COMPUTATIONAL COMPLEXITY OF INNER AND OUTER j-RADII OF POLYTOPES IN FINITE-DIMENSIONAL NORMED SPACES

By

Peter Gritzmann

and

Victor Klee

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COMPUTATIONAL COMPLEXITY OF INNER AND OUTER j–RADII OF POLYTOPES IN FINITE-DIMENSIONAL NORMED SPACES*

PETER GRITZMANN† AND VICTOR KLEE‡

Abstract. This paper studies the complexity of computing (or approximating, or bounding) the various inner and outer radii of a convex polytope in the space $\mathbb{R}^n$ equipped with an ℓ_p norm or a polytopal norm. The polytope P is assumed to be presented as the convex hull of finitely many points with rational coordinates ($\mathcal{V}$-presented) or as the intersection of finitely many closed halfspaces defined by linear inequalities with rational coefficients ($\mathcal{H}$-presented). The inner j-radius of P is the radius of a largest j-ball contained in P; it is P's inradius when $j = n$ and half of P's diameter when $j = 1$. The outer j-radius measures how well P can be approximated, in a minimax sense, by an $(n - j)$-flat; it is P's circumradius when $j = n$ and half of P's width when $j = 1$. The binary (Turing machine) model of computation is employed. The primary concern is not with finding optimal algorithms, but with establishing polynomial-time computability or $\mathbb{NP}$-hardness. Special attention is paid to the case in which P is centrally symmetric. When the dimension n is permitted to vary, the situation is roughly as follows: (a) for general $\mathcal{H}$-presented polytopes in ℓ_p spaces with $1 < p < \infty$, all outer radius computations are $\mathbb{NP}$-hard; (b) in the remaining cases (including symmetric $\mathcal{H}$-presented polytopes), some radius computations can be accomplished in polynomial time and others are $\mathbb{NP}$-hard. These results are obtained by using a variety of tools from the geometry of convex bodies, from linear and nonlinear programming, and from the theory of computational complexity. Applications of the results to various problems in Mathematical Programming, Computer Science and other fields are included.

Key words. computational convexity, computational complexity, polynomial-time algorithms, $\mathbb{NP}$-hardness, mathematical programming, computational geometry, ellipsoid method, linear programming, sensitivity analysis, quadratic programming, robotics, convex body, polarity, polytope, convex hull, breadth, width, diameter, radius, insphere, circumsphere

AMS(MOS) subject classifications. 90C30, 68Q15, 52A25, 90C05, 90C20, 90C25, 11J72, 11J81

Introduction and Basic Definitions.

This paper is concerned with the complexity of computing the various inner and outer radii of polytopes. In particular, we establish polynomial-time computability or $\mathbb{NP}$-hardness of computing the diameter, the width, the inradius and the circumradius of polytopes. Motivation for this work arises from various problems in Mathematical Programming and Computer Science. Some of these applications are outlined in Section 6.

The present study uses the results on geometric and combinatorial properties of radii given in [GK90a], and some complexity results of [BGKV90] for the related problem of norm-maximization. Some perspective is added by results of [GHK90] on the algebraic

*Part of this paper was written while both authors were visiting the Institute for Mathematics and its Applications, 206 Church Street S.E., Minneapolis, MN 55455.

†Universität Augsburg, Institut für Mathematik, Universitätsstr. 8, D-8900 Augsburg, Federal Republic of Germany. Research supported in part by the Alexander-von-Humboldt foundation and by the Deutsche Forschungsgemeinschaft.

‡Department of Mathematics, University of Washington, Seattle, Washington 98195, U.S.A. Research supported in part by the National Science Foundation.

tractability of radii. However, the present paper is self-contained in the sense that we will refer to these preliminary results explicitly, rather than merely implicitly.

In view of the applications given in Section 6, our main interest is in dealing with the various radii in the spaces $\mathbb{R}^n$ which are endowed with the Euclidean norm, with the maximum norm, or with the ℓ_1 norm. However, there has also been some interest in polytope containment problems (cf. [EF82], [FO85]) which for symmetric polytopes amount to finding inradii or circumradii in spaces that carry a norm whose unit ball is itself a polytope. To cover all these cases we formulate our radii problems in the general framework of Minkowski spaces.

Let $\mathbb{M}$ denote a *Minkowski space* — a normed finite-dimensional vector space over the real field $\mathbb{R}$. The dimension of $\mathbb{M}$ is denoted by n and the norm by $\| \quad \|$. The *unit ball* and *unit sphere* of $\mathbb{M}$ are the sets $\mathbb{B} = \{x : \|x\| \leq 1\}$ and $\mathbb{S} = \{x : \|x\| = 1\}$ respectively. The term *polytope* will mean an n-dimensional subset of $\mathbb{M}$ that is the convex hull of a finite set of points or, equivalently, is the intersection of a finite collection of closed halfspaces.

The most important cases — for which the results of the present paper are almost complete — are the Minkowski spaces with polytopal norms or with ℓ_p norms. A norm is *polytopal* if the unit ball is a polytope. For $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, the ℓ_p norm $\|x\|_p$ is defined as follows for $1 \leq p \leq \infty$:

$$\|x\|_p := \left(\sum_{i=1}^{n} |\xi_i|^p \right)^{1/p} \qquad (\text{for } 1 \leq p < \infty); \qquad \|x\|_\infty := \max_{1 \leq i \leq n} |\xi_i|.$$

Note that the norms $\| \quad \|_1$ and $\| \quad \|_\infty$ are polytopal. The Minkowski space that is $\mathbb{R}^n$ endowed with an ℓ_p norm will be denoted by $\mathbb{R}^n_p$, its unit ball and unit sphere by $\mathbb{B}^n_p$ and $\mathbb{S}^n_p$ respectively.

With each Minkowski space $\mathbb{M}$ there is an associated *conjugate space* $\mathbb{M}^*$. The points of $\mathbb{M}^*$ are the linear functionals on $\mathbb{M}$, and the norm of a functional is the maximum of its values on the unit ball $\mathbb{B}$ of $\mathbb{M}$. The norm, unit ball and unit sphere of $\mathbb{M}^*$ are denoted by $\mathbb{B}^*$ and $\mathbb{S}^*$ respectively. As is customary, the norms in both $\mathbb{M}$ and $\mathbb{M}^*$ are denoted by $\| \quad \|$ when there is no danger of confusion. Of course, polytopes are defined for $\mathbb{M}^*$ as for $\mathbb{M}$.

For each $p \in [0, \infty]$, the number $\overline{p} \in [0, \infty]$ is defined by the condition that

$$\frac{1}{p} + \frac{1}{\overline{p}} = 1.$$

Thus the spaces $\mathbb{R}^n_p$ and $\mathbb{R}^n_{\overline{p}}$ are conjugate to each other. The most important spaces of this sort are the self-conjugate space $\mathbb{R}^n_2$ (Euclidean n-space) and the mutually conjugate spaces $\mathbb{R}^n_1$ and $\mathbb{R}^n_\infty$.

The usual bilinear form on $\mathbb{M} \times \mathbb{M}^*$ is denoted by $\langle \, , \, \rangle$, so that for $x \in \mathbb{M}$ and $y \in \mathbb{M}^*$, $\langle x, y \rangle$ denotes the value of the functional y at the point x. Thus, in particular, if $\mathbb{M} = \mathbb{R}^n_p$

and $\mathbb{M}^* = \mathbb{R}^n_p$, and (as we usually assume) if coordinates are given with respect to the standard (dual) bases, then

$$\langle x, y\rangle = \sum_{i=1}^{n} \xi_i \eta_i$$

for each $x = (\xi_1, \ldots, \xi_n) \in \mathbb{M}$ and $y = (\eta_1, \ldots, \eta_n) \in \mathbb{M}^*$.

For $X \subset \mathbb{M}$, the *polar* of X is the set $X^\circ \subset \mathbb{M}^*$ given by

$$X^\circ = \{y \in \mathbb{M}^* : \langle x, y\rangle \leq 1 \text{ for all } x \in X\}.$$

Polars are defined in the same way for subsets of $\mathbb{M}^*$, and by identifying $(\mathbb{M}^*)^*$ with $\mathbb{M}$ in the usual way, these polars are regarded as subsets of $\mathbb{M}$.

Prefixes often indicate dimension. For example, the j-dimensional linear ⟨resp. affine⟩ subspaces of $\mathbb{M}$ are called *j-subspaces* ⟨resp. *j-flats*⟩, and a *j-ball of radius* ρ in $\mathbb{M}$ is a set of the form

$$(q + \rho\mathbb{B}) \cap F = \{x \in F : \|x - q\| \leq \rho\}$$

for some j-flat F in $\mathbb{M}$ and point $q \in F$.

For $1 \leq j \leq n$, the *inner j-radius* $r_j(P)$ of a polytope $P \subset \mathbb{M}$ is the maximum of the radii of the j-balls contained in P.

The *outer j-radius* $R_j(P)$ of P measures how well P can be approximated, in a minimax sense, by an $(n-j)$-flat. Specifically, $R_j(P)$ is the minimum of the positive numbers ρ such that $\mathbb{M}$ contains an $(n-j)$-flat F for which $P \subset F + \rho\mathbb{B}$.

The numbers $r_n(P)$ and $R_n(P)$ are respectively the radius of a largest n-ball contained in P and of a smallest n-ball containing P. They are called, respectively, the *inradius* and the *circumradius* of P, and the center of each such ball is called an *incenter* ⟨resp. a *circumcenter*⟩ of P. The number $2r_1(P)$ is the *diameter* of P—the maximum distance that is realized between two points of P. The number $2R_1(P)$ is the *width* of P—the smallest of the distances between pairs of parallel supporting hyperplanes of P.

Approximation theorists have computed the radii of particular bodies of special interest, and have estimated the radii when precise computation proved to be too difficult (cf. [Ti60], [Si70], [Pi85] and their references). The present study has a somewhat different focus, related to the fact that motivation for the computation of polytope radii has arisen more recently from problems in computer graphics, robotics, global optimization, and the sensitivity analysis of linear programming (see Section 6). We are concerned here with the intrinsic complexity of computing (or approximating or bounding) the various inner and outer radii of an arbitrary polytope or of an arbitrary centrally symmetric polytope. These are basic problems in computational geometry, and our approach is that of the theory of computational complexity.

We are interested in the complexity of radius computations for both fixed dimension and variable dimension, but our emphasis is on the latter. We employ the standard *binary* or *Turing machine* model of computation [GJ79], in which the *size of the input* is defined as the length of the binary encoding needed to present the input data to a Turing machine and the *time-complexity* of an algorithm is also defined in terms of the operation of a Turing machine. For this model, each computation begins with a presentation of a polytope P in terms of integers, and this is necessarily related to a coordinatization of the space. It is essential to distinguish between polytopes that are $\mathcal{V}$-presented and polytopes that are $\mathcal{H}$-presented.

A *$\mathcal{V}$-presentation* of a polytope $P \subset \mathbb{R}^n$ consists of integers m and n with $m > n \geq 1$, and an m-tuple $v_1, \ldots, v_m$ of rational points of $\mathbb{R}^n$ such that

$$P = \operatorname{conv}\{v_1, \ldots, v_m\}.$$

Note that if the presentation is irredundant—i.e., if the convex hull is diminished when any one of the m points v_i is omitted—then the m points $v_1, \ldots, v_m$ are precisely the vertices of P.

An *$\mathcal{H}$-presentation* of a polytope P consists of integers m and n with $m > n \geq 1$, a rational $m \times n$ matrix A, and a rational m-vector b such that

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}.$$

Note that if the presentation is irredundant—i.e., the intersection

$$\bigcap_{i=1}^{m} \{x = (\xi_1, \ldots, \xi_n) : \sum_{j=1}^{n} \alpha_{ij}\xi_j \leq \beta_i\}$$

is enlarged when any one of the m halfspaces $\{x : \sum_{j=1}^{n} \alpha_{ij}\xi_j \leq \beta_i\}$ is omitted—then the m hyperplanes $\{x : \sum_{j=1}^{n} \alpha_{ij}\xi_j = \beta_i\}$ are precisely the affine hulls of the facets of P.

As usual (see e.g. [GLS88]) the *size* of the input is defined as the number L of binary digits necessary to encode all input data. We may assume without loss of generality that for each of the relevant rational numbers γ/δ, the numerator γ and the denominator δ are relatively prime. That is because when this assumption is not satisfied, the Euclidean algorithm can be applied to produce (in polynomial time) a presentation of P that does satisfy the assumption. (The Euclidean algorithm runs in polynomial time but not in strongly polynomial time, so the assumption of relative primality of γ and δ would be a genuine added restriction if we were aiming at strongly polynomial algorithms. See pp. 36–40 of [GLS88] for a discussion of this point in the context of Gaussian elimination.)

As is well-known, each polytope P admits both a $\mathcal{V}$-presentation and an $\mathcal{H}$-presentation. However, since P may have many more vertices than facets (or vice-versa), it may happen that the minimum size for one sort of presentation is much larger than the minimum size for the other sort. That is why the complexity results for $\mathcal{V}$-presentations differ from those for $\mathcal{H}$-presentations. However, some of our results do apply to both $\mathcal{V}$-presented and $\mathcal{H}$-presented polytopes, and in those cases we refer simply to *presented polytopes.*

It is known (see [Kh80], [PS82] for the corresponding results stated for polytopes that are presented by means of integer data) that if P is presented in either of the above manners, then with respect to the Euclidean norm, $R_n(P) \le 2^{4L}$—in fact, $P \subset 2^{4L}\mathbb{B}_2^n$. ¿From this it follows at once that

$$P \subset 2^{4L}\sqrt{n}\mathbb{B}_p^n \qquad \text{for } 1 \le p \le \infty.$$

Also, if $\operatorname{int} P \ne \emptyset$ (as is implied by our condition that $\dim P = \dim \mathbb{M}$), then with respect to the Euclidean norm, $r_n(P) \ge 2^{-4L}$. In fact, the ellipsoid method will produce, in time bounded by a polynomial in the size of the input, a rational point q such that $q + 2^{-4L}\mathbb{B}_2^n \subset P$, whence

$$q + (2^{-4L}/n)\mathbb{B}_p^n \subset P \qquad \text{for } 1 \le p \le \infty.$$

These facts add perspective to our general assumption that P is bounded and has interior points.

When precise computation of a polytope radius ρ turns out to be difficult, it is of interest to estimate it by approximating it closely or bounding it above or below. This involves, as an additional part of the input data, a positive rational number λ, where λ is the bound to be tested or is the desired closeness of approximation. Of course, the size of λ is then part of the size of the input, which is $L + \text{size}(\lambda)$, with L as defined earlier. The output desired for the *approximation problem* is a rational number μ such that $|\mu - \rho(P)| < \lambda$, and for *lower-bounding* ⟨*upper-bounding*⟩ it is a correct answer to the question "Is $\rho(P) \ge \lambda$?" ⟨"Is $\rho(P) \le \lambda$?"⟩. In the following, we will use the same symbol L to denote the size of the input in all these different situations, and it is understood that the bound λ is part of L in case of approximation, lower-bounding or upper-bounding.

It should be emphasized that the proofs of $\mathbb{NP}$-hardness do not involve exotic polytopes. In particular, parallelotopes and simplices suffice for the hardness results indicated in the tables at the end of this section. The tables describe the situation for the four most important radii and the three most important values of p.

Our section headings are as follows:

In addition to providing detailed statements of our main results on the *computational* complexity of the various radii, Section 1 contains an informal description of some of the difficulties encountered in attempting to compute these radii.

As a service to the reader, the Appendix provides an overview of the basic properties of radii that are needed in establishing the computational results. Even though some of the results have been well-known for a long time, we refer to [GK90a] as a convenient single source for these and various related results.

The present paper belongs to a series that includes [GK89], [BGKV90], [GHK90], [GK90a], [GK90b], and other papers in preparation. Their purpose is to study the intrinsic complexity of various computations involving fundamental geometric measures of convex polytopes and other convex bodies, with emphasis on the case of variable dimension. This direction of research might be called *computational convexity*, since in many contexts the term *computational geometry* seems to imply that attention is to be restricted to a fixed low dimension.

As basic references for the topics involved in this paper, we suggest [AHU74] and [GJ79] for computational complexity, [PS85] and [Ed87] for computational geometry , [Gr67] and [Br83] for convex polytopes, and [Sc'87] for linear programming.

For a study of some related problems see [Me90].

The following tables provide a rapid indication of our main results for the four most important radii r_1, R_1, r_n, R_n and the three most important values of p (1, 2 and ∞). The designations $\mathbb{P}$, $\mathbb{NPC}$ and $\mathbb{NPH}$ indicate respectively polynomial-time computability, $\mathbb{NP}$-completeness, and $\mathbb{NP}$-hardness. (For the listing of $\mathbb{NPH}$, we do not know whether the problem belongs to the class $\mathbb{NP}$.) These results refer to the case of variable finite dimension. As is described at the end of Section 1, the picture changes when the dimension is fixed.

The reader should be warned that the tables are in some respects imprecise, and are intended only to convey a rough impression of our main results. Detailed, precise statements of all of our main results are provided in Section 1.

$\mathbb{R}_2^n$		$\mathcal{H}$-polytopes		$\mathcal{V}$-polytopes	
		general	symmetric	general	symmetric
Diameter	r_1^2	$\mathbb{NPC}$	$\mathbb{NPC}$	$\mathbb{P}$	$\mathbb{P}$
Inradius	r_n^2	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{NPH}$	$\mathbb{NPC}$
Width	R_1^2	$\mathbb{NPC}$	$\mathbb{P}$	$\mathbb{NPC}$	$\mathbb{NPC}$
Circumradius	R_n^2	$\mathbb{NPC}$	$\mathbb{NPC}$	$\mathbb{P}$	$\mathbb{P}$

$\mathbb{R}_1^n$		$\mathcal{H}$-polytopes		$\mathcal{V}$-polytopes	
		general	symmetric	general	symmetric
Diameter	r_1	$\mathbb{NPC}$	$\mathbb{NPC}$	$\mathbb{P}$	$\mathbb{P}$
Inradius	r_n	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$
Width	R_1	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$	$\mathbb{P}$
Circumradius	R_n	$\mathbb{NPC}$	$\mathbb{NPC}$	$\mathbb{P}$	$\mathbb{P}$

$\mathbb{R}^n_\infty$		$\mathcal{H}$-polytopes		$\mathcal{V}$-polytopes	
		general	symmetric	general	symmetric
Diameter	r_1	**P**	**P**	**P**	**P**
Inradius	r_n	**P**	**P**	**NPC**	**NPC**
Width	R_1	**NPC**	**P**	**NPC**	**NPC**
Circumradius	R_n	**P**	**P**	**P**	**P**

The tables provide only a rough indication of results. They are imprecise in the following respects: (i) the diameter and width are actually equal to $2r_1$ and $2R_1$ respectively; (ii) the results for ℓ_2 involve the square of the radius rather than the radius itself; (iii) some of the **P** entries are based on polynomial-time approximability rather than polynomial-time computability; (iv) the designations **NPC** and **NPH** do not refer to computability per se, but to the appropriately related decision problems involving the establishment of lower or upper bounds for the radii in question.

1. Statement of Main Results.

Before stating our main results, we want to mention three possible sources of difficulty in computing a given inner or outer radius of a polytope. One source is the possible algebraic complexity of the desired number itself. This particular difficulty is in a sense an artifact of our model of computation, and it vanishes if we ask for approximation (rather than precise computation) in the Turing machine model. For example, there is an obvious algorithm for "computing" the diameter of a $\mathcal{V}$-polytope in an ℓ_p space—simply compute distances between vertices, and take the maximum number thus obtained. However, for the binary model of computation it must be required that p is an integer (or ∞), and what is really output is the p^{th} power of the diameter rather than the diameter itself. A crucial fact is that the p^{th} power of the diameter is rational. Now suppose that p is an integer and P is a presented polytope in an ℓ_p space. Suppose also that $p \neq 2$ and ρ is the circumradius of P, or that $p = 2$ and ρ is the inradius of P. Then, as is shown in [GHK90], there is no rational polynomial q such that $q(\rho)$ is guaranteed, under the specified circumstances, to be a rational number. This does not entirely rule out the possibility that the number ρ can in some indirect way be determined by a Turing machine, but it does make it more difficult to frame an appropriate definition of such determination. (In some cases it would suffice to augment the Turing machine by an "oracle" that computes norms of rational vectors.)

Another difficulty in computing (or even approximating) radii arises from the fact that the optimizing positions of the relevant flat (and, for inner radii, of a point in that flat) may not be sufficiently strongly related to the facial structure of the polytope. This is the case for the outer 2-radii of polytopes in Euclidean 3-space, as D. Larman (private communication) has shown that there are infinitely many smallest cylinders *that contain* a regular simplex. Also, there are 3-polytopes with arbitrarily many vertices that do not have a unique minimum containing cylinder [GK90a]. Further, [GK90a] shows that there is no "Helly-type" characterization of optimal cylinders, for there are 3-polytopes with arbitrarily many vertices which have the property that when any vertex is omitted the convex hull of the remaining vertices has smaller radius R_2 than the polytope itself.

Finally, even when there must be a strong geometric relationship of the optimizing flat to the polytope, there may be too many positions of the flat that satisfy this relationship and hence are candidates for being the optimizing position. This is the classically difficult situation of trying to optimize a function that has, in addition to its global optimum, many local optima that are not global optima. For some examples of this phenomenon, see [GK90a].

Each of our complexity results involves the following ingredients:

a collection $\mathcal{M}$ of Minkowski spaces;

a manner $\mathcal{W}$ of presenting polytopes (where $\mathcal{W}$ is $\mathcal{V}$ or $\mathcal{H}$);

for each $\mathbf{M} \in \mathcal{M}$, a measurement $\mu_{\mathbf{M}}$ that is applicable to polytopes in $\mathbf{M}$.

Typically, $\mu_{\mathbf{M}}(P)$ is a power of the inner or outer j-radius of P, where $1 \le j \le \dim \mathbf{M}$ and where, when the collection $\mathcal{M}$ is infinite, there are some restrictions on the rate at which j or $n - j$ grows as $\dim \mathbf{M} \to \infty$.

The following conventions serve to clarify or simplify certain statements:

we speak of computation *in ℓ_p spaces* or *in an ℓ_p space* to refer to computation in the spaces $\mathbb{R}_p^n$ where p is fixed but the dimension n is variable;

we speak of computation *in fixed $\mathbb{R}_p^n$* to indicate that both p and the dimension n are fixed;

for each real α, $\alpha^\infty := \alpha$; thus *the p^{th} power of α* has its usual meaning when $p \in [1, \infty[$, and when $p = \infty$ it merely means the number α.

Note that for polytopes in ℓ_p spaces, some of the results below involve p^{th} powers and others involve $\overline{p}^{\text{th}}$ powers.

1.1 MEMBERSHIP IN $\mathbb{P}$. *For polytopes in ℓ_p spaces, each of the following can be computed in polynomial time:*

the p^{th} power of the diameter —

of a $\mathcal{V}$-polytope when $p \in \mathbb{N} \cup \infty$;

of an $\mathcal{H}$-polytope when $p = \infty$;

the p^{th} power of the circumradius —

of a symmetric $\mathcal{V}$-polytope when $p \in \mathbb{N} \cup \infty$;

of a $\mathcal{V}$-polytope when $p \in \{1, 2, \infty\}$;

of an $\mathcal{H}$-polytope when $p = \infty$;

the $\overline{p}^{\text{th}}$ power of the width —

of a symmetric $\mathcal{H}$-polytope when $\overline{p} \in \mathbb{N} \cup \infty$;

of an $\mathcal{H}$-polytope when $p = 1$;

of a $\mathcal{V}$-polytope when $p = 1$;

the $\overline{p}^{\text{th}}$ power of the inradius —

of a symmetric $\mathcal{H}$-polytope when $\overline{p} \in \mathbb{N} \cup \infty$;

of an $\mathcal{H}$-polytope when $p \in \{1, \infty\}$;

of a $\mathcal{V}$-polytope when $p = 1$.

The statement about diameters is obvious, but the other results appear to be new in the form stated. It is interesting that all of the powers mentioned in Theorem 1.1 are rational—obvious for diameters but perhaps not entirely so in the other cases.

1.2 APPROXIMATING THE INRADIUS AND THE CIRCUMRADIUS.
There are polynomial-time algorithms that do the following for polytopes in ℓ_p spaces:

for $\mathcal{V}$-polytopes, when $p \in \mathbb{N} \cup \{\infty\}$, approximate the circumradius;

for $\mathcal{H}$-polytopes, when $p \in \mathbb{N} \cup \{\infty\}$, approximate the inradius.

1.3 MEMBERSHIP IN $\mathbb{NP}$. *For presented polytopes in ℓ_p spaces, each of the following problems belongs to the class $\mathbb{NP}$ —*
when p is an integer or ∞:

lower-bounding the p^{th} power of the diameter;

upper-bounding the $\overline{p}^{\text{th}}$ power of the width;

when $p \in \{1, 2, \infty\}$,

lower-bounding the p^{th} power of the circumradius;

when $p \in \{1, \infty\}$,

upper-bounding the $\overline{p}^{\text{th}}$ power of the inradius.

In the other cases, we do not know whether the corresponding circumradius or inradius problem is in $\mathbb{NP}$.

1.4 QUESTION ABOUT THE INRADIUS AND THE CIRCUMRADIUS. *For presented polytopes in ℓ_p spaces, which of the following problems belongs to the class $\mathbb{NP}$:*
when $3 \le p < \infty$,

lower-bounding the p^{th} power of the circumradius?

when $2 \le p < \infty$,

upper-bounding the $\overline{p}^{\text{th}}$ power of the inradius?

Some additional remarks and results concerning the "algebraic intractability" of these radii and some related questions are contained in Section 4.

In proving the $\mathbb{NP}$-hardness of certain radius computations, we proceed by transformation from two problems that are known to be $\mathbb{NP}$-hard, MAXNORM$_p$ and PARTITION. For each fixed $p \in \mathbb{N}$, MAXNORM$_p$ is the problem of lower-bounding the function $\| \ \|_p^p$ (the p^{th} power of the ℓ_p norm) on an $\mathcal{H}$-presented parallelotope centered at the origin in $\mathbb{R}^n$. The $\mathbb{NP}$-hardness of this problem is proved in [BGKV90] by means of a sequence of transformations from the problem NOT-ALL-EQUAL-3SAT of [Sc78]. PARTITION is the well-known problem from the list of [Ka′72], proved to be $\mathbb{NP}$-hard by a transformation from the 3-SATISFIABILITY of [Co71] by way of 3-dimensional matching.

1.5 $\mathbb{NP}$-Hardness for Radii with Low Indices. *For each fixed $p \in \mathbb{N}$ and each fixed $j \in \mathbb{N}$, each of the following problems is $\mathbb{NP}$-hard for $\mathcal{H}$-presented parallelotopes in ℓ_p spaces:*

> *lower-bounding $r_j(P)^p$;*
>
> *upper-bounding $R_{n+1-j}(P)^p$.*

For each fixed $\overline{p} \in \mathbb{N}$ and each fixed $j \in \mathbb{N}$, each of the following problems is $\mathbb{NP}$-hard for $\mathcal{V}$-presented cross-polytopes in ℓ_p spaces:

> *upper-bounding $R_j(P)^{\overline{p}}$;*
>
> *lower-bounding $r_{n+1-j}(P)^{\overline{p}}$.*

For $p \in \{2, \infty\}$ and each fixed $j \in \mathbb{N}$ the following problem is $\mathbb{NP}$-hard in ℓ_p spaces:

> *upper-bounding $R_j^p(P)$ for presented simplices.*

1.6 $\mathbb{NP}$-Hardness for Radii with High Indices. *For each fixed $p \in \mathbb{N}$, and fixed $\beta, \gamma \in \mathbb{N}$ with $\beta < \gamma$, each of the following problems is $\mathbb{NP}$-hard for $\mathcal{H}$-presented parallelotopes in the spaces $\mathbb{R}_p^{\gamma n}$:*

> *lower-bounding $r_{\beta n}(P)^p$;*
>
> *upper-bounding $R_{\beta n}(P)^p$.*

For each fixed $\overline{p} \in \mathbb{N}$, and fixed $\beta, \gamma \in \mathbb{N}$ with $\beta < \gamma$, each of the following problems is $\mathbb{NP}$-hard for $\mathcal{V}$-presented cross-polytopes in the spaces $\mathbb{R}_p^{\gamma n}$:

> *upper-bounding $R_{\beta n}(P)^{\overline{p}}$;*
>
> *lower-bounding $r_{\beta n}(P)^{\overline{p}}$.*

In fact, Theorems 5.7 and 5.10 contain slightly stronger results.

Our results for the circumradius solve problems proposed in [Ka″89]. The main problem that remains open is to determine the computational complexity of the width for $\mathcal{H}$-polytopes in ℓ_p spaces when $p \in \mathbb{N} \setminus \{1, 2\}$.

1.7 QUESTION ABOUT THE WIDTH. *For $p \in \mathbb{N}$ but $p \geq 3$, what is the computational complexity of upper-bounding $R_1(P)$ for $\mathcal{H}$-presented polytopes in ℓ_p spaces?*

Some remarks on this problem appear in Section 5.

Theorems 1.1 – 1.3 and 1.5 – 1.6 apply to polytopes in ℓ_p spaces of unrestricted dimension. There is also interest in the existence of polynomial-time algorithms for polytopes in $\mathbb{R}^n_p$ for a fixed dimension n. For fixed n, the conversion between $\mathcal{V}$-presentations and $\mathcal{H}$-presentations can be accomplished in polynomial time, so in this case the conclusions of Theorems 1.1 and 1.3 apply without specifying the manner of presentation. Hence, for the most part the following theorem is a corollary to Theorem 1.1. In addition, the symmetry restriction can be removed in the case of the width.

1.8 MEMBERSHIP IN $\mathbb{P}$ FOR FIXED DIMENSION. *For presented polytopes in fixed $\mathbb{R}^n_p$, each of the following can be computed in polynomial time:*

the p^{th} power of the diameter —

when $p \in \mathbb{N} \cup \infty$;

the p^{th} power of the circumradius —

when $p \in \mathbb{N} \cup \infty$ and the polytopes are symmetric;

when $p \in \{1, 2, \infty\}$;

the $\overline{p}^{\text{th}}$ power of the width —

when $\overline{p} \in \mathbb{N} \cup \infty$;

the $\overline{p}^{\text{th}}$ power of the inradius —

when $\overline{p} \in \mathbb{N} \cup \infty$ and the polytopes are symmetric;

when $\overline{p} \in \{1, \infty\}$.

For presented polytopes in fixed $\mathbb{R}^n_p$ there are polynomial-time algorithms that do the following:

when $p \in \mathbb{N} \cup \{\infty\}$, approximate the circumradius;

when $\overline{p} \in \mathbb{N} \cup \{\infty\}$, approximate the inradius.

For low-dimensional spaces, there are algorithms more efficient than the ones presented here. However, since our emphasis lies only on deciding polynomial time computability or $\mathbb{NP}$-hardness we do not go into details here. The interested reader might want to look at [Ed87], [HT85], [PS85], [KD82], [Me83] and at the literature cited in these books and papers.

All of the preceding statements have dealt with the $\mathbb{P}$- or $\mathbb{NP}$-computability, or the $\mathbb{NP}$-hardness, of various radii in spaces of arbitrary (though sometimes fixed) dimension. From the viewpoint of applications, there is particular interest in the complexity of radius computations in $\mathbb{R}_1^n$, $\mathbb{R}_\infty^n$ and especially $\mathbb{R}_2^n$ for $n = 2$ and $n = 3$. The principal open question along these lines is the following:

1.9 QUESTION ABOUT INNER AND OUTER 2-RADII IN 3-SPACE. *What is the complexity of computing $r_2(P)$ and $R_2(P)$ for polytopes P in $\mathbb{R}_2^3$?*

The answer to this question is unclear even when P is a 3-simplex (tetrahedron). [GK90a] contains some results that are related to Problem 1.9, and the relationship of $R_2(P)$ to robotics is discussed in Section 6.

2. Computational Preliminaries.

In studying the complexity of computing radii, we make frequent use of the fact ([Kh80], [Ka84]) that there is a polynomial-time algorithm for linear programming (both for optimization and for feasibility tests). One of the consequences of this is that, whether we are dealing with $\mathcal{V}$-polytopes or $\mathcal{H}$-polytopes, it may be assumed that the presentation is irredundant.

2.1 REMOVING REDUNDANCY. *When a polytope is $\mathcal{V}$-presented or $\mathcal{H}$-presented, an irredundant presentation of the same sort can be obtained in polynomial time.*

Proof. When a polytope P is given by means of a $\mathcal{V}$-presentation,

$$P = \text{conv}\{v_1, \dots, v_m\}, \quad \text{with } v_i = (\nu_{i1}, \dots, \nu_{in}),$$

the following algorithm uses a number of linear feasibility tests to produce an irredundant $\mathcal{V}$-presentation $P = \text{conv}\{v_i : i \in K\}$.

begin
  $K \leftarrow \{1, \dots, m\}$;
  for $k \leftarrow 1$ **until** m **do**
    begin
      if there exist coefficients $\gamma_i \geq 0$ such that
        $\sum_{i \in K \setminus \{k\}} \gamma_i = 1$ and $\nu_{kj} = \sum_{i \in K \setminus \{k\}} \gamma_i \nu_{ij}$ for $1 \leq j \leq n$
      then $K \leftarrow K \setminus \{k\}$
    end
end.

When a polytope P is given by means of an $\mathcal{H}$-presentation,

$$P = \bigcap_{i=1}^{m} \{x : \sum_{j=1}^{n} \alpha_{ij} \xi_j \leq \beta_i\},$$

the following algorithm solves a number of linear programs to produce an irredundant $\mathcal{H}$-presentation

$$P = \bigcap_{i\in K} \{x : \sum_{j=1}^{n} \alpha_{ij}\xi_j \leq \beta_j\}.$$

begin
  $K \leftarrow \{1, \ldots, m\}$;
  for $k \leftarrow 1$ **until** m **do**
    begin
      compute the maximum δ_k of $\sum_{j=1}^{n} \alpha_{kj}\xi_j$
        over the intersection $\bigcap_{i\in K\setminus\{k\}} \{x : \sum_{j=1}^{n} \alpha_{ij}\xi_j \leq \beta_j\}$;
      if $\delta_k \leq \beta_k$ **then** $K \leftarrow K \setminus \{k\}$
    end
end.

□

Symmetric polytopes play a special role in our study. In this connection, it should be noted that from either sort of presentation of a polytope P, it is possible to decide in polynomial time whether P is symmetric and, if so, to find its center of symmetry.

2.2 FINDING THE CENTER. *There are algorithms which, when applied to a presented polytope P, determine in polynomial time whether P is symmetric, and find its center of symmetry if it has one.*

Proof. Suppose first that P is given by means of an irredundant $\mathcal{V}$-presentation. Using the square of the ℓ_2 norm, find two vertices v_j and v_k at maximum distance, and let $w = \frac{1}{2}(v_j + v_k)$. If P is symmetric, then w must be the center of symmetry (see A.6). Hence the polytope P is symmetric if and only if it is true for each vertex v_i that the point $2w - v_i$ is also in the list of vertices.

Now consider the case of an irredundant $\mathcal{H}$-presentation, $Ax \leq b$. Each of the inequalities $a_i x \leq b_i$ (where the a_i's are the row-vectors of A) defines a halfspace whose bounding hyperplane is the affine hull of a facet of P. For each i, solve the linear programming problem of minimizing $a_i x$ over P, denote the minimum by γ_i and set $c = (\gamma_1, \ldots, \gamma_m)$. Then P is symmetric if and only if the linear system $Ax = \frac{1}{2}(b + c)$ has a solution. If the solution exists, it is unique and is the center of symmetry. □

The following well-known facts are used in some proofs of membership in the class $\mathbb{NP}$.

2.3 GUESSING A VERTEX. *Suppose that P is an $\mathcal{H}$-presented polytope in $\mathbb{R}^n$, defined by a system $Ax \leq b$ of linear inequalities. For each set I consisting of n indices in $1, \ldots, m$, it can be determined in polynomial time whether the system of equations*

$$a_i x = \beta_i \qquad (i \in I)$$

has a unique solution v. Each such v can be tested in polynomial time for membership in P, and P's vertex-set consists precisely of the unique solutions v that pass this test.

2.4 GUESSING A FACET-INDUCING INEQUALITY. *Suppose that P is a $\mathcal{V}$-presented polytope in $\mathbb{R}^n$, given as $\operatorname{conv}\{v_1, \ldots, v_m\}$. For each set I consisting of n indices in $1, \ldots, m$, it can be determined in polynomial time whether the points of $\{v_i : i \in I\}$ are affinely independent. If this is the case a rational normal vector $z \in (\mathbb{R}^n)^*$ and a rational "right-hand side" β can be computed in polynomial time (such that $\langle v_i, z\rangle = \beta$ for $i \in I$) and it can be tested in polynomial time whether all points $v_1, \ldots, v_m$ satisfy the inequality $\langle v, z\rangle \leq \beta$.*
If the pair (z, β) passes this test it gives a facet-inducing inequality of P.

For each fixed dimension n, Theorems 2.3 and 2.4 provide the basis for polynomial-time algorithms that converts an $\mathcal{H}$-presentation of an n-polytope P into a $\mathcal{V}$-presentation of P and vice versa. Much better algorithms are available for that purpose, but this one is adequate for our present needs. Note, however, that when the dimension is unrestricted and a conversion from a $\mathcal{V}$-presentation to an $\mathcal{H}$-presentation (or vice-versa) is needed, the size of the output is not in general bounded by any polynomial in the size of the input. Indeed,

$$\binom{k - \lfloor (n+1)/2 \rfloor}{k-n} + \binom{k - \lfloor (n+2)/2 \rfloor}{k-n}$$

is the maximum number of vertices of n-polytopes with k facets and also the maximum number of facets of n-polytopes with k vertices [Mc70]. Thus it is not surprising that the complexity of some radius computations depends on the manner in which the polytope is presented. For studies of the computational complexity of passing from one sort of polytope presentation to the other, see [Dy83], [Sw85] and [Se87].

In some cases when computing certain radii is difficult it is still possible to approximate or to bound them. The following discusses the close relationship between bounding and approximation of polytope radii.

2.5 FROM BOUNDING TO APPROXIMATION. *Suppose that p and j are positive integers, and let ρ denote the inner j-radius r_j or the outer j-radius R_j, defined for each polytope P in a space $\mathbb{R}^n_p$ with $n \geq j$. Let the term "$\mathcal{W}$-polytope" mean either $\mathcal{V}$-presented polytope or $\mathcal{H}$-presented polytope. If there is a polynomial-time algorithm for upper-bounding or for lower-bounding ρ on $\mathcal{W}$-polytopes, then there is a polynomial-time algorithm for approximating ρ on these polytopes.*

Proof. We want to construct an algorithm that finds, for each $\mathcal{W}$-polytope P and each positive rational λ, a rational number μ with $|\mu - \rho(P)| < \lambda$, and such that the algorithm's running time is bounded by a polynomial in the size of (P, λ). Note first that from either sort of presentation of P, the same sort of presentation of the polytope $Q = (1/\lambda)P$ can be obtained in polynomial time; and since $P \subset 2^{4L}\sqrt{n}\mathbb{B}^n_p$ it is true that

$\rho(Q) \leq (1/\lambda)2^{4L}\sqrt{n}$. Hence by combining the procedure for lower-bounding or upper-bounding ρ with an appropriate number of bisections of intervals, we can compute in polynomial time an integer τ with $\tau \leq \rho(Q) < \tau + 1$ $\langle \tau < \rho(Q) \leq \tau + 1$, respectively $\rangle$. The rational number $\mu = \frac{1}{2}\lambda(2\tau + 1)$ is then a desired approximation of $\rho(P)$. □

2.6 FROM APPROXIMATION TO BOUNDING. *With notation and hypotheses as in 2.5, suppose in addition that there are strictly increasing polynomials ϕ and ψ on $[0, \infty[$ for which the following is true —*

> *for each $\mathcal{W}$-presentation of size L of a polytope P in $\mathbb{R}^n_p$ there is a polynomial q that satisfies the following conditions:*
>
> *the degree of q is at most $\phi(L)$;*
>
> *q's coefficients are integers of absolute value at most $\psi(2^L)$;*
>
> *$\rho(P)$ is the only positive root of q.*

If there is a polynomial algorithm for approximating ρ on $\mathcal{W}$-polytopes, there is also a polynomial algorithm for lower-bounding and upper-bounding ρ on these polytopes.

Proof. It is assumed that p and j are fixed positive integers, $\rho = r_j$ or $\rho = R_j$, and $\mathcal{W} = \mathcal{V}$ or $\mathcal{W} = \mathcal{H}$. For each positive integer β, let $\mathcal{P}_\beta$ denote the set of all $\mathcal{W}$-polytopes of size $L \leq \beta$ situated in spaces $\mathbb{R}^n_p$ with $n \geq j$.

Let $\alpha \in \mathbb{Q}$, and let $\alpha_1, \alpha_2 \in \mathbb{Z}$ such that $\alpha = \alpha_1/\alpha_2$. Further, let $q(\xi) = \sum_{i=0}^k \gamma_i \xi^i$ and suppose that $q(\alpha) \neq 0$ but $q(\xi_0) = 0$. With $t(\xi) = q(\xi + \alpha)$ and $\xi_1 = \xi_0 - \alpha$, we have $t(0) \neq 0$ and $t(\xi_1) = 0$. If $|\xi_1| < 1$ then

$$\frac{1}{|\alpha_2|^k} \leq |t(0)| = |t(\xi_1) - t(0)| = \left| \sum_{i=1}^k \left(\sum_{h=i}^k \binom{h}{i} \gamma_h \alpha^{h-i} \right) \xi_1^i \right|$$

$$\leq |\xi_1| \sum_{i=1}^k \sum_{h=i}^k \binom{h}{i} |\gamma_h| |\alpha^{h-i}| < |\xi_0 - \alpha| |\alpha_1|^{\phi(L)} \psi(2^L) 2^{\phi(L)+1}.$$

Now, set

$$\tau_\beta = 2^{\beta\phi(\beta)} \psi(2^\beta) 2^{\phi(\beta)+1}.$$

This implies that

> whenever $P \in \mathcal{P}_\beta$ and $\rho(P)$ is not a rational of size at most β, the number $\rho(P)$ differs from the nearest such rational by more than $1/\tau_\beta$.

Now use the approximation algorithm to find in polynomial time a rational number μ such that $|\mu - \rho(P)| < \frac{1}{2\tau_\beta}$. Then, for a prospective bound λ for the upper-bounding problem we proceed as follows: For $\lambda < \mu - 1/2\tau_\beta$ we report "no", while for $\lambda \in [\mu - 1/2\tau_\beta, \mu + 1/2\tau_\beta]$ we have $\rho(P) = \lambda$ and hence report "yes"; and for $\lambda > \mu + 1/2\tau_\beta$ we also report "yes". This solves the problem of upper-bounding ρ correctly in polynomial time.

The same arguments apply also to the case of lower-bounding. □

3. Radius Computations for Polytopal Norms.

If the norm in the Minkowski space M is unrestricted the use of certain methods (similar to those provided by A.11 and A.12) for computing or bounding the radii of a polytope $P \subset \mathsf{M}$ is rather limited. However, such methods are applicable when M's unit ball $\mathbb{B}$ is itself a polytope. It is assumed in the present section that $\mathbb{B}$ is given irredundantly in the form

$$(\mathrm{B_{ver}}) \qquad \mathbb{B} = \mathrm{conv}\{q_1, \ldots, q_k\}$$

or the form

$$(\mathrm{B_{hyp}}) \qquad \mathbb{B} = \bigcap_{i=1}^{k} \{x \in \mathsf{M} : \langle x, y_i \rangle \leq 1\},$$

and that the polytope $P \subset \mathsf{M}$ is given irredundantly in the form

$$(\mathrm{P_{ver}}) \qquad P = \mathrm{conv}\{v_1, \ldots, v_m\}$$

or the form

$$(\mathrm{P_{hyp}}) \qquad P = \bigcap_{i=1}^{m} \{x \in \mathsf{M} : \langle x, a_i \rangle \leq \beta_i\}.$$

Of course, $\mathbb{B}$ is symmetric, though some of the results hold even for asymmetric unit balls. When P is assumed to be symmetric, that is indicated by writing ($\mathrm{symP_{ver}}$) or ($\mathrm{symP_{hyp}}$). It is also assumed, where convenient, that bases for M and M^* are specified and points of these spaces are given by means of their coordinates with respect to the bases.

In this section only, our model of computation is infinite-precision real arithmetic, but because we use only polytopal calculations the methods turn out to be applicable to the Turing machine model as well. As the term is used here, a *polytopal calculation* is any of the following:

(i) forming the sum, difference, product or quotient of two real numbers; deciding whether a given real number is nonnegative;

(ii) multiplying a point of M by a real scalar; forming the sum or difference of two points of M; evaluating a linear functional at a point of M;

(iii) testing the consistency of a finite system of linear inequalities,

$$\langle x, \varphi_i \rangle \leq \gamma_i \qquad (1 \leq i \leq m),$$

where the γ_i's are real numbers and the φ_i's are linear functionals on M; maximizing a linear functional $\varphi_0(x)$ subject to the inequalities (when the maximum is known to exist).

The *level* of a polytopal calculation is defined to be 1 in case (i), n (= $\dim \mathbb{M}$) in case (ii), and mn in case (iii). These crude measures suffice for our purposes because we are concerned only with polynomial-time computability. With respect to each of the situations (B_{ver},P_{ver}), (B_{ver},P_{hyp}), (B_{hyp},P_{ver}), and (B_{hyp},P_{hyp}), the term *efficient polytopal algorithm* means a sequence of polytopal calculations, the sum of whose levels is bounded by a fixed polynomial in k, m and n.

3.1 POLYTOPAL ALGORITHMS FOR THE DIAMETER. *There is an efficient polytopal algorithm that computes P's diameter in the situations (B_{ver}, P_{ver}), (B_{hyp}, P_{ver}), (B_{hyp}, P_{hyp}).*

Proof. Suppose first that (B_{hyp}) holds. Then the diameter of P is given by

$$2r_1(P) = \max\{\delta_1, \ldots, \delta_k\},$$

where $\delta_j = \max_{u,x \in P}\langle u - x, y_j\rangle$. For ($B_{hyp}$,$P_{ver}$), observe that

$$\delta_j = \max_{h,i \in \{1,\ldots,m\}} \langle v_h - v_i, y_j\rangle$$

and hence the main step in computing P's diameter is $O(km^2)$ evaluations of linear functionals on $\mathbb{M}$. The sum of the levels of the necessary polytopal calculations is $O(km^2n)$.

For (B_{hyp},P_{hyp}), observe that the "hyp" representation of P in $\mathbb{M}$ yields a similar representation (using $2m$ linear inequalities) of $P \times P$ in $\mathbb{M} \times \mathbb{M}$. Since

$$\delta_j = \max_{(u,x) \in P \times P}(\langle u, y_j\rangle - \langle x, y_j\rangle),$$

the main step in computing P's diameter is the solution of k linear programs over $P \times P$. The sum of the levels is $O(kmn)$. (The fact that the sum of the levels for (B_{hyp},P_{hyp}) is asymptotically smaller than for (B_{hyp},P_{ver}) is an artifact of our crude definition of level, and has no significance for practical computation.)

Now suppose that (B_{ver},P_{ver}) holds, and let

$$W = \{w_1, \ldots, w_r\} = \{v_h - v_i : 1 \leq h, i \leq m\}.$$

Then

$$2r_1(P) = \min\{\lambda > 0 : W \subset \lambda\mathbb{B}\}.$$

Since $\mathbb{B} = \operatorname{conv}\{q_1, \ldots, q_k\}$, the diameter of P is the inverse of the solution of the problem of maximizing μ subject to the constraints

$$\sum_{j=1}^{k} \tau_{ij} q_j = \mu w_i \qquad (1 \leq i \leq r)$$

$$\sum_{j=1}^{k} \tau_{ij} = 1 \qquad (1 \leq i \leq r)$$

$$\tau_{ij} \geq 0 \qquad (1 \leq i \leq r, 1 \leq j \leq k).$$

Here we have $(n + 1)r \leq (n + 1)m^2$ equality constraints and $kr \leq km^2$ nonnegativity constraints for kr variables, so the total level of the polytopal calculations is $O(k^2m^4n)$. □

3.2 POLYTOPAL ALGORITHMS FOR THE WIDTH. *There is an efficient polytopal algorithm that computes P's width in the situations* $(B_{\mathrm{ver}}, P_{\mathrm{ver}})$, $(B_{\mathrm{ver}}, P_{\mathrm{hyp}})$, $(B_{\mathrm{hyp}}, \mathit{sym}P_{\mathrm{hyp}})$.

Proof. With $D = \frac{1}{2}(P - P)$, ($\mathrm{P}_{\mathrm{ver}}$) yields ($\mathrm{D}_{\mathrm{ver}}$) by means of polytopal calculations the sum of whose levels is $O(m^2n)$, and ($\mathrm{symP}_{\mathrm{hyp}}$) is ($\mathrm{D}_{\mathrm{hyp}}$). Using polarity, ($\mathrm{B}_{\mathrm{ver}}$), ($\mathrm{B}_{\mathrm{hyp}}$), ($\mathrm{D}_{\mathrm{ver}}$), and ($\mathrm{D}_{\mathrm{hyp}}$) are equivalent respectively to ($\mathrm{B}^*_{\mathrm{hyp}}$), ($\mathrm{B}^*_{\mathrm{ver}}$), ($\mathrm{D}^\circ_{\mathrm{hyp}}$), and ($\mathrm{D}^\circ_{\mathrm{ver}}$). Since $R_1(P) = R_1(D) = 1/r_1(D^\circ)$ by A.2 and A.3, the entries ($\mathrm{B}_{\mathrm{hyp}}$,$\mathrm{P}_{\mathrm{hyp}}$), ($\mathrm{B}_{\mathrm{hyp}}$,$\mathrm{P}_{\mathrm{ver}}$), and ($\mathrm{B}_{\mathrm{ver}}$,$\mathrm{P}_{\mathrm{ver}}$) in Theorem 3.1 justify the respective entries ($\mathrm{B}_{\mathrm{ver}}$,$\mathrm{P}_{\mathrm{ver}}$), ($\mathrm{B}_{\mathrm{ver}}$,$\mathrm{symP}_{\mathrm{hyp}}$), and ($\mathrm{B}_{\mathrm{hyp}}$,$\mathrm{symP}_{\mathrm{hyp}}$) in Theorem 3.2. There remains the consideration of ($\mathrm{B}_{\mathrm{ver}}$,$\mathrm{P}_{\mathrm{hyp}}$) for asymmetric P.

Now suppose that ($\mathrm{B}_{\mathrm{ver}}$,$\mathrm{P}_{\mathrm{hyp}}$) holds, and recall from A.5 that

$$2R_1(P) = \min_{s \in \mathbb{S}} l_s(P),$$

where $l_s(P)$ is the length of the longest segment in P that is parallel to the line $\mathbb{R}s$. This implies that the following three statements are equivalent:

(i) $2R_1(P) \geq \lambda$;

(ii) for each $z \in \mathbb{M}$ with $\|z\| \leq \lambda$ there exists $x \in P$ such that $x + z \in P$;

(iii) $\lambda\mathbb{B}$ is contained in the set of all $z \in \mathbb{M}$ for which the system

$$\langle x, a_i \rangle \leq \min\{\beta_i, \beta_i - \langle z, a_i \rangle\} \qquad (1 \leq i \leq m)$$

admits a solution x.

The set of all such z is easily seen to be convex (in fact, it is a polytope), and hence when $\mathbb{B} = \mathrm{conv}\{q_1, \ldots, q_k\}$ the conditions are equivalent to the following:

(iv) for $1 \leq j \leq k$, the system

$$\langle x, a_i \rangle \leq \min\{\beta_i, \beta_i - \lambda\langle q_i, a_i \rangle\} \qquad (1 \leq i \leq m)$$

is consistent.

Hence the width is the solution to the following linear program

$$\begin{aligned} &\max \lambda \\ \langle x_j, a_i \rangle &\leq \beta_i \qquad (1 \leq i \leq m;\ 1 \leq j \leq k) \\ \langle x_j, a_i \rangle + \lambda\langle q_i, a_i \rangle &\leq \beta_i \qquad (1 \leq i \leq m;\ 1 \leq j \leq k). \end{aligned}$$

This linear program can be solved by polytopal calculations the sum of whose levels is $O(k^2mn)$. □

3.3 Polytopal Algorithms for the Inradius. *There is an efficient polytopal algorithm that computes P's inradius in the situations $(B_{\text{ver}}, P_{\text{ver}})$, $(B_{\text{ver}}, P_{\text{hyp}})$, $(B_{\text{hyp}}, P_{\text{hyp}})$.*

Proof. For $(\mathrm{B}_{\text{ver}}, \mathrm{P}_{\text{ver}})$ the inradius $r_n(P)$ is the solution of the following linear program:

$$\begin{aligned} &\max \rho \\ \sum_{j=1}^{m} \tau_{ij} v_j - \rho q_i - c = 0 \qquad &(1 \le i \le k) \\ \sum_{j=1}^{m} \tau_{ij} = 1 \qquad &(1 \le i \le k) \\ \tau_{ij} \ge 0 \qquad &(1 \le i \le k, 1 \le j \le m). \end{aligned}$$

Thus $r_n(P)$ can be computed by means of polytopal calculations the sum of whose levels is $O(k^2 m^2 n^2)$. Notice that in an optimal solution c is the center of an insphere.

For the two remaining cases we use the fact that by A.12, $r_n(P)$ is the solution of the linear program

$$\begin{aligned} &\max \rho \\ \langle x, a_i \rangle + \rho \|a_i\|^* \le \beta_i \qquad &(1 \le i \le m) \end{aligned}$$

which can be solved by means of polytopal calculations the sum of whose levels is $O(mn)$ provided we know the norms $\|a_1\|^*, \ldots, \|a_m\|^*$. Under the assumption $(\mathrm{B}_{\text{ver}})$,

$$\|a\|^* = \max_{1 \le i \le k} \langle a, q_i \rangle,$$

and under the assumption $(\mathrm{B}_{\text{hyp}})$, $\|a\|^*$ is the solution of the linear program

$$\begin{aligned} &\max \langle a, x \rangle \\ \langle x, y_i \rangle \le 1 \qquad &(1 \le i \le k). \end{aligned}$$

Thus the total level of polytopal calculations is $O(kmn)$. $\square$

3.4 Polytopal Algorithms for the Circumradius. *There is an efficient polytopal algorithm that computes P's circumradius in the situations $(B_{\text{hyp}}, P_{\text{ver}})$, $(B_{\text{hyp}}, P_{\text{hyp}})$. Furthermore, in the situation $(B_{\text{ver}}, P_{\text{ver}})$ there is an efficient polytopal algorithm that upper bounds P's circumradius.*

Proof. Suppose first that $(\mathrm{B}_{\text{ver}}, \mathrm{P}_{\text{ver}})$ holds, and consider an arbitrary $\lambda > 0$. Then $R_n(P) \le \lambda$ if and only if the following linear system is consistent:

$$\begin{aligned} \sum_{j=1}^{k} \tau_{ij} \lambda q_j + c = v_i \qquad &(1 \le i \le m) \\ \sum_{j=1}^{k} \tau_{ij} = 1 \qquad &(1 \le i \le m) \\ \tau_{ij} \ge 0 \qquad &(1 \le i \le m, 1 \le j \le k). \end{aligned}$$

In the case of consistency, c becomes the center of a ball of radius λ that contains P. The total level of the polytopal calculations is $O(m^2k^2n^2)$.

Now consider the case of $(\mathrm{B}_{\mathbf{hyp}}, \mathrm{P}_{\mathbf{ver}})$. The circumradius of P is the solution of the following linear program, which can be solved with the aid of polytopal calculations of total level $O(kmn)$:

$$\min \rho$$
$$\rho + \langle c, y_i \rangle \geq \langle v_j, y_i \rangle \qquad (1 \leq i \leq k, 1 \leq j \leq m).$$

Finally, in the case of $(\mathrm{B}_{\mathbf{hyp}}, \mathrm{P}_{\mathbf{hyp}})$, for $1 \leq i \leq k$ let δ_i denote the solution of the linear program

$$\max \langle x, y_i \rangle$$
$$\langle a_j, x \rangle \leq \beta_j \qquad (1 \leq j \leq m).$$

Then $R_n(P)$ is the solution of the linear program

$$\min \rho$$
$$\rho + \langle y_i, c \rangle \geq \delta_i \qquad (1 \leq i \leq k).$$

In this case, the total level of the polytopal calculations is $O(kmn)$. $\square$

In closing this section, we mention once more that the results given here are closely related to the parts of Theorem 1.5 that deal with ℓ_1 or ℓ_∞ spaces. In fact, the assumptions $(\mathrm{B}_{\mathrm{ver}})$ and $(\mathrm{B}_{\mathbf{hyp}})$ correspond to the use of the 1-norm and the ∞-norm respectively. This will be used in Section 4.

Let us further remark that the results here are best possible in the sense that the situations that are not listed in Theorems 3.1 – 3.4 contain instances of NP-hard problems.

4. Radius Computations in P and NP.

We begin with some remarks concerning the algebraic tractability of certain radii of rational polytopes in ℓ_p spaces. Attention is confined to the case in which p or $\overline{p}$ is a positive integer or ∞. We focus on the following question, which is intimately related to computational aspects of the radii:

For which p and for which radii $\rho = r_j$ or $\rho = R_j$ does there exist a nonconstant rational polynomial q such that the number $q(\rho(P))$ is rational whenever P is a rational polytope (or a symmetric rational polytope) in an ℓ_p space?

It seems the answer is "Not many!", at least when P is permitted to be asymmetric. For each $p, \alpha \in \mathbb{N}$ let $r_2(p; \alpha)$ denote the inradius of the triangle $\mathrm{conv}\{(0,0),(1,0),(0,\alpha)\}$ and let $R_2(p;\alpha)$ denote the circumradius of the triangle $\mathrm{conv}\{(-1,0),(0,\alpha),(1,0)\}$, both radii being measured in the Minkowski plane $\mathbb{R}^2_p$. It is proved in [GHK90] that each nonconstant rational polynomial q has the following two properties:

for each $p \geq 3$, $q(R_2(p;\alpha))$ is irrational for infinitely many α;

$q(r_2(2;\alpha))$ is irrational for all but finitely many α.

Hence the circumradii and inradii in question are in a sense computationally intractable, so far as precise computation (even of an implicit nature) in the binary model is concerned. It is conjectured in [GHK90] that in the result for inradii, $r_2(2,\alpha)$ may be replaced with $r_2(p;\alpha)$ for $p \geq 2$. The intractability of circumradii and inradii extends to higher dimensions.

On the other hand, there are a few pairs (ρ, p) for which the powers ρ^p or $\rho^{\overline{p}}$ must be rational, thus providing very simple rationalizing polynomials. The pairs that we know about are described in A.13, and the sizes of the radii in question are all bounded by at most $16pn^4L$ or $16\overline{p}n^4L$. These bounds are used later when it is shown that the radii in question can be computed precisely by approximation followed by suitable rounding.

Now let us turn to Theorem 1.1. For a $\mathcal{V}$-polytope P, an irredundant $\mathcal{V}$-presentation can be produced in polynomial time and this yields P's vertex-set. Since P's diameter is attained as the distance between two vertices (A.6), 1.1's assertion about the diameter of a $\mathcal{V}$-polytope is obvious. In view of A.3, this also implies 1.1's statement about the circumradius of a symmetric $\mathcal{V}$-polytope.

In view of A.2, Theorem 1.1's statements about the width and inradius of symmetric $\mathcal{H}$-polytopes follow by polarity. For ℓ_1 spaces and ℓ_∞ spaces, the assertions of Theorem 1.1 follow from Theorems 3.1 – 3.4 — for the circumradius in ℓ_1 spaces, with the additional aid of Theorem 2.5 and A.13. Thus the only part of Theorem 1.1 that remains to be established is its assertion about the circumradius of an asymmetric $\mathcal{V}$-polytope in an ℓ_2 space.

4.1 USING THE ELLIPSOID METHOD TO APPROXIMATE THE CIRCUMRADIUS IN ℓ_p SPACES. *For a finite subset W of an ℓ_p space, the circumradius can be approximated in polynomial time by means of the ellipsoid algorithm.*

Proof. For $w \in W$ and $x \in \mathbb{R}^n_p$, let $\varphi_w(x) = \|x - w\|_p$ and set $\Phi(x) = \sup_{w \in W} \varphi_w(x)$. By A.11, Φ is a convex contraction. The ellipsoid method, in the form originally suggested [Sh'85] for the minimization of convex functions, can be applied to the function Φ^p for an arbitrary integer $p \geq 1$. However, we have seen already that the circumradius can be computed in polynomial time in ℓ_1 spaces, so for technical simplicity we assume in the following that $p \geq 2$. Since Φ^p is continuous and piecewise differentiable, to find the subgradient of Φ^p at a point x it suffices to find a w with $\varphi_w(x) = \Phi(x)$ and then take the gradient of φ^p at x. Let $\delta \in \mathbb{N}$. We want to approximate $R_n(P)^p$ up to an error at most $1/\delta$.

ELLIPSOID-ALGORITHM FOR Φ^p.

Initialize: $x_0 = 0, B_0 = I_n, \rho_0 = 2^{8L}, \kappa_0 = \frac{2^{4L}}{n+1}$, $k_t = \lceil 4n^2(6pL + \log(\delta) \rceil$.

Iteration: *For* $k = 0, \dots, k_t$

determine a w *such that* $\Phi(x_k) = \varphi_w(x_k)$

set $v' = B_k^T \nabla \varphi_w^p(x_k)$, $v = v'/\|v'\|_2$,

$$x_{k+1} = x_k - \kappa_k B_k v$$

$$B_{k+1} = B_k \left(I_n - \left[1 - \left(\tfrac{n-1}{n+1}\right)^{1/2} \right] vv^T \right)$$

$$\kappa_{k+1} = \kappa_k \left(\tfrac{n^2}{n^2-1}\right)^{1/2}.$$

$$\rho_{k+1} = \min\{\rho_k, \varphi_w^p(x_k)\}.$$

The iteration of the algorithm is well defined since B_k is nondegenerate and $v \neq 0$. The first statement follows from the fact that the orthogonal complement of $\mathrm{lin}\{v\}$ is the eigenspace of the matrix

$$I_n - \left(1 - \sqrt{\frac{n-1}{n+1}}\right) vv^T$$

with eigenvalue 1 and v is an eigenvector with eigenvalue $\sqrt{(n-1)/(n+1)}$. Hence, the determinant is nonzero. The second statement can be verified as follows. Since $v' = B_k^T \nabla \varphi_w^p(x_k)$ we have $v = 0$ if and only if $\nabla \varphi_w^p(x_k) = 0$ which, in turn, is equivalent to $x_k = w$. By the choice of φ_w, this means $R_n(P) = \Phi(x_k) = 0$, which is a contradiction.

Since the method converges linearly and not too slowly, we obtain the desired approximation. More precisely, using the results of [Sh′85] (see also [Ak84]) we obtain (with some calculation)

$$\rho_{k_t} \leq \alpha 2^{8L} e^{-\frac{k_t}{4n^2}},$$

where α is a uniform bound for the Euclidean norm of all gradients that may occur. Since $P \subset 2^{4L}\mathbb{B}_2^n$ and $\nabla \varphi_w(x) = p\|x - w\|_{p-1}^{p-1}$ we obtain readily $\alpha < 2^{(6p-1)L}$ and hence $\rho_{k_t} \leq 1/\delta$.

Let us finally point out that in our binary model of computation the normalization step requires rounding. This can, however, be done in such a way that the complexity of each step of the algorithm is polynomial and the rate of the convergence remains essentially the same (see [Lo′86], [GLS88]). □

4.2 Using the Ellipsoid Method to Compute the Circumradius in ℓ_2 Spaces. *For a finite subset W of an ℓ_2 space, the circumradius can be computed in polynomial time by means of the ellipsoid algorithm.*

Proof. Using the approach given in the proof of Theorem 4.1, the main thing that remains to be shown is that when we stop the regular iteration after a certain (polynomial) number of steps, we can then obtain the exact solution by rounding. But this follows from A.13's result that the size of the square of P's circumradius (as measured for the binary model of computation) is bounded by a polynomial in the size of the input. □

That completes the proof of Theorem 1.1 and also of the first part of Theorem 1.2. Note that, because of the algebraic intractability established in [GHK90], for $p \geq 3$ the result of Theorem 4.1 cannot be sharpened from "approximating" to "computing" as was done in Theorem 4.2 for the Euclidean case.

Next we deal with the inradius.

4.3 Approximating the Inradius. *In ℓ_p spaces for $\overline{p} \in \mathbb{N} \cup \{\infty\}$, the $\overline{p}^{\text{th}}$ power of the inradius of an $\mathcal{H}$-polytope can be approximated in polynomial time.*

Proof. Let $P = \{x : a_1 x \leq \beta_1, \ldots, a_m x \leq \beta_m\}$. By A.12, $r_n(P)$ is the solution of the linear program

$$\max \xi$$
$$a_i x + \xi \|a_i\|_{\overline{p}} \leq \beta_i \qquad (i = 1, \ldots, m).$$

Assume, without loss of generality, that $\|a_i\|_{\overline{p}} \geq 1$ for all i. In order to obtain an approximative algorithm in the binary model of computation, we approximate the norms of the a_i's and then solve the perturbed linear program. Denote by $\alpha_1, \ldots, \alpha_m$ the rationals obtained from $\|a_1\|_{\overline{p}}, \ldots, \|a_m\|_{\overline{p}}$, respectively, by rounding to k binary digits.

Furthermore, let ρ be the solution of the perturbed linear program

$$\max \xi$$
$$a_i x + \xi \alpha_i \leq \beta_i \qquad (i = 1, \ldots, m).$$

Observe that for every $x \in P$ and $i = 1, \ldots, m$

$$\begin{aligned} 0 &\leq \left(\frac{\beta_i}{\alpha_i} - \frac{a_i x}{\alpha_i}\right) - \left(\frac{\beta_i}{\|a_i\|_{\overline{p}}} - \frac{a_i x}{\|a_i\|_{\overline{p}}}\right) \\ &= (\beta_i - a_i x)\left(\frac{1}{\alpha_i} - \frac{1}{\| a_i \|_{\overline{p}}}\right) \\ &\leq 2^{2L-k}. \end{aligned}$$

Hence,

$$\rho + 2^{8L-k} \leq r_n(P) \leq \rho.$$

With $k = 8L + \lceil \log(\delta) \rceil$ the perturbed linear program gives the insphere radius up to an additive error $1/\delta$. □

That completes the proof of Theorem 1.2, and we turn next to the proof of Theorem 1.3.

4.4 Membership in NP. *For each $p \in \mathbb{N} \cup \{\infty\}$, each of the following problems belongs to the class* NP*:*

> *lower-bounding the p^{th} power of the diameter of a presented polytope in an ℓ_p space;*
>
> *upper-bounding the p^{th} power of the width of a presented polytope in an $\ell_{\overline{p}}$ space.*

Proof. The diameter of a $\mathcal{V}$-presented polytope does not require further consideration, for we have seen that its p^{th} power can be computed in polynomial time. Each of the remaining problems requires a suitable guessing algorithm and a checking algorithm whose running time is bounded by a polynomial in the size of the original input.

To lower-bound $r_1(P)^p$ for an $\mathcal{H}$-presented P, guess pairs $\{v, w\}$ of vertices of P (see Theorem 2.3) and note that by A.6, $r_1(P)^p \geq \lambda$ if and only if $\|v - w\|_p^p \geq 2^p \lambda$ for some $\{v, w\}$.

To upper-bound $R_1(P)^p$ for a $\mathcal{V}$-presented n-polytope

$$P = \operatorname{conv}\{v_1, \ldots, v_m\} \subset \mathbb{R}_p^n,$$

set $W = \{\frac{1}{2}(v_i - v_j) : 1 \leq i, j \leq m\}$ and note that by A.3, the symmetric polytope $Q = \frac{1}{2}(P - P) = \operatorname{conv}(W)$ has the same width as P.

Since $Q = -Q$, the polar

$$Q^\circ = \{x \in \mathbb{R}_p^n : \langle x, w \rangle \leq 1 \text{ for all } w \in W\}$$

is a symmetric $\mathcal{H}$-presented polytope in $\mathbb{R}_p^n$. Hence by A.2,

$$R_1(P) = R_1(Q) = 1/r_1(Q^\circ).$$

Then apply the method of the preceding paragraph to Q°, noting that $R_1(P)^p \leq \lambda$ if and only if $r_1(P)^p \geq \lambda$.

When an asymmetric P is $\mathcal{H}$-presented, the method of the preceding paragraph does not apply explicitly, because a polynomial-time symmetrization is not available. However, it is possible to proceed by guessing vertices in a way that will now be described. This method can also be applied directly to a $\mathcal{V}$-presented polytope.

Let $\mathcal{G}$ consist of all ordered $2n$-tuples of vertices of P. When the guessing algorithm has produced a member

$$G = (v_1, \ldots, v_{2n}) \in \mathcal{G},$$

proceed as follows:

(a) let A_G be the $n \times n$ matrix whose i^{th} row lists the coordinates of the vector $v_i - v_{i+n}$;

(b) for each $x = (\xi_1, \ldots, \xi_n)$, let $\varphi_G(x)$ denote the determinant of the following matrix:

$$\begin{bmatrix} & & \xi_1 \\ & A_G & \vdots \\ & & \xi_n \\ 1 & \ldots & 1 \end{bmatrix};$$

(c) for the linear functional φ_G, compute the coefficient of each of the ξ_i's as the value of an $n \times n$ determinant formed from the row of 1's and $n-1$ of A_G's rows;

(d) compute $\|\varphi_G\|_p^p$ for the functional $\varphi_G \in \mathbf{R}_p^n = (\mathbf{R}_{\bar{p}}^n)^*$;

(e) by means of linear programming (when P is $\mathcal{H}$-presented) or straightforward evaluation at vertices (when P is $\mathcal{V}$-presented), compute the minimum and the maximum of φ_G on P.

(f) if
$$\left(\frac{\max_{x\in P}\varphi_G(x) - \min_{x\in P}\varphi_G(x)}{\|\varphi_G\|_p}\right)^p \leq \lambda,$$
report that $(2R_1(P))^p \leq \lambda$.

Note that for each guess G, all of the prescribed computations can be accomplished in polynomial time. Hence the proof of membership in $\mathbb{NP}$ is completed by observing that by A.5, the actual width of P is equal to the maximum of the quotients whose p^{th} powers appear on the left side of (f). □

The previous results take care of the first two statements of Theorem 1.3. The rest of 1.3's assertions is contained in the following theorem.

4.5 More on Membership in $\mathbb{NP}$. *For presented polytopes in ℓ_p spaces, each of the following problems belongs to the class $\mathbb{NP}$:*

> *lower-bounding the p^{th} power of the circumradius when $p \in \{1, 2, \infty\}$;*
> *upper-bounding the inradius when $p \in \{1, \infty\}$.*

Proof. Since the circumradius of $\mathcal{V}$-presented polytopes in ℓ_p spaces can actually be computed in polynomial time for $p = 1, 2, \infty$, we only have to deal with $\mathcal{H}$-presented polytopes to prove the first statement. To lower-bound $R_n(P)^p$ for an $\mathcal{H}$-presented polytope $P \subset \mathbb{R}_p^n$, guess $(n+1)$-tuples $\{v_0, \ldots, v_n\}$ of vertices of P. By A.9, $R_n(P)^p \geq \lambda$ if and only if $R_n(\{v_0, \ldots, v_n\})^p \geq \lambda$ for some guess. And by Theorem 1.1, $R_n(\{v_0, \ldots, v_n\})^p$ can be computed in polynomial time for each guess.

For the inradius, by Theorem 1.1, the only remaining case is that of a $\mathcal{V}$-polytope P in $\mathbb{R}_\infty^n$. Here, we can apply Theorem 2.4 $n+1$ times; i.e., we guess $n+1$ facet-hyperplanes of P. Then we use the Helly-type characterization of the inradius given in A.10, and Theorem 1.1's result concerning the inradius of $\mathcal{H}$-presented polytopes. □

The asymmetry of the results in Theorems 4.4 and 4.5 appears to be related to the algebraic natures of circumradii and inradii. Even though 4.1 and 4.3 show that we can approximate suitable powers of both inradius and circumradius in polynomial time, it is not clear that the additional assumptions in Theorem 2.6 are satisfied. Hence, it is not clear whether, in general ℓ_p spaces, the problems of upper-bounding a suitable power of the inradius or of lower-bounding a suitable power of the circumradius belong to the clas $\mathbb{NP}$.

In [GHK90] it is shown that certain circumradii and certain inradii are in a sense computationally intractable for the binary model of computation employed here. The specific results were mentioned at the beginning of this section. However, many other questions about the *algebraic tractability of radii* are of interest from a geometric viewpoint and — as we have seen — are also of some importance for computational issues.

4.6 PROBLEM: COMPUTATIONAL TRACTABILITY OF INRADII. *Let p be a fixed integer greater than 2. Does there exist a rational polynomial q such that $q(r_n(P))$ is rational whenever P is a rational polytope in $\mathbb{R}^n_p$?*

In [GK90a] it is shown that the diameter and the width of rational polytopes are computationally tractable. It is not clear what happens with the "intermediate" radii.

4.7 PROBLEM: COMPUTATIONAL TRACTABILITY OF INTERMEDIATE RADII. *For which triples (p, n, j) does there exist a rational polynomial q such that $q(r_j(P))$ is rational whenever P is a rational polytope in $\mathbb{R}^n_p$? For which triples (p, n, j) does there exist a rational polynomial q such that $q(R_j(P))$ is rational whenever P is a rational polytope in $\mathbb{R}^n_p$?*

In Problems 4.6 and 4.7, a q that is strictly increasing on $[0, \infty[$ would be of special interest, because it would facilitate the comparison of radii between different polytopes.

Now suppose that p is a positive integer and P is a rational $\mathcal{H}$-presented polytope in $\mathbb{R}^n_{\overline{p}}$. Then, by A.12, $r_n(P)$ is the maximum of ξ subject to the constraints

$$\langle x, a_i\rangle + \|a_i\|_p \xi \leq \beta_i \quad \text{for} \quad i = 1, \ldots, m,$$

where the a_i's and β_i's are the parameters of the $\mathcal{H}$-presentation of P. The maximum is attained at a vertex of the relevant $(n+1)$-polytope, and hence there are indices – say, for notational convenience, $1, \ldots, n+1$ – such that v is the solution of the system

$$\begin{pmatrix} a_1 & \|a_1\|_p \\ \vdots & \vdots \\ a_{n+1} & \|a_{n+1}\|_p \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n+1} \end{pmatrix}.$$

Then by Cramer's rule the inradius is equal to the quotient

$$\begin{vmatrix} a_1 & \beta_1 \\ \vdots & \vdots \\ a_{n+1} & \beta_{n+1} \end{vmatrix} \begin{vmatrix} a_1 & \|a_1\|_p \\ \vdots & \vdots \\ a_{n+1} & \|a_{n+1}\|_p \end{vmatrix}^{-1}.$$

Here the numerator is an integer (bounded by 2^L) and the denominator is of the form

$$\sum_{i=1}^{n+1} \alpha_i \|a_i\|_p,$$

where the numbers α_i and $\|a_i\|_p^p$ are integers. Hence, the following problem is of interest for the task of deciding whether upper bounding (suitable powers) of the inradius is in the class $\mathbb{NP}$.

4.8 Problem: Upper bounding of Sums of Norms. *Does the following problem belong to the class* $\mathbb{NP}$?

> Instance: *Positive integers* n, m; *integer vectors* $s_0, \ldots, s_m$ *in* $\mathbb{R}_p^n$; *a positive integer* λ.
>
> Question: *Is* $\sum_{i=0}^{m} \|s_i\|_p \leq \lambda$?

Problem 4.8 is open even for the special case $m = n$, which, as we have seen, is relevant to the algebraic nature of the inradius. The problem of deciding whether upper bounding the inradius is in $\mathbb{NP}$ is intimately related to the question of whether, for arbitrary rational polytopes $P \in \mathbb{R}_{\overline{p}}^n$, the quantity $r_n(P)$ is an algebraic number of a degree that is polynomial in n (cf. Theorem 2.6).

4.9 Problem: Algebraic Nature of Radii. *For* $p \in \mathbb{N} \cup \{\infty\}$ *or* $\overline{p} \in \mathbb{N} \cup \{\infty\}$, *are all inner and outer radii of rational polytopes algebraic numbers? For those radii that are algebraic numbers, what is their degree?*

We close this section with the proof of Theorem 1.8. Most of its statements follow immediately from Theorems 1.1 and 1.2, which we have already proved. The only remaining case is that of the width for asymmetric polytopes.

4.10 Radius Computations in Fixed Dimension. *For* $\overline{p} \in \mathbb{N} \cup \{0\}$, $R_1^{\overline{p}}(P)$ *can be computed in polynomial time for presented polytopes* P *in fixed* $\mathbb{R}_p^n$.

Proof. We can use the part of the proof of Theorem 4.4 which showed that upper bounding $R_1^{\overline{p}}(P)$ for (asymmetric) $\mathcal{H}$-polytopes P in $\mathbb{R}_p^n$ is in $\mathbb{NP}$. Just observe that for fixed n we can replace the guessing algorithm by a deterministic polynomial-time algorithm that computes all $2n$-tuples of vertices of P. □

5. $\mathbb{NP}$-Hard Radius Computations.

This section contains two basic $\mathbb{NP}$-hardness results and derives others from them. Here are the two basic results:

> For each fixed $p \in \mathbb{N}$, lower bounding the p^{th} power of the circumradius of a full-dimensional $\mathcal{H}$-parallelotope in an ℓ_p space is $\mathbb{NP}$-hard.
>
> At least for $p = 2$ and $p = \infty$, upper-bounding the $\overline{p}^{\text{th}}$ power of the width of a presented full-dimensional simplex in an ℓ_p space is $\mathbb{NP}$-hard.

There is no need to specify whether the simplex is $\mathcal{V}$-presented or $\mathcal{H}$-presented, because either sort of presentation of a simplex can be derived from the other in polynomial time.

A result equivalent to the first one has already been proved in [BGKV90]. And for $p = 2$, the second result can be derived from the first one by means of a geometric construction outlined in [GK89]. Alternatively, it can also be derived from 3-SAT by a transformation given in [AK90] — again via geometric constructions of [GK89]. However, here we provide a detailed proof of the second result by a different argument that works in ℓ_∞ spaces as well as in ℓ_2 spaces. Since we know already that in ℓ_1 spaces the width of an arbitrary presented polytope can be computed in polynomial time, this argument settles the problem in the three most important ℓ_p spaces. Furthermore, the proof given below seems to have a better chance of being extended to additional values of p.

5.1 HARDNESS OF CERTAIN RADII OF SYMMETRIC POLYTOPES. *If $p \in \mathbb{N}$ then each of the following problems is* **NP**-*complete for symmetric polytopes in ℓ_p spaces:*

> *lower-bounding $r_1(P)^p$ for an $\mathcal{H}$-polytope P;*
>
> *lower-bounding $R_n(P)^p$ for an $\mathcal{H}$-polytope P.*

If $\overline{p} \in \mathbb{N}$ then each of the following problems is **NP**-*complete for symmetric polytopes in ℓ_p spaces:*

> *upper-bounding $R_1(P)^{\overline{p}}$ for a $\mathcal{V}$-polytope P;*
>
> *upper-bounding $r_n(P)^{\overline{p}}$ for a $\mathcal{V}$-polytope P.*

The **NP**-*hardness persists even when the symmetric $\mathcal{H}$-polytopes are required to be parallelotopes and the symmetric $\mathcal{V}$-polytopes are required to be cross-polytopes.*

Proof. The paper [BGKV90] establishes the **NP**-completeness of the problem of lower-bounding the maximum of the function $\|\ \|_p^p$ on a full-dimensional parallelotope P centered at the origin in an ℓ_p space. By A.1, the maximum of $\|\ \|_p^p$ on P is equal to $R_n(P)^p$, and by A.3, $R_n(P) = r_1(P)$. That takes care of the first two claims in Theorem 5.1.

Now consider a $\mathcal{V}$-presentation of a full-dimensional cross-polytope Q centered at the origin in $\mathbb{R}_p^n$. This may be interpreted also as an $\mathcal{H}$-presentation of the polar polytope $P = Q^\circ \subset \mathbb{R}_{\overline{p}}^n$, and P is a full-dimensional parallelotope centered at the origin. To complete the proof, use the fact that by A.2, $R_1(Q) = 1/r_1(P)$ and $r_n(Q) = 1/R_n(P)$. □

Now we want to show that upper-bounding the width of certain simplices is **NP**-hard. First we need a result that plays an intermediate role in the reduction from the known **NP**-hard problem, PARTITION, and then a computation of widths for certain simplices.

5.2 RATIONAL POWER SUMS. *For each fixed positive integer p, the following problem is* **NP**-*complete:*

> Instance: *Sequence $\tau_1, \ldots, \tau_r$ of positive rational p^{th} powers.*
>
> Question: *With $\tau = \sum_{i=1}^r \tau_i$, does there exist $I \subset \{1, \ldots, r\}$ such that*

$$\left| \frac{\tau}{2} - \sum_{i \in I} \tau_i \right| \leq \frac{1}{4}?$$

Proof. Membership in **NP** is obvious. The proof of **NP**-hardness is by reduction from PARTITION. Consider an arbitrary instance $\sigma_1, \ldots, \sigma_r$ of PARTITION, let

$$m = \max\{4pr, \sigma_1, \ldots, \sigma_r\}, \quad a = m^{p+1}, \quad b = m^p,$$

and consider the following sequence of positive rational p^{th} powers:

$$\left(\frac{1}{b}\right)^p, \left(\frac{2}{b}\right)^p, \ldots, \left(\frac{a-1}{b}\right)^p, \left(\frac{a}{b}\right)^p = m^p \geq m.$$

The largest gap in this sequence is

$$\left(\frac{a}{b}\right)^p - \left(\frac{a-1}{b}\right)^p \leq \frac{pa^{p-1}}{b^p} = \frac{pm^{(p^2-1)}}{m^{(p^2)}} = \frac{p}{m} \leq \frac{1}{4r}.$$

Hence, using binary search, it is easy to produce $\tau_1, \ldots, \tau_r$ in the sequence such that

$$\sigma_i = \tau_i + z_i \quad \text{with} \quad |z_i| \leq \frac{1}{8r}.$$

Thus when

$$\sigma = \sum_{i \in I}^{r} \sigma_i \quad \text{and} \quad \tau = \sum_{i=1}^{r} \tau_i,$$

it is true that

$$|\sigma - \tau| \leq \sum_{i=1}^{r} |z_i| \leq \frac{1}{8}.$$

Now consider an arbitrary subset I of $\{1, \ldots, r\}$. If $\sum_{i \in I} \sigma_i = \sigma/2$ then

$$\left|\frac{\tau}{2} - \sum_{i \in I} \tau_i\right| \leq \left|\frac{\tau}{2} - \frac{\sigma}{2}\right| + \left|\frac{\sigma}{2} - \sum_{i \in I} \sigma_i\right| + \left|\sum_{i \in I} (\sigma_i - \tau_i)\right| \leq \frac{1}{16} + 0 + \frac{1}{8} < \frac{1}{4}.$$

And if

$$\left|\frac{\tau}{2} - \sum_{i \in I} \tau_i\right| \leq \frac{1}{4}$$

then

$$\left|\frac{\sigma}{2} - \sum_{i \in I} \sigma_i\right| \leq \left|\frac{\sigma}{2} - \frac{\tau}{2}\right| + \left|\frac{\tau}{2} - \sum_{i \in I} \tau_i\right| + \left|\sum_{i \in I} (\tau_i - \sigma_i)\right| \leq \frac{1}{16} + \frac{1}{4} + \frac{1}{8} < \frac{1}{2}.$$

Since σ and $\sum_{i \in I} \sigma_i$ are both integers, it follows that

$$\sum_{i \in I} \sigma_i = \frac{\sigma}{2}.$$

That completes the reduction and the proof. □

The following result appears in a more detailed form as Theorem 4.1 of [GK90a].

5.3 Width of Certain Simplices. *With $\{e_0, \ldots, e_n\}$ denoting the standard bases for $\mathbb{R}^{n+1}$, suppose that $\eta = (\eta_0, \ldots, \eta_n)$ is a sequence of positive numbers and let $v_i = \frac{1}{\eta_i} e_i$ for each i. Let*

$$S(\eta) = \operatorname{conv}\{v_0, \ldots, v_n\},$$

an n-simplex in $\mathbb{R}^{n+1}$ whose k-faces for $0 \leq k \leq n$ are precisely the sets of the form

$$F_I = \operatorname{conv}\{v_i : i \in I\}$$

with $I \subset \{0, \ldots, n\}$ and $|I| = k + 1$. For $1 \leq p \leq \infty$, let $w_p(\eta)$ denote the width of $S(\eta)$ with respect to its own affine hull in the normed space $\mathbb{R}_p^{n+1}$. Then $w_p(\eta)$ is equal to the minimum, over all pairs (I, J) of complementary proper subsets of $\{0, \ldots, n\}$, of the quantities $b_p(I, J)$ defined as follows:

$$b_1(I, J) = \frac{1}{\max_{i \in I} \eta_i} + \frac{1}{\max_{j \in J} \eta_j}; \qquad b_\infty(I, J) = \max\left\{\frac{1}{\sum_{i \in I} \eta_i}, \frac{1}{\sum_{j \in J} \eta_j}\right\};$$

for $1 < p < \infty$,

$$b_p(I, J) = \left(\frac{1}{(\sum_{i \in I} \eta_i^{\overline{p}})^{p-1}} + \frac{1}{(\sum_{j \in J} \eta_j^{\overline{p}})^{p-1}}\right)^{1/p}.$$

Note that the minimum of the numbers $b_1(I, J)$ is easily computed. That is to be expected, for we have already seen that in ℓ_1 spaces, polytope widths can be computed in polynomial time.

5.4 Upper-Bounding Simplex Width in a Hyperplane. *Let $e_0, \ldots, e_n$ be the standard basis for the space $\mathbb{R}^{n+1}$, and for each sequence $\zeta_0, \ldots, \zeta_n$ of positive reals let*

$$T(\zeta_0, \ldots, \zeta_n) = \operatorname{conv}\{\zeta_i e_i : 0 \leq i \leq n\}.$$

Then the following problem is $\mathbb{NP}$-complete for $p = 2$ and $p = \infty$:

> Instance: *Positive integers n and λ, a sequence $\zeta_0, \ldots, \zeta_n$ of positive integers;*
>
> Question: *Is λ an upper bound for the $\overline{p}^{\text{th}}$ power of the width of the n-simplex $T(\zeta_0, \ldots, \zeta_n)$ relative to its own affine hull in the spaces $\mathbb{R}_p^{n+1}$?*

Proof. If, instead of being integers, the ζ_i's and λ are merely rational, with $\zeta_i = \alpha_i / \beta_i$ and $\lambda = \delta / \epsilon$ for positive integers $\alpha_i, \beta_i, \delta$ and ϵ, and if $\mu = \beta_0 \beta_1 \cdots \beta_n \epsilon$, then

$$\text{width of } T(\zeta_0, \ldots, \zeta_n) \leq \lambda \quad \iff \quad \text{width of } T(\mu\zeta_0, \ldots, \mu\zeta_n) \leq \mu\lambda.$$

Hence the rational version of this simplex width problem is polynomially equivalent to the the integer version, and we can establish the $\mathbb{NP}$-completeness of the latter by dealing with the former.

Suppose, then, that $\zeta_0, \ldots, \zeta_n$ and λ are positive rationals, and let $\eta_i = \frac{1}{\zeta_i}$. Then the simplex F_N of Theorem 5.3 is the same as $T(\zeta_0, \ldots, \zeta_n)$. Hence the $\overline{p}^{\text{th}}$ power of the width of $T(\zeta_0, \ldots, \zeta_n)$ is less than or equal to λ if and only if there are complementary proper subsets I and J of N such that

$$\max\left\{\frac{1}{\sigma_I}, \frac{1}{\sigma_J}\right\} \leq \lambda \quad \text{for } p = \infty, \qquad \frac{1}{\sigma_I} + \frac{1}{\sigma_J} \leq \lambda \quad \text{for } p = 2,$$

where

$$\sigma_I = \sum_{i \in I} \eta_i^{\overline{p}}.$$

Thus the desired membership in the class **NP** is evident.

Suppose now that $p = \infty$ and set $\lambda = 2(\sum_{i=0}^{n} \eta_i)^{-1}$. Then the width of $T(\zeta_0, \ldots, \zeta_n)$ is less than or equal to λ if and only if there is a pair (I, J) of proper complementary subsets of N such that

$$\sum_{i \in I} \eta_i \geq \frac{1}{\lambda} \quad \text{and} \quad \sum_{j \in J} \eta_j \geq \frac{1}{\lambda}.$$

Since

$$\sum_{i \in I} \eta_i + \sum_{j \in J} \eta_j = \sum_{i=0}^{n} \eta_i = \frac{2}{\lambda}$$

the above conditions can only be satisfied with equality, hence the **NP**-hardness follows by reduction from PARTITION.

Now suppose that $p = 2$. It follows that $\overline{p} = 2$ and $\sigma_I = \sum_{i \in I} \eta_i^2$. With $y = \sigma_N$ and $\psi = \frac{1}{\lambda}$, the following three conditions are equivalent:

$$\frac{1}{\sigma_I} + \frac{1}{\sigma_J} \leq \lambda; \qquad \psi y \leq \sigma_I(y - \sigma_I); \qquad |\frac{y}{2} - \sigma_I| \leq \frac{\sqrt{y^2 - 4\psi y}}{2}.$$

When $\psi = \frac{4y^2 - 1}{16y}$, the last inequality becomes

$$|\frac{y}{2} - \sigma_I| \leq \frac{1}{4}.$$

Hence the claimed **NP**-hardness follows by reduction from the **NP**-hardness of RATIONAL POWER SUMS. □

As it stands, Theorem 5.4 does not establish the **NP**-hardness of the problem of upper-bounding the width of a presented polytope. As subsets of $\mathbb{R}^{n+1}$, the n-simplices of the theorem have integral vertices but they are not polytopes in our special sense because they are not full-dimensional. Of course, each of the n-simplices has nonempty interior relative to its own n-dimensional affine hull E. And even though this n-dimensional subspace E

depends on the choice of $(k_0, \ldots, k_n)$, it is true when $p = 2$ that E is always isometric to the space $\mathbb{R}_2^n$. However, it may happen that no integral simplex in $\mathbb{R}_2^n$ is similar to the simplex $T(k_0, \ldots, k_n)$ (see [GK90a] for a brief discussion of this), and hence it may be impossible to use simplices of precisely the latter form to establish the $\mathbb{NP}$-hardness of upper-bounding the width of a presented polytope in an ℓ_2-space. This difficulty will now be overcome by an extension of the above construction.

5.5 UPPER-BOUNDING SIMPLEX WIDTH IS $\mathbb{NP}$-HARD. *The following problem is $\mathbb{NP}$-hard for $p = 2$ and $p = \infty$:*

> Instance: *A positive integer n; a $\mathcal{V}$- or an $\mathcal{H}$-presented simplex T in $\mathbb{R}_p^n$; a positive integer λ.*
>
> Question: *Is λ an upper bound for the $\overline{p}^{\text{th}}$ power of the width of T?*

Proof. We prove the first assertion by transforming the problem in Theorem 5.4 to our given problem. So, let – almost as before – $e_1, \ldots, e_n$ be an orthonormal basis for the space $\mathbb{R}_2^n$, let $\zeta_1, \ldots, \zeta_n \in \mathbb{N}$ and set $T = T(\zeta_1, \ldots, \zeta_n) = \operatorname{conv}\{\zeta_i e_i : 1 \leq i \leq n\}$. Compute an integer vector z orthogonal to $\operatorname{aff}(T)$ with all coordinates negative and of size bounded by a polynomial in L. Then set

$$T' = 2^{68n^4 L} z + T \quad \text{and} \quad C = \operatorname{conv}(T \cup T').$$

Furthermore, let $a \in T'$ of size bounded by a polynomial in L and set

$$S = \operatorname{conv}(\{a\} \cup T).$$

Observe, first, that

$$\text{width of } T \text{ in } \operatorname{aff}(T) \quad = \quad 2R_1(C).$$

This follows from A.8 together with the fact that

$$\text{width of } T \text{ in } \operatorname{aff}(T) \leq 2R_n(T) \leq 2\max\{\zeta_1, \ldots, \zeta_n\}.$$

Now let $s^* \in (\mathbb{R}_2^n)^*$ be any unit vector for which the breadth function is minimum for S, i.e. $b_{s^*}(S) = 2R_1(S)$. Let H_+ and H_- be the corresponding supporting hyperplanes and let $S_\pm = S \cap H_\pm$. Clearly, none of these sets is equal to $\{a\}$. By A.8 there are points $s_\pm \in S_\pm$ such that the segment $\operatorname{conv}\{s_+, s_-\}$ is of length $2R_1(S)$. Now consider the 2-dimensional plane $E = \operatorname{aff}\{a, s_+, s_-\}$, which intersects S in the triangle $D = \operatorname{conv}\{a, s_+, s_-\}$. The height ζ of D at s_- (which is equal to $2R_1(S)$) is minimum when $a - s_- \in \mathbb{R}z$. In this case, the height ζ_0 is given by $\zeta_0 = \alpha\beta/\gamma$, where α denotes the length of $T \cap E$, $\beta = 2^{68n^4 L}\|z\|_2$ and γ is the length of the hypotenuse of D. Note that α is an upper bound on the width of C. Hence, since

$$\alpha \leq R_n(T) \leq 2\max\{\zeta_1, \ldots, \zeta_n\} \leq 2^{2L},$$

we have

$$\alpha - \zeta \le \alpha - \zeta_0 = \alpha\left(1 - \sqrt{1 - \frac{\alpha^2}{\gamma^2}}\right) < \alpha\left(1 - \sqrt{1 - \frac{\alpha}{\gamma}}\right) < \frac{\alpha^2}{\gamma}$$
$$\le \frac{2^{4L}}{2^{68n^4L}} = 2^{-64n^4L}.$$

Thus

$$R_1(C) - 2^{-64n^4L} < R_1(S) \le R_1(C).$$

Further, by A.13, the size of the square of the width of T in $\mathrm{aff}(T)$ is bounded by $32n^4L$, whence

$$R_1(T) \le \lambda \iff R_1(S) \le \lambda,$$

and the problem of Theorem 5.4 is transformed to our present problem.

To deal with the width of full-dimensional simplices when $p = \infty$, we bypass Theorem 5.4 and describe a direct transformation from PARTITION. Suppose that $\eta_1, \dots, \eta_n$ are positive integers, and let $\gamma = \sum_{i=1}^n \eta_i$. With

$$w = \sum_{i=1}^n e_i \qquad \text{and} \qquad v_i = \frac{1}{\eta_i} e_i \quad \text{for} \quad 1 \le i \le n,$$

let

$$S = \mathrm{conv}\{v_1, \dots, v_n, -\mu w\},$$

where μ is a positive integer still to be specified. Then S is a rationally presented n-simplex in $\mathbb{R}^n$, and the origin is interior to S. Set

$$\rho = \frac{10(\gamma\mu - 1)}{5\gamma^2\mu + 10\gamma - 4\mu}.$$

To establish the transformation of PARTITION to the problem of upper-bounding the width of simplices in ℓ_∞ spaces, we will show that for an appropriate choice of μ, the following two statements are equivalent:

there is a subset I of the set $N = \{1, \dots, n\}$ such that

$$\sum_{i \in I} \eta_i = \frac{1}{2}\gamma;$$

in the space $\mathbb{R}^n_\infty$, $R_1(S) \le \rho$.

For each proper subset I of N there is a unique triple $(\varphi_I, \alpha_I, \beta_I)$ that satisfies the following conditions

φ_I is a linear functional on $\mathbb{R}^n_\infty$, with $\|\varphi_I\|_1 = 1$;

α_I and β_I are real numbers, with $\alpha_I > -\beta_I$;

$\varphi_I(-\mu w) = \alpha_I$;

$\varphi_I(v_i) = \alpha_I$ for all $i \in I$, and $\varphi_I(v_j) = -\beta_I$ for all $j \notin I$.

Since $\|\varphi_I\|_1 = 1$, the φ_I-breadth of S is equal to $\alpha_I + \beta_I$.

Now let us focus on a fixed choice of I, and temporarily suppress the subscript I. Since α and $-\beta$ are respectively the maximum and the minimum of φ on S, and since the origin belongs to S, it is clear that $\alpha > 0 > -\beta$. Let $\varphi = \sum_{i=1}^n \varphi_i \overline{e}_i$. Then

$$\varphi_i = \alpha\eta_i \quad \text{for all } i \in I \qquad \text{and} \qquad \varphi_j = -\beta\eta_j \quad \text{for all } j \notin I.$$

Set

$$\sigma = \sum_{i\in I} \eta_i \qquad \text{and} \qquad \tau = \sum_{j\notin I} \eta_j$$

(for $p = \infty$, the σ_I and σ_J of Theorem 5.4). The condition that $\|\varphi\|_1 = 1$ may be written as

$$\sigma\alpha + \tau\beta = 1.$$

Also,

$$\varphi(-\mu w) = -\mu\varphi(w) = -\mu\left(\sum_{i\in I}\varphi_i + \sum_{j\notin I}\varphi_j\right) = -\mu(\alpha\sigma - \beta\tau),$$

and with $\varphi(-\mu w) = \alpha$ this yields

$$(1+\mu\sigma)\alpha - \mu\tau\beta = 0.$$

Solving the two linear equations for α and β, we obtain

$$\alpha = \frac{\mu}{1+2\mu\sigma} \qquad \text{and} \qquad \beta = \frac{1+\mu\sigma}{(1+2\mu\sigma)\tau}.$$

Using the fact that $\sigma + \tau = \gamma$, we see that $\alpha + \beta \le \rho$ if and only if

$$(1+\gamma\mu) \le (1+2\mu\sigma)(\gamma-\sigma)\rho.$$

This inequality may be rewritten as

$$\left(\sigma - \frac{1}{2}\gamma\right)^2 \le \frac{\gamma\rho - 1 - \mu\gamma}{2\mu\rho} + \frac{\gamma^2}{4} - \frac{\sigma}{2\mu},$$

and with the value for ρ specified earlier this becomes

$$\left(\sigma - \frac{1}{2}\gamma\right)^2 \le \frac{1}{5} - \frac{\sigma}{2\mu}.$$

Now set $\mu = 3\gamma$, so that the right side of the last inequality is positive for each σ, and the inequality is satisfied when $\sigma = \gamma/2$.

Conversely, if

$$\left(\sigma - \frac{1}{2}\gamma\right)^2 \leq \frac{1}{5} - \frac{\sigma}{2\mu}$$

then

$$|\sigma - \frac{1}{2}\gamma| < \frac{1}{2},$$

and since σ and γ are both integers this implies that $\sigma = \frac{1}{2}\gamma$.

Now recalling the description of width provided by A.8, we see that the width of S is at most ρ if and only if either there is a

proper subset I of $\{1, \ldots, n\}$ for which $\sigma_I = \frac{1}{2}\gamma$, or the distance between the remaining pair of candidate parallel hyperplanes is at most ρ. That remaining distance is equal to $\delta + \epsilon$, where these numbers are associated with a linear functional $\varphi = \sum_{i=1}^n \varphi_i \bar{e}_i$ such that $\|\varphi\|_1 = 1$, $\varphi(-\mu w) = \delta$, and $\varphi(v_i) = -\epsilon$ for all i. The last condition implies that $\varphi_i = -\epsilon\eta_i$ for all i, whence $\epsilon\gamma = 1$. And

$$\delta = \varphi(-\mu w) = -\mu\varphi(w) = -\mu \sum_{i=1}^{n} \varphi_i = \mu,$$

so $\delta + \epsilon = \mu + 1/\gamma > \rho$ and the proof is complete. □

The proof of Theorem 5.4, for $p = 2$ and $p = \infty$, relies partly on the fact that for these values of p the numbers $b_p(I, J)^{\overline{p}}$ of 5.3 are algebraically tractable. We do not know how to extend the analysis to other values of p.

In Theorems 5.1 and 5.5, the dimension n is arbitrary but the index of the radius in question is fixed at 1 or at n. Now we want to extend the hardness results to cases in which the radius index is permitted to vary between 1 and n. It is possible to prove a theorem in which p is also permitted to vary, but we choose to keep p fixed because we do not know of a situation in which it seems natural to vary p. However, it does seem natural, for fixed $\beta, \gamma \in \mathbb{N}$ with $\beta < \gamma$, to consider the minimax approximation of polytopes in $\mathbb{R}_p^{\gamma n}$ by affine subspaces of dimension βn, and that is one of the cases covered by the results below.

We begin by utilizing the results of Theorem 5.1. In the following proofs, a crucial property of the norm $\| \quad \|_p$ is that when $\mathbb{R}^n$ is canonically embedded in $\mathbb{R}^{n+k}$, the canonical projection of $\mathbb{R}^{n+k}$ onto $\mathbb{R}^n$ is a linear transformation of norm 1.

As an aid in considering radii of variable indices, we require some calculations that relate certain radii of a symmetric $\mathcal{H}$-polytope $P \subset \mathbb{R}_p^n$ to those of the product of P by a suitable cube. The following result is used only for the case in which P is a parallelotope, and then of course the product is also a parallelotope.

5.6 Computational Lemma. *Suppose that the n-polytope $P \subset \mathbb{R}_p^n$ is symmetric about the origin and admits an $\mathcal{H}$-presentation of size L. For $k, \alpha \in \mathbb{N}$, let $\mathbb{M}^n = \mathbb{R}^n \times \{0\}^k$, $\mathbb{M}^k = \{0\}^n \times \mathbb{R}^k$,*

$$C^k = \{0\}^n \times [-1, 1]^k \subset \mathbb{M}^n \oplus \mathbb{M}^k = \mathbb{R}_p^{n+k},$$

and

$$Q(P,k,\alpha) = P \times \{0\}^k + \alpha C^k \subset \mathbb{R}_p^{n+k}.$$

Then the polytope $Q(P,k,\alpha)$ is symmetric about the origin and admits an $\mathcal{H}$-presentation of size $L + 2k(1 + \log(\lceil\alpha\rceil))$. If P is a parallelotope then so is $Q(P,k,\alpha)$.
For each k and α,

$$r_1(P)^p = r_1(Q(P,k,\alpha))^p - \alpha^p k,$$

and for $\alpha \geq r_1(P)$ (in particular, for $\alpha \geq \sqrt{n}2^{4L}$),

$$r_1(P) \; = \; r_{1+k}(Q(P,k,\alpha)).$$

If $\epsilon > 0$ and $\alpha \geq \sqrt{n}2^{4L}\left(1 + \frac{\sqrt{n}2^{4L}}{\epsilon}\right)$ then

$$R_n(Q(P,k,\alpha)) \; \leq \; R_n(P) \; \leq \; R_n(Q(P,k,\alpha)) + \epsilon.$$

Proof. It is obvious that the polytope $Q(P,k,\alpha)$ is symmetric when P is symmetric and is a parallelotope when P is a parallelotope. For the statement about presentation size, it suffices to observe that if $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ and $e = (1,\ldots,1)^T \in \mathbb{R}^k$, then $Q(P,k,\alpha)$ may be regarded as the set of all pairs $(x,y) \in \mathbb{R}^n \times \mathbb{R}^k$ satisfying the inequalities

$$\begin{aligned} Ax \; &\leq \; b \\ -\alpha e \; \leq y \; &\leq \; \alpha e. \end{aligned}$$

Since p is fixed, let us denote the norm $\| \quad \|_p$ simply by $\| \quad \|$, and let $\mathbb{B}^n$ and $\mathbb{B}^{n+k}$ denote the unit balls of $\mathbb{M}^n$ and $\mathbb{R}^{n+k}$ respectively. Then of course $\mathbb{B}^n = \mathbb{B}^{n+k} \cap \mathbb{M}^n$.

We turn now to the statements about inner radii. The fact that

$$r_1(P)^p = r_1(Q(P,k,\alpha))^p - k\alpha^p$$

follows at once from the definition of the norm and the way in which P and C^k are embedded in $\mathbb{R}^{n+k}$. To prove the other assertion about inner radii, let $x_0 \in P$ such that $\| x_0 \| = r_1(P)$. Obviously, $[-x_0, x_0] + \alpha C^k \subset Q(P,k,\alpha)$. Now suppose that $\alpha \geq r_1(P)$, which by P's symmetry is equivalent to requiring that $P \subset \alpha\mathbb{B}^n$. Since the vector sum $S = \mathbb{R}x_0 + \mathbb{M}^k$ is a $(k+1)$-dimensional subspace of $\mathbb{R}^{n+k}$, and $S \cap (r_1(P)\mathbb{B}^{n+k}) \subset Q(P,k,\alpha)$, it follows that

$$r_1(P) \leq r_{1+k}(Q(P,k,\alpha)).$$

Note that by an observation in the introduction, $P \subset \sqrt{n}2^{4L}\mathbb{B}^n$, and hence this reasoning applies to all $\alpha \geq \sqrt{n}2^{4L}$.

Now let L^{1+k} be any $(1+k)$-dimensional subspace of $\mathbb{R}^{n+k}$ such that

$$L^{1+k} \cap r_{1+k}(Q(P,k,\alpha))\mathbb{B}^{n+k} \subset Q(P,k,\alpha),$$

and let $L = L^{1+k} \cap \mathbb{M}^n$. Then

$$L \cap r_{1+k}(Q(P,k,\alpha))\mathbb{B}^{n+k} \subset Q(P,k,\alpha) \cap \mathbb{M}^n = P,$$

which implies that

$$r_1(P) \geq r_{\dim L} \geq r_{1+k}(Q(P,k,\alpha)).$$

This completes the proof of the lemma's assertion about inner radii.

We turn now to the outer radii, where the first inequality is easy. Since $P \subset R_n(P)\mathbb{B}^n$ we have $Q(P,k,\alpha) \subset R_n(P)\mathbb{B}^n + \mathbb{M}^k \subset R_n(P)\mathbb{B}^{n+k} + \mathbb{M}^k$ and thus $R_n(Q(P,k,\alpha)) \leq R_n(P)$.

It remains to show that $R_n(P) \leq R_n(Q(P,k,\alpha)) + \epsilon$, and for this purpose it is convenient to set up some notation. Let S be any k dimensional subspace of $\mathbb{R}^{n+k}$ with $S \cap \mathbb{M}^n = \{0\}$. Let N be a $k \times (n+k)$ matrix and let D be an $(n+k) \times k$ matrix such that $\mathbb{M}^n$ is the nullspace of N and S is the column space of D. Moreover, let $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ denote the parallel projection along S onto $\mathbb{M}^n$. Then, for every point $z \in \mathbb{R}^{n+k}$ we have

$$\pi(z) = z - s_z \quad \text{where} \quad s_z = D(ND)^{-1}Nz.$$

To make the calculations easier let us choose

$$N = (0, I_k) \quad \text{and} \quad D = \begin{pmatrix} D' \\ I_k \end{pmatrix},$$

where (for any $j \in \mathbb{N}$) I_j denotes the $j \times j$ identity matrix. In these special representations we get

$$\pi(z) = Pz = \begin{pmatrix} I_n & -D' \\ 0 & 0 \end{pmatrix} z \quad \text{and} \quad s_z = \begin{pmatrix} 0 & D' \\ 0 & I_k \end{pmatrix}.$$

Using the above notation, we can now prove the remaining statement of Lemma 5.6—namely, that $R_n(P) \leq R_n(Q(P,k,\alpha)) + \epsilon$. Let J be any k-dimensional subspace of $\mathbb{R}^{n+k}$ such that $Q(P,k,\alpha) \subset R_n(Q(P,k,\alpha))\mathbb{B}^{n+k} + J$. Since the width of C^k, taken in $\mathbb{M}^k$, is at least 2α it is easy to show that $J \cap \mathbb{M}^n = \{0\}$. Let us abbreviate $R_n(Q(P,k,\alpha))$ by R. To derive the inequality observe that for $z = \begin{pmatrix} x \\ y \end{pmatrix} \in R\mathbb{B}^{n+k}$ we have

$$\| \pi(z) \| = \| Pz \| = \| x - D'y \| \leq \| x \| + \| D'y \| \leq R + \| D'y \| .$$

Thus it suffices to show that $\| D'y \| \leq \epsilon$. Obviously,

$$\pi(\alpha c) = \alpha Pc = -\alpha D'c' \in \pi(R\mathbb{B}^{n+k}) \quad \text{for all} \quad c = \begin{pmatrix} 0 \\ c' \end{pmatrix} \in C^k.$$

This means, in particular, whenever $z = \begin{pmatrix} x \\ y \end{pmatrix} \in R\mathbb{B}^{n+k}$ and hence $\begin{pmatrix} 0 \\ \frac{\alpha}{R} y \end{pmatrix} \in \alpha C^k$, that there exists a vector $\tilde{z} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \in R\mathbb{B}^{n+k}$ with

$$\frac{\alpha}{R} D' y = \tilde{x} - D'\tilde{y}.$$

Thus we have

$$\| D'y \| = \frac{R}{\alpha} \| \tilde{x} - D'\tilde{y} \| \leq \frac{R}{\alpha} (\| \tilde{x} \| + \| D'\tilde{y} \|).$$

Now let β denote the maximum of $\| D'y \|$ taken over all those vectors y. Then

$$\beta \left(1 - \frac{R}{\alpha}\right) \leq \frac{R^2}{\alpha}, \quad \text{i.e.} \quad \beta \leq \frac{R^2}{\alpha - R}.$$

Since, furthermore, $R \leq \sqrt{n} 2^{4L}$, the desired assertion follows from our choice of α. □

5.7 HARDNESS OF r_j IN THE SPACES $\mathbb{R}^m_p$. *Suppose that $\mathcal{A}$ is a polynomial-time algorithm that accepts as input an arbitrary $n \in \mathbb{N}$, and for each n outputs a pair $\mathcal{A}(n) = (j, m) \in \mathbb{N} \times \mathbb{N}$ such that $m - j \geq n - 1$. Then the following problem is $\mathbb{NP}$-hard for each $p \in \mathbb{N}$:*

> Instance: *A positive integer n, a pair $(j, m) = \mathcal{A}(n)$, an $\mathcal{H}$-presentation of a parallelotope P in $\mathbb{R}^m_p$, a rational λ.*
>
> Question: *Is $r_j(P)^p \geq \lambda$?*

Proof. In view of Theorem 5.1, we may show that the problem is $\mathbb{NP}$-hard by transforming to it the problem of lower-bounding the inner 1-radius of an $\mathcal{H}$-presented parallelotope in $\mathbb{R}^n_p$. Consider an instance of the latter problem with parallelotope P. Run the algorithm $\mathcal{A}$ to produce $\mathcal{A}(n) = (j, m)$, and then set

$$t = (m - j) - (n - 1) \geq 0, \qquad P' = Q(P, t, 1).$$

Then $r_1(P)^p = r_1(P')^p - t$ by Lemma 5.6. By the observation in the Introduction, the circumradius of P is at most $\sqrt{n} 2^{4L}$, whence the same is true of $r_1(P)$ and it follows that the number

$$\delta = t + n^{\frac{p}{2}} 2^{pL}.$$

is an upper bound for $r_1(P')^p$. Hence for $\alpha \geq \delta^{\frac{1}{p}}$, it follows with the aid of Lemma 5.6 (applied for $k = j - 1$) that

$$r_1(P') = r_j(Q(P', j - 1, \alpha))^p.$$

Hence $r_1(P)^p \geq \lambda$ if and only if $r_j(Q(P', j - 1, \alpha)^p \geq \lambda + t^p$. That completes the transformation and the proof. □

The following two results are probably the most interesting corollaries of Theorem 5.7. They are both concerned with spaces of variable dimension.

5.8 HARDNESS OF r_j AND R_j IN THE SPACES $\mathbb{R}_p^n$. *For each fixed $j \in \mathbb{N}$, each of the following problems is* $\mathbb{NP}$*-hard in ℓ_p spaces:*

for fixed $p \in \mathbb{N}$, lower bounding the p^{th} power of the inner j-radius of an $\mathcal{H}$-presented parallelotope;

for fixed $\overline{p} \in \mathbb{N}$, upper bounding the $\overline{p}^{\text{th}}$ power of the outer j-radius of a $\mathcal{V}$-presented cross-polytope.

Proof. For each $n \in \mathbb{N}$, let $\mathcal{A}(n) = (j, j+n-1)$. Then apply Theorem 5.7 to deal with the inner j-radius. For the outer j-radius, use polarity and A.3 as in the proof of Theorem 5.1. □

5.9 HARDNESS OF $r_{\beta n}$ AND $R_{\beta n}$ IN THE SPACES $\mathbb{R}_p^{\gamma n}$. *For fixed $\beta, \gamma \in \mathbb{N}$ with $\beta < \gamma$, each of the following problems is* $\mathbb{NP}$*-hard:*

for fixed $p \in \mathbb{N}$, lower-bounding the p^{th} power of the inner βn-radius of an $\mathcal{H}$-presented parallelotope in the spaces $\mathbb{R}_p^{\gamma n}$;

for fixed $\overline{p} \in \mathbb{N}$, upper-bounding the $\overline{p}^{\text{th}}$ power of the outer βn-radius of a $\mathcal{V}$-presented cross-polytope in the spaces $\mathbb{R}_p^{\gamma n}$.

Proof. For each $n \in \mathbb{N}$, let $\mathcal{A}(n) = (\beta n, \gamma n)$. Then apply Theorem 5.7, polarity, and A.3. □

Theorem 5.7 led to $\mathbb{NP}$-hardness results for inner radii of $\mathcal{H}$-presented polytopes and – by polarity – for outer radii of $\mathcal{V}$-presented polytopes. The next theorem is analogous to 5.7 and deals with outer radii of $\mathcal{H}$-presented polytopes.

5.10 HARDNESS OF R_j IN THE SPACES $\mathbb{R}_p^m$. *Suppose that $\mathcal{A}$ is a polynomial-time algorithm that accepts as input an arbitrary $n \in \mathbb{N}$, and for each n outputs a pair $\mathcal{A}(n) = (j, m) \in \mathbb{N} \times \mathbb{N}$ such that $m \geq j \geq n$. Then the following problem is* $\mathbb{NP}$*-hard for each $p \in \mathbb{N}$:*

Instance: *A positive integer n, a pair $(j, m) = \mathcal{A}(n)$, an $\mathcal{H}$-presentation of a parallelotope P in $\mathbb{R}_p^m$, a positive rational λ.*

Question: *Is $R_j(P)^p \leq \lambda$?*

Proof. In view of Theorem 5.1, we may show that the problem is $\mathbb{NP}$-hard by transforming to it the problem of upper-bounding the outer n-radius of an $\mathcal{H}$-presented parallelotope in $\mathbb{R}_p^n$. Consider an instance of the latter problem with a parallelotope P. Run the algorithm $\mathcal{A}$ to produce $\mathcal{A}(n) = (j, m)$, and then set

$$t = j - n \geq 0, \qquad P' = Q(P, t, 1).$$

Then $r_1(P)^p = r_1(P')^p - t$ by Lemma 5.6, whence by A.3, $R_n(P)^p = R_j(P')^p - t$. By A.13, $R_j(P')$ is of size at most $16pj^4L$. Now, set

$$k = m - j, \quad \epsilon \le 2^{-38pj^4L}, \quad \alpha = \left\lceil \sqrt{j}2^{4L}\left(1 + \frac{\sqrt{j}2^{4L}}{\epsilon}\right)\right\rceil, \quad P'' = Q(P', k, \alpha).$$

By Lemma 5.6 we have

$$R_j(P'')^p \le R_j(P')^p \le (R_j(P'') + \epsilon)^p \le R_j(P'')^p + \epsilon(2\sqrt{j}2^{4L})^p < R_j(P'')^p + \frac{1}{2^{32pj^4L}},$$

and hence

$$R_j(P')^p \le \lambda \qquad \Longleftrightarrow \qquad R_j(P'')^p \le \lambda.$$

This implies the assertion just as in the proof of 5.7. □

Again, the following two results are probably the most interesting corollaries of Theorem 5.10.

5.11 HARDNESS OF R_j AND r_j IN THE SPACES $\mathbb{R}_p^n$. *For each fixed $j \in \mathbb{N} \cup \{0\}$, each of the following problems is* NP*-hard in ℓ_p spaces:*

for fixed $p \in \mathbb{N}$, upper bounding the p^{th} power of the outer $(n-j)$-radius of an $\mathcal{H}$-presented parallelotope;

for fixed $\overline{p} \in \mathbb{N}$, lower bounding the $\overline{p}^{\text{th}}$ power of the inner $(n-j)$-radius of a $\mathcal{V}$-presented cross-polytope.

Proof. For each $n \in \mathbb{N}$, let $\mathcal{A}(n) = (n, j+n)$. Then apply Theorem 5.10 to deal with the outer $n-j$-radius. For the inner $n-j$-radius, use polarity and A.3 as in the proof of Theorem 5.7. □

5.12 HARDNESS OF $R_{\beta n}$ AND $r_{\beta n}$ IN THE SPACES $\mathbb{R}_p^{\gamma n}$. *For fixed $\beta, \gamma \in \mathbb{N}$ with $\beta < \gamma$, each of the following problems is* NP*-hard:*

for fixed $p \in \mathbb{N}$, upper-bounding the p^{th} power of the outer βn-radius of an $\mathcal{H}$-presented parallelotope in the spaces $\mathbb{R}_p^{\gamma n}$;

for fixed $\overline{p} \in \mathbb{N}$, lower-bounding the $\overline{p}^{\text{th}}$ power of the inner βn-radius of a $\mathcal{V}$-presented cross-polytope in the spaces $\mathbb{R}_p^{\gamma n}$.

Proof. For each $n \in \mathbb{N}$, let $\mathcal{A}(n) = (\beta n, \gamma n)$. Then apply Theorem 5.6, polarity, and A.3. □

Observe that the statements of Theorems 5.8 and 5.9 are best possible, since the inradius problem, the "right end of the spectrum," becomes "easy" even for arbitrary $\mathcal{H}$-presented polytopes (see Theorem 4.3). Theorems 5.11 and 5.12, however, can be extended.

Of course, for parallelotopes the width, the "left end of the spectrum," can be computed quite easily, but – as shown in Theorem 5.5 – at least for $p \in \{2, \infty\}$ the width problem is NP-hard for simplices.

This gives rise to some additional NP-hardness results. First, we need again a computational lemma.

5.13 COMPUTATIONAL LEMMA. *For $p \in \mathbb{N} \cup \{\infty\}$, let P be a polytope in $\mathbb{R}^n_p$ that admits an $\mathcal{H}$-presentation of size L, and suppose $\frac{1}{2}r_n(P)\mathbb{B}^n_p \subset P$. For $k \in \mathbb{N}$, let $\mathbb{M}^n = \mathbb{R}^n \times \{0\}^k$, $\mathbb{M}^k = \{0\}^n \times \mathbb{R}^k$, and*

$$S_k = \{0\}^n \times \operatorname{conv}\{e_{n+1}, \ldots, e_{n+k}\} \subset \mathbb{M}^n \oplus \mathbb{M}^k = \mathbb{R}^{n+k}_p,$$

where $e_1, \ldots, e_{n+k}$ denotes the standard basis of $\mathbb{R}^{n+k}$, and for $\alpha \in \mathbb{Q}$, $\alpha \geq 0$ set

$$T(P, k, \alpha) = \operatorname{conv}(P \times \{0\}^k, \alpha S_k) \subset \mathbb{R}^{n+k}_p.$$

Then the polytope $T(P, k, \alpha)$ admits an $\mathcal{H}$-presentation of size bounded by a polynomial in L, k and the size of α. If P is a simplex then so is $T(P, k, \alpha)$ — in this case P and $T(P, k, \alpha)$ admit $\mathcal{V}$-presentations of sizes which are bounded by a polynomial in L or a polynomial in L, k and the size of α, respectively. For $\alpha \leq \frac{1}{2}r_n(P)$ we have

$$R_1(P) = R_{1+k}(T(P, k, \alpha)).$$

Proof. Observe, first, that $T(P, k, \alpha)$ is the set of all points (x, y) with $x \in \mathbb{R}^n_p$, $y \in \mathbb{R}^k_p$ that satisfy the following system of linear inequalities.

$$\begin{aligned} Ax &\leq b \\ \alpha \sum_{i=1}^{k} \tau_i e_{n+i} - y &= 0 \\ \sum_{i=1}^{k} \tau_i &= 1 \\ \tau_i &\geq 0 \qquad (1 \leq i \leq k). \end{aligned}$$

This, together with the fact that for simplices either kind of presentation can be converted into the other in polynomial time, takes care of the statements about the sizes of P and $T(P, k, \alpha)$.

Let F be a hyperplane in $\mathbb{M}^n$ such that $P \subset F + R_1(P)\mathbb{B}^n_p$. Since $\frac{1}{2}r_n(P)\mathbb{B}^n_p \subset P$ we have

$$\alpha S_k \subset \frac{1}{2}\alpha\mathbb{B}^{n+k}_p \subset \frac{1}{2}r_n(P)\mathbb{B}^{n+k}_p \subset (F \times \{0\}^k) + R_1(P)\mathbb{B}^{n+k}_p$$

and hence

$$T(P,k,\alpha) = \mathrm{conv}(P \times \{0\}^k, S_k) \subset (F \times \{0\}^k) + R_1(P)\mathbb{B}_p^{n+k}.$$

This implies $R_{1+k}(T(P,k,\alpha)) \leq R_1(P)$.

Now let G be an $(n-1)$-flat in $\mathbb{M}^{n+k}$ such that $T(P,k,\alpha) \subset G + R_{1+k}(T(P,k,\alpha))\mathbb{B}_p^{n+k}$. Let $g \in G$ and set

$$S = g + [0,1]\left(\left[(g + R_{1+k}(T(P,k,\alpha))\mathbb{B}_p^{n+k}) \cap \mathrm{bd}\left(G + R_{1+k}(T(P,k,\alpha))\mathbb{B}_p^{n+k}\right)\right] - g\right).$$

Then S is of dimension $k+1$, is symmetric about g, and is connected. Hence, $\mathbb{M}^n \cap S \neq \emptyset$. Now, let H_1, H_2 be parallel $(n+k-1)$-flats that support the set $G + R_{k+1}(T(P,k,\alpha))\mathbb{B}_p^{n+k}$ at two points s_1, s_2 of $\mathbb{M}^n \cap S$ that are symmetric about g. Then

$$P \subset T(P,k,\alpha) \cap \mathbb{M}^n \subset \left(G + R_{k+1}(T(P,k,\alpha))\mathbb{B}_p^{n+k}\right) \cap \mathbb{M}^n \subset \mathrm{conv}(H_1 \cap \mathbb{M}^n, H_2 \cap \mathbb{M}^n).$$

Since the distance of $H_1 \cap \mathbb{M}^n$ and $H_2 \cap \mathbb{M}^n$ is equal to $\|s_1 - s_2\|_p = 2R_{1+k}(T(P,k,\alpha))$ we have $R_{1+k}(T(P,k,\alpha)) \geq R_1(P)$. □

5.14 Hardness of R_j in the spaces $\mathbb{R}_p^n$. *For each fixed $p \in \{2, \infty\}$ and each fixed $j \in \mathbb{N}$, the problem of upper bounding the $\overline{p}^{\text{th}}$ power of the outer j-radius of a presented simplex is* $\mathbb{NP}$*-hard in ℓ_p spaces.*

Proof. We prove the assertion by reducing the problem of upper bounding the $\overline{p}^{\text{th}}$ power of the width of presented simplices to the problem of upper bounding the $\overline{p}^{\text{th}}$ power of the outer j-radius. The $\mathbb{NP}$-hardness of the latter then follows from Theorem 5.5. Let (P,λ) be an instance of the former problem. By the observation in the Introduction, a rational point q with $q + 2^{-4L}\mathbb{B}_p^n \subset P$ can be computed in polynomial time. Since, further, the width is invariant under translations we can assume without loss of generality that P satisfies the requirements of Lemma 5.13. Hence, by applying Lemma 5.13 to P with $k = j-1$ and $\alpha = 2^{-6L}$ we obtain in time that is bounded by a polynomial in L an equivalent instance $(T(P,k,\alpha),\lambda)$ of the latter problem. (Observe that, here, k is a constant and α is of size $O(L)$.) □

6. Some Applications of Radii.

Since the subsections of this section make little contact with each other, each has its own subheading.

6.A Global Optimization. In many iterative algorithms for optimizing a function over a feasible region C, a lower bound on C's inradius serves to guarantee the numerical stability of the algorithm. As we have seen in Theorem 4.3, the inradius of an $\mathcal{H}$-presented polytope can be approximated in polynomial time.

For practical purposes it is often relevant to have some additional information when the inradius is small. It would be useful to know a hyperplane H such that $C \subset H + \epsilon\mathbb{B}^n$, with small ϵ. But this is the problem of approximating the width $2R_1(C)$ of C and finding a corresponding hyperplane.

As we have seen in Theorem 5.5 the problem of computing $R_1(C)^2$ in $\mathbb{R}_2^n$ is NP-complete even for simplices. However, the width can be approximated by polynomial time algorithms (even in a much more general situation, where C is a convex body that is given only by some "oracles") with an error of order n^α, where α is an arbitrary real greater then $\frac{1}{2}$ [GK90b]. A weaker version of this result and some related theorems can be found in [GLS88].

6.B Sensitivity Analysis of Linear Programs. In the Introduction, we mentioned the relevance of certain polytope radii to the sensitivity analysis of linear programs. The basic problem of linear programming asks for the maximization of a linear objective function over an $\mathcal{H}$-presented feasible region P. *Sensitivity analysis* is concerned with the way in which small changes in the constraints may affect feasibility or the optimum value. As the following theorem shows, the width and the inradius of P provide information concerning the effect on feasibility of varying the right-hand side b. It follows from (a) and (b) below that if the representation has been normalized (by rescaling A and b) in such a way that $\|a_i\|^* = 1$ for all i, then without destroying feasibility, any single constraint halfspace can be moved (parallel to itself) by any distance not exceeding $2R_1(P)$, or all the constraint halfspaces can be moved (parallel to themselves), by distances not exceeding $r_n(P)$.

6.1 Width, Inradius and Sensitivity Analysis. *Consider an $\mathcal{H}$-presented polytope $P = \{x \in \mathbb{R}^n : Ax \le b\}$, where A is an $m \times n$ matrix $[\alpha_{ij}]$ and $b = (\beta_1, \dots, \beta_m)$. For $1 \le i \le m$, let $\|a_i\|^*$ denote the norm of the i^{th} row of A interpreted as a linear functional on $\mathbb{M}_n$. For each $e = (\epsilon_1, \dots, \epsilon_m)^T \in \mathbb{R}^m$, let $P_e = \{x \in \mathbb{R}^n : Ax \le b - e\}$, the result of perturbing the right-hand side b by an amount e. Then P_e is nonempty if either of the following conditions is satisfied:*

(a) *$\epsilon_k = 0$ for all $k \ne i$, and*

$$\epsilon_i \le \frac{2R_1(P)}{\|\alpha_i\|^*};$$

(b) *for $1 \le i \le m$,*

$$\epsilon_i \le \frac{r_n(P)}{\|\alpha_i\|^*}.$$

Furthermore, the system

$$Ax \le b, \quad Ax \le b - Aq$$

is feasible for all $q \in \mathbb{R}^n$ with $\|q\| \le \alpha$ if and only if $\alpha \le 2R_1(P)$.

Proof. For (a), note that if feasibility is destroyed then P lies between two hyperplanes whose distance apart is less than $2R_1(P)$, contradicting the definition of width. For (b), note that P contains a ball of radius $r_n(P)$, and if each constraint halfspace is moved by an amount less than this radius then the center of the ball will still lie in all the halfspaces. For the last statement, note that if $\|q\| \leq 2R_1(P)$ then by A.5, the translate $P + q$ intersects P. This says that there exists $x \in P$ such that $A(x+q) \leq b$, whence $Ax \leq b - Aq$. □

Theorem 6.1 speaks only of feasibility, but it can also be applied to analyze the sensitivity to changes in the right-hand side b of the maximum over P of a linear objective function y. Suppose, for example, that

$$\lambda < \mu = \max_{x \in P} \langle x, y \rangle,$$

and the question is how much the i^{th} component of b can be varied without forcing the maximum to be less than λ. Then useful information can be obtained by applying 6.1(a) to the polytope obtained by intersecting P with the halfspace $\{x : \langle x, -y \rangle \leq -\lambda\}$.

6.C Orthogonal Minimax Regression. The most familiar form of regression involves a real-valued function g of n causal variables (i.e. $\operatorname{dom} g \subset \mathbb{R}^n$), g's graph

$$G = \{(x, g(x)) : x \in \operatorname{dom} g\} \subset \mathbb{R}^{n+1},$$

and the observation of a finite number of points of G — say $z_1 = (x_1, g(x_1)), \ldots, z_k = (x_k, g(x_k))$. In a typical situation, the observer wants to find an affine function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ that best represents g in the sense that some measure of the deviation of f's graph

$$F = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$$

from G is as small as possible. In *least squares regression*, the deviation is measured by $\sum_{i=1}^{k} |f(x_i) - g(x_i)|^2$, the sum of the squares of the "vertical distances" of the observed points z_i from F. While least squares regression is by far the most common, there are also situations that call for *minimax regression* — taking $\max_i |f(x_i) - g(x_i)|$ as the measure of deviation (see [AD81], [PD83]).

Now suppose that causal relationships are obscure, so that each observed point $z_i \in \mathbb{R}^{n+1}$ is regarded merely as representing some (not necessarily functional) relationship (still denoted by G) among the $n+1$ coordinates. If it is then desired to approximate G by an affine relation, hence by a hyperplane H in $\mathbb{R}^{n+1}$, the observer may choose to employ *orthogonal least squares regression.* Then the deviation of H from G is measured by $\sum_{i=1}^{k} \operatorname{dist}(z_i, H)^2$, the sum of the squares of the Euclidean distances (or "orthogonal distances") of the observed points z_i from H. A useful alternative in some cases is *orthogonal minimax regression*, in which $\max_i \operatorname{dist}(z_i, H)$ measures the deviation. The hyperplanes H for which the orthogonal minimax deviation is minimum are precisely those that lie midway between two parallel supporting hyperplanes H_- and H_+ of the set $P = \mathrm{conv}\{z_1, \ldots, z_k\}$ such that the distance between H_- and H_+ is the width of P. Hence finding H implies determining the width of the set of observed points.

As we have seen, width determination is **NP**-hard for $\mathcal{V}$-presented polytopes and for asymmetric $\mathcal{H}$-presented polytopes in Euclidean spaces of unrestricted dimension. However, width determination in the Euclidean plane is involved in the applications to computer vision and chromosome classification described in subsections 6.F and 6.G. It would be of interest, for the case of unrestricted dimension, to find conditions (significantly weaker than central symmetry which is sufficient for $\mathcal{H}$-presented polytopes) under which the width of a polytope in $\mathbb{R}_2^n$ can be computed in polynomial time.

6.D Collision Avoidance. Knowing the outer radii of a polytope in Euclidean 3-space would be useful in dealing with collision-avoidance problems in robotics. Consider, for example, the case of a symmetric body C with center c. Let ψ and ρ denote respectively the outer 2-radius and the outer 3-radius of C, so that $\psi \leq \rho$. Suppose that we want to move the center c from a starting position s to a terminal position t in such a way that during the motion, C does not collide with a given fixed obstacle Z. For the route to be followed by c, we could use any path W from s to t such that Z is disjoint from the ρ-neighborhood of W. In moving the center c along such a path, arbitrary rotations of C about c can be permitted, for they will not cause C to collide with Z. However, if we can suitably restrict the rotations of C as c is moved, and if ρ is much greater than ψ (as will be the case when C is long but narrow), we may be able to find a much shorter admissible path from s to t by using information about ψ. For let L be a line through c such that $C \subset L + \psi\mathbb{B}$, and suppose that $s = v_0, v_1, \ldots, v_k = t$ is a sequence of points such that the obstacle Z is disjoint from the ρ-neighborhood of the set $\{v_0, \ldots, v_{k-1}\}$ and from the ψ-neighborhood of the set $\bigcup_{i=1}^{k}[v_{i-1}, v_i]$. Then C can be moved from s to t in the following manner, without colliding with Z:

with c fixed at v_0, rotate C about c to align L with the segment $[v_0, v_1]$;

slide c along the segment from v_0 to v_1, maintaining the alignment of L;

with c fixed at v_1, rotate C about c to align L with the segment $[v_1, v_2]$;

...

slide c along the segment from v_{k-1} to v_k, maintaining the alignment of L.

Now consider a body C in Euclidean 3-space, and for each positive λ let $R(C, \lambda)$ denote the radius of the smallest circular cylinder of length λ through which C can be passed. For each λ that exceeds C's diameter δ, $R(C, \lambda)$ is equal to the outer 2-radius of C. However, when $\lambda < \delta$ it may happen that $R(C, \lambda) < R(C, \delta)$. In particular, $R(C, 0)$ is the radius of the smallest circle through which C can be passed, and the inequality $R(C, 0) < R_2(C)$ is established by [Zi19] and [St82] for certain bodies C. The problem of computing $R(P, 0)$ for a 3-polytope P is of interest, but appears to be difficult.

6.E Smallest Covering Cone. The following problem, involving circumspheres, appears to be of interest for all n. For $n = 3$ it arose from an investigation of multiple airborne target tracking with a ground based radar [GEL65].

> In $\mathbb{R}_2^n$, let X be a fixed finite set and a a point sufficiently remote from X so that all of X lies in some open halfspace whose bounding hyperplane passes through a. Find the right circular cone C with apex a of minimum vertex angle containing on or in it all the points of X.

We may assume without loss of generality that a is the origin. Let $Y = \mathbb{S}_2^n \cap [0, \infty[X$, the set of all points of $\mathbb{S}_2^n$ that lie on rays from a through X. As is observed in [GEL65], the desired cone C is asymptotically (i.e. for $\mathrm{dist}(a, Y) \to \infty$) the set $[0, \infty[B$, where B is the smallest ball containing Y.

6.F Computer Graphics, Computer Vision. In this subsection we give a short account (more intuitive than rigorous) of some applications of the width problem in computer graphics and computer vision. The approach described appears in [KD82], and additional information on related methods and shape analysis techniques can be found in [Pa77].

One of the basic procedures in Computer Vision is the approximation of digitalized curves by polygons in order to preprocess images for shape analysis. In the easiest case each image is represented by an $m \times n$ 0-1-matrix and the goal is to approximate a "region" given by the 1 entries by a polygon. The conflicting objectives are, on the one hand, to find a polygon with as few edges as possible and, on the other hand, to maintain as much information as necessary to be able to perform the subsequent shape analysis.

Let us in the following assume that each "region" is connected; i.e., we approximate it with just one polygon. The first step is usually to compute the *contour line* of the "region", thus obtaining an ordered curve which contains the "boundary information" of the 0-1-matrix.

The next step associates with the contour line the point set that is obtained by associating with every 1 entry of the matrix the center of the corresponding pixel. This yields an ordered (according to the corresponding position on the contour line) point set S in the plane. The task is, now, to approximate this point set by a polygonal curve. [KD82] give an efficient algorithm for this purpose. A crucial iterative step of this algorithm is to approximate a subset S' of S by a line. But this is nothing else than computing $R_1(S')$ and a corresponding line F such that $S' \subset F + R_1(S')\mathbb{B}^2$.

6.G Chromosome Classification. Computing the width of a convex polygon is involved in a procedure for banded chromosome classification proposed in [PG89]. The procedure involves measurements with respect to the axis A of a chromosome C, and the first step in finding A consists of defining C's orientation. The orientation is represented by a line L that provides a best minimax approximation of C or, equivalently, of C's convex hull. After L has been found, the smallest rectangle R that contains C and has sides parallel and perpendicular to L is determined. In effect, R is regarded as a first approximation of C and R's axis $R \cap L$ is regarded as a first approximation of A. Quoting from [PG89]: "Most chromosomes are not exactly straight, but there is generally no diagnostic significance in how bent they are ... so we aim to extract features that are *invariant* to chromosome bending. In order to do this, feature measurements are ... related to the longitudinally medial axis of the chromosome about which the chromatids are symmetrical. We also require the *orientation* and *polarity* of chromosomes so that they may be displayed with proper orientation in the karyogram." [To define the orientation] "we use the orientation of the minimum width enclosing rectangle. ... It can be shown that this rectangle is parallel to the chord" [i.e., edge] "of the convex hull, which is used elsewhere in the MRC system ... and so must be computed anyway. Obtaining the orientation in this way is computationally very fast, though ... it is not invariably optimum as judged by a human observer (though we know of no consistantly superior method)."

Appendix: Results on Radii used here.

The following results are all used in the present paper. They are formulated for polytopes P, even though they hold in greater generality, since this is the case in which they are needed here. The general statements, the proofs and additional results are contained in [GK90a].

In the following P always denotes a polytope in some Minkowski space $\mathbb{M}$; n denotes the dimension of $\mathbb{M}$ and j is an integer with $1 \leq j \leq n$.

A.1: If P is symmetric about the origin then P contains a j-ball of radius $r_j(P)$ centered at the origin; and there is an $(n-j)$-subspace F of $\mathbb{M}$ such that $P \subset F + R_j(P)\mathbb{B}$.

A.2: If P is symmetric about the origin then $r_j(P)R_j(P^\circ) = 1$ and $R_j(P)r_j(P^\circ) = 1$.

A.3: $r_1(P) \leq R_n(P)$ and $r_n(P) \leq R_1(P)$, with equality if P is symmetric.

A.4: Let $s \in \mathbb{S}$ and $s^* \in \mathbb{S}^*$, let $l_s(P)$ denote the length of the longest segment in P that is parallel to the line $\mathbb{R}s$, and let $b_{s^*}(P) = \max_{c \in P}\langle c, s^*\rangle - \min_{c \in P}\langle c, s^*\rangle$. If $Q = \frac{1}{2}(P - P)$, then $l_s(P) = l_s(Q)$ and $b_{s^*}(P) = b_{s^*}(Q)$.

A.5: $2r_1(P) = \max_{s \in \mathbb{S}} l_s(P) = \max_{s^* \in \mathbb{S}^*} b_{s^*}(P)$,
$2R_1(P) = \min_{s \in \mathbb{S}} l_s(P) = \min_{s^* \in \mathbb{S}^*} b_{s^*}(P)$.

A.6: There is a diametral pair $\{v, w\}$ that consists of extreme points of P, and if $P = -P$ it may be chosen so that $w = -v$. When the unit ball $\mathbb{B}$ of $\mathbb{M}$ is rotund, each diametral pair in P consists of exposed points of P, and if, in addition, $P = -P$, then each point of a diametral pair is the negative of the other.

A.7: If P is symmetric about the origin then P admits parallel supporting hyperplanes H_- and H_+ such that: (a) the distance between H_- and H_+ is equal to the width of P; (b) there are antipodal smooth points $-q$ and q of P such that $-q \in H_-$ and $q \in H_+$. If the ball $\mathbb{B}$ is smooth, every pair of hyperplanes that satisfies (a) must satisfy (b) as well.

A.8: Let H_- and H_+ be parallel supporting hyperplanes of P whose distance is equal to the width of P. Let $P_- = P \cap H_-$ and $P_+ = P \cap H_+$. Then $\dim P_- + \dim P_+ \geq n-1$, with $\dim P_- = \dim P_+ = n-1$ when P is symmetric. Further, let H denote the hyperplane that is parallel to and equidistant from H_- and H_+. Suppose that $q \in H$ and $s \in \mathbb{S}$ are such that $q \pm R_1(P)s \in H_\pm$. Then the sets $P^\pm = P_\pm \mp R_1(P)s$ are subsets of H that are not strictly separated in H, and when the ball $\mathbb{B}$ is smooth they are not even weakly separated in H.

A.9: Let $V \in \mathbb{M}$ be finite and let $P = \operatorname{conv}\{v : v \in V\}$. Then

$$R_n(P) = \sup_{v_0,\ldots,v_n \in V} R_n(\operatorname{conv}\{v_0,\ldots,v_n\}).$$

A.10: Let $Y \in \mathbb{M}^*$ be finite, let $\beta_y \in \mathbb{R}$ for $y \in Y$ and let $P = \operatorname{conv}\{x : \langle x, y\rangle \leq \beta_y \text{ for all } y \in Y\}$. Then

$$r_n(P) = \min_{y_0,\ldots,y_n \in Y} r_n\left(\bigcap_{i=0}^{n}\{x : \langle x, y_i\rangle \leq \beta_{y_i}\}\right).$$

A.11: Suppose that W is a finite subset of $\mathbb{M}$. For each $w \in W$ and $x \in \mathbb{M}$, let $\varphi_w(x) = \|x - w\|$, and then set $\Phi(x) = \sup_{w \in W} \varphi_w(x)$. The function Φ is a convex contraction whose global minimum is the circumradius of the set W.

A.12: Let $P = \{x : a_1 x \leq \beta_1, \ldots, a_m \leq \beta_m\}$. Then the inradius of P is the solution of the optimization problem:

$$\begin{array}{ll} \sup \xi & \\ a_i x + \xi\|a_i\|^* \leq \beta_i & (i = 1,\ldots,m). \end{array}$$

A. 13: Let P be a presented polytope in $\mathbb{R}^n_p$ of size L. Then the size of the following numbers is bounded by $16pn^4L$ or $16\overline{p}n^4L$ respectively:

$$\begin{array}{cc} r_1(P)^p \text{ for } p \in \mathbb{N} \cup \{\infty\} & R_1(P)^{\overline{p}} \text{ for } \overline{p} \in \mathbb{N} \cup \{\infty\} \\ R_n(P)^p \text{ for } p \in \mathbb{N} \cup \{\infty\},\ P \text{ symmetric} & R_n(P)^p \text{ for } p \in \{1,2,\infty\} \\ r_n(P)^{\overline{p}} \text{ for } \overline{p} \in \{1,\infty\} & r_n(P)^{\overline{p}} \text{ for } p \in \mathbb{N} \cup \{\infty\},\ P \text{ symmetric} \end{array}$$

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