Analysis and numerics of the mechanics of gels

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Abstract

In this thesis a mathematical model of polymer gel dynamics is proposed and analyzed. This work is motivated by problems in biomedical device manufacturing. The goal of this thesis is to develop and analyze models of gels consisting of balance laws in the form of systems of partial differential equations and boundary conditions. The model based on mixture theory accounts for nonlinear elasticity, viscoelasticity, transport, and diffusion. The derived model includes as limiting cases incompressible elasticity, viscous incompressible fluid, and Doi's stress diffusion equations.

Two classes of problems are considered. The first class addresses nonlinear problems in special domains and the second class addresses linear problems in arbitrary domains. Special emphasis is placed on linear problems with the goal of studying and implementing finite element methods. The first class of problems includes a one dimensional free boundary problem analyzed in terms of one-dimensional hyperbolic theory. The second class includes coupled elasticity and fluid flow problems. One challenging issue is accounting for the fact that, although the gel may be incompressible, the polymer may experience large changes of volume.

Numerical analysis of elastic solids and polymer gels is carried out. The Taylor-Hood algorithm for Stokes flow is applied to linearly visco-hyperelastic polymers. The simulations show the presence of stress concentrations at the boundary which relax over time. In the case of a gel, the conditionally stable mixed finite element method proposed by Feng and He for Doi's stress-diffusion coupling model is modified to handle the case of polymer viscosity. The modified numerical scheme is shown to be unconditionally stable and convergent.

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Chapter 1

Introduction

Polymeric networks as well as melts may form gels in the presence of a solvent such as water. The polymer confines the solvent and, in turn, the solvent prevents the gel from collapsing into a dry material. In addition to polymers, many types of molecules may form gel states in the presence of solvent. Gels are present in nature and occur in the manufacturing of biomedical devices such as pacemakers, artificial skin and bone prostheses. Gels may have the appearance and behavior of very hard solids.

Ideally, models of mechanical gel behavior should account for solvent diffusion, transport of polymer and solvent, friction and viscosity, elasticity, and time relaxation. The goal of this thesis is to develop and analyze models of gels consisting of balance laws in the form of systems of partial differential equations and boundary conditions.

In some gels electric and chemical effects may dominate and interact with the mechanical ones. One type of these gels, known as hydrogels, consists of polymer networks that can experience large changes in volume and exhibit complex temperature or chemically induced phase transition behavior between collapsed and swollen states [59].

The current work addresses gels where the mechanical effects dominate. These are relevant in applications to biomedical devices. Such devices often consist of more than one material, for example polymer and metal. Upon implantation of the device into the body, body fluids turns the material into a gel. This causes a buildup of stress, particularly at the interfaces between different materials, and can lead to device failure. Mathematical models that account for time evolution of stresses within the lifetime of the device are crucial to the biomedical industry.

The governing system of equations which are derived provides the framework for the problems to analyze. One feature of the governing system is the coupling of the equations of elasticity and Navier-Stokes. It is well known that open problems in each of these theories remain and are inherited by the present model. This together with physical goals motivates the choice of problems addressed here. Analytic and numerical studies of equilibrium states carried out by Zhang [72] and Micek, Rognes and Calderer [11] are relevant to the analysis of the elasticity operator in the model. However, gel behavior is inherently a time evolution problem due to the combined effects of transport, diffusion and dissipation. On the other hand, scaling arguments justify neglecting early dynamics. For instance, the size of polymer dissipation effects with respect to inertia results in the latter being dominant at time scales smaller than 10^{-7} seconds [12]. Such time scales are negligible for biomedical devises which often have lifetimes of 20 years. From a different point of view, the model proposed here shares analogies with deformable porous media flow models but has the additional complication of accounting for interaction between fluid and polymer through the Flory Huggins energy.

The research in this thesis consists of two main groups of topics. The first one includes special geometries and regimes, and the second one addresses the whole system in arbitrary domains and neglects inertia. Special emphasis is placed on linearized problems with the goal of studying and implementing finite element discretizations. To the author's knowledge, these are all open problems in the literature. The first class of problems includes a one dimensional free boundary problem transformed to a fixed boundary and analyzed in terms of one-dimensional hyperbolic theory. The second class includes the analysis of coupled elasticity and fluid flow problems for both viscous and non-viscous solvents, particularly the study of a finite element discretization. One challenging issue is accounting for the fact that, although the gel may be incompressible, the polymer may experience large changes of volume.

In this thesis a model of the dynamics of gels is derived from the point of view of the theory of mixtures. One goal is to compare the derivation method, assumptions and resulting equations with other models available in the literature, and determine their regimes of validity. The selection of constraints plays a main role in this development. The free energy of the model consists of the Flory-Huggins mixing energy together with an isotropic nonlinearly elastic contribution. The model accounts for viscoelastic behavior of the polymer yielding a fully nonlinear reversible stress on time-independent deformations. Physically realistic boundary conditions are listed that make the governing system mathematically well posed, and describe the free boundary evolution for special geometries. The Second Law of Thermodynamics yields an energy dissipation inequality for the total energy of the system. This inequality motivates the definition of a Rayleighian of the system that, in turn, yields a variational principle to derive governing equations. This is, in fact, the method employed by Yamaue and Doi in deriving the stress-diffusion coupling model ([68], [69], [70] and [19]). The gel is assumed to be non-ionic, and thermal effects are neglected [71].

The model is developed in chapter 2 following the approach of mixture theory. It is assumed that the gel is an incompressible and immiscible mixture of polymer and fluid [66], [13], [21]. Laws of balance of mass and linear momentum are postulated for a system with two components: solvent and elastic solid. The free energy is of the Flory-Huggins type with nonlinearly elastic and isotropic stored energy function ([23], [48], [47] and [50]). Action-reaction forces between polymer and solvent particles acting as friction forces are included. The Second Law of Thermodynamics is postulated in the form of the Clausius-Duhem inequality, accounting for the appropriate constraints. This yields information on the form of the constitutive equations, in particular, the dependence of the reversible components of the stress on derivatives of the free energy, the friction forces as depending on the relative velocity, and the residual inequality to be satisfied by the dissipative part of the stress.

The polymer and solvent in the gel form an immiscible mixture since the Flory-Huggins free energy depends explicitly on volume fractions. Recall that a mixture is miscible if the corresponding equations do not allow distinguishing between components, that is, they are independent of volume fractions [66]. Another characterization of mixtures is in terms of the compressibility of its components, which is formulated in terms of its component densities. For this, it is relevant to distinguish between the *bulk* density, ρ, and the *true* or *intrinsic* density, γ, of each component, the first one representing the mass of the component per unit volume of mixture, and the second one the mass per unit volume of the component, that is the density of the material in its isolated state. The ratio between the two densities defines the volume fraction, $\phi = \gamma\rho^{-1}$. A component is incompressible if its intrinsic density, γ is constant. Likewise, an incompressible mixture is defined as consisting of incompressible components. The role of such conditions is explained next.

The equations of balance of mass and linear momentum provide two scalar equations for the densities, ρ_1 and ρ_2 and two vector equations for the velocity fields $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. A third scalar equation is given by the constraint that the sum of volume fractions adds up to one (2.1.18), since no third material component or void is present. Altogether, in the case of a compressible mixture, these equations are not sufficient to determine the independent fields

$\{\rho_1, \rho_2, \phi_1, \phi_2, \mathbf{v}_1, \mathbf{v}_2\}$ of the problem. A closure condition is needed for the problem to be mathematically well-posed. The choice of such an additional equation is a main challenge arising in the modeling of swelling systems where compressibility of the components is a relevant property of the material, such as in the case of clays ([49], [5] and [6]).

The situation in polymeric gels is somehow different since a natural candidate for the closure condition is the incompressibility of the solvent. Accordingly, $\rho_2 = \phi_2$ is set in this study. The resulting system consists of equations (2.1.5), (2.1.9), (2.1.7), (2.1.8) and (2.1.18) for the fields $\{\rho_1, \phi_1, \phi_2, \mathbf{v}_1, \mathbf{v}_2\}$. Moreover, in this chapter, the main focus is on the case that the polymer is also incompressible, and further set $\rho_1 = \phi_1$. This is a reasonable assumption for many types of linear entangled polymers, and it precludes the well-known mathematical difficulties related to the formation of shocks ([15]). The fields of the problem now include the pressure p instead of the density ρ_1. Note that the assumption of polymer incompressibility might not be appropriate near the threshold of a volume phase transition where the system may become highly compressible [51].

Chapter 3 is devoted to the survey of models related to the one studied here and related mathematical problems. Related gel models include the approaches of Tanaka, Doi and Suo. A lot of current mathematical activity is devoted to the analysis of nonlinear elasticity and Oldroyd-B models for viscoelastic fluids and solids. These are models for single phase materials but pose the same challenge as the current work in that they combine Lagrangian and Eulerian dynamics. Indeed, polymer elasticity is naturally modeled in the Lagrangian frame, with constitutive equations depending on the deformation gradient, while fluid flow is typically set in the Eulerian frame, with dependence of the stress tensor on the gradient of the velocity field in the case of Newtonian dynamics. The compatibility condition relating the time derivative of the deformation gradient tensor with the velocity gradient brings a new transport equation into the model for which there is no existence theory. From a numerical perspective, it is relevant to emphasize the lack of finite element error estimates for incompressible elasticity [37].

Chapter 4 is devoted to special problems and regimes. First, the swelling of a polymer initially occupying a one-dimensional strip $(-L, L)$ and surrounded by fluid is considered. Initial data is prescribed. Upon transforming the problem to one in a fixed domain, the Cauchy problem is analyzed for periodic boundary conditions. The resulting system of equations is nonlinear and of mixed hyperbolic/parabolic type. The analysis of the inviscid case is presented. The system is then first order, quasilinear, and symmetric hyperbolic. Although formation of shocks is not an issue in the analysis, the hyperbolic nature of the equations poses difficulties. The formulation of the problem in $\mathbf{R}^n$ is

$$\begin{aligned} &A^0(u)u_t + \sum_{\gamma=1}^{n} A^\gamma(u)\partial_\gamma u + F(x,t,u) = 0, \\ &u(x,0) = u_0(x), \end{aligned}$$

where u is a vector function in $\mathbf{R}^n$, $A^\gamma(u)$ are smooth, symmetric, bounded matrix functions for $\gamma = 0, \ldots, n$, and A^0 is uniformly positive definite. It is shown using Picard iterations that under appropriate assumptions, a unique classical solution of the general system in $\mathbf{R}^n$ exists for small times.

The second special geometry is plane shear Newtonian flow along the x direction, and with fields depending on y. A pressure gradient along the flow direction is prescribed. The convection terms vanish yielding a parabolic system. The pressure can be determined directly from the system of equations up to a function of time. Except for the case of neo-hookean elastic energy, the system of equations is nonlinear. The Galerkin method is used to prove long-time existence and uniqueness of weak solutions in the case of periodic boundary conditions.

Assuming Newtonian viscosity of the polymer only, the equations of time-dependent incompressible linear elasticity follow. In this approach, the fluid component is neglected. The main goal of this analysis is to set the stage for the study of the gel problem by summarizing some known results and techniques. The unknowns of the elastic system are the displacement $\mathbf{u}$ and the pressure p. Initial conditions are imposed on the displacement. For boundary conditions, the domain is divided into two parts: Γ_0 and Γ_1. On Γ_0, a given displacement is imposed. For example, if Γ_0 is a surface to which the material is clamped or chemically glued, such as occurs in the case of biomedical devices implanted in the human body [11], the displacement on Γ_0 is zero. On Γ_1, a given normal stress is imposed. Existence of solutions is proved using two methods: Laplace transforms and Galerkin approximations.

The previously described elasticity problem is one of the limiting regimes of the gel model. Another limiting regime is the Navier Stokes system, which emerges in the limit of zero polymer volume fraction. A third special regime arises when neglecting viscosities, inertia, and linearizing about the identity. The system then yields Doi's stress-diffusion coupling model [69], which was analyzed by Feng and He [22] in the case of an impermeable boundary.

In chapters 5 and 6, the governing system is considered in the case of negligible inertia and is linearized about a constant deformation gradient. In particular, the linearization allows for swelling in the reference configuration, with the presence of residual stresses that now appear as forcing terms in the balance of linear momentum. It is assumed that the stress of the polymer is of Oldroyd type, that is, it includes elasticity and Newtonian viscosity. Two separate cases cases are considered: inviscid solvent and viscous solvent.

The analyses of Feng and He [22] are relevant to the case of inviscid solvent, presented in the first part of chapter 5. However, the system proposed in this thesis differs from theirs in that it accounts for viscous effects, includes the mixing energy and allows for residual stress in the polymer. Moreover, Feng and He only consider pure traction problems whereas the approach described here allows for mixed displacement-traction and pure displacement boundary conditions, both cases relevant to biomedical applications. The existence and uniqueness of solution for the inviscid fluid problem is proved using a Galerkin method combined with energy estimates. The mixed finite element method proposed in [22] for discretization in space adapted to this case is shown to be well-posed and convergent to the weak solution with the same rates of convergence as found in [22]. The time discretization scheme in [22] is modified slightly because of the polymer viscosity terms. This modification makes the scheme unconditionally stable at the price of not being able to decouple the computation of an auxiliary variable that represents polymer volume change from the computation of the pressure and displacement. This time discretization scheme is also shown to converge. This work is shown in chapter 6.

The case of viscous polymer and fluid is studied in the second part of chapter 5. The boundary conditions for the elasticity equation are of mixed displacement-traction, with part of the boundary being impermeable, whereas permeability is allowed elsewhere on the boundary. The case of pure displacement can be obtained as a special case of this analysis and it is omitted. Global in time existence of weak solutions is proved by two different methods: Laplace transform and energy methods.

Chapter 2

Derivation of model

In this chapter, a polymer gel dynamics model is derived from the points of view of continuum mechanics and mixture theory. In section 1, the governing equations of an incompressible mixture are derived. The constraint (2.1.18), in addition to bringing a Lagrange multiplier, p, into the constitutive equations of the stress and of the diffusive forces, establishes a relation between the laws of balance of mass of the components, reducing them to one single equation in the system. Such an equation can be presented in three different forms: the Eulerian equation of balance of mass of the polymer (2.1.11), the one for the solvent (2.1.12), and the Lagrangian form for the polymer (2.1.16). The latter can also be viewed as a constraint on the variables F, the polymer gradient of deformation matrix, and ϕ_1, allowing for the elimination of ϕ_1 from the system, with the model becoming purely mechanical. The choice between these forms depends on the initial and boundary conditions of the problem at hand.

Mathematically, in the case that the polymer viscosity is Newtonian, the system is of mixed hyperbolic and parabolic type. A special challenge comes from the fact that the system combines the Eulerian description of fluids with the Lagrangian one of solids. This results in a chain rule equation relating the time derivative of the deformation gradient with the spatial gradient of velocity (2.5.69). The analysis of the latter equation is presented in the article by P.L. Lions and J.R. DiPerna [17]. The model presented here bears analogies with that studied by Liu and Walkington, for a mixture of elastic particles in a viscous fluid. However, the latter does not involve diffusion effects [46].

In section 2, the energy of the system is studied and the equations of the relaxation part of the stress tensor of the polymer are derived. This consists of a nonlinear elastic Cauchy contribution together with osmotic pressure coming from the mixing energy. The stress tensor of the fluid involves the corresponding osmotic pressure; in addition, both stress tensors involve the hydrostatic pressure due to the saturation constraint. In section 3, the energy decay of the system is studied. The formulation of the viscoelastic stress of the polymer is presented in section 4. The Kaye-BKZ model for gels is formulated since it yields the (elastic-osmotic) relaxation stress at the equilibrium limit.

The balance laws for the components of the mixture yield a system of equations for the gel, as presented in section 5. Specifically, they give equations of balance of linear momentum of the center of mass velocity $\mathbf{V}$, and the relative velocity $\mathbf{U}$, together with one of the equations of conservation of mass. Whereas reference to individual components may be avoided in dealing with bulk phenomena, their separate roles re-emerge on the boundary, with either the solvent component or the solid one becoming relevant, according to the boundary conditions.

From the governing system, equations for special regimes are obtained. One of the regimes is purely diffusive, with no net transport taking place, such as the case of swelling of a gel in its own solvent. A system with only transport and no diffusion also results from the equations. For instance, a gel subject to a shearing flow and with non-permeable boundary. The mixture behaves as a single material with elastic and viscous properties. The problem becomes mathematically analogous to viscoelastic flow models analyzed in [41].

In the one-dimensional geometry corresponding to a strip domain, and neglecting viscous stress (by a scaling argument, this corresponds to addressing the early dynamics, rather than the better understood relaxation regimes), the system is of hyperbolic type with a dissipative source term depending on the diffusive velocity. The extensional stress is given by the function $G(\phi)$; the problem is mathematically well-posed provided G is monotonically decreasing (increasing when regarded as a function of the elastic deformation). In particular, this guarantees the propagation of the swelling interface separating gel from pure solvent [12]. It is found that the monotonicity property of the stress holds for the polymer data, whereas loss of monotonicity occurs in polysaccharides. Specifically, an interval of volume fraction in a high swelling regime was found where the stress changes monotonicity. This may suggest the onset of de-swelling or a volume phase transition taking place. The weak elasticity is responsible for the loss of monotonicity of the stress. From a related point of view, the study of the early dynamics of a dry polymer slab as it comes into contact with its own solvent shows that type II diffusion is a hyperbolic phenomena, rather than a diffusion process as its name would otherwise indicate ([12], [64]).

In the last part of section 5, it is explained how the model that was derived relates to the quasi-static *stress-diffusion coupling model* presented by Yamaue and Doi, and analyzed in some of their articles ([68], [69] and [70]). Omitting mass inertia terms and the Flory-Huggins contribution to the energy yields the latter; in their applications, they further take the linear version of the model with equations that include the elastic equilibrium equation of the gel and Darcy's law for the solvent. This generalizes earlier work by Tanaka and Filmore [60] based on the elasticity equation. The stress-diffusion coupling model, in turn, results from modeling work by Doi [18] based on a variational principle for the Rayleighian. The section is concluded by illustrating such a derivation in the case that the functional includes the constraint of the model and the mixing energy.

Although both approaches lead to the same system of equations, one advantage of the method based on the Second Law of Thermodynamics is that it readily gives the constitutive equations of the reversible regime.

2.1 Balance of mass, transport equations and constraints

It is assumed that the components of the gel occupy domains $\Omega_a \subset \mathbf{R}^3$, $a = 1, 2$ in the reference (Lagrangian) configuration, with φ_0 representing the reference volume fraction of polymer. Here the sub-indices *1* and *2* refer to polymer and solvent, respectively. The Lagrangian domains Ω_a, $a = 1, 2$ may be distinct, since the polymer and fluid occupy different locations in space, previous to mixing. However, both components share a common domain, Ω, in the Eulerian representation of the gel. The variables $\mathbf{X} \in \Omega_a$ and $\mathbf{x} \in \Omega$ represent Lagrangian and Eulerian coordinates, respectively. The deformation maps of polymer and solvent are given by sufficiently smooth invertible functions

$$\mathbf{x} = \mathcal{M}(\mathbf{X}, t), \quad \mathbf{X} \in \Omega_1, \tag{2.1.1}$$

$$\mathbf{x} = \mathcal{N}(\mathbf{X}, t), \quad \mathbf{X} \in \Omega_2 \tag{2.1.2}$$

with $F = \nabla_{\mathbf{X}} M(\mathbf{X}, t)$ denoting the gradient of deformation of the polymer. The material velocities of polymer and solvent, respectively, are given by

$$\tilde{\mathbf{v}}_1(\mathbf{X}, t) = \frac{\partial \mathcal{M}}{\partial t}(\mathbf{X}, t), \quad \mathbf{X} \in \Omega_1,$$
$$\tilde{\mathbf{v}}_2(\mathbf{X}, t) = \frac{\partial \mathcal{N}}{\partial t}(\mathbf{X}, t) \quad \mathbf{X} \in \Omega_2.$$

The corresponding velocity fields are

$$\mathbf{v}_1(\mathbf{x}, t) = \tilde{\mathbf{v}}_1(\mathcal{M}^{-1}(\mathbf{x}, t), t), \quad \mathbf{v}_2(\mathbf{x}, t) = \tilde{\mathbf{v}}_2(\mathcal{N}^{-1}(\mathbf{x}, t), t). \tag{2.1.3}$$

Let $\mathcal{T}_1(\mathbf{x}, t)$ and $\mathcal{T}_2(\mathbf{x}, t)$ denote Cauchy stress tensors. Each one includes elastic and dissipative contributions. Also take into account friction forces $\mathbf{f}_a$ per unit volume that the polymer exerts upon the fluid, and vice-verse.

2.1.1 Governing equations for a mixture of incompressible solvent and compressible polymer

According to the theory of mixtures, polymer and solvent particles may occupy the same location, with volume fractions $\phi_1(\mathbf{x}, t)$, $\phi_2(\mathbf{x}, t)$, respectively. Assume also that no other material is present in the region, so

$$\phi_1(\mathbf{x}, t) + \phi_2(\mathbf{x}, t) = 1 \tag{2.1.4}$$

holds. Let ρ_1 and ρ_2 denote the mass densities of the components per unit volume in space. These are related to the *true* or *intrinsic densities* $(\frac{\text{mass of component}}{\text{volume of component}})$, γ_1 and γ_2 as follows:

$$\rho_1 = \gamma_1 \phi_1, \quad \rho_2 = \gamma_2 \phi_2.$$

The volume fractions ϕ_1 and ϕ_2 are independent fields of the constitutive equations postulated later, and consequently, the mixture is immiscible [66]. The local forms of the laws of balance of mass and linear momentum in Eulerian coordinates are given by

$$\frac{\partial \rho_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla)\rho_1 + \rho_1 \nabla \cdot \mathbf{v}_1 = 0 \tag{2.1.5}$$

$$\frac{\partial \rho_2}{\partial t} + (\mathbf{v}_2 \cdot \nabla)\rho_2 + \rho_2 \nabla \cdot \mathbf{v}_2 = 0 \tag{2.1.6}$$

$$\rho_1 \frac{\partial \mathbf{v}_1}{\partial t} + \rho_1(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 = \nabla \cdot \mathcal{T}_1 + \mathbf{f}_1 \tag{2.1.7}$$

$$\rho_2 \frac{\partial \mathbf{v}_2}{\partial t} + \rho_2(\mathbf{v} \cdot \nabla)\mathbf{v}_2 = \nabla \cdot \mathcal{T}_2 + \mathbf{f}_2 \tag{2.1.8}$$

The independent fields of the problem are $\{\rho_1, \rho_2, \phi_1, \phi_2, \mathbf{v}_1, \mathbf{v}_2\}$ ($\mathcal{T}_1$, $\mathcal{T}_2$, $\mathbf{f}_1$ and $\mathbf{f}_2$ will be given in terms of the former). Observe that the previous system together with (2.1.4) is not sufficient to determine all the independent variables since one additional scalar equation is needed. Therefore, one closure condition should be added to the system to make it well-posed. There is no general physical law dictating such a condition and, in some cases, the criterion is to recover well known phenomenological relations of the theory. For instance, this is the case of Darcy's Law in porous media [49], [5] and [6]. Another choice found in the literature is to require that the total equilibrium stress is zero which leads to a relation between the pressure part of the stress tensors. The resulting equations which hold at equilibrium only justify the validity of the linear theory.

In the case of polymer gels, it is reasonable to assume that the solvent is incompressible, leading to the relation $\gamma_2 = 1$. (For simplicity, the constant intrinsic density is taken to be 1.) Consequently, the system (2.1.4)-(2.1.8) is well posed. Note that in such a case, the governing system does not involve any Lagrange multipliers. This is indeed the case for compressible elasticity, and it is consistent with the fact that γ_1 is allowed to be free. On the other hand, it does not correspond to the gel models by Doi and Yamaue in which the Lagrange multiplier plays a prominent role. It is argued here that the additional constraint of polymer incompressibility is at play. Two cases are distinguished according to this observation.

The balance laws in the case of incompressible solvent and compressible polymer consist of (2.1.4)-(2.1.8) but replacing ρ_2 with ϕ_2 in (2.1.6). The constitutive equations for $\mathcal{T}_i$ and $\mathbf{f}_i$, $i = 1, 2$, depend on the fields $\{\rho_1, \phi_1, \phi_2, \mathbf{v}_1, \mathbf{v}_2\}$. Moreover, using equation (2.1.4) to eliminate ϕ_2, a governing system is left for the variables $\{\rho_1, \phi_1, \mathbf{v}_1, \mathbf{v}_2\}$ consisting of (2.1.5)-(2.1.8) with equation (2.1.6) replaced with

$$\frac{\partial \phi_1}{\partial t} - \nabla \cdot \big((1 - \phi_1)\mathbf{v}_2\big) = 0. \tag{2.1.9}$$

The assumption of polymer compressibility may be justified in the following situations:

1. Experiments involving volume transitions in gels suggest a highly compressible regime near the critical volume fraction ϕ_c where the first order transition takes place. The transition from a swollen regime to a collapsed one (or vice versa) may involve a significant expansion (or compression) of the polymer.

2. Incompressibility is itself an ideal property; indeed, it may be more realistic to think of the polymer as being almost incompressible. The incompressible regime arises then as a limit of almost incompressible ones. This is a condition that will be formulated in terms of the free energy.

Away from the phase transition, the polymer may then be considered as being incompressible with compressibility manifesting itself at the transition volume fraction.

2.1.2 Governing equations for a mixture of incompressible solvent and polymer

For simplicity, take $\gamma_1 = \gamma_2 = 1$. In this case, the densities and volume fractions coincide, and henceforth only the latter will be utilized,

$$\rho_1 = \phi_1, \quad \rho_2 = \phi_2. \tag{2.1.10}$$

Incompressibility of mixture components allows for deformations with Jacobian different than one. Indeed, changes in density of one component occur by changes of volume fraction. In particular, an extension of the polymer network that decreases the volume fraction ϕ_1 may occur at the expense of increasing the volume fraction ϕ_2.

The local forms of the laws of balance of mass and linear momentum in Eulerian coordinates are

$$\frac{\partial \phi_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla)\phi_1 + \phi_1 \nabla \cdot \mathbf{v}_1 = 0 \tag{2.1.11}$$

$$\frac{\partial \phi_2}{\partial t} + (\mathbf{v}_2 \cdot \nabla)\phi_2 + \phi_2 \nabla \cdot \mathbf{v}_2 = 0 \tag{2.1.12}$$

$$\phi_1 \frac{\partial \mathbf{v}_1}{\partial t} + \phi_1(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 = \nabla \cdot \mathcal{T}_1 + \mathbf{f}_1 \tag{2.1.13}$$

$$\phi_2 \frac{\partial \mathbf{v}_2}{\partial t} + \phi_2(\mathbf{v} \cdot \nabla)\mathbf{v}_2 = \nabla \cdot \mathcal{T}_2 + \mathbf{f}_2 \tag{2.1.14}$$

where $\nabla \equiv \nabla_{\mathbf{x}}$ refers to gradient with respect to the Eulerian variables. The kinematic compatibility condition between the time derivative of the deformation gradient and the velocity gradient results from the chain rule:

$$F_t + (\mathbf{v}_1 \cdot \nabla)F = (\nabla \mathbf{v}_1)F. \tag{2.1.15}$$

The Lagrangian form of the equation of balance of mass of the polymer component is

$$\begin{aligned} \int_\Omega \phi_1 \, d\,\mathbf{x} = & \int_{\Omega_0} \varphi_0 \, d\mathbf{X} \\ = & \int_{\Omega_0} \phi_1 \det \, F \, d\mathbf{X}, \end{aligned}$$

where $0 \leq \varphi_0 \leq 1$ denotes the volume fraction of polymer in the reference configuration. It is assumed that it is satisfied for all parts of the body Ω_0, yielding the local constraint

$$\phi_1 \det \, F = \varphi_0. \tag{2.1.16}$$

The latter relation is equivalent to equation (2.1.11). Defining the center of mass velocity

$$\mathbf{V} = \sum_{a=1,2} \phi_a \mathbf{v}_a, \tag{2.1.17}$$

it follows that addition of equations (2.1.11) and (2.1.12), taking (2.1.4) into account yields

$$\nabla \cdot \mathbf{V} = \sum_a \nabla \cdot (\phi_a \mathbf{v}_a) = \sum_a \mathbf{v}_a \cdot \nabla \phi_a + \phi_a \nabla \cdot \mathbf{v}_a = 0. \tag{2.1.18}$$

This suggests formulating the governing system as consisting of equations (2.1.4), (2.1.13), (2.1.14) and one of the following additional sets:

- equations (2.1.11) and (2.1.12),
- equations (2.1.11) and (2.1.18), or (2.1.12) and (2.1.18),
- equations (2.1.16) and (2.1.18).

Remark. A special case of mathematical interest arises by adding an additional constraint to the system, that is, requiring that the fluid component can only experience isochoric deformations,

$$\nabla \cdot \mathbf{v}_2 = 0. \tag{2.1.19}$$

It is easy to see that in one-dimensional geometry, the only change in volume fraction resulting from such dynamics is purely hyperbolic, in the sense that the initial volume fraction profile travels through the domain undisturbed. However, in two and three dimensional geometries, it may still allow for non-trivial swelling dynamics. It is also an appropriate model of how rheological properties of the polymer depend on the amount of solvent held by the polymer.

The previous set of equations do not yet fully determine the system, since constitutive equations and additional kinematic relations need to be prescribed.

The balance of energy is now formulated. Letting ε_a, $a = 1, 2$ denote component internal energy densities, write the equation of balance of energy:

$$\sum_a (\phi_a \frac{\partial \varepsilon_a}{\partial t} + \phi_a (\mathbf{v}_a \cdot \nabla) \varepsilon_a) = \sum_a (\mathrm{t}r(\nabla \mathbf{v}_a \mathcal{T}_a) + \nabla \cdot \mathbf{q}_a + \phi_a r_a + \hat{\varepsilon}_a). \tag{2.1.20}$$

Here $\hat{\varepsilon}_a(\mathbf{x}, t)$ represent the internal energy rate production in each component, $\mathbf{q}_a$, $a = 1, 2$ the heat flux vectors and r_a radiation terms. The friction forces and the internal production rate quantities are required to satisfy

$$\mathbf{f}_1 = -\mathbf{f}_2 \tag{2.1.21}$$

$$\sum_{a=1,2} \hat{\varepsilon}_a(\mathbf{x}, t) + (\mathbf{v}_a - \mathbf{V}) \cdot \mathbf{f}_a = 0. \tag{2.1.22}$$

The free energy density and the entropy densities of the mixture, respectively, can be written in terms of their components as follows:

$$\Psi = \phi_1 \psi_1(\phi_1, \phi_2, F) + \phi_2 \psi_2(\phi_1, \phi_2) \tag{2.1.23}$$

$$\eta = \phi_1 \eta_1(\phi_1, \phi_2, F) + \phi_2 \eta_2(\phi_1, \phi_2). \tag{2.1.24}$$

Moreover, use is made of the relations

$$\psi_a = \varepsilon_a - T_a \eta_a, \tag{2.1.25}$$

defining the free energy density ψ_a of the components in terms of the corresponding internal energy ε_a and entropy η_a; T_a denotes absolute temperature. The free energy Ψ of the system accounts for the Flory-Huggins and elastic contributions given in the next subsection.

Remark. It may seem more reasonable to directly postulate Ψ rather than appealing to components ψ_a, $a = 1, 2$ as in (2.1.23), and likewise for the entropy of the system. For a justification of the approach discussed above, see [26].

Expressions of the component free energies [26] that yield (2.2.46) are

$$\psi_1 = \frac{K_B T}{V_m} \chi \phi_2^2 + \frac{K_B T}{N_1 V_m} \log \phi_1 + W(F), \tag{2.1.26}$$

$$\psi_2 = \frac{K_B T}{V_m} \chi \phi_1^2 + \frac{K_B T}{N_2 V_m} \log \phi_2. \tag{2.1.27}$$

2.1.3 Thermodynamics and constitutive equations

The second law of thermodynamics for the mixture is now postulated:

$$\sum_{a=1,2} \phi_a \dot{\eta}_a - \nabla \cdot (\frac{\mathbf{q}_a}{T_a}) - \phi_a \frac{r_a}{T_a} \geq 0. \tag{2.1.28}$$

For any quantity u_a associated to component a, the symbol $\dot{u}_a$ denotes the material time derivative of u_a, which is given in Eulerian coordinates by the expression

$$\dot{u}_a = \frac{\partial u_a}{\partial t} + \mathbf{v}_a \cdot \nabla u_a.$$

Substituting (2.1.25) into (2.1.28) and the balance of energy (2.1.20), the following is obtained

$$\sum_a \frac{1}{T_a} \{\mathrm{t}r(\mathcal{T}_a^T \nabla \mathbf{v}_a) - \dot{T}_a \phi_a \eta_a - \phi_a \dot{\psi}_a + \frac{1}{T_a} \mathbf{q}_a \cdot \nabla T_a - \mathbf{f}_a \cdot (\mathbf{v}_a - \mathbf{V})\} \geq 0. \tag{2.1.29}$$

From now on, thermal effects are neglected and isothermal regimes only are considered, so that the previous inequality expresses mechanical dissipation,

$$\sum_a \mathrm{t}r(\mathcal{T}_a^T \nabla \mathbf{v}_a) - \phi_a \dot{\psi}_a - \mathbf{f}_a \cdot \mathbf{v}_a \geq 0. \tag{2.1.30}$$

The entropy inequality is reformulated when the fields satisfy the constraint (2.1.18). The following relations, based on the chain-rule, will be used in simplifying the entropy inequality:

$$\begin{aligned} \nabla \mathbf{v}_1 &= \dot{F} F^{-1} \\ \nabla \cdot \mathbf{v}_1 &= \mathrm{t}r(\nabla \mathbf{v}_1) = \mathrm{t}r(F^{-1} \dot{F}). \end{aligned} \tag{2.1.31}$$

Let p denote the Lagrange multiplier associated with constraint (2.1.18). The dissipation inequality should hold for those processes such that the constraints are approximately satisfied,

$$\sum_a \mathrm{t}r(\mathcal{T}_a^T \nabla \mathbf{v}_a) - \phi_a \dot{\psi}_a - \mathbf{f}_a \cdot \mathbf{v}_a - p \nabla \cdot (\phi_a \mathbf{v}_a) \geq 0. \tag{2.1.32}$$

Using (2.1.31), the latter can be written as follows

$$\begin{aligned}(2.1.33)\qquad & tr\{\big(F^{-1}\mathcal{T}_1^T + \phi_1^2(\frac{\partial\psi_1}{\partial\phi_1} - \frac{\partial\psi_1}{\partial\phi_2})F^{-1} - \phi_1\frac{\partial\psi_1}{\partial F}^T + p\phi_1 F^{-1}\big)\dot{F}\} \\ & +\mathrm{tr}\{\big(\mathcal{T}_2^T + \phi_2^2(-\frac{\partial\psi_2}{\partial\phi_1} + \frac{\partial\psi_2}{\partial\phi_2} + p)I\big)\nabla\mathbf{v}_2\} \\ & -(\mathbf{f}_1 - p\nabla\phi_1)\cdot\mathbf{v}_1 - (\mathbf{f}_2 - p\nabla\phi_1)\cdot\mathbf{v}_2 \geq 0.\end{aligned}$$

Requiring the inequality to be satisfied for all choices $F, \dot{F}, \phi_1, \phi_2, \mathbf{v}_1 - \mathbf{v}_2, \nabla\mathbf{v}_2$ yields the forms of $\mathcal{T}_a^r$, the reversible contributions to the stress tensor,

$$(2.1.34)\qquad \mathcal{T}_1^r = \phi_1\frac{\partial\psi_1}{\partial F}F^T - \big(\phi_1 p + \pi_1\big)I$$

$$(2.1.35)\qquad \mathcal{T}_2^r = -\big(\phi_2 p + \pi_2\big)I,$$

where

$$(2.1.36)\qquad \pi_1 = \quad \phi_1^2\big(\frac{\partial\psi_1}{\partial\phi_1} - \frac{\partial\psi_1}{\partial\phi_2}\big),$$

$$(2.1.37)\qquad \pi_2 = \quad \phi_2^2\big(-\frac{\partial\psi_2}{\partial\phi_1} + \frac{\partial\psi_2}{\partial\phi_2}\big)$$

denote the mixing components of the osmotic pressure. Substituting (2.1.34) and (2.1.35) into (2.1.33), the residual inequality for $\mathbf{f}_a$, and the dissipative parts $\mathcal{T}_1^d$ and $\mathcal{T}_2^d$ of the stress tensors follows:

$$(2.1.38)\qquad -\mathrm{tr}(\dot{F}^T\mathcal{T}_1^d F^{-T} + (\nabla\mathbf{v}_2)^T\mathcal{T}_2^d) + \sum_a(\mathbf{f}_a - p\nabla\phi_a)\cdot\mathbf{v}_a \leq 0.$$

Using once more the arbitrariness in the choice of $\dot{F}$, $\nabla\mathbf{v}_2$, and $\mathbf{v}_1 - \mathbf{v}_2$,

$$(\mathbf{f}_1 - p\nabla\phi_1)\cdot(\mathbf{v}_1 - \mathbf{v}_2) \leq 0.$$

In order for the previous inequality to hold, take

$$(2.1.39)\qquad \mathbf{f}_1 = p\nabla\phi_1 - \beta(\mathbf{v}_1 - \mathbf{v}_2), \quad \mathbf{f}_2 = p\nabla\phi_2 + \beta(\mathbf{v}_1 - \mathbf{v}_2),$$

where $\beta = \beta(\phi_1, \phi_2) > 0$ is related to the diffusion coefficient in Darcy's law as shown later. Using (2.1.39), inequality (2.1.38) finally yields,

$$(2.1.40)\qquad tr(\dot{F}^T\mathcal{T}_1^d F^{-T}) \geq 0$$

$$(2.1.41)\qquad tr((\nabla\mathbf{v}_2)^T\mathcal{T}_2^d) \geq 0.$$

Remark. In viscoelastic gels, the stress $\mathcal{T}_1$ depends on the past history of the deformation gradient. In such cases, the reversible stress component corresponds to the relaxation limit of $\mathcal{T}_1$, with the dissipative component being the difference between the total stress and its relaxation limit.

Remark. For the mixture with incompressible solvent and compressible polymer, the equations for the equilibrium part of the stress tensor are given by

$$\mathcal{T}_1 = \rho_1(\frac{\partial \psi_1}{\partial F})F^T - \rho_1^2 \frac{\partial \psi_1}{\partial \phi_1} I \tag{2.1.42}$$

$$\mathcal{T}_2 = -\big(\rho_1 \phi_2 (\frac{\partial \psi_1}{\partial \phi_2} - \frac{\partial \psi_1}{\partial \phi_1}) + \phi_2^2 (\frac{\partial \psi_2}{\partial \phi_2} - \frac{\partial \psi_2}{\partial \phi_1})\big) I. \tag{2.1.43}$$

2.2 Elastic and mixing energies

The assumption that the free energy $W(F)$ is isotropic implies that

$$W(F) = \hat{w}(I_1, I_2, I_3), \quad \text{with } B = FF^T \tag{2.2.44}$$

where

$$I_1 = \mathrm{tr}\, B, \quad 2I_2 = (\mathrm{tr}^2 B - \mathrm{tr} B^2), \quad I_3 = \mathrm{d}et\, B. \tag{2.2.45}$$

Combining the elastic free energy of the polymer and Flory-Huggins energy of the mixture gives the total free energy (2.1.23) of the mixture

$$\Psi = \frac{K_B T}{V_m}(\chi \phi_1 \phi_2 + \frac{1}{N_1}\phi_1 \log \phi_1 + \frac{1}{N_2}\phi_2 \log \phi_2) + \phi_1 W(F). \tag{2.2.46}$$

where

$$W(F) = \frac{3K_B T}{2N_x V_m}((\det B)^{\frac{1}{3}} - 1 - \frac{1}{6}\log(\det B)) + \mu w(I_1, I_2, I_3). \tag{2.2.47}$$

The following neo-Heokean elasticity is considered:

$$w(I_1, I_2, I_3) = \mathrm{tr}\, B - \mathrm{tr}\, I. \tag{2.2.48}$$

The parameters of the previous equations correspond to the following:

- V_m is the volume occupied by one monomer;
- K_B is the Boltzmann constant, and T is the absolute temperature;
- N_1, N_2 denote the number of lattice sites occupied by the polymer and the solvent, respectively;
- N_x is the number of monomers between entanglement points;
- χ is the Flory interaction parameter;
- β is the polymer drag coefficient;
- μ is a scaling parameter related to the shear modulus.

Parameter values appropriate to semi-dry polymers [7] are given next.

(2.2.49)

N_x	20
N_1	1000
N_2	1
V_m	$.1\text{n}m^3$
χ	.5
μ	$2 \times 10^2 - 2 \times 10^6 \frac{\text{pN}}{\text{n}m^2}$
β	$2.4 \times 10^{10} \frac{\text{pN}s}{\text{nm}^4}$
T	300^oK

Parameter values for polysaccharides [30] are as in the previous table with the following exceptions:

(2.2.50)

μ	$2 \times 10^{-5} \frac{\text{pN}}{\text{n}m^2}$
β	$2.4 \times 10^3 \frac{\text{pN}s}{\text{nm}^4}$

This section is concluded by calculating the expressions of the stress tensor.

$$\frac{\partial W(F)}{\partial F} = \frac{K_B T}{N_x V_m}((\det F)^{-\frac{1}{3}} - \frac{1}{2 \det F})\text{a}dj(F) + 2\mu F,$$

where $\mathrm{a}djF$ denotes the adjoint of F. Moreover,

$$\frac{\partial W(F)}{\partial F}F^T = \frac{K_BT}{V_mN_x}((\det F)^{\frac{2}{3}} - \frac{1}{2})\mathbf{I} + 2\mu FF^T. \tag{2.2.51}$$

Notice that the terms multiplying the identity matrix are elastic contributions to the osmotic pressure, which are denoted

$$\Pi_1 = \pi_1 + \frac{K_BT}{N_xV_m}(\frac{1}{2} - (\det B)^{\frac{1}{3}}), \;\; \Pi_2 = \pi_2, \tag{2.2.52}$$

where π_i denote the mixing contributions to the osmotic pressure (2.1.36) and (2.1.37). Using the calculation (2.2.51) in equations (2.1.34) and (2.1.35), the equilibrium stresses when the polymer is a neo-Hookean solid are obtained:

$$\mathcal{T}_1^r = -(\Pi_1 + p\phi_1)I + 2\mu\phi_1 B \tag{2.2.53}$$

$$\mathcal{T}_2^r = -(\phi_2 p + \Pi_2)I. \tag{2.2.54}$$

This section is concluded with the expression of the stress tensors when the elastic stored energy function of the polymer is as in (2.2.47). A calculation as in [50] gives

$$\mathcal{T}_1^r = \quad \phi_1\mathcal{H}(B) - (\Pi_1 + \phi_1 p)I, \quad \text{with} \tag{2.2.55}$$

$$\mathcal{H} = \quad 2\frac{\mu}{\det\, B^{\frac{1}{2}}}\big(\frac{\partial w}{\partial I_1}B + \frac{\partial w}{\partial I_2}(I_1B - B^2) + \det\, B\frac{\partial w}{\partial I_3}I\big). \tag{2.2.56}$$

Remark.

Observe that substituting equations (2.1.4) and (2.1.16) into (2.2.53) and (2.2.54) gives a fully mechanical form of the stresses, with dependence on gradient of deformation and its invariants, and on the Lagrange multipliers.

Notation. The extra stresses σ_1 and σ_2 denote the constitutive contributions in the total stress components, $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, that is

$$\sigma_1 = \mathcal{T}_1 + \phi_1 pI, \;\; \sigma_2 = \mathcal{T}_2 + \phi_2 pI. \tag{2.2.57}$$

2.3 Energy Dissipation

It is now shown that the rate of change of the total energy is bounded above by the work per unit time done by the boundary forces (i.e., the power of the forces acting on the boundary of the domain occupied by the gel).

Let $\Omega(t) \subset \mathbf{R}^3$ denote the region occupied by the gel at time t. Let ν represent the unit outer normal to the boundary, $\partial\Omega(t)$.

Suppose that $\mathbf{v}_1(\mathbf{x}, t)$, $\mathbf{v}_2(\mathbf{x}, t)$, $\phi_1(\mathbf{x}, t)$, $\phi_2(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ is a smooth solution of the governing equations (2.1.11), (2.1.12), (2.1.13), (2.1.14), (2.1.34), (2.1.35), (2.1.23) and (2.1.4) for prescribed initial and boundary conditions. Then the following inequality holds:

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega(t)} \frac{1}{2}(\phi_1|\mathbf{v}_1|^2 + \phi_2|\mathbf{v}_2|^2 + \Psi)\, d\mathbf{x} \\ &\leq \int_{\partial\Omega_t} \mathbf{t}_1 \cdot \mathbf{v}_1 + \mathbf{t}_2 \cdot \mathbf{v}_2, \end{aligned} \tag{2.3.58}$$

where $\mathbf{t}_1 = \mathcal{T}_1\nu$ and $\mathbf{t}_2 = \mathcal{T}_2\nu$ and ν denotes normal to the boundary.

To derive the previous inequality, carry out the dot product of equations (2.1.13) and (2.1.14) by $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, add up the two equations and integrate the resulting expression on Ω:

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega(t)} (\frac{\phi_1}{2}|\mathbf{v}_1|^2 + \frac{\phi_2}{2}|\mathbf{v}_2|^2)\, d\mathbf{x} = \\ &\int_{\Omega(t)} (\mathbf{v}_1 \cdot \mathrm{d}iv\mathcal{T}_1 + \mathbf{v}_2 \cdot \mathrm{d}iv\mathcal{T}_2 + \mathbf{v}_1 \cdot \mathbf{f}_1 + \mathbf{v}_2 \cdot \mathbf{f}_2)\, d\mathbf{x}. \end{aligned}$$

Application of the divergence theorem to the terms involving $\mathcal{T}$ gives

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega(t)} (\frac{\phi_1}{2}|\mathbf{v}_1|^2 + \frac{\phi_2}{2}|\mathbf{v}_2|^2)\, d\mathbf{x} = \\ &-\int_{\Omega(t)} tr(\mathcal{T}_1^T(\nabla\mathbf{v}_1) + \mathcal{T}_2^T(\nabla\mathbf{v}_2))\, d\mathbf{x} + \int_{\Omega(t)} (\mathbf{v}_1 \cdot \mathbf{f}_1 + \mathbf{v}_2 \cdot \mathbf{f}_2)\, d\mathbf{x} \\ &+\int_{\partial\Omega(t)} (\mathbf{t}_1 \cdot \mathbf{v}_1 + \mathbf{t}_2 \cdot \mathbf{v}_2)\, dS. \end{aligned}$$

Equivalently,

$$\frac{d}{dt}\int_{\Omega(t)}(\frac{\phi_1}{2}|\mathbf{v}_1|^2+\frac{\phi_2}{2}|\mathbf{v}_2|^2)\,d\mathbf{x}-\int_{\partial\Omega(t)}(\mathbf{t}_1\cdot\mathbf{v}_1+\mathbf{t}_2\cdot\mathbf{v}_2)\,dS=$$
$$-\int_{\Omega(t)}tr(\mathcal{T}_1^T(\nabla\mathbf{v}_1)+\mathcal{T}_2^T(\nabla\mathbf{v}_2))\,d\mathbf{x}+\int_{\Omega(t)}(\mathbf{v}_1\cdot\mathbf{f}_1+\mathbf{v}_2\cdot\mathbf{f}_2)\,d\mathbf{x}.$$

Applying inequality (2.1.32) to the previous equation yields

$$\frac{d}{dt}\int_{\Omega(t)}(\frac{\phi_1}{2}|\mathbf{v}_1|^2+\frac{\phi_2}{2}|\mathbf{v}_2|^2)\,d\mathbf{x}+\int_{\Omega(t)}(\phi_1\dot{\psi}_1+\phi_2\dot{\psi}_2)\,d\mathbf{x}$$
$$-\int_{\partial\Omega(t)}(\mathbf{t}_1\cdot\mathbf{v}_1+\mathbf{t}_2\cdot\mathbf{v}_2)\,dS\le 0.$$

By the transport theorem, the previous inequality becomes

$$\frac{d}{dt}\int_{\Omega(t)}[(\frac{\phi_1}{2}|\mathbf{v}_1|^2+\frac{\phi_2}{2}|\mathbf{v}_2|^2)+(\phi_1\psi_1+\phi_2\psi_2)]-\int_{\partial\Omega(t)}(\mathbf{t}_1\cdot\mathbf{v}_1+\mathbf{t}_2\cdot\mathbf{v}_2)\le 0,$$

giving (2.3.58) by taking into account (2.1.23).

2.4 Viscoelasticity

Assume that the dissipation of the gel stems mostly from the viscosity of the polymer, with the solvent contribution taken to be Newtonian. The focus now is on modeling the dissipative stress of the polymer.

In the case of steady flow and small strain rate, Newtonian behavior is recovered, and the dissipative and total stress expressions are, then

$$\mathcal{T}_1^d=2\eta_1\mathbf{D}_1+\mu_1\nabla\cdot\mathbf{v}_1 I, \tag{2.4.59}$$

$$\mathcal{T}_1=\mathcal{T}_1^r+2\eta_1\mathbf{D}_1+\mu_1\nabla\cdot\mathbf{v}_1 I, \tag{2.4.60}$$

with $\mathbf{D}_1=\frac{1}{2}(\nabla\mathbf{v}_1+(\nabla\mathbf{v}_1)^T)$, $\eta_1>0$, $\mu_1>0$. Likewise, for a viscous solvent,

$$\mathcal{T}_2^d=2\eta_2\mathbf{D}_2+\mu_2\nabla\cdot\mathbf{v}_2 I, \tag{2.4.61}$$

$$\mathcal{T}_2=-\Pi_2 I+2\eta_2\mathbf{D}_2+\mu_2\nabla\cdot\mathbf{v}_2 I, \tag{2.4.62}$$

with $\mathbf{D}_2 = \frac{1}{2}(\nabla \mathbf{v}_2 + (\nabla \mathbf{v}_2)^T)$, $\eta_2 > 0$, $\mu_2 > 0$. In general $\eta_1 >> \eta_2 > 0$ and $\mu_1 >> \mu_2 > 0$ hold.

Increasing the strain rate to achieve a weak departure from Newtonian behavior yields a higher order of approximation of the stress

$$(2.4.63) \qquad \begin{aligned} \mathcal{T}_1 &= 2\eta_1 \mathbf{D}_1 - 2\epsilon\eta_1[\dot{\mathbf{D}}_1 - (\nabla \mathbf{v}_1)^T \mathbf{D}_1 - \mathbf{D}_1(\nabla \mathbf{v}_1)] + \mu_1 \nabla \cdot \mathbf{v}_1 I \\ &\quad -2\epsilon\mu_1[\nabla \cdot \dot{\mathbf{v}}_1 I - 2\mathbf{D}_1 \nabla \cdot \mathbf{v}_1 - \mathrm{t}r(\nabla \mathbf{v}_1)^2 I] + \mathcal{T}_1^r, \end{aligned}$$

with $\dot{\mathbf{D}}$ denoting material time derivative, and $\epsilon > 0$ small. If multiple relaxation times are present, it is then appropriate to assume that the dissipative stress depends on the past history of the deformation. There are many models in the literature appropriate to specific regimes and materials, solids and fluids. Applications to viscoelastic solids, in contrast to fluids, involve the additional issue of how to accurately account for both viscous and elastic properties. Two criteria that must be satisfied by the chosen model are:

- the viscoelastic stress yields the correct instantaneous elastic response when the deformation experiences a discontinuity in time,
- the elastic relaxation response is recovered, that is, when the deformation history is set constant, the viscoelastic stress reduces to the reversible contribution.

For this, a viscoelastic constitutive equation is proposed based on the Kaye-BKZ model ([36], page 75), ([47], page 158). Such a model is based on postulating a functional of the form

$$w(\mathbf{x}, t) = \hat{w}(I_1, I_2, I_3, t - t'), \; I_i = I_i(\mathbf{x}, t), \, t > t',$$

and define

$$(2.4.64) \qquad \begin{aligned} \mathcal{T}_1(\mathbf{x}, t) &= \int_{-\infty}^{t} 2\det\, B(t,t')^{-\frac{1}{2}} \left[\hat{w}_{I_1} B(t,t') + \hat{w}_{I_2}(I_1 B(t,t') \right. \\ &- \left. B(t,t')^2) + I_3\, \hat{w}_{I_3} I\right] dt' - (\pi_1 \phi_1 p) I, \end{aligned}$$

with π_1 as in (2.1.36). Here

$$B(t,t') = F(t,t')F^T(t,t'), \quad \text{and} \quad F(t,t') = \frac{\partial \mathcal{M}(\mathbf{X},t)}{\partial \mathcal{M}(\mathbf{X},t')},$$

with $\mathbf{x} = \mathcal{M}(\mathbf{X},t)$ as in (2.1.1).

Remark. The Kaye-BKZ model is often formulated for incompressible materials and involving the tensor $C = F^T F$ instead of B.

The dependence on past history enters through the energy functional w. In applications, the assumption of factorability is often made, that is,

$$\hat{w}(I_1, I_2, I_3, s) = m(s)w(I_1, I_2, I_3), \tag{2.4.65}$$

where $m(.) \geq 0$ and $m'(.) < 0$ and w is the elastic stored energy function.

In the case that the elasticity is neo-Hookean (2.2.48), and (2.4.65) holds, the stress in (2.4.64) reduces to

$$\begin{aligned} \mathcal{T}_1(\mathbf{x},t) = & \int_{-\infty}^{t} \frac{2}{(\det\, B(t,t'))^{\frac{1}{2}}}(m(t-t')B(t,t')\, dt' \\ - & (\pi_1 + \phi_1 p)I). \end{aligned} \tag{2.4.66}$$

The viscoelastic relation in (2.4.66) is known as the Lodge model.

2.5 Governing equations of gels

Equations are derived for the mixture from those for the individual components. The fields of the gel consist of

$$\{\mathbf{V}, \mathbf{U}, \mathcal{T}, \phi_2, p\},$$

where $\mathbf{V}$ denotes the previously defined center of mass velocity, and $\mathbf{U} = \mathbf{v}_1 - \mathbf{v}_2$ represents the diffusion velocity. The total stress is given by

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 - (1-\phi_2)\phi_2 \mathbf{U} \otimes \mathbf{U},$$

where $\mathcal{T}_i$ include the reversible and dissipative contributions to the stress.

The first two of the equations shown below follow by adding and subtracting, respectively, the equations of balance of linear momentum of the components (in the latter case, first divide the individual equations by ϕ_1 and ϕ_2, respectively). The third equation is a form of the chain rule. The last equation is a consequence of (2.1.18). In addition, one of the three equations (2.1.11), (2.1.12) or (2.1.16) has to be satisfied as well. These equations are summarized as follows,

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = \nabla\cdot\mathcal{T}, \tag{2.5.67}$$

$$\begin{aligned}\frac{\partial \mathbf{U}}{\partial t} &+ (\phi_2-\phi_1)(\nabla\mathbf{U})\mathbf{U} + (\mathbf{U}\otimes\mathbf{U})\nabla\phi_2 + (\nabla\mathbf{V})\mathbf{U} + (\nabla\mathbf{U})\mathbf{V} \\ &= \frac{1}{\phi_1}\nabla\cdot\mathcal{T}_1 - \frac{1}{1-\phi_1}\nabla\cdot\mathcal{T}_2 - \frac{\beta}{\phi_1(1-\phi_1)}\mathbf{U}\end{aligned} \tag{2.5.68}$$

$$\frac{\partial F}{\partial t} + (\mathbf{V}+(1-\phi_1)\mathbf{U})\cdot\nabla F = \nabla(\mathbf{V}+(1-\phi_1)\mathbf{U})F \tag{2.5.69}$$

$$\nabla\cdot\mathbf{V} = 0. \tag{2.5.70}$$

Note that the first equation gives the balance of linear momentum for the mixture, with $\mathcal{T}$ denoting the total stress. The second equation governing the evolution of the diffusive velocity $\mathbf{U}$ describes the microstructure of the system. This set of equations should be supplemented with initial and boundary conditions.

Expressions of the stress tensors are now given for the Newtonian and viscoelastic cases, respectively.

1. Newtonian stress tensors.

The equations for the stress tensors of the system in the case that both polymer and solvent admit Newtonian dissipation are

$$\begin{aligned}\mathcal{T}_1 = \;& \phi_1\mathcal{H}(B) - (\Pi_1 + p\phi_1) + \eta_1[\mathbf{U}\otimes\nabla\phi_2 + \nabla\phi_2\otimes\mathbf{U} \\ & +\nabla\mathbf{V} + \nabla\mathbf{V}^T + \phi_2(\nabla\mathbf{U}+\nabla\mathbf{U}^T)] + \mu_1\nabla\cdot(\mathbf{V}+\phi_2\mathbf{U})I,\end{aligned} \tag{2.5.71}$$

$$\begin{aligned}\mathcal{T}_2 = \;& -(\phi_2 p + \Pi_2)I + \eta_2[\nabla\mathbf{V} + \nabla\mathbf{V}^T - \mathbf{U}\otimes\nabla\phi_1 - \nabla\phi_1\otimes\mathbf{U} \\ & -\phi_1(\nabla\mathbf{U}+\nabla\mathbf{U}^T)] + \mu_2\nabla\cdot(\mathbf{V}-\phi_1\mathbf{U})I.\end{aligned} \tag{2.5.72}$$

The total stress now gives

$$
\begin{aligned}
(2.5.73) \quad \mathcal{T} = \quad & \phi_1 \mathcal{H}(B) - (\Pi_1 + \Pi_2 + p)I + (\eta_1 + \eta_2)(\nabla \mathbf{V} + \nabla \mathbf{V}^T) \\
& + (\eta_1 \phi_2 - \eta_2 \phi_1)(\nabla \mathbf{U} + \nabla \mathbf{U}^T) + \mathbf{U} \otimes (\eta_1 \nabla \phi_2 - \eta_2 \nabla \phi_1) \\
& + (\eta_1 \nabla \phi_2 - \eta_2 \nabla \phi_1) \otimes \mathbf{U} - (1 - \phi_2)\phi_2 \mathbf{U} \otimes \mathbf{U} \\
& + [\mu_1 \nabla (\mathbf{V} + \phi_2 \mathbf{U}) + \mu_2 \nabla \cdot (\mathbf{V} - \phi_1 \mathbf{U})] I.
\end{aligned}
$$

2. Viscoelastic stress tensors.

In the case that the polymer response is viscoelastic with stress tensor as in (2.4.64), the total stress tensor, assuming Newtonian solvent with stress $\mathcal{T}_2$ as in (2.5.72), is given by

$$
\begin{aligned}
(2.5.74) \qquad & \mathcal{T} = \int_{-\infty}^{t} 2 \det\, B(t,t')^{-\frac{1}{2}} [\hat{w}_I B(t,t') + \hat{w}_{I_2}(I_1 B(t,t') - B(t,t')^2) \\
& + I_3\, \hat{w}_{I_3} I]\, dt' + \eta_2 [\nabla \mathbf{V} + \nabla \mathbf{V}^T - \mathbf{U} \otimes \nabla \phi_1 - \nabla \phi_1 \otimes \mathbf{U} \\
& - \phi_1 (\nabla \mathbf{U} + \nabla \mathbf{U}^T)] + \mu_2 \nabla \cdot (\mathbf{V} - \phi_1 \mathbf{U}) I - (1 - \phi_2)\phi_2 \mathbf{U} \otimes \mathbf{U} \\
& - (\Pi_1 + \Pi_2 + p) I
\end{aligned}
$$

2.5.1 Special regimes

Two distinctive regimes emerge according to the following observations.

Purely diffusive regime

If $\mathbf{V}(\mathbf{x}, t) = 0$ at t=0, and also $\mathbf{V}(\mathbf{x}, t) = 0$ for $\mathbf{x} \in \Omega$, $t > 0$, then there is a solution to the equations satisfying $\mathbf{V} = 0$ for all time. In fact, setting $\mathbf{V} = 0$ in equation (2.5.67), a first order equation for p is obtained that can be solved by direct integration in terms of the remaining fields of the problem. The resulting problem involving the independent fields $\{\mathbf{U}, \phi_1, F\}$ is formulated as follows:

$$\begin{aligned} &\frac{\partial \mathbf{U}}{\partial t} + (\phi_2 - \phi_1)(\nabla \mathbf{U})\mathbf{U} + (\mathbf{U} \otimes \mathbf{U})\nabla \phi_2 \\ &= \frac{1}{\phi_1}\nabla \cdot \mathcal{T}_1 - \frac{1}{1-\phi_1}\nabla \cdot \mathcal{T}_2 - \frac{\beta}{\phi_1(1-\phi_1)}\mathbf{U}, \end{aligned} \tag{2.5.75}$$

$$\frac{\partial F}{\partial t} + (1-\phi_1)\mathbf{U} \cdot \nabla F = \nabla((1-\phi_1)\mathbf{U})F, \tag{2.5.76}$$

and either one of the balance laws (2.1.11), (2.1.12) or (2.1.16).

Diffusion equation. A special case of the former results when the inertia terms in (2.5.75) are neglected, giving

$$\mathbf{U} = \beta^{-1}\phi_1(1-\phi_1)\big(\frac{1}{\phi_1}\nabla \cdot \mathcal{T}_1 - \frac{1}{1-\phi_1}\nabla \cdot \mathcal{T}_2\big). \tag{2.5.77}$$

A simple calculation shows that in the case $\mathbf{V} = 0$, $\mathbf{v}_1 = (1-\phi_1)\mathbf{U}$ which substituted into equation (2.1.11) permits rewriting it as

$$\frac{\partial \phi_1}{\partial t} + \nabla \cdot (\phi_1(1-\phi_1)\mathbf{U}) = 0.$$

Substitution of $\mathbf{U}$ from (2.5.77) into the latter yields

$$\frac{\partial \phi_1}{\partial t} = -\nabla \cdot \big(\beta^{-1}\phi_1^2(1-\phi_1)^2[\frac{1}{\phi_1}\nabla \cdot \mathcal{T}_1 - \frac{1}{1-\phi_1}\nabla \cdot \mathcal{T}_2]\big). \tag{2.5.78}$$

The right hand side of the previous equation contains a term of the form $\nabla \cdot (\beta^{-1}\kappa(\phi_1, F, \nabla \mathbf{U})\nabla \phi_1)$ (here the dependence on $\nabla \mathbf{U}$ holds for the special case that the polymer dissipative stress is Newtonian). Moreover, if one could further assert that the coefficient multiplying $\nabla \phi_1$ in such a term is positive, that is $\kappa < 0$, then it would follow that (2.5.78) is a (nonlinear) diffusion equation for ϕ_1. This is a model widely used in the studies of gels and is known as the diffusion model. The quantity β^{-1} corresponds to the diffusion constant D used in characterization

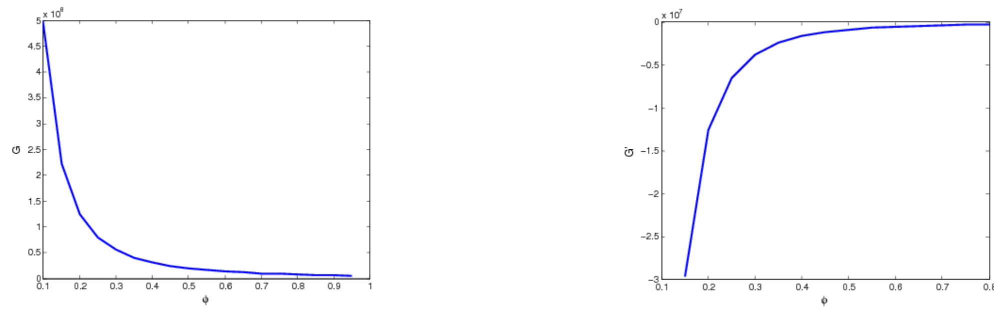

Figure 2.1: Graphs of $G(\phi)$ and $G'(\phi)$ for polymer data.

of gels [7]. The inequality $\kappa < 0$ in general does not follow from thermodynamics. Consider the one-dimensional geometry to interpret such an inequality. The governing system (2.1.11) and (2.5.75) in the interval $(-1, 1)$ in the one-dimensional geometry reduces to [12]

$$\frac{\partial \phi_1}{\partial t} + \frac{\partial(\phi_1(1-\phi_1)U)}{\partial x} = 0, \tag{2.5.79}$$

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(\frac{1}{2}U^2(1-2\phi_1) - G) = -\frac{\beta}{\phi_1(1-\phi_1)}U, \tag{2.5.80}$$

where $\mathbf{U} = (U(x,t), 0, 0)$, $\phi_1 = \phi_1(x,t)$ and

$$\begin{aligned} G(\phi_1) = {} & \frac{K_B T}{V_m N_x}(-\frac{1}{2}\varphi_0^{2/3}\phi_1^{-\frac{2}{3}} - (\frac{1}{2} + \frac{N_x}{N_1})\log\phi_1) \\ & + \mu\varphi_0^2\phi_1^{-2} - \frac{K_B T \chi}{V_m}\phi_1 + \frac{K_B T}{N_2 V_m}\log(1-\phi_1). \end{aligned} \tag{2.5.81}$$

The analog to (2.5.77) obtained from (2.5.80) and its substitution into (2.5.79) yield

$$\frac{\partial \phi_1}{\partial t} = -\frac{\partial(\beta^{-1}G'(\phi)\phi'(x)))}{\partial x}. \tag{2.5.82}$$

The function $G(\phi)$ represents the extensional stress in terms of the polymer volume fraction. Equation (2.5.82) has a solution provided $G'(\phi_1) < 0$ holds. This monotonicity condition holds for the polymer data shown in section 2. However, in the case of polysaccharides, there is a critical value $\phi_c = \phi_c(\mu)$, $0 < \phi_c < 1$ such that $G'(\phi_c) = 0$. Moreover, the value ϕ_c corresponds to a critical point of the volume phase transition between swollen and collapsed states [39]. Such properties are illustrated by the graphs of $G(\phi_1)$ and $G'(\phi_1)$. The graphs in figure 2.1 correspond to polymer data with $\mu = 2 * 10^6 \, \frac{\text{pN}}{\text{n}m^2}$, whereas those in figure 2.2 are for polysaccharide data with $\mu = 2 * 10^{-5} \, \frac{\text{pN}}{\text{n}m^2}$. Note the change of sign of $G'(\phi_1)$ in the latter case. Later in this subsection, the monotonicity condition from the dynamics point of view will be discussed.

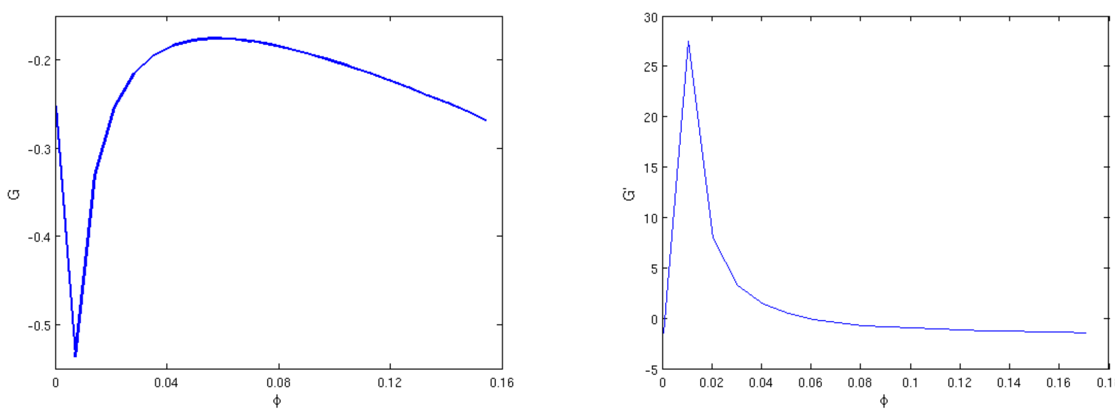

Figure 2.2: Graph of $G(\phi)$ and $G'(\phi)$ for polysaccharide data.

Transport regime

On the other hand, if $\mathbf{U} = 0$ initially and on the boundary for all time, then the system becomes that of an incompressible viscoelastic fluid for the fields

$\{\mathbf{V}, p, \phi_1, F\}$, with no diffusion present,

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = \nabla\cdot\mathcal{T}, \tag{2.5.83}$$

$$\frac{\partial F}{\partial t} + \mathbf{V}\cdot\nabla F = (\nabla\mathbf{V})F \tag{2.5.84}$$

$$\nabla\cdot\mathbf{V} = 0. \tag{2.5.85}$$

together with equations (2.1.18) and (2.1.16).

2.5.2 Initial and boundary conditions

On the boundary of the gel region, it is relevant to distinguish again between solid and solvent components of the material. Different types of boundary conditions motivated by applications will be presented. First of all, two main classes emerge according to whether the boundary is free, fixed, or a combination of both types. For the sake of this presentation, it will be assumed that the polymer has Newtonian stress and the solvent is inviscid (that is, $\eta_2 = 0$, $\mu_2 = 0$ in (2.4.62)). The governing system reduces then to equations (2.1.13), (2.1.14), (2.1.18), (2.1.16) and (2.1.15), together with constitutive equations (2.2.55), (2.2.56), (2.2.54), (2.1.39), (2.4.60), (2.1.36), (2.1.37), (2.2.52). The unknown fields of the problem are $\{\mathbf{v}_1, \mathbf{v}_2, F, p\}$. The governing system in the current domain Ω reduces to

$$\dot{\mathbf{v}}_1 = -\nabla p + \frac{\det F}{\varphi_0}\{\nabla\cdot\mathcal{T}_1(F, \nabla\mathbf{v}_1) + \mathbf{f}_1\} \tag{2.5.86}$$

$$\dot{\mathbf{v}}_2 = -\nabla p + \frac{\det F}{\det F - \varphi_0}\{\nabla\cdot\mathcal{T}_2(\det F) + \mathbf{f}_2\} \tag{2.5.87}$$

$$\frac{\partial F}{\partial t} + \mathbf{v}_1\cdot\nabla F = (\nabla\mathbf{v}_1)F \tag{2.5.88}$$

$$\nabla\cdot\big(\frac{\varphi_0}{\det F}\mathbf{v}_1 + (1 - \frac{\varphi_0}{\det F})\mathbf{v}_2\big) = 0. \tag{2.5.89}$$

To recover the deformation map of the polymer $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, it is necessary to further solve the first order system

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}}(\mathbf{X}, t) = F(\mathbf{x}(\mathbf{X}, t), t), \tag{2.5.90}$$

in the reference domain Ω_0.

Gel confined to a given domain Ω

This corresponds to the case that the polymer is chemically glued to the boundary $\partial\Omega$. This corresponds to requiring

$$\mathbf{v}_1(\mathbf{x}, t) = 0, \;\; \mathbf{x} \in \partial\Omega, \;\; t > 0. \tag{2.5.91}$$

Another boundary condition is needed to characterize the degree of permeability of the membrane. Following Yamaue and Doi [68], [69], [70], [19], three type of membrane boundaries are distinguished:

$$\mathbf{v}_2 \cdot \mathbf{n} = \mathbf{v}_1 \cdot \mathbf{n}, \;\; \text{impermeable membrane} \tag{2.5.92}$$

$$p(\mathbf{x}, t) + \Pi_2(\mathbf{x}, t) = P, \;\; \text{fully permeable membrane} \tag{2.5.93}$$

$$P - \big(p(\mathbf{x}, t) + \Pi_2(\mathbf{x}, t)\big) = \kappa(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{n}, \;\; \text{semipermeable membrane,} \tag{2.5.94}$$

where P denotes the pressure of the external solvent, κ is a permeability constant, and $\mathbf{n}$ is the unit normal vector to $\partial\Omega$. This type of boundary condition occurs in applications to body implantable biomedical devices where the polymer is glued to its support but the surface allows for penetrability of solvent. Since the polymer boundary is maintained fixed a build-up of stress occurs near the contact surface causing eventual failure of the device. The amount of normal stress acting on the boundary is that measured in the experiment by Suzuki and Hara [58] where the gel is confined between two circular disks with the bottom one maintained fixed and the top one allowed to move in order to relax the stress [68].

Initial data should be prescribed as well

$$\mathbf{v}_1(\mathbf{x}, 0) = 0, \; \mathbf{v}_2(\mathbf{x}, 0) = 0. \tag{2.5.95}$$

In addition, take the reference configuration to be Ω in which case $F = I$ at $t = 0$, allowing for integration of (2.5.88).

Gel subject to prescribed boundary force

Consider a gel with deformable boundary subject to a prescribed force. For instance, this is the case of the gel immersed in its own solvent subject to a constant pressure P. In addition to one of the three boundary conditions (2.5.93), (2.5.94) or (2.5.92) being prescribed, the boundary condition (2.5.91) is now replaced with

$$-p\mathbf{n} + (\sigma_1 + \sigma_2)\mathbf{n} = -P\mathbf{n}, \ \ x \in \partial\Omega(t), \tag{2.5.96}$$

where σ_1 and σ_2 are defined by (2.2.57). In the case that the membrane is permeable, combining (2.5.93) with (2.5.96) to eliminate p gives the relation

$$\Pi_2\mathbf{n} + (\sigma_1 + \sigma_2)\mathbf{n} = 0. \tag{2.5.97}$$

This relation involves the volume fraction at the interface. In order to study the motion of the free boundary with arbitrary geometry, it is necessary to employ the phase field or the level surface methods. For two special domains, one can directly supplement the previous boundary conditions with equations of evolution of the free boundary.

One-dimensional slab. Consider a gel initially confined in the strip $|x| < 1$. Let $x = S(t)$ denote the location of the point initially at $x = 1$ (by symmetry, $x = -S(t)$ describes the evolution of the left boundary). Assume that at time $t = 0$, the polymer enters into contact with its own solvent, initially located at $|x| > 1$. The evolution of the gel in the strip $|x| \leq S(t)$ is given by equations (2.5.79) and (2.5.80) for $\{\phi_1, U\}$, subject to initial and boundary conditions described next. The evolution equation of the interface between gel and pure solvent is given by

$$\frac{dS(t)}{dt} = v_1(S(t), t), \ \ S(0) = 1. \tag{2.5.98}$$

Assuming that boundary conditions (2.5.96) and (2.5.93) hold, then (2.5.97) gives an equation for ϕ_1 on the boundary $x = S(t)$. The value of ϕ^* that solves the equation is known as *saturation volume fraction*. Note that such a value is only meaningful in one-dimensional geometries since then F and its determinant, det F coincide. Moreover, one can then use the balance of mass relation $\phi_1 \det F = \varphi_0$ to fully determine ϕ_1 at the boundary from (2.5.97). This problem is analyzed in ([12]).

In particular, it is found that the system is hyperbolic provided that the inequality $G'(\phi_1) < 0$ holds. This guarantees the propagation of the interface. As previously stated, such a condition holds for the polymer data shown in section 2, and it can be interpreted as follows. A (dry) polymer slab brought into contact with its own solvent at the initial time will keep swelling with the speed of the interface decreasing with time. However, in the case of polysaccharides, there is a critical value $\phi_c = \phi_c(\mu)$, $0 < \phi_c < 1$ such that $G'(\phi_c) = 0$; if $\phi^* > \phi_c$, the interface will evolve, and the gel will continue swelling until ϕ_1 reaches the critical value. The analysis suggests that ϕ_c may indicate the onset of de-swelling.

Spherical domain. In the case of a spherical domain (2.5.98) is represented by

$$\frac{dr(t)}{dt} = v_1(r(t), t), \quad r(0) = R, \tag{2.5.99}$$

where $R > 0$ denotes the initial radius of the spherical gel region, and v_1 represents the radial polymer velocity [12]. This case may be appropriate to the study of drug release devices.

2.5.3 The stress-diffusion coupling model

In this subsection, the quasi-static version of the previously developed model is presented under a special choice of the free energy. This leads to the stress-diffusion coupling model proposed by Yamaue and Doi [69].

The assumption of no voids present in the material yields (2.5.70). Neglecting the Flory-Huggins energy in (2.1.23), assuming the latter in the form of $\Psi =$

$\phi_1\mu\text{tr}\,(F^TF)$, the stress tensors in (2.1.34) and (2.1.35) become

$$\mathcal{T}_1 = \phi_1\frac{\partial\psi_1}{\partial F}F^T - \phi_1 p \tag{2.5.100}$$

$$\mathcal{T}_2 = -\phi_2 p \tag{2.5.101}$$

The corresponding system of balance laws of linear momentum is

$$\phi_1\big(\frac{\partial\mathbf{v}_1}{\partial t} + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1\big) = \nabla\cdot(\phi_1\frac{\partial\psi_1}{\partial F}F^T) - \phi_1\nabla p - \beta(\mathbf{v}_1 - \mathbf{v}_2) \tag{2.5.102}$$

$$\phi_2\big(\frac{\partial\mathbf{v}_2}{\partial t} + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\big) = -\phi_2\nabla p + \beta(\mathbf{v}_1 - \mathbf{v}_2) \tag{2.5.103}$$

Now, consider the equations of the steady state obtained by neglecting acceleration terms, that is, setting $\frac{\partial\mathbf{v}_2}{\partial t} + (\mathbf{v}_2\cdot\nabla)\mathbf{v}_2 = 0$ and $\frac{\partial\mathbf{v}_1}{\partial t} + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1 = 0$. Then the equations of balance of linear momentum reduce to

$$\nabla\cdot\mathcal{T}_1 + \mathbf{f}_1 = 0$$
$$\nabla\cdot\mathcal{T}_2 + \mathbf{f}_2 = 0.$$

Since $\mathbf{f}_1 = p\nabla\phi_1 - \beta(\mathbf{v}_1 - \mathbf{v}_2)$ (2.1.39), and $\mathbf{f}_1 + \mathbf{f}_2 = 0$, the former become

$$\nabla\cdot(\phi_1\frac{\partial W(F)}{\partial F}F^T) - \phi_1\nabla p - \beta(\mathbf{v}_1 - \mathbf{v}_2) = 0$$
$$-\phi_2\nabla p + \beta(\mathbf{v}_1 - \mathbf{v}_2) = 0,$$

and upon adding them, the following is obtained

$$\nabla\cdot(\phi_1\frac{\partial W(F)}{\partial F}F^T - pI) = 0 \tag{2.5.104}$$

where $\frac{\partial W(F)}{\partial F}$ is the Piola-Kirchhoff stress tensor. Taking into account the balance of mass equation $\phi_1\det F = 1$, (2.5.104) can be rewritten as

$$\nabla\cdot(\det F^{-1}\frac{\partial W(F)}{\partial F}F^T - pI) = 0. \tag{2.5.105}$$

Thus, the corresponding stress tensor is given by the equation

$$\sigma = \det F^{-1}\frac{\partial W(F)}{\partial F}F^T. \tag{2.5.106}$$

Summarizing, the resulting governing system is

$$\nabla \cdot (\sigma - pI) = 0 \tag{2.5.107}$$

$$-\phi_2 \nabla p + \beta(\mathbf{v}_1 - \mathbf{v}_2) = 0 \tag{2.5.108}$$

$$\nabla \cdot ((1-\phi_2)\mathbf{v}_1 + \phi_2 \mathbf{v}_2) = 0 \tag{2.5.109}$$

$$\frac{\partial \phi_2}{\partial t} + \nabla \cdot (\phi_2 \mathbf{v}_2) = 0. \tag{2.5.110}$$

Observe that this system governs quasi-static regimes, with (2.5.108) corresponding to Darcy's law.

2.5.4 Energy time rate and the Rayleighian

Consider the two-component formulation given at the beginning of the chapter. Here assume that only the laws of balance of mass hold and the constraint (2.1.4) hold. The goal is to provide an alternate derivation of the equations. Define the Rayleighian functional

$$\begin{aligned} R(t) = {}& \frac{d}{dt}\int_{\Omega(t)} \big(\frac{1}{2}\phi_1 \mathbf{v}_1 \cdot \mathbf{v}_1 + \frac{1}{2}\phi_2 \mathbf{v}_2 \cdot \mathbf{v}_2 + \Psi\big)\, d\mathbf{x} \\ + {}& \int_{\Omega(t)} p\nabla \cdot (\phi_1 \mathbf{v}_1 + \phi_2 \mathbf{v}_2)\, d\mathbf{x} + W(t), \end{aligned} \tag{2.5.111}$$

$$\begin{aligned} W(t) = {}& \frac{1}{2}\int_{\Omega(t)} \beta(\phi_1,\phi_2)(\mathbf{v}_1 - \mathbf{v}_2)^2\, d\mathbf{x} \\ + {}& \int_{\partial\Omega(t)} ((\pi_1 + \phi_1 p)\mathbf{v}_1 + (\pi_2 + \phi_2 p)\mathbf{v}_2) \cdot \mathbf{n} - \sigma \mathbf{n} \cdot \mathbf{v}_1 .\, dS \end{aligned} \tag{2.5.112}$$

The quantity $W(t)$ represents the power dissipated by the friction forces plus that by the boundary forces. In addition, variations on $R(t)$ have to be carried out under the constraint $\nabla \cdot (\phi_1 \mathbf{v}_1 + \phi_2 \mathbf{v}_2) = 0$. Consequently, the power associated with maintaining the constraint is included. Compute the time derivative of the total energy.

$$\frac{d}{dt}\mathcal{E}(t) = \int_{\Omega(t)} \big(\phi_1 \mathbf{v}_1 \cdot \dot{\mathbf{v}}_1 + \phi_2 \mathbf{v}_2 \cdot \dot{\mathbf{v}}_2 + \phi_1 \dot{\psi}_1 + \phi_2 \dot{\psi}_2\big)\, d\,\mathbf{x}$$

Now,

$$\begin{aligned}\phi_1\dot{\psi}_1 + \phi_2\dot{\psi}_2 = & \quad \phi_1(\frac{\partial\psi_1}{\partial\phi_1} - \frac{\partial\psi_1}{\partial\phi_2})\dot{\phi}_1 + \phi_2(-\frac{\partial\psi_2}{\partial\phi_1} + \frac{\partial\psi_2}{\partial\phi_2})\dot{\phi}_2 \\ + & \quad \phi_1\frac{\partial\psi_1}{\partial F}\cdot\dot{F} \\ = & \quad -\pi_1(\nabla\cdot\mathbf{v}_1) - \pi_2(\nabla\cdot\mathbf{v}_2) + \phi_1\frac{\partial\psi_1}{\partial F}\cdot\dot{F} \\ = & \quad -\nabla\cdot\big(\pi_1\mathbf{v}_1\big) - \nabla\cdot\big(\pi_2\mathbf{v}_2\big) \\ + & \quad \nabla\pi_1\cdot\mathbf{v}_1 + \nabla\pi_2\cdot\mathbf{v}_2 + \phi_1\frac{\partial\psi_1}{\partial F}\cdot(\nabla\mathbf{v}_1)F,\end{aligned}$$

where (2.1.11) and (2.5.69) have been used. Using the definition of stress tensor

$$\sigma = \phi_1\frac{\partial\psi_1}{\partial F}F^T,$$

rewrite

$$\phi_1\frac{\partial\psi_1}{\partial F}\cdot(\nabla\mathbf{v}_1)F = \nabla\cdot(\sigma\mathbf{v}_1) - (\nabla\cdot\sigma)\cdot\mathbf{v}_1. \tag{2.5.113}$$

Integrating and applying the divergence theorem,

$$\begin{aligned}\int_{\Omega(t)}(\phi_1\dot{\psi}_1 + \phi_2\dot{\psi}_2)d\,\mathbf{x} = & \quad \int_{\Omega(t)}\big[\nabla\pi_1\cdot\mathbf{v}_1 + \nabla\pi_2\cdot\mathbf{v}_2 - (\nabla\cdot\sigma)\cdot\mathbf{v}_1\big]\,d\mathbf{x} \\ & \quad -\int_{\partial\Omega}\big[\pi_1\mathbf{v}_1\cdot\mathbf{n} + \pi_2\mathbf{v}_2\cdot\mathbf{n} - \sigma\mathbf{n}\cdot\mathbf{v}_1\big]\,dS.\end{aligned} \tag{2.5.114}$$

Also,

$$\int_{\Omega(t)}p(\phi_1\nabla\cdot\mathbf{v}_1 + \phi_2\nabla\cdot\mathbf{v}_2) = -\int_{\Omega(t)}\nabla p\cdot(\phi_1\mathbf{v}_1 + \phi_2\mathbf{v}_2) + \int_{\partial\Omega(t)}p(\phi_1\mathbf{v}_1 + \phi_2\mathbf{v}_2)\cdot\mathbf{n}.$$

Taking the previous calculations into account, write

$$\begin{aligned}R(t) = & \quad \int_{\Omega(t)}[(\phi_1\dot{\mathbf{v}}_1 - \nabla\cdot\sigma + \nabla\pi_1 + \phi_1\nabla p)\cdot\mathbf{v}_1 \\ & \quad +(\phi_2\dot{\mathbf{v}}_2 + \nabla\pi_2 + \phi_2\nabla p)\cdot\mathbf{v}_2 + \beta(\mathbf{v}_1 - \mathbf{v}_2)^2]\,d\mathbf{x}.\end{aligned} \tag{2.5.115}$$

Apply the variational principle of minimizing the Rayleighian with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$ giving

$$\phi_1\dot{\mathbf{v}}_1 - \nabla\cdot\sigma + \nabla\pi_1 + \phi_1\nabla p = \beta(\mathbf{v}_1 - \mathbf{v}_2)$$

$$\phi_2\dot{\mathbf{v}}_2 + \nabla\pi_2 + \phi_2\nabla p = -\beta(\mathbf{v}_1 - \mathbf{v}_2).$$

Note that the terms of the left hand side of the previous equations involving stress do not appear as perfect divergence. However, adding the term $\phi_1 \nabla p$ to both sides of the first equation, and likewise, the term $\phi_2 \nabla p$ to the second one give

$$\phi_1 \dot{\mathbf{v}}_1 - \nabla \cdot \sigma + \nabla \pi_1 + \nabla(\phi_1 p) = \beta(\mathbf{v}_1 - \mathbf{v}_2) + \phi_1 \nabla p$$
$$\phi_2 \dot{\mathbf{v}}_2 + \nabla \pi_2 + \nabla(\phi_2 p) = -\beta(\mathbf{v}_1 - \mathbf{v}_2) + \phi_2 \nabla p$$

These now agree with the equations of balance of linear momentum with the stress and forcing terms obtained by imposing the Second Law of Thermodynamics.

Chapter 3

Review of related work

This chapter gives a summary of previous mathematical works in the literature that relate to and help motivate the problems of this thesis and the methods used to address them. Section 1 gives a historical overview of the modeling of polymer gel swelling beginning with the early experiments of Tanaka and Fillmore up to the recent work by Doi and Suo. Section 2 provides a summary of a paper by Calderer et al on equilibrium models for polymer gels. A mixed finite element method is proposed and analyzed for numerically calculating residual stresses [11], which may have deleterious effects on biomedical devices.

Section 3 provides a review of previous mathematical work done on the theory of viscoelastic fluids. The literature on the subject is very extensive and varied. The topics reviewed here were chosen according to analogies shared with different aspects of the present work, such as coupling of Eulerian and Lagrangian dynamics, transport, coexistence of solid elasticity with Newtonian viscosity, and porous media effects. The work of Liu and Walkington presents an existence theory for a fluid transporting visco-hyperelastic solid particles [46]. An analogy shared with the current work is that governing equations involve elastic forces and Newtonian viscosity for both solid and fluid. However, the framework of Liu and Walkington's research is not that of mixure theory. Instead, it makes use of a labelling function to denote locations occupied by the fluid and by solid, respectively. In particular, their approach does not address gel modeling since it neglects mixing and friction forces between solvent and polymer components. Guillopé and Saut considered Oldroyd-type viscoelastic fluids with small initial data [29]. In their paper, the extra stress (total stress minus pressure) is given by a differential equation instead of an algebraic expression in terms of the other variables. Lin, Liu, and Zhang prove existence of solutions for a linear Oldroyd viscoelastic fluid [41] and consider a model coupling macroscopic and microscopic effects [42]. Lions and Masmoudi use compactness results to prove existence of solutions for an Oldroyd system of non-Newtonian fluids [45]. DiPerna and Lions study scalar linear transport equations with nonsmooth coefficients and address questions of weak convergence of approximating sequences to weak solutions [17].

Section 4 gives a brief review of papers by Douglas and Showalter on porous media flow. In Douglas's paper, the velocity of the fluid is governed by Darcy's law [20], analagous to the case of the stress-diffusion coupling model [22], as well as a special case of the model studied in chapter 5 of this thesis. In [56] Showalter gives an overview of previously developed models of fluid flow through porous media and discusses some recent results. In [55] the same author discusses the mathematical theory (variational form, existence of solutions) of porous media models. In [57], Showalter and Shi develop the mathematical theory for models of plasticity. Section 5 gives a review of work done by Temam [63] on the linear Navier-Stokes equations. Section 6 describes the numerical method proposed by Feng and He for the stress diffusion coupling model of Doi [22].

3.1 Experimental results and modeling of gels

The dynamics of the swelling or shrinking of a polymer gel placed in solution were first analyzed by Tanaka and Fillmore [60]. The equation that they proposed (the TF equation) to describe the kinetics of spherical gel swelling explained a characteristic feature of swelling phenomena of spherical gels: the relaxation time of the gel is directly proportional to the square of the gel diameter and inversely proportional to the diffusion constant. This has been verified experimentally [65]. However, Suzuki and Hara demonstrated experimentally that for a thin-plate gel constrained between two rectangular surfaces, the time evolution of the thickness of the gel is described by a single exponential and that the relaxation time depends on the lengths of the rectangular surfaces but not on the gel's thickness [58]. This effect cannot be explained by the TF equation. In fact, the TF equation cannot describe general anisotropic deformations of gels such as the free swelling of long cylindrical or large disklike gels [38].[1] Doi proposed the stress-diffusion coupling model in order to describe general anisotropic deformations [69]. One main difficulty of the model by Tanaka is that it treats gels as liquids, since the free energy depends only on volume changes ($\det F$), whereas Doi allows for solid-type deformations by considering an energy that depends on $\mathrm{tr} FF^T$. The equations of

[1] A word of caution about the use of the terms *isotropic* and *anisotropic* is in order. The usage of the words by Tanaka and Doi does not follow the conventional definitions in mechanics. An elastic solid is isotropic if its symmetry group is the whole set proper orthogonal rotation matrices. This implies that the elastic energy depends on the deformation gradient F through the three princial invariants of $B = FF^T$. In this sense, the stress diffusion model by Doi is isotropic. The term *anisotropic* as used in [69] refers to the preferred direction of an applied deformation, say, stretching a cylindrical gel along the direction of its axis.

the model by Doi and Yamaue are

$$\beta(\mathbf{v} - \mathbf{u}_t) = -(1 - \phi)\nabla p,$$
$$\nabla \cdot [\sigma - p\mathcal{I}] = \mathbf{0},$$
$$\nabla \cdot [\phi\mathbf{u}_t + (1 - \phi)\mathbf{v}] = 0,$$

where $\mathbf{u}$ is the polymer displacement, $\mathbf{v}$ is the solvent velocity, p is the pressure, σ is the stress of the gel network, ϕ is the polymer volume fraction, and β is the friction coefficient. In the linear analysis, ϕ is approximately constant, and β is also assumed to be constant. Assuming that the polymer exhibits linear elasticity, the stress on the gel is given by the equation

$$\sigma = (K - \frac{2}{3}G)\nabla \cdot \mathbf{u}\mathcal{I} + G(\nabla\mathbf{u} + \nabla\mathbf{u}^T),$$

where K is the bulk modulus and G is the shear modulus of the gel. Doi and Yamaue applied this theoretical model to the swelling of a thin-plate gel constrained between two rectangular surfaces and showed that their theory was in agreement with the experiments of Suzuki and Hara [69].

Suo et al formulated a field theory of coupled mass transport and large deformation in polymer gels [31]. The gel consists of an elastomer, a species of long polymers cross-linked into a three-dimensional network, in contact with a solvent that migrates into the network. The gel swells and shrinks reversibly as the small molecules of the solvent migrate in and out. The theory is formulated in terms of nonequilibrium thermodynamics. Additionally, a free energy function and a kinetic law are specified independently since they are material specific and thus cannot be specified from purely thermodynamic considerations. The thermodynamic part of the theory is derived by considering the compromise between the configurational entropy of the network and the configurational entropy of the mixture as small molecules enter the network and cause it to swell. The kinetic law is given by considering two modes of deformation: local rearrangement of molecules and long-range migration of small molecules. The first mode occurs on a much faster time scale than the second and is thus taken to be instantaneous. Long-range migration is modeled as a time-dependent process.

The basic field equations of the theory come from continuum mechanics in material coordinates. The first two equations come from the equations of force balance in the gel volume and at the solvent-gel interfaces. Let s denote the nominal stress, i.e. the first Piola-Kirchhoff stress tensor or the stress in the undeformed domain. Let $\mathbf{B}$ denote the force density in the gel bulk, $\mathbf{n}$ the outward unit normal on an interface, $\mathbf{T}$ the normal stress on an interface, and s^-, s^+ the limits of nominal stress on an interface as the interface is approached from the exterior and interior of the gel, respectively. The force balance equations in material coordinates are

$$\frac{\partial s_{ik}}{\partial X_k} + \mathbf{B}_i = \mathbf{0} \tag{3.1.1}$$

in the volume and

$$(s_{ik}^- - s_{ik}^+)\mathbf{n}_k = \mathbf{T}_i \tag{3.1.2}$$

on an interface. Since small molecules are injected into the gel, governing equations for this process must be given. It is assumed that no chemical reactions occur, so the number of small molecules is conserved and is thus governed by

$$\frac{\partial C}{\partial t} + \frac{\partial \mathbf{J}_k}{\partial X_k} = r \tag{3.1.3}$$

in the volume and

$$(\mathbf{J}_k^+ - \mathbf{J}_k^-)\mathbf{n}_k = i \tag{3.1.4}$$

on an interface. C is the concentration of small molecules in the gel, $\mathbf{J}$ is the flux of molecules crossing an interface, r is the rate of molecules injected into the volume, and i is the rate injected into an interface. The deformation gradient of the polymer is given by

$$F_{ik} = \frac{\partial x_i}{\partial X_k}, \tag{3.1.5}$$

where $\mathbf{x}(\mathbf{X}, t)$ is the spatial coordinate.

The equations (3.1.1)-(3.1.5) do not give a closed system of equations, so balance of energy is postulated to achieve closure. Take the free energy density to be a function of the deformation gradient and the concentration and denote it by $W(F, C)$. The rate of change of free energy is given by the rate of work done by external volume and interfacial forces and the rate of work done by injecting molecules. Balance of energy requires that the total free energy of the system should never increase. This combined with the assumption of instantaneous local rearrangement of molecules implies that

$$s_{ik} = \frac{\partial W}{\partial F_{ik}}. \tag{3.1.6}$$

Let μ denote the chemical potential of a pump which injects small molecules into the gel. The rate of work due to injection depends on μ. Local equilibrium between the molecules in each pump and those in the gel near the pump is assumed. This local equilibrium assumption implies that

$$\mu = \frac{\partial W}{\partial C}. \tag{3.1.7}$$

Long-range migration of small molecules dissipate energy and thus should be modeled as a time-dependent process. This is imposed by adopting the kinetic law

$$\mathbf{J}_k = -M_{kl}\frac{\partial \mu}{\partial X_l}, \tag{3.1.8}$$

where the mobility tensor $M = M(F, C)$ is symmetric and positive definite.

In order to close the system, special forms for the free energy density W and the mobility tensor M specific to the material being studied must be chosen.

First, recall that under most types of loads, the polymers and small molecules can undergo large configurational changes without appreciable volume changes. Thus it is assumed that the polymers and solvent are incompressible. Since the gel is a condensed matter with no voids, this incompressibility assumption is given by the constraint equation

$$1 + vC = \det F, \tag{3.1.9}$$

where v is the volume per small molecule. This constraint is imposed on the equation system using the Lagrange multiplier Π, which corresponds to the osmotic pressure of the mixture. Consequently, incompressibility requires that the equations of state (3.1.6) and (3.1.7) be modified:

$$s_{ik} = \frac{\partial W}{\partial F_{ik}} - \Pi F_{ik}^{-T} \det F, \tag{3.1.10}$$

$$\mu = \frac{\partial W}{\partial C} + \Pi v. \tag{3.1.11}$$

The free energy comes from two molecular processes: stretching of the polymer network and mixing of the polymers and small molecules. Following Flory and Rehner [25], assume that the free energy has the form

$$W(F, C) = W_s(F) + W_m(C), \tag{3.1.12}$$

where W_s and W_m are contributions from stretching and mixing, respectively. Following [23], the free energy due to stretching is taken to be

$$W_s(F) = \frac{1}{2} NkT(|F|^2 - 3 - 2\log(\det F)), \tag{3.1.13}$$

where N is the number of polymer chains divided by the volume of dry polymers, $|F|^2 = \mathrm{tr} FF^T$, and kT is the thermal energy of the gel. Following [24], the mixing free energy is taken to be

$$W_m(C) = -\frac{kT}{v}[vC \log(1 + \frac{1}{vC}) + \frac{\chi}{1 + vC}], \tag{3.1.14}$$

where χ is a dimensionless parameter. Finally, a kinetic law gives the mobility tensor in the form

$$M_{ml} = \frac{D}{vkT} F_{im}^{-T} F_{il}^{-T} [\det F - 1], \tag{3.1.15}$$

where D is the diffusion coefficient which for simplicity is assumed to be isotropic and independent of F and C. Equations (3.1.1)-(3.1.5), (3.1.8)-(3.1.15) give a complete system of equations once initial and boundary conditions are specified.

The model of Suo et al described here differs substantially from the gel model derived in Chapter 2. Apart from formulating their model in Lagrangian instead of Eulerian coordinates and using different variables, Suo's model does not account for any form of viscosity in the gel. Also, energy is dissipated in a different way. In the gel model proposed in Chapter 2, energy is dissipated by viscosity in the two components as well as by friction between the components (i.e., relative motion). In Suo's model, energy is dissipated by long-range migration of small molecules driven by chemical potential gradients. This is similar to the phenomenon of mass diffusion in a two-fluid mixture driven by chemical potential gradients discussed by Landau and Lifschitz [35, p. 219].

3.2 Static gel models

In 2009 Calderer et al studied equilibrium configurations of gels from the point of view of compressible elasticity [11]. In Lagrangian coordinates, the polymer stress tensor in [11] depends only on ϕ_1 and F. The equation of balance of linear momentum of the polymer is

$$\nabla \cdot \mathcal{S}(F, \phi_1) = \mathbf{f}, \tag{3.2.16}$$

where $\mathcal{S}$ is the first Piola-Kirchhoff stress of the polymer and $\mathbf{f}$ is an external body force which may be non-zero. The Lagrangian equation of balance of mass (2.1.16) applies as well. In order to complete the system of equations, a constitutive equation for $\mathcal{S}$ must be postulated and boundary conditions must be prescribed.

An expression for the Piola-Kirchhoff stress is derived from the Euler-Lagrange equations of the total free energy. The gel is assumed to be hyperelastic, so the free energy is written as the sum of an elastic contribution and a Flory-Huggins mixing contribution. Let $\mathbf{x}$ denote the deformation map between Lagrangian and Eulerian coordinates. The total free energy in the reference configuration Ω is given by the expression

$$\mathcal{E}(\mathbf{x}, \phi_1) = \int_{\Omega} [\varphi_0 W(\nabla \mathbf{x}) + \det F(\mathcal{W}_{FH}(\phi_1) + c_{FH})] d\mathbf{X}. \tag{3.2.17}$$

Recall that $F = \nabla_{\mathbf{X}} \mathbf{x}$, where the gradient is taken with respect to reference coordinates, and $W(F)$ is the elastic energy of the polymer. $\mathcal{W}_{FH}(\phi_1)$ is the Flory-Huggins mixing energy of the gel and satisfies the expression

$$\mathcal{W}_{FH}(\phi_1) = G(\phi_1, 1 - \phi_1)$$

where

$$G(\phi_1, \phi_2) = \frac{K_B T}{V_m} [\chi \phi_1 \phi_2 + \frac{1}{N_1} \phi_1 \log \phi_1 + \frac{1}{N_2} \phi_2 \log \phi_2].$$

The expression c_{FH} is a non-negative scalar which depends on the parameters of $\mathcal{W}_{FH}$ and is chosen in such a way that the expression $\mathcal{W}_{FH}(\phi_1) + c_{FH}$ is non-negative for all volumer fractions $\phi_1 \in [0, 1]$. The last term in the integral has the effect of penalizing expansions, as should be the case when modeling compressible elasticity. Since the determinant is bounded away from zero [11, Remark 2.1], terms in the free energy penalizing collapse are not needed. The isotropic elastic energy density is

$$W(F) = \mu_E [\frac{1}{2s} (|F|^{2s} - |I|^{2s}) + \frac{|I|^{2s-2}}{r} ((\det F)^{-r} - 1)], \tag{3.2.18}$$

where $s \geq 1$ and $r \geq 0$ are dimensionless parameters, $|F|^2 = \mathrm{tr} F F^T$, and μ_E is the elastic stiffness modulus of the polymer. Lagrangian balance of mass (2.1.16) acts as a constraint condition on the total energy. This constraint is substituted into $\mathcal{E}$ to remove ϕ_1 from the problem. The following energy, which depends only on $\mathbf{x}$, is obtained:

$$\mathcal{E}(\mathbf{x}) = \int_\Omega [\varphi_0 W(\nabla\mathbf{x}) + \det\nabla\mathbf{x}(\mathcal{W}_{FH}(\frac{\varphi_0}{\det\nabla\mathbf{x}}) + c_{FH})]d\mathbf{X}. \tag{3.2.19}$$

The energy given in (3.2.19) is shown to have a minimizer in an appropriately chosen function space. Let Ω be an open, bounded subset of $\mathbf{R}^d$ with Lipschitz boundary $\partial\Omega$, and let $\partial\Omega = \Gamma_0 \bigcup \Gamma_1$ with $\Gamma_0 \bigcap \Gamma_1 = \emptyset$. Assume that Γ_0 has positive Lebesgue measure. Impose some value of $\mathbf{x}$ on Γ_0 and let $\mathbf{x}_0$ be an extension of this boundary condition to all of Ω. Define the function spaces

$$W_0^{1,2s} = \{x \in W^{1,2s}(\Omega) : \mathbf{x}|_{\Gamma_0} = \mathbf{0}\},$$
$$\mathcal{A} = \{\mathbf{x} \in \mathbf{x}_0 + W_0^{1,2s} : (\det\nabla\mathbf{x})\nabla\mathbf{x}^{-1} \in L^{2q}(\Omega), \det\nabla\mathbf{x} \geq \varphi_0 a.e.\}$$

for $2s > d$ and $q(2s-1) \geq s$. Suppose that $\mathbf{x}_0 \in W^{1,2s}(\Omega)$ and $\det\nabla\mathbf{x}_0 > \varphi_0 > 0$. Then there exists $\mathbf{x} \in \mathcal{A}$ that minimizes the energy functional (3.2.19) [11, Theorem 3.2]. The existence of a minimizer implies the existence of a solution to the Euler-Lagrange equations of the energy functional. The energy includes the work done by a prescribed surface force $\mathbf{s}_0$ on Γ_1. The Piola-Kirchhoff stress is given by the first variational derivative of $\mathcal{E}$ in (3.2.19) with respect to F. The governing system of equations is

$$\mathcal{S} = \nu(\nabla\mathbf{x})\nabla\mathbf{x} - \kappa(\nabla\mathbf{x})\nabla\mathbf{x}^{-T}, \tag{3.2.20}$$

$$\nabla \cdot \mathcal{S} = \mathbf{f}, \tag{3.2.21}$$

$$\mathbf{x}|_{\Gamma_0} = \mathbf{x}_0,\ \mathcal{S}\cdot\mathbf{n}|_{\Gamma_1} = \mathbf{s}_0, \tag{3.2.22}$$

where ν and κ are scalar functions of a matrix variable that take the forms

$$\nu(F) = \mu_E\varphi_0|F|^{2s-2},$$
$$\kappa(F) = \mu_E\varphi_0|I|^{2s-2}(\det F)^{-r} - r(\frac{\varphi_0}{\det F})\det F$$

with

$$r(\phi) = \mathcal{W}_{FH}(\phi) + c_{FH} - \phi \mathcal{W}'_{FH}(\phi).$$

Note that the residual stress, that is, the stress in the reference domain (i.e. when $F = I$) vanishes if and only if $r(\varphi_0) = 0$. When $r(\varphi_0) \neq 0$, there is an residual stress on the gel.

One of the implications of residual stress can be seen from linearizing the Piola-Kirchhoff stress. Define the displacement $\mathbf{u}$ by the equation $\mathbf{x} = \mathbf{x}_0 + \mathbf{u}$. $\mathcal{S}(F)$ is linearized about $F = F_0 = \nabla \mathbf{x}_0$:

$$\mathcal{S}_{ij}(F) \approx \mathcal{S}_{ij}(F_0) + \frac{\partial \mathcal{S}_{ij}}{\partial F_{kl}}(F_0)\frac{\partial \mathbf{u}_k}{\partial \mathbf{x}_l}.$$

The linear small strain equations are obtained by taking $F_0 = I$. The linearized stress is given by the equation

$$\mathcal{S}_{ij} = C_r[\nabla \mathbf{u}]_{ij} + r(\varphi_0)\delta_{ij}, \tag{3.2.23}$$

where $C_r[\nabla \mathbf{u}]$ is the residual-dependent stiffness tensor given by

$$C_r[\nabla \mathbf{u}]_{ij} = \frac{\partial \mathcal{S}_{ij}}{\partial F_{kl}}(I)\frac{\partial \mathbf{u}_k}{\partial \mathbf{x}_l} = \mu(\varphi_0)(\nabla \mathbf{u} + \nabla \mathbf{u}^T)_{ij} - r(\varphi_0)\frac{\partial \mathbf{u}_j}{\partial \mathbf{x}_i} + \lambda(\varphi_0)\nabla \cdot \mathbf{u}\delta_{ij}$$

for generalized Lamé coefficients μ and λ. Notice that when $r(\varphi_0) = 0$, the stiffness tensor reduces to the standard linear elasticity operator and the stress is symmetric. Otherwise, the stress tensor is not symmetric. This nonsymmetry is one of the main consequences of residual stress, and this leads to significant mathematical and numerical difficulties.

3.3 Viscoelastic fluids

A great deal of mathematical theory has been developed for viscoelastic fluids governed by model equations similar to those developed in the previous chapter. Liu and Walkington developed existence theory for a system describing a fluid containing visco-hyperelastic solid particles exhibiting small strain and arbitrary rotation [46]. Guillopé and Saut proved global existence of regular solutions to a system of equations describing the flow of viscoelastic fluids which obey an Oldroyd-type differential constitutive law under the assumption of small data [29]. In one paper, Lin, Liu, and Zhang proved local and global existence and uniqueness of solutions for the linear viscoelastic fluid system of the Oldroyd model in two dimensions under the assumption of small data [41]. In a different paper, the same authors developed a model coupling microscopic and macroscopic descriptions of polymeric fluids, established energy estimates, and proved existence of solutions for initial data not far from equilibrium [42]. Lin and Zhang proved global existence of solutions for an incompressible viscoelastic fluid system of Oldroyd type with small data in a smooth bounded domain in two or three dimensions [43]. Lions and Masmoudi proved existence of global solutions of Oldroyd models of non-Newtonian flows [45]. DiPerna and Lions prove existence and uniqueness of weak solutions for linear transport equations with nonsmooth coefficients [17]. This paper does not focus on viscoelastic fluids, however a review of it is being included here because the equation being studied is similar to the deformation gradient equation (2.5.69).

Liu and Walkington [46] considered the behavior of a fluid containing viscoelastic particles restricted to a domain Ω with smooth boundary. Their system of equations is similar to the transport regime (2.5.83)-(2.5.85). There is only one velocity for the mixture: the center of mass velocity $\mathbf{V}$. Instead of volume fractions, a phase function $\phi(\mathbf{x}, t)$ is used to mark whether the point $\mathbf{x}$ at time t contains fluid ($\phi(\mathbf{x}, t) = +1$) or solid ($\phi(\mathbf{x}, t) = -1$). This function satisfies the transport equation

$$\phi_t + \mathbf{V} \cdot \nabla \phi = 0. \tag{3.3.24}$$

The mixture is incompressible, so balance of mass implies

$$\nabla \cdot \mathbf{V} = 0. \tag{3.3.25}$$

Homogeneous Dirichlet boundary conditions are imposed on $\mathbf{V}$ and balance of linear momentum is expressed in the following weak form since it allows discontinuities in stresses and densities:

$$\int_\Omega [\rho \mathbf{V}_t \cdot \mathbf{w} + \rho(\mathbf{V} \cdot \nabla \mathbf{V}) \cdot \mathbf{w} + \mathcal{T}_{ij} \mathbf{D}(\mathbf{w})_{ij}] = \int_\Omega \rho \mathbf{f} \cdot \mathbf{w} \tag{3.3.26}$$

for smooth vector fields $\mathbf{w}$ vanishing on $\partial\Omega$. ρ is the mixture density, $\mathbf{f}$ is external force, and $\mathcal{T}$ is the Cauchy stress tensor. Assuming that the mixture consists of an incompressible Newtonian fluid and incompressible visco-hyperelastic solid particles, the Cauchy stress tensor has the form

$$\mathcal{T} = -pI + \eta \mathbf{D}(\mathbf{V}) + \frac{\partial W}{\partial F}(F)F^T, \tag{3.3.27}$$

where η is the Newtonian viscosity and $W(F)$ is the elastic energy. One technical but very important assumption made about the elastic energy function in [46] is that the residual stress $\frac{\partial W}{\partial F}(I)$ vanishes. This is necessary in order to obtain energy estimates. Both ρ and η are known functions of ϕ. The deformation gradient F satisfies the equation

$$F_t + \mathbf{V} \cdot \nabla F = \nabla \mathbf{V} F. \tag{3.3.28}$$

The elastic stress $\frac{\partial W}{\partial F}(F)F^T$ is in general a nonlinear function of the variable F which satisfies the nonlinear hyperbolic equation (3.3.28). In order to analyze this problem, some simplifying approximations must be made. In classical linear elasticity, F is expressed in the form $F = I + \nabla \mathbf{u}$ for displacement $\mathbf{u}$ and the elastic energy is linearized about $F = I$ under the assumption of small displacements. However, this assumption is not plausible for elastic particles being transported in a fluid medium. Instead, allow arbitrarily large rotations but assume small strains. The polar decomposition of F takes the form $F = R(I + E)$ where R is a rotation tensor and $E = E^T$ is small. R and E satisfy the evolution equations

$$R_t + \mathbf{V} \cdot \nabla R = \mathbf{S}(\mathbf{V})R, \tag{3.3.29}$$

$$E_t + \mathbf{V} \cdot \nabla E = R^T \mathbf{D}(\mathbf{V})R, \tag{3.3.30}$$

where $\mathbf{S}(\mathbf{V}) = \frac{1}{2}(\nabla\mathbf{V} - \nabla\mathbf{V}^T)$ is the spin tensor. In order to obtain a satisfactory existence theory, the spin tensor $\mathbf{S}(\mathbf{V})$ is replaced by its smooth mollification $\mathbf{S}_\epsilon(\mathbf{V}) = \mathbf{S}(\mathbf{V}_\epsilon)$. The rotation tensor satisfies

$$R_t + \mathbf{V} \cdot \nabla R = \mathbf{S}_\epsilon(\mathbf{V})R, \tag{3.3.31}$$

Assuming zero residual stress, the linearized elastic stress satisfies

$$\frac{\partial W}{\partial F}(F)F^T \approx R\mathcal{C}[E]R^T, \tag{3.3.32}$$

where $\mathcal{C}[E]$ is a linear operator acting on E. The final system of equations under consideration consists of (3.3.24), (3.3.25), (3.3.26), (3.3.27), (3.3.30), (3.3.31), and (3.3.32). In order to close the system, the following initial and boundary conditions are imposed:

- $\phi|_{t=0} = \phi_0$;
- $\mathbf{V}|_{t=0} = \mathbf{V}_0$, $\nabla \cdot \mathbf{V}_0 = 0$, $\mathbf{V}|_{\partial\Omega} = \mathbf{0}$;
- $E|_{t=0} = E_0$;
- $R|_{t=0} = R_0$.

Here is the main result of the paper:

Theorem 3.1 ([46], Theorem 4.4)**.** *Equations (3.3.24)-(3.3.27), (3.3.30)-(3.3.32) with the above-stated assumptions on the boundary and initial data have a weak solution.*

The theorem is proved using Galerkin approximations and compactness results from [17].

Guillopé and Saut established existence results for incompressible viscoelastic fluids which obey an Oldroyd type constitutive law of the form

$$\tau + \lambda_1 \frac{D_a \tau}{Dt} = 2\eta \left[\mathbf{D}(\mathbf{v}) + \lambda_2 \frac{D_a \mathbf{D}(\mathbf{v})}{Dt} \right], \tag{3.3.33}$$

where τ is the symmetric extra-stress tensor (the total Cauchy stress tensor is $\sigma = -pI + \tau$); $\mathbf{v}$ is fluid velocity; λ_1 is the relaxation time, λ_2 the retardation time, $0 \leq \lambda_2 < \lambda_1$; p is the pressure; and η is the fluid viscosity [29]. The symbol D_a/Dt is a frame indifferent time derivative defined by

$$\frac{D_a \tau}{Dt} = \frac{\partial \tau}{\partial t} + \mathbf{v} \cdot \nabla \tau + \tau \mathbf{S}(\mathbf{v}) - \mathbf{S}(\mathbf{v})\tau - a(\mathbf{D}(\mathbf{v})\tau + \tau \mathbf{D}(\mathbf{v})),$$

where $-1 \leq a \leq 1$. Note that the case $\lambda_1 > 0$, $\lambda_2 = 0$ corresponds to a purely elastic fluid and the case $\lambda_1 = \lambda_2 = 0$ corresponds to a purely Newtonian fluid. Guillopé and Saut consider only $\lambda_2 > 0$. The velocity and pressure satisfy conservation of momentum and mass:

$$\rho(\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p = \nabla \cdot \tau + \mathbf{f}, \tag{3.3.34}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{3.3.35}$$

where ρ is the (constant) density and $\mathbf{f}$ is external body force. Equations (3.3.33), (3.3.34), and (3.3.35) are defined over a bounded domain Ω with smooth boundary $\partial\Omega$. The velocity satisfies nonslip conditions on the boundary and initial conditions are imposed on $\mathbf{v}$ and τ :

- $\mathbf{v}|_{\partial\Omega} = \mathbf{0}$;
- $\mathbf{v}|_{t=0} = \mathbf{v}_0$, $\tau|_{t=0} = \tau_0$.

Guillopé and Saut nondimensionalize their system. Introducing the Reynolds number Re, the Weissenberg number We, and the retardation parameter $\omega =$

$1-\frac{\lambda_2}{\lambda_1}$ satisfying $0<\omega<1$, the dimensionless system takes the form

$$Re(\mathbf{v}_t+\mathbf{v}\cdot\nabla\mathbf{v})+\nabla p=(1-\omega)\Delta\mathbf{v}+\nabla\cdot\tau+\mathbf{f}, \tag{3.3.36}$$

$$\nabla\cdot\mathbf{v}=0, \tag{3.3.37}$$

$$\tau+We\frac{D_a\tau}{Dt}=2\omega\mathbf{D}(\mathbf{v}), \tag{3.3.38}$$

$$\mathbf{v}|_{\partial\Omega}=\mathbf{0}, \tag{3.3.39}$$

$$\mathbf{v}|_{t=0}=\mathbf{v}_0,\ \tau|_{t=0}=\tau_0. \tag{3.3.40}$$

The main results of the paper are given in three theorems, one concerning local-in-time existence of solutions for arbitrary initial data, a second concerning uniqueness, and the third concerning global-in-time solutions. In order to state these theorems, some notation is needed. Let H be the space of solenoidal L^2 vector functions whose normal trace on $\partial\Omega$ is zero. Let P be the orthogonal projection of L^2 onto H. Let V be the space of solenoidal H^1_0 vector functions. Let $D(A)$ be the domain of the Stokes operator $A=-P\Delta$, $D(A)=V\bigcap H^2(\Omega)$. The following local existence result holds:

Theorem 3.2 ([29],Theorem 2.4)**.** *Assume* $\partial\Omega\in C^3$, $\mathbf{f}\in L^2_{loc}(0,\infty;H^1(\Omega))$, $\mathbf{f}_t\in L^2_{loc}(0,\infty;H^{-1}(\Omega))$, $\mathbf{v}_0\in(A)$, $\tau_0\in H^2(\Omega)$. *Then there exists a* $T>0$, $\mathbf{v}\in L^2(0,T;H^3(\Omega))\bigcap C([0,T];D(A))$, $\mathbf{v}_t\in L^2(0,T;V)\bigcap C([0,T];H)$, $p\in L^2(0,T;H^2(\Omega))$, *and* $\tau\in C([0,T];H^2(\Omega))$ *such that* $(\mathbf{v},p,\tau)$ *is a solution to (3.3.36)-(3.3.40).*

Uniqueness of local-in-time regular solutions follows from the following theorem:

Theorem 3.3 ([29], Theorem 2.5)**.** *Let* $T>0$. *The system (3.3.36)-(3.3.40) admits at most one solution* $(\mathbf{v},\tau)$ *in the class* $L^2(0,T;H^3(\Omega))\bigcap C([0,T];D(A))\times C([0,T];H^2(\Omega))$. *The pressure* p *is unique up to an additive constant in the space* $L^2(0,T;H^2(\Omega))$.

These two theorems, with initial data of arbitrary size, are proved using Schauder's fixed point theorem and results from Temam [63]. Note that as with the case of the Navier-Stokes equations, the pressure p is unique only up to an additive constant [34]. Standard energy methods are used to obtain a priori bounds on the local solution. These estimates are used to prove that for sufficiently small initial data, the local solution can be extended to a regular solution that exists for all time, as given more precisely in the following theorem:

Theorem 3.4 ([29], Theorem 3.3)**.** *Let $\partial\Omega \in C^4$. There exists some ω_0, $0 < \omega_0 < 1$, depending on Ω, such that, if $0 < \omega < \omega_0$, and if $\mathbf{v}_0 \in D(A)$, $\tau_0 \in H^2(\Omega)$, $\mathbf{f} \in L^\infty(0,\infty; H^1(\Omega))$, and $\mathbf{f}_t \in L^\infty(0,\infty; H^{-1}(\Omega))$ are small enough in their spaces, then (3.3.36)-(3.3.40) admits a unique solution $(\mathbf{v}, \tau)$ defined for all time t, and satisfying*

$$\begin{aligned}
&\mathbf{v} \in C_b([0,\infty]; D(A)) \bigcap L^2_{loc}(0,\infty; H^3(\Omega)),\\
&\mathbf{v}_t \in C_b([0,\infty]; H) \bigcap L^2_{loc}(0,\infty; V),\\
&\tau \in C_b([0,\infty]; H^2(\Omega)), \quad \tau_t \in C_b([0,\infty]; H^1(\Omega)).
\end{aligned}$$

Lin, Liu, and Zhang (LLZ) investigate the existence of solutions for equations modeling an incompressible linear viscoelastic fluid [41]. The elastic energy satisfies $W(F) = |F|^2$. The equation system takes the form

$$\begin{aligned}
&F_t + \mathbf{v} \cdot \nabla F = \nabla \mathbf{v} F,\\
&\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \eta \Delta \mathbf{v} + \nabla \cdot (F F^T),\\
&\nabla \cdot \mathbf{v} = 0.
\end{aligned}$$

The following initial conditions are imposed:

$$F(x,0) = F_0(x), \quad \det F_0 = 1, \quad \mathbf{v}(x,0) = \mathbf{v}_0(x).$$

The domain Ω can be either a bounded domain with smooth boundary, all of $\mathbf{R}^2$ or $\mathbf{R}^3$, or a periodic box. In the case of a bounded domain, the Dirichlet boundary condition

$$\mathbf{v}|_{\partial\Omega} = \mathbf{0}$$

is imposed. It is noted that for an incompressible fluid, the deformation gradient F satisfies the following transport equation:

$$(\frac{\partial F_{ij}}{\partial x_i})_t + \mathbf{v}_k \frac{\partial}{\partial k}(\frac{\partial F_{ij}}{\partial x_i}) = 0.$$

Therefore, if the initial deformation gradient satisfies $\frac{\partial (F_0)_{ij}}{\partial x_i} = 0$, then $\frac{\partial F_{ij}}{\partial x_i} = 0$ for all times and in the two-dimensional case, there exists a vector $\phi = (\phi_1, \phi_2)$ such that

$$F = \begin{bmatrix} -\frac{\partial \phi_1}{\partial x_2} & -\frac{\partial \phi_2}{\partial x_2} \\ \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_2}{\partial x_1} \end{bmatrix}.$$

ϕ is a volume-preserving map with $\det(\nabla\phi) = 1$. The Oldroyd system can be rewritten as

$$\phi_t + \mathbf{v} \cdot \nabla\phi = \mathbf{0}, \tag{3.3.41}$$

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla\mathbf{v} + \nabla p = \eta\Delta\mathbf{v} - \Delta\phi\nabla\phi, \tag{3.3.42}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{3.3.43}$$

where $\Delta\phi\nabla\phi = \sum_{i=1}^{2} \Delta\phi_i \nabla\phi_i$. This system is closely related to the Ericksen-Leslie systems for the evolution of liquid crystals [40]. The initial conditions become

$$\phi(x,0) = \phi_0(x), \quad \det(\nabla\phi_0) = 1, \quad \mathbf{v}(x,0) = \mathbf{v}_0(x). \tag{3.3.44}$$

Finally, in the case of a bounded domain, the following boundary condition is imposed:

$$\mathbf{v}|_{\partial\Omega} = \mathbf{0}. \tag{3.3.45}$$

Smooth solutions satisfy the energy law

$$\frac{d}{dt}\int_\Omega \frac{1}{2}[|\mathbf{v}|^2 + |\nabla\phi|^2] + \int_\Omega \eta|\nabla\mathbf{v}|^2 = 0. \tag{3.3.46}$$

Local existence of a unique classical solution to (3.3.41)-(3.3.45) is proved by using (3.3.46), Sobolev embeddings, and Gronwall and interpolation inequalities to establish H^2 and higher order energy estimates that can be applied to sequences of Galerkin approximations. More precisely, the theorem proved is

Theorem 3.5 ([41], Theorem 2.2)**.** *Let $k \geq 2$ be a positive integer, $\nabla\phi_0 \in H^k(\Omega)$, and $\mathbf{v}_0 \in H^k(\Omega)$. Then there exists a positive time T, which depends only on the initial conditions, such that the system (3.3.41)-(3.3.45) possesses a unique solution in the time interval $[0, T]$ with*

$$\begin{aligned} &\partial_t^j \partial_x^\alpha \mathbf{v} \in L^\infty(0, T; H^{k-2j-|\alpha|}(\Omega)) \bigcap L^2(0, T; H^{k-2j-|\alpha|+1}(\Omega)), \\ &\partial_t^j \partial_x^\alpha \nabla\phi \in L^\infty(0, T; H^{k-2j-|\alpha|}(\Omega)), \end{aligned}$$

for all j and α satisfying $2j + |\alpha| \leq k$. Moreover, if T^* is the maximal time of existence, then

$$\int_0^{T^*} ||\nabla \mathbf{v}(s)||_{H^2(\Omega)}^2 ds = +\infty. \tag{3.3.47}$$

Further, it is proved that when Ω is either a periodic box or all of $\mathbf{R}^2$ and the initial data ϕ_0 and $\mathbf{v}_0$ are sufficiently small, the system (3.3.41)- (3.3.43) will have a unique global classical solution. More precisely,

Theorem 3.6 ([41], Theorem 2.3)**.** *Let Ω be a periodic box or the whole space $\mathbf{R}^2$, $k \geq 2$ a positive integer, and $\nabla\phi_0 \in H^k(\Omega)$ with $\det \nabla\phi_0 = 1$ and $\mathbf{v}_0 \in H^k(\Omega)$. Furthermore, for some large enough constant C, assume that*

$$||\mathbf{v}||_{H^2(\Omega)} + ||\nabla(\phi_0 - \mathbf{x})||_{H^2(\Omega)} \leq \frac{\eta}{C(1 + 1/\eta)^3(1 + \eta + 1/\eta)}.$$

Then the system (3.3.41)-(3.3.45) will have a unique global classical solution such that

$$||\mathbf{v}||_{H^2(\Omega)}^2 + ||\nabla(\phi - \mathbf{x})||_{H^2(\Omega)}^2 + \eta \int_0^\infty ||\nabla \mathbf{v}(s)||_{H^2(\Omega)}^2 ds \leq \frac{\eta}{C(1 + \eta + 1/\eta)}$$

and (3.3.47) holds for $T^* = \infty$.

Finally, LLZ prove local existence and uniqueness of a classical solution to (3.3.41)-(3.3.45) for the inviscid case ($\eta = 0$), when the system is hyperbolic.

In [42], LLZ consider a system which couples the microscopic and macroscopic behavior of incompressible polymeric fluids. Elastic effects are modeled in term of the behavior of molecules. This is done probabilistically using a probability distribution function ψ. This function depends on the time variable t, the spatial coordinate $\mathbf{x}$, and the microscopic variable $\mathbf{Q}$, which is the end-to-end vector of a molecule and describes molecular orientations. $\psi(\mathbf{x}, \mathbf{Q}, t)$ is governed by a Fokker-Planck equation. Being a probability density, ψ naturally satisfies the conditions $\psi \geq 0$ and $\int_{\mathbf{R}^3} \psi d\mathbf{Q} = 1$. At the macroscopic level, the velocity of the fluid is governed by the linear momentum equation of continuum mechanics with a stress tensor that depends on the microstructure as described by the distribution function ψ and a molecular potential $U(\mathbf{Q})$. Global existence of classical solutions is proved for initial data near hydrodynamic equilibrium, i.e. for velocity near zero and probability close to the Maxwellian distribution.

The model system is derived using an energy variation principle and statistical mechanics. Modeling the polymer molecules as one-dimensional springs and assuming that the microscopic time scale is much faster than the macroscopic scale, the microscopic variable $\mathbf{Q}$ is governed by the evolution equation

$$\gamma \mathbf{Q}_t = -\nabla_{\mathbf{Q}} U \tag{3.3.48}$$

for a given microscopic elastic potential $U(\mathbf{Q})$. Here $\nabla_{\mathbf{Q}}$ denotes the gradient with respect to the $\mathbf{Q}$ variable and γ is a constant which describes the linear damping mechanism. Incorporating isotropic Brownian motion into the dynamics of $\mathbf{Q}$ and using incompressibility, the fact that the polymer molecules are transported by the macroscopic velocity $\mathbf{v}$, and (3.3.48), the equation governing the evolution of the distribution function ψ takes the form

$$\psi_t + \mathbf{v} \cdot \nabla \psi + \nabla_{\mathbf{Q}} \cdot (\nabla \mathbf{v} \mathbf{Q} \psi) = \sigma \Delta_{\mathbf{Q}} \psi + \frac{1}{\gamma} \nabla_{\mathbf{Q}} \cdot (\nabla_{\mathbf{Q}} U \psi), \tag{3.3.49}$$

where $\sigma = K_B T$ is the thermal energy. The incompressibility condition

$$\nabla \cdot \mathbf{v} = 0 \tag{3.3.50}$$

has been used to derive (3.3.49). The momentum equation is derived by minimizing an energy functional that incorporates microscopic elastic effects. Including Newtonian viscosity in the energy and using (3.3.50) as a constraint, the velocity equation takes the form

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \eta \Delta \mathbf{v} + \frac{\lambda}{\gamma} \nabla \cdot \left[\int_{\mathbf{R}^3} \nabla_{\mathbf{Q}} U \otimes \mathbf{Q} \psi d\mathbf{Q} \right], \tag{3.3.51}$$

where λ here is the ratio between the kinetic and elastic energy. In order to obtain a global existence theory for this multiscale problem, consider only perturbations of ψ of the form

$$\psi = M + \sqrt{M} f, \tag{3.3.52}$$

where $M = e^{-U}$ is the pure Maxwellian stationary solution. Plugging this into (3.3.49) and (3.3.51), a new system for the variables $(\mathbf{v}, f, p)$ is obtained after setting all constants except σ and η equal to 1:

$$f_t + \mathbf{v} \cdot \nabla f + \nabla \mathbf{v} \mathbf{Q} \cdot \nabla_{\mathbf{Q}} f - \sigma(\Delta_{\mathbf{Q}} f + \frac{\Delta_{\mathbf{Q}} U}{2} f - \frac{|\nabla_{\mathbf{Q}} U|^2}{4} f) \tag{3.3.53}$$

$$= \sqrt{M} \nabla \mathbf{v} \mathbf{Q} \cdot \nabla_{\mathbf{Q}} U + \frac{1}{2} \nabla \mathbf{v} \mathbf{Q} \cdot \nabla_{\mathbf{Q}} U f,$$

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \eta \Delta \mathbf{v} + \nabla \cdot (\int_{\mathbf{R}^3} \nabla_{\mathbf{Q}} U \otimes \mathbf{Q} \sqrt{M} f d\mathbf{Q}). \tag{3.3.54}$$

The following initial condition are imposed:

$$f|_{t=0} = f_0, \quad \mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \nabla \cdot \mathbf{v}_0 = 0, \quad \int_{\mathbf{R}^3} \sqrt{M} f d\mathbf{Q} = 0. \tag{3.3.55}$$

LLZ derive an energy law for the system and use it along with standard inequalities and interpolations to obtain higher order energy estimates which ensure global existence and uniqueness of a smooth solutions to (3.3.50), (3.3.53), and (3.3.54) assuming sufficiently small initial data in (3.3.55) and making certain assumptions on the microscopic potential U.

Lin and Zhang [43] consider the following initial-boundary value problem for the Oldroyd model for viscoelastic fluids:

$$F_t + \mathbf{V} \cdot \nabla F = \nabla \mathbf{V} F, \tag{3.3.56}$$

$$\mathbf{V}_t + \mathbf{V} \cdot \nabla \mathbf{V} + \nabla p = \Delta \mathbf{V} + \nabla \cdot (FF^T), \tag{3.3.57}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{3.3.58}$$

$$F|_{t=0} = F_0, \quad \mathbf{V}|_{t=0} = \mathbf{V}_0, \tag{3.3.59}$$

$$\mathbf{V}|_{\partial\Omega} = \mathbf{0}. \tag{3.3.60}$$

Note that this system is identical to (2.5.83), (2.5.84), (2.5.85) with total Cauchy stress

$$\mathcal{T} = -pI + 2\mathbf{D}(\mathbf{V}) + FF^T$$

with neo-Hookean elastic energy $W(F) = \frac{1}{2}|F|^2$. Also note that (3.3.56)-(3.3.60) is identical to the equation system studied in [41]. One difference between [41] and [43] is that [41] considers only the two dimensional case, whereas [43] considers both two and three dimensions. Recall from hydrodynamics that the flow map is given by $\mathbf{x}(t, \mathbf{X})$, where $\mathbf{X}$ is the material coordinate, and that the deformation gradient in spatial coordinates is given by

$$F(t, \mathbf{x}) = \frac{\partial \mathbf{x}(t, \mathbf{X}^{-1}(t, \mathbf{x}))}{\partial \mathbf{X}},$$

where $\mathbf{X}^{-1}(t, \mathbf{x})$ is the inverse of the flow map. In an effort to bypass the mathematical difficulties posed by the nonlinear hyperbolic nature of (3.3.56), Lin and Zhang define the quantities

$$G = \frac{\mathbf{X}^{-1}(t, \mathbf{x})}{\partial \mathbf{x}}, \quad U = G - I.$$

U has the important property of being a curl-free matrix function. This implies that there exists a vector-valued function ψ taking values in $\mathbf{R}^d$ for $d = 2, 3$ such that $(U_{j,1}, U_{j,2}, \ldots, U_{j,d}) = \nabla\psi_j$ for $j = 1, \ldots, d$. ψ satisfies the transport equation and boundary condition

$$\psi_t + \mathbf{V} = -\mathbf{V} \cdot \nabla\psi, \quad \psi|_{\partial\Omega} = \mathbf{0}, \tag{3.3.61}$$

which is equivalent to (3.3.56). (3.3.61) is used to prove global existence of classical solutions to (3.3.56)-(3.3.60) for small initial data.

In order to state the main results of the paper, some notation must be introduced. Let H denote the space of solenoidal L^2 vector functions with normal trace on $\partial\Omega$ equal to $\mathbf{0}$. Let P be the orthogonal projection of vector L^2 functions into H. Let V denote the space of solenoidal H_0^1 functions and let $D(A)$ be the domain of the operator $A = -P\Delta$. The main results of the paper are two existence theorems. The first theorem concerns local existence and uniqueness of a classical solution with general data:

Theorem 3.7 ([43], Theorem 1.1)**.** *Let Ω be a bounded open set of $\mathbf{R}^d$, $d = 2, 3$. Assume that $\partial\Omega \in C^3$, $\mathbf{V}_0 \in D(A)$, and $F_0 \in H^2(\Omega)$. Then there exists a positive constant T such that (3.3.56)-(3.3.60) has a unique solution $(F, \mathbf{V}, p)$ satisfying $F \in C^{1-i}([0,T]; H^{2-i}(\Omega))$ for $i = 0, 1$; $\mathbf{V} \in C([0,T]; D(A)) \bigcap L^2(0,T; H^3(\Omega))$; $\mathbf{V}_t \in C([0,T]; H) \bigcup L^2(0,T; V)$, and $\nabla p \in L^2(0,T; H^1(\Omega))$.*

This theorem is proved by establishing a priori estimates which can be used in conjunction with Galerkin approximations and the compactness method or an iteration scheme and Schauder's fixed point theorem. The a priori estimates also establish uniqueness. The second main result establishes global existence for small data:

Theorem 3.8 ([43], Theorem 1.2)**.** *Under the assumptions of the above theorem, assume further that $U_0 = F_0^{-1} - I$ is curl-free and*

$$\det F_0 = 1, \quad ||F_0 - I||_{L^2(\Omega)} + ||\mathbf{V}_0||_{L^2(\Omega)} \leq c_0, \tag{3.3.62}$$

for some c_0 sufficiently small. Then the solution constructed in the above theorem is global with $\mathbf{V} \in L^2(0,\infty; H^3(\Omega))$, $\mathbf{V}_t \in L^2(0,\infty; V)$, and $\nabla p \in L^2(0,\infty; H^1(\Omega))$. Assume further that $\mathbf{V}_t(0) \in H^1(\Omega)$ and $||\mathbf{V}_t(0)||_{L^1(\Omega)} \leq c_0$. Then there exists some C, $\omega > 0$ such that

$$[||\mathbf{V}(t)||_{H^2(\Omega)} + ||\mathbf{V}_t(t)||_{L^2(\Omega)} + ||U(t)||_{H^2(\Omega)} + ||U_t(t)||_{L^2(\Omega)}] \exp(\omega t)$$
$$+ \int_0^t [||\mathbf{V}_t(s)||^2_{H^2(\Omega)} + ||\mathbf{V}_{tt}(s)||^2_{L^2(\Omega)}] \exp(\omega s) ds \leq C c_0$$

for all $t > 0$.

The theorem is proved by establishing higher order energy estimates and using (3.3.61) in place of (3.3.56).

Lions and Masmoudi [45] prove existence of global weak solutions for an Oldroyd model of incompressible non-Newtonian flows. The system which they study is

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \eta \Delta \mathbf{v} + \nabla p = \nabla \cdot \tau, \tag{3.3.63}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{3.3.64}$$

$$\tau_t + \mathbf{v} \cdot \nabla \tau + \tau \mathbf{S}(\mathbf{v}) - \mathbf{S}(\mathbf{v}) \tau + a\tau = b\mathbf{D}(\mathbf{v}), \tag{3.3.65}$$

where $a, b \geq 0$ are constants and τ is the symmetric extra-stress tensor. The authors consider three types of boundary conditions:

1. periodic boundary conditions where all functions are periodic in $\mathbf{x}_i$ with period $T_i > 0$ for $1 \leq i \leq n$;

2. the whole space $\mathbf{R}^n$ for $n = 2, 3$ with the unknowns $\mathbf{v}, \tau, p$ vanishing at infinity;

3. homogeneous Dirichlet conditions: Ω is a smooth, bounded, open domain and $\mathbf{v}$ vanishes on $\partial\Omega$.

The initial conditions

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \tau|_{t=0} = \tau_0, \quad \nabla \cdot \mathbf{v}_0 = 0 \tag{3.3.66}$$

are imposed. For Dirichlet boundary conditions, the compatibility condition $\mathbf{v}_0 \cdot \mathbf{n} = 0$ on $\partial\Omega$ must be imposed. The concept of global weak solution is precisely defined for each type of boundary condition being considered and for the cases $n = 2$ and $n = 3$. The main result is that a global weak solution exists provided $\mathbf{v}_0, \tau_0 \in L^2$. Solutions of (3.3.63)-(3.3.65) satisfy the energy law

$$\frac{d}{dt}\int [\frac{|\mathbf{v}|^2}{2} + \frac{|\tau|^2}{2b}] + \int [\frac{a}{b}|\tau|^2 + \eta|\nabla \mathbf{v}|^2] = 0. \tag{3.3.67}$$

This energy law is used to obtain a priori bounds for $\mathbf{v}$ and τ in various Lebesgue and Sobolev spaces. Some of these bounds depend upon the equicontinuity of τ in L^2, so it is proved that if τ_0 belongs to an equicontinuous set in L^2, then τ is equicontinuous uniformly in t. These results are used to prove compactness of sequences of approximations, so that they converge strongly to weak solutions in appropriate function spaces. Here is the main compactness result:

Theorem 3.9 ([45], Theorem 4.1)**.** *Let τ_0^n be a sequence approximating the initial condition τ_0 and let τ^n be a sequence of approximate solutions to (3.3.65) in the weak sense. If τ_0^n converges strongly to τ_0 in L^2, then τ^n converges strongly to τ in $C([0,T]; L^2)$ and $(\mathbf{v}, \tau)$ is a weak solution of (3.3.63)-(3.3.66).*

Since the system (3.3.63)-(3.3.65) is nonlinear, this compactness result is needed in order to ensure that a sequence of approximations converges to a weak solution of the system.

DiPerna and Lions prove existence and uniqueness of weak solutions of linear scalar transport equations under very general assumptions on the data [17]. The equation being studied in this paper is

$$u_t - \mathbf{b} \cdot \nabla u + cu = 0 \text{ in } (0,T) \times \mathbf{R}^N \tag{3.3.68}$$

where $T > 0$ is given, $N \geq 1$, and $\mathbf{b}$, c are known functions that satisfy at least

$$\mathbf{b} \in L^1(0,T;(L^1_{loc}(\mathbf{R}^N))^N), \quad c \in L^1(0,T;L^1_{loc}(\mathbf{R}^N)). \tag{3.3.69}$$

Given an initial condition $u^0 \in L^p(\mathbf{R}^N)$ for some $p \in [1,\infty]$, a weak solution in $L^\infty(0,T;L^p(\mathbf{R}^N))$ is sought. This weak solution will satisfy the equation

$$-\int_0^T \int_{\mathbf{R}^N} u\phi_t - \int_{\mathbf{R}^N} u^0\phi(0,x) + \int_0^T \int_{\mathbf{R}^N} u[\nabla \cdot (\mathbf{b}\phi) + c\phi] = 0 \tag{3.3.70}$$

for all test functions $\phi \in C^\infty([0,T] \times \mathbf{R}^N)$ with compact support in $[0,T) \times \mathbf{R}^N$. This definition makes sense provided

$$c + \nabla \cdot \mathbf{b} \in L^1(0,T;L^q_{loc}(\mathbf{R}^N)), \quad \mathbf{b} \in L^1(0,T;(L^q_{loc}(\mathbf{R}^N))^N), \tag{3.3.71}$$

where q is the Sobolev conjugate of p ($\frac{1}{q} + \frac{1}{p} = 1$). DiPerna and Lions prove the following proposition:

Proposition 3.1 ([17], Proposition II.1)**.** *Let $p \in [1,\infty]$, $u^0 \in L^p(\mathbf{R}^N)$, assume (3.3.69), (3.3.71), and*

$$\begin{cases} c + \frac{1}{p}\nabla \cdot \mathbf{b} \in L^1(0,T;L^\infty(\mathbf{R}^N)) & \text{if } p > 1 \\ c, \nabla \cdot \mathbf{b} \in L^1(0,T;L^\infty(\mathbf{R}^N)) & \text{if } p = 1. \end{cases}$$

Then there exists a weak solution u of (3.3.68) in $L^\infty(0,T;L^p(\mathbf{R}^N))$ corresponding to the initial condition u^0.

The proof of this proposition is obtained by mollifying the coefficient and initial data, solving a sequence of approximate equations, and extracting a subsequence that converges weakly or weakly-$*$ to a weak solution of the problem. Formally, the following estimates are obtained. If $p = \infty$,

$$\frac{d}{dt}||u||_{L^\infty(\mathbf{R}^N)} \leq ||c||_{L^\infty(\mathbf{R}^N)}||u||_{L^\infty(\mathbf{R}^N)}.$$

If $p < \infty$,

$$\frac{d}{dt}\int_{\mathbf{R}^N} |u|^p \leq p||c + \frac{1}{p}\nabla \cdot \mathbf{b}||_{L^\infty(\mathbf{R}^N)} \int_{\mathbf{R}^N} |u|^p.$$

In both cases Gronwall's inequality gives bounds on $||u||_{L^\infty(0,T;L^p(\mathbf{R}^N))}$ under the assumptions of the proposition. To prove existence of solutions, regularize $\mathbf{b}$, c, and u^0. Take $\rho \in \mathcal{D}_+(\mathbf{R}^N)$, the space of positive, infinitely differentiable functions with compact support. Assume $\int_{\mathbf{R}^N} \rho = 1$ and take $\rho_\epsilon(x) = \frac{1}{\epsilon^N}\rho(\frac{x}{\epsilon})$ for $\epsilon > 0$. Consider the following convolutions in x : $\mathbf{b}_\epsilon = \mathbf{b} * \rho_\epsilon$, $c_\epsilon = c * \rho_\epsilon$, $u^0_\epsilon = u^0 * \rho_\epsilon$. By standard considerations, there exists a unique solution $u_\epsilon \in C([0,T]; C^1_b(\mathbf{R}^N))$ of

$$\frac{\partial u_\epsilon}{\partial t} - \mathbf{b}_\epsilon \cdot \nabla u_\epsilon + c_\epsilon u_\epsilon = 0, \quad u_\epsilon|_{t=0} = u^0_\epsilon.$$

In view of the above formal estimates, which can be proved rigorously for u_ϵ, u_ϵ is bounded in $L^\infty(0,T;L^p(\mathbf{R}^N))$ uniformly in ϵ. Therefore, extracting subsequences if neccessary, there exists $u \in L^\infty(0,T;L^p(\mathbf{R}^N))$ such that u_ϵ converges weakly-$*$ to u in $L^\infty(0,T;L^p(\mathbf{R}^N))$ as $\epsilon \to 0$. It can easily be shown that this limit is a weak solution of (3.3.68) in the sense of (3.3.70). Uniqueness of this solution follows under slightly stronger assumptions on the coefficients $\mathbf{b}$ and c. The scalar transport equation (3.3.68) is similar to the chain rule relation of elasticity

$$F_t + \mathbf{v} \cdot \nabla F = \nabla \mathbf{v} F$$

with the scalar u corresponding to the second-rank tensor F, $\mathbf{b}$ corresponding to $-\mathbf{v}$, and the scalar c corresponding to the second-rank tensor $\nabla\mathbf{v}$. If $\mathbf{v}$ and $\nabla\mathbf{v}$ satisfy the same regularity and integrability assumptions as $\mathbf{b}$ and c, respectively, then the same arguments used for scalar transport equations apply to the tensor equation and a unique weak solution to the chain rule relation exists. Unfortunately, the chain rule relation is usually coupled with a momentum equation for $\mathbf{v}$ and it is not yet known how to obtain weak solutions to the coupled system in general.

3.4 Porous media flow

The models derived in Chapter 2 also have characteristics in common with models of porous media flow. Douglas et al [20] derive a model for single-phase flow through a fractured porous media. This research has important applications to oil recovery techniques in the petroleum industry. In the model a slightly compressible fluid such as oil flows through a multiscale fractured reservoir. The reservoir exhibits a heirarchy of length scales, with different types of fractures occuring at different length scales. The fluid has viscosity η and constant (small) compressibility c. The equation of state is

$$d\rho = c\rho dp. \tag{3.4.72}$$

Fluid velocity is given in this model by Darcy's law,

$$\mathbf{v} = -\frac{K}{\eta}\nabla p, \tag{3.4.73}$$

where K is the permeability of the porous medium. K changes values at different scales and can be a second-order tensor at some length scales and a scalar at others. Taking ϕ to be the porosity of the medium, balance of mass requires

$$\phi\rho_t + \nabla \cdot (\rho\mathbf{v}) = S, \tag{3.4.74}$$

where S is an external source of mass. Using (3.4.72) in (3.4.73) and substituting into (3.4.74), the equation for the density is given by

$$\phi\rho_t - \nabla \cdot (\frac{K}{\eta c}\nabla\rho) = S. \tag{3.4.75}$$

The porosity, permeability, and mass source all have different values at different length scales. Therefore, ρ will have different values at different length scales and a system of equations must be solved in order to obtain values for the density at different scales. These equations are homogenized across different scales and the homogenization is described in detail in the case of two levels of fracture. A general $(N+1)-$scale homogenized system is given for $N \geq 2$ and well-posedness is proved using a standard Hilbert space result [54].

Showalter gives an overview of mathematical models of porous media flow used in the contexts of geomechanics and oil recovery [56]. Fluid flows through the holes of a deformable porous solid. Deformation of the solid affects the flow of the fluid and, likewise, the fluid pressure deforms the solid. The simplest model system is the classical Biot deformation-diffusion system for a homogeneous and isotropic medium:

$$\rho \mathbf{u}_{tt} - (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \Delta \mathbf{u} + \alpha \nabla p = \mathbf{f}, \tag{3.4.76}$$

$$c_0 p_t - \nabla \cdot (k \nabla p) + \alpha \nabla \cdot \mathbf{u}_t = h, \tag{3.4.77}$$

where p, the fluid pressure, and $\mathbf{u}$, the solid displacement, are the unknown variables. $\rho \geq 0$ is the density and $c_0 \geq 0$ is a physical coefficient related to the compressibility of the fluid and the porosity of the medium. Of particular interest are the quasi-static case, where $\rho = 0$ and inertia effects are ignored, and the incompressible case, where $c_0 = 0$ and the system becomes degenerate. Showalter describes how initial and mixed traction/displacement/permeability boundary conditions are prescribed to the compressible, linear, quasi-static case. For a heterogeneous medium with several distinct spatial scales, the simplest model involves combining the Barenblatt double-diffusion model, which considers the combined effects of two components in parallel, with (3.4.76)-(3.4.77). The resulting system consists of three equations: one equation each for the two pressures p_1 and p_2 corresponding to each component, and one equation for the displacement $\mathbf{u}$. A nonlinear quasi-static model for a deforming dam made of an isotropic, homogeneous material is given. The unknown variables of the system are displacement $\mathbf{u}$, pressure p, and filtration velocity $\mathbf{q}$, which is related to the pressure by Darcy's law. The density is given as a function of pressure by a thermodynamic state equation. A general model for Darcy flow through a visco-plastic medium is given as a coupled system of differential and functional equation:

$$\frac{\partial}{\partial t}(c_0 p + \alpha \nabla \cdot \mathbf{u}) - \nabla \cdot (k \nabla p) = c_0^{1/2} h_0, \tag{3.4.78}$$

$$\rho \mathbf{u}_{tt} - \nabla \mu^* (\nabla \cdot \mathbf{u}_t) - \nabla \cdot \sigma + \alpha \nabla p = \rho^{1/2} \mathbf{f}_0, \tag{3.4.79}$$

$$\sigma = \mathcal{H}(\mathbf{D}(\mathbf{u})), \tag{3.4.80}$$

where the elastic stress tensor σ is some given (possibly nonlinear) function of the linearized strain tensor $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$. Finally, Showalter describes a homogenized multi-scale porous media problem.

In [55] Showalter begins by deriving models of porous media flow. By the balance of mass principle in continuum mechanics, the basic equation for a fluid flowing through a porous medium is

$$\frac{\partial (c\rho)}{\partial t} + \nabla \cdot \mathbf{J} = f, \tag{3.4.81}$$

where ρ is the fluid density, c is the porosity of the medium, $\mathbf{J}$ is the flux, and f is a mass source density term. Darcy's law gives a flux of the form

$$\mathbf{J} = -\frac{k}{\eta} \rho \nabla p, \tag{3.4.82}$$

where k is the permeability of the medium, η is the viscosity of the fluid, and p is the fluid pressure. Density and pressure are related by an equation of state

$$\rho = s(p),$$

where s is a monotone, and usually strictly increasing, function. Denote by S the antiderivative of s. Introducing the variable $u = S(p)$ into (3.4.81) and (3.4.82) gives the generalized porous medium equation

$$\frac{\partial (ca(u))}{\partial t} - \nabla \cdot [\frac{k}{\eta} \nabla u] = f, \tag{3.4.83}$$

where $a(\cdot)$ is a known function. The permeability function k can be a function of ∇u in some situations, so (3.4.83) can be quasilinear. This model only applies to homogeneous media. For a heterogeneous medium of two components, the parallel flow model can be used. This model gives a system of two diffusion equations, one for each component. Many initial-boundary value problems of porous media flow can be reformulated as abstract initial value problems of the form

$$\frac{du(t)}{dt} + Au(t) = 0, \quad u(0) = u_0,$$

where A is an appropriately chosen operator and $u(t)$ takes values in an appropriately chosen function space. A very brief review of important functional analysis results (Banach and Hilbert spaces, scalar products and norms, linear operators, Lax-Milgram theorem, semigroup theory) is given. These theorems and lemmas are applied to an initial-boundary value problem for a distributed microstructure model, which improves upon parallel flow models by taking different time and space scales for the two distinct components. The domain Ω represents the global region of the model. At each point $x \in \Omega$, there is a cell Ω_x which is a scaled representation of the microstructure near x. One differential equation describes the global flow throughout Ω. A separate equation is specified in each cell to describe the flow internal to that cell. Any coupling between the equations occurs on the boundary of Ω_x, denoted by Γ_x. These concepts are used to model single phase flow in a fissured medium. Global flow is described in the macroscale variable x by the equation

$$\frac{\partial}{\partial t}(a(x)u(x,t)) - \nabla \cdot (A(x)\nabla u) + q(x,t) = f(x,t), \quad x \in \Omega, \tag{3.4.84}$$

where $q(x,t)$ represents the flow into Ω_x. The flow in each cell Ω_x is given by

$$\frac{\partial}{\partial t}(b(x,y)U(x,y,t)) - \nabla_y \cdot (B(x,y)\nabla_y U) = F(x,y,t), \quad y \in \Omega_x, \tag{3.4.85}$$

where y is the local (microscale) variable. The fissure pressure on the cell interface

Γ_x is imposed by the condition

$$B(x,s)\nabla_y U(x,s,t)\nabla \mathbf{n} + \frac{1}{\delta}(U(x,s,t) - u(x,t)) = 0, \quad s \in \Gamma_x, \tag{3.4.86}$$

where $\mathbf{n}$ is the outward unit normal on Γ_x. Finally, $q(x,t)$ is the average fluid flux across the interface:

$$q(x,t) = \frac{1}{|\Omega_x|} \int_{\Gamma_x} B(x,s)\nabla_y U \cdot \mathbf{n}.$$

The variational formulation of the distributed microstructure model is given and the following well-posedness result is a consequence of a theorem from semigroup theory:

Theorem 3.10 ([55], Theorem in Section 5)**.** *Define the Hilbert space* $H = L^2(\Omega) \times L^2(\Omega, L^2(\Omega_x))$*. Suppose that the initial condition* $(u,U)|_{t=0} = (u_0, U_0) \in H$ *is imposed on (3.4.84)-(3.4.85) and* $(f,F) \in C^\alpha([0,\infty), H)$ *for* $0 < \alpha < 1$*. Then for each* $\delta > 0$ *there is a unique* $(u_\delta, U_\delta) \in C([0,\infty), H) \bigcap C((0,\infty), H)$ *which satisfies (3.4.84)-(3.4.86) in the variational sense for* $(u_\delta, U_\delta)|_{t=0} = (u_0, U_0)$*.*

The distributed microstructure model itself is justified by homogenization. Finally, in the last section of [55], the generalized porous medium equation (3.4.83) is expressed as an abstract Cauchy problem taking values in a Banach space and semigroup theory and Banach space analysis are used to prove existence and uniqueness of solutions.

In [57], Showalter and Shi discuss the mathematical theory of dynamic plasticity models. Classical models for elastic-plastic materials consist of the momentum equation

$$u_{tt} + D^*\sigma = f \tag{3.4.87}$$

coupled with a constitutive law for the stress tensor

$$\sigma = F(\varepsilon) \tag{3.4.88}$$

which contains a system of variational inequalities. Here u is the displacement vector and ε is the strain rate $\varepsilon = Du \equiv \frac{1}{2}(\nabla u + \nabla u^T)$ and D^* is the dual operator of the symmetric gradient D and is defined by

$$(D^*\sigma)_i = -\frac{\partial \sigma_{ij}}{\partial x_j}.$$

The constitutive law (3.4.88) is expressed as a variational equation or inequality

$$\sigma_t + \partial\varphi(\sigma) - Dv \ni g, \tag{3.4.89}$$

which is coupled with (3.4.87). Here $v = u_t$, $\varphi(\cdot)$ denotes either the indicator function $I_K(\cdot)$ of a given closed convex set K characterizing the plasticity model or a smooth convex function for viscosity models, and $\partial\varphi$ is the corresponding subgradient or derivative, respectively. In some cases, such as the Prandtl-Ishlinski models with multi-yield surfaces [67], the total stress sigma is the sum of a collection of stress components

$$\sigma = \sum_j \sigma_j. \tag{3.4.90}$$

Substituting $v = u_t$ into (3.4.87), the dynamics of the system (3.4.87), (3.4.89), (3.4.90) is governed by an operator taking values in some L^2-type Hilbert space. The main theorem is a result from functional analysis:

Theorem 3.11 ([57], Theorem A)**.** *Let A be an m-accretive operator in the Hilbert space H. If $T > 0$, $\mathbf{x}_0$ is in the domain of A, and $\mathbf{f} \in W^{1,1}(0,T;H)$, then there exists a unique solution $\mathbf{x} \in W^{1,\infty}(0,T;H)$ of the Cauchy problem*

$$\mathbf{x}'(t) + A(\mathbf{x}(t)) \ni \mathbf{f}(t), \quad t > 0$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

with $\mathbf{x}(t)$ in the domain of A for all $0 \leq t \leq T$.

This theorem is used to prove existence and uniqueness of weak, strong, and regular solutions to a Cauchy problem for a general one-dimensional plasticity model defined on the interval $(0,1)$ under appropriate assumptions on the parameters of the model.

3.5 Navier-Stokes equations

In [63] Temam gives an in-depth study of the Navier-Stokes equations, both the linear and nonlinear cases. The overview given in this review will be restricted to the following linear Dirichlet problem defined over $\Omega \times [0,T]$:

$$\mathbf{u}_t - \eta\Delta\mathbf{u} + \nabla p = \mathbf{f}, \tag{3.5.91}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{3.5.92}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \tag{3.5.93}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \tag{3.5.94}$$

where Ω is a Lipschitz open bounded set in $\mathbf{R}^n$, $T > 0$ is fixed, and $\mathbf{f}$ and $\mathbf{u}_0$ are given. Define the following spaces:

$$\mathcal{V} = \{\mathbf{u} \in \mathcal{D}(\Omega) : \nabla \cdot \mathbf{u} = 0\},$$
$$V = \text{the closure of } \mathcal{V} \text{ in } [H_0^1(\Omega)]^n,$$
$$H = \text{the closure of } \mathcal{V} \text{ in } [L^2(\Omega)]^n,$$

where $\mathcal{D}(\Omega)$ is the set of all infinitely differentiable vector functions which have compact support in Ω. H is equipped with the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v}$$

and V is equipped with the product

$$((\mathbf{u}, \mathbf{v})) = \int_\Omega (\nabla\mathbf{u})_{ij}(\nabla\mathbf{v})_{ij}.$$

Let V' denote the dual space of V and for any linear functional $\mathbf{f} \in V'$ and any $\mathbf{v} \in V$, denote the action of $\mathbf{f}$ on $\mathbf{v}$ by $< \mathbf{f}, \mathbf{v} >$. The definition of a weak solution of (3.5.91)-(3.5.94) is

Definition 3.1 ([63], p. 253)**.** *For* $\mathbf{f} \in L^2(0,T;V')$ *and* $\mathbf{u}_0 \in H$ *given, find* $\mathbf{u} \in L^2(0,T;V)$ *satisfying*

$$\frac{d}{dt}(\mathbf{u},\mathbf{v}) + \eta((\mathbf{u},\mathbf{v})) =< \mathbf{f}, \mathbf{v} > \quad \forall \mathbf{v} \in V, \tag{3.5.95}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{3.5.96}$$

The main result of the analysis is the following theorem:

Theorem 3.12 ([63], Ch. III, Sec. 1, Theorem 1.1)**.** *For given* $\mathbf{f} \in L^2(0,T;V')$ *and* $\mathbf{u}_0 \in H$, *There exists a unique weak solution* $\mathbf{u}$. *Moreover,*

$$\mathbf{u} \in C([0,T];H).$$

The proof of existence follows from the Faedo-Galerkin method. Uniqueness follows from the linearity of the problem. Continuity in time follows from an interpolation theorem of Lions-Magenes [44]. For the weak solution $\mathbf{u}$, the pressure p exists in the following sense:

Proposition 3.2 ([63], Ch. III, Sec. 1, Proposition 1.1)**.** *Under the assumptions of the previous theorem, there exists a distribution* p *on* $\Omega \times [0,T]$ *such that the weak solution* $\mathbf{u}$ *and* p *satisfy (3.5.91) and (3.5.92) in the sense of distributions on* $\Omega \times [0,T]$. *(3.5.94) is satisfied in the sense*

$$\mathbf{u}(t) \to \mathbf{u}_0 \text{ in } L^2(\Omega) \text{ as } t \to 0.$$

Higher regularity of the weak solution and of p are obtained provided $\mathbf{f}$, $\mathbf{u}_0$, and $\partial\Omega$ are sufficiently smooth.

Often, the objective is to solve the Navier-Stokes system numerically. This is accomplished by discretizing in space and time. For the space discretization, let

V_h be a finite dimensional subspace of $[L^2(\Omega)]^n$. This space is associated with a finite element scheme and is equipped with two norms: the norm $|\cdot|$ induced by $[L^2(\Omega)]^n$ and its own norm $||\cdot||_h$. Since V_h is finite dimensional, these norms must be equivalent. Let $\mathbf{u}_h^0$ denote the orthogonal projection in $[L^2(\Omega)]^n$ of the initial condition $\mathbf{u}_0$ onto V_h. Temam lists three finite difference schemes for discretizing the linear problem in time. Starting at time $t = 0$, take time steps of size $k > 0$.

Scheme 3.1 ([63], Ch. III, Sec. 5, Scheme 5.1)**.** *When* $\mathbf{u}_h^0, \ldots, \mathbf{u}_h^{m-1}$ *are known,* $\mathbf{u}_h^m$ *is the solution in* V_h *of*

$$\frac{1}{k}(\mathbf{u}_h^m - \mathbf{u}_h^{m-1}, \mathbf{v}_h) + \eta((\mathbf{u}_h^m, \mathbf{v}_h))_h = (\mathbf{f}^m, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h.$$

Scheme 3.2 ([63], Ch. III, Sec. 5, Scheme 5.2)**.** *When* $\mathbf{u}_h^0, \ldots, \mathbf{u}_h^{m-1}$ *are known,* $\mathbf{u}_h^m$ *is the solution in* V_h *of*

$$\frac{1}{k}(\mathbf{u}_h^m - \mathbf{u}_h^{m-1}, \mathbf{v}_h) + \frac{\eta}{2}((\mathbf{u}_h^{m-1} + \mathbf{u}_h^m, \mathbf{v}_h))_h = (\mathbf{f}^m, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h.$$

Scheme 3.3 ([63], Ch. III, Sec. 5, Scheme 5.4)**.** *When* $\mathbf{u}_h^0, \ldots, \mathbf{u}_h^{m-1}$ *are known,* $\mathbf{u}_h^m$ *is the solution in* V_h *of*

$$\frac{1}{k}(\mathbf{u}_h^m - \mathbf{u}_h^{m-1}, \mathbf{v}_h) + \eta((\mathbf{u}_h^{m-1}, \mathbf{v}_h))_h = (\mathbf{f}^m, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h.$$

The inner product $((\mathbf{u}_h, \mathbf{v}_h))_h$ will depend upon the particular function space V_h that is chosen for the space discretization scheme. The first scheme is the backward Euler method and is a fully implicit scheme. The second scheme is the Crank-Nicholson method and is also implicit. Both of these schemes require the inversion of a matrix that is positive definite, nonsymmetric, and dependent on the time step m. The third scheme is explicit and requires the inversion of a matrix which is positive definite, symmetric, and independent of m. The reason why the explicit scheme requires the inversion of a matrix is due to the discrete condition $\nabla \cdot \mathbf{u} = 0$ built into the space V_h, so the determination of $\mathbf{u}_h^m$ requires inverting a matrix. Temam proves the following theorem:

Theorem 3.13 ([63], Ch. III, Sec. 5, Theorem 5.1)**.** *The functions* $\mathbf{u}_h$ *corresponding to Schemes (3.1) and (3.2) are unconditionally* $L^\infty(0, T; L^2(\Omega))$ *stable.*

Under appropriate conditions, these schemes converge strongly in $L^2(0, T; L^2(\Omega))$ as h and k, the step size in time, go to 0. In order to solve this problem numerically, the finite element space V_h must be specified. Assume Ω is a bounded domain in $\mathbf{R}^2$ and let Ω_h denote a triangulation of Ω. Take W_h to be the space of continuous vectors whose components are polynomials of degree two on each simplex $K \in \Omega_h$ and vanish on $\partial\Omega$. The discrete inner product $((\mathbf{u}_h, \mathbf{v}_h))_h$ is defined by the equation

$$((\mathbf{u}_h, \mathbf{v}_h))_h = ((\mathbf{u}_h, \mathbf{v}_h)) \ \forall \mathbf{u}_h, \mathbf{v}_h \in W_h.$$

Let V_h be the subspace of W_h consisting of all functions in W_h which satisfy

$$\int_K \nabla \cdot \mathbf{u}_h = 0 \quad \forall K \in \Omega_h.$$

In order to impose the discrete zero divergence condition, let X_h be the space of step functions of the form

$$\pi_h = \sum_{K \in \Omega_h} \eta_K \chi_K, \quad \eta_K \in \mathbf{R},$$

where χ_K is the characteristic function of K. The discrete divergence is a step function having the form

$$D_h \mathbf{u}_h = \sum_{K \in \Omega_h} \eta_K \chi_K,$$
$$\eta_K = \frac{1}{|K|} \int_K \nabla \cdot \mathbf{u}_h,$$

where $|K|$ denotes the Lebesgue measure of K. An adaptation of the Uzawa algorithm can now be used to compute the approximate velocity $\mathbf{u}_h^m \in V_h$ and the approximate pressure $p_h^m \in X_h$ at time step m if the velocity and pressure at time step $m-1$ have already been computed. The velocity and pressure are obtained as the limits of two sequences of elements

$$\mathbf{u}_h^{m,r} \in W_h, \quad p_h^{m,r} \in X_h, \quad r = 0, 1, \ldots, \infty.$$

The algorithm associated to Scheme (3.1) is as follows:

Algorithm 3.1 ([63], p. 389)**.** *Start with any* $p_h^{m,0} \in X_h$*. When* $p_h^{m,r}$ *is known define* $\mathbf{u}_h^{m,r+1} \in W_h$ *and* $p_h^{m,r+1} \in X_h$ *for* $r \geq 0$ *by*

$$(\mathbf{u}_h^{m,r+1}, \mathbf{v}_h) + k\eta((\mathbf{u}_h^{m+r+1}, \mathbf{v}_h))_h - k(p_h^{m,r}, D_h\mathbf{v}_h) = (\mathbf{u}_h^{m-1}, \mathbf{v}) + k(\mathbf{f}^m, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in W_h, \tag{3.5.97}$$

$$(p_h^{m,r+1} - p_h^{m,r}, q_h) + \rho(D_h\mathbf{u}_h^{m,r+1}, q_h) = 0 \quad \forall q_h \in X_h, \tag{3.5.98}$$

where $\rho > 0$ *is a numerical parameter.*

This algorithm is shown to converge if $0 < \rho < \frac{2\eta}{n}$.

3.6 Numerics of the stress diffusion coupling model

Feng and He proposed a mixed finite element method for simulating the behavior of polymer gels described by the stress diffusion coupling model of Doi [22]. The system of equations under consideration is

$$\begin{aligned}
&\nabla \cdot [\sigma(\mathbf{u}) - p\mathcal{I}] = \mathbf{0}, \\
&\zeta(\mathbf{v} - \mathbf{u}_t) = -(1-\phi)\nabla p, \\
&\nabla \cdot [\phi\mathbf{u}_t + (1-\phi)\mathbf{v}] = 0, \\
&\sigma(\mathbf{u}) = (K - \frac{2}{3}G)\nabla \cdot \mathbf{u}\mathcal{I} + 2G\mathbf{D}(\mathbf{u}), \\
&\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T),
\end{aligned}$$

where $\mathbf{u}$ is polymer displacement vector, $\mathbf{v}$ is solvent velocity, p is the pressure, $\sigma(\mathbf{u})$ is the stress tensor of the gel network, and $\mathbf{D}(\mathbf{u})$ is the network strain tensor. ζ, ϕ, K, and G are constant parameters. The initial and boundary conditions imposed on the system are

- $\mathbf{u}|_{t=0} = \mathbf{u}_0$;
- $[\sigma(\mathbf{u})_{ij} - p\delta_{ij}]\mathbf{n}_j = \mathbf{f}_i$ on $\partial\Omega$;
- $\nabla p \cdot \mathbf{n} = 0$ on $\partial\Omega$.

The function $\mathbf{f}$ is a normal stress imposed on the boundary and $\mathbf{n}$ denotes an outward unit normal on $\partial\Omega$. The last boundary condition is a consequence of assuming that the boundary is impermeable to solvent. In order for this system to be well-posed, the normal stress must satisfy the compatibility condition

$$\int_{\partial\Omega} \mathbf{f} = \mathbf{0}.$$

Adding a new variable q to the problem, the system can be rewritten in the form

$$\beta\Delta\mathbf{u} = \nabla\tilde{p}, \tag{3.6.99}$$

$$\nabla \cdot \mathbf{u} = q, \tag{3.6.100}$$

$$q_t = \kappa\Delta p, \tag{3.6.101}$$

where $\tilde{p} = p - \alpha q$ and β, α, and κ are constants. The variables of this problem satisfy the following conservation laws for all time:

$$\int_\Omega q = C_q := \int_\Omega \nabla \cdot \mathbf{u}_0, \tag{3.6.102}$$

$$\int_\Omega p = C_p := (\alpha + \frac{\beta}{d})C_q - \frac{1}{d}\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{x}, \tag{3.6.103}$$

$$\int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n} = C_q, \tag{3.6.104}$$

$$\int_\Omega \tilde{p} = C_{\tilde{p}} := C_p - \alpha C_q, \tag{3.6.105}$$

where $d = 2, 3$ is the dimension of the domain. In order for a finite element method to be defined, weak solutions must be defined and analyzed.

Definition 3.2 ([22], Definition 2.2)**.** *Let* $(\mathbf{u}_0, \mathbf{f}) \in [H^1(\Omega)]^d \times [H^{-1/2}(\partial\Omega)]^d$, *and* $< \mathbf{f}, 1 >_{\partial\Omega} = \mathbf{0}$, *where* $< \cdot, \cdot >_{\partial\Omega}$ *is the pairing of a linear functional in* $[H^{-1/2}(\partial\Omega)]^d$

with an element of $[H^{1/2}(\partial\Omega)]^d$. *Given* $T > 0$, *a triple* $(\mathbf{u}, \tilde{p}, q)$ *with*

$$\mathbf{u} \in L^\infty(0,T;[H^1(\Omega)]^d), \quad \tilde{p} \in L^2(0,T;L^2(\Omega)),$$
$$q \in L^\infty(0,T;L^2(\Omega)) \bigcap H^1(0,T;H^{-1}(\Omega)), \quad \alpha q + \tilde{p} \in L^2(0,T;H^1(\Omega)),$$

is called a weak solution to (3.6.99)-(3.6.101) with the boundary and initial conditions stated above if for almost every $t \in [0,T]$,

$$\int_\Omega [\beta(\nabla \mathbf{u})_{ij}(\nabla \mathbf{v})_{ij} - \tilde{p}\nabla \cdot \mathbf{v}] = < \mathbf{f}, \mathbf{v} >_{\partial\Omega} \quad \forall \mathbf{v} \in [H^1(\Omega)]^d, \tag{3.6.106}$$

$$\int_\Omega \nabla \cdot \mathbf{u}\varphi = \int_\Omega q\varphi \quad \forall \varphi \in L^2(\Omega), \tag{3.6.107}$$

$$(q_t, \psi)_\Omega + \int_\Omega \kappa \nabla(\alpha q + \tilde{p}) \cdot \nabla \psi = 0 \quad \forall \psi \in H^1(\Omega), \tag{3.6.108}$$

$$\mathbf{u}(0) = \mathbf{u}_0, , \tag{3.6.109}$$

$$q(0) = q_0 := \nabla \cdot \mathbf{u}_0, \tag{3.6.110}$$

where $(q_t, \psi)_\Omega$ *is the pairing of the linear functional* $q_t \in H^{-1}(\Omega)$ *with* $\psi \in H^1(\Omega)$.

An energy law is derived from (3.6.106)-(3.6.110) and is used to prove that a unique weak solution exists. Weak solutions are shown to satisfy the conservation laws (3.6.102)-(3.6.105). Note that the redefined pressure $\tilde{p}$ is in $L^2(\Omega)$ for almost all time whereas the original pressure $p = \tilde{p} + \alpha q$ is in $H^1(\Omega)$ for almost all time. This fact allows Feng and He to develop a finite elements method which uses low order (hence computationally cheap) finite elements to approximate q and $\tilde{p}$ but approximates p to high accuracy.

Following this weak solution analysis, Feng and He define a mixed finite element method following the procedure of Taylor and Hood [61] to discretize the problem in space. Assume Ω is a polygonal domain and let Ω_h be a quasi-uniform triangulation with mesh size h. Approximations $\mathbf{u}_h$ and $\tilde{p}_h$ are sought in the Taylor-Hood spaces

$$\mathbf{V}_h = \{\mathbf{v}_h \in [C^0(\overline{\Omega})]^d : \mathbf{v}_h|_K \in [P^2(K)]^d \forall K \in \Omega_h\},$$
$$M_h = \{\varphi_h \in C^0(\overline{\Omega}) : \varphi_h|_K \in P^1(K) \forall K \in \Omega_h\}.$$

Feng and He take the approximation space W_h for q to be equal to M_h, although any piecewise polynomial space is acceptable as long as $M_h \subseteq W_h$. The proposed semi-discrete numerical scheme is as follows: find $(\mathbf{u}_h, \tilde{p}_h, q_h) : (0,T] \to \mathbf{V}_h \times M_h \times W_h$ and $(\mathbf{u}_h(0), q_h(0)) \in \mathbf{V}_h \times W_h$ such that for all $t \in (0,T]$ there hold

$$\int_\Omega [\beta(\nabla \mathbf{u}_h)_{ij}(\nabla \mathbf{v}_h)_{ij} - \tilde{p}_h \nabla \cdot \mathbf{v}_h] =< \mathbf{f}, \mathbf{v}_h >_{\partial\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.6.111}$$

$$\int_\Omega \nabla \cdot \mathbf{u}_h \varphi_h = \int_\Omega q_h \varphi_h \quad \forall \varphi_h \in M_h, \tag{3.6.112}$$

$$(\frac{\partial q_h}{\partial t}, \psi_h)_\Omega + \int_\Omega \kappa \nabla(\alpha q_h + \tilde{p}_h) \cdot \nabla \psi_h = 0 \quad \forall \psi_h \in W_h, \tag{3.6.113}$$

$$\int_\Omega (\nabla \mathbf{u}_h(0))_{ij}(\nabla \mathbf{w}_h)_{ij} = \int_\Omega (\nabla \mathbf{u}_0)_{ij}(\nabla \mathbf{w}_h)_{ij} \quad \forall \mathbf{w}_h \in \mathbf{V}_h, \tag{3.6.114}$$

$$\int_{\partial\Omega} \mathbf{u}_h(0) \cdot \mathbf{n} = \int_{\partial\Omega} \mathbf{u}_0 \cdot \mathbf{n},$$

$$\int_\Omega q_h(0)\chi_h = \int_\Omega q_0 \chi_h, \quad \forall \chi_h \in W_h. \tag{3.6.115}$$

This numerical scheme is shown to be well-posed and numerically stable by way of an energy law. Also, the solution to this scheme satisfies the conservation laws. It is proved that if the weak solution is sufficiently regular, $\mathbf{u}_h$ converges to $\mathbf{u}$ in the $L^\infty(0,T;H^1(\Omega))$ norm and q_h converges to q in the $L^\infty(0,T;L^2(\Omega))$ norm with order h^2 and $\nabla p_h = \nabla(\tilde{p}_h + \alpha q_h)$ converges to $\nabla p = (\tilde{p} + \alpha q)$ in the $L^2(0,T;L^2(\Omega))$ norm with order h.

Finally, in order to solve the problem computationally, the system must be discretized in time as well as space. A first order backward Euler method is used to discretize in time. Choose a uniform partition $\{0 = t^0 < t^1 < \ldots < t^N = T\}$ of $[0,T]$ and denote the step size by Δt. The full discretization of the system is given by the following algorithm:

Algorithm 3.2 ([22], Fully discrete version of Algorithm 1)**.** *i: Compute* $q_h^0 \in W_h$

and $\mathbf{u}_h^0 \in \mathbf{V}_h$ *by*

$$\int_\Omega q_h^0 \chi_h = \int_\Omega q_0 \chi_h \quad \forall \chi_h \in W_h, \tag{3.6.116}$$

$$\int_\Omega (\nabla \mathbf{u}_h^0)_{ij} (\nabla \mathbf{w}_h)_{ij} = \int_\Omega (\nabla \mathbf{u}_0)_{ij} (\mathbf{w}_h)_{ij} \quad \forall \mathbf{w}_h \in \mathbf{V}_h, \tag{3.6.117}$$

$$\int_{\partial\Omega} \mathbf{u}_h^0 \cdot \mathbf{n} = \int_{\partial\Omega} \mathbf{u}_0 \cdot \mathbf{n}.$$

ii: For $n = 0, 1, 2, \ldots,$ *do the following two steps.*

Step 1: Solve for $(\mathbf{u}_h^{n+1}, \tilde{p}_h^{n+1}) \in \mathbf{V}_h \times M_h$ *such that*

$$\int_\Omega [\beta (\nabla \mathbf{u}_h^{n+1})_{ij} (\nabla \mathbf{v}_h)_{ij} - \tilde{p}_h^{n+1} \nabla \cdot \mathbf{v}_h] =< \mathbf{f}, \mathbf{v}_h >_{\partial\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.6.118}$$

$$\int_\Omega \nabla \cdot \mathbf{u}_h^{n+1} \varphi_h = \int_\Omega q_h^n \varphi_h \quad \forall \varphi_h \in M_h, \tag{3.6.119}$$

$$\int_\Omega \tilde{p}_h^{n+1} = C_{\tilde{p}}, \quad \int_{\partial\Omega} \mathbf{u}_h^{n+1} \cdot \mathbf{n} = C_q.$$

Step 2: Solve for $q_h^{n+1} \in W_h$ *such that*

$$\int_\Omega [d_t q_h^{n+1} \psi_h + \kappa \nabla (\alpha q_h^{n+1} + \tilde{p}_h^{n+1}) \cdot \nabla \psi_h] = 0 \quad \forall \psi_h \in W_h, \tag{3.6.120}$$

$$\int_\Omega q_h^{n+1} = C_q,$$

where $d_t q_h^{n+1} := (q_h^{n+1} - q_h^n)/\Delta t$ *is a finite difference operator.*

Notice that q_h^n is taken in (3.6.119) instead of q_h^{n+1}. This has the effect of decoupling the computation of q_h^{n+1}, which is computed by a convergent diffusion solver, from the computation of $\mathbf{u}_h^{n+1}$ and $\tilde{p}_h^{n+1}$, which are computed by the Taylor-Hood mixed finite element of Stokes flow. However, this decoupling requires that h and Δt be restricted in order to ensure the stability of the scheme. The mesh constraint, which is derived by way of an energy estimate, is $\Delta t = O(h^2)$. It is shown that if the weak solution is sufficiently regular, $\mathbf{u}_h^n$ converges to $\mathbf{u}$ in the $L^\infty(0,T;H^1(\Omega))$ norm and q_h^n converges to q in the $L^\infty(0,T;L^2(\Omega))$ norm with order $h^2 + \Delta t$ and $\nabla p_h^n = \nabla(\tilde{p}_h^n + \alpha q_h^n)$ converges to ∇p in the $L^2(0,T;L^2(\Omega))$ norm with order $h + \Delta t$. Two-dimensional numerical experiments were run to test the efficiency of the fully discrete method.

Chapter 4

Special geometries and regimes

In this chapter, well-posedness theory is developed for special geometries and regimes for the model system devised in Chapter 2. The system consists of laws of balance of mass and linear momentum with elastic and viscous properties, coupling Lagrangian and Eulerian dynamics, with many different sources of difficulties for which the question of existence may be beyond reach. In particular, in three dimensions, the nonlinear equation (2.5.69) is still an open problem. However, it is possible to obtain well-posedness results for special geometries and flows of interest in applications.

In section 1, one dimensional extensional flow is considered. The polymer and the solvent both move only in the z direction and both velocities depend only on z. Likewise, the volume fractions and the polymer deformation gradient are functions of z only. For this flow, it is possible to express all of the equations in the polymer reference configuration and to eliminate the pressure p. Furthermore, if both polymer and solvent are assumed to have no viscosity, the resulting model is a first order system of quasilinear symmetric hyperbolic equations. Periodic boundary conditions are taken and an energy estimate is derived. The domain is extended periodically over the whole real line $\mathbf{R}$. This problem generalizes to a first order quasilinear symmetric hyperbolic Cauchy problem over $\mathbf{R}^n$ for $n \geq 1$. Local-in-time existence of classical solutions is proved for this general system using Picard iterations.

In section 2, one dimensional shear flow is considered. Both components are assumed to exhibit Newtonian viscosity. The relevant velocities are the center of mass velocity $\mathbf{V} = \phi_1 \mathbf{v}_1 + \phi_2 \mathbf{v}_2$ and the relative velocity $\mathbf{U} = \mathbf{v}_1 - \mathbf{v}_2$. The components move only in the x direction and the velocities depend only on y. Volume fractions and polymer deformation gradient are also functions of y only, and all convection terms vanish. This results in the volume fractions being constant, thus removing them as unknowns of the problem. The unknowns of the problem are u, v and α, which represent relative and center of mass velocities and the deformation gradient, respectively. These unknowns are the solutions to a linear system of three parabolic equations. Periodic boundary conditions are taken over the symmetric interval $[-L, L]$ for $L > 0$ and existence of global-in-time weak solutions is proved using Galerkin approximations and standard compactness arguments.

In section 3, as a first approximation to the study of three-dimensional problems, it is assumed that no fluid is present in the domain and that deformations of the polymer from the reference domain are small, so that the small strain approximation is accurate. Acceleration terms are neglected. The result is a mixed parabolic-elliptic linear system of equations describing the dynamical behavior of an incompressible linearly elastic solid material with Newtonian viscosity and friction dissipation. The model equations under these assumptions are given. Initial and boundary conditions are imposed. Consistency requires that the initial displacement be divergence-free since the material is now incompressible. The boundary of the domain Ω is divided into two parts. A Dirichlet condition for the displacement $\mathbf{u}$ is imposed on one part of the boundary, and traction is prescribed on the other part. Existence, uniqueness, and regularity of global-in-time solutions is proved using Laplace transforms. Existence and uniqueness of global-in-time weak solutions is also proved under a different set of assumptions using Galerkin approximations. Finally, at the end of the chapter, a mixed finite element method for solving the problem numerically is described and results of numerical simulations are given.

4.1 One dimensional extensional flow

In this section it is assumed that at time $t = 0$, the dry polymer is separated from the fluid by a sharp boundary. They subsequently interact and the polymer deforms and expands due to absorbing fluid. These initial conditions are taken as the reference or undeformed domain. The two components of the gel thus have distinct, nonintersecting reference domains that deform differently as they mix and the gel expands. It is assumed that the mixture occupies the domain $\Omega = \{(x, y, z) : 0 \leq z \leq L\}$ for some $L > 0$ and that no deformations occur in the $x-$ and $y-$ directions. Solutions of the following form are sought:

$$\begin{aligned}
\mathbf{v_1} &= (0, 0, u(z, t)), \\
\mathbf{v_2} &= (0, 0, v(z, t)), \\
\phi_a &= \phi_a(z, t); a = 1, 2, \\
p &= p(x, y, z, t).
\end{aligned}$$

The deformation gradient under these conditions becomes

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g(z, t) \end{pmatrix},$$

where $g(z, t) = \frac{\partial M}{\partial Z}|_{Z=M^{-1}(z,t)}$. Note that $\det F = g$ and in the reference configuration, the balance of mass for the polymer is

$$\phi_1(z, t) \det F(z, t) = 1,$$

so $g = \frac{1}{\phi_1}$. In this geometry, the following four equations for the four unknowns $\{\phi_1, \phi_2, u, v\}$ are obtained:

$$(4.1.1)\ \partial_t \phi_1 + \partial_z(\phi_1 u) = 0,$$

$$(4.1.2)\ \partial_t \phi_2 + \partial_z(\phi_2 v) = 0,$$

$$(4.1.3)\ \phi_1\phi_2(u_t + uu_z - v_t - vv_z) = \phi_2\partial_z(G_1(\phi_1)) + \phi_1\partial_z(G_2(\phi_2)) - \beta(u - v),$$

$$(4.1.4)\ \phi_1 + \phi_2 = 1,$$

where $G_1(\phi_1)$ and $G_2(\phi_2)$ are functions of the indicated volume fractions which depend on the free energy densities of the components. The free energy densities taken here have the form (2.1.26), (2.1.27) with elastic energy (2.2.47). This is a free boundary problem.

4.1.1 Material coordinate formulation

The problem is transformed from a free boundary to a fixed boundary problem by changing to Lagrangian coordinates. The transformation described in this subsection was done in collaboration with Weinberger [1]. Let X denote the reference coordinates of the polymer, with $z = M(X,t)$ being the polymer deformation. Let Y denote the solvent reference coordinates, with $z = N(Y,t)$ being the solvent deformation. Initially, the components interact only at $X = 0 = Y$. The two components begin to move into each other, so there are two interfacial boundaries during the swelling process: a wet boundary where $\phi_1 = 0$, $v = 0$, $u = \frac{\partial z}{\partial t}$; and a dry boundary where $\phi_2 = 0$, $u = 0$, and $v = \frac{\partial z}{\partial t}$. The boundary positions thus move and are unknown in the spatial configuration.

In reference coordinates, the wet boundary is at $X = 0$, its position at time t is $M(0,t)$, and $N(M(0,t),t) = M(0,t)$ since the polymer does not move until the gel arrives. The dry boundary is at $Y = 0$, its position at time t is $N(0,t)$, and $M(N(0,t),t) = N(0,t)$. Also, the coordinates X and Y are not independent.

Any point (z,t) comes from a point X with $z = M(X,t)$ and also from a point Y with $z = N(Y,t)$. Thus $M(X,t) = N(Y,t)$ at time t. Let $Y = Q(X,t)$ so that

$$N(Q(X,t),t) = M(X,t).$$

On the wet and dry boundaries,

$$Q(0,t) = M(0,t),$$

$$Q(N(0,t),t) = 0.$$

An expression is now derived for $Q(X,t)$. Let $R(z,t)$ be the inverse of M and let $S(z,t)$ be the inverse of N, so that

$$M(R(z,t),t) = z,$$

$$N(S(z,t),t) = z.$$

Differentiating with respect to z:

$$\phi_1(z,t) = \frac{\partial R}{\partial z}(z,t),$$

$$\phi_2(z,t) = \frac{\partial S}{\partial z}(z,t).$$

Since $\phi_1 + \phi_2 = 1$, $\partial_z(R+S) = 1$, so

$$R + S = z + f(t).$$

However, $R = 0$ and $S = z$ for all t on the wet boundary, so $f(t) \equiv 0$, and

$$R(z,t) + S(z,t) = z.$$

Because $X = R(z,t)$ and $Y = S(z,t)$,

$$Y = Q(X,t) = z - X = M(X,t) - X.$$

Using capital letters to denote variables in the reference configuration and keeping in mind that $\Phi_1 = \Phi_1(X,t)$, $\Phi_2 = \Phi_2(Y,t)$, $M = M(X,t)$, and $N = N(Y,t)$, the system of equations becomes

$$\Phi_1 M_X = 1, \tag{4.1.5}$$

$$\Phi_2 N_Y = 1, \tag{4.1.6}$$

$$\Phi_1\Phi_2(M_{tt} - N_{tt}) = \Phi_1\Phi_2 G(\Phi_1)\frac{\partial \Phi_1}{\partial X} - \beta(M_t - N_t), \tag{4.1.7}$$

$$\Phi_1 + \Phi_2 = 1, \tag{4.1.8}$$

where G is some function of Φ_1. Taking $\Phi = \Phi_1$, this system can be simplified down to two equations in two unknowns $\{M(X,t), \Phi(X,t)\}$:

$$\Phi M_X = 1, \tag{4.1.9}$$

$$\frac{M_{tt}}{1-\Phi} + \frac{2}{(1-\Phi)^2}M_t\Phi_t + \frac{\Phi}{(1-\Phi)^3}M_t^2\Phi_X = G(\Phi)\Phi_X - \frac{\beta}{\Phi(1-\Phi)^2}M_t. \tag{4.1.10}$$

4.1.2 The Cauchy problem

Let $\Omega = [-1,1]$ be the polymer reference configuration. The variables M, Φ satisfy equations (4.1.9),(4.1.10) on the domain Ω. Existence, uniqueness, and regularity of solutions of the Cauchy problem is proved for suitable initial conditions. In order to determine what conditions should be imposed on the initial data to ensure boundedness of solutions, energy estimates are made and the domain Ω is extended periodically over all of $\mathbf{R}$.

Energy estimate

A general (possibly deformed) domain Ω is considered. Let ϕ_1, ϕ_2 denote the polymer and solvent volume fractions, respectively. Let $\mathbf{v}_1$, $\mathbf{v}_2$ be the velocities of the polymer and solvent, repectively. Let $\mathcal{T}_1$, $\mathcal{T}_2$ denote the polymer and solvent stress tensors, and let Ψ denote the Helmholtz free energy of the system. The stress tensors and free energy are given by

$$\mathcal{T}_1 = \frac{a}{N_x}\phi_1[\phi_1^{-\frac{2}{3}} - (\frac{1}{2} + \frac{N_x}{N_1}) + \chi N_x\phi_1\phi_2]I - p\phi_1 I + 2\mu\phi_1 FF^T,$$
$$\mathcal{T}_2 = -\phi_2(\frac{a}{N_2} - a\chi\phi_1\phi_2 + \lambda)I,$$
$$\Psi = a(\frac{1}{2}\chi\phi_1\phi_2 + \frac{1}{N_1}\phi_1\log\phi_1 + \frac{1}{N_2}\phi_2\log\phi_2)$$
$$+\phi_1[\frac{3a}{2N_x}(\phi_1^{-\frac{2}{3}} - 1 + \frac{1}{3}\log\phi_1) + \mu(\phi_1^{-2} - 1)],$$

where $a = \frac{K_B T}{V_m}$. The kinetic plus free energy satisfy the inequality

$$\frac{d}{dt}\int_\Omega \frac{1}{2}(\phi_1|\mathbf{v}_1|^2 + \phi_2|\mathbf{v}_2|^2 + 2\Psi)d\mathbf{x} \leq \int_{\partial\Omega}(\mathbf{t}_1\cdot\mathbf{v}_1 + \mathbf{t}_2\cdot\mathbf{v}_2)dS, \tag{4.1.11}$$

where $\mathbf{t}_1 = \mathcal{T}_1\nu$, $\mathbf{t}_2 = \mathcal{T}_2\nu$, and ν is a unit outward vector normal to the boundary of Ω. [10] If Ω is chosen to be the domain $\Omega = \{(x, y, z) : |z| \leq 1\}$, then $\nu = (0, 0, \pm 1)$. Define the functions

$$G_1(\phi_1, \phi_2) = \frac{a}{N_x}(\phi_1^{\frac{1}{3}} - \frac{1}{2}\phi_1) - \frac{a}{N_1}\phi_1 + a\chi\phi_1^2\phi_2 + 2\mu\phi_1^{-1},$$

$$G_2(\phi_1, \phi_2) = -\frac{a}{N_2}\phi_2 + a\chi\phi_1\phi_2^2.$$

Then the traction vectors are given by

$$\mathbf{t}_1 = \pm(0, 0, G_1(\phi_1, \phi_2) - p\phi_1),\ \ \mathbf{t}_2 = \pm(0, 0, G_2(\phi_1, \phi_2) - p\phi_2).$$

Let $\mathbf{v}_1 = (0, 0, u)$, $bv_2 = (0, 0, v)$. Then the energy inequality becomes

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{2}(\phi_1 u^2 + \phi_2 v^2 + 2\Psi(\phi_1, \phi_2))dz \quad &\leq \quad [G_1(\phi_1, \phi_2)u + G_2(\phi_1, \phi_2)v \\ &\qquad -p(\phi_1 u + \phi_2 v)]_{z=-1}^{z=1}.\end{aligned}$$

The expression $\phi_1 u + \phi_2 v$ is the center of mass velocity of the gel, which cannot depend on the space variable due to the incompressibility condition in one dimension. Therefore the last term on the right hand side of the inequality must be zero.

The variables are transformed to the reference configuration. Let $\phi(X) = \phi_1(M(X))$ denote polymer volume fraction in material coordinates. Then, $u = M_t$, $v = N_t$, and $N_t = -\frac{M_t\phi}{1-\phi}$. Letting $x = M(X)$ denote position in material coordinates, the energy inequality becomes

$$\frac{d}{dt}\int_\Omega \frac{1}{2}(\frac{\phi M_t^2}{1-\phi} + 2\Psi(\phi, 1-\phi))dx \leq M_t H(\phi)|_{x=-1}^{x=1},$$

where $H(\phi) = G_1(\phi, 1-\phi) - G_2(\phi, 1-\phi)\frac{\phi}{1-\phi}$. For periodic boundary conditions, the right hand side of the inequality vanishes. Therefore, for time $t > 0$,

(4.1.12)

$$\int_\Omega \frac{1}{2}[\frac{\phi(x,t)M_t^2(x,t)}{1-\phi(x,t)} + 2\Psi(\phi(x,t), 1-\phi(x,t))]dx \leq \int_\Omega C(\phi(x,0), M_t(x,0))dx.$$

From this estimate, it is clear that if there is a constant $C_1 > 0$ such that $|M_t(x,0)| \leq C_1$ for all $x \in \Omega$ and there are $0 < \delta_1, \delta_2 \ll 1$ such that $\delta_1 \leq \phi(x,0) \leq 1 - \delta_2$ for all $x \in \Omega$, then M_t is finitely bounded a.e. in Ω for all time and there are positive constants $\delta_3, \delta_4 \ll 1$ such that $\delta_3 \leq \phi(x,t) \leq 1 - \delta_4$ a.e. in Ω for all time.

Setup of the Cauchy problem

The system of equations (4.1.9),(4.1.10) is a nonlinear hyperbolic system. It is very difficult to compute its characteristics. Also, this equation system is not physically realistic until viscous dissipation is added to at least one of the components. Thus, it is not advantageous to impose boundary conditions. Instead, the domain is extended periodically over all of $\mathbf{R}$ and the Cauchy problem is considered. The system of equations can be expressed as a system of quasilinear hyperbolic equations. Let $u = M_x$, $v = M_t$. Then, since $\phi M_x = 1$, some algebraic manipulations give

(4.1.13) $$u_t - v_x = 0$$

(4.1.14) $$(1-u^{-1})^2 v_t + B(u,v)u_x + A(u,v)v_x + h(u,v) = 0,$$

where

$$\begin{aligned} B(u,v) &= (1-u^{-1})^3[\frac{a}{N_x}(\frac{1}{3}u^{-\frac{4}{3}}-\frac{1}{2}u^{-2})-2\mu-\frac{a}{N_1}u^{-2}+a\chi u^{-3}] \\ &\quad -\frac{a}{N_2}u^{-3}(1-u^{-1})^2-u^{-3}v^2, \\ A(u,v) &= -2u^{-2}v(1-u^{-1}), \\ h(u,v) &= \beta uv(1-u^{-1}). \end{aligned}$$

If equation (4.1.13) is multiplied by $-B(u,v)$, then the system can be expressed as a symmetric hyperbolic system with initial conditions:

$$(4.1.15)\quad \begin{bmatrix} -B & 0 \\ 0 & (1-u^{-1})^2 \end{bmatrix}\begin{bmatrix} u_t \\ v_t \end{bmatrix}+\begin{bmatrix} 0 & B \\ B & A \end{bmatrix}\begin{bmatrix} u_x \\ v_x \end{bmatrix}+\begin{bmatrix} 0 \\ h \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$u(x,0)=u_0(x),$$
$$v(x,0)=v_0(x).$$

Let $A^0=\begin{bmatrix} -B & 0 \\ 0 & (1-u^{-1})^2 \end{bmatrix}$, $A^1=\begin{bmatrix} 0 & B \\ B & A \end{bmatrix}$, and $F=\begin{bmatrix} 0 \\ h \end{bmatrix}$. Suppose that initial conditions imposed over the domain Ω are extended periodically over all of space and that the initial conditions are chosen so that there are constants $C_1, C_2, \delta > 0$ such that $|v(x,0)| \le C_1$, $1+\delta \le u(x,0) \le C_2$ for all $x \in \mathbf{R}$. Then A^0, A^1 are symmetric, smooth, bounded matrix functions with all derivatives bounded. F is smooth in all of its arguments and for all multiindexes α, $|\partial^{\alpha}_{u,v}F(u,v)| \le C(K)\ \forall (u,v) \in \{(u,v) : |(u,v)| \le K\}$, where $K > 0$ and $C(K) > 0$ is a constant that depends on K. Moreover, $F(0,0)=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. A^0 is assumed to be uniformly positive definite.

4.1.3 Existence of solutions for Cauchy problem

Existence of solutions is proved for a more general quasilinear symmetric hyperbolic Cauchy problem in $\mathbf{R}^n$ for $n \ge 1$. Let $x \in \mathbf{R}^n$, $t \ge 0$, $u(x,t) \in \mathbf{R}^N$ for $N \ge 1$,

$A^\gamma(x,t,u) \in \mathbf{R}^{N\times N}$ for $\gamma = 0,\dots,n$, and $F(x,t,u) \in \mathbf{R}^N$. $A^\gamma(x,t,u)$ are assumed to be smooth, symmetric, bounded matrix functions with all derivatives bounded for $\gamma = 0,\dots,n$. $F(x,t,u)$ is assumed to be smooth in all of its arguments and for all multiindexes β, $|\partial^\beta_{x,t,u}F(x,t,u)| \le C(K)\ \forall (x,t,u) \in \mathbf{R}^n \times \mathbf{R} \times \{|u| \le K\}$ for some constant $C(K) > 0$ depending on $K > 0$. Also, $F(x,t,0) = 0$. $A^0(x,t,u)$ is assumed to be uniformly positive definite for all x,t,u. Let $s > \frac{n}{2}+1$. The problem under consideration is

$$
\begin{aligned}
&A^0(x,t,u)u_t + \sum_{\gamma=1}^{n} A^\gamma(x,t,u)\partial_\gamma u + F(x,t,u) = 0, \\
&u(x,0) = u_0(x),
\end{aligned}
\tag{4.1.16}
$$

where $\partial_\gamma u = \frac{\partial u}{\partial x_\gamma}$.

Theorem 4.1. *Let all assumptions on A^γ and F made above apply. Assume $u_0 \in H^s(\mathbf{R}^n)$. Then $\exists! u \in C([0,T]; H^s(\mathbf{R}^n))$ solving (4.1.16) for some $T > 0$.*

The solution is found as a limit of Picard iterations. Let $u^{(1)} = u_0$ and for $\nu > 1$, let $u^{(\nu)}$ solve

$$
\begin{aligned}
&A^0(x,t,u^{(\nu-1)})u_t^{(\nu)} + \sum_{\gamma=1}^{n} A^\gamma(x,t,u^{(\nu-1)})\partial_\gamma u^{(\nu)} + F(x,t,u^{(\nu-1)}) = 0, \\
&u^{(\nu)}(x,0) = u_0(x).
\end{aligned}
\tag{4.1.17}
$$

By Friedrich's theorem, $\exists! u^{(\nu)} \in C([0,T]; H^s(\mathbf{R}^n))$ for all $T > 0$ solving (4.1.17) for each $\nu > 1$ [53].

Let $(w,v) = \int_{\mathbf{R}^n} wv\,dx$ denote the $L^2(\mathbf{R}^n)$ inner product and, since $A^0 = A^0(x,t,u^{(\nu-1)})$ is uniformly positive definite, define the equivalent norm $||u||_E = (u, A^0 u)$. For $\gamma = 1,\dots,n$, let $A^\gamma = A^\gamma(x,t,u^{(\nu-1)})$ and define $Gu \equiv \sum A^\gamma \partial_\gamma u$ and $Lu \equiv A^0 u_t + Gu$. Using the assumptions above, the fact that $||u||_{L^\infty} \le C||u||_{H^s}$ since $s > \frac{n}{2}$, and Gronwall's inequality,

$$
||u^{(\nu)}(t)||_{L^2} \le C \exp^{\int_0^t C(||u^{(\nu-1)}(\tau)||_{H^s})d\tau}[||u^{(\nu)}(0)||_{L^2} + \int_0^t ||Lu^{(\nu)}(\tau)||_{L^2} d\tau]
$$

for a.e. $t > 0$, where $C(\cdot)$ is a positive function. Estimating derivatives and using the Gagliardo-Nirenberg inequality,

$$||u^{(\nu)}(t)||_{H^s} \leq C_0 \exp^{\int_0^T C(||u^{(\nu-1)}(\tau)||_{H^s})d\tau}[||u^{(\nu)}(0)||_{H^s} +$$
$$\int_0^T C(||u^{(\nu-1)}(\tau)||_{H^s})||Lu^{(\nu)}(\tau)||_{H^s} d\tau]$$
$$\{1 + C_0 T(\max_{0\leq\tau\leq T} C(||u^{(\nu-1)}(\tau)||_{H^s})) \exp^{\int_0^T C(||u^{(\nu-1)}(\tau)||_{H^s})d\tau}$$
$$\exp(C_0 T(\max_{0\leq\tau\leq T} C(||u^{(\nu-1)}(\tau)||_{H^s})) \exp^{\int_0^T C(||u^{(\nu-1)}(\tau)||_{H^s})d\tau})\}.$$

Choose $R \in \mathbf{R}$ such that $R > 2C_0||u_0||_{H^s}$. Using induction on ν and Schauder's lemma, it is easy to see that for $T > 0$ small enough, $\forall \nu$ and $\forall t \in [0, T]$, $||u^{(\nu)}(t)||_{H^s} \leq R$. For $\nu \geq 2$, let $w^{(\nu)} = u^{(\nu+1)} - u^{(\nu)}$. Using Schauder's lemma and conventional estimates,

$$||w^{(\nu)}(t)||_{L^2} \leq C_0 C(R) \exp^{C(R)t} \int_0^t ||w^{(\nu-1)}(\tau)||_{L^2} d\tau. \tag{4.1.18}$$

Define $M_1 \equiv \sup_{0\leq t\leq T} ||w^{(1)}(t)||_{L^2}$ and $M_2 \equiv C_0 C(R) \exp^{C(R)T}$. It is clear by induction on ν that

$$||w^{(\nu)}(t)||_{L^2} \leq \frac{M_1 M_2^{\nu-1} t^{\nu-1}}{(\nu-1)!}$$

$\forall t \in [0, T]$ and that

$$\sum_{\nu=1}^{\infty} \sup_{0\leq t\leq T} ||w^{(\nu)}(t)||_{L^2} \leq M_1 \exp^{M_2 T} < \infty.$$

Let ν, μ be positive integers. Then,

$$||u^{(\nu+\mu)}(t) - u^{(\nu)}(t)||_{L^2} \leq \sum_{k=0}^{\mu-1} ||w^{(\nu+\mu-k-1)}(t)||_{L^2} \to 0$$

as $\nu, \mu \to \infty$ for all $t \in [0, T]$. Therefore, $u^{(\nu)}(t)$ is Cauchy in $L^2(\mathbf{R}^n)$ for all $t \in [0, T]$. L^2 is a Hilbert space, so $\exists u(t) \in L^2$ such that $u^{(\nu)}(t) \to u(t)$ in L^2 for each $t \in [0, T]$. Since all bounds are uniform in t, $u \in L^\infty(0, T; L^2)$. Using an interpolation inequality, it is clear that $\forall s' < s$, $u \in L^\infty(0, T; H^{s'})$.

Now it is shown that $u \in L^\infty(0,T;H^s(\mathbf{R}^n))$. $H^s \subseteq H^{s'}$ with H^s dense in $H^{s'}$, so $H^{-s'} \subseteq H^{-s}$ with $H^{-s'}$ dense in H^{-s}. Let $\phi \in H^{-s}$, $\psi \in H^{-s'}$ chosen so that $||\phi - \psi||_{H^{-s'}} \leq \frac{1}{2}$. Let $<\phi, v>$ denote the pairing of $\phi \in H^{-s}$ with $v \in H^s$. Consider $\phi \in H^{-s}$ with $||\phi||_{H^{-s}} = 1$. Then

$$\begin{aligned} ||u(t)||_{H^s} &\leq <\phi, u> +1 \\ &\leq <\phi - \psi, u> + <\psi, u - u^{(\nu)}> + <\psi, u^{(\nu)}> +1 \\ &\leq \frac{1}{2}||u(t)||_{H^s} + \frac{1}{2} + K + 1 \end{aligned}$$

for $K > 0$ independent of t and for large enough ν since $u^{(\nu)} \to u$ in $H^{s'}$. Thus, $||u(t)||_{H^s} \leq 3 + 2K \ \forall t$, so $u(t)$ is bounded uniformly in H^s. Therefore, $u \in L^\infty(0,T;H^s(\mathbf{R}^n))$.

Consider $0 \leq t_1 \leq t \leq T$. Let u denote $u(t)$, u_1 denote $u(t_1)$, $w = u - u_1$, $Gw = \sum A^\gamma(x,t,u)\partial_\gamma w$, and $Lw = A^0(x,t,u)w_t + Gw$. By the usual energy estimates and Gronwall's inequality,

$$||w(t)||_{L^2} \leq C_0 \exp^{\int_{t_1}^t C(||u(\tau)||_{H^s})d\tau} \int_{t_1}^t ||Lw(\tau)||_{L^2} d\tau \to 0$$

as $t \to t_1$. Therefore, $u \in C([0,T];L^2)$. Now fix $s' < s$. Then, $\exists \gamma, \theta > 0$ such that

$$||w(t)||_{H^{s'}} \leq C||w(t)||_{L^2}^\theta ||w(t)||_{H^s}^\gamma \to 0$$

as $t \to t_1$. Therefore, $u \in C([0,T];H^{s'})$.

$\forall \epsilon > 0$, let $J_\epsilon : H^s \to \bigcap_{k<\infty} H^k$ be a Friedrichs mollifier. Using the usual estimates and estimates on commutators found in Taylor [62, p.3], it is found that $\forall \epsilon > 0$, $J_\epsilon u \in C([0,T];H^s)$. Let $0 \leq t_1 \leq t \leq T$. Then,

$$\begin{aligned} ||u(t) - u(t_1)||_{H^s} &\leq ||u(t) - J_\epsilon u(t)||_{H^s} + ||J_\epsilon u(t) - J_\epsilon u(t_1)||_{H^s} \\ &\quad + ||J_\epsilon u(t_1) - u(t_1)||_{H^s} \to 0 \end{aligned}$$

as $t \to t_1$ since $J_\epsilon u(t) \to u(t)$ in H^s as $\epsilon \to 0$ independent of $t \in [0,T]$ and $J_\epsilon u \in C([0,T];H^s)\ \forall \epsilon > 0$. Therefore, $u \in C([0,T];H^s)$.

Now let $u_1, u_2 \in C([0,T]; H^s)$ be two solutions of the quasilinear system. Let $w = u_1 - u_2$. Using conventional estimates, Schauder's lemma, and Gronwall's inequality, it is clear that $||w(t)||_{L^2} = 0$ for all $t \in [0,T]$. Therefore, $u_1(t) = u_2(t)$ in L^2 for a.e. $t \in [0,T]$, so the solution is unique in $C([0,T]; H^s)$. □

By the Sobolev embedding theorem, $u \in C([0,T]; C_B^1(\mathbf{R}^n))$, where $C_B^1(\mathbf{R}^n)$ is the set of all functions that are bounded and continuous everywhere in $\mathbf{R}^n$, whose first derivatives exist, and whose first derivatives are bounded and continuous everywhere [3].

4.2 One-dimensional shear flow

In this section Newtonian viscosity is added to both components and one dimensional shear flow is analyzed. Existence of weak solutions is proved for periodic boundary conditions. The system is homogenized, meaning that the polymer and solvent are assumed to be so well-mixed that individual component velocities cannot be measured. The physically relevant velocities in this situation are the barycentric velocity $\mathbf{V} = \phi_1 \mathbf{v}_1 + \phi_2 \mathbf{v}_2$ and the relative or diffusive velocity $\mathbf{U} = \mathbf{v}_1 - \mathbf{v}_2$. The basic equations are

$$\frac{\partial \phi_1}{\partial t} + \nabla \cdot (\phi_1 \mathbf{V} + \phi_1 \phi_2 \mathbf{U}) = 0, \tag{4.2.19}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{4.2.20}$$

$$\begin{aligned} \mathbf{V}_t + \mathbf{V} \cdot \nabla \mathbf{V} + \mathbf{U} \nabla \cdot (\phi_1 \phi_2 \mathbf{U}) + \phi_1 \phi_2 \mathbf{U} \cdot \nabla \mathbf{U} &= \nabla \cdot (\phi_1 \frac{\partial \psi_1}{\partial F} F^T) \\ &- \nabla (p + p_1(\phi_1)) + \nabla \cdot [2\eta_1 \mathbf{D}(\mathbf{V} + \phi_2 \mathbf{U}) + 2\eta_2 \mathbf{D}(\mathbf{V} - \phi_1 \mathbf{U})] \\ &+ \nabla [\mu_1 \nabla \cdot (\mathbf{V} + \phi_2 \mathbf{U}) + \mu_2 \nabla \cdot (\mathbf{V} - \phi_1 \mathbf{U})], \end{aligned} \tag{4.2.21}$$

$$(4.2.22)\quad \phi_1\phi_2[\mathbf{U}_t + \mathbf{U}\cdot\nabla(\phi_2\mathbf{U}) - \phi_1\mathbf{U}\cdot\nabla\mathbf{U} + \mathbf{V}\cdot\nabla\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{V}] =$$

$$\phi_2\nabla\cdot[\phi_1\frac{\partial\psi_1}{\partial F}F^T + 2\eta_1\mathbf{D}(\mathbf{V}+\phi_2\mathbf{U})] - \phi_1\nabla\cdot[2\eta_2\mathbf{D}(\mathbf{V}-\phi_1\mathbf{U})]$$

$$+\phi_2\nabla[\mu_1\nabla\cdot(\mathbf{V}+\phi_2\mathbf{U})] - \nabla p_2(\phi_1) - \beta\mathbf{U} - \phi_1\nabla[\mu_2\nabla\cdot(\mathbf{V}-\phi_1\mathbf{U})],$$

$$(4.2.23)\quad F_t + (\mathbf{V}+\phi_2\mathbf{U})\cdot\nabla F = \nabla(\mathbf{V}+\phi_2\mathbf{U})F,$$

where $p_1(\phi_1)$, $p_2(\phi_1)$ are osmotic pressure terms that depend on the free energies of the components. As before, $\phi_2 = 1 - \phi_1$. For one-dimensional shear flow, the volume fractions, pressure, and velocities take the form

$$\phi_1 = \phi_1(y,t),\ \phi_2 = \phi_2(y,t),\ p = p(x,y,z,t),$$
$$\mathbf{U} = (u(y,t),0,0),\ \mathbf{V} = (v(y,t),0,0).$$

The deformation gradient takes the form

$$F = \begin{pmatrix} 1 & \alpha(y,t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As a consequence of shear flow, the equations become

$$(4.2.24)\qquad \frac{\partial\phi_1}{\partial t} = 0,$$

$$(4.2.25)\qquad v_t = \frac{\partial}{\partial y}[\eta_1\frac{\partial}{\partial y}(v+\phi_2 u) + \eta_2\frac{\partial}{\partial y}(v-\phi_1 u) + \phi_1\frac{\partial\psi_1}{\partial F_{12}}] - p_x,$$

$$(4.2.26)\qquad 0 = \frac{\partial}{\partial y}[\phi_1\frac{\partial\psi_1}{\partial F_{22}} - p_1(\phi_1) - p],$$

$$(4.2.27)\qquad 0 = \frac{\partial}{\partial y}[\phi_1\frac{\partial\psi_1}{\partial F_{32}}] - p_z,$$

$$(4.2.28)\qquad \phi_1\phi_2 u_t = \phi_2\frac{\partial}{\partial y}[\phi_1\frac{\partial\psi_1}{\partial F_{12}} + \eta_1\frac{\partial}{\partial y}(v+\phi_2 u)]$$
$$-\phi_1\frac{\partial}{\partial y}[\eta_2\frac{\partial}{\partial y}(v-\phi_1 u)] - \beta u,$$

$$0 = \phi_2 \frac{\partial}{\partial y}[\phi_1 \frac{\partial \psi_1}{\partial F_{22}}] + \frac{\partial p_2}{\partial y}, \tag{4.2.29}$$

$$0 = \phi_2 \frac{\partial}{\partial y}[\phi_1 \frac{\partial \psi_1}{\partial F_{32}}], \tag{4.2.30}$$

$$\alpha_t = \frac{\partial}{\partial y}(v + \phi_2 u). \tag{4.2.31}$$

Equation (4.2.24) implies that $\phi_1 = \phi_1(y)$, $\phi_2 = \phi_2(y)$. Thus the volume fractions can be set by initial conditions. It is assumed henceforth that all variables are periodic in the y direction. Thus the domain is taken to be the symmetric interval $[-L, L]$ for some $L > 0$. Equations (4.2.26), (4.2.27), (4.2.29), (4.2.30), give an equation for p and compatibility conditions for the free energies ψ_1, ψ_2. To ensure consistency, it is assumed that $p_x = f(t)$ for some function of t, and that $\frac{\partial}{\partial y}[\phi_1 \frac{\partial \psi_1}{\partial F_{32}}] \equiv 0$. f can be set, for example, in the case of Poiseuille flow, where pressure gradients are applied. Assume that $\psi_1 = H(\phi_1, \phi_2) + W(F)$ for some known functions H and W. The equations become

$$v_t = \frac{\partial}{\partial y}[\phi_1 \frac{\partial W}{\partial F_{12}} + \eta_1 \frac{\partial}{\partial y}(v + \phi_2 u) + \eta_2 \frac{\partial}{\partial y}(v - \phi_1 u)] - f(t), \tag{4.2.32}$$

$$\begin{aligned} \phi_1 \phi_2 u_t = {} & \phi_2 \frac{\partial}{\partial y}[\phi_1 \frac{\partial W}{\partial F_{12}} + \eta_1 \frac{\partial}{\partial y}(v + \phi_2 u)] \\ & -\phi_1 \frac{\partial}{\partial y}[\eta_2 \frac{\partial}{\partial y}(v - \phi_1 u)] - \beta u, \end{aligned} \tag{4.2.33}$$

$$\alpha_t = \frac{\partial}{\partial y}(v + \phi_2 u). \tag{4.2.34}$$

Multiplying (4.2.33) by u and (4.2.32) by v, using (4.2.34), integrating over $[-L, L]$, and assuming periodic boundary conditions, the following energy law is obtained:

$$\begin{aligned} & \frac{d}{dt} \int_{-L}^{L} [\frac{1}{2}(v^2 + \phi_1 \phi_2 u^2) + \phi_1 W] = \\ & - \int_{-L}^{L} [\eta_1 (\frac{\partial}{\partial y}(v + \phi_2 u))^2 + \eta_2 (\frac{\partial}{\partial y}(v - \phi_1 u))^2 + \beta u^2] - \int_{-L}^{L} f(t) v. \end{aligned} \tag{4.2.35}$$

Recall that $B = FF^T$ in (2.2.44). For simplicity assume $W(F) = \mu(\mathrm{tr} B - 3)$, where $\mu > 0$ is related to the polymer shear modulus. With this choice of elastic energy and the fact that ϕ_1 is independent of time, it can be shown using the constraint $\frac{\partial}{\partial y}[\phi_1 \frac{\partial \psi_1}{\partial F_{32}}] \equiv 0$ that ϕ_1 must in fact be constant. Therefore, choose $\phi_1 \in (0,1)$. $\frac{\partial W}{\partial F_{12}}$ is linear in α and the system of equations (4.2.32)-(4.2.34) is linear. Let W' denote $\frac{\partial W}{\partial F_{12}}$. In the purely inviscid case $(\eta_1, \eta_2 = 0)$, it can be shown that for this choice of the elastic free energy $W(F)$, the system is symmetric hyperbolic. This guarantees local in time existence for classical solutions for appropriately chosen initial conditions in both the viscous and inviscid cases [32].

Here is the main result of this section.

Theorem 4.2. *Let $\phi_1, \phi_2 \in (0,1)$ and let u_0, v_0, α_0 be initial conditions for the system of equations with periodic boundary conditions. Assume $u_0, v_0, \alpha_0 \in L^2(-L, L)$ are periodic. Assume μ, β, η_1, η_2 are positive constants; and W' is a linear function of α. Assume $f \in L^2(0,T)$ for some $T > 0$. Then $\exists u, v \in L^\infty(0,T; L^2(-L,L)) \cap L^2(0,T; H^1(-L,L))$ and $\exists \alpha \in L^\infty(0,T; L^2(-L,L))$ such that u, v, α are weak solutions of (4.2.32), (4.2.33), (4.2.34) that exist up to time T. Moreover, this weak solution is unique.*

Proof: The proof uses Galerkin approximations and follows the ideas of Liu and Walkington [46] and of Temam [63]. Since the boundary conditions are periodic, the finite dimensional Galerkin function space is taken to be $V_N = span\{1, \sin(\frac{n\pi}{L}y), \cos(\frac{n\pi}{L}y)\}_{n=1}^N$. Let $u^N(t,\cdot), v^N(t,\cdot), \alpha^N(t,\cdot) \in V_N$ satisfy $\forall w^N \in V_N$ the equations

$$(4.2.36)\int_{-L}^{L} v_t^N w^N = -\int_{-L}^{L} [\eta_1(v^N + \phi_2 u^N)_y + \phi_1 W'(\alpha^N) + \eta_2(v^N - \phi_1 u^N)_y] w_y^N - \int_{-L}^{L} f(t) w^N,$$

$$(4.2.37)\int_{-L}^{L} \phi_1\phi_2 u_t^N w^N = -\int_{-L}^{L} [\phi_1 W'(\alpha^N) + \eta_1(v^N + \phi_2 u^N)_y](\phi_2 w^N)_y + \int_{-L}^{L} \eta_2(v^N - \phi_1 u^N)_y(\phi_1 w^N)_y - \int_{-L}^{L} \beta u^N w^N,$$

$$(4.2.38)\qquad \int_{-L}^{L}\phi_1\alpha_t^N w^N = \int_{-L}^{L}\phi_1(v^N+\phi_2u^N)_y w^N,$$
$$\int_{-L}^{L} v^N(y,0)w^N = \int_{-L}^{L} v_0 w^N,$$
$$\int_{-L}^{L} u^N(0,y)w^N = \int_{-L}^{L} u_0 w^N,$$
$$\int_{-L}^{L} \alpha^N(0,y)w^N = \int_{-L}^{L} \alpha_0 w^N.$$

Since $u^N(t,\cdot), v^N(t,\cdot), \alpha^N(t,\cdot) \in V_N$, they can be written as linear combinations of the basis functions with coefficients depending on t. These expressions are substituted into the above equations and a linear system of ODEs in time with a full set of initial conditions is obtained. Since the spanning functions of V_N form a linearly independent basis which is orthogonal in the $L^2_{per}(-L,L)$ inner product, the stiffness matrix on the left hand side is invertible. By standard ODE theory, there exists a unique solution $u^N(t), v^N(t), \alpha^N(t) \in V_N$ up to time T for each $N \geq 1$.

This approximation must be shown to converge to a weak solution of the system. This is achieved by deriving an energy estimate. Take $w^N = v^N$ in (4.2.36), $w^N = u^N$ in (4.2.37), $w^N = W'(\alpha^N)$ in (4.2.38). u^N, v^N, α^N satisfy, after integrating over time from 0 to T and using Poincaré's and Hölder's inequalities, the energy estimate

$$(4.2.39)\qquad \int_{-L}^{L}[\frac{1}{2}((v^N)^2(T)+\phi_1\phi_2(u^N)^2(T))+\phi_1 W(\alpha^N(T))]$$
$$+\int_0^T\int_{-L}^{L}[\eta_1(v^N+\phi_2u^N)_y^2+\eta_2(v^N-\phi_1u^N)_y^2+\beta(u^N)^2] \leq$$
$$|\Omega|||f||_{L^2(0,T)}||(v^N)_\Omega||_{L^2(0,T)}+\int_{-L}^{L}[\frac{1}{2}((v_0)^2+(u_0)^2)+W(\alpha_0)],$$

where $\Omega = [-L,L]$ and $(v^N)_\Omega$ is the average of v^N over Ω. Since v^N is periodic in Ω, $(v^N)_\Omega$ is independent of N. By (4.2.39), u^N, v^N are bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and α^N is bounded in $L^\infty(0,T;L^2(\Omega))$. Thus,

$\exists u, v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and $\exists \alpha \in L^\infty(0,T;L^2(\Omega))$ such that $u^N \overset{*}{\rightharpoonup} u$, $v^N \overset{*}{\rightharpoonup} v$, $\alpha^N \overset{*}{\rightharpoonup} \alpha$ in $L^\infty(0,T;L^2(\Omega))$ and $u^N \rightharpoonup u$, $v^N \rightharpoonup v$ in $L^2(0,T;H^1(\Omega))$ after passing to a subsequence if necessary.

Let $w \in C_0^\infty([0,T);C_{per}^\infty(-L,L))$. By denseness, $\exists \{w^N\}_{N=1}^\infty$ with $w^N \in C^1(0,T;V_N)$ such that $w^N \longrightarrow w$ in $C^1(0,T;W^{1,q}(-L,L))$ for $q \geq 1$. It is possible to select $w^N(T) \equiv 0$. For all $w \in C_0^\infty([0,T);C_{per}^\infty(-L,L))$, the Galerkin approximations satisfy

$$
\begin{aligned}
-\int_0^T \int_{-L}^L v^N w_t^N &= \int_{-L}^L v^N(0) w^N(0) - \int_0^T \int_{-L}^L \eta_1 (v^N + \phi_2 u^N)_y w_y^N \\
&- \int_0^T \int_{-L}^L \phi_1 W'(\alpha^N) w_y^N - \int_0^T \int_{-L}^L [\eta_2 (v^N - \phi_1 u^N)_y w_y^N + f(t) w^N], \\
-\int_0^T \int_{-L}^L \phi_1 \phi_2 u^N w_t^N &= \int_{-L}^L \phi_1 \phi_2 u^N(0) w^N(0) - \int_0^T \int_{-L}^L \beta u^N w^N \\
&- \int_0^T \int_{-L}^L [\eta_1 (v^N + \phi_2 u^N)_y + \phi_1 W'(\alpha^N)](\phi_2 w^N)_y \\
&- \int_0^T \int_{-L}^L \eta_2 (v^N - \phi_1 u^N)_y (\phi_1 w^N)_y, \\
-\int_0^T \int_{-L}^L \phi_1 \alpha^N w_t^N &= \int_{-L}^L \phi_1 \alpha^N(0) w^N(0) + \int_0^T \int_{-L}^L \phi_1 (v^N + \phi_2 u^N)_y w^N.
\end{aligned}
$$

Since $u^N(0), v^N(0), \alpha^N(0)$ converge in $L^2(-L,L)$ to u_0, v_0, α_0, respectively, it can be shown using compactness theorems that each term in the above equations converges in the limit. The limit system satisfies (4.2.32)-(4.2.34) in the sense of distributions for a.e. $t \in [0,T]$ and is thus a weak solution. Uniqueness follows from the linearity of the system and the energy law (4.2.35). □

4.3 Elasticity

In this section, the behavior of a linearly elastic solid with Newtonian viscosity in a two or three dimensional domain Ω ($\Omega \subset \mathbf{R}^d$ for $d = 2, 3$) with no fluid present is studied. Inertia is neglected, $\phi_1 \equiv 1$, and $\phi_2 \equiv 0$. Let $\tilde{\mathbf{v}}_1$ denote solid velocity.

The equations are

$$\nabla \cdot \tilde{\mathbf{v}}_1 = 0, \qquad (4.3.40)$$
$$\nabla \cdot [\frac{\partial W}{\partial F} F^T + 2\eta_1 \mathbf{D}(\tilde{\mathbf{v}}_1) - (\pi_1 + p)I] - \beta \tilde{\mathbf{v}}_1 = 0.$$

The polymer osmotic pressure π_1 is a constant since $\phi_1 \equiv 1$. Let $\tilde{\mathbf{u}}$ denote the displacement of the solid. The solid velocity $\tilde{\mathbf{v}}_1$ then satisfies $\tilde{\mathbf{v}}_1 = \tilde{\mathbf{u}}_t$. Take as elastic energy the function $W(F) = a(I_1^{\alpha_1} - \mathrm{tr}^{\alpha_1} I) + bI_2^{\alpha_2} + c(I_3^{\gamma_1} - 1) + d(I_3^{-\gamma_2} - 1)$, where $a, b, c, d \geq 0$, $\alpha_2, \gamma_1, \gamma_2 > 0$, $\alpha_1 \geq 1$, and I_1, I_2, I_3 are defined by (2.2.45). The displacement is defined by the equation $F = I + \nabla \tilde{\mathbf{u}}$. Linearize the system about the deformation gradient $F = I$. As shown in [14],

$$B = FF^T \approx I + 2\mathbf{D}(\tilde{\mathbf{u}}).$$

It can be shown that this linearization gives the following result:

$$\frac{\partial W}{\partial F} F^T \approx J_1 I + J_2 \nabla \cdot \tilde{\mathbf{u}} + 4\mu \mathbf{D}(\tilde{\mathbf{u}}), \qquad (4.3.41)$$

where J_1, J_2, and μ are constants that depend on the parameters of the elastic energy and μ is assumed to be positive. The constant term $J_1 I$ can be absorbed into the pressure variable p. For the divergence term, recall the Lagrangian balance of mass equation $\phi_1 = \frac{\varphi_0}{\det F}$. Under this linearization, $\frac{1}{\det F} \approx 1 - \nabla \cdot \tilde{\mathbf{u}}$, so Lagrangian balance of mass becomes

$$\phi_1 \approx \varphi_0 (1 - \nabla \cdot \tilde{\mathbf{u}}). \qquad (4.3.42)$$

The reference configuration is taken to be the configuration at time $t = 0$, so $\varphi = 1$. Since $\phi_1 = 1$ at all times, including at $t = 0$, it is clear that the condition

$$\nabla \cdot \tilde{\mathbf{u}}|_{t=0} = 0 \qquad (4.3.43)$$

must be satisfied. This condition combined with (4.3.40) imply that

$$\nabla \cdot \tilde{\mathbf{u}} = 0$$

for all times. Thus all that remains of the elastic stress is

$$\frac{\partial W}{\partial F} F^T \approx 4\mu \mathbf{D}(\tilde{\mathbf{u}}).$$

The equations become

$$\nabla \cdot \tilde{\mathbf{u}}_t = 0, \tag{4.3.44}$$

$$\nabla \cdot [4\mu \mathbf{D}(\tilde{\mathbf{u}}) + 2\eta_1 \mathbf{D}(\tilde{\mathbf{u}}_t) + pI] - \beta \tilde{\mathbf{u}}_t = 0. \tag{4.3.45}$$

μ is related to the shear modulus of the solid and p is a redefined Lagrange multiplier.

In order for this system of equations to be well-posed, initial and boundary conditions must be applied. Impose the initial condition

$$\tilde{\mathbf{u}}|_{t=0} = \tilde{\mathbf{u}}_0$$

for $\tilde{\mathbf{u}}_0$ some known displacement. Divide $\partial\Omega$ into two subsets Γ_0 and Γ_1 such that $|\Gamma_0| > 0$, $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Here $|\Gamma_0|$ denotes the Lebesque measure of Γ_0 and $\overline{\Gamma}_0$ denotes the closure of Γ_0. Dirichlet conditions are imposed on $\tilde{\mathbf{u}}$ on Γ_0. The normal stress of the solid is imposed on Γ_1. In symbols, the initial and boundary conditions are

- $\tilde{\mathbf{u}}|_{t=0} = \tilde{\mathbf{u}}_0$ with $\nabla \cdot \tilde{\mathbf{u}}_0 = 0$;
- $\tilde{\mathbf{u}}|_{\Gamma_0} = \tilde{\boldsymbol{U}}$;
- $[pI + 4\mu \mathbf{D}(\tilde{\mathbf{u}}) + 2\eta_1 \mathbf{D}(\tilde{\mathbf{u}}_t)]\mathbf{n}|_{\Gamma_1} = \tilde{\mathbf{g}}$.

Here, $\tilde{\boldsymbol{U}}$ is assumed to be a known function on Γ_0, $\mathbf{n}$ is the outward normal vector on Γ_1, and $\tilde{\mathbf{g}}$ is assumed to be a known function on Γ_1. For consistency, the condition $\tilde{\boldsymbol{U}}|_{t=0} = \tilde{\mathbf{u}}_0|_{\Gamma_0}$ is imposed on the displacement. The following lemma, which is used to transform the problem to the homogeneous Dirichlet problem, is a standard generalization of the results of Ladyzhenskaya [34, Ch.1 Sec. 2]:

Lemma 4.1. *Assume $\partial\Omega \in C^2$, $\tilde{\boldsymbol{U}}(t) \in H^{3/2}(\Gamma_0)$ is differentiable with respect to t in the sense of distributions and $\int_{\Gamma_0} \tilde{\boldsymbol{U}}_t \cdot \mathbf{n} = 0$ for a.e. $t \geq 0$. Then $\exists \boldsymbol{U}(t) \in H^2(\Omega)$ differentiable with respect to t in the sense of distributions for a.e. $t \geq 0$ such that $\boldsymbol{U}|_{\Gamma_0} = \tilde{\boldsymbol{U}}$, $\nabla \cdot \boldsymbol{U}|_{t=0} = 0$, and $\nabla \cdot \boldsymbol{U}_t = 0$, and $||\boldsymbol{U}(t)||_{H^2(\Omega)} \leq C||\tilde{\boldsymbol{U}}(t)||_{H^{3/2}(\Gamma_0)}$ for a.e. $t \geq 0$.*

Define $\mathbf{u} = \tilde{\mathbf{u}} - \boldsymbol{U}$ and $\mathbf{u}_0 = \tilde{\mathbf{u}}_0 - \boldsymbol{U}|_{t=0}$. Then the PDE system can be expressed in the form

$$\nabla \cdot \mathbf{u}_t = 0, \tag{4.3.46}$$

$$\frac{\partial}{\partial x_j}[4\mu \mathbf{D}(\mathbf{u})_{ij} + 2\eta_1 \mathbf{D}(\mathbf{u}_t)_{ij} + p\delta_{ij}] - \beta \frac{\partial \mathbf{u}_i}{\partial t} = \mathbf{f}_i. \tag{4.3.47}$$

The initial and boundary conditions satisfy

- $\mathbf{u}|_{t=0} = \mathbf{u}_0$ with $\nabla \cdot \mathbf{u}_0 = 0$, $\mathbf{u}_0|_{\Gamma_0} = 0$;
- $\mathbf{u}|_{\Gamma_0} = 0$;
- $[pI + 4\mu\mathbf{D}(\mathbf{u}) + 2\eta_1\mathbf{D}(\mathbf{u}_t)]\mathbf{n}|_{\Gamma_1} = \mathbf{g}$.

In these equations,

$$\mathbf{f} = -\nabla \cdot [4\mu\mathbf{D}(\boldsymbol{U}) + 2\eta_1\mathbf{D}(\boldsymbol{U}_t)] + \beta\boldsymbol{U}_t,$$
$$\mathbf{g} = \tilde{\mathbf{g}} - [4\mu\mathbf{D}(\boldsymbol{U}) + 2\eta_1\mathbf{D}(\boldsymbol{U}_t)]\mathbf{n}|_{\Gamma_1}.$$

Note that if $|\Gamma_0| > 0$, then any solution $\mathbf{u} \in H^1(\Omega)$ satisfies $||\mathbf{u}||_{H^1(\Omega)} \leq C||\mathbf{D}(\mathbf{u})||_{L^2(\Omega)}$ for some constant $C > 0$ independent of $\mathbf{u}$ by Korn's inequality. [14] This fact will be used throughout the rest of this thesis.

4.3.1 Laplace transform and existence of solutions

Assume that $\tilde{\boldsymbol{U}} \in W^{1,\infty}(0,\infty; H^{3/2}(\Omega))$. Then $\mathbf{f} \in L^\infty(0,\infty; L^2(\Omega))$ and $\mathbf{g} \in L^\infty(0,\infty; H^{1/2}(\Gamma_1))$ if $\tilde{\mathbf{g}} \in L^\infty(0,\infty; H^{1/2}(\Gamma_1))$. Following Arendt [4], the Laplace transform with respect to time $\mathbf{F}$ of $\mathbf{f}$ exists, is analytic in the transform parameter $\tau \in \mathbf{R}$, and takes values in $L^2(\Omega)$ provided $\tau > 0$. Also, the Laplace transform with respect to time $\mathbf{G}$ of $\mathbf{g}$ exists, is analytic in τ, and takes values in $H^{1/2}(\Gamma_1)$ provided $\tau > 0$. Take the Laplace transform in time of (4.3.46)-(4.3.47) and of the boundary conditions and let $\mathbf{U}$ and P denote the transforms of $\mathbf{u}$ and p, respectively. The system of equations becomes

$$\nabla \cdot [\tau \mathbf{U} - \mathbf{u}_0] = 0, \tag{4.3.48}$$

$$\frac{\partial}{\partial x_j}[P\delta_{ij} + 4\mu \mathbf{D}(\mathbf{U})_{ij} + 2\eta_1 \mathbf{D}(\tau\mathbf{U} - \mathbf{u}_0)_{ij}] - \beta(\tau\mathbf{U} - \mathbf{u}_0)_i = \mathbf{F}_i. \tag{4.3.49}$$

The boundary conditions are

- $\mathbf{U}|_{\Gamma_0} = \mathbf{0}$ with $\mathbf{u}_0|_{\Gamma_0} = \mathbf{0}$, $\nabla \cdot \mathbf{u}_0 = 0$;
- $[P\delta_{ij} + 4\mu \mathbf{D}(\mathbf{U})_{ij} + 2\eta_1 \mathbf{D}(\tau\mathbf{U} - \mathbf{u}_0)_{ij}]\mathbf{n}_j|_{\Gamma_1} = \mathbf{G}_i$.

This is an elliptic system of equations. Define the function space

$$\mathbf{W} = \{\mathbf{w} \in H^1(\Omega) : \mathbf{w}|_{\Gamma_0} = \mathbf{0}, \nabla \cdot \mathbf{w} = 0\}.$$

Take $\mathbf{w} \in \mathbf{W}$ and define the bilinear form

$$B[\mathbf{U}, \mathbf{w}] = \int_\Omega [(4\mu + 2\eta_1\tau)\mathbf{D}(\mathbf{U})_{ij}\mathbf{D}(\mathbf{w})_{ij} + \beta\tau\mathbf{U} \cdot \mathbf{w}].$$

Define the linear form

$$L[\mathbf{w}] = \int_{\Gamma_1} \mathbf{G} \cdot \mathbf{w} + \int_\Omega [2\eta_1 \mathbf{D}(\mathbf{u}_0)_{ij}\mathbf{D}(\mathbf{w})_{ij} + (\beta\mathbf{u}_0 - \mathbf{F}) \cdot \mathbf{w}].$$

Notice that if $\mathbf{u}_0 \in H^1(\Omega)$, then L defines a bounded linear functional on $\mathbf{W}$ for $\tau > 0$ since $\mathbf{F} \in L^2(\Omega)$ and $\mathbf{G} \in H^{1/2}(\Gamma_1)$ [14]. Also notice that for $\tau > 0$, the bilinear form B defines an inner product on $\mathbf{W}$. Define a weak solution of the system to be any $\mathbf{U} \in \mathbf{W}$ such that $B[\mathbf{U}, \mathbf{w}] = L[\mathbf{w}] \forall \mathbf{w} \in \mathbf{W}$. By the Riesz representation theorem, there exists a unique weak solution $\mathbf{U} \in \mathbf{W}$.

Theorem 4.3. *Assume* $\mathbf{F} \in L^2(\Omega)$, $\mathbf{G} \in H^{1/2}(\Gamma_1)$, *and* $\partial\Omega \in C^2$. *If* $\mathbf{u}_0 \in H^2(\Omega')$ *for some* $\Omega' \subseteq\subseteq \Omega$, *then the weak solution* $\mathbf{U} \in H^2_{loc}(\Omega')$ *and* $\exists P \in H^1_{loc}(\Omega')$ *such that (4.3.49) is satisfied a.e. in* Ω'. *Also,* $\forall \Omega'' \subseteq\subseteq \Omega'$,

$$||\mathbf{U}||_{H^2(\Omega'')} + ||\nabla P||_{L^2(\Omega'')} \leq C[||\mathbf{F}||_{L^2(\Omega)} + ||\mathbf{u}_0||_{H^2(\Omega')} + ||\mathbf{u}_0||_{H^1(\Omega)} + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)}].$$

Proof: Let Ω'' be any subdomain which lies strictly inside Ω'. Take $\mathbf{w} = (\nabla \times (\zeta^2 \nabla \times \mathbf{U}_\rho))_\rho$, where the index ρ denotes averaging over Ω' and $\zeta \in C_0^2(\Omega')$, $0 \leq \zeta \leq 1$ in Ω', and $\zeta \equiv 1$ in $\overline{\Omega}'' \subseteq\subseteq \Omega'$. Assume that the width of the boundary strip in Ω' where $\zeta \equiv 0$ is greater than ρ. After making some manipulations using the fact that $\nabla \cdot \mathbf{U}_\rho = 0$, it can be shown that

$$\begin{aligned}
&\int_{\Omega'} (2\mu + \eta_1\tau)\zeta^2|\Delta \mathbf{U}_\rho|^2 = \int_{\Omega'} [(\beta \mathbf{u}_{0\rho} - \mathbf{F}_\rho - \eta_1 \Delta \mathbf{u}_{0\rho}) \cdot (\nabla(\zeta^2) \times (\nabla \times \mathbf{U}_\rho) \\
&-\zeta^2 \Delta \mathbf{U}_\rho) - \zeta^2|\nabla \mathbf{U}_\rho|^2] + \int_{\Omega'} [(2\mu + \eta_1\tau)\Delta \mathbf{U}_\rho \cdot (\nabla(\zeta^2) \times (\nabla \times \mathbf{U}_\rho)) \\
&-\beta\tau \mathbf{U}_\rho \cdot (\nabla(\zeta^2) \times (\nabla \times \mathbf{U}_\rho)) - \mathbf{U}_{i_\rho} \frac{\partial \mathbf{U}_{i_\rho}}{\partial x_j} \frac{\partial(\zeta^2)}{\partial x_j}].
\end{aligned}$$

Furthermore, using Poincaré's inequality, it can be shown that

$$\begin{aligned}
\int_{\Omega'} \zeta^2|\Delta \mathbf{U}_\rho|^2[(2\mu + \eta_1\tau)(1 - 2\epsilon_2) - \epsilon_1] \quad \leq \quad & \int_{\Omega'} C[|\mathbf{F}_\rho|^2 + |\mathbf{u}_{0\rho}|^2 + |\Delta \mathbf{u}_{0\rho}|^2 \\
& +|\mathbf{U}_\rho|^2 + |\nabla \mathbf{U}_\rho|^2]
\end{aligned}$$

for all $\epsilon_1, \ \epsilon_2 \ \geq 0$. Choosing $\epsilon_1, \epsilon_2 \geq 0$ such that $(2\mu + \eta_1\tau)(1 - 2\epsilon_2) - \epsilon_1 > 0$, it can be shown that

$$\int_{\Omega'} \zeta^2|\Delta \mathbf{U}_\rho|^2 \leq C[||\mathbf{F}_\rho||^2_{L^2(\Omega')} + ||\mathbf{u}_{0\rho}||^2_{H^2(\Omega')} + ||\mathbf{U}_\rho||^2_{H^1(\Omega')}].$$

The equation for the weak solution with $\mathbf{w} = \mathbf{U}$ can be used to show that

$$||\mathbf{U}_\rho||^2_{H^1(\Omega')} \leq ||\mathbf{U}||^2_{H^1(\Omega)} \leq C[||\mathbf{F}||^2_{L^2(\Omega)} + ||\mathbf{u}_0||^2_{H^1(\Omega)} + ||\mathbf{G}||^2_{H^{1/2}(\Gamma_1)}].$$

Therefore,

$$\begin{aligned}
||\Delta \mathbf{U}_\rho||^2_{L^2(\Omega'')} \leq \int_{\Omega'} \zeta^2|\Delta \mathbf{U}_\rho|^2 \quad \leq \quad & C[||\mathbf{F}||^2_{L^2(\Omega)} + ||\mathbf{u}_0||^2_{H^1(\Omega)} + ||\mathbf{G}||^2_{H^{1/2}(\Gamma_1)} \\
& +||\mathbf{u}_0||^2_{H^2(\Omega')}].
\end{aligned}$$

The right hand side of this inequality is independent of ρ, so $\mathbf{U} = \lim_{\rho \to 0} \mathbf{U}_\rho$ has second derivatives, $\mathbf{U} \in H^2(\Omega'')$, and

$$||\mathbf{U}||_{H^2(\Omega'')} \leq C[||\mathbf{F}||_{L^2(\Omega)} + ||\mathbf{u}_0||_{H^1(\Omega)} + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)} + ||\mathbf{u}_0||_{H^2(\Omega')}].$$

Returning to the weak solution equation, it can be shown using integration by parts that

$$\int_\Omega [\frac{\partial}{\partial x_j}(4\mu \mathbf{D}(\mathbf{U})_{ij} + 2\eta_1 \mathbf{D}(\tau \mathbf{U} - \mathbf{u}_0)_{ij}) - \beta(\tau \mathbf{U} - \mathbf{u}_0)_i - \mathbf{F}_i]\mathbf{w}_i = \int_{\Gamma_1} \mathbf{w}_i[4\mu \mathbf{D}(\mathbf{U})_{ij} + 2\eta_1 \mathbf{D}(\tau \mathbf{U} - \mathbf{u}_0)_{ij}\mathbf{n}_j - \mathbf{G}_i] \tag{4.3.50}$$

for all $\mathbf{w} \in \mathbf{W}$. Define the function space

$$\mathbf{H} = \{\mathbf{v} \in C_0^\infty(\Omega'') : \ \nabla \cdot \mathbf{v} = 0\}.$$

Note that the closure of $\mathbf{H}$ in the $H^1(\Omega)$ norm is a subspace of $\mathbf{W}$. Take $\mathbf{w} \in \mathbf{H}$ in (4.3.50). Then

$$\int_{\Omega''} [\frac{\partial}{\partial x_j}(4\mu \mathbf{D}(\mathbf{U})_{ij} + 2\eta_1 \mathbf{D}(\tau \mathbf{U} - \mathbf{u}_0)_{ij}) - \beta(\tau \mathbf{U} - \mathbf{u}_0)_i - \mathbf{F}_i]\mathbf{w}_i = 0$$

for all $\mathbf{w} \in \mathbf{H}$. Since $\frac{\partial}{\partial x_j}(4\mu \mathbf{D}(\mathbf{U})_{ij} + 2\eta_1 \mathbf{D}(\tau \mathbf{U} - \mathbf{u}_0)_{ij}) - \beta(\tau \mathbf{U} - \mathbf{u}_0)_i - \mathbf{F}_i \in L^2(\Omega'')$, by [34] $\exists P \in H^1(\Omega'')$ such that

$$\frac{\partial}{\partial x_j}(4\mu \mathbf{D}(\mathbf{U})_{ij} + 2\eta_1 \mathbf{D}(\tau \mathbf{U} - \mathbf{u}_0)_{ij}) - \beta(\tau \mathbf{U} - \mathbf{u}_0)_i - \mathbf{F}_i = -\frac{\partial P}{\partial x_i}$$

a.e. in Ω''. From this equation it can easily be shown that

$$||\nabla P||_{L^2(\Omega'')} \leq C[||\mathbf{F}||_{L^2(\Omega)} + ||\mathbf{u}_0||_{H^1(\Omega)} + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)} + ||\mathbf{u}_0||_{H^2(\Omega')}].$$

□

With this interior regularity in mind, for $\mathbf{U}, \mathbf{w} \in \mathbf{W} \bigcap H^2_{loc}(\Omega')$, define the linear operator $A : \mathbf{W} \bigcap H^2_{loc}(\Omega') \to (\mathbf{W} \bigcap H^2_{loc}(\Omega'))'$ by the equation $< A[\mathbf{U}], \mathbf{w} >:= B[\mathbf{U}, \mathbf{w}]$, where $(\mathbf{W} \bigcap H^2_{loc}(\Omega'))'$ denotes the dual space of $\mathbf{W} \bigcap H^2_{loc}(\Omega')$ and

$< A[\mathbf{U}], \mathbf{w} >$ is the action of $A[\mathbf{U}]$ applied to $\mathbf{w}$. Since the weak solution is unique, A is invertible and the weak solution satisfies the functional equation $A[\mathbf{U}] = L$. Since B is analytic in τ for $\tau > 0$, so is A. Following Dautray [16], A^{-1} is analytic in τ for $\tau > 0$. L is analytic in τ for $\tau > 0$, so $\mathbf{U} = A^{-1}L$ is analytic in τ for $\tau > 0$. Therefore, following Arendt et al [4], the inverse Laplace transform $\mathbf{u}$ of $\mathbf{U}$ exists and is in $L^\infty(0, \infty; H^1(\Omega) \bigcap H^2_{loc}(\Omega'))$. In fact, $P \in H^1_{loc}(\Omega')$ is analytic in τ for $\tau > 0$, so its inverse Laplace transform p exists and is in $L^\infty(0, \infty; H^1_{loc}(\Omega'))$.

The inverse transforms $\mathbf{u}$, p are weak global-in-time solutions of (4.3.46), (4.3.47) in the following sense. Take the inverse Laplace transforms of (4.3.48) and (4.3.49) to get back (4.3.46) and (4.3.47). Let ϕ, $\mathbf{v} \in W^{1,1}(0, T)$ for any $T > 0$ and take $\phi(T) = 0$, $\mathbf{v}(T) = \mathbf{0}$. Multiply (4.3.46) by ϕ, take the dot product of (4.3.47) and $\mathbf{v}$, and integrate by parts in time over $(0, T)$. The resulting equations are

$$\int_0^T (\nabla \cdot \mathbf{u}) \frac{\partial \phi}{\partial t} = 0,$$

$$\int_0^T \{ \frac{\partial}{\partial x_j} [4\mu \mathbf{D}(\mathbf{u})_{ij} + p\delta_{ij}] \mathbf{v}_i - \frac{\partial}{\partial x_j} (2\eta_1 \mathbf{D}(\mathbf{u})_{ij}) \frac{\partial \mathbf{v}_i}{\partial t} + \beta \mathbf{u} \cdot \frac{\partial \mathbf{v}}{\partial t} \}$$
$$= \int_0^T [\mathbf{f} \cdot \mathbf{v} + \frac{\partial}{\partial x_j} (2\eta_1 \mathbf{D}(\mathbf{u}_0)_{ij}) \mathbf{v}_i(0) - \beta \mathbf{u}_0 \cdot \mathbf{v}(0)]$$

for all ϕ, $\mathbf{v} \in W^{1,1}(0, T)$ a.e. in Ω'.

Now, return to the Laplace-transformed problem (4.3.48), (4.3.49).

Theorem 4.4. *Let $m \geq 0$ be an integer. Assume $\partial\Omega \in C^{m+2}$, $\mathbf{F}(\tau) \in H^m(\Omega) \forall \tau > 0$, $\mathbf{u}_0 \in H^{m+2}(\Omega')$, and $\mathbf{G}(\tau) \in H^{1/2}(\Gamma_1) \forall \tau > 0$ for some $\Omega' \subseteq \Omega$. Then $\mathbf{U} \in H^{m+2}_{loc}(\Omega')$, $P \in H^{m+1}_{loc}(\Omega')$. Also, $\forall \Omega'' \subseteq\subseteq \Omega'$,*

$$||\mathbf{U}||_{H^{m+2}(\Omega'')} + ||\nabla P||_{H^m(\Omega'')} \leq C[||\mathbf{F}||_{H^m(\Omega)} + ||\mathbf{u}_0||_{H^{m+2}(\Omega')} + ||\mathbf{u}_0||_{H^1(\Omega)} + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)}].$$

Proof: The proof is achieved by induction on m. The $m = 0$ case was done in the previous theorem. Given any integer $m \geq 0$, assume $\partial\Omega \in C^{m+2}$, $\mathbf{U} \in$

$H^{m+2}_{loc}(\Omega')$, $P \in H^{m+1}_{loc}(\Omega')$, and $\forall \Omega'' \subseteq\subseteq \Omega'$,

$$\begin{aligned} ||\mathbf{U}||_{H^{m+2}(\Omega'')} + ||\nabla P||_{H^m(\Omega'')} &\leq C[||\mathbf{F}||_{H^m(\Omega)} + ||\mathbf{u}_0||_{H^{m+2}(\Omega')} + ||\mathbf{u}_0||_{H^1(\Omega)} \\ &\quad + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)}]. \end{aligned}$$

Suppose $\partial\Omega \in C^{m+3}$, $\mathbf{F} \in H^{m+1}(\Omega)$, and $\mathbf{u}_0 \in H^{m+3}(\Omega')$. By hypothesis,

$$\begin{aligned} ||\mathbf{U}||_{H^{m+2}(\Omega''')} + ||\nabla P||_{H^m(\Omega''')} &\leq C[||\mathbf{F}||_{H^m(\Omega)} + ||\mathbf{u}_0||_{H^{m+2}(\Omega')} + ||\mathbf{u}_0||_{H^1(\Omega)} \\ &\quad + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)}] \end{aligned}$$

$\forall \Omega''' \subseteq\subseteq \Omega'$. Fix $\Omega'' \subseteq\subseteq \Omega''' \subseteq\subseteq \Omega'$. Define the function space

$$\mathbf{H} = \{\mathbf{v} \in C_0^\infty(\Omega''') : \nabla \cdot \mathbf{v} = 0\}.$$

Let α be any multi-index with $|\alpha| = m+1$ and choose any test function $\tilde{\mathbf{w}} \in \mathbf{H}$. Insert $\mathbf{w} = (-1)^{|\alpha|} D^\alpha \tilde{\mathbf{w}}$ into the equation for weak solutions. Then, after integrating by parts and defining $\tilde{\mathbf{U}} = D^\alpha \mathbf{U} \in H^1(\Omega''')$, $\tilde{\mathbf{F}} = D^\alpha \mathbf{F} \in L^2(\Omega)$, $\tilde{\mathbf{u}}_0 = D^\alpha \mathbf{u}_0 \in H^2(\Omega''')$, it can be shown that

$$\text{(4.3.51)} \quad \int_\Omega [(4\mu + 2\eta_1\tau)\mathbf{D}(\tilde{\mathbf{U}})_{ij} \frac{\partial \tilde{\mathbf{w}}_i}{\partial x_j} + \beta\tau\tilde{\mathbf{U}}\cdot\tilde{\mathbf{w}}] = \int_\Omega [(\beta\tilde{\mathbf{u}}_0 - \tilde{\mathbf{F}})\cdot\tilde{\mathbf{w}} + 2\eta_1\mathbf{D}(\tilde{\mathbf{u}}_0)_{ij}\frac{\partial \tilde{\mathbf{w}}_i}{\partial x_j}]$$

$\forall \tilde{\mathbf{w}} \in \mathbf{H}$. Thus $\tilde{\mathbf{U}}$ is a weak solution of

$$\begin{aligned} &\nabla \cdot (\tau\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_0) = 0, \\ &\nabla \cdot [4\mu\mathbf{D}(\tilde{\mathbf{U}}) + 2\eta_1\mathbf{D}(\tau\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_0) + \tilde{P}I] = \tilde{\mathbf{F}} \end{aligned}$$

in Ω'''. In view of the previous theorem, $\tilde{\mathbf{U}} \in H^2(\Omega'')$ and

$$||\tilde{\mathbf{U}}||_{H^2(\Omega'')} \leq C[||\mathbf{F}||_{H^{m+1}(\Omega)} + ||\mathbf{u}_0||_{H^{m+3}(\Omega')} + ||\mathbf{u}_0||_{H^1(\Omega)} + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)}].$$

This holds for every multi-index α with $|\alpha| = m+1$ and $\tilde{\mathbf{U}} = D^\alpha \mathbf{U}$. Thus, $\mathbf{U} \in H^{m+3}(\Omega'')$ and

$$||\mathbf{U}||_{H^{m+3}(\Omega'')} \leq C[||\mathbf{F}||_{H^{m+1}(\Omega)} + ||\mathbf{u}_0||_{H^{m+3}(\Omega')} + ||\mathbf{u}_0||_{H^1(\Omega)} + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)}]$$

$\forall \Omega'' \subseteq\subseteq \Omega'$. Therefore, if $\mathbf{F} \in H^m(\Omega)$ and $\mathbf{u}_0 \in H^{m+2}(\Omega')$ for some $\Omega' \subseteq\subseteq \Omega$ and $m \geq 0$ is an integer, then the weak solution $\mathbf{U}$ is in $H^{m+2}_{loc}(\Omega')$ and

$$||\mathbf{U}||_{H^{m+2}(\Omega'')} \leq C[||\mathbf{F}||_{H^m(\Omega)} + ||\mathbf{u}_0||_{H^{m+2}(\Omega')} + ||\mathbf{u}_0||_{H^1(\Omega)} + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)}]$$

$\forall \Omega'' \subseteq\subseteq \Omega'$. Note that since $\nabla \cdot [4\mu \mathbf{D}(\mathbf{U}) + 2\eta_1 \mathbf{D}(\tau \mathbf{U} - \mathbf{u}_0)] - \beta(\tau \mathbf{U} - \mathbf{u}_0) - \mathbf{F} \in H^m_{loc}(\Omega')$, $P \in H^{m+1}_{loc}(\Omega')$. Let γ be any multi-index with $|\gamma| \leq m$. It is easy to show that for any $\Omega'' \subseteq\subseteq \Omega'$,

$$||D^\gamma \nabla P||_{L^2(\Omega'')} \leq C[||\mathbf{F}||_{H^m(\Omega)} + ||\mathbf{u}_0||_{H^{m+2}(\Omega')} + ||\mathbf{u}_0||_{H^1(\Omega)} + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)}].$$

Therefore, $\forall \Omega'' \subseteq\subseteq \Omega'$,

$$\begin{aligned} ||\mathbf{U}||_{H^{m+2}(\Omega'')} + ||\nabla P||_{H^m(\Omega'')} \quad \leq \quad & C[||\mathbf{F}||_{H^m(\Omega)} + ||\mathbf{u}_0||_{H^{m+2}(\Omega')} + ||\mathbf{u}_0||_{H^1(\Omega)} \\ & + ||\mathbf{G}||_{H^{1/2}(\Gamma_1)}]. \end{aligned}$$

□

Recall that in the untransformed system, $\mathbf{f}$ depends on $\tilde{\boldsymbol{U}}$, the original solid displacement on Γ_0. For any integer $m \geq 0$, if $\partial\Omega \in C^{m+2}$ and $\tilde{\mathbf{U}} \in W^{1,\infty}(0,\infty; H^{m+3/2}(\Gamma_0))$, then $\mathbf{f} \in L^\infty(0,\infty; H^m(\Omega))$. In this case, the Laplace transform F exists and $F(\tau) \in H^m(\Omega)$ $\forall \tau > 0$. Therefore, if $\tilde{\boldsymbol{U}} \in W^{1,\infty}(0,\infty; H^{m+3/2}(\Gamma_0))$ and $\mathbf{u_0} \in H^1(\Omega) \bigcap H^{m+2}(\Omega')$ for some $\Omega' \subseteq\subseteq \Omega$, then the inverse Laplace transforms $\mathbf{u}$, p of $\mathbf{U}$, P exist and $\mathbf{u} \in L^\infty(0,\infty; H^{m+2}_{loc}(\Omega') \bigcap H^1(\Omega))$, $p \in L^\infty(0,\infty; H^{m+1}_{loc}(\Omega'))$.

4.3.2 Galerkin approximations and weak solutions

Global-in-time weak solutions of (4.3.46)-(4.3.47) with the boundary and initial conditions given above are sought. In order to prove existence of such solutions, an energy equation for the system is sought. Take the dot product of $\mathbf{u}_t$ and (4.3.47), integrate by parts over Ω, and apply (4.3.46), the given boundary conditions, and the fact that $\mathbf{u}_t|_{\Gamma_0} = \mathbf{0}\forall t$. The result is

$$\int_\Omega \frac{\partial}{\partial t}(2\mu|\mathbf{D}(\mathbf{u})|^2) + \int_\Omega [2\eta_1|\mathbf{D}(\mathbf{u}_t)|^2 + \beta|\mathbf{u}_t|^2] = \int_{\Gamma_1} \mathbf{g}\cdot\mathbf{u}_t - \int_\Omega \mathbf{f}\cdot\mathbf{u}_t.$$

Now, integrate in time over the interval $[0, T]$ for some $T > 0$. Use initial conditions.

$$\begin{aligned}(4.3.52) \quad & \int_\Omega 2\mu|\mathbf{D}(\mathbf{u}(T))|^2 + \int_0^T \int_\Omega [2\eta_1|\mathbf{D}(\mathbf{u}_t)|^2 + \beta|\mathbf{u}_t|^2] = \int_\Omega 2\mu|\mathbf{D}(\mathbf{u}_0)|^2 \\ & + \int_0^T \int_{\Gamma_1} \mathbf{g}\cdot\mathbf{u}_t - \int_0^T \int_\Omega \mathbf{f}\cdot\mathbf{u}_t.\end{aligned}$$

This energy expression motivates the following theorem.

Theorem 4.5. *Assume* $\mathbf{u}_0 \in H^1(\Omega)$, $\mathbf{f} \in L^2(0,T;L^2(\Omega))$, *and* $\mathbf{g} \in L^2(0,T;$ $H^{1/2}(\Gamma_1))$. *Then there exists a unique weak solution* $\mathbf{u} \in H^1(0,T;$ $H^1(\Omega)) \bigcap L^\infty(0,T;H^1(\Omega))$ *that exists up to time* T.

Proof: Define the function space

$$\mathbf{W} = \{\mathbf{w} \in H^1(\Omega) : \mathbf{w}|_{\Gamma_0} = \mathbf{0}, \nabla\cdot\mathbf{w} = 0\}.$$

$\mathbf{W}$ is seperable, so there exists a sequence of linearly independent smooth functions $\{\mathbf{w}^k\}_{k=1}^\infty$ which is dense in $\mathbf{W}$. Define the finite dimensional subspace $\mathbf{W}_N = span\{\mathbf{w}^k\}_{k=1}^N$. Let $\mathbf{u}^N(t,\cdot) \in \mathbf{W}_N$ satisfy $\forall \mathbf{w} \in \mathbf{W}_N$ the equations

$$\begin{aligned}&\int_\Omega [(4\mu\mathbf{D}(\mathbf{u}^N)_{ij} + 2\eta_1\mathbf{D}(\mathbf{u}_t^N)_{ij})\mathbf{D}(\mathbf{w})_{ij} + \beta\mathbf{u}_t^N\cdot\mathbf{w}] = \int_{\Gamma_1}\mathbf{g}\cdot\mathbf{w} - \int_\Omega \mathbf{f}\cdot\mathbf{w}, \\ &\int_\Omega \mathbf{D}(\mathbf{u}^N(0,\cdot))_{ij}\mathbf{D}(\mathbf{w})_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij}\mathbf{D}(\mathbf{w})_{ij}.\end{aligned}$$

Since $\mathbf{u}^N(t,\cdot) \in \mathbf{W}_N$, it can be written as a linear combination of basis functions with coefficients depending on t. These expressions are substituted into the above equations and a linear system of ODEs in time with a full set of initial conditions is obtained by setting $\mathbf{w}$ equal to each of the basis functions. By standard ODE theory, $\exists\mathbf{u}^N(t) \in \mathbf{W}_N$ for $t \in [0,T]$. Note that $\mathbf{u}_t^N \in \mathbf{W}_N$ as well.

The Galerkin approximation $\mathbf{u}^N$ satisfies $\forall \mathbf{w}^N \in \mathbf{W}_N$ the equation

$$\begin{aligned}(4.3.53)\qquad &\int_\Omega [(4\mu \mathbf{D}(\mathbf{u}^N)_{ij} + 2\eta_1 \mathbf{D}(\mathbf{u}_t^N)_{ij})\mathbf{D}(\mathbf{w}^N)_{ij} + \beta \mathbf{u}_t^N \cdot \mathbf{w}^N] \\ &= \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{w}^N - \int_\Omega \mathbf{f} \cdot \mathbf{w}^N.\end{aligned}$$

An energy estimate for $\mathbf{u}^N$ is obtained from this equation. Take $\mathbf{w}^N = \mathbf{u}_t^N$ and substitute back into the equation.

$$\int_\Omega \frac{\partial}{\partial t}(2\mu |\mathbf{D}(\mathbf{u}^N)|^2) + \int_\Omega [2\eta_1 |\mathbf{D}(\mathbf{u}_t^N)|^2 + \beta |\mathbf{u}_t^N|^2] = \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{u}_t^N - \int_\Omega \mathbf{f} \cdot \mathbf{u}_t^N$$

Integrating over time and using Young's inequality and embedding theorems, it can be shown that

$$\begin{aligned}&\int_\Omega |\mathbf{D}(\mathbf{u}^N(T))|^2 + \int_0^T \int_\Omega [|\mathbf{D}(\mathbf{u}_t^N)|^2 + |\mathbf{u}_t^N|^2] \leq C[||\mathbf{D}(\mathbf{u}_0)||^2_{L^2(\Omega)} \\ &+||\mathbf{g}||^2_{L^2(0,T;H^{1/2}(\Gamma_1))} + ||\mathbf{f}||^2_{L^2(0,T;L^2(\Omega))}].\end{aligned}$$

The right hand side of the inequality is bounded uniformly in N. From this inequality, it is clear that

- $\mathbf{u}_t^N$ is bounded in $L^2(0,T;H^1(\Omega))$,
- $\mathbf{u}^N$ is bounded in $L^\infty(0,T;H^1(\Omega))$.

Since $T < \infty$ and Ω is bounded, $\mathbf{u}^N$ is bounded in $H^1(0,T;H^1(\Omega))$. Therefore, after possibly passing to subsequences,
$\exists \mathbf{u} \in H^1(0,T;H^1(\Omega)) \bigcap L^\infty(0,T;H^1(\Omega))$ such that $\mathbf{u}^N \rightharpoonup \mathbf{u}$ in $H^1(0,T;H^1(\Omega))$ and $\mathbf{u}^N \overset{*}{\rightharpoonup} \mathbf{u}$ in $L^\infty(0,T;H^1(\Omega))$.

Define the function space

$$C_W^\infty(\Omega) = \{\mathbf{w} \in C^\infty(\Omega) : \mathbf{w}|_{\Gamma_0} = \mathbf{0}, \nabla \cdot \mathbf{w} = 0\}.$$

Let $\mathbf{w} \in C_0^\infty([0,T); C_W^\infty(\Omega))$. By denseness, $\exists \{\mathbf{w}^N\}_{N=1}^\infty$ with $\mathbf{w}^N \in C^1(0,T;\mathbf{W}_N)$ such that $\mathbf{w}^N \to \mathbf{w}$ in $C^1(0,T;W^{1,\delta}(\Omega))$ for $\delta \geq 1$. Select $\mathbf{w}^N(T) = \mathbf{0}$. The

Galerkin approximations satisfy

$$\text{(4.3.54)} \quad \int_0^T \int_\Omega [4\mu \mathbf{D}(\mathbf{u}^N)_{ij}\mathbf{D}(\mathbf{w}^N)_{ij} - 2\eta_1 \mathbf{D}(\mathbf{u}^N)_{ij}\mathbf{D}(\mathbf{w}_t^N)_{ij}$$
$$-\beta \mathbf{u}^N \cdot \mathbf{w}_t^N] = \int_\Omega [2\eta_1 \mathbf{D}(\mathbf{u}^N(0))_{ij}\mathbf{D}(\mathbf{w}^N(0))_{ij} + \beta \mathbf{u}^N(0)\cdot \mathbf{w}^N(0)]$$
$$+\int_0^T \int_{\Gamma_1} \mathbf{g}\cdot \mathbf{w}^N - \int_0^T \int_\Omega \mathbf{f}\cdot \mathbf{w}^N.$$

Taking weak and weak-$*$ limits, it is clear that $\forall \mathbf{w} \in C^1(0,T;\mathbf{W})$,

$$\text{(4.3.55)} \int_0^T \int_\Omega [4\mu \mathbf{D}(\mathbf{u})_{ij}\mathbf{D}(\mathbf{w})_{ij} - 2\eta_1 \mathbf{D}(\mathbf{u})_{ij}\mathbf{D}(\mathbf{w}_t)_{ij} - \beta \mathbf{u}\cdot \mathbf{w}_t] =$$
$$\int_\Omega [2\eta_1 \mathbf{D}(\mathbf{u}_0)_{ij}\mathbf{D}(\mathbf{w}(0))_{ij} + \beta \mathbf{u}_0 \cdot \mathbf{w}(0)] + \int_0^T \int_{\Gamma_1} \mathbf{g}\cdot\mathbf{w} - \int_0^T \int_\Omega \mathbf{f}\cdot \mathbf{w}.$$

Since this equation holds $\forall \mathbf{w} \in C_0^\infty([0,T);C_W^\infty(\Omega))$, $\mathbf{u} \in H^1(0,T;H^1(\Omega)) \bigcap L^\infty(0,T;H^1(\Omega))$ is a weak solution of (4.3.46)-(4.3.47).

To prove uniqueness, it suffices to take $\mathbf{u}_0 \equiv 0 \equiv \mathbf{f} \equiv \mathbf{g}$ and show that $\mathbf{u} = 0$ a.e. in $[0,T]\times\Omega$. Weak solutions in this case satisfy the equation

$$\int_\Omega [(4\mu \mathbf{D}(\mathbf{u})_{ij} + 2\eta_1 \mathbf{D}(\mathbf{u}_t)_{ij})\mathbf{D}(\mathbf{w})_{ij} + \beta \mathbf{u}_t \cdot \mathbf{w}] = 0$$

$\forall \mathbf{w} \in \mathbf{W}$ and a.e. $t \in [0,T]$. Take $\mathbf{w} = \mathbf{u}_t(t)$ for some time t where this equation is satisfied. Integrate in time over $[0,T]$.

$$\int_\Omega 2\mu |\mathbf{D}(\mathbf{u}(T))|^2 + \int_0^T \int_\Omega [2\eta_1 |\mathbf{D}(\mathbf{u}_t)|^2 + \beta |\mathbf{u}_t|^2] = 0.$$

From this it is clear that $\mathbf{u}_t = 0$ a.e. in $[0,T]\times\Omega$, so $\mathbf{u}$ is constant a.e. in $[0,T]\times\Omega$. Since $\mathbf{u}_0 \equiv 0$ in Ω by assumption and $\mathbf{u} \in C([0,T];H^1(\Omega))$ by standard embedding theorems, $\mathbf{u} = 0$ a.e. in $[0,T]\times\Omega$. This proves uniqueness and the theorem. □

4.3.3 Numerical simulations

In this subsection a mixed finite element method is described and used to calculate stresses for an incompressible linearly elastic material with Newtonian dissipation in two dimensions. The domain Ω is taken to be a square. The boundary is divided into two parts, Γ_0 and Γ_1, so that $\partial\Omega = \Gamma_0 \bigcup \Gamma_1$. The problem under consideration is

$$(4.3.56) \qquad \begin{aligned} &\frac{\partial}{\partial x_j}[4\mu\mathbf{D}(\mathbf{u})_{ij} + 2\eta_1\mathbf{D}(\mathbf{u}_t)_{ij} + p\delta_{ij}] - \beta\frac{\partial \mathbf{u}_i}{\partial t} = 0, \\ &\nabla \cdot \mathbf{u}_t = 0, \end{aligned}$$

with initial and boundary conditions satisfying

- $\mathbf{u}|_{t=0} = \mathbf{u}_0$ with $\nabla \cdot \mathbf{u}_0 = 0$;
- $\mathbf{u}|_{\Gamma_0} = \mathbf{0}$;
- $[4\mu\mathbf{D}(\mathbf{u})_{ij} + 2\eta_1\mathbf{D}(\mathbf{u}_t)_{ij} + p\delta_{ij}]\mathbf{n}_j|_{\Gamma_1} = \mathbf{g}_i$.

Following the method proposed by Boffi, Brezzi and Fortin in [8] for the Stokes problem, define

$$\mathbf{V} = \{\mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v}|_{\Gamma_0} = \mathbf{0}\}, \quad Q = L^2(\Omega),$$

and

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega [4\mu\mathbf{D}(\mathbf{u})_{ij} + 2\eta_1\mathbf{D}(\mathbf{u}_t)_{ij}]\mathbf{D}(\mathbf{v})_{ij} + \int_\Omega \beta\frac{\partial \mathbf{u}_i}{\partial t}\mathbf{v}_i, \\ b(\mathbf{v}, q) &= \int_\Omega q\nabla \cdot \mathbf{v}, \end{aligned}$$

Problem (4.3.56) can clearly be written in the form: find $\mathbf{u} \in \mathbf{V}$ and $p \in Q$ such that

$$(4.3.57) \qquad \begin{aligned} &a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V}, \\ &b(\mathbf{u}_t, q) = 0 \quad \forall q \in Q, \end{aligned}$$

which is a mixed problem.

In order to solve this problem numerically, let Ω_h be a uniform mesh of Ω consisting of triangles with height h. Following the method of Hood and Taylor in [61] for the Navier-Stokes equation, let $\mathbf{V}_h$ denote the subspace of $\mathbf{V}$ defined by

$$\mathbf{V}_h = \{\mathbf{v} \in [C(\bar{\Omega})]^2 : \mathbf{v}|_K \in [P^2(K)]^2 \forall K \in \Omega_h\}.$$

Let Q_h denote the subspace of Q defined by

$$Q_h = \{q \in C(\bar{\Omega}) : q|_K \in P^1(K) \forall K \in \Omega_h\}.$$

Boffi, Brezzi, and Fortin showed that this method is numerically stable for the steady state Stokes problem [8]. The numerical scheme becomes: find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in Q_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) &= \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ b(\frac{\partial \mathbf{u}_h}{\partial t}, q) &= 0 \quad \forall q \in Q_h. \end{aligned} \tag{4.3.58}$$

Time discretization is accomplished using the backward Euler finite difference method. The stress is obtained by L^2-projecting $\nabla \mathbf{u}_h$ and p_h into the space of discontinuous, piecewise linear functions and using backward Euler to approximate the time derivative. The variables $\mathbf{u}_h$, p_h, and the stress σ_h all converge with order 1 for the full space and time discretization scheme.

For the purpose of simulations, the domain is taken to be the unit square $\Omega = [0,1] \times [0,1]$ where the length of a side is $L = 1$ cm. The Dirichlet and Neumann sections of the boundary, Γ_0 and Γ_1 respectively, are given by

$$\begin{aligned} \Gamma_0 &= \{(x, y) \in \partial\Omega : x = 0, 1\}, \\ \Gamma_1 &= \{(x, y) \in \partial\Omega : y = 0, 1\}. \end{aligned}$$

Positions and displacements are normalized by the length scale $L = 1$ cm, stresses are normalized by the pressure scale $P = \mu$, and times are normalized by the

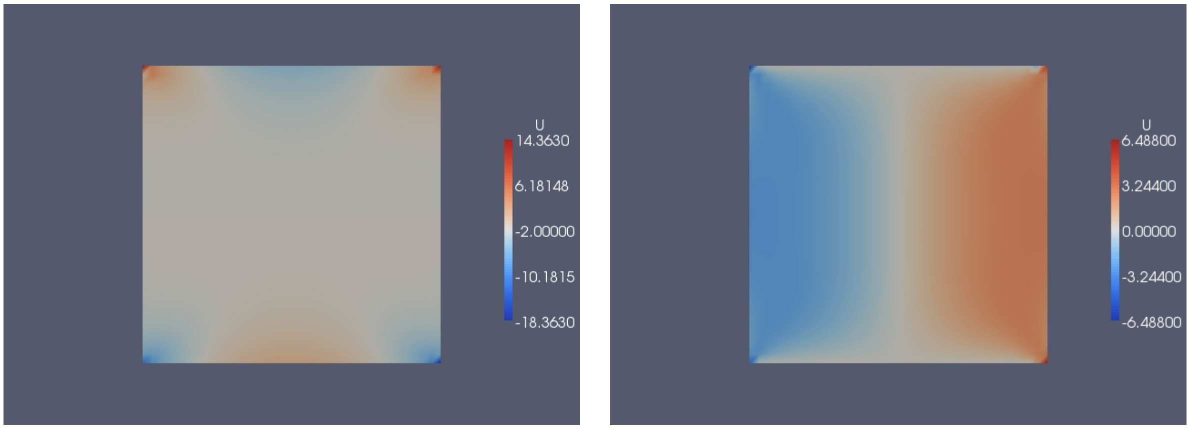

Figure 4.1: Left: Normal stress profile at time $t = 0.1$ s. Right: Shear stress profile at $t = 0.1$ s.

time scale $T = \frac{\eta_1}{\mu}$. The solid polymer under consideration in these simulations is polyethylene, so parameter values consistent with polyethylene are used. Following [33] and [52], the viscosity is 10^8 Pa-s, the shear modulus is 10^9 Pa, and the friction is 10^{12} Pa-s/m^2. Following the numerical experiments of Suo et al in [73], a traction of $(0, 2\mu)$ Pa is imposed where $y = 0$ and $(0, -2\mu)$ Pa is imposed where $y = 1$. The normalized initial displacement is taken to be $\mathbf{u}_0 = (0, -5\sin(\pi x))$. Notice that this vector function is divergence-free, which is required by the conservation of mass. A uniform mesh of 2048 triangles, each with height $h = 2^{-5}$, is taken. The time interval $[0, t]$ is divided into $N = 160$ equally spaced subintervals. Transient and equilibrium normal and shear stresses are computed. Figure 4.1 shows normal and shear stress profiles at time $t = 0.1$ s, which for these parameters equals the time scale $T = \frac{\eta_1}{\mu}$. Notice that both the normal and shear stress profiles have peaks at the corners, indicating boundary layer-type effects. The maximum and minimum stress values, which occur at the corners, grow unbounded as the mesh becomes finer, thus the Taylor-Hood algorithm is unreliable for calculating maximum stresses. This is not unexpected since there are no pointwise bounds for the stresses. However, established energy estimates require that the stresses

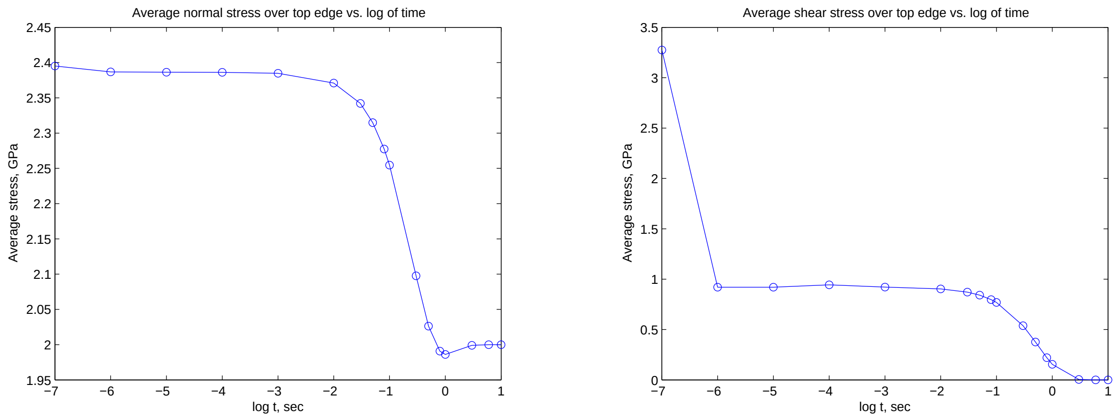

Figure 4.2: Left: Average normal stress over top edge vs. log time. Right: Average shear stress over top edge vs. log time.

be bounded in the L^2 norm. Thus, average stresses appropriately defined over $\partial\Omega$ should give a reliable measure of the maximum stress. For the sake of brevity, averages are reported only over the top edge of the square. The average normal and shear stresses over the top edge at time t are given, respectively, by the equations

$$\sigma_{yy}^{avg}(t) = \sqrt{\frac{1}{\int_{\{y=1\}} 1 ds} \int_{\{y=1\}} \sigma_{yy}^2(t) ds}, \ \sigma_{xy}^{avg}(t) = \sqrt{\frac{1}{\int_{\{y=1\}} 1 ds} \int_{\{y=1\}} \sigma_{xy}^2(t) ds}.$$

Figure 4.2 gives plots of average normal and shear stresses over the top edge of the unit square versus log of time. Notice that there is a fairly steep decrease in normal stress between $t = .01$ s and $t = 1$ s corresponding to $\log t = -2$ and $\log t = 0$ in the figure. This range contains the time scale of the problem $T = .1$ s. Thus, the time scale corresponds roughly to the time at which the normal stress relaxes to its equilibrium value of 2 GPa for this initial-boundary value problem. Note from the second plot in Figure 4.2 that the time scale also corresponds to the time at which the shear stress begins to relax to its equilibrium value of 0.

Chapter 5

Analysis of linearized gel equations

In this chapter, the equations of a gel (i.e. a two component mixture) will be studied in a general two or three dimensional domain. Earlier work on equilibrium equations for gels was carried out by Calderer et al [11]. Working in the gel reference domain, they gave an expression for the total energy of the gel and used variational methods to derive the Euler-Lagrange equations which govern the equilibrium behavior of the gel. It was shown that the given energy, which depends on polymer volume fraction and deformation gradient, has a minimizer in an appropriate function space under the constraint (2.1.16). The Euler-Lagrange system consists of an equation for the first Piola-Kirchhoff stress tensor of the gel and the equation of balance of linear momentum of the gel in material coordinates. The Piola-Kirchhoff stress, which using the constraint depends only on the deformation gradient and the reference polymer volume fraction, is not symmetric. This system is linearized about a constant tensor F_0, so the displacement is defined by the equation $F = F_0 + \nabla \mathbf{u}$. Due to the form of the elastic energy for an isotropic material, it is possible to choose F_0 in such a way that the linearized Piola-Kirchhoff stress is symmetric. In such a case, the elliptic operator is symmetric and standard elliptic methods can be used to analyze the problem. However, if $F_0 = I$, the resulting Piola-Kirchhoff stress is symmetric only for a single value of the reference polymer volume fraction. This nonsymmetry is found to be caused by the presense of a nonzero residual stress in the gel reference domain.

This chapter will focus on linearized dynamics. The model proposed in this thesis has been presented in Eulerian coordinates, so the (Cauchy) stress tensors will always be symmetric. All inertia terms will be neglected. In section 1, the case of an elastic polymer mixed with an inviscid solvent will be considered. Here it will be possible to solve (2.1.14) for fluid velocity $\mathbf{v}_2$ in terms of the other variables and thus remove this variable from the system. Using the same elastic energy as Calderer et al and following their approach, the system will be linearized about some tensor F_0 [11]. It is seen that when $F_0 = I$, residual stress in Eulerian coordinates acts as a body force on the system. The special case where F_0 represents a dilation or compression ($F_0 = f_0 I$ for $f_0 > 0$) will be considered in detail. The result will be a mixed parabolic-elliptic system for displacement $\mathbf{u}$ and pressure p. However, this system will involve third order mixed derivatives for $\mathbf{u}$, making the problem very difficult to solve. Following the work of Feng and He [22], a new auxiliary variable will be defined and existence and uniqueness of global-in-time weak solutions will be proved using Galerkin approximations. In a later chapter, the work of Feng and He will be used to define a mixed finite element method for the problem. Numerical stability and convergence of the method to the weak solution will be proved.

In section 2, the case of an elastic polymer mixed with a viscous fluid will be considered. In contrast to the inviscid case, it is not possible to solve explicitly for $\mathbf{v}_2$ in terms of the other variables. The elastic energy given by (2.2.47) will be taken and the resulting system will be linearized about the identity. This corresponds to the Eulerian formulation of gel with nonzero residual stress. Under appropriate initial and boundary conditions, existence and uniqueness of solutions will be proved using both Laplace transforms and Galerkin approximations.

5.1 Inviscid fluid linearized gel equations

In this section the time-dependent behavior of polymer gels is studied. The governing equations will be linearized and it will be assumed that $\eta_2 \ll \eta_1$ and $\mu_2 \ll \mu_1$, i.e. that the solvent is effectively inviscid.

Neglecting inertial terms, the governing equations for a gel consisting of a viscous, elastic polymer mixed with an inviscid fluid are

$$\phi_1 = (\det F)^{-1}\varphi_0, \tag{5.1.1}$$

$$\nabla \cdot [\phi_1 \tilde{\mathbf{v}}_1 + (1-\phi_1)\tilde{\mathbf{v}}_2] = 0, \tag{5.1.2}$$

$$\nabla \cdot [2\eta_1 \mathbf{D}(\tilde{\mathbf{v}}_1) + (\mu_1 \nabla \cdot \tilde{\mathbf{v}}_1 - \tilde{p})I + \phi_1 \frac{\partial W}{\partial F} F^T - (\pi_1 + \pi_2)I] = \mathbf{0}, \tag{5.1.3}$$

$$\tilde{\mathbf{v}}_2 = \tilde{\mathbf{v}}_1 - \frac{1}{\beta}\nabla \pi_2 - \frac{1}{\beta}(1-\phi_1)\nabla \tilde{p}, \tag{5.1.4}$$

$$\tilde{\mathbf{u}}_t = \tilde{\mathbf{v}}_1. \tag{5.1.5}$$

Here the fluid momentum equation has been used to solve for $\tilde{\mathbf{v}}_2$, which has been substituted into the solid equation. Notice that (5.1.3) is the linear momentum equation for the total mixture, not the polymer component only. The only nonlinear term in the total stress is $\phi_1 \frac{\partial W}{\partial F} F^T - (\pi_1 + \pi_2)I$. Note that (5.1.1) can be used to express the volume fraction in the deformed domain, ϕ_1, in terms of the deformation gradient F and the volume fraction φ_0 in the undeformed domain, which is assumed to be constant. Thus the nonlinear part of the stress can be expressed solely in terms of F. Moreover, it is shown in [11, Remark 2.1] that $\det F \geq \varphi_0$. Define the elastic energy by the equation

$$W(F) = \mu_E[\frac{1}{2s}(|F|^{2s} - |I|^{2s}) + \frac{|I|^{2(s-1)}}{r}((\det F)^{-r} - 1)],$$

where μ_E is the elastic modulus of the polymer and $r > 0$, $s \geq 1$ are dimensionless parameters. Here, $|F|^2 = \text{t}rFF^T$. Calderer et al [11] proved that this energy has a minimizer for $s > 1$. The osmotic pressures π_1 and π_2 are defined by (2.1.26), (2.1.27), (2.1.36), and (2.1.37). Define the tensor $\mathcal{S}$ to be the nonlinear part of the total Cauchy stress:

$$\mathcal{S} = \phi_1 \frac{\partial W}{\partial F} F^T - (\pi_1 + \pi_2)I.$$

Substituting (5.1.1) into the equations for the elastic energy and osmotic pressures, it is found that

$$\mathcal{S} = \nu(F)B - \kappa(F)I,$$

where $B = FF^T$ and

$$\begin{aligned} \nu(F) &= \varphi_0 \mu_E \frac{|F|^{2(s-1)}}{\det F}, \\ \kappa(F) &= \varphi_0 \mu_E |I|^{2(s-1)} (\det F)^{-r-1} + \frac{K_B T}{V_m}[\frac{\varphi_0}{N_1 \det F} + \frac{1}{N_2}(1 - \frac{\varphi_0}{\det F}) \\ & \quad -2\chi \frac{\varphi_0}{\det F}(1 - \frac{\varphi_0}{\det F})]. \end{aligned}$$

Note that $\mathcal{S}$ is symmetric for all F. Define the polymer displacement by the equation $\tilde{\mathbf{u}} = \mathbf{x} - F_0\mathbf{X}$ for any prescribed constant tensor F_0, so the deformation gradient is given by $F = F_0 + \nabla \mathbf{u}$. Linearize $\mathcal{S}$ about $F = F_0$, so

$$\mathcal{S}_{ij}(F) \approx \mathcal{S}_{ij}(F_0) + \frac{\partial \mathcal{S}_{ij}}{\partial F_{kl}}(F_0)\nabla \tilde{\mathbf{u}}_{kl},$$

where $\nabla \tilde{\mathbf{u}}_{kl} = \frac{\partial \tilde{\mathbf{u}}_k}{\partial \mathbf{x}_l}$. Calculating the derivatives, it can be shown that

$$\mathcal{S}(\nabla \tilde{\mathbf{u}}) \approx \mathcal{S}(F_0) + 2\nu(F_0)\overline{\mathbf{D}}(\tilde{\mathbf{u}}; F_0) + \text{tr}(F_0 \nabla \tilde{\mathbf{u}})A_1(F_0) + \text{tr}(F_0^{-1}\nabla \tilde{\mathbf{u}})A_2(F_0),$$

where

$$\overline{\mathbf{D}}(\tilde{\mathbf{u}}; F_0) = \frac{1}{2}[\nabla \tilde{\mathbf{u}} F_0^T + (\nabla \tilde{\mathbf{u}} F_0^T)^T]$$

is symmetric and $A_1(F_0)$ and $A_2(F_0)$ are constant symmetric tensors. Linearizing $(\det F)^{-1}$ and $\pi_2(F) := \pi_2(\frac{\varphi_0}{\det F})$ about $F = F_0$ gives

$$(\det F)^{-1} \approx (\det F_0)^{-1}[1 - \mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}})],$$
$$\pi_2 \approx \pi_2(\frac{\varphi_0}{\det F_0}) + \lambda_1(F_0)\mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}}),$$

where $\lambda_1(F_0)$ is a scalar constant that depends on F_0 and the parameters of the problem. The linearized gel equations are

$$\begin{aligned}
&\phi_1 = \frac{\varphi_0}{\det F_0}[1 - \mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}})],\\
&\nabla\cdot[\frac{\varphi_0}{\det F_0}\tilde{\mathbf{v}}_1 + (1 - \frac{\varphi_0}{\det F_0})\tilde{\mathbf{v}}_2] = 0,\\
&\nabla\cdot[2\eta_1\mathbf{D}(\tilde{\mathbf{v}}_1) + 2\nu(F_0)\overline{\mathbf{D}}(\tilde{\mathbf{u}};F_0) + (\mu_1\nabla\cdot\tilde{\mathbf{v}}_1 - \tilde{p})I + \mathcal{S}(F_0)\\
&+\mathrm{tr}(F_0\nabla\tilde{\mathbf{u}})A_1(F_0) + \mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}})A_2(F_0)] = \mathbf{0},\\
&\tilde{\mathbf{v}}_2 = \tilde{\mathbf{v}}_1 - \frac{1}{\beta}\nabla[\pi_2(F_0) + \lambda_1(F_0)\mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}})] - \frac{1}{\beta}(1 - \frac{\varphi_0}{\det F_0})\nabla\tilde{p}, \qquad (5.1.6)\\
&\tilde{\mathbf{u}}_t = \tilde{\mathbf{v}}_1.
\end{aligned}$$

Substituting $\tilde{\mathbf{v}}_2$ from (5.1.6) and $\tilde{\mathbf{u}}_t = \tilde{\mathbf{v}}_1$ into the other equations, the following linearized system is obtained:

$$\begin{aligned}
&\phi_1 = \frac{\varphi_0}{\det F_0}[1 - \mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}})],\\
&\nabla\cdot\{\tilde{\mathbf{u}}_t - \frac{1}{\beta}(1 - \frac{\varphi_0}{\det F_0})\nabla[\pi_2(F_0) + \lambda_1(F_0)\mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}})] \qquad (5.1.7)\\
&-\frac{1}{\beta}(1 - \frac{\varphi_0}{\det F_0})^2\nabla\tilde{p}\} = 0,\\
&\nabla\cdot[2\eta_1\mathbf{D}(\tilde{\mathbf{u}}_t) + 2\nu(F_0)\overline{\mathbf{D}}(\tilde{\mathbf{u}};F_0) + (\mu_1\nabla\cdot\tilde{\mathbf{u}}_t - \tilde{p})I + \mathcal{S}(F_0) \qquad (5.1.8)\\
&+\mathrm{tr}(F_0\nabla\tilde{\mathbf{u}})A_1(F_0) + \mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}})A_2(F_0)] = \mathbf{0}.
\end{aligned}$$

The following initial and boundary conditions are imposed:

- $\tilde{\mathbf{u}}|_{t=0} = \tilde{\mathbf{u}}_0$ with $\mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}}_0) = 1 - \det F_0$;
- $\tilde{\mathbf{u}}|_{\Gamma_0} = \tilde{\boldsymbol{U}}$, $\tilde{\mathbf{u}}_0|_{\Gamma_0} = \tilde{\boldsymbol{U}}|_{t=0}$;

- $(\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2) \cdot \mathbf{n}|_{\partial\Omega} = 0$;
- $[2\eta_1 \mathbf{D}(\tilde{\mathbf{u}}_t)_{ij} + 2\nu(F_0)\overline{\mathbf{D}}(\tilde{\mathbf{u}}; F_0)_{ij} + (\mu_1 \nabla \cdot \tilde{\mathbf{u}}_t - \tilde{p})\delta_{ij} + S(F_0)_{ij} + \mathrm{tr}(F_0 \nabla \tilde{\mathbf{u}}) A_1(F_0)_{ij} + \mathrm{tr}(F_0^{-1} \nabla \tilde{\mathbf{u}}) A_2(F_0)_{ij}] \mathbf{n}_j|_{\Gamma_1} = \mathbf{g}_i$.

The third condition given implies impermeability of the entire boundary to the solvent. Since the fluid viscosities are negligible, the fluid linear momentum equation is similar to Euler's equation. Therefore, only the normal component of the relative velocity need be specified in order to ensure well-posedness of the problem. Using (5.1.6), the impermeability condition becomes

$$\nabla[\pi_2(F_0) + \lambda_1(F_0)\mathrm{tr}(F_0^{-1}\nabla\tilde{\mathbf{u}}) + (1 - \frac{\varphi_0}{\det F_0})\tilde{p}] \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

In order to analyze this system, it must be rewritten as a system with homogeneous Dirichlet boundary conditions. This can be done because of the following standard generalization of a lemma of Ladyzhenskaya [34, p. 68]:

Lemma 5.1. *If* $\tilde{\boldsymbol{U}} \in H^{3/2}(\Gamma_0)$ *and* $\Gamma_0 \in C^2$, *then* $\exists \boldsymbol{U} \in H^2(\Omega)$ *such that* $\boldsymbol{U}|_{\Gamma_0} = \tilde{\boldsymbol{U}}$ *and* $||\boldsymbol{U}||_{H^2(\Omega)} \leq C||\tilde{\boldsymbol{U}}||_{H^{3/2}(\Gamma_0)}$.

Take $\tilde{\boldsymbol{U}} \in H^{3/2}(\Gamma_0)$ and $\Gamma_0 \in C^2$ and apply the lemma. Since $\tilde{\boldsymbol{U}}$ is independent of time, so is its extension $\boldsymbol{U}$. Define $\mathbf{u} = \tilde{\mathbf{u}} - \boldsymbol{U}$ and $\mathbf{u}_0 = \tilde{\mathbf{u}}_0 - \boldsymbol{U}|_{t=0}$. The system may be rewritten as

$$\phi_1 = \frac{\varphi_0}{\det F_0}[1 - \mathrm{tr}(F_0^{-1}\nabla(\mathbf{u} + \boldsymbol{U}))],$$

$$\nabla \cdot \{\mathbf{u}_t - \frac{1}{\beta}(1 - \frac{\varphi_0}{\det F_0})\nabla[\lambda_1(F_0)\mathrm{tr}(F_0^{-1}\nabla\mathbf{u}) + (1 - \frac{\varphi_0}{\det F_0})\tilde{p}]\} = = -\nabla \cdot \boldsymbol{U}_t + \nabla \cdot \{\frac{1}{\beta}(1 - \frac{\varphi_0}{\det F_0})\nabla[\lambda_1(F_0)\mathrm{tr}(F_0^{-1}\nabla\boldsymbol{U})]\} \tag{5.1.9}$$

$$\nabla \cdot [2\eta_1\mathbf{D}(\mathbf{u}_t) + 2\nu(F_0)\overline{\mathbf{D}}(\mathbf{u}; F_0) + (\mu_1\nabla \cdot \mathbf{u}_t - \tilde{p})I + \mathrm{tr}(F_0\nabla\mathbf{u})A_1(F_0) + \mathrm{tr}(F_0^{-1}\nabla\mathbf{u})A_2(F_0)] = -\nabla \cdot [2\eta_1\mathbf{D}(\boldsymbol{U}_t) + 2\nu(F_0)\overline{\mathbf{D}}(\boldsymbol{U}; F_0) + \mathcal{S}(F_0) \mu_1\nabla \cdot \boldsymbol{U}_t I + \mathrm{tr}(F_0\nabla\boldsymbol{U})A_1(F_0) + \mathrm{tr}(F_0^{-1}\nabla\boldsymbol{U})A_2(F_0)] \tag{5.1.10}$$

The rewritten initial and boundary conditions have the following form:

- $\mathbf{u}|_{t=0} = \mathbf{u}_0$ with $\mathrm{tr}(F_0^{-1}\nabla \mathbf{u}_0) = 1 - \det F_0 - \mathrm{tr}(F_0^{-1}\nabla \boldsymbol{U})|_{t=0}$;

- $\mathbf{u}|_{\Gamma_0} = \mathbf{0}$, $\mathbf{u}_0|_{\Gamma_0} = \mathbf{0}$;

- $\nabla[\lambda_1(F_0)\mathrm{tr}(F_0^{-1}\nabla \mathbf{u}) + (1 - \frac{\varphi_0}{\det F_0})\tilde{p}] \cdot \mathbf{n}|_{\partial\Omega} = -\nabla[\lambda_1(F_0)\mathrm{tr}(F_0^{-1}\nabla \boldsymbol{U})] \cdot \mathbf{n}|_{\partial\Omega}$;

- $[2\eta_1\mathbf{D}(\mathbf{u}_t)_{ij} + 2\nu(F_0)\overline{\mathbf{D}}(\mathbf{u}; F_0)_{ij} + (\mu_1\nabla \cdot \mathbf{u}_t - \tilde{p})\delta_{ij} + \mathrm{tr}(F_0\nabla\mathbf{u})A_1(F_0)_{ij} + \mathrm{tr}(F_0^{-1}\nabla\mathbf{u})A_2(F_0)_{ij}]\mathbf{n}_j|_{\Gamma_1} = \mathbf{g}_i - [2\eta_1\mathbf{D}(\boldsymbol{U}_t)_{ij} + 2\nu(F_0)\overline{\mathbf{D}}(\boldsymbol{U}; F_0)_{ij} + \mu_1\nabla \cdot \boldsymbol{U}_t\delta_{ij} + \mathrm{tr}(F_0\nabla\boldsymbol{U})A_1(F_0)_{ij} + \mathrm{tr}(F_0^{-1}\nabla\boldsymbol{U})A_2(F_0)_{ij} + \mathcal{S}(F_0)_{ij}]\mathbf{n}_j|_{\Gamma_1}$.

If $F_0 = I$, it is clear from (5.1.10) that the residual stress, $\mathcal{S}(I)$, plays the role of a body force.

5.1.1 Dilated-compressed reference domain

The case where $F_0 = f_0 I$ for $f_0 > 0$ is consistent with the behavior of gels since F_0 as a deformation gradient represents a dilation or contraction. The case of general F_0 requires a reorganization of the estimates. Note that for $F_0 = f_0 I$,

$$\overline{\mathbf{D}}(\mathbf{u}; F_0) = f_0\mathbf{D}(\mathbf{u}).$$

The system can be rewritten in the form

$$\begin{aligned}
&\phi_1 = \frac{\varphi_0}{f_0^d}[1 - f_0^{-d}\nabla \cdot (\mathbf{u} + \boldsymbol{U})],\\
&\nabla \cdot [\mathbf{u}_t - \kappa\nabla(\tilde{\alpha}\nabla \cdot \mathbf{u} + \gamma\tilde{p})] = -\nabla \cdot \mathbf{f}(\boldsymbol{U}),\\
&\nabla \cdot [2\eta_1\mathbf{D}(\mathbf{u}_t) + (\mu_1\nabla \cdot \mathbf{u}_t - \tilde{p} + \lambda_2\nabla \cdot \mathbf{u})I + 2\nu(F_0)f_0\mathbf{D}(\mathbf{u})] = -\nabla \cdot \Sigma(\boldsymbol{U}),
\end{aligned}$$

where κ, $\tilde{\alpha}$, γ, and λ_2 are scalar constants that depend on the parameter group $\frac{\varphi_0}{f_0^d}$, the friction coefficient β, and the parameters of the gel free energy given by (2.1.23). By Remark 2.1 of [11], it can be seen that κ and γ are positive. Moreover, assume that the free energy parameters, φ_0, and f_0 are chosen in such a way that $\tilde{\alpha}$ and λ_2 are both positive. $\Sigma(\boldsymbol{U})$ is a tensor defined by

$$\Sigma(\boldsymbol{U}) = 2\eta_1\mathbf{D}(\boldsymbol{U}_t) + (\mu_1\nabla \cdot \boldsymbol{U}_t + \lambda_2\nabla \cdot \boldsymbol{U})I + 2\nu(F_0)f_0\mathbf{D}(\boldsymbol{U}) + \mathcal{S}(F_0) \tag{5.1.11}$$

and $\mathbf{f}(\boldsymbol{U})$ is a vector function given by

$$\mathbf{f}(\boldsymbol{U}) = \boldsymbol{U}_t - \kappa\nabla(\tilde{\alpha}\nabla\cdot\boldsymbol{U}). \tag{5.1.12}$$

This system is difficult to solve numerically because of the third order derivative for $\mathbf{u}$ and because there is no initial condition provided for the pressure $\tilde{p}$. To overcome this difficulty, the problem is reformulated. Define

$$q := \nabla\cdot\mathbf{u}. \tag{5.1.13}$$

Physically, q measures the volume change of the solid network of the gel. Define the constant $\alpha = \tilde{\alpha} + \gamma\lambda_2$ and the new pressures

- $p := \tilde{p} - \lambda_2 q$,
- $\overline{p} = \gamma p + \alpha q$

Note that α is positive by the assumptions on $\tilde{\alpha}$ and λ_2. Using the new variable q and the new pressures p and $\overline{p}$, the system can be rewritten as

$$q_t - \kappa\Delta\overline{p} = -\nabla\cdot\mathbf{f}(\boldsymbol{U}), \tag{5.1.14}$$

$$\nabla\cdot[2\eta_1\mathbf{D}(\mathbf{u}_t) + 2\nu(F_0)f_0\mathbf{D}(\mathbf{u}) + (\mu_1 q_t - p)I] = -\nabla\cdot\Sigma(\boldsymbol{U}) \tag{5.1.15}$$

$$\nabla\cdot\mathbf{u} = q. \tag{5.1.16}$$

The initial and boundary conditions for this reformulation are

- $\mathbf{u}|_{t=0} = \mathbf{u}_0$;
- $q|_{t=0} = q_0 \equiv \nabla\cdot\mathbf{u}_0 = f_0 - f_0^{d+1} - \nabla\cdot\boldsymbol{U}|_{t=0}$;
- $\mathbf{u}|_{\Gamma_0} = \mathbf{0}$, $\mathbf{u}_0|_{\Gamma_0} = \mathbf{0}$;
- $\nabla\overline{p}\cdot\mathbf{n}|_{\partial\Omega} = -\nabla(\tilde{\alpha}\nabla\cdot\boldsymbol{U})\cdot\mathbf{n}|_{\partial\Omega}$;
- $[2\eta_1\mathbf{D}(\mathbf{u}_t)_{ij} + 2\nu(F_0)f_0\mathbf{D}(\mathbf{u})_{ij} + (\mu_1 q_t - p)\delta_{ij}]\mathbf{n}_j|_{\Gamma_1} = \mathbf{g}_i - \Sigma(\boldsymbol{U})_{ij}\mathbf{n}_j|_{\Gamma_1}$.

In order to obtain an energy law for this system, take the dot product of (5.1.15) and $\gamma\mathbf{u}_t$, multiply (5.1.14) by $\overline{p}$, integrate both equations by parts over Ω, apply boundary conditions, and add the result. Take the time derivative of (5.1.16), multiply it by $\gamma(\mu_1 q_t - p)$, integrate over Ω, and substitute into the sum. Integrating in time over $[0,T]$ for $T > 0$, the following energy law is obtained:

$$(5.1.17)\quad J(T) + \int_0^T \int_\Omega [\kappa|\nabla\overline{p}|^2 + 2\eta_1\gamma|\mathbf{D}(\mathbf{u}_t)|^2 + \mu_1\gamma q_t^2] = J(0) + \int_0^T \int_{\Gamma_1} \gamma\mathbf{g}\cdot\mathbf{u}_t$$
$$+ \int_0^T \int_\Omega [\gamma\Sigma(\boldsymbol{U})_{ij}\mathbf{D}(\mathbf{u}_t)_{ij} + \mathbf{f}(\boldsymbol{U})\cdot\nabla\overline{p}] - \int_0^T \int_{\partial\Omega} \overline{p}\boldsymbol{U}_t\cdot\mathbf{n},$$

where

$$(5.1.18)\qquad J(t) := \int_\Omega [\frac{1}{2}\alpha q^2(t) + \gamma\nu(F_0)f_0|\mathbf{D}(\mathbf{u}(t))|^2].$$

$\nu(F_0)$ is positive by the defining equation for ν since $\det F_0 > 0$. Notice that the last term on the right hand side of the energy law prohibits obtaining a uniform bound on the L^2 norms of the variables. Therefore, it will be assumed that the imposed displacement $\tilde{\boldsymbol{U}}$ on Γ_0 is independent of time. Then its extension $\boldsymbol{U}$ over Ω is also independent of time.

5.1.2 Pure displacement boundary conditions

In this subsection, only pure displacement and impermeability boundary conditions are considered. No normal stress is imposed anywhere on $\partial\Omega$, so $|\Gamma_1| = 0$. Notice that upon integrating (5.1.14) over Ω, integrating the right hand side by parts, and using the Neumann boundary condition on $\overline{p}$, the variable q satisfies the conservation law

$$\int_\Omega q_t = 0.$$

Therefore,

$$(5.1.19)\qquad \int_\Omega q(x,t) = \int_\Omega q(x,0) = \int_\Omega q_0,$$

which is a constant. Define the variable $\tilde{q} = q - (q)_\Omega$ where $(q)_\Omega = \frac{1}{|\Omega|}\int_\Omega q$. In order to ensure uniqueness of weak solutions, it is necessary to impose the condition

$$\int_\Omega \gamma\tilde{p} = -(\alpha - \lambda_2)(q)_\Omega$$

on the original pressure $\tilde{p}$ of the inviscid-solvent gel. Then $\int_\Omega \overline{p} = \int_\Omega [\gamma\tilde{p} + (\alpha - \lambda_2)(\tilde{q} + (q)_\Omega) = 0$. (5.1.14)-(5.1.16) can be rewritten in the form

$$\tilde{q}_t - \nabla \cdot (\kappa\nabla\overline{p}) = \nabla \cdot [\kappa\nabla(\tilde{\alpha}\nabla \cdot \boldsymbol{U})], \tag{5.1.20}$$

$$\nabla \cdot [2\eta_1 \mathbf{D}(\mathbf{u}_t) + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{u}) + (\mu_1\tilde{q}_t - p)I] = -\nabla \cdot \Sigma, \tag{5.1.21}$$

$$\nabla \cdot \mathbf{u} = \tilde{q} + (q)_\Omega, \tag{5.1.22}$$

satisfying the initial and boundary conditions

- $\mathbf{u}|_{t=0} = \mathbf{u}_0$, $\mathbf{u}_0|_{\partial\Omega} = \mathbf{0}$;
- $\tilde{q}|_{t=0} = 0$;
- $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$;
- $\nabla\overline{p} \cdot \mathbf{n}|_{\partial\Omega} = -\nabla(\tilde{\alpha}\nabla \cdot \boldsymbol{U}) \cdot \mathbf{n}|_{\partial\Omega}$.

With (5.1.17) in mind, define the following function spaces:

$$\begin{aligned} &\mathbf{W} = [H_0^1(\Omega)]^d, \\ &Q = \{\phi \in L^2(\Omega) : \int_\Omega \phi = 0\}, \\ &P = \{\psi \in H^1(\Omega) : \int_\Omega \psi = 0\}. \end{aligned}$$

Using these function spaces, define the concept of weak solutions of this system.

Definition 5.1. *For $0 < T < \infty$, a triple $(\mathbf{u}, \tilde{q}, \overline{p}) \in \mathbf{W} \times Q \times P$ is called a weak* *solution for a.e.* $t \in [0,T]$ *if for all* $(\mathbf{w}, \phi, \psi) \in \mathbf{W} \times Q \times P$,

$$\text{(5.1.23)} \quad \int_\Omega [2\eta_1 \mathbf{D}(\mathbf{u}_t)_{ij} + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{u})_{ij} + (\mu_1 \tilde{q}_t - \frac{1}{\gamma}\overline{p} + \frac{\alpha}{\gamma}\tilde{q})\delta_{ij}] \frac{\partial \mathbf{w}_i}{\partial x_j} = - \int_\Omega \tilde{\Sigma}_{ij} \frac{\partial \mathbf{w}_i}{\partial x_j},$$

$$\text{(5.1.24)} \quad \int_\Omega \phi \nabla \cdot \mathbf{u} = \int_\Omega \phi \tilde{q},$$

$$\text{(5.1.25)} \quad \int_\Omega \tilde{q}_t \psi + \int_\Omega \kappa \nabla \overline{p} \cdot \nabla \psi = \int_\Omega \mathbf{f} \cdot \nabla \psi,$$

$$\text{(5.1.26)} \quad \int_\Omega \mathbf{D}(\mathbf{u}(0))_{ij} \mathbf{D}(\mathbf{w})_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij} \mathbf{D}(\mathbf{w})_{ij},$$

$$\text{(5.1.27)} \quad \tilde{q}(0) = 0,$$

where $\tilde{\Sigma} = \Sigma + (q)_\Omega I$ *for* Σ *defined by (5.1.11) and* $\mathbf{f}$ *defined by (5.1.12) with* $\boldsymbol{U}_t \equiv 0$.

The following theorem holds:

Theorem 5.1. *Let* $\mathbf{u}_0 \in \mathbf{W}$, $\mathbf{f} \in L^2(0,T;L^2(\Omega))$, *and* $\tilde{\Sigma} \in L^2(0,T;H^1(\Omega))$. *Then* $\exists!(\mathbf{u}, \tilde{q}, \overline{p})$ *which is a weak solution. Also,*

$$\mathbf{u} \in L^\infty(0,T;H^1(\Omega)),\ \mathbf{u}_t \in L^2(0,T;H^1(\Omega))$$
$$\tilde{q} \in L^\infty(0,T;L^2(\Omega)),\ \tilde{q}_t \in L^2(0,T;L^2(\Omega))$$
$$\overline{p} \in L^2(0,T;H^1(\Omega)),\ p = \frac{1}{\gamma}(\overline{p} - \alpha\tilde{q} - \alpha(q)_\Omega) \in L^2(0,T;L^2(\Omega)).$$

Proof: The theorem is proved using the Faedo-Galerkin method. Note that $\mathbf{W}$ can be expressed as the orthogonal decomposition $\mathbf{W} = \mathbf{W}_1 \oplus \mathbf{W}_2$ where $\mathbf{W}_1$ is the set of all divergence-free vectors in $\mathbf{W}$ and $\mathbf{W}_2$ is its orthogonal complement under the inner product $\int_\Omega \mathbf{D}(\mathbf{w})_{ij}\mathbf{D}(\mathbf{v})_{ij}$ for $\mathbf{w}, \mathbf{v} \in \mathbf{W}$. $\mathbf{W}_1$ and $\mathbf{W}_2$ are seperable Hilbert spaces, so there exists a sequence of linearly independent smooth functions $\{\mathbf{w}^{(1),k}\}_{k=1}^\infty$ which is dense in $\mathbf{W}_1$ and there exists a sequence of linearly independent smooth functions $\{\mathbf{w}^{(2),k}\}_{k=1}^\infty$ which is dense in $\mathbf{W}_2$. The sequence $\{\mathbf{w}^{(1),k}, \mathbf{w}^{(2),k}\}_{k=1}^\infty$ forms a linearly independent dense set in $\mathbf{W}$. Define the sequence $\{\phi^k = \nabla \cdot \mathbf{w}^{(2),k}\}_{k=1}^{\infty}$. This sequence forms a linearly independent dense set in Q. Also, since $P \subseteq Q$, the sequence $\{\phi^k\}_{k=1}^{\infty}$ defines a linearly independent dense set in P as well. For any integer $N \geq 1$, define the finite dimensional Galerkin spaces $\mathbf{W}_N = span\{\mathbf{w}^{(1),k}, \mathbf{w}^{(2),k}\}_{k=1}^{N}$, $Q_N = span\{\phi^k\}_{k=1}^{N} := \nabla \cdot \mathbf{W}_N$, and $P_N = span\{\phi^k\}_{k=1}^{N} := Q_N$. Find $(\mathbf{u}^N, \tilde{q}^N, \overline{p}^N) \in \mathbf{W}_N \times Q_N \times P_N$ satisfying for all $(\mathbf{w}^N, \phi^N, \psi^N) \in \mathbf{W}_N \times Q_N \times P_N$ the equations

$$(5.1.28)\int_\Omega [2\eta_1 \mathbf{D}(\mathbf{u}_t^N)_{ij} + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{u}^N)_{ij} + (\mu_1 \tilde{q}_t^N - \frac{1}{\gamma}\overline{p}^N + \frac{\alpha}{\gamma}\tilde{q}^N)\delta_{ij}] \frac{\partial \mathbf{w}_i^N}{\partial x_j} = - \int_\Omega \tilde{\Sigma}_{ij} \mathbf{D}(\mathbf{w}^N)_{ij},$$

$$(5.1.29)\int_\Omega \phi^N \nabla \cdot \mathbf{u}^N = \int_\Omega \phi^N \tilde{q}^N,$$

$$(5.1.30)\int_\Omega \tilde{q}_t^N \psi^N + \int_\Omega \kappa \nabla \overline{p}^N \cdot \nabla \psi^N = \int_\Omega \mathbf{f} \cdot \nabla \psi^N,$$

$$(5.1.31)\int_\Omega \mathbf{D}(\mathbf{u}^N(0))_{ij} \mathbf{D}(\mathbf{w}^N)_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij} \mathbf{D}(\mathbf{w}^N)_{ij},$$

$$(5.1.32)\tilde{q}^N(0) = 0.$$

This leads to a system of linear ODEs in time for the coefficients of $\mathbf{u}^N$, $\tilde{q}^N$, and $\overline{p}^N$. By linear ODE theory, $\exists!(\mathbf{u}^N(\cdot,t), \tilde{q}^N(\cdot,t), \overline{p}^N(\cdot,t)) \in \mathbf{W}_N \times Q_N \times P_N$ for all $t \in [0,T]$. Take $\mathbf{w}^N = \gamma \mathbf{u}_t^N$, $\phi^N = \mu_1 \gamma \tilde{q}_t^N - \overline{p}^N + \alpha \tilde{q}^N$, and $\psi^N = \overline{p}^N$. The approximations satisfy the following energy law:

$$\int_\Omega [\gamma \nu(F_0) f_0 |\mathbf{D}(\mathbf{u}^N(T))|^2 + \frac{1}{2}\alpha(\tilde{q}^N(T))^2] + \int_0^T \int_\Omega [2\eta_1 \gamma |\mathbf{D}(\mathbf{u}_t^N)|^2 + \kappa |\nabla \overline{p}^N|^2 + \mu_1 \gamma (\tilde{q}_t^N)^2] = \int_\Omega [\gamma \nu(F_0) f_0 |\mathbf{D}(\mathbf{u}^N(0))|^2 + \frac{1}{2}\alpha(\tilde{q}^N(0))^2] + \int_0^T \int_\Omega [\mathbf{f} \cdot \nabla \overline{p}^N - \gamma \tilde{\Sigma}_{ij} \mathbf{D}(\mathbf{u}_t^N)_{ij}].$$

Using this energy law, initial conditions, and Poincaré's inequality, it can be shown that

- $\mathbf{u}^N$ is bounded uniformly (independent of N) in $L^\infty(0,T;H^1(\Omega))$,
- $\mathbf{u}_t^N$ is bounded uniformly in $L^2(0,T;H^1(\Omega))$,

- $\tilde{q}^N$ is bounded uniformly in $L^\infty(0,T;L^2(\Omega))$,
- $\tilde{q}_t^N$ is bounded uniformly in $L^2(0,T;L^2(\Omega))$,
- $\overline{p}^N$ is bounded uniformly in $L^2(0,T;H^1(\Omega))$.

After passing to subsequences if necessary, it can be seen that

- $\exists \mathbf{u} \in H^1(0,T;H^1(\Omega)) \bigcap L^\infty(0,T;H^1(\Omega))$ such that $\mathbf{u}^N \rightharpoonup \mathbf{u}$ in $H^1(0,T;$ $H^1(\Omega))$ and $\mathbf{u}^N \overset{*}{\rightharpoonup} \mathbf{u}$ in $L^\infty(0,T;H^1(\Omega))$,
- $\exists \tilde{q} \in H^1(0,T;L^2(\Omega)) \bigcap L^\infty(0,T;L^2(\Omega))$ such that $\tilde{q}^N \rightharpoonup \tilde{q}$ in $H^1(0,T;$ $L^2(\Omega))$ and $\tilde{q}^N \overset{*}{\rightharpoonup} \tilde{q}$ in $L^\infty(0,T;L^2(\Omega))$,
- $\exists \overline{p} \in L^2(0,T;H^1(\Omega))$ such that $\overline{p}^N \rightharpoonup \overline{p}$ in $L^2(0,T;H^1(\Omega))$.

Using these weak limits, it can be shown that the triple $(\mathbf{u}, \tilde{q}, \overline{p})$ is a weak solution of the system. Also, it can easily be seen that $p = \frac{1}{\gamma}(\overline{p} - \alpha\tilde{q} - \alpha(q)_\Omega) \in L^2(0,T;L^2(\Omega))$. Finally, uniqueness is a consequence of the energy law for weak solutions. $\square$

5.1.3 Displacement-traction boundary conditions

In this subsection, both displacement and normal stress are imposed on different parts of the boundary, so $|\Gamma_0| > 0$ and $|\Gamma_1| > 0$. Since the pressure p can be determined on the traction boundary Γ_1, there is no need to impose $\int_\Omega \overline{p} = 0$ on the system for these boundary conditions. For simplicity, take the imposed displacement $\tilde{\boldsymbol{U}}$ on Γ_0 to be identically zero, so $\boldsymbol{U} \equiv \mathbf{0} \equiv \mathbf{f}$ and $\Sigma = \mathcal{S}(F_0)$ in (5.1.14)-(5.1.16). The case of nonzero $\tilde{\boldsymbol{U}}$ is a straightforward generalization. The energy law becomes

$$(5.1.33)\quad J(T) + \int_0^T \int_\Omega [\kappa|\nabla\overline{p}|^2 + 2\eta_1\gamma|\mathbf{D}(\mathbf{u}_t)|^2 + \mu_1\gamma q_t^2] = J(0) + \int_0^T \int_{\Gamma_1} \gamma\mathbf{g}\cdot\mathbf{u}_t - \int_0^T \int_\Omega \gamma\mathcal{S}(F_0)_{ij}\frac{\partial \mathbf{u}_{ti}}{\partial x_j},$$

where $J(t)$ is defined by (5.1.18). With this in mind, define the following function spaces:

$$\begin{aligned}
&\mathbf{W} = \{\mathbf{w} \in [H^1(\Omega)]^d : \mathbf{w}|_{\Gamma_0} = 0\},\\
&Q = L^2(\Omega),\\
&P = H^1(\Omega).
\end{aligned}$$

In order to obtain existence of weak solutions, the following result is needed:

Lemma 5.2. *There exists* $\alpha_0 > 0$ *such that*

$$\sup_{\mathbf{w}\in\mathbf{W}} \frac{\int_\Omega \phi\nabla\cdot\mathbf{w}}{||\mathbf{D}(\mathbf{w})||_{L^2(\Omega)}} \geq \alpha_0||\phi||_{L^2(\Omega)} \quad \forall\phi \in Q. \tag{5.1.34}$$

Proof: The proof presented here is due to Sayas [2] but is a special case of the general LBB condition [9]. Since $Q = Q_0 \oplus \mathbf{R}$ (Q_0 is the orthogonal complement of $\mathbf{R}$ under the L^2 inner product) where $Q_0 = \{q \in Q : \int_\Omega q = 0\}$, it is clear that the inequality (5.1.34) is equivalent to

$$\sup_{\mathbf{w}\in\mathbf{W}} \frac{\int_\Omega(\phi_0 + c)\nabla\cdot\mathbf{w}}{||\mathbf{D}(\mathbf{w})||_{L^2(\Omega)}} \geq \alpha_0[||\phi_0||_{L^2(\Omega)} + |c|] \quad \forall\phi_0 \in Q_0,\ \forall c \in \mathbf{R}. \tag{5.1.35}$$

Define the function spaces

$$\begin{aligned}
\mathbf{W}_1 &= \{\mathbf{w} \in \mathbf{W} : \int_\Omega q_0\nabla\cdot\mathbf{w} = 0\forall q_0 \in Q_0\},\\
\mathbf{W}_2 &= \{\mathbf{w} \in \mathbf{W} : \int_\Omega c\nabla\cdot\mathbf{w} = 0\forall c \in \mathbf{R}\}.
\end{aligned}$$

By [27], (5.1.35) holds if and only if the following are valid:

1. There exists some $\alpha_1 > 0$ such that

$$\sup_{\mathbf{w}\in\mathbf{W}} \frac{\int_\Omega \phi_0\nabla\cdot\mathbf{w}}{||\mathbf{D}(\mathbf{w})||_{L^2(\Omega)}} \geq \alpha_1||\phi_0||_{L^2(\Omega)} \quad \forall\phi_0 \in Q_0,$$

2. There exists some $\alpha_2 > 0$ such that

$$\sup_{\mathbf{w}\in\mathbf{W}} \frac{\int_\Omega c\nabla\cdot\mathbf{w}}{||\mathbf{D}(\mathbf{w})||_{L^2(\Omega)}} \geq \alpha_2|c| \quad \forall c \in \mathbf{R},$$

3. $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2$.

Note that since $[H_0^1(\Omega)]^d \in \mathbf{W}$, the first item holds if

$$\sup_{\mathbf{w}\in[H_0^1(\Omega)]^d} \frac{\int_\Omega \phi_0 \nabla\cdot\mathbf{w}}{||\mathbf{D}(\mathbf{w})||_{L^2(\Omega)}} \geq \alpha_1 ||\phi_0||_{L^2(\Omega)} \forall \phi_0 \in Q_0.$$

This is a well-known result [28]. In order to prove the validity of the second item, note that for $\mathbf{w} \in \mathbf{W}$, $\int_\Omega c\nabla\cdot\mathbf{w} = \int_{\Gamma_1} c\mathbf{w}\cdot\mathbf{n}$. Thus the item holds if it is possible to find some $\mathbf{w} \in \mathbf{W}$ satisfying $\int_{\Gamma_1} \mathbf{w}\cdot\mathbf{n} \neq 0$. Assuming Γ_1 is at least Lipschitz, take $\tilde{\Gamma_1} \subset \Gamma_1$ with nonzero measure and a fixed vector $\mathbf{m} \in \mathbf{R}^d$ such that for some $\delta_0 > 0$,

$$\mathbf{m}\cdot\mathbf{n}(\mathbf{x}) \geq \delta_0$$

for a.e. $\mathbf{x} \in \tilde{\Gamma_1}$. Choose any $\varphi \in C^\infty(\Gamma_1)$ with $\varphi \geq 0$, supp$\varphi \subset \tilde{\Gamma_1}$, and $\int_{\tilde{\Gamma_1}} \varphi > 0$. The function $\varphi : \Gamma_1 \to \mathbf{R}$ can be lifted to an element $w \in H^1(\Omega)$ that has it as its trace on Γ_1. Take $\mathbf{w} = w\mathbf{m} \in \mathbf{W}$. Then

$$\int_{\Gamma_1} \mathbf{w}\cdot\mathbf{n} = \int_{\tilde{\Gamma_1}} \varphi\mathbf{m}\cdot\mathbf{n} \geq \delta_0 \int_{\tilde{\Gamma_1}} \varphi > 0.$$

Therefore the second item holds. Finally, for the third item, it must be shown that for any $\mathbf{w} \in \mathbf{W}$, $\exists \mathbf{w}_1 \in \mathbf{W}_1$ and $\exists \mathbf{w}_2 \in \mathbf{W}_2$ such that $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$. Note that $\mathbf{W}_1$ is equivalent to the set of vectors in $\mathbf{W}$ with constant divergence. Also, $\mathbf{W}_2$ is equivalent to the set of vectors in $\mathbf{W}$ with normal component on Γ_1 equal to 0. Since $[H_0^1(\Omega)]^d \subseteq \mathbf{W}_2$, select $\mathbf{w}_2 \in [H_0^1(\Omega)]^d$ satisfying $\nabla\cdot\mathbf{w}_2 = \nabla\cdot\mathbf{w} - \frac{1}{|\Omega|}\int_\Omega \nabla\cdot\mathbf{w}$. Set $\mathbf{w}_1 = \mathbf{w} - \mathbf{w}_2$. This implies that (5.1.35) holds and thus completes the proof of the lemma. $\square$

Using the function spaces given above, define the concept of weak solutions of (5.1.14)-(5.1.16).

Definition 5.2. *For* $0 < T < \infty$, *a triple* $(\mathbf{u}, q, \overline{p}) \in \mathbf{W} \times Q \times P$ *is called a weak* *solution for a.e.* $t \in [0, T]$ *if for all* $(\mathbf{w}, \phi, \psi) \in \mathbf{W} \times Q \times P$,

$$(5.1.36)\int_\Omega [2\eta_1 \mathbf{D}(\mathbf{u}_t)_{ij} + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{u})_{ij} + (\mu_1 q_t - \frac{1}{\gamma}\overline{p} + \frac{\alpha}{\gamma} q)\delta_{ij}] \frac{\partial \mathbf{w}_i}{\partial x_j} = \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{w} - \int_\Omega \mathcal{S}(F_0)_{ij} \frac{\partial \mathbf{w}_i}{\partial x_j},$$

$$(5.1.37)\int_\Omega \phi \nabla \cdot \mathbf{u} = \int_\Omega \phi q,$$

$$(5.1.38)\int_\Omega q_t \psi + \int_\Omega \kappa \nabla \overline{p} \cdot \nabla \psi = 0,$$

$$(5.1.39)\int_\Omega \mathbf{D}(\mathbf{u}(0))_{ij} \mathbf{D}(\mathbf{w})_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij} \mathbf{D}(\mathbf{w})_{ij},$$

$$(5.1.40)\int_\Omega q(0)\phi = \int_\Omega q_0 \phi.$$

Note that (5.1.36) combined with (5.1.34) and (5.1.33) implies that the pressure $\overline{p}$ is bounded in $L^2(0, T; H^1(\Omega))$.

Theorem 5.2. *Let* $\mathbf{u}_0 \in \mathbf{W}$ *and* $\mathbf{g} \in L^2(0, T; H^{1/2}(\Gamma_1))$. *Then* $\exists!(\mathbf{u}, q, \overline{p})$ *which is a weak solution. Also,*

$$\mathbf{u} \in L^\infty(0, T; H^1(\Omega)),\ \mathbf{u}_t \in L^2(0, T; H^1(\Omega))$$
$$q \in L^\infty(0, T; L^2(\Omega)),\ q_t \in L^2(0, T; L^2(\Omega))$$
$$\overline{p} \in L^2(0, T; H^1(\Omega)),\ p = \frac{1}{\gamma}(\overline{p} - \alpha q) \in L^2(0, T; L^2(\Omega)).$$

Proof: As in the pure displacement boundary condition case, the Faedo-Galerkin method is used to prove the theorem. Define finite dimensional subspaces $\mathbf{W}_N$, Q_N, and P_N of $\mathbf{W}$, Q, and P for $N \geq 1$ in the same way as in the pure displacement case. By [27] the following inf-sup condition holds: for the same $\alpha_0 > 0$ as in (5.1.34) and for each $N \geq 1$,

$$(5.1.41) \qquad \sup_{\mathbf{w}^N \in \mathbf{W}_N} \frac{\int_\Omega \phi^N \nabla \cdot \mathbf{w}^N}{||\mathbf{D}(\mathbf{w}^N)||_{L^2(\Omega)}} \geq \alpha_0 ||\phi^N||_{L^2(\Omega)} \quad \forall \phi^N \in Q_N.$$

Find $(\mathbf{u}^N, q^N, \overline{p}^N) \in \mathbf{W}_N \times Q_N \times P_N$ satisfying for all $(\mathbf{w}^N, \phi^N, \psi^N) \in \mathbf{W}_N \times Q_N \times$

P_N the equations

$$(5.1.42)\int_\Omega [2\eta_1 \mathbf{D}(\mathbf{u}_t^N)_{ij} + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{u}^N)_{ij} + (\mu_1 q_t^N - \frac{1}{\gamma}\overline{p}^N + \frac{\alpha}{\gamma} q^N)\delta_{ij}] \frac{\partial \mathbf{w}_i^N}{\partial x_j} =$$
$$\int_{\Gamma_1} \mathbf{g} \cdot \mathbf{w}^N - \int_\Omega \mathcal{S}(F_0)_{ij} \frac{\partial \mathbf{w}_i^N}{\partial x_j},$$
$$(5.1.43)\int_\Omega \phi^N \nabla \cdot \mathbf{u}^N = \int_\Omega \phi^N q^N,$$
$$(5.1.44)\int_\Omega q_t^N \psi^N + \int_\Omega \kappa \nabla \overline{p}^N \cdot \nabla \psi^N = 0,$$
$$(5.1.45)\int_\Omega \mathbf{D}(\mathbf{u}^N(0))_{ij} \mathbf{D}(\mathbf{w}^N)_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij} \mathbf{D}(\mathbf{w}^N)_{ij},$$
$$(5.1.46)\int_\Omega q^N(0)\phi^N = \int_\Omega q_0 \phi^N.$$

This leads to a system of linear ODEs in time for the coefficients of $\mathbf{u}^N$, q^N, and $\overline{p}^N$. By linear ODE theory, $\exists!(\mathbf{u}^N(\cdot,t), q^N(\cdot,t), \overline{p}^N(\cdot,t)) \in \mathbf{W}_N \times Q_N \times P_N$ for all $t \in [0,T]$. Take $\mathbf{w}^N = \gamma \mathbf{u}_t^N$, $\phi^N = \mu_1 \gamma q_t^N - \overline{p}^N + \alpha q^N$, and $\psi^N = p^N$. The approximations satisfy the following energy law:

$$\int_\Omega [\nu(F_0) f_0 \gamma |\mathbf{D}(\mathbf{u}^N(T))|^2 + \frac{1}{2}\alpha (q^N(T))^2] + \int_0^T \int_\Omega [2\eta_1 \gamma |\mathbf{D}(\mathbf{u}_t^N)|^2 + \kappa |\nabla p^N|^2$$
$$+ \mu_1 \gamma (q_t^N)^2] = \int_\Omega [\nu(F_0) f_0 \gamma |\mathbf{D}(\mathbf{u}^N(0))|^2 + \frac{1}{2}\alpha (q^N(0))^2] + \int_0^T \int_{\Gamma_1} \gamma \mathbf{g} \cdot \mathbf{u}_t^N$$
$$- \int_0^T \int_\Omega \gamma \mathcal{S}(F_0)_{ij} \frac{\partial \mathbf{u}_{ti}}{\partial x_j}.$$

Moreover, this energy law, (5.1.42), and the Galerkin inf-sup condition (5.1.41) imply that $\overline{p}^N$ is bounded independent of N in $L^2(0,T;H^1(\Omega))$. Using the energy law, the uniform $L^2(0,T;H^1(\Omega))$ bound on $\overline{p}^N$, initial conditions, and Poincaré's inequality, it can be shown that

- $\mathbf{u}^N$ is bounded uniformly (independent of N) in $L^\infty(0,T;H^1(\Omega))$,
- $\mathbf{u}_t^N$ is bounded uniformly in $L^2(0,T;H^1(\Omega))$,
- q^N is bounded uniformly in $L^\infty(0,T;L^2(\Omega))$,

- q_t^N is bounded uniformly in $L^2(0,T;L^2(\Omega))$,
- $\overline{p}^N$ is bounded uniformly in $L^2(0,T;H^1(\Omega))$.

After passing to subsequences if necessary, it can be seen that

- $\exists \mathbf{u} \in H^1(0,T;H^1(\Omega)) \bigcap L^\infty(0,T;H^1(\Omega))$ such that $\mathbf{u}^N \rightharpoonup \mathbf{u}$ in $H^1(0,T;H^1(\Omega))$ and $\mathbf{u}^N \overset{*}{\rightharpoonup} \mathbf{u}$ in $L^\infty(0,T;H^1(\Omega))$,
- $\exists q \in H^1(0,T;L^2(\Omega)) \bigcap L^\infty(0,T;L^2(\Omega))$ such that $q^N \rightharpoonup q$ in $H^1(0,T;L^2(\Omega))$ and $q^N \overset{*}{\rightharpoonup} q$ in $L^\infty(0,T;L^2(\Omega))$,
- $\exists \overline{p} \in L^2(0,T;H^1(\Omega))$ such that $\overline{p}^N \rightharpoonup \overline{p}$ in $L^2(0,T;H^1(\Omega))$.

Using these weak limits, it can be shown that the triple $(\mathbf{u}, q, \overline{p})$ is a weak solution of the system. Also, it can easily be seen that $p = \frac{1}{\gamma}(\overline{p} - \alpha q) \in L^2(0,T;L^2(\Omega))$. Finally, uniqueness of the weak solution follows from the uniqueness of weak limits. □

5.2 Viscous fluid linearized gel equations

In this section, the equations describing the behavior of a linearly elastic solid mixed with a viscous fluid are considered. Both components have Newtonian dissipation. Let $\tilde{\mathbf{v}}_1$ denote solid velocity and $\tilde{\mathbf{v}}_2$ denote fluid velocity. Neglecting inertia, the basic equations are

$$\begin{aligned}
&\phi_1 = \varphi_0(\det F)^{-1},\\
&\nabla \cdot (\phi_1 \tilde{\mathbf{v}}_1 + \phi_2 \tilde{\mathbf{v}}_2) = 0,\\
&\nabla \cdot [\phi_1 \frac{\partial W}{\partial F} F^T + 2\eta_1 \mathbf{D}(\tilde{\mathbf{v}}_1) + (\mu_1 \nabla \cdot \tilde{\mathbf{v}}_1 - \pi_1) I] - \phi_1 \nabla p - \beta(\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2) = 0,\\
&\nabla \cdot [2\eta_2 \mathbf{D}(\tilde{\mathbf{v}}_2) + (\mu_2 \nabla \cdot \tilde{\mathbf{v}}_2 - \pi_2) I] - \phi_2 \nabla p + \beta(\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2) = 0,
\end{aligned}$$

where $\varphi_0 \in (0,1)$ is the initial volume fraction. The elastic energy used in this expression is given by (2.2.47) and (2.2.48). As in the previous section, the osmotic pressures π_1 and π_2 are defined by (2.1.26), (2.1.27), (2.1.36), and (2.1.37). Define the tensor

$$\mathcal{S} = \phi_1 \frac{\partial W}{\partial F} F^T - \pi_1 I.$$

By the Lagrangian balance of mass equation, this tensor can be expressed as a function of F, and so can π_2. Let $\tilde{\mathbf{u}}$ denote the displacement of the solid. The solid velocity $\tilde{\mathbf{v}}_1$ then satisfies $\tilde{\mathbf{v}}_1 = \tilde{\mathbf{u}}_t$. Following the same procedure as in the inviscid fluid case and without loss of generality, the system is linearized about $F = I$. The equations become

$$\phi_1 = \varphi_0(1 - \nabla \cdot \tilde{\mathbf{u}}), \tag{5.2.47}$$

$$\nabla \cdot [\varphi_0 \tilde{\mathbf{u}}_t + (1 - \varphi_0)\tilde{\mathbf{v}}_2] = 0, \tag{5.2.48}$$

$$\begin{aligned} &\nabla \cdot [(\mu_1 \nabla \cdot \tilde{\mathbf{u}}_t + (r(I) - 2\mu\varphi_0)\nabla \cdot \tilde{\mathbf{u}} - \varphi_0 p)I + 4\mu\varphi_0 \mathbf{D}(\tilde{\mathbf{u}}) \\ &+2\eta_1 \mathbf{D}(\tilde{\mathbf{u}}_t) + \mathcal{S}(I)] - \beta(\tilde{\mathbf{u}}_t - \tilde{\mathbf{v}}_2) = \mathbf{0}, \end{aligned} \tag{5.2.49}$$

$$\begin{aligned} &\nabla \cdot [(\mu_2 \nabla \cdot \tilde{\mathbf{v}}_2 - \lambda_1(I)\nabla \cdot \tilde{\mathbf{u}} - (1 - \varphi_0)p - \pi_2(I))I + 2\eta_2 \mathbf{D}(\tilde{\mathbf{v}}_2)] \\ &+\beta(\tilde{\mathbf{u}}_t - \tilde{\mathbf{v}}_2) = \mathbf{0}. \end{aligned} \tag{5.2.50}$$

μ is the shear modulus of the solid and $r(F)$, $\lambda_1(F)$ are scalar functions of the deformation gradient that depend on φ_0 and the parameters of the elastic energy and the free energies of the components.

In order for this system of equations to be well-posed, initial and boundary conditions must be applied. Impose the initial condition

$$\tilde{\mathbf{u}}|_{t=0} = \tilde{\mathbf{u}}_0$$

for $\tilde{\mathbf{u}}_0$ some known displacement. Since φ_0 is the initial volume fraction of the solid, consistency requires that

$$\nabla \cdot \tilde{\mathbf{u}}_0 = 0.$$

Let the bounded open set $\Omega \subset \mathbf{R}^3$ be the domain of the mixture. Divide $\partial\Omega$ into two subsets Γ_0 and Γ_1 such that $|\Gamma_0| > 0$, $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Here $|\Gamma_0|$

denotes the Lebesque measure of Γ_0 and $\overline{\Gamma}_0$ denotes the closure of Γ_0. Dirichlet conditions are imposed on both $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}_2$ on Γ_0. The component velocities are assumed to be equal on Γ_1, i.e. the mixture is impermeable at Γ_1. Also, the traction of the mixture is imposed on Γ_1. In symbols, the initial and boundary conditions are

- $\tilde{\mathbf{u}}|_{t=0} = \tilde{\mathbf{u}}_0$ with $\nabla \cdot \tilde{\mathbf{u}}_0 = 0$;
- $\tilde{\mathbf{u}}|_{\Gamma_0} = \tilde{\boldsymbol{U}}$, $\tilde{\mathbf{v}}_2|_{\Gamma_0} = \tilde{\boldsymbol{V}}$, $(\tilde{\mathbf{u}}_t - \tilde{\mathbf{v}}_2)|_{\Gamma_1} = \mathbf{0}$;
- $[\mathcal{S}_{ij}(I) + 2\eta_1 \mathbf{D}(\tilde{\mathbf{u}}_t)_{ij} + 2\eta_2 \mathbf{D}(\tilde{\mathbf{v}}_2)_{ij} + 4\mu\varphi_0 \mathbf{D}(\tilde{\mathbf{u}})_{ij} + \{\mu_1 \nabla \cdot \tilde{\mathbf{u}}_t + \mu_2 \nabla \cdot \tilde{\mathbf{v}}_2 - \pi_2(I) + (r(I) - 2\mu\varphi_0 - \lambda_1(I))\nabla \cdot \tilde{\mathbf{u}} - p\}\delta_{ij}]\mathbf{n}_j|_{\Gamma_1} = \tilde{\mathbf{g}}_i$.

Here, $\tilde{\boldsymbol{U}}$, $\tilde{\boldsymbol{V}}$ are prescribed functions on Γ_0, $\mathbf{n}$ is the outward normal vector on Γ_1, and $\tilde{\mathbf{g}}$ is a prescribed function on Γ_1. For consistency, the condition $\tilde{\boldsymbol{U}}|_{t=0} = \tilde{\mathbf{u}}_{\mathbf{0}}|_{\Gamma_0}$ is imposed on the displacement. Notice that if $\tilde{\boldsymbol{U}}$ or $\tilde{\boldsymbol{V}}$ is nonzero, the Dirichlet conditions are inhomogeneous. The problem is transformed to the analogous homogeneous Dirichlet problem by the following lemma, which is a standard generalization of results of Ladyzhenskaya [34, Ch. 1 Sec. 2]:

Lemma 5.3. *Assume* $\partial\Omega \in C^1$, $\tilde{\boldsymbol{U}}(t) \in H^{1/2}(\Gamma_0)$ *is differentiable with respect to* t *in the sense of distributions,* $\tilde{\boldsymbol{V}}(t) \in H^{1/2}(\Gamma_0)$, *and* $\int_{\Gamma_0}[\varphi_0 \tilde{\boldsymbol{U}}_t + (1-\varphi_0)\tilde{\boldsymbol{V}}] \cdot \mathbf{n} = 0$ *for a.e.* $t \geq 0$. *Then* $\exists\, \boldsymbol{U}(t) \in H^1(\Omega)$ *differentiable with respect to* t *in the sense of distributions and* $\exists\, \boldsymbol{V}(t) \in H^1(\Omega)$ *for a.e.* $t \geq 0$ *such that* $\boldsymbol{U}|_{\Gamma_0} = \tilde{\boldsymbol{U}}$, $\boldsymbol{V}|_{\Gamma_0} = \tilde{\boldsymbol{V}}$, $\boldsymbol{V}|_{\Gamma_1} = \boldsymbol{U}_t|_{\Gamma_1}$, $\nabla \cdot \boldsymbol{U}|_{t=0} = \mathbf{0}$, *and* $\nabla \cdot [\varphi_0 \boldsymbol{U}_t + (1-\varphi_0)\boldsymbol{V}_2] = 0$ *for a.e.* $t \geq 0$.

Define $\mathbf{u} = \tilde{\mathbf{u}} - \boldsymbol{U}$, $\mathbf{v} = \tilde{\mathbf{v}}_2 - \boldsymbol{V}$, and $\mathbf{u}_0 = \tilde{\mathbf{u}}_0 - \boldsymbol{U}|_{t=0}$. Also, define the tensors

$$
\begin{aligned}
\mathcal{F}(\boldsymbol{U}) &= [(r(I) - 2\mu\varphi_0)\nabla \cdot \boldsymbol{U} + \mu_1 \nabla \cdot \boldsymbol{U}_t]I + 4\mu\varphi_0 \mathbf{D}(\boldsymbol{U}) \\
&\quad + 2\eta_1 \mathbf{D}(\boldsymbol{U}_t) + \mathcal{S}(I), \\
\mathcal{G}(\boldsymbol{U}, \boldsymbol{V}) &= 2\eta_2 \mathbf{D}(\boldsymbol{V}) + [\mu_2 \nabla \cdot \boldsymbol{V} - \lambda_1(I)\nabla \cdot \boldsymbol{U} - \pi_2(I)]I.
\end{aligned}
$$

The PDE system can be expressed in the form

$$(5.2.51) \qquad \nabla\cdot[\varphi_0\mathbf{u}_t+(1-\varphi_0)\mathbf{v}]=0,$$

$$(5.2.52) \qquad \frac{\partial}{\partial x_j}[(\mu_1\nabla\cdot\mathbf{u}_t+(r(I)-2\mu\varphi_0)\nabla\cdot\mathbf{u}-\varphi_0 p)\delta_{ij}+4\mu\varphi_0\mathbf{D}(\mathbf{u})_{ij}$$
$$+2\eta_1\mathbf{D}(\mathbf{u}_t)_{ij}]-\beta((\mathbf{u}_t)_i-\mathbf{v}_i)=-\frac{\partial}{\partial x_j}\mathcal{F}_{ij}(\boldsymbol{U})+\beta(\boldsymbol{U}_t-\boldsymbol{V})_i,$$

$$(5.2.53) \qquad \frac{\partial}{\partial x_j}[(\mu_2\nabla\cdot\mathbf{v}-\lambda_1(I)\nabla\cdot\mathbf{u}-(1-\varphi_0)p)\delta_{ij}+2\eta_2\mathbf{D}(\mathbf{v})_{ij}]$$
$$+\beta((\mathbf{u}_t)_i-\mathbf{v}_i)=-\frac{\partial}{\partial x_j}\mathcal{G}_{ij}(\boldsymbol{U},\boldsymbol{V})-\beta(\boldsymbol{U}_t-\boldsymbol{V})_i.$$

The initial and boundary conditions satisfy

- $\mathbf{u}|_{t=0}=\mathbf{u}_0$ with $\nabla\cdot\mathbf{u}_0=0$, $\mathbf{u}_0|_{\Gamma_0}=\mathbf{0}$;
- $\mathbf{u}|_{\Gamma_0}=\mathbf{0}$, $\mathbf{v}|_{\Gamma_0}=\mathbf{0}$, $(\mathbf{u}_t-\mathbf{v})|_{\Gamma_1}=\mathbf{0}$;
- $[(\mu_1\nabla\cdot\mathbf{u}_t+(r(I)-2\mu\varphi_0-\lambda_1(I))\nabla\cdot\mathbf{u}-p+\mu_2\nabla\cdot\mathbf{v})\delta_{ij}+4\mu\varphi_0\mathbf{D}(\mathbf{u})_{ij}+2\eta_1\mathbf{D}(\mathbf{u}_t)_{ij}+2\eta_2\mathbf{D}(\mathbf{v})_{ij}]\mathbf{n}_j|_{\Gamma_1}=\tilde{\mathbf{g}}_i-\mathcal{F}_{ij}(\boldsymbol{U})-\mathcal{G}_{ij}(\boldsymbol{U},\boldsymbol{V})$.

5.2.1 Energy estimates

In this subsection, energy estimates for $\mathbf{u}$ and $\mathbf{v}$ are derived. Consider $\mathbf{w}_1$, $\mathbf{w}_2$ such that $\mathbf{w}_1,\mathbf{w}_2|_{\Gamma_0}=\mathbf{0}$, $(\mathbf{w}_1-\mathbf{w}_2)|_{\Gamma_1}=\mathbf{0}$, and $\nabla\cdot[\varphi_0\mathbf{w}_1+(1-\varphi_0)\mathbf{w}_2]=0$ in Ω. Take the dot products of $\mathbf{w}_1$ with (5.2.52) and $\mathbf{w}_2$ with (5.2.53), sum them, integrate over Ω, and impose boundary conditions. The weak formulation of the problem is

$$(5.2.54) \int_\Omega[(r(I)-2\mu\varphi_0+\frac{\varphi_0\lambda_1(I)}{1-\varphi_0})(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{w}_1)+4\mu\varphi_0\mathbf{D}(\mathbf{u})_{ij}\mathbf{D}(\mathbf{w}_1)_{ij}$$
$$+\mu_1(\nabla\cdot\mathbf{u}_t)(\nabla\cdot\mathbf{w}_1)]+\int_\Omega[2\eta_1\mathbf{D}(\mathbf{u}_t)_{ij}\mathbf{D}(\mathbf{w}_1)_{ij}+\mu_2(\nabla\cdot\mathbf{v})(\nabla\cdot\mathbf{w}_2)$$
$$+2\eta_2\mathbf{D}(\mathbf{v})_{ij}\mathbf{D}(\mathbf{w}_2)_{ij}+\beta(\mathbf{u}_t-\mathbf{v})\cdot(\mathbf{w}_1-\mathbf{w}_2)]=\int_{\Gamma_1}\tilde{\mathbf{g}}\cdot\mathbf{w}_2$$
$$-\int_\Omega[\mathcal{F}_{ij}(\boldsymbol{U})\mathbf{D}(\mathbf{w}_1)_{ij}+\mathcal{G}_{ij}(\boldsymbol{U},\boldsymbol{V})\mathbf{D}(\mathbf{w}_2)_{ij}+\beta(\boldsymbol{U}_t-\boldsymbol{V})\cdot(\mathbf{w}_1-\mathbf{w}_2)].$$

Setting $\mathbf{w}_1 = \mathbf{u}_t$, $\mathbf{w}_2 = \mathbf{v}$, and integrating in time over the interval $[0, T]$ for $T > 0$, the energy equation becomes

$$\int_\Omega [\frac{1}{2}(r(I) - 2\mu\varphi_0 + \frac{\varphi_0\lambda_1(I)}{1-\varphi_0})(\nabla \cdot \mathbf{u}(T))^2 + 2\mu\varphi_0|\mathbf{D}(\mathbf{u}(T))|^2]$$
$$+ \int_0^T \int_\Omega [\mu_2(\nabla \cdot \mathbf{v})^2 + 2\eta_2|\mathbf{D}(\mathbf{v})|^2 + \mu_1(\nabla \cdot \mathbf{u}_t)^2 + 2\eta_1|\mathbf{D}(\mathbf{u}_t)|^2 + \beta|\mathbf{u}_t - \mathbf{v}|^2]$$
$$= \int_\Omega 2\mu\varphi_0|\mathbf{D}(\mathbf{u}_0)|^2 + \int_0^T \int_{\Gamma_1} \tilde{\mathbf{g}} \cdot \mathbf{v} - \int_0^T \int_\Omega [\mathcal{F}_{ij}(\boldsymbol{U})\mathbf{D}(\mathbf{u}_t)_{ij}$$
$$+\mathcal{G}_{ij}(\boldsymbol{U}, \boldsymbol{V})\mathbf{D}(\mathbf{v})_{ij} + \beta(\boldsymbol{U}_t - \boldsymbol{V}) \cdot (\mathbf{u}_t - \mathbf{v})],$$

where the initial condition on $\mathbf{u}$ has been imposed. Using Poincaré's inequality and standard embedding theorems (see [14]), the following L^2 inequality for $\mathbf{u}_t$ and $\mathbf{v}$ is obtained:

$$\int_\Omega [\frac{1}{2}(r(I) - 2\mu\varphi_0 + \frac{\varphi_0\lambda_1(I)}{1-\varphi_0})(\nabla \cdot \mathbf{u}(T))^2 + 2\mu\varphi_0|D(\mathbf{u}(T))|^2]$$
$$+ \int_0^T \int_\Omega [\mu_2(\nabla \cdot \mathbf{v})^2 + (2\eta_2 - \epsilon_1)|\mathbf{D}(\mathbf{v})|^2 + \mu_1(\nabla \cdot \mathbf{u}_t)^2$$
$$+(2\eta_1 - \epsilon_2)|D(\mathbf{u}_t)|^2 + \beta(1-\epsilon_3)|\mathbf{u}_t - \mathbf{v}|^2] \leq \int_\Omega 2\mu\varphi_0|D(\mathbf{u}_0)|^2$$
$$+C[||\tilde{\mathbf{g}}||^2_{L^2(0,T;H^{1/2}(\Gamma_1))} + ||\mathcal{F}||^2_{L^2(0,T;L^2(\Omega))} + ||\mathcal{G}||^2_{L^2(0,T;L^2(\Omega))}$$
$$+||\boldsymbol{U}_t - \boldsymbol{V}||^2_{L^2(0,T;L^2(\Omega))}]. \quad (5.2.55)$$

Here, $C > 0$ and $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ are chosen so that $2\eta_2 - \epsilon_1 > 0$, $2\eta_1 - \epsilon_2 > 0$, and $1 - \epsilon_3 > 0$.

5.2.2 Laplace transform and existence of weak solution

Assume now that $\tilde{\boldsymbol{U}} \in W^{1,\infty}(0,\infty; H^{1/2}(\Gamma_0))$ and $\tilde{\boldsymbol{V}} \in L^\infty(0,\infty; H^{1/2}(\Gamma_0))$. Then $\boldsymbol{U} \in W^{1,\infty}(0,\infty; H^1(\Omega))$ and $\boldsymbol{V} \in L^\infty(0,\infty; H^1(\Omega))$. Assume $\tilde{\mathbf{g}} \in L^\infty(0,\infty; H^{1/2}(\Gamma_1))$. Following Arendt [4], Laplace transforms with respect to time $\mathcal{F}_\tau$ and $\mathcal{G}_\tau$ of $\mathcal{F}$ and $\mathcal{G}$, respectively, exist, are analytic in the transform parameter $\tau \in \mathbf{R}$, and take values in $L^2(\Omega)$ provided $\mathbf{u}_0 \in H^1(\Omega)$ and $\tau > 0$. Also, the Laplace transform with respect to time $\tilde{\mathbf{g}}_\tau$ of $\tilde{\mathbf{g}}$ exists, is analytic in τ, and takes values in $H^{1/2}(\Gamma_1)$ provided $\tau > 0$. Take the Laplace transform in time of (5.2.51)-(5.2.53) and of the boundary conditions, let $\mathbf{U}_\tau$, $\overline{\mathbf{V}}_\tau$, and P_τ denote the transforms of $\mathbf{u}$, $\mathbf{v}$, and p, respectively, and set $\mathbf{V}_\tau = \overline{\mathbf{V}}_\tau + \mathbf{u}_0$. The system of equations becomes

(5.2.56) $\nabla \cdot [\varphi_0 \tau \mathbf{U}_\tau + (1-\varphi_0)\mathbf{V}_\tau] = 0,$

(5.2.57) $$\frac{\partial}{\partial x_j}[((\mu_1\tau + r(I) - 2\mu\varphi_0)\nabla\cdot\mathbf{U}_\tau - \varphi_0 P_\tau)\delta_{ij} + (4\mu\varphi_0 + 2\eta_1\tau)\mathbf{D}(\mathbf{U})_{ij}]$$
$$-\beta(\tau\mathbf{U}_\tau - \mathbf{V}_\tau)_i = \frac{\partial}{\partial x_j}[2\eta_1\mathbf{D}(\mathbf{u}_0)_{ij} - \mathcal{F}_\tau(\boldsymbol{U})_{ij}] + \beta\mathcal{L}[(\boldsymbol{U}_t - \boldsymbol{V})_i],$$

(5.2.58) $$\frac{\partial}{\partial x_j}[(\mu_2\nabla\cdot\mathbf{V}_\tau - \lambda_1(I)\nabla\cdot\mathbf{U}_\tau - (1-\varphi_0)P_\tau)\delta_{ij} + 2\eta_2\mathbf{D}(\mathbf{V}_\tau)_{ij}]$$
$$+\beta(\tau\mathbf{U}_\tau - \mathbf{V}_\tau)_i = \frac{\partial}{\partial x_j}[2\eta_2\mathbf{D}(\mathbf{u}_0)_{ij} - \mathcal{G}_\tau(\boldsymbol{U},\,\boldsymbol{V})_{ij}] - \beta\mathcal{L}[(\boldsymbol{U}_t - \boldsymbol{V})_i],$$

where here, $\mathcal{L}[\cdot]$ denotes the Laplace transform of the argument. The boundary conditions are

- $\mathbf{U}_\tau|_{\Gamma_0} = \mathbf{0}$, $\mathbf{V}_\tau|_{\Gamma_0} = \mathbf{0}$, $(\tau\mathbf{U}_\tau - \mathbf{V}_\tau)|_{\Gamma_1} = \mathbf{0}$;
- $[((\mu_1\tau + r(I) - 2\mu\varphi_0 - \lambda_1(I))\nabla\cdot\mathbf{U}_\tau + \mu_2\nabla\cdot\mathbf{V}_\tau - P_\tau)\delta_{ij} + (4\mu\varphi_0 + 2\eta_1\tau)\mathbf{D}(\mathbf{U}_\tau)_{ij}$ $+2\eta_2\mathbf{D}(\mathbf{V}_\tau)_{ij}]\mathbf{n}_j|_{\Gamma_1} = (\tilde{\mathbf{g}}_\tau)_i - [\mathcal{F}_\tau(\boldsymbol{U})_{ij} + \mathcal{G}_\tau(\boldsymbol{U},\,\boldsymbol{V})_{ij} - 2(\eta_1+\eta_2)\mathbf{D}(\mathbf{u}_0)_{ij}]\mathbf{n}_j|_{\Gamma_1}$.

Define the function space

$$\mathbf{W}_\tau = \{(\mathbf{w}_1, \mathbf{w}_2) \in H^1(\Omega) \times H^1(\Omega) : \mathbf{w}_1, \mathbf{w}_2|_{\Gamma_0} = \mathbf{0};$$
$$(\tau\mathbf{w}_1 - \mathbf{w}_2)|_{\Gamma_1} = \mathbf{0}; \nabla\cdot[\varphi_0\tau\mathbf{w}_1 + (1-\varphi_0)\mathbf{w}_2] = 0\}.$$

$\mathbf{W}_\tau$ is a subspace of $H^1(\Omega)\times H^1(\Omega)$. Take $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{W}_\tau$ and define the bilinear form

$$B[(\mathbf{U}_\tau, \mathbf{V}_\tau), (\mathbf{w}_1, \mathbf{w}_2)] = \int_\Omega [(\mu_1\tau + r(I) - 2\mu\varphi_0 + \frac{\varphi_0\lambda_1(I)}{1-\varphi_0})\tau(\nabla\cdot\mathbf{U}_\tau)(\nabla\cdot\mathbf{w}_1)$$
$$+(4\mu\varphi_0 + 2\eta_1\tau)\tau\mathbf{D}(\mathbf{U}_\tau)_{ij}\mathbf{D}(\mathbf{w}_1)_{ij} + \mu_2(\nabla\cdot\mathbf{V}_\tau)(\nabla\cdot\mathbf{w}_2)$$
$$+2\eta_2\mathbf{D}(\mathbf{V}_\tau)_{ij}\mathbf{D}(\mathbf{w}_2)_{ij} + \beta(\tau\mathbf{U}_\tau - \mathbf{V}_\tau)\cdot(\tau\mathbf{w}_1 - \mathbf{w}_2)].$$

Define the linear form

$$L[(\mathbf{w}_1, \mathbf{w}_2)] = \int_{\Gamma_1} \tilde{\mathbf{g}}_\tau \cdot \mathbf{w}_2 + \int_\Omega [2\mathbf{D}(\mathbf{u}_0)_{ij}(\eta_1 \tau \mathbf{D}(\mathbf{w}_1)_{ij} + \eta_2 \mathbf{D}(\mathbf{w}_2)_{ij})$$
$$-\mathcal{F}_\tau(\boldsymbol{U})_{ij} \tau \mathbf{D}(\mathbf{w}_1)_{ij} - \mathcal{G}_\tau(\boldsymbol{U}, \boldsymbol{V})_{ij} \mathbf{D}(\mathbf{w}_2)_{ij} - \beta \mathcal{L}[(\boldsymbol{U}_t - \boldsymbol{V})] \cdot (\tau \mathbf{w}_1 - \mathbf{w}_2)]$$
$$- \int_{\Gamma_1} 2\mathbf{D}(\mathbf{u}_0)_{ij} \mathbf{n}_j [\tau \mathbf{w}_{1i} \eta_1 + \mathbf{w}_{2i} \eta_2].$$

For $\tau > 0$ and $r(I) - 2\mu\varphi_0 + \mu_1\tau + \frac{\varphi_0 \lambda_1(I)}{1-\varphi_0} > 0$, the bilinear form B defines an inner product on $\mathbf{W}_\tau$. Also, for $\tilde{\mathbf{g}}_\tau \in H^{1/2}(\Gamma_1,\ \boldsymbol{U} \in W^{1,\infty}(0, \infty; H^1(\Omega)),\ \boldsymbol{V} \in L^\infty(0, \infty; H^1(\Omega))$, and $\mathbf{u}_0 \in H^2(\Omega)$, L defines a bounded linear functional on $\mathbf{W}_\tau$ for $\tau > 0$ [14]. Define a weak solution of the system to be any $(\mathbf{U}_\tau, \mathbf{V}_\tau) \in \mathbf{W}_\tau$ such that $B[(\mathbf{U}_\tau, \mathbf{V}_\tau), (\mathbf{w}_1, \mathbf{w}_2)] = L[(\mathbf{w}_1, \mathbf{w}_2)] \forall (\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{W}_\tau$. By the Riesz representation theorem, there exists a unique weak solution $(\mathbf{U}_\tau, \mathbf{V}_\tau) \in \mathbf{W}_\tau$. Let ${\mathbf{W}_\tau}'$ denote the dual space of $\mathbf{W}_\tau$. Let the linear operator $A : \mathbf{W}_\tau \rightarrow {\mathbf{W}_\tau}'$ define the linear functional $< A[(\mathbf{U}_\tau, \mathbf{V}_\tau)], (\mathbf{w}_1, \mathbf{w}_2) >:= B[(\mathbf{U}_\tau, \mathbf{V}_\tau), (\mathbf{w}_1, \mathbf{w}_2)]$. A is a bounded linear operator. Since the bilinear form B is analytic in τ for $\tau > 0$, so is A. Since the weak solution is unique in $\mathbf{W}_\tau$, A is invertible. The weak solution is determined by the equation $A[(\mathbf{U}_\tau, \mathbf{V}_\tau)] = L$, so $(\mathbf{U}_\tau, \mathbf{V}_\tau) = A^{-1}L$. Following Dautray and Lions [16], A^{-1} is analytic in τ for $\tau > 0$. Therefore, $(\mathbf{U}_\tau, \mathbf{V}_\tau)$ is analytic in τ for $\tau > 0$ since L is analytic in τ for $\tau > 0$. Following Arendt et al [4], the inverse Laplace transforms $\mathbf{u}$ and $\mathbf{v}$ of $\mathbf{U}$ and $\overline{\mathbf{V}}_\tau = \mathbf{V}_\tau - \mathbf{u}_0$ exist and are in $L^\infty(0, \infty; H^1(\Omega))$.

5.2.3 Galerkin approximations and weak solutions

As in the case of solid elasticity, it is possible to prove the existence of global-in-time weak solutions of (5.2.51)-(5.2.53) using Galerkin approximations. Assume that for some $T > 0$, $\boldsymbol{U} \in H^1(0, T; H^{1/2}(\Gamma_0))$ and $\boldsymbol{V} \in L^2(0, T; H^{1/2}(\Gamma_0))$. Then $\mathcal{F}$, $\mathcal{G} \in L^2(0, T; L^2(\Omega))$. The energy estimate above motivates the following theorem.

Theorem 5.3. *Assume* $\mathbf{u}_0 \in H^1(\Omega)$, $\mathcal{F}, \mathcal{G} \in L^2(0,T;L^2(\Omega))$, *and* $\tilde{\mathbf{g}} \in L^2(0,T;$ $H^{1/2}(\Gamma_1))$. *Also assume that* $r(I) - 2\mu\varphi_0 + \frac{\varphi_0\lambda_1(I)}{1-\varphi_0} > 0$. *Then the system of equations (5.2.51)-(5.2.53) has a unique weak solution* $\mathbf{u} \in H^1(0,T;H^1(\Omega))$, $\mathbf{v} \in L^2(0,T;H^1(\Omega))$ *which exists up to time* T.

Proof: Define the function space

$$\mathbf{W} = \{(\mathbf{w}_1, \mathbf{w}_2) \in H^1(\Omega) \times H^1(\Omega) : \mathbf{w}_1, \mathbf{w}_2|_{\Gamma_0} = \mathbf{0};$$
$$(\mathbf{w}_1 - \mathbf{w}_2)|_{\Gamma_1} = \mathbf{0}; \nabla \cdot [\varphi_0 \mathbf{w}_1 + (1-\varphi_0)\mathbf{w}_2] = 0\}.$$

Define a weak solution to be any $(\mathbf{u}, \mathbf{v})$ with $(\mathbf{u}_t, \mathbf{v}) \in \mathbf{W}$ that satisfies (5.2.54) for all $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{W}$. Since $\mathbf{W}$ is seperable, there exists a sequence of linearly independent functions $\{(\phi_k^1, \phi_k^2)\}_{k=1}^{\infty}$ which is dense in $\mathbf{W}$. For $N \geq 1$, define the finite dimensional subspace $\mathbf{W}_N = span\{(\phi_k^1, \phi_k^2)\}_{k=1}^{N}$. Let $(\mathbf{u}^N(t,\cdot), \mathbf{v}^N(t,\cdot)) \in \mathbf{W}_N$ satisfy for all $(\mathbf{w}_1, \mathbf{w}_2) \in \mathbf{W}_N$ the equations

$$(5.2.59) \int_\Omega [(r(I) - 2\mu\varphi_0 + \frac{\varphi_0\lambda_1(I)}{1-\varphi_0})(\nabla \cdot \mathbf{u}^N)(\nabla \cdot \mathbf{w}_1) + 4\mu\varphi_0 \mathbf{D}(\mathbf{u}^N)_{ij}\mathbf{D}(\mathbf{w}_1)_{ij}]$$
$$+ \int_\Omega [\mu_1(\nabla \cdot \mathbf{u}_t^N)(\nabla \cdot \mathbf{w}_1) + 2\eta_1 \mathbf{D}(\mathbf{u}_t^N)_{ij}\mathbf{D}(\mathbf{w}_1)_{ij} + \mu_2(\nabla \cdot \mathbf{v}^N)(\nabla \cdot \mathbf{w}_2)]$$
$$+ \int_\Omega [2\eta_2 \mathbf{D}(\mathbf{v}^N)_{ij}\mathbf{D}(\mathbf{w}_2)_{ij} + \beta(\mathbf{u}_t^N - \mathbf{v}^N)\cdot(\mathbf{w}_1 - \mathbf{w}_2)] = \int_{\Gamma_1} \tilde{\mathbf{g}} \cdot \mathbf{w}_2$$
$$- \int_\Omega [\mathcal{F}_{ij}(\boldsymbol{U})\mathbf{D}(\mathbf{w}_1)_{ij} + \mathcal{G}_{ij}(\boldsymbol{U}, \boldsymbol{V})\mathbf{D}(\mathbf{w}_2)_{ij} + \beta(\boldsymbol{U}_t - \boldsymbol{V})\cdot(\mathbf{w}_1 - \mathbf{w}_2)],$$
$$(5.2.60) \int_\Omega \mathbf{D}(\mathbf{u}^N(0,x))_{ij}\mathbf{D}(\mathbf{w}_1)_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij}\mathbf{D}(\mathbf{w}_1)_{ij}.$$

Since $(\mathbf{u}^N(t,\cdot), \mathbf{v}^N(t,\cdot)) \in \mathbf{W}_N$, it can be expressed as a linear combination of basis functions with coefficients depending on t. These expressions are substituted into the above equations and a linear system of ODEs in time with a full set of initial conditions is obtained by setting $(\mathbf{w}_1, \mathbf{w}_2)$ equal to each of the basis functions. By standard ODE theory, $\exists(\mathbf{u}^N(t), \mathbf{v}^N(t)) \in \mathbf{W}_N$ for $t \in [0,T]$. Note that $(\mathbf{u}_t^N, \mathbf{v}^N) \in \mathbf{W}_N$ as well.

To get an energy estimate on the approximations, take $(\mathbf{w}_1, \mathbf{w}_2) = (\mathbf{u}_t^N, \mathbf{v}^N)$ in (5.2.59). Integrating over time and using standard inequalities and embedding theorems, it can be shown that

$$\int_\Omega [\frac{1}{2}(r(I) - 2\mu\varphi_0 + \frac{\varphi_0\lambda_1(I)}{1-\varphi_0})(\nabla \cdot \mathbf{u}^N(T))^2 + 2\mu\varphi_0|\mathbf{D}(\mathbf{u}^N(T))|^2]$$
$$+ \int_0^T \int_\Omega [\mu_1(\nabla \cdot \mathbf{u}_t^N)^2 + \eta_1|\mathbf{D}(\mathbf{u}_t^N)|^2 + \mu_2(\nabla \cdot \mathbf{v}^N)^2 + \eta_2|\mathbf{D}(\mathbf{v}^N)|^2$$
$$+\frac{1}{2}\beta|\mathbf{u}_t^N - \mathbf{v}^N|^2] \le C[||\mathbf{D}(\mathbf{u}_0)||^2_{L^2(\Omega)} + ||\tilde{\mathbf{g}}||^2_{L^2(0,T;H^{1/2}(\Gamma_1))} + ||\mathcal{F}||^2_{L^2(0,T;L^2(\Omega))}$$
$$+||\mathcal{G}||^2_{L^2(0,T;L^2(\Omega))} + ||\boldsymbol{U}_t - \boldsymbol{V}||^2_{L^2(0,T;L^2(\Omega))}].$$

The right hand side of this inequality is bounded uniformly in N. Thus, it is clear that the following is true:

- $\mathbf{u}^N$ is uniformly bounded in $L^\infty(0,T;H^1(\Omega))$,
- $\mathbf{u}_t^N$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$,
- $\mathbf{v}^N$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$.

Since $T < \infty$ and Ω is bounded, $\mathbf{u}^N$ is uniformly bounded in $H^1(0,T;$ $H^1(\Omega))$. Therefore, after possibly passing to subsequences, $\exists \mathbf{u} \in H^1(0,T;H^1(\Omega))$ $\bigcap L^\infty(0,T;H^1(\Omega))$, $\exists \mathbf{v} \in L^2(0,T;H^1(\Omega))$ such that $\mathbf{u}^N \rightharpoonup \mathbf{u}$ in $H^1(0,T;H^1(\Omega))$, $\mathbf{v}^N \rightharpoonup \mathbf{v}$ in $L^2(0,T;H^1(\Omega))$, and $\mathbf{u}^N \overset{*}{\rightharpoonup} \mathbf{u}$ in $L^\infty(0,T;H^1(\Omega))$.

Define the function space

$$C_W^\infty(\Omega) = \{(\mathbf{w}_1, \mathbf{w}_2) \in C^\infty(\Omega) \times C^\infty(\Omega) : \mathbf{w}_1, \mathbf{w}_2|_{\Gamma_0} = \mathbf{0},$$
$$\mathbf{w}_1 - \mathbf{w}_2|_{\Gamma_1} = \mathbf{0}, \nabla \cdot [\varphi_0\mathbf{w}_1 + (1-\varphi_0)\mathbf{w}_2] = 0\}.$$

Let $(\mathbf{w}_1, \mathbf{w}_2) \in C_0^\infty([0,T), C_W^\infty(\Omega))$. By denseness, $\exists\{(\mathbf{w}_1^N, \mathbf{w}_2^N)\}_{N=1}^\infty$ with $(\mathbf{w}_1^N, \mathbf{w}_2^N) \in C^1([0,T];\mathbf{W}_N)$ such that $(\mathbf{w}_1^N, \mathbf{w}_2^N) \to (\mathbf{w}_1, \mathbf{w}_2)$ in $C^1([0,T];$ $W^{1,\delta}(\Omega) \times W^{1,\delta}(\Omega))$ for $\delta \ge 1$. Select $(\mathbf{w}_1^N, \mathbf{w}_2^N)(T) \equiv (\mathbf{0}, \mathbf{0})$. After integrating by parts with respect to time, it is clear that the Galerkin approximations satisfy

$$\int_0^T \int_\Omega [(r(I) - 2\mu\varphi_0 + \frac{\varphi_0\lambda_1(I)}{1-\varphi_0})(\nabla\cdot\mathbf{u}^N)(\nabla\cdot\mathbf{w}_1^N) + 4\mu\varphi_0\mathbf{D}(\mathbf{u}^N)_{ij}\mathbf{D}(\mathbf{w}_1^N)_{ij}$$
$$+\mu_2(\nabla\cdot\mathbf{v}^N)(\nabla\cdot\mathbf{w}_2^N) + 2\eta_2\mathbf{D}(\mathbf{v}^N)_{ij}\mathbf{D}(\mathbf{w}_2^N)_{ij} - \beta\mathbf{v}^N\cdot(\mathbf{w}_1^N - \mathbf{w}_2^N)$$
$$-\beta\mathbf{u}^N\cdot(\mathbf{w}_{1_t}^N - \mathbf{w}_{2_t}^N) - \mu_1(\nabla\cdot\mathbf{u}^N)(\nabla\cdot\mathbf{w}_{1_t}^N) - 2\eta_1\mathbf{D}(\mathbf{u}^N)_{ij}\mathbf{D}(\mathbf{w}_{1_t}^N)_{ij}] =$$
$$\int_\Omega [\beta\mathbf{u}^N(0)\cdot\mathbf{w}^N(0) + 2\eta_1 D(\mathbf{u}^N(0))_{ij}\mathbf{D}(\mathbf{w}^N(0))_{ij}] + \int_0^T\int_{\Gamma_1}\tilde{\mathbf{g}}\cdot\mathbf{w}_2^N$$
$$-\int_0^T\int_\Omega [\mathcal{F}_{ij}(\boldsymbol{U})\mathbf{D}(\mathbf{w}_1^N)_{ij} + \mathcal{G}_{ij}(\boldsymbol{U},\boldsymbol{V})\mathbf{D}(\mathbf{w}_2^N)_{ij} + \beta(\boldsymbol{U}_t - \boldsymbol{V})\cdot(\mathbf{w}_1^N - \mathbf{w}_2^N)].$$

Taking the infinite limit in N, it is clear that the weak and weak-$*$ limits satisfy

$$\int_0^T \int_\Omega [(r(I) - 2\mu\varphi_0 + \frac{\varphi_0\lambda_1(I)}{1-\varphi_0})(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{w}_1) + 4\mu\varphi_0\mathbf{D}(\mathbf{u})_{ij}\mathbf{D}(\mathbf{w}_1)_{ij}$$
$$+\mu_2(\nabla\cdot\mathbf{v})(\nabla\cdot\mathbf{w}_2) + 2\eta_2\mathbf{D}(\mathbf{v})_{ij}\mathbf{D}(\mathbf{w}_2)_{ij} - \beta\mathbf{v}\cdot(\mathbf{w}_1 - \mathbf{w}_2)$$
$$-\beta\mathbf{u}\cdot(\mathbf{w}_{1_t} - \mathbf{w}_{2_t}) - \mu_1(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{w}_{1_t}) - 2\eta_1\mathbf{D}(\mathbf{u})_{ij}\mathbf{D}(\mathbf{w}_{1_t})_{ij}] =$$
$$\int_\Omega [\beta\mathbf{u}_0\cdot\mathbf{w}(0) + 2\eta_1 D(\mathbf{u}_0)_{ij}\mathbf{D}(\mathbf{w}(0))_{ij}] + \int_0^T\int_{\Gamma_1}\tilde{\mathbf{g}}\cdot\mathbf{w}_2$$
$$-\int_0^T\int_\Omega [\mathcal{F}_{ij}(\boldsymbol{U})\mathbf{D}(\mathbf{w}_1)_{ij} + \mathcal{G}_{ij}(\boldsymbol{U},\boldsymbol{V})\mathbf{D}(\mathbf{w}_2)_{ij} + \beta(\boldsymbol{U}_t - \boldsymbol{V})\cdot(\mathbf{w}_1 - \mathbf{w}_2)].$$

This equation holds $\forall(\mathbf{w}_1, \mathbf{w}_2) \in C^1([0,T]; W^{1,\delta}(\Omega)\times W^{1,\delta}(\Omega))$. In particular, it holds $\forall(\mathbf{w}_1,\mathbf{w}_2) \in C^1([0,T];\mathbf{W})$. Therefore, $\mathbf{u} \in H^1(0,T;H^1(\Omega)) \bigcap L^\infty(0,T; H^1(\Omega))$, $\mathbf{v} \in L^2(0,T;H^1(\Omega))$ is a weak solution.

To prove uniqueness, it suffices to take $\mathbf{u}_0 \equiv \mathbf{0} \equiv \mathcal{F} \equiv \mathcal{G} \equiv \boldsymbol{U} \equiv \boldsymbol{V}$ and show that $\mathbf{u} = 0$, $\mathbf{v} = 0$ a.e. in $[0,T]\times\Omega$. Weak solutions in this case satisfy the equation

$$\int_\Omega [4\mu\varphi_0\mathbf{D}(\mathbf{u})_{ij}\mathbf{D}(\mathbf{w}_1)_{ij} + 2\eta_1\mathbf{D}(\mathbf{u}_t)_{ij}\mathbf{D}(\mathbf{w}_1)_{ij} + \mu_1(\nabla\cdot\mathbf{u}_t)(\nabla\cdot\mathbf{w}_1)$$
$$+(r(I) - 2\mu\varphi_0 + \frac{\varphi_0\lambda_1(I)}{1-\varphi_0})(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{w}_1) + \mu_2(\nabla\cdot\mathbf{v})(\nabla\cdot\mathbf{w}_2)$$
$$+2\eta_2\mathbf{D}(\mathbf{v})_{ij}\mathbf{D}(\mathbf{w}_2)_{ij} + \beta(\mathbf{u}_t - \mathbf{v})\cdot(\mathbf{w}_1 - \mathbf{w}_2)] = 0$$

$\forall(\mathbf{w}_1,\mathbf{w}_2) \in W$ and a.e. $t \in [0,T]$. Take $\mathbf{w}_1 = \mathbf{u}_t(t)$, $\mathbf{w}_2 = \mathbf{v}(t)$ for some time t

where this equation is satisfied. Integrate in time over $[0, T]$.

$$\int_\Omega [2\mu\varphi_0 |\mathbf{D}(\mathbf{u}(T))|^2 + \frac{1}{2}(r(I) - 2\mu\varphi_0 + \frac{\varphi_0 \lambda_1(I)}{1-\varphi_0})(\nabla \cdot \mathbf{u}(T))^2]$$
$$+ \int_0^T \int_\Omega [2\eta_1 |\mathbf{D}(\mathbf{u}_t)|^2 + 2\eta_2 |\mathbf{D}(\mathbf{v})|^2 + \beta |\mathbf{u}_t - \mathbf{v}|^2 + \mu_1 (\nabla \cdot \mathbf{u}_t)^2$$
$$+ \mu_2 (\nabla \cdot \mathbf{v})^2] = 0.$$

From this it is clear that $\mathbf{u}_t = 0 = \mathbf{v}$ a.e. in $[0, T] \times \Omega$, so $\mathbf{u}$ is constant a.e. in $[0, T] \times \Omega$. Since $\mathbf{u}_0 \equiv 0$ in Ω by assumption and $\mathbf{u} \in C([0, T]; H^1(\Omega))$ by standard embedding theorems, $\mathbf{u} = 0$ a.e. in $[0, T] \times \Omega$. This proves uniqueness and the theorem. □

Chapter 6

Numerical method for linearized gel equations

In this chapter, numerical methods are defined for solving the inviscid solvent linearized gel equations (5.1.20)-(5.1.22) for the pure displacement boundary problem and (5.1.14)-(5.1.16) for mixed displacement-traction boundary conditions. For simplicity, it is assumed in this chapter that $\Omega \subseteq \mathbf{R}^d$ is polygonal if $d = 2$ or polyhedral if $d = 3$. The numerical method described here was first proposed by Feng and He in [22] and used to solve Doi's stress-diffusion coupling model. At each time step, Feng and He were able to decouple the problem of solving for displacement and pressure, which satisfy a Stokes-type system, from the problem of computing q, which satisfies a diffusion equation. However, their algorithm is only conditionally convergent. The time step size must be restricted. When this method is applied to the equations of a gel composed of a polymer with Newtonian viscosity, the algorithm is unconditionally convergent. Unfortunately, this comes with the price of not being able to decouple the Stokes solver from the diffusion solver. A mixed finite element method is used to discretize the system in space. Approximations $\mathbf{u}_h$ and $\overline{p}_h$ for the displacement $\mathbf{u}$ and the pressure $\overline{p}$, respectively, are sought using the Taylor-Hood method ([61], [8]). A diffusion solver is used to solve for approximations $\tilde{q}_h$ of $\tilde{q}$ in (5.1.23)-(5.1.25) and q_h of q in (5.1.36)-(5.1.38). The semi-discrete method is shown to be well-posed, unconditionally numerically stable, and convergent to the weak solution. Rates of convergence with respect to the mesh size h are derived. The backward Euler finite difference method is used to discretize in time, giving a fully discrete algorithm for solving the problem. This algorithm is shown to be well-posed, unconditionally stable, and convergent to the solution of the semi-discrete problem, and thus convergent to the weak solution. Rates of convergence are derived with respect to the step size Δt^n at time step $n \geq 1$. Section 1 describes the details and convergence theory for the pure displacement problem while Section 2 covers the displacement/traction problem.

6.1 Discretization scheme for pure displacement problem

Referring back to Chabpter 5, it can be seen that $\mathbf{u}$ and $\overline{p}$ satisfy a Stokes-type system and $\tilde{q}$ satisfies a diffusion-type equation. Therefore, in order to solve (5.1.20)-(5.1.22) numerically, Taylor-Hood elements will be used to approximate $\mathbf{u}$ and $\overline{p}$ and a convergent diffusion solver will be used to approximate $\tilde{q}$. These two solvers (Taylor-Hood and diffusion) are coupled because one objective is to define an unconditionally stable numerical method. Let Ω_h be a quasi-uniform triangulation of Ω with mesh size h. Define the finite element spaces

$$
\begin{aligned}
&\mathbf{W}_h = \{\mathbf{w}_h \in [C^0(\overline{\Omega})]^d : \mathbf{w}_h|_K \in [P^2(K)]^d \forall K \in \Omega_h;\ \mathbf{w}_h|_{\partial\Omega} = \mathbf{0}\},\\
&P_h = \{\phi_h \in C^0(\overline{\Omega}) : \phi_h|_K \in P^1(K) \forall K \in \Omega_h;\ \int_\Omega \phi_h = 0\},\\
&Q_h = P_h.
\end{aligned}
$$

Q_h can actually be chosen to be any piecewise polynomial space as long as $P_h \subseteq Q_h$. Define a semi-discrete mixed finite element algorithm as follows: Find $(\mathbf{u}_h, \overline{p}_h, \tilde{q}_h) : (0,T] \to \mathbf{W}_h \times P_h \times Q_h$ and $(\mathbf{u}_h(0), \tilde{q}_h(0)) \in \mathbf{W}_h \times Q_h$ such that for all $(\mathbf{w}_h, \phi_h, \psi_h) \in \mathbf{W}_h \times Q_h \times P_h$,

$$(6.1.1)\int_\Omega [2\eta_1 \mathbf{D}(\frac{\partial \mathbf{u}_h}{\partial t})_{ij} + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{u}_h)_{ij} + (\mu_1 \frac{\partial \tilde{q}_h}{\partial t} - \frac{1}{\gamma}\overline{p}_h + \frac{\alpha}{\gamma}\tilde{q}_h)\delta_{ij}] \frac{\partial (\mathbf{w}_h)_i}{\partial x_j} = -\int_\Omega \tilde{\Sigma}_{ij} \frac{\partial \mathbf{w}_{hi}}{\partial x_j},$$

$$(6.1.2)\int_\Omega \phi_h \nabla \cdot \mathbf{u}_h = \int_\Omega \phi_h \tilde{q}_h,$$

$$(6.1.3)\int_\Omega \frac{\partial \tilde{q}_h}{\partial t}\psi_h + \int_\Omega \kappa \nabla \overline{p}_h \cdot \nabla \psi_h = \int_\Omega \mathbf{f} \cdot \nabla \psi_h,$$

$$(6.1.4)\int_\Omega \mathbf{D}(\mathbf{u}_h(0))_{ij} \mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij} \mathbf{D}(\mathbf{w}_h)_{ij},$$

$$(6.1.5)\int_\Omega \tilde{q}_h(0)\phi_h = 0.$$

This algorithm leads to a linear system of ODEs in time with a complete set of initial conditions. By standard linear ODE theory, it can be seen that there exists a unique solution in the finite element spaces. To prove stability of the discretization, take $\mathbf{w}_h = \gamma \frac{\partial \mathbf{u}_h}{\partial t}$ in (6.1.1), differentiate (6.1.2) with respect to time and take $\phi_h = \gamma\mu_1 \frac{\partial \tilde{q}_h}{\partial t} - \overline{p}_h + \alpha \tilde{q}_h$, and take $\psi_h = \overline{p}_h$ in (6.1.3). Upon adding (6.1.1) and (6.1.3), using the time derivative of (6.1.2), and integrating in time, it can be shown that the finite element approximation satisfies the following energy law:

$$J_h(T) + \int_0^T \int_\Omega [\kappa |\nabla \overline{p}_h|^2 + 2\eta_1 \gamma |\mathbf{D}(\frac{\partial \mathbf{u}_h}{\partial t})|^2 + \mu_1 \gamma (\frac{\partial \tilde{q}_h}{\partial t})^2] =$$
$$J_h(0) + \int_0^T \int_\Omega [\mathbf{f} \cdot \nabla \overline{p}_h - \gamma \tilde{\Sigma}_{ij} \mathbf{D}(\frac{\partial \mathbf{u}_h}{\partial t})_{ij}],$$

where

$$J_h(t) := \int_\Omega [\frac{1}{2}\alpha \tilde{q}_h^2(t) + \gamma \nu(F_0) f_0 |\mathbf{D}(\mathbf{u}_h(t))|^2].$$

Thus this scheme is numerically stable with the only restriction being that Ω_h must be quasi-uniform.

Convergence of this mixed method is considered. Let $(\mathbf{u}, \tilde{q}, \overline{p})$ be the weak solution. Define the error functions

$$\mathbf{Eu} = \mathbf{u} - \mathbf{u}_h, \; E\tilde{q} = \tilde{q} - \tilde{q}_h, \; E\overline{p} = \overline{p} - \overline{p}_h.$$

$\forall(\mathbf{w}_h, \phi_h, \psi_h) \in \mathbf{W}_h \times Q_h \times P_h$, these functions satisfy the error equations

$$\begin{aligned}
&\int_\Omega [2\eta_1 \mathbf{D}(\mathbf{Eu}_t)_{ij} + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{Eu})_{ij} + (\mu_1 E\tilde{q}_t \\
&-\frac{1}{\gamma} E\overline{p} + \frac{\alpha}{\gamma} E\tilde{q})\delta_{ij}] \mathbf{D}(\mathbf{w}_h)_{ij} = 0, \\
&\int_\Omega \phi_h \nabla \cdot \mathbf{Eu} = \int_\Omega \phi_h E\tilde{q}, \\
&\int_\Omega E\tilde{q}_t \psi_h + \int_\Omega \kappa \nabla E\overline{p} \cdot \nabla \psi_h = 0, \\
&\int_\Omega \mathbf{D}(\mathbf{Eu}(0))_{ij} \mathbf{D}(\mathbf{w}_h)_{ij} = 0, \\
&\int_\Omega E\tilde{q}(0)\phi_h = 0.
\end{aligned}$$

In order to estimate the error functions, the weak solution needs to be projected into the finite element spaces. Define the projection operators $Q^h : Q \to Q_h$, $R^h : \mathbf{W} \to \mathbf{W}_h$, and $S^h : P \to P_h$ by

$$\begin{aligned}
&\int_\Omega Q^h \phi \chi_h = \int_\Omega \phi \chi_h \; \forall \chi_h \in Q_h, \\
&\int_\Omega \mathbf{D}(R^h \mathbf{v})_{ij} \mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega \mathbf{D}(\mathbf{v})_{ij} \mathbf{D}(\mathbf{w}_h)_{ij} \; \forall \mathbf{w}_h \in \mathbf{W}_h, \\
&\int_\Omega \nabla S^h \varphi \cdot \nabla \psi_h = \int_\Omega \nabla \varphi \cdot \nabla \psi_h \; \forall \psi_h \in P_h.
\end{aligned}$$

Notice that $\mathbf{u}_h(0) = R^h \mathbf{u}_0$ and $\tilde{q}_h(0) = Q^h(0) = 0$. These projections satisfy the following estimates [22]: For any $\phi \in H^s(\Omega) \bigcap Q$, $\mathbf{v} \in H^s(\Omega) \bigcap \mathbf{W}$, and $\varphi \in H^s(\Omega) \bigcap P$ for $s \geq 1$,

$$\begin{aligned}
&||Q^h \phi - \phi||_{L^2(\Omega)} + h||\nabla(Q^h \phi - \phi)||_{L^2(\Omega)} \leq Ch^l ||\phi||_{H^l(\Omega)}, \\
&h||\mathbf{D}(R^h \mathbf{v} - \mathbf{v})||_{L^2(\Omega)} \leq Ch^m ||\mathbf{v}||_{H^m(\Omega)}, \\
&||S^h \varphi - \varphi||_{L^2(\Omega)} + h||\nabla(S^h \varphi - \varphi)||_{L^2(\Omega)} \leq Ch^l ||\varphi||_{H^l(\Omega)}
\end{aligned}$$

for $l = \min\{2, s\}$ and $m = \min\{3, s\}$. The following error function decompositions are introduced:

$$\mathbf{Eu} = \mathbf{\Lambda u} + \mathbf{\Theta u},\ \mathbf{\Lambda u} = \mathbf{u} - R^h\mathbf{u},\ \mathbf{\Theta u} = R^h\mathbf{u} - \mathbf{u}_h;$$
$$E\tilde{q} = \Lambda\tilde{q} + \Theta\tilde{q},\ \Lambda\tilde{q} = \tilde{q} - Q^h\tilde{q},\ \Theta\tilde{q} = Q^h\tilde{q} - \tilde{q}_h;$$
$$E\overline{p} = \Psi\overline{p} + \Phi\overline{p},\ \Psi\overline{p} = \overline{p} - S^h\overline{p},\ \Phi\overline{p} = S^h\overline{p} - \overline{p}_h.$$

Using these decompositions and the definitions of the projections, the error equations can be expressed as:

$$\int_\Omega [2\eta_1 \mathbf{D}(\mathbf{\Theta u}_t)_{ij} + 2\nu(F_0)f_0\mathbf{D}(\mathbf{\Theta u})_{ij} + (\mu_1\Theta\tilde{q}_t - \frac{1}{\gamma}\Phi\overline{p} + \frac{\alpha}{\gamma}\Theta\tilde{q})\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega (\mu_1\Lambda\tilde{q}_t - \frac{1}{\gamma}\Psi\overline{p} + \frac{\alpha}{\gamma}\Lambda\tilde{q})\nabla\cdot\mathbf{w}_h, \tag{6.1.6}$$

$$\int_\Omega \phi_h\nabla\cdot\mathbf{\Theta u} = \int_\Omega \phi_h\Theta\tilde{q} - \int_\Omega \phi_h\nabla\cdot\mathbf{\Lambda u}, \tag{6.1.7}$$

$$\int_\Omega \Theta\tilde{q}_t\psi_h + \int_\Omega \kappa\nabla\Phi\overline{p}\cdot\nabla\psi_h = 0, \tag{6.1.8}$$

$$\mathbf{\Theta u}(0) = \mathbf{0}, \tag{6.1.9}$$

$$\Theta\tilde{q}(0) = 0 \tag{6.1.10}$$

for all $(\mathbf{w}_h, \phi_h, \psi_h) \in \mathbf{W}_h \times Q_h \times P_h$. Take $\mathbf{w}_h = \gamma\mathbf{\Theta u}_t$ in (6.1.6) and $\psi_h = \Phi\overline{p}$ in (6.1.8). (6.1.7) is differentiated with respect to t and $\phi_h = \gamma(\mu_1\Theta\tilde{q}_t - \frac{1}{\gamma}\Phi\overline{p} + \frac{\alpha}{\gamma}\Theta\tilde{q})$ is taken. Adding (6.1.6) and (6.1.8) and using the time derivative of (6.1.7), it can be shown upon integrating in time that

$$\int_\Omega [\gamma\nu(F_0)f_0|\mathbf{D}(\mathbf{\Theta u}(T))|^2 + \frac{1}{2}\alpha\Theta\tilde{q}^2(T)] + \int_0^T \int_\Omega [2\eta_1\gamma|\mathbf{D}(\mathbf{\Theta u}_t)|^2 + \gamma\mu_1\Theta\tilde{q}_t^2 + \kappa|\nabla\Phi\overline{p}|^2] = \int_\Omega \alpha\Theta\tilde{q}(T)\nabla\cdot\mathbf{\Lambda u}(T) + \int_0^T \int_\Omega [(\mu_1\gamma\Theta\tilde{q}_t - \Phi\overline{p})\nabla\cdot\mathbf{\Lambda u}_t - (\mu_1\gamma\Lambda\tilde{q}_t - \Psi\overline{p} + \alpha\Lambda\tilde{q})\nabla\cdot\mathbf{\Theta u}_t - \alpha\Theta\tilde{q}_t\nabla\cdot\mathbf{\Lambda u}].$$

Using Cauchy, Poincaré, and Hölder inequalities and projection estimates, it can be shown that

$$\int_\Omega [\nu(F_0) f_0 \gamma |\mathbf{D}(\mathbf{\Theta u}(T))|^2 + \frac{1}{4}\alpha\Theta\tilde{q}^2(T)] + \int_0^T \int_\Omega [\eta_1\gamma|\mathbf{D}(\mathbf{\Theta u}_t)|^2 + \frac{1}{2}\gamma\mu_1\Theta\tilde{q}_t^2$$
$$+\frac{1}{2}\kappa|\nabla\Phi\overline{p}|^2] \leq Ch^4[||\mathbf{u}(T)||^2_{H^3(\Omega)} + ||\tilde{q}||^2_{H^1(0,T;H^2(\Omega))} + ||\overline{p}||^2_{L^2(0,T;H^2(\Omega))}$$
$$+||\mathbf{u}||^2_{H^1(0,T;H^3(\Omega))}]$$

for a sufficiently regular weak solution. From this estimate it is clear that all approximations converge optimally. In addition, notice that $||\nabla\Phi\overline{p}||_{L^2(0,T;L^2(\Omega))}$ converges with order h^2 and thus enjoys superconvergence. Finally, the errors satisfy

$$||\mathbf{D}(\mathbf{u}-\mathbf{u}_h)||_{L^\infty(0,T;L^2(\Omega))} + ||\tilde{q}-\tilde{q}_h||_{L^\infty(0,T;L^2(\Omega))} + ||\tilde{q}_t - \frac{\partial\tilde{q}_h}{\partial t}||_{L^2(0,T;L^2(\Omega))}$$
$$+||\mathbf{D}(\mathbf{u}_t - \frac{\partial\mathbf{u}_h}{\partial t})||_{L^2(0,T;L^2(\Omega))} \leq C_1(T;\mathbf{u},\tilde{q},\overline{p})h^2$$

and

$$||\nabla(\overline{p}-\overline{p}_h)||_{L^2(0,T;L^2(\Omega))} \leq C_2(T;\mathbf{u},\tilde{q},\overline{p})h.$$

Now it is time to discretize with respect to t. The backward Euler method is used to accomplish this. Let $0 = t^0 < t^1 < \cdots < t^N = T$ be a partition of the interval $[0,T]$. Define $\Delta t^n = t^n - t^{n-1}$ for $1 \leq n \leq N$. Define the finite difference operators

$$d_{t^n}\tilde{q}_h^n := \frac{1}{\Delta t^n}(\tilde{q}_h^n - \tilde{q}_h^{n-1}),\ d_{t^n}\mathbf{u}_h^n := \frac{1}{\Delta t^n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})$$

and let $\mathbf{f}^n = \mathbf{f}(t^n)$, $\tilde{\mathbf{\Sigma}}^n = \tilde{\Sigma}(t^n)$. Define the following fully discrete algorithm:

Algorithm 6.1. *Step i: Compute* $\tilde{q}_h^0 \in Q_h$, $\mathbf{u}_h^0 \in \mathbf{W}_h$ *satisfying* $\forall \mathbf{w}_h \in \mathbf{W}_h$, $\forall \chi_h \in Q_h$ *the equations*

(6.1.11) $$\int_\Omega \mathbf{D}(\mathbf{u}_h^0)_{ij}\mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij}\mathbf{D}(\mathbf{w}_h)_{ij},$$

(6.1.12) $$\int_\Omega \tilde{q}_h^0\chi_h = 0.$$

Step ii: For $n = 1, 2, \cdots, N$, *compute* $(\mathbf{u}_h^n, \tilde{q}_h^n, \overline{p}_h^n) \in \mathbf{W}_h \times Q_h \times P_h$ *satisfying* $\forall(\mathbf{w}_h, \phi_h, \psi_h) \in \mathbf{W}_h \times Q_h \times P_h$ *the equations*

$$\int_\Omega [2\eta_1 \mathbf{D}(d_{t^n}\mathbf{u}_h^n)_{ij} + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{u}_h^n)_{ij} + (\mu_1 d_{t^n}\tilde{q}_h^n - \frac{1}{\gamma}\overline{p}_h^n + \frac{\alpha}{\gamma}\tilde{q}_h^n)\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij} = -\int_\Omega \tilde{\Sigma}_{ij}^n \cdot \frac{\partial \mathbf{w}_{hi}}{\partial x_j}, \tag{6.1.13}$$

$$\int_\Omega \phi_h \nabla \cdot \mathbf{u}_h^n = \int_\Omega \phi_h \tilde{q}_h^n, \tag{6.1.14}$$

$$\int_\Omega d_{t^n}\tilde{q}_h^n \psi_h + \int_\Omega \kappa \nabla \overline{p}_h^n \cdot \nabla \psi_h = \int_\Omega \mathbf{f}^n \cdot \nabla \psi_h. \tag{6.1.15}$$

The first step in analyzing this discretization is determining what restrictions must be placed on Δt^n in order for the method to be numerically stable. In order to do this, an energy law is derived analogous to the one found for the semi-discrete case. Take $\mathbf{w}_h = \gamma d_{t^n}\mathbf{u}_h^n$ in (6.1.13), apply the difference operator to (6.1.14), take $\phi_h = \gamma(\mu_1 d_{t^n}\tilde{q}_h^n - \frac{1}{\gamma}\overline{p}_h^n + \frac{\alpha}{\gamma}\tilde{q}_h^n)$, and substitute this into (6.1.13). Take $\psi_h = \overline{p}_h^n$ in (6.1.15), add this to (6.1.13) and apply the operator $\sum_{n=1}^N \Delta t^n$ to the sum. The discrete energy law is

$$J_h^N + \sum_{n=1}^N \Delta t^n \int_\Omega [\gamma(2\eta_1 + \nu(F_0) f_0 \Delta t^n)|\mathbf{D}(d_{t^n}\mathbf{u}_h^n)|^2 + (\gamma\mu_1 + \frac{1}{2}\alpha\Delta t^n)(d_{t^n}\tilde{q}_h^n)^2]$$
$$+ \sum_{n=1}^N \Delta t^n \int_\Omega \kappa |\nabla \overline{p}_h^n|^2 = J_h^0 + \sum_{n=1}^N \Delta t^n \int_\Omega [\mathbf{f}^n \cdot \nabla \overline{p}_h^n - \gamma \tilde{\Sigma}_{ij}^n \mathbf{D}(d_{t^n}\mathbf{u}_h^n)_{ij}],$$

where

$$J_h^n := \int_\Omega [\nu(F_0) f_0 \gamma |\mathbf{D}(\mathbf{u}_h^n)|^2 + \frac{1}{2}\alpha(\tilde{q}_h^n)^2].$$

It is clear from this discrete energy law that the algorithm given above is numerically stable for any $\Delta t^n > 0$.

To study convergence of the algorithm, the solution of the fully discrete problem is compared to the solution of the semi-discrete problem. Then, by the triangle inequality, convergence to the weak solution is obtained. Denote the solution to the semi-discrete problem by $(\mathbf{u}_h, \tilde{q}_h, \overline{p}_h)$. Define the following error functions:

$$\mathbf{E}\mathbf{u}^n = \mathbf{u}_h(t^n) - \mathbf{u}_h^n, \; E\tilde{q}^n = \tilde{q}_h(t^n) - \tilde{q}_h^n, \; E\overline{p}^n = \overline{p}_h(t^n) - \overline{p}_h^n.$$

Also, define the following remainder functions:

$$R\mathbf{u}_h^n = -\frac{1}{\Delta t^n}\int_{t^{n-1}}^{t^n}(s-t^{n-1})(\mathbf{u}_h)_{tt}(s)ds,\ R\tilde{q}_h^n = -\frac{1}{\Delta t^n}\int_{t^{n-1}}^{t^n}(s-t^{n-1})(\tilde{q}_h)_{tt}(s)ds.$$

Using Taylor's formula, the semi-discrete problem can be reformulated as follows:

$$(6.1.16)\int_\Omega[2\eta_1\mathbf{D}(d_{t^n}\mathbf{u}_h(t^n))_{ij} + 2\nu(F_0)f_0\mathbf{D}(\mathbf{u}_h(t^n))_{ij} + (\mu_1 d_{t^n}\tilde{q}_h(t^n) - \frac{1}{\gamma}\overline{p}_h(t^n) + \frac{\alpha}{\gamma}\tilde{q}_h(t^n))\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij} = -\int_\Omega \tilde{\Sigma}_{ij}^n\mathbf{D}(\mathbf{w}_h)_{ij} + \int_\Omega[2\eta_1\mathbf{D}(R\mathbf{u}_h^n)_{ij} + \mu_1 R\tilde{q}_h^n\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij}\ \forall\mathbf{w}_h\in\mathbf{W}_h,$$

$$(6.1.17)\int_\Omega \phi_h\nabla\cdot\mathbf{u}_h(t^n) = \int_\Omega \phi_h\tilde{q}_h(t^n)\ \forall\phi_h\in Q_h,$$

$$(6.1.18)\int_\Omega d_{t^n}\tilde{q}_h(t^n)\psi_h + \int_\Omega \kappa\nabla\overline{p}_h(t^n)\cdot\nabla\psi_h = \int_\Omega[\mathbf{f}^n\cdot\nabla\psi_h + R\tilde{q}_h^n\psi_h]\ \forall\psi_h\in P_h,$$

$$(6.1.19)\int_\Omega \mathbf{D}(\mathbf{u}_h(0))_{ij}\mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij}\mathbf{D}(\mathbf{w}_h)_{ij}\ \forall\mathbf{w}_h\in\mathbf{W}_h,$$

$$(6.1.20)\int_\Omega \tilde{q}_h(0)\chi_h = 0\ \forall\chi_h\in Q_h.$$

Subtracting the equations for the fully discrete problem from (6.1.16)-(6.1.20), the following error equations are obtained:

$$(6.1.21)\qquad \int_\Omega[2\eta_1\mathbf{D}(d_{t^n}\mathbf{E}\mathbf{u}^n)_{ij} + 2\nu(F_0)f_0\mathbf{D}(\mathbf{E}\mathbf{u}^n)_{ij} + (\mu_1 d_{t^n}E\tilde{q}^n - \frac{1}{\gamma}E\overline{p}^n + \frac{\alpha}{\gamma}E\tilde{q}^n)\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega[2\eta_1\mathbf{D}(R\mathbf{u}_h^n)_{ij} + \mu_1 R\tilde{q}_h^n\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij}\ \forall\mathbf{w}_h\in\mathbf{W}_h,$$

$$(6.1.22)\qquad \int_\Omega \phi_h\nabla\cdot\mathbf{E}\mathbf{u}^n = \int_\Omega \phi_h E\tilde{q}^n\ \forall\phi_h\in Q_h,$$

$$(6.1.23)\qquad \int_\Omega d_{t^n}E\tilde{q}^n\psi_h + \int_\Omega \kappa\nabla E\overline{p}^n\cdot\nabla\psi_h = \int_\Omega R\tilde{q}_h^n\psi_h\ \forall\psi_h\in P_h,$$

$$(6.1.24)\qquad \mathbf{E}\mathbf{u}^0 = 0,$$

$$(6.1.25)\qquad E\tilde{q}^0 = 0.$$

Following a procedure that mimics the steps taken to obtain the energy law for the fully discrete problem, the following energy law for the error equations is obtained:

$$G_h^N + \sum_{n=1}^{N} \Delta t^n \int_\Omega [\gamma(2\eta_1 + \nu(F_0) f_0 \Delta t^n)|\mathbf{D}(d_{t^n}\mathbf{E}\mathbf{u}^n)|^2 + (\gamma\mu_1 + \frac{1}{2}\alpha\Delta t^n)$$

$$(d_{t^n} E\tilde{q}^n)^2 + \kappa|\nabla E\overline{p}^n|^2] = G_h^0 + \sum_{n=1}^{N} \Delta t^n \int_\Omega [2\eta_1\gamma \mathbf{D}(R\mathbf{u}_h^n)_{ij}\mathbf{D}(d_{t^n}\mathbf{E}\mathbf{u}^n)_{ij}$$

$$+ E\overline{p}^n R\tilde{q}_h^n + \mu_1\gamma R\tilde{q}_h^n \nabla \cdot d_{t^n}\mathbf{E}\mathbf{u}^n],$$

where

$$G_h^n := \int_\Omega [\nu(F_0) f_0 \gamma |\mathbf{D}(\mathbf{E}\mathbf{u}^n)|^2 + \frac{1}{2}\alpha(E\tilde{q}^n)^2]$$

for $0 \le n \le N$. Note that $\mathbf{E}\mathbf{u}^0 = \mathbf{0}$ and $E\tilde{q}^0 = 0$, so $G_h^0 = 0$. Using Hölder's inequality and the definitions of the remainder functions, it can be shown that

$$\int_\Omega (R\tilde{q}_h^n)^2 \le \frac{1}{3}\Delta t^n ||(\tilde{q}_h)_{tt}||^2_{L^2(t^{n-1},t^n;L^2(\Omega))},$$

$$\int_\Omega |\mathbf{D}(R\mathbf{u}_h^n)|^2 \le \frac{1}{3}\Delta t^n ||\mathbf{D}((\mathbf{u}_h)_{tt})||^2_{L^2(t^{n-1},t^n;L^2(\Omega))}.$$

Using these results and Poincaré's inequality, the following estimate is obtained:

$$(6.1.26)\quad G_h^N + \sum_{n=1}^{N} \Delta t^n \int_\Omega [\gamma(\frac{1}{2}\eta_1 + \nu(F_0) f_0 \Delta t^n)|\mathbf{D}(d_{t^n}\mathbf{E}\mathbf{u}^n)|^2 + \frac{1}{2}\kappa|\nabla E\overline{p}^n|^2$$

$$+(\gamma\mu_1 + \frac{\alpha}{2}\Delta t^n)(d_{t^n}E\tilde{q}^n)^2] \le \sum_{n=1}^{N} \frac{(\Delta t^n)^2}{3}[\eta_1\gamma||\mathbf{D}((\mathbf{u}_h)_{tt})||^2_{L^2(t^{n-1},t^n;L^2(\Omega))}$$

$$+C||(\tilde{q}_h)_{tt}||^2_{L^2(t^{n-1},t^n;L^2(\Omega))}],$$

where C is a positive constant depending on η_1, μ_1, κ, and constants obtained from applying Poincaré's inequality. This proves that the fully discrete scheme converges.

6.2 Discretization scheme for displacement-traction problem

Let $\mathbf{u}$, q, $\overline{p}$ be the weak solution of (5.1.14)-(5.1.16) with mixed displacement -traction boundary conditions with $\boldsymbol{U} \equiv \mathbf{0}$ for simplicity. As with the pure displacement problem, Taylor-Hood elements are used to approximate $\mathbf{u}$ and $\overline{p}$ while a convergent diffusion solver is used to approximate q. Let Ω_h be a quasi-uniform triangulation of Ω with mesh size h. Define the finite element spaces

$$\mathbf{W}_h = \{\mathbf{w}_h \in [C^0(\overline{\Omega})]^d : \mathbf{w}_h|_K \in [P^2(K)]^d \forall K \in \Omega_h;\ \mathbf{w}_h|_{\Gamma_0} = \mathbf{0}\},$$
$$P_h = \{\phi_h \in C^0(\overline{\Omega}) : \phi_h|_K \in P^1(K) \forall K \in \Omega_h\},$$
$$Q_h = P_h.$$

The following finite element inf-sup condition holds for $\mathbf{W}_h \times Q_h$: $\exists \beta_0 > 0$ independent of h such that

$$\sup_{\mathbf{w}_h \in \mathbf{W}_h} \frac{\int_\Omega \phi_h \nabla \cdot \mathbf{w}_h}{||\mathbf{D}(\mathbf{w}_h)||_{L^2(\Omega)}} \geq \beta_0 ||\phi_h||_{L^2(\Omega)} \quad \forall \phi_h \in Q_h. \tag{6.2.27}$$

As in the pure displacement case, define a semi-discrete mixed finite element algorithm as follows: Find $(\mathbf{u}_h, q_h, \overline{p}_h) : (0, T] \to \mathbf{W}_h \times Q_h \times P_h$ and $(\mathbf{u}_h(0), q_h(0)) \in \mathbf{W}_h \times Q_h$ such that for all $(\mathbf{w}_h, \phi_h, \psi_h) \in \mathbf{W}_h \times Q_h \times P_h$,

$$\int_\Omega [2\eta_1 \mathbf{D}(\frac{\partial \mathbf{u}_h}{\partial t})_{ij} + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{u}_h)_{ij} + (\mu_1 \frac{\partial q_h}{\partial t} - \frac{1}{\gamma}\overline{p}_h + \frac{\alpha}{\gamma} q_h)\delta_{ij}] \frac{\partial \mathbf{w}_{hi}}{\partial x_j} = \int_{\Gamma_1} \mathbf{g} \cdot \mathbf{w}_h - \int_\Omega \mathcal{S}(F_0)_{ij} \frac{\partial \mathbf{w}_{hi}}{\partial x_j}, \tag{6.2.28}$$

$$\int_\Omega \phi_h \nabla \cdot \mathbf{u}_h = \int_\Omega \phi_h q_h, \tag{6.2.29}$$

$$\int_\Omega \frac{\partial q_h}{\partial t} \psi_h + \int_\Omega \kappa \nabla \overline{p}_h \cdot \nabla \psi_h = 0, \tag{6.2.30}$$

$$\int_\Omega \mathbf{D}(\mathbf{u}_h(0))_{ij} \mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij} \mathbf{D}(\mathbf{w}_h)_{ij}, \tag{6.2.31}$$

$$\int_\Omega q_h(0) \phi_h = \int_\Omega q_0 \phi_h. \tag{6.2.32}$$

This algorithm leads to a linear system of ODEs in time with a complete set of initial conditions. By standard linear ODE theory, it can be seen that there exists a unique solution in the finite element spaces. To prove stability of the discretization, take $\mathbf{w}_h = \gamma \frac{\partial \mathbf{u}_h}{\partial t}$ in (6.2.28), differentiate (6.2.29) with respect to time and take $\phi_h = \gamma(\mu_1 \frac{\partial q_h}{\partial t} - \frac{1}{\gamma}\overline{p}_h + \frac{\alpha}{\gamma} q_h)$, and take $\psi_h = \overline{p}_h$ in(6.2.30). Upon adding (6.2.28) and (6.2.30), using the time derivative of (6.2.29), and integrating in time, it can be shown that the finite element approximation satisfies the following energy law:

$$
\begin{aligned}
&J_h(T) + \int_0^T \int_\Omega [\kappa|\nabla \overline{p}_h|^2 + 2\eta_1\gamma|\mathbf{D}(\frac{\partial \mathbf{u}_h}{\partial t})|^2 + \mu_1\gamma(\frac{\partial q_h}{\partial t})^2] = \\
&J_h(0) + \int_0^T \int_{\Gamma_1} \gamma \mathbf{g} \cdot \frac{\partial \mathbf{u}_h}{\partial t} - \int_0^T \int_\Omega \gamma \mathcal{S}(F_0)_{ij} \mathbf{D}(\frac{\partial \mathbf{u}_h}{\partial t})_{ij},
\end{aligned}
$$

where

$$
J_h(t) := \int_\Omega [\frac{1}{2}\alpha q_h^2(t) + \gamma\nu(F_0) f_0 |D(\mathbf{u}_h(t))|^2].
$$

This energy law combined with (6.2.27) applied to $\overline{p}_h$ implies that the scheme is numerically stable with the only restriction being that Ω_h must be quasi-uniform.

Convergence of this mixed method is considered. Let $(\mathbf{u}, q, p)$ be the weak solution. Define the error functions

$$
\mathbf{Eu} = \mathbf{u} - \mathbf{u}_h, \ Eq = q - q_h, \ E\overline{p} = \overline{p} - \overline{p}_h.
$$

$\forall (\mathbf{w}_h, \phi_h, \psi_h) \in \mathbf{W}_h \times Q_h \times P_h$, these functions satisfy the error equations

$$
\begin{aligned}
&\int_\Omega [2\eta_1 \mathbf{D}(\mathbf{Eu}_t)_{ij} + 2\nu(F_0) f_0 \mathbf{D}(\mathbf{Eu})_{ij}] \mathbf{D}(\mathbf{w}_h)_{ij} + \int_\Omega [\mu_1 Eq_t - \frac{1}{\gamma} E\overline{p} \\
&+ \frac{\alpha}{\gamma} Eq] \nabla \cdot \mathbf{w}_h = 0, \\
&\int_\Omega \phi_h \nabla \cdot \mathbf{Eu} = \int_\Omega \phi_h Eq, \\
&\int_\Omega Eq_t \psi_h + \int_\Omega \kappa \nabla E\overline{p} \cdot \nabla \psi_h = 0, \\
&\int_\Omega \mathbf{D}(\mathbf{Eu}(0))_{ij} \mathbf{D}(\mathbf{w}_h)_{ij} = 0, \\
&\int_\Omega Eq(0)\phi_h = 0.
\end{aligned}
$$

In order to estimate the error functions, the weak solution needs to be projected into the finite element spaces. Define the same projection operators $Q^h : Q \to Q_h$

and $R^h : \mathbf{W} \to \mathbf{W}_h$ as in the pure displacement case, with Q and Q_h, P and P_h redefined appropriately. As in the pure displacement case, $\mathbf{u}_h(0) = R^h\mathbf{u}_0$ and $q_h(0) = Q^h q_0$. Define the projection $S^h : P \to P_h$ by

$$\int_\Omega \nabla S^h\varphi \cdot \nabla\psi_h = \int_\Omega \nabla\varphi \cdot \psi_h \quad \forall\psi_h \in P_h,$$
$$\int_\Omega S^h\varphi = \int_\Omega \varphi.$$

Recall the following estimates on these projections: For any $\phi \in H^s(\Omega)\bigcap Q$, $\mathbf{v} \in H^s(\Omega)\bigcap \mathbf{W}$, and $\varphi \in H^s(\Omega)\bigcap P$ for $s \geq 1$,

$$||Q^h\phi - \phi||_{L^2(\Omega)} + h||\nabla(Q^h\phi - \phi)||_{L^2(\Omega)} \leq Ch^l||\phi||_{H^l(\Omega)},$$
$$h||\mathbf{D}(R^h\mathbf{v} - \mathbf{v})||_{L^2(\Omega)} \leq Ch^m||\mathbf{v}||_{H^m(\Omega)},$$
$$||S^h\varphi - \varphi||_{L^2(\Omega)} + h||\nabla(S^h\varphi - \varphi)||_{L^2(\Omega)} \leq Ch^l||\varphi||_{H^l(\Omega)}$$

for $l = \min\{2, s\}$ and $m = \min\{3, s\}$. The following error function decompositions are introduced:

$$\mathbf{Eu} = \mathbf{\Lambda u} + \mathbf{\Theta u},\ \mathbf{\Lambda u} = \mathbf{u} - R^h\mathbf{u},\ \mathbf{\Theta u} = R^h\mathbf{u} - \mathbf{u}_h;$$
$$Eq = \Lambda q + \Theta q,\ \Lambda q = q - Q^hq,\ \Theta q = Q^hq - q_h;$$
$$E\overline{p} = \Psi\overline{p} + \Phi\overline{p},\ \Psi\overline{p} = \overline{p} - S^h\overline{p},\ \Phi\overline{p} = S^h\overline{p} - \overline{p}_h.$$

Using these decompositions and the definitions of the projections, the error equations can be expressed as:

$$\int_\Omega [2\eta_1\mathbf{D}(\mathbf{\Theta u}_t)_{ij} + 2\nu(F_0)f_0\mathbf{D}(\mathbf{\Theta u})_{ij} + (\mu_1\Theta q_t - \frac{1}{\gamma}\Phi\overline{p} + \frac{\alpha}{\gamma}\Theta q)\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij} = -\int_\Omega (\mu_1\Lambda q_t - \frac{1}{\gamma}\Psi\overline{p} + \frac{\alpha}{\gamma}\Lambda q)\nabla \cdot \mathbf{w}_h, \tag{6.2.33}$$

$$\int_\Omega \phi_h\nabla \cdot \mathbf{\Theta u} = \int_\Omega \phi_h\Theta q - \int_\Omega \phi_h\nabla \cdot \mathbf{\Lambda u}, \tag{6.2.34}$$

$$\int_\Omega \Theta q_t\psi_h + \int_\Omega \kappa\nabla\Phi\overline{p} \cdot \nabla\psi_h = 0, \tag{6.2.35}$$

$$\mathbf{\Theta u}(0) = \mathbf{0}, \tag{6.2.36}$$

$$\Theta q(0) = 0 \tag{6.2.37}$$

for all $(\mathbf{w}_h, \phi_h, \psi_h) \in \mathbf{W}_h \times Q_h \times P_h$. Upon taking $\mathbf{w}_h = \gamma\mathbf{\Theta u}_t$ in (6.2.33) and $\psi_h = \Phi\overline{p}$ in (6.2.35), differentiating (6.2.34) with respect to t and taking $\phi_h = \gamma(\mu_1\Theta q_t - \frac{1}{\gamma}\Phi\overline{p} + \frac{\alpha}{\gamma}\Theta q)$, adding (6.2.33) and (6.2.35), and using the time derivative of (6.2.34), it can be shown upon integrating in time that

$$\int_\Omega [\gamma\nu(F_0)f_0|\mathbf{D}(\mathbf{\Theta u}(T))|^2 + \frac{1}{2}\alpha\Theta q^2(T)] + \int_0^T \int_\Omega [2\eta_1\gamma|\mathbf{D}(\mathbf{\Theta u}_t)|^2 + \gamma\mu_1\Theta q_t^2$$
$$+\kappa|\nabla\Phi\overline{p}|^2] = \int_\Omega \alpha\Theta q(T)\nabla\cdot\mathbf{\Lambda u}(T) + \int_0^T \int_\Omega [(\mu_1\gamma\Theta q_t - \Phi\overline{p})\nabla\cdot\mathbf{\Lambda u}_t$$
$$-(\mu_1\gamma\Lambda q_t - \Psi\overline{p} + \alpha\Lambda q)\nabla\cdot\mathbf{\Theta u}_t - \alpha\Theta q_t\nabla\cdot\mathbf{\Lambda u}].$$

Using Cauchy, Poincaré, and Hölder inequalities, projection estimates, and the discrete inf-sup condition (6.2.27) for $\Phi\overline{p} \in P_h = Q_h$, it can be shown that

$$\sup_{0\le t\le T}\int_\Omega [\nu(F_0)f_0\gamma|\mathbf{D}(\mathbf{\Theta u}(t))|^2 + \frac{1}{4}\alpha\Theta q^2(t)] + \int_0^T \int_\Omega [\eta_1\gamma|\mathbf{D}(\mathbf{\Theta u}_t)|^2$$
$$+\frac{1}{2}\gamma\mu_1\Theta q_t^2 + \kappa|\nabla\Phi\overline{p}|^2] \le Ch^4[||\mathbf{u}||^2_{L^\infty(0,T;H^3(\Omega))} + ||q||^2_{H^1(0,T;H^2(\Omega))}$$
$$+||\overline{p}||^2_{L^2(0,T;H^2(\Omega))} + ||\mathbf{u}||^2_{H^1(0,T;H^3(\Omega))}]$$

for a sufficiently regular weak solution. From this estimate it is clear that all approximations converge optimally. In addition, notice that $||\nabla\Phi\overline{p}||_{L^2(0,T;L^2(\Omega))}$ converges with order h^2 and thus enjoys superconvergence. Finally, the errors satisfy

$$||\mathbf{D}(\mathbf{u} - \mathbf{u}_h)||_{L^\infty(0,T;L^2(\Omega))} + ||q - q_h||_{L^\infty(0,T;L^2(\Omega))} + ||q_t - \frac{\partial q_h}{\partial t}||_{L^2(0,T;L^2(\Omega))}$$
$$+||\mathbf{D}(\mathbf{u}_t - \frac{\partial \mathbf{u}_h}{\partial t})||_{L^2(0,T;L^2(\Omega))} \le C_1(T;\mathbf{u},q,\overline{p})h^2$$

and

$$||\nabla(\overline{p} - \overline{p}_h)||_{L^2(0,T;L^2(\Omega))} \le C_2(T;\mathbf{u},q,\overline{p})h.$$

Now it is time to discretize with respect to t. The backward Euler method is used to accomplish this. Adopt the same terminology n the time discretization on the pure displacement problem. Let $\mathbf{g}^n = \mathbf{g}(t^n)$. Define the following fully discrete algorithm:

Algorithm 6.2. *Step i: Compute* $q_h^0 \in Q_h$, $\mathbf{u}_h^0 \in \mathbf{W}_h$ *satisfying* $\forall \mathbf{w}_h \in \mathbf{W}_h$, $\forall \chi_h \in Q_h$ *the equations*

$$\int_\Omega \mathbf{D}(\mathbf{u}_h^0)_{ij}\mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij}\mathbf{D}(\mathbf{w}_h)_{ij}, \tag{6.2.38}$$

$$\int_\Omega q_h^0 \chi_h = \int_\Omega q_0 \chi_h. \tag{6.2.39}$$

Step ii: For $n = 1, 2, \cdots, N$, *compute* $(\mathbf{u}_h^n, q_h^n, \overline{p}_h^n) \in \mathbf{W}_h \times Q_h \times P_h$ *satisfying* $\forall(\mathbf{w}_h, \phi_h, \psi_h) \in \mathbf{W}_h \times Q_h \times P_h$ *the equations*

$$\begin{aligned}\int_\Omega [2\eta_1 \mathbf{D}(d_{t^n}\mathbf{u}_h^n)_{ij} + 2\nu(F_0)f_0\mathbf{D}(\mathbf{u}_h^n)_{ij} + (\mu_1 d_{t^n} q_h^n - \frac{1}{\gamma}\overline{p}_h^n \\ + \frac{\alpha}{\gamma} q_h^n)\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij} = \int_{\Gamma_1} \mathbf{g}^n \cdot \mathbf{w}_h - \int_\Omega \mathcal{S}(F_0)_{ij}\mathbf{D}(\mathbf{w}_h)_{ij},\end{aligned} \tag{6.2.40}$$

$$\int_\Omega \phi_h \nabla \cdot \mathbf{u}_h^n = \int_\Omega \phi_h q_h^n, \tag{6.2.41}$$

$$\int_\Omega d_{t^n} q_h^n \psi_h + \int_\Omega \kappa \nabla \overline{p}_h^n \cdot \nabla \psi_h = 0. \tag{6.2.42}$$

In order to derive an energy law for this discretization, take $\mathbf{w}_h = \gamma d_{t^n}\mathbf{u}_h^n$ in (6.2.40). Apply the difference operator to (6.2.41) and then take $\phi_h = \gamma(\mu_1 d_{t^n} q_h^n - \frac{1}{\gamma}\overline{p}_h^n + \frac{\alpha}{\gamma} q_h^n)$ and substitute this into (6.2.40). Take $\psi_h = \overline{p}_h^n$ in (6.2.42) and add this to (6.2.40). Apply the operator $\sum_{n=1}^N \Delta t^n$ to the sum. The discrete energy law is

$$\begin{aligned}J_h^N + \sum_{n=1}^N \Delta t^n \int_\Omega [\gamma(2\eta_1 + \nu(F_0)f_0\Delta t^n)|\mathbf{D}(d_{t^n}\mathbf{u}_h^n)|^2 + (\gamma\mu_1 + \frac{1}{2}\alpha\Delta t^n)(d_{t^n} q_h^n)^2 \\ + \kappa|\nabla \overline{p}_h^n|^2] = J_h^0 + \sum_{n=1}^N \Delta t^n [\int_{\Gamma_1} \gamma \mathbf{g}^n \cdot d_{t^n}\mathbf{u}_h^n - \int_\Omega \gamma \mathcal{S}(F_0)_{ij}\mathbf{D}(d_{t^n}\mathbf{u}_h^n)_{ij}],\end{aligned}$$

where

$$J_h^n := \int_\Omega [\gamma\nu(F_0)f_0|\mathbf{D}(\mathbf{u}_h^n)|^2 + \frac{1}{2}\alpha(q_h^n)^2].$$

It is clear from this discrete energy law that the algorithm given above is numerically stable for any $\Delta t^n > 0$.

To study convergence of the algorithm, the solution of the fully discrete problem is compared to the solution of the semi-discrete problem. Then, by the triangle inequality, convergence to the weak solution is obtained. The solution to the semi-discrete problem is denoted by $(\mathbf{u}_h, q_h, \overline{p}_h)$. Define the following error functions:

$$\mathbf{Eu}^n = \mathbf{u}_h(t^n) - \mathbf{u}_h^n,\ Eq^n = q_h(t^n) - q_h^n,\ E\overline{p}^n = \overline{p}_h(t^n) - \overline{p}_h^n.$$

Also, define the following remainder functions:

$$R\mathbf{u}_h^n = -\frac{1}{\Delta t^n}\int_{t^{n-1}}^{t^n}(s - t^{n-1})(\mathbf{u}_h)_{tt}(s)ds,\ Rq_h^n = -\frac{1}{\Delta t^n}\int_{t^{n-1}}^{t^n}(s - t^{n-1})(q_h)_{tt}(s)ds.$$

Using Taylor's formula, the semi-discrete problem can be reformulated as follows:

$$\begin{aligned}(6.2.43)\quad & \int_\Omega [2\eta_1 \mathbf{D}(d_{t^n}\mathbf{u}_h(t^n))_{ij} + 2\nu(F_0)f_0\mathbf{D}(\mathbf{u}_h(t^n))_{ij} + (\mu_1 d_{t^n} q_h(t^n) \\ & -\frac{1}{\gamma}\overline{p}_h(t^n) + \frac{\alpha}{\gamma}q_h(t^n))\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij} = \int_{\Gamma_1}\mathbf{g}^n\cdot\mathbf{w}_h + \int_\Omega [2\eta_1\mathbf{D}(R\mathbf{u}_h^n)_{ij} \\ & +\mu_1 Rq_h^n\delta_{ij} - \mathcal{S}(F_0)_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij}\ \forall \mathbf{w}_h \in \mathbf{W}_h,\end{aligned}$$

$$(6.2.44)\quad \int_\Omega \phi_h \nabla\cdot\mathbf{u}_h(t^n) = \int_\Omega \phi_h q_h(t^n)\ \forall\phi_h \in Q_h,$$

$$(6.2.45)\quad \int_\Omega d_{t^n}q_h(t^n)\psi_h + \int_\Omega \kappa\nabla\overline{p}_h(t^n)\cdot\nabla\psi_h = \int_\Omega Rq_h^n\psi_h\ \forall\psi_h\in P_h,$$

$$(6.2.46)\quad \int_\Omega \mathbf{D}(\mathbf{u}_h(0))_{ij}\mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega \mathbf{D}(\mathbf{u}_0)_{ij}\mathbf{D}(\mathbf{w}_h)_{ij}\ \forall\mathbf{w}_h\in\mathbf{W}_h,$$

$$(6.2.47)\quad \int_\Omega q_h(0)\chi_h = \int_\Omega q_0\chi_h\ \forall\chi_h\in Q_h.$$

Subtracting the equations for the fully discrete problem from (6.2.43)-(6.2.47), the following error equations are obtained:

$$\int_\Omega [2\eta_1 \mathbf{D}(d_{t^n}\mathbf{Eu}^n)_{ij} + 2\nu(F_0)f_0\mathbf{D}(\mathbf{Eu}^n)_{ij} + (\mu_1 d_{t^n}Eq^n - \frac{1}{\gamma}E\overline{p}^n + \frac{\alpha}{\gamma}Eq^n)\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij} = \int_\Omega [2\eta_1\mathbf{D}(R\mathbf{u}_h^n)_{ij} + \mu_1 Rq_h^n\delta_{ij}]\mathbf{D}(\mathbf{w}_h)_{ij} \quad \forall \mathbf{w}_h \in \mathbf{W}_h, \tag{6.2.48}$$

$$\int_\Omega \phi_h \nabla\cdot\mathbf{Eu}^n = \int_\Omega \phi_h Eq^n \ \forall\phi_h\in Q_h, \tag{6.2.49}$$

$$\int_\Omega d_{t^n}Eq^n\psi_h + \int_\Omega \kappa\nabla E\overline{p}^n\cdot\nabla\psi_h = \int_\Omega Rq_h^n\psi_h \ \forall\psi_h\in P_h, \tag{6.2.50}$$

$$\mathbf{Eu}^0 = 0, \tag{6.2.51}$$

$$Eq^0 = 0. \tag{6.2.52}$$

Following a procedure that mimics the steps taken to obtain the energy law for the fully discrete problem, the following energy law for the error equations is obtained:

$$G_h^N + \sum_{n=1}^N \Delta t^n \int_\Omega [\gamma(2\eta_1 + \nu(F_0)f_0\Delta t^n)|\mathbf{D}(d_{t^n}\mathbf{Eu}^n)|^2 + (\gamma\mu_1 + \frac{1}{2}\alpha\Delta t^n)(d_{t^n}Eq^n)^2 + \kappa|\nabla E\overline{p}^n|^2] = G_h^0 + \sum_{n=1}^N \Delta t^n \int_\Omega [2\eta_1\gamma\mathbf{D}(R\mathbf{u}_h^n)_{ij}\mathbf{D}(d_{t^n}\mathbf{Eu}^n)_{ij} + \mu_1\gamma Rq_h^n\nabla\cdot d_{t^n}\mathbf{Eu}^n + E\overline{p}^n Rq_h^n],$$

where

$$G_h^n := \int_\Omega [\gamma\nu(F_0)f_0|\mathbf{D}(\mathbf{Eu}^n)|^2 + \frac{1}{2}\alpha(Eq^n)^2]$$

for $0 \le n \le N$. Note that $\mathbf{Eu}^0 = \mathbf{0}$ and $Eq^0 = 0$, so $G_h^0 = 0$. Using Hölder's inequality and the definitions of the remainder functions, it can be shown that

$$\int_\Omega (Rq_h^n)^2 \le \frac{1}{3}\Delta t^n||(q_h)_{tt}||^2_{L^2(t^{n-1},t^n;L^2(\Omega))},$$

$$\int_\Omega |\mathbf{D}(R\mathbf{u}_h^n)|^2 \le \frac{1}{3}\Delta t^n||\mathbf{D}((\mathbf{u}_h)_{tt})||^2_{L^2(t^{n-1},t^n;L^2(\Omega))}.$$

Using these results and Poincaré's inequality, the following estimate is obtained:

$$(6.2.53) G_h^N + \sum_{n=1}^{N} \Delta t^n \int_\Omega [\gamma(\frac{1}{2}\eta_1 + \nu(F_0) f_0 \Delta t^n)|\mathbf{D}(d_{t^n}\mathbf{E}\mathbf{u}^n)|^2 + \frac{1}{2}\kappa|\nabla E\overline{p}^n|^2$$
$$+(\gamma\mu_1 + \frac{\alpha}{2}\Delta t^n)(d_{t^n}Eq^n)^2] \leq \sum_{n=1}^{N} \frac{(\Delta t^n)^2}{3}[\eta_1\gamma||\mathbf{D}((\mathbf{u}_h)_{tt})||^2_{L^2(t^{n-1},t^n;L^2(\Omega))}$$
$$+C||(q_h)_{tt}||^2_{L^2(t^{n-1},t^n;L^2(\Omega))}],$$

where C is a positive constant depending on η_1, μ_1, κ, and constants obtained from applying Poincaré's inequality. This proves convergence of the fully discrete scheme.

Chapter 7

Conclusions

A model of gel dynamics using the approaches of mixture theory and thermodynamics is derived for a two-component mixture: polymer and solvent. The governing equations consist of balance of mass and linear momentum for the individual components. The model couples nonlinear elasticity, which is usually formulated in material coordinates, with fluid flow, which is formulated in spatial coordinates. A compatibility condition is needed in order to reconcile these two formulations. This condition is a nonlinear hyperbolic equation which makes the full problem very difficult to solve over a three dimensional domain. Two types of problems are analyzed here: special geometries and regimes, and arbitrary domains neglecting inertia.

In addition to developing a physically sound mathematical model of gels amenable to methods of partial differential equations and analyzing nonlinear problems in special geometries, the work fully addresses well-posedness of the linear problems and the numerical discretizations needed for numerical simulations of interest to the biomedical industry.

Bibliography

[1] H. Weinberger, personal communication.

[2] F.-J. Sayas, personal communication.

[3] R. A. Adams, *Sobolev Spaces*, Academic Press, 1975.

[4] W. Arendt, C. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Birkhäuser Verlag, 2001.

[5] L. S. Bennethum and J. H. Cushman, *Multiscale, hybrid mixture theory for swelling systems–i: Balance laws*, Int. J. Eng. Sci, 34 (1996), pp. 125–145.

[6] ——, *Multiscale, hybrid mixture theory for swelling systems–ii: Constitutive theory*, Int. J. Eng. Sci, 34 (1996), pp. 147–169.

[7] R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, vol. 1, John Wiley & Sons, New York, 2 ed., 1987.

[8] D. Boffi, F. Brezzi, and M. Fortin, *Finite elements for the stokes problem*, in Lecture Notes in Mathematics, vol. 1939, 2008, pp. 45–100.

[9] F. Brezzi, *On the existence, uniqueness and approximation of saddle point problems arising from Lagrange multipliers*, RAIRO Numerical Analysis, 8 (1974), pp. 129–151.

[10] M. C. Calderer, B. Chabaud, S. Lyu, and H. Zhang, *Modeling approaches to the dynamics of hydrogel swelling.* Accepted, 2008.

[11] M. C. Calderer, M. E. Rognes, and C. A. Micek, *Mixed finite element methods for gels with biomedical applications.* Submitted, 2009.

[12] M. C. Calderer and H. Zhang, *Incipient dynamics of swelling of gels*, SIAM J. Appl. Math., 68 (2008), pp. 1641–1664.

[13] J. M. Caruthers, S. R. Lustig, and N. A. Peppas, *Continuum thermodynamics and transport theory for polymer-fluid mixtures*, Chemical Engineering Science, 47 (1992), pp. 3037–3057.

[14] P. G. Ciarlet, *Mathematical Elasticity, Vol 1*, North-Holland, 1987.

[15] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer, 2005.

[16] R. Dautray and J. L. Lions, *Mathematical analysis and numerical methods for science and technology: Volume 5*, Springer-Verlag, 2000.

[17] R. J. Diperna and P. L. Lions, *Ordinary differential equations, transport theory and sobolev spaces*, Inventiones Mathematicae, 98 (1989), pp. 511–547.

[18] M. Doi, *Dynamics and Patterns in Complex Fluids*, Springer, 1990.

[19] M. Doi and A. Onuki, *Dynamic coupling between stress and composition in polymer solutions and blends*, J. Phys. II France, 2 (1992), pp. 1631–1656.

[20] J. Douglas, M. Kischinhevsky, P. J. Paes-Leme, and A. M. Spagnuolo, *A multiple-porosity model for a single-phase flow through naturally-fractured porous media*, Comp. Appl. Math., 17 (1998), pp. 19–48.

[21] N. T. Dunwoody, *A thermomechanical theory of diffusion in solid-fluid mixtures*, Arch. Rat. Mech. Anal., 38 (1970), pp. 348–371.

[22] X. Feng and Y. He, *Analysis of fully discrete finite element methods for a system of differential equations modeling swelling dynamics of polymer gels.* Submitted, 2009.

[23] P. Flory, *Principles of Polymer Chemistry*, Cornell University Press, 1953.

[24] P. J. Flory, *Thermodynamics of high polymer solutions*, J. Chem. Phys., 10 (1942), pp. 51–61.

[25] P. J. Flory and J. Rehner, *Statistical mechanics of cross-linked polymer networks, ii. swelling*, J. Chem. Phys., 11 (1943), pp. 521–526.

[26] D. R. Gaskell, *Introduction to the Thermodynamics of Materials*, Taylor & Francis, 1995.

[27] G. Gatica and F.-J. Sayas, *Characterizing the inf-sup condition on product spaces*, Numer. Math., 109 (2008), pp. 209–231.

[28] V. Girault and P. A. Raviart, *Finite Element Approximation of the Navier Stokes Equations*, no. 749 in Lecture Notes in Mathematics, Springer Verlag, Berlin, Heidelbert, New York, 1979.

[29] C. Guillope and J. C. Saut, *Existence results for the flow of viscoelastic fluids with a differential constitutive law*, Nonlinear Anal., 15 (1990), pp. 849–869.

[30] E. Hoiczyk, D. Kaiser, C. Wolgemuth, and G. Oster, *How myxobacteria glide*, Curr. Biol., 12 (2002), pp. 369–377.

[31] W. Hong, X. Zhao, J. Zhou, and Z. Suo, *A theory of coupled diffusion and large deformation in polymeric gels*, J. Mech. Phys. Solids, 56 (2008), pp. 1779–1793.

[32] S. Kawashima, *Systems of a hyperbolic-parabolic composite type,with applications to the equations of magnetohydrodynamics*, PhD thesis, Kyoto University, 1983.

[33] M. Kinzl, G. Luft, R. Horst, and B. A. Wolf, *Viscosity of solutions of low-density polyethylene in ethylene as a function of temperature and pressure*, J. Rheol., 47 (2003), pp. 869–877.

[34] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, 1969.

[35] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, 1966.

[36] R. Larson, *Constitutive Equations for Polymer Melts and Solutions*, Butterworths, 1988.

[37] P. Le Tallec, *Numerical methods for nonlinear three-dimensional elasticity*, in Handbook of Numerical Analysis, Vol. III, North-Holland, Amsterdam, 1994, pp. 465–622.

[38] Y. Li and T. Tanaka, *Kinetics of swelling and shrinking of gels*, J. Chem. Phys., 92 (1990), pp. 1365–1371.

[39] Y. Li and T. Tanaka, *Phase transitions of gels*, Ann. Rev. Mater. Sci., 22 (1992), pp. 243–277.

[40] F.-H. Lin and C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Comm. Pure Appl. Math., 48 (1995), pp. 501–537.

[41] F.-H. Lin, C. Liu, and P. Zhang, *On hydrodynamics of viscoelastic fluids*, Comm. Pure Appl. Math., LVIII (2005), pp. 1437–1471.

[42] F.-H. Lin, C. Liu, and P. Zhang, *On a micro-macro model for polymeric fluids near equilibrium*, Comm. Pure Appl. Math., LX (2007), pp. 838–866.

[43] F.-H. Lin and P. Zhang, *On the initial-boundary value problem of the incompressible viscoelastic fluid system*, Comm. Pure Appl. Math., LXI (2008), pp. 539–558.

[44] J. L. Lions and E. Magenes, *Nonhomogeneous boundary value problems and applications*, Springer-Verlag, Berlin, 1972.

[45] P. L. Lions and N. Masmoudi, *Global solutions for some oldroyd models of non-newtonian flows*, Chin. Ann. of Math., 21B (2000), pp. 131–146.

[46] C. Liu and N. J. Walkington, *An eulerian description of fluids containing visco-hyperelastic particles*, Arch. Ration. Mech. Anal., 159 (2001), pp. 229–252.

[47] C. Macosko, *Rheology: Principles, Measurements and Applications*, VCH Publishers, 1993.

[48] S. Mafe, J. Manzanares, A.E. English, and et.al., *Equilibrium swelling properties of polyampholytic hydrogels*, J.Chem. Phys., 104 (1996), pp. 8713–8720.

[49] M. A. Murad, L. S. Bennethum, and J. H. Cushman, *Macroscale thermodynamics and the chemical potential of swelling porous media*, Transport in Porous Media, 39 (2000), pp. 187–225.

[50] R. Ogden, *Non-Linear Elastic Deformations*, Dover, 1997.

[51] A. Onuki, *Theory of pattern formation in gels: Surface folding in highly compressible elastic bodies*, Phys Rev A, 39 (1989), pp. 5932–5948.

[52] J. Patel and P. J. Phillips, *The young's modulus of polyethylene*, J. Polymer Sci.: Polymer Letters Edition, 11 (1973), pp. 771–776.

[53] J. Rauch, *Lectures on geometric optics*, in Hyperbolic Equations and Frequency Interactions, L. Caffarelli and W. E., eds., AMS, 1999, pp. 383–466.

[54] R. E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, 1979.

[55] ——, *Lectures on flow in porous media*, in III Coloquio sobre Ecuaciones Diferenciales y Aplicaciones, A. D. Rueda and J. Guinez, eds., Maracaibo, Venezuela, 1998, Universidad del Zulia, pp. 53–97.

[56] ——, *Diffusion in deforming porous media*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 10 (2003), pp. 661–678.

[57] R. E. Showalter and P. Shi, *Dynamic plasticity models*, Comput. Methods Appl. Mech. Engrg., 151 (1998), pp. 501–511.

[58] A. Suzuki and T. Hara, *Kinetics of one-dimensional swelling and shrinking of polymer gels under mechanical constraint*, J.Chem.Phys., 114 (2001), p. 5012.

[59] T. Tanaka, *Collapse of gels and the critical endpoint*, Phys. Rev. Lett., 40 (1978), pp. 820–823.

[60] T. Tanaka and D. Filmore, *Kinetics of swelling of gels*, J. Chem. Phys., 70 (1979), pp. 1214–1218.

[61] C. Taylor and P. Hood, *A numerical solution of the navier-stokes equations using the finite element technique*, Comput. fluids, 1 (1973), pp. 73–100.

[62] M. E. Taylor, *Partial Differential Equations III: Nonlinear Equations*, Springer-Verlag, 1996.

[63] R. Temam, *The Navier-Stokes Equations: Theory and Numerical Analysis*, 2nd edn. North-Holland, Amsterdam, 1977.

[64] N. L. Thomas and A. H. Windle, *A theory of case ii diffusion*, Polymer, 23 (1982), pp. 529–542.

[65] T. Tomari and M. Doi, *Hysteresis and incubation in the dynamics of volume transition of spherical gels*, Macromolecules, 28 (1995), pp. 8334–8343.

[66] C. Truesdell, *Rational Thermodynamics*, Springer Verlag, second edition, 1984.

[67] A. Visintin, *Rheological models and hysteresis effects*, Rend. Sem. Mat. Univ. Padova, 77 (1987), pp. 213–243.

[68] T. Yamaue and M. Doi, *Swelling dynamics of constrained thin-plate gels under an external force*, Phys. Rev. E, 70 (2004), p. 011401.

[69] ——, *Theory of one-dimensional swelling dynamics of polymer gels under mechanical constraint*, Phys. Rev. E, 69 (2004), p. 041402.

[70] ——, *The stress diffusion coupling in the swelling dynamics of cylindrical gels*, J. Chem. Phys., 122 (2005), p. 084703.

[71] T. Yamaue, H. Mukai, K. Asaka, and M. Doi, *Electrostress diffusion coupling model for polyelectrolyte gels*, Macromolecules, 38 (2005), pp. 1349–1356.

[72] H. Zhang, *Static and dynamical problems of hydrogel swelling: Modeling and analysis*, PhD thesis, University of Minnesota, 2007.

[73] J. Zhang, X. Zhao, Z. Suo, and H. Jiang, *A finite element method for transient analysis of concurrent large deformation and mass transport in gels.* Submitted, 2008.