REGULARITY OF SOLUTIONS FOR A TWO-PHASE
DEGENERATE STEFAN PROBLEM

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IMA Preprint Series # 804
May 1991
Regularity of Solutions for a Two-Phase Degenerate Stefan Problem

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Abstract
For a one-dimensional two-phase degenerate Stefan problem, we prove that the free boundary is \(C^\infty\) smooth and the solutions are \(C^\infty\) smooth up to the boundary. The proof is based on performing the hodograph transformation to fix the free boundary and establishing a nonlinear a priori estimate for the solution.

1 Introduction

In this paper, we study the regularity of solutions for the following one-dimensional two-phase degenerate Stefan problem.

Let \(T > 0\). Find functions \(u^-(x,t), u^+(x,t)\) and \(s(t)\), which are defined on \([-1, s(t)] \times [0, T], [s(t), 1] \times [0, T]\) and \([0, T]\) respectively, and satisfy

\[
\begin{align*}
\alpha^- (u) u^-_t - u^-_{xx} &= 0, \quad -1 < x < s(t), 0 < t < T, \\
\alpha^+ (u) u^+_t - u^+_xx &= 0, \quad s(t) < x < 1, 0 < t < T. \\
u^\pm (x,0) &= u^\pm_0 (x), \quad 0 \leq \pm x \leq 1, \quad s(0) = 0. \\
u^- (-1,t) &= g^-(t) < 0, \quad u^+(1,t) = g^+(t) > 0, \quad 0 \leq t \leq T.
\end{align*}
\]

The basic assumptions of the problem (1.1)-(1.4) are the following

*1980 Mathematics Subject Classification (1985 Revision): 35K65, 35R35.
Keywords: degenerate parabolic equations, two-phase Stefan problem.
†This work was supported in part by ONR under Grant N00014-91-J-1291, WVU Senate Grant, and IMA at University of Minnesota.
(H1) $\alpha^\pm(u) \in C^\infty(\bar{R}_\pm^1)$, $\alpha^\pm(u) \geq 0$, $\alpha^\pm(u) = 0$ if and only if $u = 0$.

(H2) $\partial_x u^\pm_0(0) > 0$, $\pm g^\pm(t) \geq \delta_0$ with $\delta_0 > 0$ and $u^\pm_0(x) \geq 0$ where the equality holds if and only if $x = 0$.

The problem (1.1)-(1.4) is a two-phase degenerate Stefan problem which arises in a number of physical processes [13]. The problem has been studied by several authors. The uniqueness of the weak solution was demonstrated by Crowley [3]. The existence of weak solution and its regularity were studied by [2,14]. The existence of weak solution was obtained and the free boundary was shown to be Lipschitz continuous [2], and the boundary condition was satisfied in the classical sense almost everywhere [14]. The main result of [2,14] is the following [2, Theorem 1]:

**Theorem 1.1** Assume the data in (1.1)-(1.4) satisfy (H1)(H2) and

- $g^\pm(t) \in C^2[0,T]$, $u^\pm_0(x) \in C^3[\pm1,0]$;
- $g^\pm(0) = u^\pm_0(\pm1)$, $u^\pm_0(0) = 0$.

Then there is a unique weak solution $(u^\pm,s)$ to the problem (1.1)-(1.4) such that

- $x = s(t)$ is Lipschitz continuous in $[0,T]$;
- $u^\pm(x,t) \in C^{2,1}([\pm1,s(t)] \times [0,T])$, satisfy (1.1)(1.2)(1.3) in the classical sense, satisfy (1.4) almost everywhere;
- $\pm u^\pm(x,t) > 0$ in $\pm s(t) < \pm x < 1$, $0 \leq t \leq T$.

In this paper, based upon the result obtained in [] and under a similar assumption, we prove the $C^\infty$ regularity of both the free boundary $x = s(t)$ and the solutions $u^\pm(x,t)$ up to the free boundary.

First we introduce the following concept of compatibility.

**Definition 1.1** The data $u^\pm_0(x)$ and $g^\pm(t)$ are called $k$-order compatible at $(x,t) = (0,0)$ and $(\pm1,0)$ if $u^\pm_0(x) \in C^{2k}[\pm1,0]$, $g^\pm(t) \in C^k[0,T]$ and there exist functions $\tilde{s}(t) \in C^k[0,T]$ and $\tilde{u}^\pm(x,t) \in C^{2k,k}([\tilde{s}(t),\pm1] \times [0,T])$ such that

\[
\tilde{u}^\pm(\pm1,t) = g^\pm(t), \quad \tilde{u}^\pm(\tilde{s}(t),t) = 0; \tag{1.5}
\]

\[
\tilde{f}^\pm(x,t) \equiv \alpha(\tilde{u}^\pm)\tilde{u}^\pm_x - \tilde{u}^\pm_{xx} = O(t^k); \tag{1.6}
\]

\[
\tilde{s}(0) = 0, \quad \tilde{g}(t) \equiv \tilde{s}'(t) - \tilde{u}^\pm_x(\tilde{s}(t) - 0, t) + \tilde{u}^\pm_x(\tilde{s}(t) + 0, t) = O(t^k). \tag{1.7}
\]

If $u^\pm_0(x) \in C^{\infty}[\pm1,0]$, $g^\pm(t) \in C^{\infty}[0,T]$ and (1.5)-(1.7) are satisfied for any $k$, then the initial data are called to be $C^\infty$ compatible.
The main result of this paper is the following theorem:

**Theorem 1.2** For the problem (1.1)-(1.4) under the assumptions (H1)(H2),

1. if the data \( u_0^\pm(x) \in C^4[\pm 1, 0] \) and \( g^\pm(t) \in C^2[0, T] \) satisfy the 2-order compatibility conditions (1.5)-(1.7), then the unique solution has the following regularity:

   \[
   s(t) \in C^\infty(0, T], \quad u^\pm(x, t) \in C^\infty((\pm 1, s(t)) \times (0, T]).
   \]  

(1.8)

2. if the data \( u_0^\pm(x) \in C^\infty[\pm 1, 0] \) and \( g^\pm(t) \in C^\infty[0, T] \) satisfy the \( C^\infty \) compatibility conditions (1.5)-(1.7), then the unique solution is also \( C^\infty \) smooth up to the boundaries \( t = 0 \) and \( x = \pm 1 \).

The \( C^\infty \) compatibility requirement in the second part of theorem 1.2 is obviously necessary for the solutions to be \( C^\infty \) smooth up to \( t = 0 \) and \( x = \pm 1 \).

We prove Theorem 1.2 in the following sections. In section 2, we perform the hodograph transformation to reduce the free boundary problem (1.1)-(1.4) into a fixed boundary problem. Then a priori estimates for the resulting problem are derived in section 3. Section 4 establishes the existence of required smooth solution by linear iteration.

It is easy to see that all the proofs in this paper can also applied to the case with nondegenerate \( \alpha^\pm(u) \geq \delta > 0 \). Therefore, this paper also provides another proof to the result in [4,12] and other previous works. The author is thankful to Professors Avner Friedman, Daniel Phillips and Lihe Wang for helpful conversations.

## 2 Transformed Problem

In the following, we will always treat the point \( t = T \) as if it is an interior point where all the boundary conditions as well as interior equations (1.1) are satisfied. This is indeed true because we can always extend the definitions of \( g^\pm \) into \([0, T + \epsilon]\) and consider instead the problem in the larger domain \([-1, 1] \times [0, T + \epsilon]\).

### 2.1 Preparatory propositions

First, we state some easy consequences of Theorem 1.1.

**Proposition 2.1** The solution \( u^\pm(x, t) \) in Theorem 1.1 is \( C^\infty \) in \([\pm 1, s(t)] \times (0, T]\).

**Proof:** In \([\pm 1, s(t)] \times (0, T]\), \( \alpha^\pm(u) \geq 0 \) since \( u \neq 0 \). Hence \( C^\infty \) smoothness of the solution follows from applying repeatedly the standard interior Schauder estimate.
Proposition 2.2 Let $u^\pm(x,t)$ be the solution of (1.1)-(1.4) in Theorem 1.1. Then there exists $\beta_0 > 0$ such that for any $\epsilon$, $0 \leq \epsilon \leq \beta_0$, the set

$$S_{\pm \epsilon} = \{(x,t) : u^\pm(x,t) = \pm \epsilon, \ 0 \leq t \leq T\}$$

intersects with any line $t = t'$ $(0 \leq t' \leq T)$ only at one point. and the mapping

$$(t,x) \mapsto (\tilde{t}, \tilde{x}) = (t, u(x,t))$$

is a bijection

$$R_{\beta_0} = \bigcup_{0 \leq \epsilon \leq \beta_0} S_{\pm \epsilon} \to [-\beta_0, \beta_0] \times [0,T].$$

PROOF: The case of $\epsilon = 0$ is proved in [2]. From the assumption (H2), for $\beta_0 \ll 1$, the case for $0 < \epsilon \leq \beta_0$ is derived readily from maximum principle.

The following proposition improves the estimate obtained in [2].

Proposition 2.3 Let $u^\pm(x,t)$ be the solution of (1.1)-(1.4) obtained in [2]. Then there exists $\beta > 0$ such that for any $\epsilon$, $0 < \epsilon \leq \beta$, the set

$$S_{\pm \epsilon} = \{(x,t) : u^\pm(x,t) = \pm \epsilon, \ 0 \leq t \leq T\}$$

can be written as

$$x = s_{\pm \epsilon}(t), \ s_{\pm \epsilon}(t) \in C^\infty(0,T] \cap C^1[0,T].$$

In addition, there exists $\delta > 0$ such that for the derivatives $u_x^\pm(x,t)$ in $R_\beta$ where they exist, we have

$$\frac{1}{\delta} \geq u_x^\pm(x,t) \geq \delta > 0, \ \text{in} \ R_\beta.$$

PROOF: First of all, we can choose $\beta \ll 1$ so that $\beta \leq \beta_0$ in the Proposition 2.2 and $\pm \beta$ are not critical values for $u^\pm(x,t)$. Consequently by Sard lemma and following the same argument as in [2],

$$x = s_{\pm \beta}(t) \in C^\infty(0,T] \cap C^1[0,T], \ u^\pm(s_{\pm \beta}(t),t) = \pm \beta.$$

Besides, by assumption (H2), we can also choose $\beta$ such that

$$\delta_0^{-1} \geq \partial_x u_0^\pm(x) \geq \delta_0 > 0, \ -\beta \leq x \leq \beta.$$

For the above chosen $\beta$, we need only to show (2.3), because (2.3) implies that none of the values $\epsilon \in [-\beta,0) \cup (0,\beta]$ is critical value of $u^\pm(x,t)$ and (2.2) follows by Sard lemma.
(2.3) can be shown by similar argument in [2] as follows. Let \( c(u) \) be defined by
\[
c(u) = \begin{cases} 
\int_0^u \alpha^+(\xi) \, d\xi \equiv c^+(u), & u > 0, \\
[-1, 0], & u = 0, \\
-1 + \int_0^u \alpha^-(\xi) \, d\xi \equiv c^-(u), & u < 0.
\end{cases}
\]
Construct smooth approximate sequences
\[
c_n(u) \in C^\infty(R^1), \quad U_{0n}(x) \in C^2[-\beta, \beta]
\]
such that
- \( \lim_{n \to \infty} c_n(u) = c(u) \) in \( L^2(R^1) \);
- \( c'_n(u) \geq 1/n \) and \( c'_n(u) = c'(u) \) when \( |u| \geq \delta_n \) for small \( \delta_n > 0 \) and \( \delta_n \to 0 \) as \( n \to \infty \);
- \( U_{0n}(x) = u_0^+(x) \) in \([x_-, x_0/2], [x_{+0}/2, x_+0]\) with \( x_{\pm0} = s_{\pm0}(0) \);
- \( |U_{0n}(x)| \leq \beta, U_{0n}(x) \to u_0^+(x) \) in \([x_-, x_0/2]\) uniformly and
\[
\frac{2}{\delta_0} \geq \partial_x U_{0n}(x) \geq \frac{\delta_0}{2}.
\]
(2.6)
Consider the approximation problem as [2] in \( R_\beta = [s_{-\beta}, s_{+\beta}] \times [0, T] \):
\[
(P_n) \left\{ \begin{array}{l}
\partial_t c_n(U_n) = \partial_x^2 U_n \text{ in } (s_{-\beta}(t), s_{+\beta}(t)) \times (0, T), \\
U_n(s_{\pm\beta}(t), t) = \pm \beta, \quad 0 \leq t \leq T, \\
U_n(x, 0) = U_{0n}(x), \quad x_- \leq x \leq x_+. \end{array} \right.
\]
The problem \( P_n \) has a unique classical solution \( U_n(x, t) \in C^{2,1}(R_\beta) \) and there is a subsequence of \( \{U_n(x, t)\} \) converging uniformly to the solution \( u^\pm(x, t) \) of (1.1)-(1.4) in \( R_\beta \). It was established in [2, Lemma 3, 4] that
\[
0 < N_n \leq U_{nx}(x, t) \leq M \text{ in } R_\beta,
\]
(2.7)
with \( M \) independent of \( n \), but \( N_n \) may not be uniform with respect of \( n \). However, with our choice of the domain \( R_\beta \), we are going to show that actually \( N_n \) can be chosen independent of \( n \)
\[
N_n \geq \delta > 0.
\]
(2.8)
Once (2.8) is proven, we derive for the limit function \( u^\pm(x,t) \) the estimate (2.3) by the well-known theorem in functional analysis that the strong closure and weak closure for convex set coincide.

To prove (2.8), first we notice that on the boundaries \( x = s_{\pm \beta}(t) \), \( U_n(x,t) \) takes its maximum and minimum value \( \pm \beta \). The boundaries \( x = s_{\pm \beta}(t) \) are smooth so that we can apply the strong maximum principle to derive

\[
U_{nx}(s_{\pm \beta}(t), t) \geq \delta'.
\]  

(2.9)

We show that \( \delta' \) can be chosen independent of \( n \) and \( t \). Since \([0,T]\) is compact, we need only to show that for each fixed \( t \) \((0 \leq t \leq T)\), \( \delta' \) is independently of \( n \). Since at \( t = 0 \), \( U_{nx}(s_{\pm \beta}(t), t) = U_{0x}(x_{\pm 0}) \geq \delta_0/2 \), we may assume \( t > 0 \). In the following we consider the point on \( x = s_{-\beta}(t) \), the case \( x = s_{+\beta}(t) \) can be discussed in the same way.

For any fixed point \( P(x^*, t^*) \equiv (s_{+\beta}(t^*), t^*) \), \((t^* > 0)\), we can construct a sphere \( B \) contained in \( R_\beta \), centered at \((\bar{x}, \bar{t}) \) \((\bar{x} \neq x^*)\) and tangent to \( x = s_{+\beta}(t) \) only at \( P(x^*, t^*) \).

As in the standard proof of the strong maximum principle [5], divide the plane into two parts \( \pi^- \) and \( \pi^+ \) by a straight line \( \pi \) such that

\[
(\bar{x}, \bar{t}) \in \pi^-, \ (x^*, t^*) \in \pi^+; \ B^+ \equiv \pi^+ \cap B \neq \emptyset;
\]

\[
|\bar{x} - x| \geq \text{const} > 0, \ (x, t) \in B^+.
\]

Denote the boundary of \( B^+ \) as \( \partial B^+ = S_1 \cup S_2 \),

\[
S_1 = \partial B^+ \cap \partial B, \ S_2 = \partial B^+ \cap \pi.
\]

Construct barrier function

\[
h(x, t) = e^{-\kappa(\|x-x^\| + |t-t^*|)} - e^{-\kappa r^2}
\]

where \( r \) is the radius of \( B \). Then \( h = 0 \) on \( S_1 \), \( h \geq 0 \) on \( \tilde{B}^+ \). For \( \kappa \gg 1 \), we have

\[
(\partial_\bar{x}^2 - c_n(U_n) \partial_\bar{t})h > 0, \ \text{in} \ B^+
\]

independent of \( n \). Taking \( \epsilon \ll 1 \), the function \( V_n(x,t) = U_n(x,t) + \epsilon h(x,t) < \beta \) on \( S_2 \) uniformly in \( n \) for large \( n \) because

\[
u^+(x,t) \leq \beta - \epsilon_1, \ \text{for} \ \epsilon \ll 1 \ \text{on} \ S_2
\]

and \( U_n \) converges uniformly to \( u^+ \). On \( S_1 \), \( V_n(x,t) = U_n(x,t) < \beta \) for \((x,t) \neq (x^*, t^*)\) and \( V_n(x^*, t^*) = U_n(x^*, t^*) = \beta \). By the strong maximum principle, \( V_n(x,t) < \beta \) in \( B^+ \). Therefore \( \partial_\bar{x} V_n(x^*, t^*) \geq 0 \). But \( h(x^*, t^*) < 0 \), so

\[
\partial_\bar{x} U_n(x^*, t^*) > -h(x^*, t^*) > 0
\]

which is uniformly true for all \( n \). Applying \( \partial_\bar{x} \) to the problem \((P_n)\) and employing the strong maximum principle as in [2] to \( \partial_\bar{x} U_n \) in \( R_\beta \), we obtain (2.8). This concludes the proof of Proposition 2.3.
2.2 Hodograph transformation

Since $u^\pm(x,t)$ are known to be $C^\infty$ smooth away from $x = s(t), \pm 1$, and near the fixed boundaries $x = \pm 1$, the regularity results of the solutions are classical. In the following we will consider the solution $u^\pm(x,t)$ only in the domain $R$.

We perform the hodograph transformation [6,9] as in (2.1) to reduce the problem (1.1)-(1.4) inside $R$ into a problem in $[-\beta, \beta] \times [0, T]$. Applying hodograph transformation in the discussion of multi-dimensional nondegenerate Stefan problem was previously found in [10]. Introduce new variables

$$\tilde{x} = u(x,t), \quad \tilde{t} = t.$$ \hspace{1cm} (2.10)

By Proposition 2.3, the transformation (2.10) is invertible in $R^\circ = [s^{-\beta}(t), s(t)]$ and $R^\circ = (s(t), s^\circ(t))$ respectively. Denote the inverse transformation as:

$$x = v(\tilde{x}, \tilde{t}), \quad t = \tilde{t}.$$ \hspace{1cm} (2.11)

Taking derivative of $x = v(u(x,t), t)$ with respect to $x$ and $t$, we have

$$u_t = -\frac{v_t}{v_x}, \quad u_x = \frac{1}{v_x}.$$ \hspace{1cm} (2.12)

Hence

$$u_{xx} = \frac{1}{v_x} \partial_x u_x = \frac{1}{v_x} \partial_x \left( \frac{1}{v_x} \right) = -\frac{v_{xx}}{v_x^3}.$$ \hspace{1cm} (2.13)

From (2.11), obviously

$$s'(t) = v_t(0,t).$$ \hspace{1cm} (2.14)

Omitting bar in new coordinates, we obtain a new transformed problem in $[-\beta, \beta] \times [0, T]$:

1. Interior equations:

$$\begin{align*}
\alpha_-(x)v^-_t + \partial_x \left( \frac{1}{v^-_x} \right) &= 0, \quad -\beta < x < 0, \quad 0 < t \leq T, \\
\alpha_+(x)v^+_t + \partial_x \left( \frac{1}{v^+_x} \right) &= 0, \quad 0 < x < \beta, \quad 0 < t \leq T. 
\end{align*}$$ \hspace{1cm} (2.15)

2. Initial conditions:

$$v^\pm(x,0) = v_0^\pm(x), \quad 0 \leq \pm x \leq \beta; \quad v_{0x}^\pm(0) > 0.$$ \hspace{1cm} (2.16)
3. Boundary conditions:

\[ v^\pm(\pm \beta, t) = s_{\pm \beta}(t) \in C^\infty(0, T); \]  \hspace{1cm} (2.17)

\[ v^+(0, t) - v^-(0, t) = 0, \quad 0 < t \leq T; \]  \hspace{1cm} (2.18)

\[ v_t(0, t) = \frac{1}{v^-_t(0, t)} - \frac{1}{v^+_t(0, t)}, \quad 0 < t \leq T. \]  \hspace{1cm} (2.19)

According to the Proposition 2.3, we see that

\[ 0 < \delta \leq v^\pm_t(x, t) \leq \delta^{-1}, \quad \text{uniformly in \ } [\pm \beta, 0) \times [0, T]. \]  \hspace{1cm} (2.20)

Therefore the smooth solutions \( v^\pm \) for the problem (2.15)-(2.19) in \( \{[-\beta, 0] \cup [0, \beta]\} \times [0, T] \) are equivalent to the smooth solutions \( (u^\pm, s) \) of the original problem (1.1)-(1.4). Consequently the proof of Theorem 1.2 is reduced to the proof of the following equivalent

**Theorem 2.1** In the problem (2.15)-(2.19),

- assume that \( v^+_0(x) \in C^4[\pm \beta, 0], \ s_{\pm \beta}(t) \in C^2[0, T] \) satisfy the \( 2^{\text{nd}} \)-order compatibility conditions, there is a unique solution \( v^\pm(x, t) \in C^\infty((\pm \beta, 0) \times (0, T]) \).

- If the data \( v^+_0(x) \in C^k[\pm \beta, 0] \) and \( s_{\pm \beta}(t) \in C^\infty[0, T] \) satisfy the \( C^\infty \) compatibility conditions, then the unique solution is also \( C^\infty \) smooth up to the boundaries \( t = 0 \) and \( x = \pm \beta \).

Here the \( k^{\text{th}} \)-order and \( C^\infty \) compatibility conditions are similarly defined as in (1.5)-(1.7) of the Definition 1.1.

We prove the theorem 2.1 in section 3 and 4.

### 3 A Priori Estimate for Nonlinear Problem

#### 3.1 A priori estimate

For simplicity, we will denote briefly \( v(x, t) = v^\pm(x, t) \) and \( \alpha(x) = \alpha^\pm(x) \) in \( \pm x \geq 0 \). We denote \( (\cdot, \cdot) \) the inner product of \( L^2[-\beta, \beta] \) and \( \| \cdot \| \) the corresponding norm. Therefore

\[ \| \sqrt{\alpha} v(t) \|^2 \equiv \int_{-\beta}^{0} \alpha_{-}(x)|v^-(x, t)|^2 dx + \int_{0}^{\beta} \alpha_{+}(x)|v^+(x, t)|^2 dx \]

and so on.

First we prove the standard energy estimate for solutions \( v^\pm(x, t) \in C^\infty([\pm \beta, 0] \times [0, T]) \) of (2.15)-(2.19) when the \( C^\infty \) compatibility conditions are satisfied. The main result of this section is the following theorem.
Theorem 3.1 Let \( v^\pm(x, t) \in C^\infty([\pm \beta, 0] \times [0, T]) \) be a solution of (2.15)-(2.19). Then for any nonnegative integer \( k \) and \( t \in [0, T] \), it satisfies the following estimate

\[
\sum_{j=0}^{k} \partial_t \left( \|\sqrt{\alpha} \partial_{x}^{j} v(t)\|^2 + |\partial_{x}^{j} v(0, t)|^2 \right) + \sum_{2k_1 + k_2 \leq 2k} \|\partial_{x}^{k_1} \partial_{x}^{k_2+1} v(t)\|^2 \leq C_k. \tag{3.1}
\]

Here the constant \( C_k \) depends upon the known initial and boundary values at \( t = 0 \) and \( x = \pm \beta \) up to the order related to \( k \) and is independent of the time \( t \in [0, T] \).

Proof: The proof of the theorem is carried out by induction on \( k \).

1. \( k = 0 \):
   
   Taking inner product of (2.15) with \( v(x, t) \) and integrating by parts over \( -\beta \leq x \leq \beta \), we have
   
   \[
   \frac{1}{2} \partial_t \|\sqrt{\alpha} v(t)\|^2 - \frac{v^+(0, t)}{v_x^+(0, t)} + \frac{v^-(0, t)}{v_x^-(0, t)} - 2 \leq C_0.
   \]

   Making use of the boundary conditions (2.18)(2.19), we obtain
   
   \[
   \partial_t \left( \|\sqrt{\alpha} v(t)\|^2 + |v(0, t)|^2 \right) \leq C. \tag{3.2}
   \]

   The estimate for \( \partial_{x} v \) follows directly from (2.20).

2. \( k = 1 \):
   
   Applying \( \partial_t \) to (2.15), taking its inner product with \( v_t(x, t) \) and integrating by parts over \( -\beta \leq x \leq \beta \), we have
   
   \[
   \frac{1}{2} \partial_t \|\sqrt{\alpha} v_t(t)\|^2 - (\partial_{x}^{-1} v_{tx} - v_{tx}^+(0, t) \left( \frac{1}{v_x^+} \right)_t (0, t) + v_{tx}^-(0, t) \left( \frac{1}{v_x^-} \right)_t (0, t) \leq C_1.
   \]

   Making use of the boundary conditions (2.18)(2.19) and noticing (2.20), we obtain
   
   \[
   \partial_t \left( \|\sqrt{\alpha} v_t(t)\|^2 + |v_t(0, t)|^2 \right) + \|v_{tx}(t)\|^2 \leq C_1. \tag{3.3}
   \]

   From (2.15) and (2.20), we obtain the estimate for \( \|v_{xx}(t)\| \) and \( \|v_{xxx}\| \) since
   
   \[
   \|v_{xxx}(t)\| = \|\partial_x (v_x^2 \alpha(x) v_t)(t)\| \leq C \left( \|v_t(t)\| + \|v_{xx}(t)\| + \|v_{tx}(t)\| \right).
   \]

3. \( k = 2 \):
Applying $\partial^2_t$ to (2.9), taking its inner product with $v_{tt}(x,t)$ and integrating by parts over $-1 \leq x \leq 1$, we have

$$\frac{1}{2} \partial_t \|\sqrt{\alpha} v_{tt}(t)\|^2 - (\partial^2_t v_x^{-1}, v_{tx}) - v^+_t(0,t) \left( \frac{1}{v_x^+} \right)_{tt} - v^+_t(0,t) \left( \frac{1}{v_x^-} \right)_{tt} \leq C_2.$$ 

Making use of the boundary conditions (2.18)(2.19) and noticing (2.20), we obtain

$$\partial_t \left( \|\sqrt{\alpha} v_{tt}(t)\|^2 + |v_{tt}(0,t)|^2 \right) + \|v_{tx}(t)\|^2 \leq C_2 + 2|\langle v^2_x v_x^{-3}, v_{tx} \rangle|. \quad (3.4)$$

Since

$$2|\langle v^2_x v_x^{-3}, v_{tx} \rangle| \leq \frac{1}{2} \|v_{tx}(t)\|^2 + 4 \|v_x^2(t)\|^2 \leq \frac{1}{2} \|v_{tx}(t)\|^2 + 4 \|v_x(t)\|_{L^\infty} \|v_{tx}(t)\|^2$$

and

$$|v_{tx}(t)|_{L^\infty} \leq C(1 + \|v_{txx}(t)\|) = C(1 + \|\partial_t (v_x^2 \alpha v_t)(t)\|) \leq C(1 + \|v_{tx}\| + \|\sqrt{\alpha} v_{tt}(t)\|) \leq C_1(1 + \|\sqrt{\alpha} v_{tt}(t)\|),$$

Noticing (3.3) we obtain the following

$$\partial_t \left( \|\sqrt{\alpha} v_{tt}(t)\|^2 + |v_{tt}(0,t)|^2 \right) + \|v_{tx}\|^2 \leq C_2 (1 + \|\sqrt{\alpha} v_{tt}(t)\|^2).$$

Employing Gronwall inequality, we have

$$\partial_t \left( \|\sqrt{\alpha} v_{tt}(t)\|^2 + |v_{tt}(0,t)|^2 \right) + \|v_{tx}\|^2 \leq C_2. \quad (3.6)$$

Using (2.20) and the equations (2.15), we derive readily from (3.6) the estimates

$$\sum_{2k_1 + k_2 \leq 4} \|\partial^{k_1}_t \partial^{k_2+1}_x v(t)\|^2 \leq C_2.$$

Therefore (3.1) is proved for $k = 2$.

4. Induction on $k$:

Assume (3.1) true for all $j \leq k$, $k \geq 2$. Applying $\partial^{k+1}_t$ to (2.15), taking its inner product with $\partial^{k+1}_t v(x,t)$ and integrating by parts over $-\beta \leq x \leq \beta$, we have

$$\frac{1}{2} \partial_t \|\sqrt{\alpha} \partial^{k+1}_t v(t)\|^2 - (\partial^{k+1}_t v_x^{-1}, \partial^{k+1}_t v_x)

- \partial^{k+1}_t v^+(0,t) \partial^{k+1}_t \left( \frac{1}{v^+_x} \right)(0,t) + \partial^{k+1}_t v^-(0,t) \partial^{k+1}_t \left( \frac{1}{v^-_x} \right)(0,t) \leq C_{k+1}. \quad (3.7)$$
Making use of the boundary conditions (2.11)(2.12) and noticing $v_x$ is bounded, we obtain
\[ \partial_t \left( \| \sqrt{\alpha} \partial_t^{k+1} v(t) \|^2 + | \partial_t^{k+1} v(0, t) |^2 \right) + \| \partial_t^{k+1} v_x(t) \|^2 \leq C_{k+1} (1 + | (F_k, \partial_t^{k+1} v_x) |) \] (3.8)

where $F_k$ has the form
\[ F_k = \partial_t^{k} \left( \frac{v_{xt}}{v_x^2} \right) - \frac{\partial_t^k v_x^2}{v_x^2} = \sum_{k_1 + k_2 = k-1} A_{k_1, k_2} \left( \partial_t^{k_1+1} v_x \right) \partial_t^{k_2+1} \left( \frac{1}{v_x^2} \right) . \] (3.9)

(a) To estimate $F_k$.
- For $k_1 \leq k - 2$ and $k_2 \leq k - 2$, by induction we have
  \[ \| \partial_t^{k_1+1} v_x \partial_t^{k_2+1} v_x^{-2} \| \leq C_{k+1} | \partial_t^{k_1+1} v_x |_{L^\infty} | \partial_t^{k_2+1} v_x |_{L^\infty} \]
  \[ \leq C_{k+1} \left( 1 + \| \partial_t^{k_1+1} v_{xx} \| \right) \left( 1 + \| \partial_t^{k_2+1} v_{xx} \| \right) \leq C_{k+1} . \]

- For $k_1 = k - 1$, $k_2 = 0$ or $k_1 = 0$, $k_2 = k - 1$, we have
  \[ \| \partial_t^{k_1+1} v_x \partial_t^{k_2+1} v_x^{-2} \| \leq C_{k+1} | \partial_t v_x |_{L^\infty} \| \partial_t^k v_x \| \]
  \[ \leq C_{k+1} \left( 1 + \| \partial_t^k v_x \| \right) \leq C_{k+1} . \]

Therefore $\| F_k \| \leq C_{k+1}$. Combining with (3.8), we have
\[ \partial_t \left( \| \sqrt{\alpha} \partial_t^{k+1} v(t) \|^2 + | \partial_t^{k+1} v(0, t) |^2 \right) + \| \partial_t^{k+1} v_x(t) \|^2 \leq C_{k+1} . \] (3.11)

(b) To prove the Theorem 3.1, we need to estimate all the terms of
\[ \partial_t^{k_1} \partial_x^{k_2+1} v, \quad 2k_1 + k_2 \leq 2(k + 1) . \] (3.12)

These terms can be estimated by induction on $k_2$. The case for $k_2 = 0$ is (3.11). We will assume (3.12) is estimated for $k_2 \leq 2m$, $m > 0$ and derive the estimate for $k_2 = 2m + 1$ and $k_2 = 2m + 2$.

- For $k_2 = 2m + 1$, then $k_1 \leq k - m$.
  If $k_1 < k - m$, then $2k_1 + k_2 \leq 2k$. So the terms in (3.12) are estimated in (3.1) by the induction assumption on $k$.
  In $k_1 = k - m$, $k_2 = 2m + 1$,
  \[ \partial_t^{k-m} \partial_x^{2m+2} v = \partial_x^{2m} \partial_t^{k-m} (v_x^2 \alpha v_t) \]
  \[ = \partial_x^{2m} \sum_{j_1 + j_2 = k-m} B_{ji,j_j} \left( \partial_t^{j_1} (v_x^2) \right) \left( \partial_t^{j_2} (\alpha v_t) \right) . \] (3.13)
By the Nirenberg inequality [11], we have
\[
\|\partial_t^{k-m} \partial_x^{2m+2} v\|
\leq C_{k+1} \sum_{j_1 + j_2 = k-m} \left( |\partial_t^{j_2} (\alpha v_t)|_{L^\infty} \|\partial_t^{j_2} \partial_x^{2m} v_x^2\| + |\partial_t^{j_1} v_x^2|_{L^\infty} \|\partial_t^{j_2} \partial_x^{2m} (\alpha v_t)\| \right)
\tag{3.14}
\]

By the induction assumption on \(k_2\) we derive from (3.14) the estimate
\[
\|\partial_t^{k-m} \partial_x^{2m+2} v\| \leq C_{k+1}.
\tag{3.15}
\]

• For \(k_2 = 2m + 2\), then \(k_1 \leq k - m\). If \(k_2 < k - m\), the terms are estimated by the induction assumption on \(k\). Consider \(k_2 = 2m + 2\) and \(k_1 = k - m\). Similar as (3.13)-(3.15), we have
\[
\|\partial_t^{k-m} \partial_x^{2m+3} v\| = \|\partial_x^{2m+1} \partial_t^{k-m} (v_x^2 \alpha v_t)\|
\leq C_{k+1} \sum_{j_1 + j_2 = k-m} \left( |\partial_t^{j_2} (\alpha v_t)|_{L^\infty} \|\partial_t^{j_1} \partial_x^{2m+1} v_x^2\| + |\partial_t^{j_1} v_x^2|_{L^\infty} \|\partial_t^{j_2} \partial_x^{2m+1} (\alpha v_t)\| \right)
\]
which can be estimated by (3.15) and the induction assumption on \(k_2 = 2m\). This finished the induction proof on \(k_2\).

### 3.2 Truncated estimate

Now we assume that the solutions \(v^\pm(x, t)\) be \(C^\infty\) only in \(t > 0\) and be \(C^2\) up to \(t = 0\). Then we have the following theorem of the truncated estimate for (2.15)-(2.19).

**Theorem 3.2** Let \(v^\pm(x, t) \in C^\infty([-\beta, \beta] \times (0, T)]\) be a solution of (2.15)-(2.19) and being \(C^2\) up to \(t = 0\). Then for any nonnegative integer \(k\) and \(0 \leq t \leq T\), the solution satisfies the following estimate
\[
\sum_{j=0}^{k} \partial_t \left( \|\sqrt{\alpha} D_t^j \partial_x^2 v(t)\|^2 + |D_t^j \partial_t^2 v(0, t)|^2 \right)
+ \sum_{2k_1 + k_2 = 2k \atop 2j_1 + j_2 \leq 4} \|D_t^{k_1} D_x^{k_2} \partial_t^{j_1} \partial_x^{j_2+1} v(t)\|^2 \leq C'_k,
\tag{3.16}
\]

where
\[
D_t \equiv t \partial_t, \quad D_x \equiv t \partial_x.
\]

Here in (3.16) the constant \(C'_k\) depends upon the \(C^4\) norms of the initial data at \(t = 0\) and \(C^{k+2}\) truncated norms of the boundary value on \(x = \pm \beta\).
PROOF: The case $k = 0$ is included in (3.1). We derive (3.16) by induction on $k$ as in the proof of Theorem 3.1.

1. Applying $D_{t}^{k+1} \partial_t^2$ to (2.15) and integrating by parts its inner product with $D_{t}^{k+1} \partial_t^2 v$ over $[-\beta, \beta]$, we have

$$\frac{1}{2} \partial_t \left( \| \sqrt{\alpha} D_{t}^{k+1} \partial_t^2 v(t) \|^2 + \| D_{t}^{k+1} \partial_t^2 v(0, t) \|^2 \right) + \| D_{t}^{k+1} \partial_t^2 v_x(t) \|^2 \leq G_k$$

(3.17)

where

$$G_k \equiv G'_k + G''_k + C_{k+1}$$

$$= (\alpha [\partial_t, D_{t}^{k+1}] \partial_t^2 v, D_{t}^{k+1} \partial_t^2 v) + \langle [\partial_t, D_{t}^{k+1}] \partial_t^2 v, D_{t}^{k+1} \partial_t^2 v \rangle_{x=0}$$

$$- (\langle D_{t}^{k+1} \partial_t, v_x^{-2} \rangle v_{tx}, D_{t}^{k+1} \partial_t^2 v_x) + C_{k+1}$$

(3.18)

- Estimate $G'_k$. Using the equation (2.15), integrating by parts and using the boundary conditions (2.18)(2.19), we have

$$G'_k = (\alpha [\partial_t, D_{t}^{k+1}] \partial_t^2 v, D_{t}^{k+1} \partial_t^2 v) + \langle [\partial_t, D_{t}^{k+1}] \partial_t^2 v, D_{t}^{k+1} \partial_t^2 v \rangle_{x=0}$$

$$= - (\langle [\partial_t, D_{t}^{k+1}] \partial_t v_x^{-1}, D_{t}^{k+1} \partial_t^2 v \rangle + \langle [\partial_t, D_{t}^{k+1}] \partial_t^2 v, D_{t}^{k+1} \partial_t^2 v \rangle_{x=0}$$

$$= (\langle [\partial_t, D_{t}^{k+1}] \partial_t v_x^{-1}, D_{t}^{k+1} \partial_t^2 v_x \rangle).$$

Therefore, we obtain

$$\| G'_k \|^2 \leq \frac{1}{4} \| D_{t}^{k+1} \partial_t^2 v \|^2 + 4 \| \partial_t, D_{t}^{k+1} \partial_t v_x^{-1} \|^2 \leq \frac{1}{4} \| D_{t}^{k+1} \partial_t^2 v \|^2 + C'_k$$

(3.19)

which follows from

$$\| \partial_t, D_{t}^{k+1} \partial_t v_x^{-1} \| \leq C'_k \sum_{j \leq k} \| D_{i}^j \partial_t \left( \frac{v_{tx}}{v_x^2} \right) \| \leq C'_k \sum_{j_1, j_2 \leq k} \| D_{i}^j v_x \| \| D_{i}^j v_{tx} \| \leq C'_k.$$

- Estimate $G''_k$. From (3.18)

$$\| G''_k \|^2 \leq \frac{1}{4} \| D_{t}^{k+1} \partial_t^2 v_x \|^2 + 4 \| [D_{t}^{k+1} \partial_t, v_x^{-2}] v_{tx} \|^2.$$

(3.20)

It is easy to derive

$$\| [D_{t}^{k+1} \partial_t, v_x^{-2}] v_{tx} \|^2 \leq C \left( \| D_{t}^{k+1}(v_{tx}^2 v_x^{-3}) \| + \sum_{k_1 + k_2 = k} \| D_{t}^{k_1+1} v_x^{-2} D_{t}^{k_2} v_{tx} \| \right)$$

$$\leq C'_k \| D_{t}^{k} v_{tx} \|^2 \leq C'_k.$$
Combining (3.17)-(3.21), we have
\[ \partial_t \left( \| \sqrt{\alpha} D_t^{k+1} \partial_t^2 v(t) \|^2 + |D_t^{k+1} \partial_t^2 v(0, t)|^2 \right) + \| D_t^{k+1} \partial_t^2 v_x(t) \|^2 \leq C_k'. \quad (3.22) \]

2. It remains to estimate the following terms to finish the proof of the Theorem 3.2
\[ \sum_{2k_1 + k_2 \leq 2k + 2} \| D_t^{k_1} D_x^{k_2} \partial_t^{j_1} \partial_x^{j_2+1} v(t) \|^2 \leq C_k'. \quad (3.23) \]

(3.23) can proven similarly as in the proof of the Theorem 3.1 by induction on \( k_2 \).
The operations involve using the equation (2.15) and employing (3.22). We omit the straightforward details.

This finishes the proof of the Theorem 3.2.

4 EXISTENCE OF \( C^\infty \) SOLUTION

Once the a priori estimates in the Theorem 3.1, Theorem 3.2 are obtained, there are a number of ways to establish the existence of the solution.

1. We need only to show the existence for the case when \( C^\infty \) compatibility conditions are satisfied because we can take an approximating sequence of \( C^\infty \) compatible initial data and the estimate in the Theorem 3.2 guarantees the convergence of the solution sequence to the desired solution.

2. For the \( C^\infty \) compatible data, the existence of smooth solution for (2.15)-(2.19) can be reduced to the existence of smooth solutions for a nondegenerate problem since (3.1) is also valid for \( \alpha^\pm(x) > 0 \).

Let \( \chi(x) \in C_0^\infty (\mathbb{R}^1) \) such that
\[ \chi(x) = \begin{cases} 
1, & \text{if } |x| < \beta/4, \\
0, & \text{if } |x| > \beta/2. 
\end{cases} \]

Choose a sequence of functions \( \alpha^\pm_{\epsilon}(x) \equiv \alpha^\pm(x) + \epsilon \chi(x) \) so that \( \alpha^\pm_{\epsilon}(x) > 0 \) and \( \alpha^\pm_{\epsilon}(x) \to \alpha^\pm(x) \) in \( C^\infty[\pm \beta, 0] \) as \( \epsilon \to 0 \).

Then choose a sequence of initial data \( v^\pm_{\epsilon 0}(x) \) such that \( v^\pm_{\epsilon 0}(x) \to v^\pm_0(x) \) in \( C^\infty[\pm \beta, 0] \) and \( v^\pm_{\epsilon 0}(x) \) is \( C^\infty \) compatible with respect to \( \alpha^\pm_{\epsilon}(x) \).
The initial data $v_{0x}^\pm(x)$ can be chosen as follows.
\[ v_{0x}^\pm(x) = v_0^\pm(x) + \epsilon \chi(x) \sigma^\pm(x), \]
where $\sigma^\pm(x) \in C^\infty[\pm \beta, 0]$ are constructed from there traces at $x = 0$. Actually, we choose $\sigma^\pm(0) = \sigma_x^\pm(0) = 0$ and determine all the higher order traces from
\[ v_x^\pm = (v_x^\pm)^2(\alpha(x) + \epsilon) \left( \frac{1}{v_x^-} - \frac{1}{v_x^+} \right). \]

Consider the following perturbed nondegenerate problem of (2.15)-(2.19):

(a) Interior equations:
\[ \begin{cases} 
\alpha_-(x)v_t^- + \partial_x \left( \frac{1}{v_x^-} \right) = 0, & -\beta < x < 0, \ 0 \leq t \leq T, \\
\alpha_+(x)v_t^+ + \partial_x \left( \frac{1}{v_x^+} \right) = 0, & 0 < x < \beta, \ 0 \leq t \leq T.
\end{cases} \] (4.1)

(b) Initial conditions:
\[ v^\pm(x,0) = v_{0x}^\pm(x), \ 0 \leq \pm x \leq \beta. \] (4.2)

(c) Boundary conditions:
\[ v^+(0,t) - v^-(0,t) = 0, \] (4.3)
\[ v_t(0,t) = \frac{1}{v_x^-(0,t)} - \frac{1}{v_x^+(0,t)}. \] (4.4)

If we can find $C^\infty$ smooth solutions $v_x^\pm$ for (4.1)-(4.4), then let $\epsilon \to 0$, There is a subsequence converging to a smooth solution $v^\pm$ for (2.9)-(2.12), by the Theorem 4.

3. To prove the existence of $C^\infty$ smooth solutions for (4.1)-(4.4), one can transform it back to the original free boundary problem and apply the result of the $C^\infty$ smooth solutions for nondegenerate Stefan problem [1,4,12]. We can also prove directly the existence for (4.1)-(4.4) by linear iteration. The proof can be sketched as follows.

Let $E_T^k$ be the space of functions $v(x,t)$ in $(-\beta, \beta) \times (0, T)$ such that
\[ \|v\|_{k,T} \equiv \sup_{0 \leq t \leq T} \sum_{0 \leq k \leq k_1 + k_2 \leq 2k+1} \left( \|\partial^{k_1}_t \partial^{k_2}_x v(t)\| \right)^{\frac{1}{2}} \leq \infty. \] (4.5)

We need to show that for any given integer $k$, the problem (4.1)-(4.4) has the solution $v(x,t) \in E_T^k$. 

15
By the estimate of Theorem 3.1, we need only to show that for any given \( k \), there is a \( t_0 > 0 \) such that in \( (0, t_0) \), (4.1)-(4.4) has a solution \( v(x, t) \in E^k_{t_0} \). Because once this is proved, the solution at \( t = t_0 \) has finite norm in \( E^k_{t_0} \) and obviously the \((k-1)\)-order compatibility conditions are satisfied at \( x = 0, \ t = t_0 \). So one can solve (4.1)-(4.4) again from the time \( t = t_0 \) and extend the solution \( v(x, t) \) to the time \( t_1 > t_0 \). This procedure can be repeated until \( t = T \).

The existence of solution in \([0, t_0]\) for (4.1)-(4.4) can be proved by linear iteration. Because the nonlinear terms in the boundary condition (4.4) contain \( v_x^{\pm}(0, t) \) and we are unable to obtain their estimate for the linearized problem, the Nash-Moser iteration can be employed. Let \( H^k(0, T) \) be the space defined by the norm \( \|v\|_{k,T} \):

\[
\|v\|_{k,T}^2 = \sum_{2k_1 + k_2 \leq 2k+1} \int_0^T \| \partial_t^{k_1} \partial_x^{k_2} v(t) \|^2 dt \leq \infty.
\]

Denote briefly

\[
L(v) v \equiv \left( \alpha(x) \partial_t - \frac{1}{v_x^2} \partial_{xx} \right) v, \quad B(v) v \equiv \partial_t v - \frac{1}{v_x^-} + \frac{1}{v_x^+}. \tag{4.6}
\]

Then the linearized operators of (4.6) at \( v \) are

\[
\ell(v) \hat{v} \equiv L(v) \hat{v} + \frac{2v_{xx}}{v_x^3} \partial_x \hat{v} \tag{4.7}
\]

and

\[
b(v) \hat{v} \equiv \partial_t \hat{v} + \frac{1}{(v_x^-)^2} \partial_x \hat{v}^- - \frac{1}{(v_x^+)^2} \partial_x \hat{v}^+. \tag{4.8}
\]

Consider the linearized problem

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\ell(v^-) \hat{v}^- = \hat{F}^-, & \text{in } -\beta < x < 0, \\
\ell(v^+) \hat{v}^+ = \hat{F}^+, & \text{in } 0 < x < \beta.
\end{array}
\right. \tag{4.9}
\end{aligned}
\]

\[
\left\{ \begin{array}{ll}
b(v) \hat{v} = \hat{G}, & \text{on } x = 0, \ \hat{v}^\pm(\pm \beta, t) = 0. \\
\hat{v}^+ - \hat{v}^- = 0 & \text{on } x = 0.
\end{array}
\right. \tag{4.10}
\]

\[
\hat{v}^\pm(x, 0) = 0. \tag{4.11}
\]

For the linearized problem (4.9)-(4.11), we can easily derive the following energy estimate

\[
\|\hat{v}\|_{k,T}^2 \leq C_k \left( \|\hat{F}\|_{k,T}^2 + \|\hat{G}\|_{k,T}^2 \right). \tag{4.12}
\]

16
by the usual integration by parts and the Gronwall inequality. This is the “tame” estimate [7] required in applying the Nash-Moser inverse function theorem. We omit the details here and the reader is referred to [8]. Thus the existence of smooth solutions is established for (4.1)-(4.4). This finishes the existence proof for the Theorem 3.1.

References


<table>
<thead>
<tr>
<th>#</th>
<th>Title</th>
<th>Author/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>721</td>
<td>The convergence of semidiscrete methods for a system of reaction-diffusion equations</td>
<td>Ling Ma</td>
</tr>
<tr>
<td>722</td>
<td>Adelina Georgescu, Models of asymptotic approximation</td>
<td>C. Foias and J.C. Saut</td>
</tr>
<tr>
<td>723</td>
<td>On bounded and harmonizable solutions on infinite order arma systems</td>
<td>A. Makagon and H. Salehi</td>
</tr>
<tr>
<td>724</td>
<td>An upwind finite-volume scheme with a triangular mesh for conservation laws</td>
<td>San-Yih Lin and Yan-Shin Chin</td>
</tr>
<tr>
<td>725</td>
<td>On the dynamics of fine structure</td>
<td>J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego &amp; P.J. Swart</td>
</tr>
<tr>
<td>726</td>
<td>Lubrication theory and long waves</td>
<td>KangPing Chen and Daniel D. Joseph</td>
</tr>
<tr>
<td>727</td>
<td>Local bifurcation theory for thermoelastic Bravais lattices</td>
<td>J.L. Ericksen</td>
</tr>
<tr>
<td>728</td>
<td>Some stability results for perturbed semilinear parabolic equations</td>
<td>Mario Taboada and Yuncheng You</td>
</tr>
<tr>
<td>729</td>
<td>The homogenization of flow in fractured porous media</td>
<td>A.J. Lawrence</td>
</tr>
<tr>
<td>730</td>
<td>Scattering of acoustic wave by obstacle in stratified medium</td>
<td>Bogdan Vernescu</td>
</tr>
<tr>
<td>731</td>
<td>Global existence for a thermodynamically consistent model of phase field type</td>
<td>Songmu Zheng</td>
</tr>
<tr>
<td>732</td>
<td>A numerical study of a rotationally degenerate hyperbolic system I: the Riemann problem</td>
<td>Heinrich Freistühler and E. Bruce Pitman</td>
</tr>
<tr>
<td>733</td>
<td>New variational problems in the statics of liquid crystals</td>
<td>Epifanio G. Virga</td>
</tr>
<tr>
<td>734</td>
<td>Geometric evolution of phase-boundaries</td>
<td>Yoshikazu Giga and Shun'ichi Goto</td>
</tr>
<tr>
<td>735</td>
<td>Analysis of the finite element approximation of microstructure in micromagnetics</td>
<td>Mitchell Luskin and Ling Ma</td>
</tr>
<tr>
<td>736</td>
<td>The dam problem with leaky boundary conditions</td>
<td>J. Carrillo and M. Chipot</td>
</tr>
<tr>
<td>737</td>
<td>Efficient hybrid shock capturing schemes</td>
<td>Eduard Harabetian and Robert Pego</td>
</tr>
<tr>
<td>738</td>
<td>Multisummability and Stokes multipliers of linear meromorphic differential equations</td>
<td>B.L.J. Braaksma</td>
</tr>
<tr>
<td>739</td>
<td>A central limit theorem for non-linear vector functionals of vector Gaussian processes</td>
<td>Tae Il Jeon and Tze-Chien Sun</td>
</tr>
<tr>
<td>740</td>
<td>Solutions to evolution equations with near-equilibrium initial values</td>
<td>Chris Grant</td>
</tr>
<tr>
<td>741</td>
<td>Invariant manifolds for retarded semilinear wave equations</td>
<td>Mario Taboada and Yuncheng You</td>
</tr>
<tr>
<td>742</td>
<td>Unique solvability of nonlinear Volterra equations in weighted spaces</td>
<td>Peter Rejto and Mario Taboada</td>
</tr>
<tr>
<td>743</td>
<td>Holder regularity for the gradient of solutions of certain singular parabolic equations</td>
<td>Hi Jun Choe</td>
</tr>
<tr>
<td>744</td>
<td>Existence of standing pulse solutions for an excitable activator-inhibitory system</td>
<td>Jack D. Dockery</td>
</tr>
<tr>
<td>745</td>
<td>Existence of travelling wave solutions for a bistable evolutionary ecology model</td>
<td>Jack D. Dockery and Roger Lui</td>
</tr>
<tr>
<td>746</td>
<td>Singular perturbation problems with a compact support semilinear term</td>
<td>Giovanni Alberti, Luigi Ambrosio and Giuseppe Buttazzo</td>
</tr>
<tr>
<td>747</td>
<td>Numerical schemes for constrained minimization problems</td>
<td>Emad A. Fatemi</td>
</tr>
<tr>
<td>748</td>
<td>Slowly oscillating periodic solutions of autonomous state-dependent delay equations</td>
<td>Y. Kuang and H.L. Smith</td>
</tr>
<tr>
<td>749</td>
<td>A new splitting method for scalar conservation laws with stiff source terms</td>
<td>Emad A. Fatemi</td>
</tr>
<tr>
<td>750</td>
<td>A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities</td>
<td>Hi Jun Choe</td>
</tr>
<tr>
<td>751</td>
<td>A vanishing viscosity approach on the dynamics of phase transitions in Van Der Waals fluids</td>
<td>Haitao Fan</td>
</tr>
<tr>
<td>752</td>
<td>The Wigner–Weyl transform on tori and connected graph propagator representations</td>
<td>T.A. Osborn and F.H. Molzahn</td>
</tr>
<tr>
<td>753</td>
<td>A free boundary problem arising in superconductor modeling</td>
<td>Avner Friedman and Bei Hu</td>
</tr>
<tr>
<td>754</td>
<td>An augmented drift-diffusion model in semiconductor device</td>
<td>Avner Friedman and Wenxiong Liu</td>
</tr>
<tr>
<td>755</td>
<td>Extinction and positivity for a system of semilinear parabolic variational inequalities</td>
<td>Avner Friedman and Miguel A. Herrera</td>
</tr>
<tr>
<td>756</td>
<td>The time-harmonic Maxwell equations in a doubly periodic structure</td>
<td>David Dobson and Avner Friedman</td>
</tr>
<tr>
<td>757</td>
<td>Interior behaviour of minimizers for certain functionals with nonstandard growth</td>
<td>Hi Jun Choe</td>
</tr>
<tr>
<td>758</td>
<td>Axis-symmetric boundary-value problems for nematic liquid crystals with variable degree of orientation</td>
<td>Vincenzo M. Tortorelli and Epifanio G. Virga</td>
</tr>
<tr>
<td>759</td>
<td>Geometric parameters and the relaxation of multwell energies</td>
<td>Nikan B. Firoozey and Robert V. Kohn</td>
</tr>
<tr>
<td>760</td>
<td>The Riemann problem for systems of conservation laws of mixed type</td>
<td>Haitao Fan and Marshall Slemrod</td>
</tr>
<tr>
<td>761</td>
<td>Analysis and application of a continuation method for a self-similar coupled Stefan system</td>
<td>Joseph D. Fehribach</td>
</tr>
<tr>
<td>762</td>
<td>Dissipativity of numerical schemes</td>
<td>C. Foias, M.S. Jolly, I.G. Kevrekidis and E.S. Titi</td>
</tr>
<tr>
<td>763</td>
<td>Kelvin–Helmholtz mechanism for side branching in the displacement of light with heavy fluid under gravity</td>
<td>D.D. Joseph, T.Y.J. Liao and J.-C. Saut</td>
</tr>
</tbody>
</table>
Chris Grant, Solutions to evolution equations with near-equilibrium initial values
B. Cockburn, F. Coquel, Ph. LeFloch and C.W. Shu, Convergence of finite volume methods
N.G. Lloyd and J.M. Pearson, Computing centre conditions for certain cubic systems
João Palhoto Matos, Young measures and the absence of fine microstructures in the $\alpha - \beta$ quartz phase transition
L.A. Peletier & W.C. Troy, Self-similar solutions for infiltration of dopant into semiconductors
H. Scott Dumas and James A. Ellison, Nekhoroshev's theorem, ergodicity, and the motion of energetic charged particles in crystals
Stathis Filippas and Robert V. Kohn, Refined asymptotics for the blowup of $u_t - \Delta u = u^p$.
Patricia Bauman, Nicholas C. Owen and Daniel Phillips, Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity
Patricia Bauman, Nicholas C. Owen and Daniel Phillips, Maximal smoothness of solutions to certain Euler–Lagrangian equations from nonlinear elasticity
Jack Carr and Robert Pego, Self-similarity in a coarsening model in one dimension
J.M. Greenberg, The shock generation problem for a discrete gas with short range repulsive forces
George R. Sell and Mario Taboada, Local dissipativity and attractors for the Kuramoto–Sivashinsky equation in thin 2D domains
T. Subba Rao, Analysis of nonlinear time series (and chaos) by bispectral methods
Nicholas Baumann, Daniel D. Joseph, Paul Mohr and Yuriko Renardy, Vortex rings of one fluid in another free fall
Oscar Bruno, Avner Friedman and Fernando Reitich, Asymptotic behavior for a coalescence problem
Johannes C.C. Nitsche, Periodic surfaces which are extremal for energy functionals containing curvature functions
F. Abergel and J.L. Bona, A mathematical theory for viscous, free-surface flows over a perturbed plane
Gunduz Caginalp and Xinfu Chen, Phase field equations in the singular limit of sharp interface problems
Robert P. Gilbert and Yongzhi Xu, An inverse problem for harmonic acoustics in stratified oceans
Roger Fosdick and Eric Volkmann, Normality and convexity of the yield surface in nonlinear plasticity
H.S. Brown, I.G. Kevrekidis and M.S. Jolly, A minimal model for spatio–temporal patterns in thin film flow
Chao–Nien Chen, On the uniqueness of solutions of some second order differential equations
Xinfu Chen and Avner Friedman, The thermistor problem for conductivity which vanishes at large temperature
Xinfu Chen and Avner Friedman, The thermistor problem with one-zero conductivity
E.G. Kalnin and W. Miller, Jr., Separation of variables for the Dirac equation in Kerr Newman space time
E. Knobloch, M.R.E. Proctor and N.O. Weiss, Finite-dimensional description of doubly diffusive convection
V.V. Pukhnachov, Mathematical model of natural convection under low gravity
M.C. Knaap, Existence and non-existence for quasi-linear elliptic equations with the $p$-laplacian involving critical Sobolev exponents
Stathis Filippas and Wenxiong Liu, On the blowup of multidimensional semilinear heat equations
A.M. Meirmanov, The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution
Bo Guan and Joel Spruck, Interior gradient estimates for solutions of prescribed curvature equations of of parabolic type
Hi Jun Choe, Regularity for solutions of nonlinear variational inequalities with gradient constraints
Peter Shi and Yongzhi Xu, Quasistatic linear thermoelasticity on the unit disk
Satyanad Kichenassamy and Peter J. Olver, Existence and non-existence of solitary wave solutions to higher order model evolution equations
Dening Li, Regularity of solutions for a two-phase degenerate Stefan Problem
Marek Fila, Bernhard Kawohl and Howard A. Levine, Quenching for quasilinear equations
Yoshikazu Giga, Shun’ichi Goto and Hitoshi Ishii, Global existence of weak solutions for interface equations coupled with diffusion equations
Mark J. Friedman and Eusebius J. Doedel, Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study
Mark J. Friedman, Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds
Peter W. Bates and Songmu Zheng, Inertial manifolds and inertial sets for the phase-field equations
J. López Gómez, V. Márquez and N. Wolanski, Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition
Xinfu Chen and Fahuai Yi, Regularity of the free boundary of a continuous casting problem
Eden, A., Foias, C., Nicolaenko, B. and Temam, R., Inertial sets for dissipative evolution equations Part I: Construction and applications
Jose–Francisco Rodrigues and Boris Zaltzman, On classical solutions of the two-phase steady-state Stefan problem in strips
Viorel Barbu and Srdjan Stojanovic, Controlling the free boundary of elliptic variational inequalities on a variable domain