i^* E \xrightarrow{\sim} E_0$.

Theorem 0.3.19 (Deformations of Vector Bundles)**.** *Let* $i : A' \to A$ *be a square-zero extension with* $I = \ker(A' \to A)$ *and let* X_0 *be a smooth superscheme over* $\operatorname{Spec} A$.

Let E_0 be a vector bundle on X_0 and let (X, π, i) denote a deformation of X_0 over $\operatorname{Spec} A'$*. There is an obstruction*

$$o(E_0, i) \in H^2(X_0, f_0^* I \otimes \mathcal{E}nd(E_0))$$

whose vanishing is necessary and sufficient for the existence of a deformation E of E_0 over X. When the obstruction vanishes, the set of deformations of E_0 over X is a torsor for $H^1(X_0, f_0^* I \otimes \mathcal{E}nd(E_0))$ and the group of automorphisms of a given deformation E is identified by $a \mapsto a - \mathrm{Id}$ with $H^0(X_0, \mathcal{E}nd(E_0) \otimes f_0^* I)$.

We omit the proof of Theorem 0.3.19 because it follow exactly as in the classical case (e.g Theorem 8.5.3(b) in [11]).

Vector Bundle Map Extension. Let (X, π, i) be a deformation of X_0 and let E and F be vector bundles on X such that $i^E = E_0$ and $i^* F = F_0$. We call an $\mathcal{O}_X$-linear map $u : E \to F$ restricting to an $\mathcal{O}_{X_0}$-linear map $u_0 : E_0 \to F_0$ an *extension of* u_0.

Theorem 0.3.20 (Extensions of Vector Bundle Maps)**.** *Let $i : A' \to A$ be a square-zero extension with $I = \ker(A' \to A)$ and let X_0 be a smooth superscheme over* $\operatorname{Spec} A$*. Let E and F be vector bundles on X such that $i^* = E_0, i^* F = F_0$, and $u_0 : E_0 \to F_0$ and $\mathcal{O}_{X_0}$-linear map. There is an obstruction*

$$o(u_0, i) \in H^1(X_0, f_0^* I \otimes \mathcal{H}om(E_0, F_0))$$

to the existence of an $\mathcal{O}_X$-linear map $u : E \to F$ extending u_0. When the obstruction vanishes, the set of u extending u_0 is a torsor for $H^0(X_0, f_0^* I \otimes \mathcal{H}om(E_0, F_0))$.

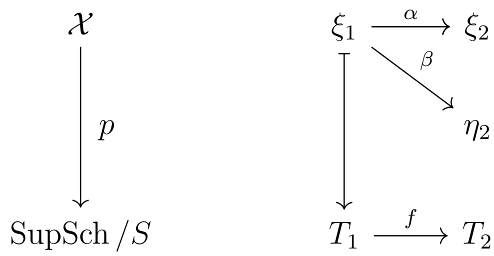
We omit the proof of Theorem 0.3.20 because it follow exactly as in the classical case (e.g Theorem 8.5.3(a) in [11]).

0.3.2 Superstacks

In this section, we define algebraic superspaces, superstacks, and Deligne-Mumford superstacks since this is the language we will use in the construction of $\mathfrak{M}_{0,n_R}$. Since there exist a number of approaches to constructing moduli spaces of curves, we should describe our motivation for using superstacks. One method to construct a moduli space is by proving that a certain "moduli functor" is representable. However, most moduli spaces (and all supermoduli spaces) or, rather their functors, are not representable by any space possessing a well-known geometric structure, like that of a (super)scheme. It, therefore, became necessary to find an object which, in some sense, would be generic enough to reflect some fundamental property of moduli functors. Grothendieck found that the main obstruction to representability is the existence of non-trivial automorphisms and that this obstruction may be remedied by including the automorphisms as part of the geometric data of the objects. The geometric object introduced to include the data of automorphisms is called a stack. Since every SUSY curve has at least one non-trivial automorphism, called the *canonical automorphism* which acts as the identity on the underlying bosonic space and as multiplication by -1 in the odd directions, supermoduli spaces will *always* be stacky.

Categories Fibered in Groupoids

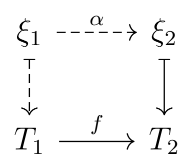
The definition of a category fibered in groupoids is motivated by deformation theory and, although, the definition itself is quite abstract, its really just about describing deformations using categorical language. Let S be a superscheme. Let $\mathcal{X}$ be a category and $p : \mathcal{X} \to \mathrm{SupSch}\,/S$ be a functor which we visualize as follows:



where ξ_1 and ξ_2, η_2 are objects in the category $\mathcal{X}$ over the S-superschemes T_1 and T_2, and the morphisms α and β are over f, where f is a morphism of superschemes.

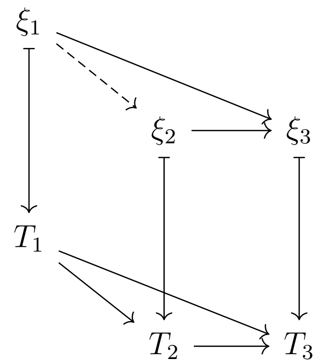
Definition 0.3.21 (Category Fibered in Groupoids)**.** *A category fibered in groupoids over* SupSch *is a category* $\mathcal{X}$ *together with a functor* $p : \mathcal{X} \to \mathrm{SupSch}\,/S$ *such that:*

(1) For every morphism of S-superschemes $T_1 \to T_2$ and an object ξ_2 of $\mathcal{X}$ over T_2, there exists an object ξ_1 over T_1 completing the diagram



where the morphism α is over f, i.e $p(\alpha) = f$. In other words, the category has pullbacks. Recall that in deformation theory the object we are deforming is a pullback of its deformation—this fact is why a category fibered in groupoids is defined to have pullbacks.

(2) Thus, for all diagrams



we require there to exist a unique arrow $\xi_1 \to \xi_2$ *over* $T_1 \to T_2$ *filling in the diagram. Recall that in deformation theory, a morphism of deformations is an isomorphism—this fact is why a category fibered in groupoids is defined to have a unique pullback, i.e the requirement that the pullback be unique immediately implies that all arrows* $\xi_1 \to \xi_2$ *are isomorphisms.*

Condition (1) and (2) imply that the fiber category $\mathcal{X}(T)$ is a groupoid, i.e all morphisms $\xi_1 \to \xi_2$ over the identity morphism $\mathrm{id}_T : T \to T$ are isomorphisms.

Superstacks

Decent theory is what we study when we consider gluing constructions over more general sites than that of schemes with the Zariski topology. Algebraic superspaces and superstacks are generalizations to sites of the gluing conditions one encounters in the study of sheaves. An algebraic superspace is a sheaf of sets on the category of superschemes with the étale topology which is locally representable by a superscheme. In particular, an algebraic superspace is glued together from these étale local subsets and this is why we say that an algebraic superspace is a superscheme with an étale equivalence relation. A superstack is a sheaf of groupoids on the category of superschemes with the étale topology which is locally representable by an algebraic superspace.

Definition 0.3.22 (Algebraic Superspace). *Let S be a superscheme. An algebraic superspace over S is a functor $X : \mathrm{SupSch} \to \mathrm{Set}$ such that the following hold:*

(a) X is a sheaf with respect to the étale topology.

(b) $\triangle\colon X \to X \times_S X$ is representable by superschemes.

(c) There exists an S-superscheme $U \to S$ and a surjective, étale morphism $U \to X$.

Definition 0.3.23 (Superstack). *An (étale)* superstack *over a superscheme S is a category $p : \mathcal{X} \to \mathrm{SupSch}$ fibered in groupoids over S such that the assignment*

$$\begin{aligned} \mathrm{SupSch}/S &\to \mathrm{Set} \\ U &\mapsto \mathcal{X}(U) = p^{-1}(U) \end{aligned} \tag{1}$$

is a sheaf of groupoids, *i.e*

1. (Isomorphisms are a Sheaf) *For all $U \in \mathrm{SupSch}/S$ and for all $x, y \in \mathcal{X}(U)$, the functor*

$$\begin{aligned} \mathrm{Iso}_U(x, y) : \mathrm{SupSch}/U &\to \mathrm{Set} \\ V &\mapsto \{\alpha : x|_V \xrightarrow{\sim} y|_V \text{ is an isomorphism in } \mathcal{X}(V)\} \end{aligned} \tag{2}$$

   is a sheaf in the étale topology (Definition 0.3.14) on SupSch*. This means that for all coverings $\{U_i \to U\}$ of U, and all isomorphisms $\alpha_i : x|_{U_i} \xrightarrow{\sim} y|_{U_i}$ such that $\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$ where U_{ij} denotes the intersection $U_i \times_U U_j$, there exists a unique isomorphism $\alpha : x \xrightarrow{\sim} y$ on U such that $\alpha|_{U_i} = \alpha_i$.*

2. (All Descent Data is Effective) *For all open covers* $\{U_i \to U\}$*, and all* $x_i \in \mathcal{X}(U_i)$*, and all* $\alpha_{ij} : x_i|_{U_{ij}} \xrightarrow{\sim} x_j|_{U_{ij}}$ *satisfying the cocycle condition over* U_{ijk}*, there exists an* $x \in \mathcal{X}(U)$ *and* $\alpha_i : x|_{U_i} \xrightarrow{\sim} x_i$ *on* U_i *such that* $\alpha_{ij} = \alpha_j|_{U_{ij}} \circ (\alpha_i|_{U_{ij}})^{-1}$.

A morphism of superstacks is a functor $F : \mathcal{X} \to \mathcal{Y}$ *such that* $p_{\mathcal{X}} = p_{\mathcal{Y}} \circ F$*. The functor* F *is an isomorphism if it is an equivalence of categories.*

Note that what we have defined as a superstack is technically an étale superstack. We would like to describe a morphism of superstacks as being smooth, étale, flat, or surjective. To do this one needs to define a representable morphism of superstacks,

Definition 0.3.24 (Representable Morphism)**.** *We call a morphism of superstacks* $F : \mathcal{X} \to \mathcal{Y}$ *a representable morphism if for all* S*-superschemes* $Y \to \mathcal{Y}$*, the fiber products* $\mathcal{X} \times_{\mathcal{Y}} Y$ *is an algebraic superspace. We say that a morphism* $F : \mathcal{X} \to \mathcal{Y}$ *is smooth, étale, flat, or surjective if the morphism* $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ *is smooth, étale, flat, or surjective.*

Definition 0.3.25 (Deligne-Mumford Superstack)**.** *A superstack* $\mathcal{X}$ *is called algebraic if the following two conditions hold:*

(i) The diagonal $\triangle : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ *is representable, and*

(ii) There exists a superscheme X *and a representable, étale, surjective, morphism* $X \to \mathcal{X}$.

If the morphism in condition (ii) is smooth instead of étale, we say that $\mathcal{X}$ *is an algebraic, or Artin, superstack.*

0.3.3 Artin Algebraization

In this section we will give a proof of Artin algebraization in the super case. In the classical setting, one proves Artin algebraization via the following series of implications: Neron-Popescu desingularization $\Longrightarrow$ Artin approximation $\Longrightarrow$ Conrad de-Jong approximation $\Longrightarrow$ Artin algebraization. The first theorem in this series of implications, namely that of Neron-Popescu desingularization is a very difficult result which actually directly implies Artin approximation; Neron-Popescu desingularization was not known at time Artin proved his approximation theorems. Fortunately, since Artin approximation has already been proved for analytic superalgebras ([13], Theorem 1.11) we will not need to generalize the proof of Neron-Popescu desingularization to the super case. We are therefore left to prove a super version of Conrad de-Jong approximation and Artin algebraization. The proof of these follow exactly as in the classical case; however, since a super version of Artin algebraization seems to be missing from the literature, we will provide a proof of it in this section. Our definitions and proofs follow almost verbatim those in [3].

Artin Approximation

Let us begin by setting up the problem of Artin Approximation. Let $S = \operatorname{Spec} A$ be an affine superscheme of finite type over $\mathbf{k}$ and let $X = \operatorname{Spec} A[x_1, \ldots, x_n | \theta_1, \ldots, \theta_m]/(f_1, \ldots, f_k)$. Let

$$h_X : \operatorname{SupAffSch}/S \to \operatorname{Set}$$

denote the functor of points of the superscheme X. Any element $\xi \in h_X(T)$ is a S-superscheme $T = \operatorname{Spec} R$ with an S-linear morphism $g : A[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \to R$ with

$a = (g(x_1), \ldots, g(x_n))|g(\theta_1), \ldots, g(\theta_m)) \in R^{\oplus n+m}$ such that $f_k(a) = 0$ for all k. Let $\mathfrak{m} \subset A$ be a maximal ideal and let $\widehat{A}$ denote the completion of A with respect to $\mathfrak{m}$. A super version of Artin approximation should give an answer to the following questions,

(Q1). *If $\widehat{g} : \operatorname{Spec} \widehat{A} \to X$ is a morphism of S-superschemes, does there exist a morphism $g : \operatorname{Spec} A_{\mathfrak{m}} \to X$?*

(Q2). *If so, how are $\widehat{g}$ and g related?*

The answer to the first question is "Yes—but only étale locally". Saying that the statement is true étale locally means that there exists such a g once we replace $A_{\mathfrak{m}}$ with its *henselization*,

$$A^h_{\mathfrak{m}} = \varinjlim_{A_{\mathfrak{m}} \to B_\lambda} B_\lambda$$

where $A_{\mathfrak{m}} \to B_\lambda$ is an étale. We answer the second question by giving the full statement of Artin Approximation.

Theorem 0.3.26 (Artin Approximation)**.** *Let $(A, \mathfrak{m})$ be a local, henselian superalgebra which is the henselization of a superalgebra of finite type over $\boldsymbol{k}$ at a $\boldsymbol{k}$-point. Let $f_1, \ldots, f_k \in A[x_1, \ldots, x_n|\theta_1, \ldots, \theta_m]$ be homogeneous polynomials and suppose $\widehat{a} = (\widehat{a_1}, \ldots, \widehat{a_n}, \widehat{\eta_1}, \ldots, \widehat{\eta_m}] \in \widehat{A}^{\oplus n+m}$ is such that $f_k(\widehat{a}) = 0$ for all k. Then for any integer $N \geq 0$, there exists a solution $a = (a_1, \ldots, a_n, \eta_1, \ldots, \eta_m) \in A^{\oplus n+m}$ such that $f(a) = 0$ for all k and such that $a \equiv \widehat{a}$ mod $\mathfrak{m}^{N+1}$.*

Proof. See Theorem 1.11 in [13] and replace analytic superalgebras with Henselian ones. The proof of Theorem 0.3.26 follows verbatim.

□

The following restatement of super Artin approximation for groupoids can be proven directly from Theorem 0.3.26.

Theorem 0.3.27 (Groupoid version of super Artin Approximation)**.** *Let S be a superscheme of finite type over $\boldsymbol{k}$ and let $\mathcal{X}$ be a limit preserving category fibered in groupoids over the category of superschemes over S. Let $s \in S$ be a $\mathbf{k}$-point and $\widehat{\xi}$ be an object of $\mathcal{X}$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s})$. For any integer $N \geq 0$, there exist an étale morphism $(S, S) \to (S, s)$ and an element ξ' of $\mathcal{X}$ over S such that restrictions $\widehat{\xi}|_{\operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1}}$ and $\xi'|_{\operatorname{Spec} \mathcal{O}_{S,S}/\mathfrak{m}_S^{N+1}}$ are equal under the identification $\mathcal{O}_{S,S}/\mathfrak{m}_S^{N+1} \cong \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1}$.*

Conrad-de Jong Approximation

In the non-super version of Artin approximation ([3], Thoerem 1.19) it is assumed that the objects $\widehat{\xi}$ of $\mathcal{X}$ we are approximating are over the spectra of completions of the local rings of a scheme S at s. In [7], it is shown that an analogous approximation will hold when loosening the hypothesis of Artin Approximation to include objects $\widehat{\xi}$ of $\mathcal{X}$ over the spectrum of a local, complete, Noetherian rings $(R, \mathfrak{m}_R)$. We restate the statement of Conrad de-Jong approximation in [3] (Theorem 2.1) for **k**-superalgebras.

Theorem 0.3.28 (Super Conrad-de Jong Approximation)**.** *Let $\mathcal{X}$ be a limit preserving category fibered in groupoids over $\boldsymbol{k}$. Let $(R, \mathfrak{m})$ be a complete, local, Noetherian $\boldsymbol{k}$-superalgebra and let $\widehat{\xi}$ be an object of $\mathcal{X}$ over $\operatorname{Spec} R$. Then for every integer $N \geq 0$, there exists*

(1) an affine superscheme $\operatorname{Spec} A$ of finite type over $\boldsymbol{k}$ and a $\boldsymbol{k}$-point of $u \in \operatorname{Spec} A$,

(2) an object ξ_A of $\mathcal{X}$ over $\operatorname{Spec} A$,

(3) an isomorphism $\alpha_{N+1} : R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$,

(4) *an isomorphism of* $\widehat{\xi}|_{\mathrm{Spec}(R/\mathfrak{m}^{N+1}}$ *and* $\xi_A|_{\mathrm{Spec}(A/\mathfrak{m}_u^{N+1}}$ *via the isomorphism* α_{N+1}*, and*

(5) *an isomorphism* $\mathrm{Gr}_{\mathfrak{m}}(R) \cong \mathrm{Gr}_{\mathfrak{m}_u}(A)$ *as graded* $\boldsymbol{k}$*-superalgebras.* [3]

The proof of Conrad de-Jong approximation given in [3] (Theorem 2.1) goes through in the super case essentially without change.

Artin Algebraization

It would obviously be useful if we could strengthen the conclusion of Conrad-de Jong approximation to give an object ξ_A of $\mathcal{X}$ over $\mathrm{Spec}\, A$ agreeing with the object $\widehat{x}$ for all N. Artin algebraization tells us that this is indeed possible, IF we are willing to impose an extra condition on the object $\widehat{\xi}$, called *formal versality.*

Definition 0.3.29 (Formal Versality)**.** *An object* $\widehat{\xi}$ *in* $\mathcal{X}$ *over* $\mathrm{Spec}\, R$*, with* R *a complete, local, noetherian* $\boldsymbol{k}$*-superalgebra is formally versal at* $x \in \mathrm{Spec}\, R$ *of for every commutative diagram*

$$\begin{array}{ccccc} \mathrm{Spec}\, k(x) & \hookrightarrow & \mathrm{Spec}\, B & \longrightarrow & \mathrm{Spec}\, R \\ & & \big\downarrow & \nearrow & \big\downarrow \widehat{\xi} \\ & & \mathrm{Spec}\, B' & \xrightarrow{\ \xi\ } & \mathcal{X} \end{array}$$

where $B' \to B$ *is a surjection of Artinian* $\boldsymbol{k}$*-superalgebras, there exists a lift* $\mathrm{Spec}\, B' \to \mathrm{Spec}\, R$ *filling in the dotted arrow in the above diagram.*

To prove Theorem 0.3.31 we need the following lemma,

[3]Here the isomorphism respects both the $\mathbb{Z}_2$ and $\mathbb{Z}$-grading.

Lemma 0.3.30. *Let* $(A, \mathfrak{m}_A)$ *and* $(B, \mathfrak{m}_B)$ *be local, Noetherian, complete superrings. If* $A \to B$ *is a local morphism such that* $A/\mathfrak{m}_A^2 \to B/\mathfrak{m}_B^2$ *is surjective, then* $A \to B$ *is surjective.*

Theorem 0.3.31 (Artin algebraization)**.** *Let* $\mathcal{X}$ *be a limit preserving category fibered in groupoids over* $\boldsymbol{k}$*. Let* $(R, \mathfrak{m})$ *be a complete, local noetherian* $\boldsymbol{k}$*-superalgebra and let* $\widehat{\xi}$ *be a formally versal object over of* $\mathcal{X}$ *over* $\operatorname{Spec} R$*. There exists*

(1) an affine superscheme $\operatorname{Spec} A$ *of finite type over* $\boldsymbol{k}$ *and a* $\boldsymbol{k}$*-point* $u \in \operatorname{Spec} A$*,*

(2) an object ξ_A *of* $\mathcal{X}$ *over* $\operatorname{Spec} A$*,*

(3) an isomorphism $\alpha : R \xrightarrow{\sim} \widehat{A}_{\mathfrak{m}_u}$ *of* $\boldsymbol{k}$*-superalgebras, and*

(4) a compatible family of isomorphism $\widehat{\xi}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}} \cong \xi_A|_{\operatorname{Spec} A/m_u^{n+1}}$ *under the identification* $R/\mathfrak{m}^{n+1} \cong A/\mathfrak{m}_u^{n+1}$ *for all* $n \geq 0$*.*

Proof. The proof again exactly mimics that of the classical statement, however, we will provide its generalization to the super case since it does seem to be missing from the literature. Let $N = 1$ and apply Conrad-de Jong approximation to get an affine superscheme $\operatorname{Spec} A$ and a point $u \in \operatorname{Spec} A$ and isomorphisms

$$\begin{gathered} \alpha_2 : \operatorname{Spec} A/\mathfrak{m}_u^2 \xrightarrow{\sim} \operatorname{Spec} R/\mathfrak{m}_R^2 \\ i_2 : \xi_1|_{\operatorname{Spec} A/\mathfrak{m}_u^2} \xrightarrow{\sim} \widehat{x}|_{\operatorname{Spec} R/\mathfrak{m}_R^2} \\ \operatorname{Gr}_{\mathfrak{m}_u}(A) \cong \operatorname{Gr}_{\mathfrak{m}}(R). \end{gathered} \tag{3}$$

By the formal versality of $(\operatorname{Spec} R, \widehat{\xi})$, we then have, for all n map $\alpha_{n+1} : \operatorname{Spec} A/\mathfrak{m}_u^{n+1} \to \operatorname{Spec} R$ filling in the dotted arrow in the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A/m_u^n & \longrightarrow & \operatorname{Spec} R \\ \big\downarrow & \overset{i_{n+1}}{\nearrow} & \big\downarrow \widehat{\xi} \\ \operatorname{Spec} A/m_u^{n+1} & \longrightarrow & \mathcal{X}. \end{array}$$

We can then take the limit $\varinjlim_n (i_{n+1} : \operatorname{Spec} A/\mathfrak{m}_u^n \to \operatorname{Spec} R)$ to get a map $R \to \widehat{A}_{\mathfrak{m}_u}$, which is surjective because it restricts to the isomorphism α_2. It is an isomorphism because for each n, the super vector spaces $\mathfrak{m}^N/\mathfrak{m}^{N+1}$ and $\mathfrak{m}_u^N/\mathfrak{m}_u^{N+1}$ have the same dimension since $\mathrm{Gr}_{\mathfrak{m}_u}(A) \cong \mathrm{Gr}_{\mathfrak{m}}(R)$. □

0.4 SUSY Curves

SUSY curves, or super Riemann surfaces, were first introduced by string theorists to describe superstring worlsheets. They have since become interesting mathematical objects in their own right. In this section, we begin by defining supercurves and give a classification of genus zero supercurves. The remainder of the section is dedicated to discussing SUSY curves, and their punctures. In the last sections, we focus on genus zero SUSY curves with Ramond punctures and discuss their automorphisms.

0.4.1 Supercurves

In this section we provide some basic results and definitions regarding supercurves. We will also show in Lemma 0.4.4 that genus zero $(1|1)$-supercurves are quite distinct from their ordinary counterpart, the projective line, as they are not all isomorphic and, furthermore, are not rigid, *i.e.*, have nontrivial moduli.

Definition 0.4.1 (Super Curve)**.** *A smooth superscheme is a superscheme X of finite type over $\boldsymbol{k}$ such that $X \to \operatorname{Spec} \boldsymbol{k}$ is a smooth morphism of superschemes. If* $\dim X =$

$(1|N)$, we say that X is a $(1|N)$-supercurve. The genus of a supercurve X is equal to the genus of X_b.

Remark 0.4.2. Even though the underlying superschemes of SUSY curves are supercurves, the literature on supercurves is somewhat sparse. This is in part due to supercurves not possessing some of the standard features of classical curves, e.g on supercurves there does not exist a correspondence between points and divisors unless that supercurve is equipped with a SUSY structure, i.e unless that supercurve is given the structure of a SUSY curve.

It is a standard result in classical algebraic geometry that a smooth scheme is étale locally a subscheme of affine space. The same result holds for smooth superschemes,

Lemma 0.4.3 ([16], Proposition 1.2.3)**.** *Let $X \to Y$ be a smooth morphism and let $y \in Y$ be a closed point of Y and let $x \in X_y$ be in the fiber over y such that $\operatorname{Kdim}_{k(y)} \mathcal{O}_{X_y,x} = (m|n)$. Then there exists a Zariski open subset $U \subset X$ containing x and an étale morphism of Y-superschemes $\phi : U \to \mathbb{A}_Y^{m|n}$.*

The utility of the above lemma will become apparent in Lemma 0.4.6 in which a SUSY structure is described explicitly in (étale) local coordinates.

Lemma 0.4.4. *Let X be a genus zero $(1|1)$-supercurve. Then there exists $m \in \mathbb{Z}$ and an isomorphism $X \xrightarrow{\sim} \mathbb{WP}^{1|1}(1, 1 \,|\, m)$.*

Proof. Since $\dim X = (1|1)$, we must have that $\mathcal{O}_X^-$ is a rank 1-sheaf of $\mathcal{O}_X^+$-modules and thus generated by a single odd term θ. Since $\theta^2 = 0$, we must have that $(\mathcal{O}_X^-)^2 = 0$, and thus $\mathcal{J} = \mathcal{O}_X^-$. Furthermore, the fact that $(\mathcal{O}_X^-)^2$ vanishes implies that $\mathcal{O}_X^+ = \mathcal{O}_{X_b}$. Therefore, $\mathcal{O}_X = \mathcal{O}_{X_b} \oplus \mathcal{J}$, and X is split. Since X is split there exists a line bundle $\mathcal{L}$ on X_b such that $\mathcal{O}_X \cong \mathcal{S}(\Pi\mathcal{L}^\vee)$. Since X_b is isomorphic to $\mathbb{P}^1$ for any such line bundle $\mathcal{L}$ there exists an integer m such that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(m)$. Therefore, for any genus zero (1|1)-supercurve there exists an integer m such that $\mathcal{O}_X$ is isomorphic to $\mathcal{S}\Pi\mathcal{O}_{\mathbb{P}^1}(-m)$ which implies that X is isomorphic to $\mathbb{WP}^{1|1}(1,1\,|\,m)$.

□

Further setting apart genus zero (1|1)-supercurves from the projective line is the fact that there exists integers m for which $\mathbb{WP}^{1|1}(1,1\,|\,m)$ is not rigid. In fact, a Cech cohomology computation shows that

$$\dim H^1(\mathcal{T}\mathbb{WP}^{1|1}(1,1\,|\,m)) = \begin{cases} 0|-m-1 & \text{for } m<1 \\ 0|0 & \text{for } 0\le m\le 3 \\ 0|m-3 & \text{for } m>3. \end{cases} \tag{4}$$

The above computation for $m = 1-n_R/2$ will play a critical role in our construction of the moduli space of genus zero SUSY curves with Ramond punctures. Indeed, plugging in $1-n_R/2$ into (4), shows that the dimension of $H^1(\mathcal{T}\mathbb{WP}^{1|1}(1,1\,|\,1-n_R/2))$ is equal to $0|n_R/2-2$, which is exactly the odd dimension of $\mathfrak{M}_{0,n_R}$. In particular, we find that the somewhat mysterious $n_R/2-2$ odd moduli of $\mathfrak{M}_{0,n_R}$ were hidden within the supermoduli theory of genus zero supercurves!

0.4.2 SUSY Curves

In this section we review the basics of super Riemann surface (SUSY curves) theory. Our main sources are [12, 27, 9].

Definition 0.4.5 (SUSY Curve)**.** *A genus g SUSY curve $(X/S, \mathcal{D})$ is the data of a smooth, proper morphism $\pi : X \to S$ of superschemes of relative dimension $(1|1)$ together with a rank $(0|1)$ sub-bundle $\mathcal{D} \subset \mathcal{T}_{X/S}$, called a SUSY structure on X, which is as non-integrable as possible. An isomorphism of SUSY curves is an isomorphism $f : X \xrightarrow{\sim} X'$ of superschemes over T preserving the SUSY structure, i.e $f^*\mathcal{D}' = \mathcal{D}$ and where the genus g is equal to the genus of X_b.*

Notation and Convention. We will write $(X, \mathcal{D})$ for a genus g SUSY curve over Spec $\mathbf{k}$.

The maximal non-integrability condition on the SUSY structure $\mathcal{D}$ is expressed by identifying $\mathcal{T}_{X/S}/\mathcal{D}$ with $\mathcal{D}^{\otimes 2}$ via the supercommutator

$$D_1 \otimes D_2 \mapsto \frac{1}{2}[D_1, D_2]. \tag{5}$$

We can dualize $\mathcal{D} \subset \mathcal{T}_{X/S}$ as

$$\Omega^1_{X/S} \to \mathcal{D}^\vee \tag{6}$$

and, the isomorphism $\mathcal{T}_{X/S}/\mathcal{D} \cong \mathcal{D}^{\otimes 2}$ dualizes to the SES,

$$0 \longrightarrow (\mathcal{D}^\vee)^{\otimes 2} \longrightarrow \Omega^1_{X/S} \longrightarrow \mathcal{D}^\vee \longrightarrow 0 \tag{7}$$

and, thereby, to an isomorphism $\mathrm{Ber}(\Omega^1_{X/S}) \cong \mathcal{D}^\vee$ and a surjective morphism

$$\delta : \Omega_{X/S} \to \omega_{X/S} := \mathrm{Ber}(\Omega^1_{X/S}) \tag{8}$$

which gives a derivation $\delta : \mathcal{O}_X \to \omega_{X/S}$.

Local Structure

It is useful to describe the SUSY structure on a SUSY curve $(X, \mathcal{D})$ explicitly in étale local coordinates. A standard example of a SUSY structure on $\mathbb{A}^{1|1}_{\mathbf{k}}$ is the sub-bundle of $\mathcal{T}_X$ generated by the odd vector field

$$D_\theta = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} \tag{9}$$

where

$$D_\theta^2 = \frac{1}{2}[D_\theta, D_\theta] = \frac{\partial}{\partial z}. \tag{10}$$

In some sense, we may, in view of 10, think of a SUSY structure $\mathcal{D}$ as a square root of the derivative. Using Lemma 0.4.3, it turns out that we can always pick coordinates on $\mathbb{A}_{\mathbf{k}}^{1|1}$ such that any SUSY structure on $\mathbb{A}_{\mathbf{k}}^{1|1}$ is generated by the vector field in 9.

Lemma 0.4.6 ([12], Lemma 4.4)**.** *Let $(X, \mathcal{D})$ be a SUSY curve and let p be a closed point of X_b.* [4] *Then there exists an (étale) open subset $U \ni p$ and a coordinate system $W = (z, \theta)$ on U, such that $\mathcal{D}|_U$ is generated by the odd vector field*

$$D_\theta = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}. \tag{11}$$

Moreover, up to SUSY isomorphism, there is a unique SUSY structure on $\mathbb{A}^{1|1}$, namely the one generated by the odd vector field D_θ.

SUSY Curves as Spin Curves

The data describing a family of SUSY curves parameterized over an ordinary scheme—as opposed to over a superscheme—is equivalent to the data describing a family of spin curves parameterized over the same scheme. The theory of spin curves and their moduli have been widely studied (e.g [14, 1, 15, 8]) and is, therefore, helpful in developing intuition for the geometry of SUSY curves.

Spin Curves. A genus g spin curve over a scheme T is the data $(\mathcal{C} \to T, \mathcal{L}, c)$ with

[4] A closed point $p \in X_b$ is the image of a morphism $\operatorname{Spec} \mathbf{k} \to X$ of superschemes.

(a) $\mathcal{C} \to T$ a curve of genus g,

(b) a line bundle $\mathcal{L}$ on $\mathcal{C}$, and

(c) an isomorphism $c : \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathcal{C}}$.

Theorem 0.4.7. *Let $(X/T, \mathcal{D})$ be a genus g SUSY curve over T. Then,*

(a) X is split.

(b) $(\Omega^1_{X/T})_b \cong \Omega^1_{X_b/T} \oplus \mathcal{J}$, and

(c) there exist isomorphisms of $\mathcal{O}_{X_b}$-modules,

$$f : (\mathcal{D}_b^\vee)^{\otimes 2} \xrightarrow{\sim} \Omega^1_{X_b/T}$$

and

$$g : \mathcal{J} \xrightarrow{\sim} \mathcal{D}_b^\vee.$$

In particular, $(X_b/T, \mathcal{D}_b^\vee, f)$ is a spin curve over T.

Remark 0.4.8. *The converse to Theorem 0.4.7— i.e that any spin curve gives rise to a SUSY curve— is also true ([12], Corollary 4.9).*

Proof of Theorem 0.4.7. Since $\dim X = 1|1$, we must have that $\mathcal{O}_X^-$ is a rank 1 sheaf of $\mathcal{O}_X^+$-modules and thus generated by a single odd term, say θ. Since $\theta^2 = 0$, we must have that $(\mathcal{O}_X^-)^2 = 0$. Recall, that

$$\mathcal{J} = \mathcal{O}_X^- \oplus (\mathcal{O}_X^-)^2$$

and so, from the above conclusion, we have that $\mathcal{J} = \mathcal{O}_X^-$. Furthermore, since $(\mathcal{O}_X^-)^2 = 0$, we have that $\mathcal{O}_X^+ = \mathcal{O}_{X_b}$. Thus $\mathcal{O}_X = \mathcal{O}_{X_b} \oplus \mathcal{J}$ and thus equal to $\mathcal{O}_{\mathrm{Gr}\, X}$ and, therefore, split.

To prove part b, recall that the surjection $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{J}$ induces a closed immersion $X_b \hookrightarrow X$. Since X is smooth, we have a short exact sequence of $\mathcal{O}_{X_b}$-modules.

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega^1_{X/T} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{J} \to \Omega^1_{X_b/T} \to 0. \tag{12}$$

Let us denote $\Omega^1_{X/T} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{J}$ as $(\Omega^1_{X/T})_b$ and its $\mathbb{Z}_2$-grading as

$$(\Omega^1_{X/T})_b = (\Omega^1_{X/T})^+_b \oplus (\Omega^1_{X/T})^-_b.$$

Since $J^2 = 0$, we rewrite the 12 as

$$0 \to \mathcal{J} \to (\Omega^1_{X/T})^+_b \oplus (\Omega^1_{X/T})^-_b \to \Omega^1_{X_b/T} \to 0. \tag{13}$$

The $\mathcal{O}_{X_b}$-modules $\mathcal{J}$ and $(\Omega^1_{X/T})^-_b$ are of rank $0|1$, and $(\Omega^1_{X/T})^+_b$ is of rank $1|0$. The maps in the above sequence must preserve the $\mathbb{Z}_2$-grading on the modules and, therefore,

$$(\Omega^1_{X/T})_b = \Omega^1_{X_b/T} \oplus \mathcal{J} \tag{14}$$

Since X is smooth, $\Omega^1_{X/T}$ is locally free of rank $1|1$ we also have that $(\mathcal{T}_{X/T})_b = \mathcal{T}_{X_b/T} \oplus (\mathcal{J})^*$.

To prove part (c), tensor the short exact sequence

$$0 \longrightarrow (\mathcal{D}^\vee)^{\otimes 2} \longrightarrow \Omega^1_{X/T} \longrightarrow \mathcal{D}^\vee \longrightarrow 0 \tag{15}$$

with the sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X/\mathcal{J}$. Since $\mathcal{O}_X/\mathcal{J}$ is projective, tensoring with it will preserve exactness, resulting in short exact sequence of $\mathcal{O}_{X_b}$-modules,

$$0 \longrightarrow (\mathcal{D}^\vee_b)^{\otimes 2} \longrightarrow \Omega^1_{X_b/T} \oplus \mathcal{J} \longrightarrow \mathcal{D}^\vee_b \longrightarrow 0. \tag{16}$$

As $\mathcal{O}_{X_b}$-modules, $(\mathcal{D}^\vee_b)^{\otimes 2}$ has rank $1|0$, $\mathcal{J}$ has rank $0|1$, and $\mathcal{D}^\vee_b$ has rank $0|1$. As before, the maps in 16 must respect the $\mathbb{Z}_2$ grading. Therefore, there exist isomorphisms $g : \mathcal{J} \xrightarrow{\sim} \mathcal{D}^\vee_b$ and $f : (\mathcal{D}^\vee_b)^{\otimes 2} \xrightarrow{\sim} \Omega^1_{X_b/T}$.

□

0.4.3 Punctured SUSY Curves

In this section we will shift our attention to punctured SUSY Curves. There are two distinct types of punctures considered on SUSY curves, called Ramond and Neveu-Schwarz (NS) punctures. The necessity for considering two distinct types of punctures is seen already in M. Cornalba's construction of the moduli space of stable spin curves ([8]). In his construction, the NS punctures are analogous to the marked points we see in the compactification of $\mathcal{M}_g$, i.e they are there to "mark" the point on the normalization of a nodal curve corresponding to the node. The Ramond punctures, on the other hand, are entirely unique to the moduli of stable spin curves and, roughly, correspond to the ramification points in the projection $p : \overline{\mathcal{SM}}_g \to \overline{\mathcal{M}}_g$, where $\overline{\mathcal{SM}}_g$ denotes the moduli space of stable spin curves.

Definition 0.4.9 (Punctured SUSY Curves)**.** *A genus g SUSY curve* $(X/S, N, \mathcal{D}, R)$ *with* n_{NS} *Neveu-Schwarz (NS) and* n_R *Ramond punctures is a smooth, proper morphism* $\pi : X \to S$ *of relative dimension* $(1|1)$*, together with the data of:* n_{NS} *sections* $N = N_i : S \to X$ *(each* N_i *called a NS puncture), a closed subscheme* $R = R_1 \sqcup \cdots \sqcup R_{n_R}$ *proper, flat, and unramified over* S *of codimension* $(1|0)$ *(each* R_i *is called a Ramond puncture and* R *is called the Ramond divisor), and a rank-*$(0|1)$ *subbundle* $\mathcal{D} \subset \mathcal{T}_{X/S}$ *such that*

$$[\ ,\ ] : \mathcal{D} \otimes \mathcal{D} \xrightarrow{\sim} \mathcal{T}_{X/S}/\mathcal{D}(-R).$$

where $[\ ,\ ]$ *denotes the supercommutator.*

As is the case with unpunctured SUSY curves, we can dualize $\mathcal{D} \subset \mathcal{T}_{X/S}$ as

$$\Omega^1_{X/S} \to \mathcal{D}^\vee(R) \tag{17}$$

and, the isomorphism $\mathcal{T}_{X/S}/\mathcal{D} \cong \mathcal{D}^{\otimes 2}(R)$ dualizes to the SES,

$$0 \longrightarrow (\mathcal{D}^{\vee})\otimes 2 \longrightarrow \Omega^1_{X/S} \longrightarrow \mathcal{D}^{\vee}(R) \longrightarrow 0. \tag{18}$$

From which we find that

$$\mathrm{Ber}(\Omega^1_{X/S}) \cong \mathcal{D} \otimes \mathcal{D}^{-2}(-R) \cong \mathcal{D}^{-1}(-R).$$

This gives a surjection $\delta : \Omega^1_{X/S} \to \mathrm{Ber}(\Omega^1_{X/S})(R)$ which we can use to give an equivalent characterization of a SUSY structure on X as a rank $(1|0)$-subbundle $\mathcal{L}$ of $\Omega^1_{X/T}$ such that $\mathcal{L}$ is the kernel of a derivation

$$\delta : \mathcal{O}_X \to \mathrm{Ber}(\Omega^1_{X/S})(R) \tag{19}$$

such that $\delta f = Df \cdot [dz/d\theta]$ where $D = \partial_\theta + p(z)\theta\partial_z$

Remark 0.4.10. A super Riemann surface with Ramond punctures is not an honest super Riemann surface because it fails to meet the definition that the supercommutator identifies $\mathcal{D}^{\otimes 2}$ with $\mathcal{T}_{X/S}/\mathcal{D}$. Similarly, the Ramond punctures R_i are not punctures, but rather codimension $(1, 0)$ divisors.

Remark 0.4.11. Given a complex $(1|1)$ supermanifold X, there exists a one-to-one correspondence between punctures on X and codimension $(1|0)$ divisors on X if and only if X admits a SUSY structure. Thus, calling Ramond punctures—punctures—is also a bit of a misnomer.

Notation and Convention. From now on we will take a punctured SUSY curve to mean a SUSY curve with Ramond punctures since NS punctures do not play a role in our moduli problem and are, moreover, already well-understood.

Local Description

Lemma 0.4.12 ([27], Section 5.1.4)**.** *Let $(X, \mathcal{D}, R)$ be a punctured SUSY curve and let p be a closed point of X_b. Then there exists (étale) locally an open subset $U \ni p$ and a coordinate system $W = (z, \theta)$ on U such that $\mathcal{D}|_U$ is generated by the odd vector field*

$$D_\theta = \frac{\partial}{\partial\theta} + p(z)\theta\frac{\partial}{\partial z} \tag{20}$$

where $p(z) = \Pi_{i=1}^{n_R}(z - z_i)$. Moreover, up to SUSY-isomorphism, there is a unique SUSY structure on $\mathbb{A}^{1|1}$, namely the one generated by the odd vector field D_θ.

Remark 0.4.13. *From the dualized perspective, $\mathcal{L}$ is locally generated by the even form*

$$dz + p(z)\theta d\theta \tag{21}$$

where $p(z) = \Pi_{i=1}^{n_R}(z - z_i)$.

Punctured SUSY Curves as Pointed Spin Curves

As in the case of unpunctured SUSY curves, there exists a useful relationship between families of punctured SUSY curves over ordinary schemes and families of pointed spin curves over the same base scheme.

n-Pointed Spin Curves. A n-pointed twisted spin curve of genus g over a scheme S is the data $(\mathcal{C} \to S, s_i, \mathcal{L}, c)$ with

(a) $(\mathcal{C} \to S, s_i : S \to \mathcal{C})$ is a n-pointed curve of genus g,

(b) a line bundle $\mathcal{L}$ on $\mathcal{C}$, and

(c) an isomorphism $c : \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathcal{C}/S}(\sum_{i=1}^n S_i)$, where the S_i are the images of the s_i in $\mathcal{C}$.

Theorem 0.4.14. *Let $(X, \mathcal{D}, R)$ be a genus g, n_R-punctured SUSY curve. Then,*

(a) X is split.

(b) $(\Omega^1_X)_b \cong \Omega^1_{X_b} \oplus \mathcal{J}$.

(c) There exist isomorphisms of $\mathcal{O}_{X_b}$-modules,

$$f : (\mathcal{D}_b^\vee)^{\otimes 2} \xrightarrow{\sim} \Omega^1_{X_b}(R) \tag{22}$$
$$g : \mathcal{J} \xrightarrow{\sim} \mathcal{D}^\vee$$

In particular, $(X_b/T, R_b, (\mathcal{D}_b^\vee), f)$ is a n_R-pointed spin curve.

Proof. The proofs of part (a) and (b) are identical to the ones gives in Theorem 0.4.7. For part (c), the proof is identical except for that our short exact sequence of $\mathcal{O}_{X_b}$-module must reflect the Ramond punctures, i.e

$$0 \longrightarrow (\mathcal{D}_b^\vee)^{\otimes 2}(-R_b) \longrightarrow \Omega^1_{X_b/T} \oplus \mathcal{J} \longrightarrow \mathcal{D}_b^\vee \longrightarrow 0. \tag{23}$$

□

Corollary 0.4.15. *The number of Ramond punctures n_R is always even.*

Proof. Since $X_b \cong \mathbb{P}^1$ and $\mathcal{O}(R_b) \cong \mathcal{O}_{\mathbb{P}^1}(n_R)$ we can use Theorem 0.4.14 to find an isomorphism

$$(\mathcal{D}_b^\vee)^{\otimes 2} \xrightarrow{\sim} \mathcal{O}(2 - n_R).$$

Thus, 2 must divide the degree n_R of the Ramond divisor R_b. □

0.4.4 Weighted Projective Superspace $\mathbb{WP}^{1|1}(1,1\,|\,1-n_R/2)$

It was already known to E. Witten in [27] that the superscheme underlying a genus zero SUSY curve with n_R Ramond punctures over $\operatorname{Spec}\mathbf{k}$ is isomorphic to $\mathbb{WP}^{1|1}(1,1\,|\,1-n_R/2)$. The weighted projective superspace $\mathbb{WP}^{1|1}(1,1\,|\,1-n_R/2)$ will, therefore, play a central role in our moduli problem. We encourage the reader to read this section alongside Section 5.

Gluing Construction. We refer the reader to Example 0.3.7 for the definition of weighted projective superspace. Another way to understand $\mathbb{WP}^{1|1}(1,1\,|\,1-n_R/2)$ is via its gluing construction. Analogous to ordinary projective space, we can construct $\mathbb{WP}^{1|1}(1,1\,|\,1-n_R/2)$ by choosing the covering $\mathcal{U} = \{U \cong \operatorname{Spec} k[z,\zeta], V \cong\cong \operatorname{Spec} k[w,\chi]\}$ where U is the subset on which $u \neq 0$ and V is the subset on which $v \neq 0$ such that

$$\begin{aligned} \phi_u : \mathbb{WP}^{1|1}(1,1\,|\,1-n_R/2)|_U &\xrightarrow{\sim} \operatorname{Spec}\mathbf{k}[z,\zeta] \\ v/u &\mapsto z \\ \theta u^{n_R/2-1} &\mapsto \zeta \end{aligned} \tag{24}$$

and

$$\begin{aligned} \phi_v : \mathbb{WP}^{1|1}(1,1\,|\,1-n_R/2)|_V &\xrightarrow{\sim} \operatorname{Spec}\mathbf{k}[w,\chi] \\ u/v &\mapsto w \\ \theta v^{n_R/2-1} &\mapsto \chi \end{aligned} \tag{25}$$

and restricting to an automorphism

$$\phi_v \circ \phi_u^{-1} : \operatorname{Spec} \mathbf{k}[z, \zeta] \xrightarrow{\sim} \operatorname{Spec} \mathbf{k}[w, \chi] \tag{26}$$
$$z \mapsto 1/w$$
$$\zeta \mapsto \chi w^{1-n_R/2}.$$

which we use to patch together $\mathbb{WP}^{1|1}(1, 1 \,|\, 1 - n_R/2)$ using the standard method of gluing.

Lemma 0.4.16. *The Picard group of* $\mathbb{WP}^{1|1}(1, 1 \,|\, 1 - n_R/2)$ *is isomorphic to* $\mathbb{WP}^{1|1}(1, 1 \,|\, 1 - n_R/2)$.

Proof. Consider the following long exact sequence on $\mathbb{WP}^{1|1}(1, 1 \,|\, 1 - n_R/2)$

$$0 \longrightarrow H^0(\mathbb{Z}) \longrightarrow H^0(\mathcal{O}) \longrightarrow H^0(\mathcal{O}^*) \longrightarrow H^1(\mathbb{Z}) \longrightarrow H^1(\mathcal{O}) \longrightarrow H^1(\mathcal{O}^*) \longrightarrow H^2(\mathbb{Z}) \longrightarrow 0.$$

From an (omitted) Čech cohomology computation we find that $H^1(\mathcal{O}) = 0$, and thus $H^1(\mathcal{O}^*) \cong \mathbb{Z}$ □

Automorphisms of $\mathbb{WP}^{1|1}(1, 1 \,|\, 1 - n_R/2)$

Let S denote the $\mathbb{Z}$-graded superalgebra $\mathbf{k}[u, v, \theta]$ with $\mathbb{Z}$-grading given by $|u| = |v| = 1$ and $|\theta| = 1 - n/2$, so that $\operatorname{Proj} S$ is isomorphic to $\mathbb{WP}$.

General Strategy. Our strategy for constructing the automorphism group $\operatorname{Aut}(\mathbb{WP})$ of $\mathbb{WP}$ is to find a short exact sequence of group superschemes

$$1 \longrightarrow \Gamma^* \longrightarrow \operatorname{Aut}(S) \longrightarrow \operatorname{Aut}(\mathbb{WP}) \longrightarrow 1, \tag{27}$$

where

Definition 0.4.17 (Γ^*)**.** *the group superscheme* Γ^* *represents the functor*

$$\begin{aligned} \Gamma^* : \mathrm{SupSch} &\to \mathrm{Group}, \\ T &\mapsto \Gamma(\mathcal{O}^*_{\mathbb{WP}\times T}), \end{aligned} \tag{28}$$

Definition 0.4.18 ($\mathrm{Aut}(S)$)**.** *the group superscheme* $\mathrm{Aut}(S)$ *represents the functor*

$$\begin{aligned} \mathrm{Aut}(S) : \mathrm{SupSch} &\to \mathrm{Group}, \\ T &\mapsto \mathrm{Aut}_R(S \otimes_{\boldsymbol{k}} R), \end{aligned} \tag{29}$$

with $R := \Gamma(T, \mathcal{O}_T)$*, and*

Definition 0.4.19 ($\mathrm{Aut}(\mathbb{WP})$)**.** *the group superscheme* $\mathrm{Aut}(\mathbb{WP})$ *represents the functor*

$$\begin{aligned} \mathrm{Aut}(\mathbb{WP}) : \mathrm{SupSch} &\to \mathrm{Group} \\ T &\mapsto \mathrm{Aut}_T(\mathbb{WP}\times T) \end{aligned} \tag{30}$$

where $\mathrm{Aut}_T(\mathbb{WP}\times T)$ *is the group of automorphisms of* $\mathbb{WP}\times T$ *over* T*.*

Our next goal is to show that these functors are indeed representable.

Description of $\mathrm{Aut}(S)$*.* The set of T-points of $\mathrm{Aut}(S)$ consists of the automorphisms

$$\begin{aligned} &u \mapsto au + bv + \theta \sum_{i=0}^{n/2} \alpha_i u^{n/2-i} v^i, \quad a, b \in R^+, \alpha_i \in R^-, \\ &v \mapsto cu + dv + \theta \sum_{j=0}^{n/2} \beta_j u^{n/2-j} v^j, \quad a', b' \in R^+, \beta_j \in R^-, \\ &\theta \mapsto e\theta, \quad e \in R^*, \end{aligned} \tag{31}$$

where $R = \Gamma(T, \mathcal{O}_T)$ and $ad - bc \neq 0$. It is immediate from the above description that, as a set-valued functor, $\mathrm{Aut}(S)$ is represented by the distinguished open subset

$D(ad - bc)$ of the affine super variety.

$$W = \operatorname{Spec} \mathbf{k}[a, b, c, d, e, e^{-1} \mid \alpha_0, \cdots, \alpha_{n/2}, \beta_0, \cdots, \beta_{n/2}].$$

A Hopf superalgebra structure on $D(ad - bc)$ can then be described explicitly by composition of the automorphisms given in (31). It then immediately follows that

Lemma 0.4.20. *The underlying superscheme of* $\operatorname{Aut}(S)$ *is isomorphic to* W*.*

Description of Γ^**.* Given a superscheme T with $R := \Gamma(T, \mathcal{O}_T)$, we may compute the group $\Gamma(\mathcal{O}^*_{\mathbb{WP}\times T})$ explicitly using Čech cohomology, and describe its elements in local coordinates (z, ζ) on U as

$$a_0(1 + \zeta \sum_{i=0}^{n/2-1} \beta_i z^i) \tag{32}$$

with $a_0 \in R^*$ and $\beta_i \in R^-$. The multiplication on the T-points of Γ^* is given by

$$a_0(1 + \theta \sum_{i=0}^{n/2-1} \beta_i z^i) \cdot a_0'(1 + \zeta \sum_{i=0}^{n/2-1} \beta_i' z^i) = a_0 a_0'(1 + \zeta \sum_{i=0}^{n/2-1} (\beta_i + \beta_i') z^i) \tag{33}$$

with $a_0, a_0' \in R^*$ and $\beta_i, \beta_i' \in R^-$. It then immediately follows that

Lemma 0.4.21. *The group superscheme* Γ^* *is isomorphic to the group superscheme* $\mathbb{G}_m \times (\mathbb{G}_a^{0|1})^{n/2}$.

Proof. The multiplicative group $\mathbb{G}_m$ is standard, whereas the supergroup $\mathbb{G}_a^{0|1}$, which is a group superscheme, to be precise, may be described by its functor of points

$$\text{SupSch} \to \text{Group},$$

$$T \mapsto \Gamma(T, \mathcal{O}_T)^-,$$

with the group law given by addition. The underlying superscheme of $\mathbb{G}_a^{0|1}$ is just $\mathbb{A}_k^{0\,|\,1}$. The lemma then follows from the explicit description of $\Gamma^*(T)$ by Equations (32) and (33). ∎

Furthermore, we find that

Lemma 0.4.22. Γ^* *is a normal group subsuperscheme of* $\mathrm{Aut}(S)$.

Proof. We can describe the each element of the group $\Gamma^*(T)$ explicitly by

$$r_0(1+\theta\sum_{i=0}^{n/2-1}\beta_i v^i u^{n/2-1-i}) \tag{34}$$

with $r_0 \in R^*$ and $\beta_i \in R^-$. and identify each such element with the automorphism in $\mathrm{Aut}(S \otimes R)$ sending

$$\begin{aligned} u &\mapsto r_0(u+\theta\sum_{i=0}^{n/2-1}\beta_i v^i u^{n/2-i}) \\ v &\mapsto r_0(v+\theta\sum_{i=0}^{n/2-1}\beta_i v^{i+1} u^{n/2-1-i}) \\ \theta &\mapsto r_0^{1-n/2}\theta. \end{aligned} \tag{35}$$

□

The Superalgebra S and Line Bundles on $\mathbb{WP}$. The superalgebra S is easily seen to be equal to the $\mathbb{Z}$-graded superalgebra $\bigoplus_{i\geq 0} H^0(\mathcal{O}_{\mathbb{WP}}(1-n/2+i))$. The superalgebra $\bigoplus_{i\geq 0} H^0(\mathcal{O}_{\mathbb{WP}}(1-n/2+i))$ is generated over the global sections of $\mathcal{O}_{\mathbb{WP}}$ by the global sections of $\mathcal{O}_{\mathbb{WP}}(1)$ and $\mathcal{O}_{\mathbb{WP}}(1-n/2)$. Let T be a superscheme over $\mathrm{Spec}\,\mathbf{k}$ and $R := \Gamma(T, \mathcal{O}_T)$, denote $\mathbb{WP} \times T$ by $\mathbb{WP}_T$ and let $p : \mathbb{WP}_T \to T$ be the canonical projection. The superscheme $\mathbb{WP}_T$ is isomorphic to Proj of the superalgebra $\bigoplus_{i\geq 0} H^0(p^*\mathcal{O}_{\mathbb{WP}}(1-n/2+i))$, where $\bigoplus_{i\geq 0} H^0(p^*\mathcal{O}_{\mathbb{WP}}(1-n/2+i))$ is equal to $S \otimes R$.

Description of $\mathrm{Aut}(\mathbb{WP})(T)$. Let α be an automorphism of $\mathbb{WP}_T$ over T and consider the diagram

$$
\begin{array}{ccccc}
\mathbb{WP}_T & \xrightarrow{\alpha} & \mathbb{WP}_T & \xrightarrow{\mathrm{pr}_1} & \mathbb{WP} \\
& \searrow^{\mathrm{pr}_2} & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \pi} \\
& & T & \xrightarrow{f} & \operatorname{Spec} \mathbf{k}
\end{array} \tag{36}
$$

The line bundles on $\mathbb{WP}_T$ are isomorphic to $\mathrm{pr}_1^* \mathcal{O}_{\mathbb{WP}}(m) \otimes \mathrm{pr}_2^* \mathcal{L}$ where $\mathcal{L}$ is a line bundle on the base T (Lemma 0.4.16). Therefore, there exists an isomorphism

$$\sigma : \alpha^* \mathrm{pr}_1^* \mathcal{O}(1) \xrightarrow{\sim} \mathrm{pr}_1^* \mathcal{O}(m) \otimes \mathrm{pr}_2^* \mathcal{L} \tag{37}$$

Note that the isomorphism σ induces an isomorphism $\alpha^* \mathrm{pr}_1^* \mathcal{O}(1-n/2) \cong \mathrm{pr}_1^*(\mathcal{O}(m) \otimes \mathrm{pr}_2^* \mathcal{L})^{1-n/2}$ by taking the $(1-n/2)$ tensor product of the line bundles in (37). Using standard methods (e.g. [22], Section 5), we find that $m = 1$. Let $\mathcal{U} = \{U_i\}$ denote a covering of T on which the line bundle $\mathcal{L}$ in (37) trivializes. Then the automorphism α restricts to an automorphism α_i of $\mathbb{WP}_{U_i}$ over U_i. The automorphisms α_i induce isomorphisms

$$
\begin{aligned}
\sigma_i &: \alpha_i^* (\mathrm{pr}_1)_i^* \mathcal{O}(1) \xrightarrow{\sim} (\mathrm{pr}_1)_i^* \mathcal{O}(1) \\
\sigma_j &: \alpha_j^* (\mathrm{pr}_1)_j^* \mathcal{O}(1) \xrightarrow{\sim} (\mathrm{pr}_1)_j^* \mathcal{O}(1).
\end{aligned} \tag{38}
$$

Using the description of $S \otimes R$ as the graded-superalgebra $\bigoplus_{i \geq 0} H^0((\mathrm{pr}_1)_i^* \mathcal{O}(1-n/2+i))$ generated by the global section of $(\mathrm{pr}_1)_i^* \mathcal{O}(1)$ and $(\mathrm{pr}_1)_i^* \mathcal{O}(1-n/2)$, we see that the isomorphisms σ_i induce automorphisms of $S \otimes R_i$, where $R_i = \Gamma(\mathcal{O}_{U_i})$. Therefore, the automorphisms of $S \otimes R$ induced by α correspond to the set of automorphisms

$$\sigma_{ij} := (\sigma_j|_{U_{ij}} \circ (\sigma_i)^{-1}|_{U_{ij}} : (\mathrm{pr}_1)_{ij}^* \mathcal{O}(1) \xrightarrow{\sim} (\mathrm{pr}_1)_{ji}^* \mathcal{O}(1) \tag{39}$$

satisfying the cocycle condition on U_{ijk}. Since $\sigma_{ij} \in \Gamma(\mathcal{O}_{U_{ij}}^*) = \Gamma^*(U_{ij})$, the automorphisms of $S \otimes R$ induced by α form a $\Gamma^* \times T$-torsor. This gives a short exact sequence of group functors

$$1 \longrightarrow \Gamma^*(T) \longrightarrow \mathrm{Aut}_R(S \otimes R) \longrightarrow \mathrm{Aut}_T(\mathbb{WP} \times T) \longrightarrow 1 \tag{40}$$

and the desired short exact sequence of their representing group superschemes.

Thus, the group of T-points of $\mathrm{Aut}(\mathbb{WP})$ is isomorphic to the group of T-points of the functor $T \mapsto W(T)/\Gamma^*(T)$.

Theorem 0.4.23. *The functor $T \mapsto W(T)/\Gamma^*(T)$ is represented by the group superscheme W/Γ^*. In particular,* $\mathrm{Aut}(\mathbb{WP})$ *is a group superscheme isomorphic to W/Γ^*.*

Proof. To prove the theorem, note that the subgroup $\Gamma^*(T) \subseteq \mathrm{Aut}_R(S \otimes R)$ with R as above is the kernel of a morphism $\mathrm{Aut}_R(S \otimes R) \to \mathrm{Aut}_T(\mathbb{WP} \times T)$, and is therefore a normal subgroup. Thus Γ^* is a closed subscheme of $\mathrm{Aut}(S)$, proving the theorem. □

0.5 Moduli of Genus Zero SUSY Curves with Ramond Punctures

In this section we will describe the moduli problem of genus zero SUSY curve with $n_R \geq 4$ ordered Ramond punctures.

Generalizing the definition of a SUSY curve in [9] we define a SUSY curve with Ramond punctures as follows:

Definition 0.5.1 (SUSY Curve)**.** *A SUSY curve Σ with n_R Ramond punctures over T is given by the data $(X/T, \mathcal{L}, u, R)$ where,*

1. *X/T denotes a smooth, proper morphism of $\pi : X \to T$ of superschemes of relative dimension $(1|1)$. We refer to X/T as a family of supercurves [5] over T underlying Σ. [6]*

[5] here supercurve means a $(1|1)$-supercurve

[6] or simply as a supercurve over T if the context is clear

2. $\mathcal{L}$ *is a rank* $(1|0)$ *vector bundle on* X*, called the SUSY line bundle.* [7]

3. $u : \mathcal{L} \hookrightarrow \Omega^1_{X/T}$ *is an inclusion of vector bundles over* X*.*

4. and R *is a degree* n_R *codimension* $(1|0)$ *divisor on* X*, unramified over* T *(see also Definition 0.4.9) such that* $\operatorname{coker}(u) \cong \operatorname{Ber}(X/T) \otimes \mathscr{L}(R)$*. We call the components of* R *Ramond punctures and refer to* $(\mathcal{L}, u, R)$ *as a SUSY structure.*

Henceforth, we will write a SUSY curve to mean a genus zero SUSY curve with $n_R \geq 4$ *ordered* Ramond punctures.

From the condition $\operatorname{coker}(u) \cong \operatorname{Ber}(X/T) \otimes \mathscr{L}(R)$ we get a short exact sequence,

$$0 \longrightarrow \mathcal{L} \longrightarrow \Omega^1_{X/T} \longrightarrow \operatorname{Ber}(X/T) \otimes \mathscr{L}(R) \longrightarrow 0. \tag{41}$$

In [9] Deligned defines a SUSY structure $(\mathcal{L}, u)$ (without Ramond punctures) as the kernel of a derivation $d : \mathcal{O}_X \twoheadrightarrow \operatorname{Ber}(X/T)$. The short exact sequence (41) generalizes the definition in *loc. cit* to the case of Ramond punctures.

Definition 0.5.2 (Isomorphism of SUSY Curves)**.** *An* isomorphism of SUSY curves over T

$$f : \Sigma = (X/T, \mathcal{L}, u, R) \xrightarrow{\sim} (X'/T, \mathcal{L}', u', R') = \Sigma'$$

is an isomorphism of $X \xrightarrow{\sim} X'$ *of supercurves over* T *such that* $f^{-1}(R') = R$*, and such that the canonical isomorphism* $\Omega^1_{X/T} \xrightarrow{\sim} f^*(\Omega^1_{X'/T})$ *(center vertical arrow) induces an isomorphism* $\mathcal{L} \xrightarrow{\sim} f^*(\mathcal{L}')$ *(leftmost dotted vertical arrow) making the diagram*

[7] following no standard convention

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L} & \overset{u}{\hookrightarrow} & \Omega^1_{X/T} & \longrightarrow & \mathrm{Ber}(X/T) \otimes \mathscr{L}(R) & \longrightarrow & 0 \\ & & \vdots\wr & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \longrightarrow & f^*(\mathcal{L}') & \overset{u'}{\hookrightarrow} & f^*(\Omega^1_{X'/T}) & \longrightarrow & f^*\,\mathrm{Ber}(X'/T) \otimes \mathscr{L}(R') & \longrightarrow & 0 \end{array}$$

commute.

We define the moduli space $\mathfrak{M}_{0,n_R}$ as follows:

Definition 0.5.3 ($\mathfrak{M}_{0,n_R}$)**.** *Let $\mathfrak{M}_{0,n_R}$ denote the category fibered in groupoids over* SupSch *with fibers over T the groupoid $\mathfrak{M}_{0,n_R}(T)$ with objects SUSY curves over T and with morphisms isomorphisms of SUSY curves over T.*

Is is proved [8] in [21] that $\mathfrak{M}_{0,n_R}$ is a Deligne-Mumford superstack . Our goal is to give an explicit construction of $\mathfrak{M}_{0,n_R}$ as a Deligne-Mumford quotient superstack of dimension $(n_R - 3|n_R/2 - 2)$.

Infinitesimal Deformations of $\mathbb{WP}$

In this section we study the first order infinitesimal deformations of $\mathbb{WP}$.

A *first order infinitesimal deformation* of $\mathbb{WP}$ is a deformation of $\mathbb{WP}$ over Spec $\mathbf{k}[\epsilon|\eta]/(\epsilon^2, \eta\epsilon)$, where the $\mathbb{Z}_2$-grading on the superalgebra $D := \mathbf{k}[\epsilon|\eta]/(\epsilon^2, \eta\epsilon)$ is given by $|\epsilon| = 0$ and $|\eta| = 1$.

Theorem 0.5.4. $\mathrm{H}^1(\mathcal{T}\mathbb{WP})$ *is a $(0|n_R/2 - 2)$-dimensional super vector space over k.*

Proof. We fix one an for all for $\mathbb{WP}$ the cover $\mathcal{U} = \{U = \operatorname{Spec} k[z,\zeta], V = \operatorname{Spec} k[w,\chi]\}$.

[8] after the initial appearance of the present article

We compute a basis for $\mathrm{H}^1(\mathcal{T}\mathbb{WP})$ from the Čech cover $\mathcal{U}$:

$$\mathrm{H}^0(U,\mathcal{T}\mathbb{WP}) = \left(\sum_{k\geq 0} a_k z^k + \sum b_k z^k\zeta\right)\frac{\partial}{\partial z} + \left(\sum_{k\geq 0} c_k z^k + \sum d_k z^k\zeta\right)\frac{\partial}{\partial \zeta} \tag{42}$$

$$\mathrm{H}^0(V,\mathcal{T}\mathbb{WP}) = \left(\sum_{k\geq 0} \overline{a_k} w^k + \sum \overline{b_k} w^k\chi\right)\frac{\partial}{\partial w} + \left(\sum_{k\geq 0} \overline{c_k} w^k + \sum \overline{d_k} w^k\chi\right)\frac{\partial}{\partial \chi}$$

where $a_k, d_k, \overline{a}_k, \overline{d}_k \in k$ and $b_k = c_k = \overline{b}_k = \overline{c}_k = 0$ for the above sections to be even and vice-versa for the sections to be odd and where all summations are for $k \geq 0$, unless otherwise specified. Changing coordinates on $U \cap V$ we compute:

$$\begin{aligned}\frac{\partial}{\partial w} =& -z^2\frac{\partial}{\partial z} + (n/2-1)\zeta z\frac{\partial}{\partial \zeta},\\ \frac{\partial}{\partial \chi} =& z^{1-n/2}\frac{\partial}{\partial \zeta}.\end{aligned}$$

A basis for the super vector space $\mathrm{H}^1(\mathcal{T}\mathbb{WP})$ is given by a basis for the (nullspace) of the following system of equations:

$$\begin{aligned}\sum_{k\geq 0} a_k z^k - \sum_{k\geq 0} \overline{a}_k z^{-k+2} =& 0, && : \frac{\partial}{\partial z} \\ \sum_{k\geq 0} b_k z^k - \sum_{k\geq 0} \overline{b}_k z^{n/2+1-k} =& 0, && : \zeta\frac{\partial}{\partial z} \\ \sum_{k\geq 0} c_k z^k - \sum_{k\geq 0} \overline{c}_k z^{1-k-n/2} =& 0, && : \frac{\partial}{\partial \zeta} \\ \sum_{k\geq 0} d_k z^k - (n/2-1)\sum_{k\geq 0} \overline{a}_k z^{-k+1} - \sum_{k\geq 0} \overline{d}_k z^{-k} =& 0, && : \zeta\frac{\partial}{\partial \zeta},\end{aligned} \tag{43}$$

We find that the following vector fields form a basis for $\mathrm{H}^1(\mathcal{T}\mathbb{WP})$:

$$\left\{z^{-1}\frac{\partial}{\partial \zeta}, \ldots, z^{-(n_R/2-2)}\frac{\partial}{\partial \zeta}\right\}. \tag{44}$$

Therefore, $\mathrm{H}^1(\mathcal{T}\mathbb{WP})$ is a $(0|n_R/2-2)$-dimensional super vector space over k.

□

Any first order infinitesimal deformation $\mathcal{X}$ of $\mathbb{WP}$ admits a covering by affines $U' \cong U \times \operatorname{Spec} D, V' \cong V \times \operatorname{Spec} D$ where the isomorphism is as deformations over $\operatorname{Spec} D$. We call U' and V' local trivializations for $\mathcal{X}$. It follows that $\mathcal{X}$ isomorphic to the superscheme constructed by patching together this affine cover with the automorphism sending

$$\begin{aligned} z &\mapsto 1/w \\ \zeta &\mapsto \chi w^{n_R/2-1} + \eta \sum_{j=1}^{n_R/2-2} c_j w^{n_R/2-1-j}. \end{aligned} \tag{45}$$

where $c_j \in k$.

Superextensions

Any polynomial superalgebra $k[\theta_1, \cdots, \theta_n]$ with $|\theta_i| = 1$ is an example of a $\mathbb{Z}/2\mathbb{Z}$-graded Artin algebra over k. Equivalently, we say that $k[\theta_1, \cdots, \theta_n]$ is a superextension (extensions by purely odd variables) of k.

Let Λ be a bosonic algebra. We denote by SupExt_Λ the category of superextensions of Λ.

For $i \geq 1$, let $\mathscr{R}^i$ denote the subcategory of SupExt_Λ of those objects with sheaf of odd nilpotent vanishing in degree $i+1$.

The sequence of inclusions $\Lambda \hookrightarrow \mathscr{R}^1 \hookrightarrow \mathscr{R}^2 \hookrightarrow \mathscr{R}^3 \hookrightarrow \cdots$ gives a filtration of SupExt_Λ. If $R \in \mathrm{SupExt}_\Lambda$, then we define $R_i = R/\mathcal{J}^{i+1}$ where $\mathcal{J}$ denotes the sheaf of odd nilpotents in R. We define the associated filtration of R as the (finite) sequence $\{\Lambda = R_b, R_1, \ldots, R_m = R\}$ where $m \in \mathbb{N}$ is such that $\mathcal{J}^m = 0$. We describe the structure of this sequence as follows:

Lemma 0.5.5. *Let R and $\mathcal{J}$ be as above with $\Lambda = R_b$. The associated filtration of R is a composite of square-zero extensions by the $R/\mathcal{J}^{i+1}$-module $\mathcal{J}^i/\mathcal{J}^{i+1}$,*

$$R/\mathcal{J}^n = R \to R/\mathcal{J}^{n-1} \to \cdots \to R/\mathcal{J}^3 \to R/\mathcal{J}^2 \to R/\mathcal{J} = R_b$$

where each $R/\mathcal{J}^{i+1}$-module $\mathcal{J}^i/\mathcal{J}^{i+1}$ has the structure of an R_b-module.

Proof. Each map $R/\mathcal{J}^{i+1} \to R/\mathcal{J}^i$ in the composition corresponds to the extension

$$0 \to \mathcal{J}^i/\mathcal{J}^{i+1} \to R/\mathcal{J}^{i+1} \to R/\mathcal{J}^i \to 0$$

which is clearly square-zero since $\mathcal{J}^i/\mathcal{J}^{i+1} \cdot \mathcal{J}^i/\mathcal{J}^{i+1} = 0$. To define an R_b-module structure on $\mathcal{J}^i/\mathcal{J}^{i+1}$ let $r \in R_b$ and let $r' \in p^{-1}(r) \in R/\mathcal{J}^{i+1}$ where $p : R/\mathcal{J}^{i+1} \to \cdots \to R/\mathcal{J} = R_b$ is the above composite of surjections. Define $r \cdot j = r'j$ for all $j \in \mathcal{J}^i/\mathcal{J}^{i+1}$. This is well defined because if $r', r'' \in p^{-1}(r)$, then $r' - r'' \in \mathcal{J}/\mathcal{J}^{i+1}$, and since $\mathcal{J}/\mathcal{J}^{i+1}$ annihilates $\mathcal{J}^i/\mathcal{J}^{i+1}$ we have that $(r' - r'')j = 0$.

□

Given a superscheme X over T the factorization of T induces a factorization $X_i := X \times_T \operatorname{Spec} R/\mathcal{J}^i$ of infinitesimal deformations, in the sense that each X_i is an infinitesimal deformation of X_{i-1} over $T_i = \operatorname{Spec} R/\mathcal{J}^{i+1}$.

0.5.1 Étale Local Descriptions

It is a standard fact that any family of genus zero curves over T is étale locally isomorphic to the trivial family $\mathbb{P}^1 \times T$. We prove an analogous fact for supercurves underlying families of SUSY curves over bosonic schemes.

Theorem 0.5.6. *Any family of supercurves X over a bosonic space T underlying a SUSY curve Σ is étale locally isomorphic to $\mathbb{WP} \times T$.*

Proof. We proved in part (a) of Theorem 0.4.14 that a supercurve X over bosonic scheme (say T) is (étale locally) isomorphic to the spin curve $\mathbb{P}^1 \times T$ with spin structure the line bundle $\mathcal{J}/\mathcal{J}^2 \cong \mathcal{J}$ (the sheaf of odd nilpotents on X) and that therefore,

$$\mathcal{O}_X \cong \mathcal{S}ym_{\mathcal{O}_{\mathbb{P}^1 \times T}} (\Pi \mathcal{J}^\vee)$$

where $\mathcal{S}ym_{\mathcal{O}_{\mathbb{P}^1 \times T}}$ denotes the supersymmetric algebra over $\mathcal{O}_{\mathbb{P}^1 \times T}$. To prove the theorem it suffices to prove that there exists an étale cover over T on which $\mathcal{O}_X$ is isomorphic to $\mathcal{S}ym\ (\Pi \mathcal{O}(1 - n_R/2))$.

In part (c) of the aforementioned theorem, we showed that $\mathcal{J}$ is isomorphic to a square-root of $\Omega^1_{\mathbb{P}^1 \times T/T} \otimes \mathscr{L}(R_b) \cong \mathcal{O}(n_R - 2)$. A square-root of $\mathcal{O}(n_R - 2)$ is a line bundle on $\mathbb{P}^1 \times T$ isomorphic to

$$\mathcal{O}(n_R/2 - 1) \otimes \pi^* \mathcal{N}$$

where $\pi : \mathbb{P}^1 \times T \to T$ is the canonical projection and where $\mathcal{N}$ is a two-torsion line bundle on T.

There exists a canonical double (thus étale) cover $p : P \to T$ on which the line bundle $\mathcal{N}$ trivializes. Indeed, define P to be the affine space $\mathrm{Spec}(\mathcal{O}_T \oplus \mathcal{N})$ with structure sheaf (with multiplication is defined component-wise) $\mathcal{O}_T \oplus \mathcal{S}ym_{\mathcal{O}_T} \mathcal{N}$. We find that $p^* \mathcal{N} \cong \mathcal{O}_P$. Given any square-root of $\mathcal{O}(n_R - 2)$ there, therefore, exists an étale cover of T on which the square-root is isomorphic to $\mathcal{O}(n_R/2 - 1)$. It follows that there exists an étale cover $p : P \to T$ such that

$$\mathcal{J}^\vee \cong \mathcal{O}(1 - n_R/2)$$

and

$$\mathcal{O}_{X \times_T P} \cong \mathcal{S}ym_{\mathbb{P}^1 \times P} (\Pi \mathcal{O}(1 - n_R/2)).$$

□

Given a superscheme T, we can construct an étale cover $p : P_b \to T_b$ of its bosonic reduction. This cover extends uniquely to an étale cover $p : P \to T$ ([21]). By fixing an isomorphism $\phi : X \times_T P_b \xrightarrow{\sim} \mathbb{WP} \times P_b$, we may treat the pair $(X \times_T P/P, \phi)$ as deformation of $\mathbb{WP} \times P_b$ over P.

Definition 0.5.7. *Let A be a graded R-algebra such that $\mathbb{WP} \times \operatorname{Spec} R = \operatorname{Proj} A$. For any $n \in \mathbb{Z}$, we define the line bundles $\mathcal{O}(n)$ to be $A(n)^{\sim}$.*

Let $A = R[u^{(1)}, v^{(1)}|\theta^{(1-n_R/2)}]$ (with superscripts indicating the degree). The line bundle $\mathcal{O}(1)$ is generated by its global sections, $u, v, u^{n_R/2-i}v^i\theta$ with $0 \le i \le n_R/2$.

Corollary 0.5.8. *Any SUSY line bundle $\mathcal{L}$ on a family of supercurves X over a bosonic space T underlying a SUSY curve Σ is étale locally isomorphic to the line bundle $\mathcal{O}_X(-2)$ on X.*

Proof. Since the theorem is local, we can assume that $X \cong \mathbb{WP}$. The sheaf $\mathcal{O}_X/\mathcal{J} \cong \mathcal{O}_{\mathbb{P}^1}$ is a projective $\mathcal{O}_X$-module. Tensoring the short exact sequence (41) with $\mathcal{O}_X/\mathcal{J}$ gives a short exact sequence (preserving the grading) of $\mathcal{O}_{\mathbb{P}^1}$-modules

$$0 \longrightarrow \mathcal{L}_b \longrightarrow \Omega^1_{\mathbb{P}^1} \oplus \mathcal{J} \longrightarrow \mathcal{D}_b^{\vee} \longrightarrow 0$$

where the subscript b indicates the bosonic reduction. Comparing ranks we find that $\mathcal{L}_b \cong \Omega^1_{\mathbb{P}^1} \cong \mathcal{O}(-2)$.

Let $j : \mathbb{WP} \to \mathbb{P}^1$ denote a section of the inclusion $i : \mathbb{P}^1 \to \mathbb{WP}$. The line bundle $j^*(\mathcal{L}_b)$ on $\mathbb{WP}$ is isomorphic to $\mathcal{O}_{\mathbb{WP}}(-2)$. Since (1) $i^*\mathcal{L} = \mathcal{L}_b$ and (2) $j^*(i^*\mathcal{L}) = \mathcal{L}$, we conclude that $\mathcal{L} \cong \mathcal{O}_{\mathbb{WP}}(-2)$.

□

Any SUSY line bundle $\mathcal{L}$ restricts (étale locally) to a line bundle on $\mathbb{WP} \times P_b$. The above corollary implies that this restriction is isomorphic to $\mathcal{O}_{\mathbb{WP}\times P_b}(-2)$. Therefore, the restriction of any SUSY line bundle on X/T is isomorphic to a deformation of $\mathcal{O}_{\mathbb{WP}\times P_b}(-2)$ to $X \times_T P$.

Lemma 0.5.9. *Let X be a family of supercurves over T whose restriction to T_b is (étale locally) isomorphic to $\mathbb{WP} \times T_b$. Then the line bundle $\mathcal{O}_{\mathbb{WP}\times T_b}(-2)$ deforms uniquely to a line bundle on X.*

Proof. Since the statement is local, we may assume that $T_b = \operatorname{Spec} k$ and $X \times_T \operatorname{Spec} k \cong \mathbb{WP}$.

Since $\mathrm{H}^2(\mathcal{O}_{\mathbb{WP}}) \otimes \mathcal{J}^i/\mathcal{J}^{i+1} = 0$ and $\mathrm{H}^1(\mathcal{O}_{\mathbb{WP}}) \otimes \mathcal{J}^i/\mathcal{J}^{i+1} = 0$ for each i in the factorization of T (Lemma 0.5.5), we conclude that (1) there exists at least one deformations of $\mathcal{O}_{\mathbb{WP}}(-2)$ to X and (2) that there exists exactly one deformation of $\mathcal{O}_{\mathbb{WP}}(-2)$ to X.

□

This means that any SUSY line bundle $\mathcal{L}$ on a family of supercurves X/T underlying a SUSY curve Σ is isomorphic to the deformation of $\mathcal{O}_{\mathbb{WP}\times T_b}(-2)$ to X.

0.5.2 Automorphisms of SUSY Curves

In this section we prove that a SUSY curve has no infinitesimal automorphism and that, thereby, the automorphism group of a SUSY curve is finite.

Let $D := \mathbf{k}[\epsilon|\eta]/(\epsilon^2, \eta\epsilon)$ be the superalgebra $\mathbb{Z}_2$-grading given by $|\epsilon| = 0$ and $|\eta| = 1$. The pullback of $(\mathbb{WP}, \mathcal{D}, R)$ to $\operatorname{Spec} \mathbf{k}[\epsilon|\eta]/(\epsilon^2, \eta\epsilon)$ is the trivial first order infinitesimal deformation $(\mathbb{WP}, \mathcal{D}, R))$. The *infinitesimal automorphism group* of $(\mathbb{WP}, \mathcal{D}, R)$ is the automorphism group of the trivial first order infinitesimal deformation $(\mathbb{WP}, \mathcal{D}, R)$.

Remark 0.5.10. Let $\mathrm{Aut}(\mathbb{WP}, \mathcal{D}, R) : \mathrm{SupSch} \to \mathrm{Group}$ denote the functor sending a superscheme T to the set of automorphisms of the SUSY curve $(\mathbb{WP}, \mathcal{D}, R)$. Then the *infinitesimal automorphism group* of $(\mathbb{WP}, \mathcal{D}, R)$ is exactly the Lie superalgebra of $\mathrm{Aut}(\mathbb{WP}, \mathcal{D}, R)$.

Definition 0.5.11 (Superconformal Vector Fields)**.** *Let $(\mathbb{WP}, \mathcal{D}, R)$ be a SUSY curve. A* superconformal vector field *is a derivation $\mathcal{X} \in \Gamma(\mathcal{T}\mathbb{WP})$ which preserves the SUSY structure $(\mathcal{D}, R)$ on $\mathbb{WP}$. We denote by $\mathcal{A} \subset \mathcal{T}_{\mathbb{WP}}$ the sheaf of superconformal vector fields on $(\mathbb{WP}, \mathcal{D}, R)$.*

It follows from standard results that,

Lemma 0.5.12. *The group of infinitesimal automorphism of $(\mathbb{WP}, \mathcal{D}, R)$ is isomorphic to $H^0(\mathbb{WP}, \mathcal{A})$. In particular, the dimension of $\mathrm{Aut}(\mathbb{WP}, \mathcal{D}, R)$ is equal to $H^0(\mathcal{A})$.*

On the chart $U = \operatorname{Spec} k[z, \zeta]$ we may write the odd sections of $\mathcal{A}$ explicitly ([27], Section 4.2.1) as

$$v_f = f(z)\left(\frac{\partial}{\partial \zeta} - h(z)\zeta\frac{\partial}{\partial z}\right) \tag{46}$$

and the even sections as

$$V_g = h(z)\left(g(z)\frac{\partial}{\partial z} + \frac{g'(z)}{2}\zeta\frac{\partial}{\partial \zeta}\right). \tag{47}$$

where $h(z)$ is the local defining section of the Ramond divisor R.

Theorem 0.5.13. *If $n_R \geq 4$, $H^0(\mathcal{A}) = 0$ and so $\dim_k \mathrm{Aut}(\mathbb{WP}, \mathcal{D}, R) = (0|0)$.*

Proof. Using Cech cohomology, we find that the restriction of the global sections of $\mathcal{T}\mathbb{WP}$ to $U \cong \operatorname{Spec} k[z, \zeta_u]$ when $n_R \geq 4$ have basis

$$\{\zeta\partial_z, z\zeta\partial_z, \ldots, z^{n/2+1}\zeta\partial_z, \partial_z, z\partial_z, z^2\partial_z + z\zeta\partial_\zeta, \zeta\partial_\zeta\}.$$

and are, therefore, locally described by the vector fields

$$\mathfrak{X}^- = \sum_{i=0}^{n_R/2+1} b_i z^i \zeta \partial_z \tag{48}$$

$$\mathfrak{X}^+ = (a_0 + a_1 z + a_2 z^2)\partial_z + (b_0 + a_2 z)\zeta\partial_\zeta. \tag{49}$$

It is clear that $\mathfrak{X}^-$ is not superconformal since it is not of the form given (46). For $\mathfrak{X}^+$ to be superconformal we must have that

$$\sum_{i=0}^{n} b_i w^{-i} \left((a_0 + a_1 w + a_2 w^2)(-w^2\partial_w + \zeta_v w \partial_{\zeta_v} + (a_1 + a_2 w)\zeta_v \partial_{\zeta_v}\right) \tag{50}$$

is a vector field on $V \cong \operatorname{Spec} \mathbf{k}[w, \zeta_v]$. This is not possible unless $b_i = 0$ for all $i \geq 2$, which would mean that $w(z)$ is the local defining function of a Ramond divisor with two irreducible components, i.e $\mathfrak{X}^+$ is a superconformal vector field if and only if $n_R = 2$. This contradicts the assumption that $n_R \geq 4$ and thus $\mathfrak{X}^+ = 0$. Therefore, any SUSY curve $(\mathbb{WP}, \mathcal{D}, R)$ with $n_R \geq 4$ Ramond punctures has no non-trivial infinitisimal automorphisms.

□

0.5.3 Bosonic Moduli Space

We define the bosonic moduli space of SUSY curves as follows:

Definition 0.5.14. *Let $(\mathfrak{M}_{0,n_R})_b$ denote the stack over Sch/k with fibers over T the groupoid $(\mathfrak{M}_{0,n_R})_b(T)$ whose objects are SUSY curves over T and whose morphisms are isomorphisms of SUSY curves over T.*

In the previous section we proved that étale locally (1) any family of supercurves X/T underlying a SUSY curve over a bosonic scheme T is isomorphic to $\mathbb{WP} \times T$ and (2) that the SUSY line bundle $\mathcal{L}$ on X is isomorphic to $\mathcal{O}(-2)$. A generating section for a SUSY structure on X is, therefore, a global section of $\mathcal{H}om(\mathcal{O}(-2), \Omega) \cong \mathrm{H}^0(\Omega^1_{X/T}(2))$. To give rise to a SUSY structure an element of $\mathrm{H}^0(\Omega^1_{X/T}(2))$ must satisfy the non-integrability condition outside of the locus of R. The (1|1)-form in (21) is an example of a (local description) of a global section of $\Omega^1_{\mathbb{WP}}(2)$ generating a SUSY structure on $\mathbb{WP}$.

Definition 0.5.15 (Pre-SUSY Structure)**.** *A pre-SUSY structure on supercurve X over T is a pair $(\mathcal{L}, u)$, where $\mathcal{L}$ is a line bundle on X isomorphic to $\mathcal{O}(-2)$ and where $u : \mathcal{L} \hookrightarrow \Omega^1_{X/T}$ is an injective morphism of vector bundles on X.*

The pre-SUSY structures on a SUSY curve Σ over a bosonic space T are in bijection with the global sections of $\mathcal{H}om(\mathcal{O}(-2), \Omega) \cong \mathrm{H}^0(\Omega^1_{X/T}(2))$.

Using Čech cohomology (with respect to the cover $\mathcal{U} = \{U = \operatorname{Spec} k[z, \zeta], V = \operatorname{Spec} k[w, \chi]$ of $\mathbb{WP}$) we can give explicit local descriptions (on U) of the global section of $\Omega^1_{X/T}(2)$ as:

$$adz + \zeta \sum_{k=0}^{n_R} a_k z^k d\zeta \tag{51}$$

with $a, a_k \in \Gamma(\mathcal{O}_T)$ and where we require a to be invertible.

The sections in (51) generate a unique pre-SUSY structure up to multiplication by an invertible factor of $\mathrm{H}^0(\mathcal{O}^*_{\mathbb{WP}})$.

The super vector space $\mathrm{H}^0(\Omega^1_{\mathbb{WP}}(2))$ gives rise to an affine space

$$\mathbb{H}^0(\Omega^1_{\mathbb{WP}}(2)) := \operatorname{Spec} \mathcal{S}(H^0(\Omega^1_{\mathbb{WP}}(2))^*)$$

over $\operatorname{Spec} k$. If we choose a basis

$$\{x_1, x_2, \ldots, x_{n_R+2}\}$$

for $H^0(\Omega^1_{\mathbb{WP}}(2))^*$ dual to the basis

$$\{dz, \zeta d\zeta, z\zeta d\zeta \ldots, z^{n_R}\zeta d\zeta\}\},$$

see (51), then we can identify this affine space with a standard one:

$$\mathbb{H}^0(\Omega^1_{\mathbb{WP}}(2)) = \operatorname{Spec} \mathcal{S}(H^0(\Omega^1_{\mathbb{WP}}(2))^*) \cong \operatorname{Spec} k[x_1, \ldots, x_{n_R+2}].$$

We are interested in the part of the affine space $\mathbb{H}^0(\Omega^1_{\mathbb{WP}}(2))$ which corresponds to the framed pre-SUSY structures on Z, *i.e.* those which restrict to a one form on $\mathbb{P}^1$ We should take the first coordinate x_1 to be invertible and identify the open affine subscheme

$$W_b := \operatorname{Spec} A[x_1, x_1^{-1}, \ldots, x_{n_R+2}] \subset \mathbb{H}^0(\Omega^1_{\mathbb{WP}}(2))$$

as the space of framed pre-SUSY structures on $\mathbb{WP}$.

We show in Section 0.5.5 that the subset of $\mathrm{H}^0(\Omega^1_{\mathbb{WP}}(2))$ corresponding to SUSY structure on $\mathbb{WP}$ is an open subscheme of W_b. Let us denote this open subscheme by Y_b.

The affine space Y_b has a canonical SUSY curve given by the supercurve $\mathbb{WP} \times Y_b$ and with SUSY structure the global section ϖ of $\mathrm{H}^0(\Omega^1_{\mathbb{WP}\times Y/Y}(2))$ whose restriction to the open subset $U = \operatorname{Spec} k[z,\zeta] \times Y$ of $\mathbb{WP} \times Y$ is

$$\varpi|_U = x_1 dz + \zeta \sum_{k=2}^{n_R+2} x_k z^{k-2} d\zeta \tag{52}$$

Explicit Description of the Bosonic Moduli Space as a Deligne-Mumford Stack

Definition 0.5.16. *Let $\mathcal{X}_b$ denote the category fibered in groupoids over* Sch *with fibers over T the groupoid $\mathcal{X}(T)$ whose objects are SUSY structures $(\mathcal{L}, u, R)$ on $\mathbb{WP} \times T$ and whose morphisms are automorphisms of $\mathcal{L}$ preserving u and R.*

Theorem 0.5.17. *The functor $\mathcal{X}_b$ is represented by the algebraic space*

$$Y_b/\mathbb{G}_m$$

of dimension n_R+1, where $\mathbb{G}_m$ acts on Y_b by its identification with $\mathrm{H}^0(\mathcal{O}^*_{\mathbb{WP}})$.

Proof. The scheme Y_b has the natural structure of a category fibered in groupoids (sets) as follows: The fiber of Y_b over T is the groupoid $Y_b(T)$ with objects SUSY structures on $\mathbb{WP} \times T$ together with a fixed framing $c : \mathcal{L} \to \mathcal{O}(-2)$ and with morphisms the identity.

Let ξ denote an object in $\mathcal{X}_b(T)$ and let $\mathcal{L}$ denote its SUSY line bundle. We can fix an isomorphism $c : \mathcal{L} \to \mathcal{O}_X(-2)$.

The SUSY structure on (ξ, c) is then generated by a global section w of $\mathrm{H}^0(\Omega^1_{\mathbb{WP}\times T/T}(2))$. The restriction of w to the chart $U \times T$ is described as in (51) but with coefficients t_i in $\Gamma(\mathcal{O}_T)$. Now define $h : T \to Y_b$ so that $x_i \mapsto t_i$. Then $h^*\varpi = w$. The pair (ξ, c) corresponds to the map $h \in Y_b(T)$. This means that the category of pairs (ξ, c) is isomorphic to $Y_b(T)$.

Define a surjection $q : Y_b(T) \to \mathcal{X}_b(T)$ sending (ξ, c) to ξ and the identity to the isomorphism c. The set of isomorphisms $c : \mathcal{L} \to \mathcal{O}(-2)$ is a torsor for the group $\mathrm{H}^0(\mathcal{O}^*_X) \cong \mathbb{G}_m(T)$. This means that each fiber $q^{-1}(\xi)$ is isomorphic to the group $\mathbb{G}_m(T)$ over T and that the kernel of q is isomorphic to $\mathbb{G}_m \times T$, where the group $\mathbb{G}_m \times T = \operatorname{Spec} k[t, t^{-1}] \times T$ acts on $Y_b \times T$ by multiplication

$$t * (x_1, x_1^{-1}, \ldots, x_{n_R+2}) = (tx_1, tx_1^{-1}, \ldots, tx_{n_R+2}).$$

It then follows that

$$\mathcal{X}_b = [Y_b/\mathbb{G}_m].$$

To show $\mathcal{X}_b$ is an algebraic space, it suffices to prove that $\mathbb{G}_m$ acts freely on Y_b. The group $\mathbb{G}_m$ acts freely on Y_b since x_1 is invertible. Indeed, the only point of Y_b (a subset of affine space) on which $\mathbb{G}_m$ could have acted trivially is $(0, 0, \cdots, 0)$, but we have excluded this point from Y_b when required x_1 to be invertible.

The dimension of $\mathcal{X}_b$ is computed as the difference between the dimension of Y_b (equal to $n_R + 2$) and the dimension of $\mathbb{G}_m$ (equal to one). □

Theorem 0.5.18. *The stack* $(\mathfrak{M}_{0,n_R})_b$ *is represented by the Deligne-Mumford stack*

$$[\mathcal{X}_b / \operatorname{Aut}(\mathbb{WP})_b]$$

of dimension $n_R - 3$*, where* $\operatorname{Aut}(\mathbb{WP})_b$ *denotes the bosonic reduction of the supergroup scheme* $\operatorname{Aut}(\mathbb{WP})$.

Proof. Let $\Sigma = (X/T, \mathcal{L}, u, R)$ be a SUSY curve over T and let $p : P \to T$ be the étale cover on which $X \times_T P \cong \mathbb{WP} \times P$ as supercurves over P. Let us fix such an isomorphism with

$$\phi : X \times_T P \xrightarrow{\sim} \mathbb{WP} \times P.$$

Using ϕ, we can pullback the SUSY structure on $X \times_T P$ to one on $\mathbb{WP} \times P$. This induces an isomorphism of SUSY curves

$$\Psi : (\Sigma \times_T P, \phi) \xrightarrow{\sim} (\mathbb{WP} \times P, \phi^* \mathcal{L}, \phi^* u, \phi^{-1} R).$$

The pair $(\Sigma \times_T P, \phi)$ is, therefore, an object in $\mathcal{X}_b(P)$. The category of pairs $(\Sigma \times_T P, \phi)$ is isomorphic to $\mathcal{X}_b(P)$.

Define $q : \mathcal{X}_p(P) \to (\mathfrak{M}_{0,n_R})_p(P)$ to send $(\Sigma \times_T P, \phi)$ to $\Sigma \times_T P$. The set of isomorphisms ϕ is a torsor for $(\operatorname{Aut}(\mathbb{WP}))_b$ and, therefore, the fibers of q are then isomorphic to $(\operatorname{Aut}(\mathbb{WP}))_b(P)$. It follows that the kernel of q is isomorphic to $(\operatorname{Aut}(\mathbb{WP}))_b \times P$.

Globalizing this fact to objects in T, we find that $(\mathfrak{M}_{0,n_R})_b$ is isomorphic to the category of triples $(T, \mathcal{F}, g)$ where $\mathcal{F}$ is a $(\mathrm{Aut}(\mathbb{WP}))_b \times \mathbb{G}_m$-torsor on the big étale site of T with an $(\mathrm{Aut}(\mathbb{WP}))_b \times \mathbb{G}_m$-equivariant map $g : \mathcal{F} \to Y_b \times T$.

There exists a canonical action of the group $(\mathrm{Aut}(\mathbb{WP}))_b$ on $Y_b/\mathbb{G}_m$ (thinking of $Y/\mathbb{G}_m$ as the projective bundle $\mathrm{H}^0(\Omega^1_{\mathbb{WP}}(2))/H^0(\mathcal{O}^*_{\mathbb{WP}})$ over $\mathbb{WP}$). It follows that

$$(\mathfrak{M}_{0,n_R})_b = [\mathcal{X}_b / \mathrm{Aut}(\mathbb{WP})_b].$$

To show that $(\mathfrak{M}_{0,n_R})_b$ is Deligne-Mumford we must show that $\mathrm{Aut}(\mathbb{WP})_b$ acts on $\mathcal{X}_b$ with finite stabilizers. It suffices to check this condition at the level of points. An automorphism of $\mathbb{WP}$ acts trivially on an object of $\mathcal{X}_b(\operatorname{Spec} k)$ (a SUSY curve over $\operatorname{Spec} k$) if and only that automorphism is also an automorphism of the specified SUSY curve, *i.e.* if it preserves the SUSY structure.

We showed in Theorem 0.5.13 that a SUSY curve over $\operatorname{Spec} k$ has a finite group of automorphisms. The stack $(\mathfrak{M}_{0,n_R})_b$ is, therefore, Deligne-Mumford.

The dimension of $(\mathfrak{M}_{0,n_R})_b$ is equal to the difference of the dimension of $\mathcal{X}_b$ (equal to $n_R + 1$) and the dimension of $\mathrm{Aut}(\mathbb{WP})_b$(equal to 4 by Theorem 0.4.23).

□

0.5.4 Deformation Space of $\mathbb{WP}$

We are interested in the determining the deformation space of $\mathbb{WP}$. It follows from Lemma 0.5.4 that $\mathbb{WP}$ deforms (non-trivially) only in "fermionic " directions . To determine the deformation space of $\mathbb{WP}$ it, therefore, suffices to work over the category of superextensions (an extension consisting of only odd indeterminates) of k.

Definition 0.5.19 ($\mathrm{Def}_{\mathbb{WP}}$). *Let*

$$\mathrm{Def}_{\mathbb{WP}} : \mathrm{SupExt}_k \to \mathrm{Set}$$

denote the functor sending a superextension R of k to the set of isomorphism classes of deformations of $\mathbb{WP}$ *over* R.

Candidate for Universal Base Scheme. Our candidate for a universal base scheme for $\mathrm{Def}_{\mathbb{WP}}$ is

$$S := \operatorname{Spec} \mathcal{S}\,\mathrm{H}^1(T\mathbb{WP})^*,$$

where $\mathcal{S}$ denotes the supersymmetric algebra over $\mathbf{k}$. Choosing coordinates $\eta_1, \dots, \eta_{n_R/2-2}$ corresponding to the basis (44) of $\mathrm{H}^1(T\mathbb{WP})$, we may identify S with the affine superspace

$$\mathbb{A}^{0|n_R/2-2} = \operatorname{Spec} \mathbf{k}[\eta_1, \dots, \eta_{n_R/2-2}].$$

We will denote the superalgebra $\mathcal{S}H^1(T\mathbb{WP})^* = \mathbf{k}[\eta_1, \dots, \eta_{n_R/2-2}]$ by A.

Candidate for Universal Deformation. Our candidate for the universal deformation of $\mathbb{WP}$ is the superscheme Z with cover $\mathcal{U} = \{U = \operatorname{Spec} A[z, \zeta], V = \operatorname{Spec} A[w, \chi]\}$, and patched together along $U \cap V$ via the automorphism of $Z|_{U\cap V}$ sending

$$\begin{aligned} z &\mapsto 1/w \\ \zeta &\mapsto \chi w^{n_R/2-1} + \sum_{j=1}^{n_R/2-2} \eta_j w^{n_R/2-1-j}. \end{aligned} \tag{53}$$

The restriction (by the unique map $S \to \operatorname{Spec} k$ sending each η_i to zero) of Z to $\operatorname{Spec} k$ is by construction equal to $\mathbb{WP}$. To treat Z as deformation of $\mathbb{WP}$ over S we need to add to it the data of an isomorphism with $\mathbb{WP}$. We take this isomorphism to be the identity map.

Theorem 0.5.20. *The functor* $\mathrm{Def}_{\mathbb{WP}}$ *is represented by* S *with universal family* Z.

Proof. Let $(X/T, \phi)$ be a deformation of $\mathbb{WP}$ over $T = \operatorname{Spec} R$.

The pair $(X_i/T_i, \phi)$ denotes the pullback of the deformation $(X/T, \phi)$ to $T_i = \operatorname{Spec} R_i$ where $R_i \in \mathscr{R}_i$

Consider the map $\tau : T_1 \to \operatorname{Spec} k \to S$. Since τ factors through $\operatorname{Spec} k$, $\tau(\eta_i) = 0$. It then follows from the gluing description of Z that $Z \times_{S,\tau} T_1$ is the superscheme with cover $\mathcal{U} \times T_1$ ($\mathcal{U}$ is the cover of $\mathbb{WP}$ described in theorem 0.5.4) glued together along $U \times T_1 \cap V \times T_1$ by the automorphism sending

$$\begin{aligned} z &\mapsto 1/w \\ \zeta &\mapsto \chi w^{n_R/2-1} \end{aligned} \tag{54}$$

to $\mathbb{WP} \times T_1$.

The pair $(X_1/T_1, \phi)$ is a deformation of $\mathbb{WP}$ over T_1. The set of isomorphism classes of deformations of $\mathbb{WP}$ over T_1 is a torsor for $\mathrm{H}^1(\mathcal{T}\mathbb{WP}) \otimes \mathcal{J}/\mathcal{J}^2$. We compute $\mathrm{H}^1(\mathcal{T}\mathbb{WP})$ explicitly using the Čech cover $\mathcal{U}$ to conclude that the elements of $\mathrm{H}^1(\mathcal{T}\mathbb{WP}) \otimes \mathcal{J}/\mathcal{J}^2$ are linear combinations (with coefficients in $\mathcal{J}/\mathcal{J}^2$) of (44). In particular, there exists an element

$$d_1 w^{n_R/2-2} \frac{\partial}{\partial \zeta} + \cdots + d_{n_R/2-2} w \frac{\partial}{\partial \zeta}$$

where $d_i \in \mathcal{J}/\mathcal{J}^2$ such that $(X_1/T_1, \phi)$ is isomorphic to the deformation of $\mathbb{WP}$ over T glued together along the intersection of $\mathcal{U} \times T_1$ by the automorphism sending

$$\begin{aligned} z &\mapsto 1/w \\ \zeta &\mapsto \chi w^{n_R/2-1} + \sum_{j=1}^{n_R/2-2} d_j w^{n_R/2-1-j}. \end{aligned} \tag{55}$$

Define $f_1 : T_1 \to S$ so that $\eta_i \mapsto d_i$. It follows (again by gluing) that $Z \times_{S,f_1} T_1 \cong X_1$ as deformations of $\mathbb{WP}$ over T_1. Thus (Z, S) is universal at first order.

Consider $(X_n/T_n, \phi)$ and let $f_n : T_n \to S$ denote the unique morphism inducing an isomorphism $X \cong Z \times_{S,f_n} T_n$.

Define $\tau : T_{n+1} \to S$ to be any map such that $\tau|_{T_n} = f_n$. Then $Z \times_{S,\tau} T_{n+1}$ is a deformation of $\mathbb{WP}$ over T_n and in particular an infinitesimal deformation of X_n over T_{n+1} glued along $U \times T_{n+1} \cap V \times T_{n+1}$ by the automorphism

$$\begin{aligned} z &\mapsto 1/w \\ \zeta &\mapsto \chi w^{n_R/2-1} + \sum_{j=1}^{n_R/2-2} \tau(\eta_j) w^{n_R/2-1-j}. \end{aligned} \tag{56}$$

$Z \times_{S,\tau} T_{n+1}$ is not necessarily isomorphic to $(X_{n+1}/T_{n+1}, \phi)$. The set of isomorphism classes of deformations of $\mathbb{WP}$ (relative to T_n) over T_{n+1} are torsor for $\mathrm{H}^1(\mathcal{T}\mathbb{WP}) \otimes_{R_b} \mathcal{J}^i/\mathcal{J}^{i+1}$. Therefore, the exists $d_g \in \mathcal{J}^i/\mathcal{J}^{i+1}$ such that the superscheme (with cover $\mathcal{U} \times T_{n+1}$) and gluing formula

$$\begin{aligned} z &\mapsto 1/w \\ \zeta &\mapsto \chi w^{n_R/2-1} + \sum_{j=1}^{n_R/2-2} (\tau(\eta_j) + d_j) w^{n_R/2-1-j}. \end{aligned} \tag{57}$$

is isomorphic to X_{n+1}. Define $f_n : \eta_j \to \tau(\eta_j) + d_j$. The proof then follows by induction on i.

□

The same proof replacing k with T_b and S with $S \times_k T_b$ shows that (Z, S) is a universal deformation space for $\mathbb{WP} \times T_b$.

Any family of supercurves X over T underlying a SUSY curve is étale locally a deformation (once we affix to X/T the data of an isomorphism, say $\phi : X \times_T T_b \xrightarrow{\sim} \mathbb{WP} \times T_b$) of $\mathbb{WP} \times T_b$ over T.

The pair (Z, S) is the universal deformation space for isomorphism classes of framed supercurves.

Lemma 0.5.21. *The supercurve Z is a closed subsuperscheme of* $\mathbb{P}^1 \times \mathbb{A}^{0|n_R/2}$.

Proof. From the gluing description of the of Z as the deformation of $\mathbb{WP}$ over S with trivializing cover $\mathcal{U} = \{U = \operatorname{Spec} A[z,\zeta], V = \operatorname{Spec} A[w,\chi]\}$, and patching together along $U \cap V$ via the automorphism of $Z|_{U\cap V}$ sending

$$z \mapsto 1/w \tag{58}$$
$$\zeta \mapsto \chi w^{n_R/2-1} + \sum_{j=1}^{n_R/2-2} \eta_j w^{n_R/2-1-j}.$$

we find that

$$\zeta = \chi (u/v)^{n/2-1} + \sum_{j=1}^{n/2-2} \eta_j (u/v)^{n/2-1-j} \tag{59}$$
$$v^{n/2-1}\zeta = u^{n/2-1}\chi + \sum_{j=1}^{n/2-2} \eta_j u^{n/2-1-j} v^j$$

Therefore,

$$Z \cong \operatorname{Proj} k[u,v|\zeta,\chi,\eta_1,\ldots,\eta_{n_R/2-2}]/(v^{n/2-1}\zeta - u^{n/2-1}\chi - \sum_{j=1}^{n/2-2} \eta_j u^{n/2-1-j} v^j)$$

where the even coordinates u, v have degree one and the odd coordinates $\zeta, \chi, \eta_1, \ldots, \eta_{n_R/2-2}$ have degree zero. The proof of the corollary then follows from the identification of $\operatorname{Proj} k[u,v|\zeta,\chi,\eta_1,\ldots,\eta_{n_R/2-2}]$ with $\mathbb{P}^1 \times \mathbb{A}^{0|n_R/2}$

□

Corollary 0.5.22. *If $\mathcal{L}$ is a SUSY line bundle on Z, then $\mathcal{L} \cong \mathcal{O}_Z(-2)$.*

Proof. This follows from the gluing description (53) of Z.

□

0.5.5 SUSY structures on Z

The (1|1)-form in (21) is an example of a local generating section for a SUSY structure on $\mathbb{WP}$. In this section we will give explicit descriptions of the local generating sections for SUSY (or rather pre-SUSY) structures on Z.

It follows from Corollary 0.5.22 that a generating section of a SUSY structure on Z is a global section of $\mathrm{Hom}(\mathcal{O}(-2), \Omega) \cong \mathrm{H}^0(\Omega^1_{Z/S}(2))$. We can compute local descriptions for the elements of $\mathrm{H}^0(\Omega^1_{Z/S}(2)$ using the initial Čech cover of Z.

Definition 0.5.23 (Pre-SUSY Structure). *A pre-SUSY structure on Z is a pair $(\mathcal{L}, u)$, where $\mathcal{L}$ is a line bundle on Z isomorphic to $\mathcal{O}_Z(-2)$ and where $u : \mathcal{L} \hookrightarrow \Omega^1_{Z/S}$ is an injective morphism of vector bundles on Z.*

Definition 0.5.24 (Framed Pre-SUSY Structure). *A framed pre-SUSY structure on Z is a SUSY structure $(\mathcal{L}, u)$ together with a choice of isomorphism $c : \mathcal{L} \xrightarrow{\sim} \mathcal{O}_Z(-2)$.*

We can compute explicit local descriptions of the global section of $\mathrm{H}^0(\Omega^1_{Z/S}(2))$ using the cover $\mathcal{U} \times S$ of Z. The resulting section are local generating sections for the pre-SUSY structures (defined below) on Z are unique up to multiplication by an invertible factor of $\mathrm{H}^0(\mathcal{O}^*_Z)$.

Definition 0.5.25 (Framed SUSY-structure). *A framed SUSY structure on Z is a SUSY structure $(\mathcal{L}, u, R)$ on Z together with a choice of framing $c : \mathcal{L} \xrightarrow{\sim} \mathcal{O}_Z(-2)$.*

A framed pre-SUSY structure is a framed SUSY structure on Z if it is integrable only along the divisor R.

Framed Pre- Structures on Z

In this section we give an explicit description of the framed pre-SUSY structures on Z. We compute $\mathrm{H}^0(\Omega^1_{Z/S}(2))$ using the Čech cover $\mathcal{U} = \{U = \operatorname{Spec} A[z,\zeta], \operatorname{Spec} A[w,\chi]$ of Z.

The local sections of $\Omega^1_{Z/S}(2)$ with respect to the cover $\mathcal{U}$ are as follows:

$$H^0(U,\Omega^1_{Z/S}(2)) = \left(\sum_{k\geq 0} a_k z^k + \zeta \sum b_k z^k\right) dz + \left(\sum_{k\geq 0} c_k z^k + \zeta \sum d_k z^k\right) d\zeta \qquad (60)$$

$$H^0(V,\Omega^1_{Z/S}(2)) = \left(\sum_{k\geq -2} \overline{a_k} w^k + \chi \sum \overline{b_k} w^k\right) dw + \left(\sum_{k\geq -2} \overline{c_k} w^k + \zeta \sum \overline{d_k} w^k\right) d\chi$$

where $a_k, d_k, \overline{a}_k, \overline{d}_k \in A^+$ and $b_k, c_k, \overline{b}_k, \overline{c}_k \in A^-$ for the above sections to be even and vice-versa for the sections to be odd. On the intersection $U \cap V$, we have that

$$dw = -1/z^2 dz$$

$$d\chi = (n/2-1) z^{n/2-2} \zeta dz - \sum_{j=1}^{n/2-2} j \eta_j z^{j-1} dz + z^{n/2-1} d\zeta$$

A basis for the super module over A of global sections of $\Omega^1_{Z/S}(2)$ correspond to the solutions to the equations:

$$\begin{aligned}\sum_{k\geq -2} a_k z^k =& -\sum_{k\geq 0}\overline{a}_k z^{-k-2} + \sum_{j=1}^{n/2-2}\eta_j \sum_{k\geq 0}\overline{b}_k z^{-k+j-2} && : dz\\ &-\sum_{j=2}^{n/2-2} j\eta_j \sum \overline{c}_k z^{-k+j-1} + \sum_{j=1}^{n/2-2}\eta_j \sum_{j'=2}^{n/2-2} j'\eta_{j'} \sum \overline{d}_k z^{-k+j+j'-1}\\ \sum_{k\geq -2} b_k z^k =& -\sum \overline{b}_k z^{n/2-3-k} + (n/2-1)\sum \overline{c}_k z^{n/2-2-k} && : \zeta d_z\\ &-(n/2-1)\sum_{j=1}^{n/2-2}\eta_j \sum \overline{d}_k z^{-k+j+n/2-2} - \sum_{j=2}^{n/2-2} j\eta_j \sum \overline{d}_k z^{n/2-2-k+j}\\ \sum_{k\geq -2} c_k z^k =& \sum \overline{c}_k z^{-k+n/2-1} - \sum_{j=1}^{n/2-2}\eta_j \sum \overline{d}_k z^{-k+j+n/2-1} && : d\zeta\\ \sum_{k\geq -2} d_k z^k =& \sum \overline{d}_k z^{-k+n-2} && : \zeta d\zeta\end{aligned}$$

where all coefficients with negative subscripts as well as η_j for $j \geq 0$ are assumed to be zero. From the above equations we find that

(a) $\{a_k | k = 0\}$

(b) $\{b_k | 0 \leq k \leq n/2 - 1\}$

(c) $\{c_k | 0 \leq k \leq n/2 + 1\}$

(d) $\{d_k | 0 \leq k \leq n\}$

is a rank-$(n_R + 2|n_R + 2)$ basis for $H^0(\Omega^1_{Z/S}(2))$. Note that since $H^1(\Omega^1_{\mathbb{WP}}(2)) = 0$, $H^0(\Omega^1_{Z/S}(2))$ is a free A-module. Setting $\overline{a}_0 := \overline{a}$, the global sections of $\Omega^1_{Z/S}(2)$ are then locally on the chart U given by

$$\left(\overline{a} + \zeta \sum_{k=0}^{n_R/2-1} \overline{b}_k z^k\right) dz + \left(\sum_{k=0}^{n_R/2+1} \overline{c}_k z^k + \zeta \sum_{k=0}^{n_R} \overline{d}_k z^k\right) d\zeta \tag{61}$$

with $\overline{a}, \overline{d}_k \in A^+$ and $\overline{c}_k, \overline{b}_k \in A^-$ for this section to be even and vice versa for this section to be odd. The set of framed pre-SUSY structures on Z can then be identified with the subset of the even part $H^0(\Omega^1_{Z/S}(2))^+$ such that the coefficient $\overline{a}$ in (61) is invertible (otherwise the restriction of a global section of the form (61) with non-invertible coefficient $\overline{a}$ could equal zero. Zero is not a pre-SUSY structure on $\mathbb{WP}$ and thus such a section could not have been a pre-SUSY structure on Z).

Open Subset of Framed SUSY Structures on Z

In this section we describe the open subset of $\mathrm{H}^0(\Omega^1_{Z/S}(2))$ corresponding to framed SUSY structures on Z.

Any free A-module M of rank $(m|n)$ gives rise to an affine $(m|n)$-superspace $\mathbb{M}$ over $S = \operatorname{Spec} A$ by taking relative Spec of the supersymmetric algebra of its dual:

$$\mathbb{M} := \operatorname{Spec} \mathcal{S}(M^*).$$

In particular, $\mathrm{H}^0(\Omega^1_{Z/S}(2))$ (free of rank $(n_R+2|n_R+2)$ over A) gives rise to an affine superspace

$$\mathbb{H}^0(\Omega^1_{Z/S}(2)) := \operatorname{Spec} \mathcal{S}(H^0(\Omega^1_{Z/S}(2))^*)$$

over S. If we choose a basis

$$\{x_1, x_2, \ldots, x_{n_R+2}, \theta_1, \ldots, \theta_{n_R+2}\}$$

for $H^0(\Omega^1_{Z/S}(2))^* \cong H^0(\Omega^1_{\mathbb{WP}\times S/S}(2))^*$ dual to the basis

$$\{dz, \zeta d\zeta, \ldots, \zeta z^{n_R} d\zeta, \zeta dz, \ldots, \zeta z^{n_R/2-1} dz, d\zeta, \ldots, z^{n_R/2+1} d\zeta\},$$

see (61), then we can identify this affine superspace with a standard one:

$$\mathbb{H}^0(\Omega^1_{Z/S}(2)) = \operatorname{Spec} \mathcal{S}(H^0(\Omega^1_{Z/S}(2))^*) \cong \operatorname{Spec} A[x_1, \ldots, x_{n_R+2}|\theta_1, \ldots, \theta_{n_R+2}].$$

We are interested in the part of the affine superspace $\mathbb{H}^0(\Omega^1_{Z/S}(2))$ which corresponds to the framed pre-SUSY structures on Z, *i.e.* those which restrict to pre-SUSY structure on $Z \times_S \operatorname{Spec} k = \mathbb{WP}$. Thus, because of the last paragraph of the previous section, we should take the first coordinate x_1 to be invertible and identify the open affine subsuperscheme

$$W := \operatorname{Spec} A[x_1, x_1^{-1}, \dots, x_{n_R+2} | \theta_1, \dots, \theta_{n_R+2}] \subset \mathbb{H}^0(\Omega^1_{Z/S}(2))$$

as the superspace of framed pre-SUSY structures on Z.

Henceforth, $A[x_1, x_1^{-1}, \dots, x_{n_R+2} | \theta_1, \dots, \theta_{n_R+2}]$ will be denoted by B.

Consider the framed pre-SUSY structure ϖ_z of $Z \times_S W$ whose restriction to the coordinate chart $U = \operatorname{Spec} B[z, \zeta]$ is

$$\varpi_Z|_U = \left(x_1 + \zeta \sum_{i=0}^{n_R/2-1} \theta_{i+1} z^i \right) dz + \left(\sum_{i=0}^{n_R/2+1} \theta_{i+n_R/2+1} z^i + \zeta \sum_{i=0}^{n_R} x_{i+2} z^i \right) d\zeta, \quad (62)$$

Let $Y \subset W$ be the open superscheme defined by the open subset $|Y|$ of the underlying topological space $|W|$ over which the framed pre-SUSY structure ϖ_Z of $Z \times_S W$ is maximally nonintegrable except over a relative divisor R on $Z \times_S W$ over W with no multiplicities. Since we are looking at the underlying topological space $|W|$, we may pass to the bosonic truncation $(W)_b$ and set all the odd variables θ and η on W to zero. Then ϖ from (62) in the chart $U(z, \zeta)$ on $Z \times_S W$ will be

$$\varpi_Z|_U = x_1 dz + \zeta p(z) d\zeta, \quad (63)$$

where

$$p(z) = x_2 + x_3 z + \cdots + x_{n_R+2} z^{n_R}.$$

The corresponding distribution will be generated by the vector field

$$\partial/\partial\zeta - \zeta p(z) \partial/\partial z,$$

whose commutator with itself is

$$-2p(z)\partial/\partial z.$$

Thus, the subset $|Y|$ above is defined as the complement to the discriminant locus of $p(z)$:

$$|Y| := |W| \setminus \{\mathrm{Disc}_z\, p(z) = 0\}$$

and is therefore open. The superscheme $Y \subset W$ is just the open superscheme defined by the open set $|Y|$.

0.5.6 Moduli of SUSY structures on Z

We make precise the notion of the moduli space of SUSY structures on Z as follows:

Definition 0.5.26. *Let $\mathcal{X}$ denote the category fibered in groupoids over* $\mathrm{SupSch}\,/S$ *with fiber over T the groupoid $\mathcal{X}(T)$ whose objects are SUSY structures $(\mathcal{L}, u, R)$ on $Z \times_S T$ and whose morphisms are automorphism of $\mathcal{L}$ preserving u and R.*

Theorem 0.5.27. *The fibered category $\mathcal{X}$ is represented by the algebraic superspace over S,*

$$[Y/\mathbb{G}_m \times (\mathbb{G}_a^{0|1})^{n_R/2} \times S]$$

and is of relative dimension $(n_R+1|n_R/2+2)$ over S, where the group S-superscheme $\mathbb{G}_m \times (\mathbb{G}_a^{0|1})^{n_R/2} \times S$ acts on Y by its identification with $H^0(\mathcal{O}_Z^*)$.

Proof. We revamp Y into a category of groupoids over $\mathrm{SupSch}\,/S$ with fibers over T the groupoid $Y(T)$ whose objects are SUSY structures $(\mathcal{L}, u, R)$ on $Z \times_S T$ with a fixed isomorphism $c : \mathcal{L} \to \mathcal{O}_Z(-2)$ and whose morphisms are the identity.

Let $(\mathcal{L}, u, R)$ denote a SUSY struture on $Z \times_S T$ and denote by ξ the corresponding object in $\mathcal{X}(T)$. We may choose an isomorphism $c : \mathcal{L} \to \mathcal{O}(-2)$ so that the pair

(ξ, c) is an object in $Y(T)$. The SUSY structures on (ξ, c) are in particular framed SUSY structures on $Z \times_S T$ and can, therefore, be described by a global sections of $\mathrm{H}^0(\Omega^1_{Z\times_S T}(2))$. We can compute a basis for $\mathrm{H}^0(\Omega^1_{Z\times_S T}(2))$ using the Čech cover $\mathcal{U} \times T$. The elements of $\mathrm{H}^0(\Omega^1_{Z\times_S T}(2))$ are described locally on $U \times T$ by the sections in (61) but with coefficients in $\Gamma(\mathcal{O}_T)$. In particular the framed SUSY structure on (ξ, c) is specified by a set of coefficients $t_i, \tau_i \in \Gamma(\mathcal{O}_T)$ Define $g : T \to Y$ to send x_i, θ_i to t_i, τ_i so that the SUSY structure on (ξ, c) is equal to $g^*\varpi_z$. The morphism g is clearly unique.

The descent data for $Y(T)$ is a torsor for the group of automorphisms of the line bundle $\mathcal{O}(-2)$ on $\mathbb{WP} \times T$. The group of automorphisms of $\mathcal{O}(-2)$ is the group of T-points of the supergroup scheme $\mathbb{G}_m \times (\mathbb{G}_a^{0|1})^{n_R/2} \times S$ (henceforth Γ^*_S) representing global invertible sections on Z. It follows that $\mathcal{X}$ is equal to the quotient superstack

$$[Y/\Gamma^*_S].$$

where $\Gamma^*_S = \operatorname{Spec} A[t, t^{-1}|\tau_1, \cdots, \tau_{n_R/2}]$ acts on Y by multiplication. The see that the action is free, note that the coefficient on dz in (62) is an element of $\Gamma^*_S(S)$. The T-points (a group) of the supergroup scheme Γ^*_S act trivially on a point g of $Y(T)$ if and only if it acts trivially on the SUSY structure associated to g, *i.e.* if it acts trivially on $g^*\varpi_z$. Since the leading coeffiicient dz (of $g^*\varpi_z$) is an element of $\Gamma^*_S(T)$ (and since a group acts freely on itself) $\Gamma^*_S(T)$ has no fixed points and, therefore, acts freely on $Y(T)$. Since T was arbitrary, we conclude that $[Y/\Gamma^*_S]$ (and thus $\mathcal{X}$) is an algebraic superspace over S.

The dimension of $\mathcal{X}$ is a equal to the difference of the dimension if Y (equal to $n_R + 2|n_R + 2$) and Γ^*_S (equal to $1|n_R$). The algebraic superspace is therefore of total dimension $(n_R + 1|n_R - 2 + n_R/2 + 2)$ where $n_R/2 - 2$ is the dimension of S.

□

0.5.7 Moduli of Genus Zero SUSY curves with $n_R \geq 4$ Ramond punctures

Let $\mathbb{E}$ denote the fiber product of the diagram

$$\begin{array}{ccc} & & \mathcal{C} \\ & & \downarrow \\ Y_b & \longrightarrow & (\mathfrak{M}_{0,n_R})_b \end{array}$$

where $\mathcal{C}$ denotes the universal object (a stacky SUSY curve) over $(\mathfrak{M}_{0,n_R})_b$ and where $Y_b \to (\mathfrak{M}_{0,n_R})_b$ is the canonical smooth covering map

$$q : Y_b \to Y_b/\mathbb{G}_m \to [(Y_b/\mathbb{G}_m)/(\mathrm{Aut}(\mathbb{WP}))_b]$$

for $(\mathfrak{M}_{0,n_R})_b$.

The fiber product $\mathbb{E}$ is the SUSY curve $(\mathbb{WP} \times Y_b, \varpi)$ constructed in Theorem 0.5.17. To construct $\mathfrak{M}_{0,n_R}$ we first construct the moduli space of deformations of $\mathbb{E}$.

Definition 0.5.28. *Let $\mathcal{D}ef_{\mathbb{E}}$ denote the category fibered in groupoids over* SupExt_{Y_b} *with fibers over T the groupoid $\mathcal{D}ef_{\mathbb{E}}(T)$ whose objects are deformations of the SUSY curve $\mathbb{E}$ over T and whose morphism are isomorphism of deformations of $\mathbb{E}$ over T.*

We already showed (Theorem 0.5.20) that $\mathbb{WP}$ has a universal deformation space $S = \mathrm{Spec}\, k[\eta_1, \dots, \eta_{n_R/2-2}]$ with universal deformation Z described in (53). It is easy to see that ϖ_z is a deformation of ϖ to Y. Indeed, since the θ_i coefficients vanish when restricted to Y_b, we have that $\varpi_z|_{Y_b} = \varpi$.

Theorem 0.5.29. *The category fibered in groupoids $\mathcal{D}ef_{\mathbb{E}}$ is an algebraic superspace represented by the quotient*

$$(Y/\Gamma_S^*)/\mathcal{A}ut(Z/\mathbb{WP})$$

of dimension $(n_R - 3|n_R/2 - 2)$.

Proof. Let Σ be a deformation of $\mathbb{E}$ over T. with underlying family of supercurves X/T. The supercurve X/T is then in particular a deformation of $\mathbb{WP} \times Y_b$.

We showed in Theorem 0.5.20 (base changed to Y_b) that there exists a unique morphism $f : T \to S \times Y_b$ such that $X \cong Z \times_{S \times Y_b, f} T$ as deformations of $\mathbb{WP} \times Y_b$ over T. Let us fix with

$$\Phi : X \xrightarrow{\sim} Z \times_{S \times Y_b, f} T$$

such an isomorphism. Let us also frame with $c : \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X(-2)$ the SUSY structure on Σ.

Using Φ, we can pullback the framed SUSY structure on X to a framed SUSY structure on $Z \times_{S \times Y_b, f} T$. This induces an isomorphism

$$\Psi : (\Sigma, \Phi) \xrightarrow{\sim} (Z \times_{S \times Y_b, f} T, \Phi^* \mathcal{L}, \Phi^* u, \Phi^{-1} R, \Phi^* c)$$

of deformations of $\mathbb{E} \times Y_b$ over T. The SUSY structure $(\Phi^* \mathcal{L}, \Phi^* u, \Phi^{-1} R, \Phi^* c)$ is a framed SUSY structures on $Z \times_{S,f} T$. By Theorem 0.5.27 there then exists a unique morphism $g : T \to Y$ such that the above SUSY structure is equal to $g^* \varpi_z$ (described in (61)). The triple (Σ, Φ, c) is the object g in $Y(T)$.

The category of triples (Σ, Φ, c) is isomorphic to $Y(T)$. Define $q : Y(T) \to \mathrm{Def}_{\mathbb{E}}(T)$ to send $(\Sigma, \Phi, c) \to \Sigma$. The set of isomorphisms Φ is a torsor for the group $\mathcal{A}ut(Z/\mathbb{WP} \times Y_b)(T)$ of formal automorphisms of Z over $S \times Y_b$ (those automorphism of Z over $S \times Y_b$ that restrict to the identity on $\mathbb{WP} \times Y_b$). The group of formal automorphisms of Z over $S \times Y_b$ is representable (Proposition 2.6.2, [23]). The fibers of q are isomorphic to the groups $\mathcal{A}ut(Z/\mathbb{WP} \times Y_b)(T) \times \Gamma^*_{S \times Y_b}(T)$. The kernel of q is therefore isomorphic to the supergroup scheme $\mathcal{A}ut(Z/\mathbb{WP} \times Y_b) \times_{\Gamma_{S \times Y_b}} \times T$.

We can identity $\mathcal{X}$ with $\mathrm{H}^0(\Omega^1_{Z/S\times Y_b}(2))/\,\mathrm{H}^0(\mathcal{O}^*_Z)$.The supergroup scheme $\mathcal{A}ut(Z/\mathbb{WP}\times Y_b)$ has a canonical action on $\mathrm{H}^0(\Omega^1_{Z/S\times Y_b}(2))/\,\mathrm{H}^0(\mathcal{O}^*_Z)$.

It then follows that

$$\mathcal{D}ef_{\mathbb{E}} = [\mathcal{X}/\mathcal{A}ut(Z/\mathbb{WP}\times Y_b)].$$

To show that the quotient is an algebraic superspace we need to show that $\mathcal{A}ut(Z/\mathbb{WP}\times Y_b)$ acts freely on $\mathcal{X}$. This facts follows from Theorem 0.5.13 in which we showed that a SUSY curve (so in particular $\mathbb{E}$) has no infinitesimal automorphisms. The dimension of $\mathcal{D}ef_{\mathbb{E}}$ is equal to the different between the dimension of Y/Γ^*_S (equal to $1|n_R/2$ over S) and the dimension of the $S\times Y_b$-supergroup scheme $\mathcal{A}ut(Z/\mathbb{WP}\times Y_b)$. The supergroup scheme $\mathcal{A}ut(Z/\mathbb{WP}\times Y_b)$ has tangent space $H^0(\mathcal{T}\mathbb{WP})$ (Propsition 2.6.2, [23]) and is therefore of dimension $(4|n_R/2+2)$.

□

We proved in Theorem 0.5.6 that any family of supercurves X/T underlying a SUSY curve Σ over T_b is étale locally isomorphic to $\mathbb{WP}\times T_b$. We use this fact to prove the following theorem:

Theorem 0.5.30. *The Deligne-Mumford superstack* $\mathfrak{M}_{0,n_R}$ *may be expressed as the quotient superstack*

$$\mathcal{D}ef_{\mathbb{E}}/\mathbb{Z}/2\mathbb{Z}$$

of dimension $(n_R-3|n_R/2-2)$.

Proof. Let Σ be a SUSY curve over T and let X/T denote its underlying family of supercurves. Let $p: P_b\to T_b$ be the étale cover on which $X\times_T P_b\cong\mathbb{WP}\times P_b$. In Theorem 0.5.18 we showed that there exists a unique map $h: P_b\to Y_b$ and an isomorphism $\phi : \Sigma \times_T P_b \xrightarrow{\sim} \mathbb{E} \times_{Y_b,h} P_b$ of SUSY curves over P_b. We may, therefore, treat the pair $(\Sigma \times_T P, \phi)$ (where P is the unique extension of the étale cover $p : P_b \to T_b$ to an étale cover $p : P \to T$) as a deformation of $\mathbb{E} \times_{Y_b,h} P_b$ over P, $i.e$ as an object in $\mathcal{D}ef_{\mathbb{E}}(P)$.

The category of pairs $(\Sigma \times_T P, \phi)$ is in particular isomorphic to the groupoid $\mathcal{D}ef_{\mathbb{E}}(P)$. Define $q : \mathcal{D}ef_{\mathbb{E}}(P) \to \mathfrak{M}_{0,n_R}(P)$ to send (Σ, ϕ) to Σ and formal automorphisms to ϕ. The set of isomorphism ϕ is a torsor for $\mathbb{Z}/2\mathbb{Z}$ (the group of SUSY-automorphisms of $\mathbb{E}$) Each fiber $q^{-1}(\Sigma)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The kernel of q is isomorphic to the group superscheme $\mathbb{Z}/2\mathbb{Z} \times P$.

Globalizing the above discussion to objects in T, we find that $\mathfrak{M}_{0,n_R}(T)$ is isomorphic to the groupoid of triples $(T, \mathcal{F}, \pi)$ where $\mathcal{F}$ is a $\mathbb{Z}/2\mathbb{Z} \times T$-torsor over the big étale site of T and where $\pi : \mathcal{F} \to \mathcal{D}ef_{\mathbb{E}}$ is a $\mathbb{Z}/2\mathbb{Z} \times T$-equvariant morphisms. It then follows from standard arguments that

$$\mathfrak{M}_{0,n_R} = [\mathcal{D}ef_{\mathbb{E}}/\mathbb{Z}/2\mathbb{Z}.]$$

The quotient is Deligne-Mumford because $\mathbb{Z}/2\mathbb{Z}$ is finite and thus acts with finite stabilizers.

The dimension of $\mathfrak{M}_{0,n_R}$ is equal to the difference between the dimension of $\mathcal{D}ef_{\mathbb{E}}$ (equal to $(n_R - 3|n_R/2 - 2)$) and the dimension of $\mathbb{Z}/2\mathbb{Z}$ (equal to zero).

□

Bibliography

[1] Dan Abramovich and Tyler Jarvis. "Moduli of twisted spin curves". In: *Proceedings of the American Mathematical Society* 131.3 (2003), pp. 685–699.

[2] Tim Adamo, Michael Groechenig, et al. "Moduli stacks of maps for supermanifolds". In: *Advances in Theoretical and Mathematical Physics* 17.6 (2013), pp. 1303–1342.

[3] Jarod Alper. "Artin algebraization and quotient stacks". In: *arXiv preprint arXiv:1510.07804* (2015).

[4] Ugo Bruzzo and Daniel Hernández Ruipérez. "The Supermoduli of SUSY curves with Ramond punctures". In: *arXiv preprint arXiv:1910.12236* (2019).

[5] Claudio Carmeli, Lauren Caston, and Rita Fioresi. *Mathematical foundations of supersymmetry*. Vol. 15. European Mathematical Society, 2011.

[6] G. Codogni and F. Viviani. "Moduli and periods of supersymmetric curves". In: *arXiv preprint arXiv:1706.04910* (2017).

[7] Brian Conrad and Aise Johan de Jong. "Approximation of versal deformations". In: *Journal of Algebra* 255.2 (2002), pp. 489–515.

[8] Maurizio Cornalba. "Moduli of curves and theta-characteristics". In: *Lectures on Riemann surfaces (Trieste, 1987)* 560 (1989), p. 589.

[9] P. Deligne. *Letter to Manin on 25 September 1987*.

[10] J. A. Dominguez Perez, D. Hernandez Ruiperez, and C. Sancho De Salas. "Global structures for the moduli of (punctured) super Riemann surfaces". In: *Journal of Geometry and Physics* 21.3 (1997), pp. 199–217.

[11] Barbara Fantechi, Lothar Gottsche, and Luc Illusie. *Fundamental algebraic geometry: Grothendieck's FGA explained*. 123. American Mathematical Soc., 2005.

[12] R. Fioresi and S. D. Kwok. "On SUSY curves". In: *Advances in Lie Superalgebras*. Springer, 2014, pp. 101–119.

[13] H Flenner and D Sundararaman. "Analytic geometry of complex superspaces". In: *Transactions of the American Mathematical Society* 330.1 (1992), pp. 1–40.

[14] Tyler J Jarvis. "Geometry of the moduli of higher spin curves". In: *International Journal of Mathematics* 11.05 (2000), pp. 637–663.

[15] Tyler J Jarvis. "Torsion-free sheaves and moduli of generalized spin curves". In: *Compositio Mathematica* 110.3 (1998), pp. 291–333.

[16] Mikhail Kapranov and Eric Vasserot. "Vertex algebras and the formal loop space". In: *Publications mathématiques de l'IHÉS* 100 (2004), pp. 209–269.

[17] C. LeBrun and M. Rothstein. "Moduli of super Riemann surfaces". In: *Communications in mathematical physics* 117.1 (1988), pp. 159–176.

[18] DA Leites. "Theory of supermanifolds". In: *Karelskij filial AN SSSR, Petrozavodsk* (1983), p. 199.

[19] Yu Manin. *Complex geometry and gauge theory*. 1988.

[20] A. Masuoka and A. N. Zubkov. "On the notion of Krull super-dimension". In: *Journal of Pure and Applied Algebra* 224.5 (2020), p. 106245.

[21] Seyed Faroogh Moosavian and Yehao Zhou. "On the Existence of Heterotic-String and Type-II-Superstring Field Theory Vertices
". In: *arXiv preprint arXiv:1911.04343* (2019).

[22] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*. Vol. 34. Springer Science & Business Media, 1994.

[23] Edoardo Sernesi. *Deformations of algebraic schemes*. Vol. 334. Springer Science & Business Media, 2007.

[24] A Yu Vaintrob. "Deformation of complex superspaces and coherent sheaves on them". In: *Journal of Soviet Mathematics* 51.1 (1990), pp. 2140–2188.

[25] Veeravalli S Varadarajan. *Supersymmetry for Mathematicians: An Introduction*. Vol. 11. American Mathematical Soc., 2004.

[26] E. Witten. *E-mail message to A. S. Schwarz*. June 2010.

[27] E. Witten. "Notes on super Riemann surfaces and their moduli". In: *arXiv preprint arXiv:1209.2459* (2012).