REGULARITY OF SOLUTIONS FOR A TWO-PHASE DEGENERATE STEFAN PROBLEM

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Regularity of Solutions for a Two-Phase Degenerate Stefan Problem *

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Abstract

For a one-dimensional two-phase degenerate Stefan problem, we prove that the free boundary is C^∞ smooth and the solutions are C^∞ smooth up to the boundary. The proof is based on performing the hodograph transformation to fix the free boundary and establishing a nonlinear a priori estimate for the solution.

1 INTRODUCTION

In this paper, we study the regularity of solutions for the following one-dimensional two-phase degenerate Stefan problem.

Let $T > 0$. Find functions $u^-(x,t)$, $u^+(x,t)$ and $s(t)$, which are defined on $[-1, s(t)] \times [0,T]$, $[s(t), 1] \times [0,T]$ and $[0,T]$ respectively, and satisfy

$$\begin{cases} \alpha^-(u)u_t^- - u_{xx}^- = 0, & -1 < x < s(t), 0 < t < T, \\ \alpha^+(u)u_t^+ - u_{xx}^+ = 0, & s(t) < x < 1, 0 < t < T. \end{cases} \tag{1.1}$$

$$u^\pm(x,0) = u_0^\pm(x), \;\; 0 \le \pm x \le 1, \;\; s(0) = 0. \tag{1.2}$$

$$u^-(-1,t) = g^-(t) < 0, \;\; u^+(1,t) = g^+(t) > 0, \;\; 0 \le t \le T. \tag{1.3}$$

$$\begin{cases} u^\pm(s(t),t) = 0, \;\; 0 \le t \le T, \\ s'(t) = u_x^-(s(t)-0,t) - u_x^+(s(t)+0,t), \;\; 0 \le t \le T. \end{cases} \tag{1.4}$$

The basic assumptions of the problem (1.1)-(1.4) are the following

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(H1) $\alpha^{\pm}(u) \in C^{\infty}(\bar{R}^1_{\pm})$, $\alpha^{\pm}(u) \geq 0$, $\alpha^{\pm}(u) = 0$ if and only if $u = 0$.

(H2) $\partial_x u_0^{\pm}(0) > 0$, $\pm g^{\pm}(t) \geq \delta_0$ with $\delta_0 > 0$ and $u_0^{\pm}(x)x \geq 0$ where the equality holds if and only if $x = 0$.

The problem (1.1)-(1.4) is a two-phase degenerate Stefan problem which arises in a number of physical processes [13]. The problem has been studied by several authors. The uniqueness of the weak solution was demonstrated by Crowley [3]. The existence of weak solution and its regularity were studied by [2,14]. The existence of weak solution was obtained and the free boundary was shown to be Lipschitz continuous [2], and the boundary condition was satisfied in the classical sense almost everywhere [14]. The main result of [2,14] is the following [2, Theorem 1]:

Theorem 1.1 *Assume the data in (1.1)-(1.4) satisfy (H1)(H2) and*

- $g^{\pm}(t) \in C^2[0,T]$, $u_0^{\pm}(x) \in C^3[\pm 1, 0]$;
- $g^{\pm}(0) = u_0^{\pm}(\pm 1)$, $u_0^{\pm}(0) = 0$.

Then there is a unique weak solution $(u^{\pm}, s)$ to the problem (1.1)-(1.4) such that

- $x = s(t)$ *is Lipschitz continuous in* $[0,T]$;
- $u^{\pm}(x,t) \in C^{2,1}([\pm 1, s(t)) \times [0,T])$, *satisfy (1.1)(1.2)(1.3) in the classical sense, satisfy (1.4) almost everywhere;*
- $\pm u^{\pm}(x,t) > 0$ *in* $\pm s(t) < \pm x \leq 1$, $0 \leq t \leq T$.

In this paper, based upon the result obtained in [] and under a similar assumption, we prove the C^{∞} regularity of both the free boundary $x = s(t)$ and the solutions $u^{\pm}(x,t)$ up to the free boundary.

First we introduce the following concept of compatibility.

Definition 1.1 *The data $u_0^{\pm}(x)$ and $g^{\pm}(t)$ are called k-order compatible at $(x,t) = (0,0)$ and $(\pm 1, 0)$ if $u_0^{\pm}(x) \in C^{2k}[\pm 1, 0]$, $g^{\pm}(t) \in C^k[0,T]$ and there exist functions $\tilde{s}(t) \in C^k[0,T]$ and $\tilde{u}^{\pm}(x,t) \in C^{2k,k}([\tilde{s}(t), \pm 1] \times [0,T])$ such that*

$$\tilde{u}^{\pm}(\pm 1, t) = g^{\pm}(t), \quad \tilde{u}^{\pm}(\tilde{s}(t), t) = 0; \tag{1.5}$$

$$\tilde{f}^{\pm}(x,t) \equiv \alpha(\tilde{u}^{\pm})\tilde{u}^{\pm}_t - \tilde{u}^{\pm}_{xx} = O(t^k); \tag{1.6}$$

$$\tilde{s}(0) = 0, \quad \tilde{g}(t) \equiv \tilde{s}'(t) - \tilde{u}^{-}_x(\tilde{s}(t) - 0, t) + \tilde{u}^{+}_x(\tilde{s}(t) + 0, t) = O(t^k). \tag{1.7}$$

If $u_0^{\pm}(x) \in C^{\infty}[\pm 1, 0]$, $g^{\pm}(t) \in C^{\infty}[0,T]$ and (1.5)-(1.7) are satisfied for any k, then the initial data are called to be C^{∞} compatible.

The main result of this paper is the following theorem:

Theorem 1.2 *For the problem (1.1)-(1.4) under the assumptions (H1)(H2),*

1. *if the data* $u_0^{\pm}(x) \in C^4[\pm 1, 0]$ *and* $g^{\pm}(t) \in C^2[0,T]$ *satisfy the 2-order compatibility conditions (1.5)-(1.7), then the unique solution has the following regularity:*

$$s(t) \in C^{\infty}(0,T], \ \ u^{\pm}(x,t) \in C^{\infty}((\pm 1, s(t)] \times (0,T]). \tag{1.8}$$

2. *if the data* $u_0^{\pm}(x) \in C^{\infty}[\pm 1, 0]$ *and* $g^{\pm}(t) \in C^{\infty}[0,T]$ *satisfy the* C^{∞} *compatibility conditions (1.5)-(1.7), then the unique solution is also* C^{∞} *smooth up to the boundaries* $t = 0$ *and* $x = \pm 1$.

The C^{∞} compatibility requirement in the second part of theorem 1.2 is obviously necessary for the solutions to be C^{∞} smooth up to $t = 0$ and $x = \pm 1$.

We prove the Theorem 1.2 in the following sections. In section 2, we perform the hodograph transformation to reduce the free boundary problem (1.1)-(1.4) into a fixed boundary problem. Then a priori estimates for the resulting problem are derived in section 3. Section 4 establishes the existence of required smooth solution by linear iteration.

It is easy to see that all the proofs in this paper can also applied to the case with nondegenerate $\alpha^{\pm}(u) \geq \delta > 0$. Therefore, this paper also provides another proof to the result in [4,12] and other previous works. The author is thankful to Professors Avner Friedman, Daniel Phillips and Lihe Wang for helpful conversations.

2 Transformed Problem

In the following, we will always treat the point $t = T$ as if it is an interior point where all the boundary conditions as well as interior equations (1.1) are satisfied. This is indeed true because we can always extend the definitions of $g^{\pm}$ into $[0, T+\epsilon]$ and consider instead the problem in the larger domain $[-1,1] \times [0, T+\epsilon]$.

2.1 Preparatory propositions

First, we state some easy consequences of Theorem 1.1.

Proposition 2.1 *The solution* $u^{\pm}(x,t)$ *in Theorem 1.1 is* C^{∞} *in* $[\pm 1, s(t)) \times (0,T]$.

Proof: In $[\pm 1, s(t)) \times (0,T]$, $\alpha^{\pm}(u) > 0$ since $u \neq 0$. Hence C^{∞} smoothness of the solution follows from applying repeatedly the standard interior Schauder estimate.

Proposition 2.2 *Let $u^{\pm}(x,t)$ be the solution of (1.1)-(1.4) in Theorem 1.1. Then there exists $\beta_0 > 0$ such that for any ϵ, $0 \le \epsilon \le \beta_0$, the set*

$$\mathsf{S}_{\pm\epsilon} = \{(x,t) : u^{\pm}(x,t) = \pm\epsilon,\ 0 \le t \le T\}$$

intersects with any line $t = t'$ $(0 \le t' \le T)$ only at one point. and the mapping

$$(t,x) \mapsto (\bar{t},\bar{x}) = (t, u(x,t)) \tag{2.1}$$

is a bijection

$$R_{\beta_0} \equiv \bigcup_{0\le\epsilon\le\beta_0} \mathsf{S}_{\pm\epsilon} \to [-\beta_0,\beta_0] \times [0,T].$$

PROOF: The case of $\epsilon = 0$ is proved in [2]. From the assumption (H2), for $\beta_0 \ll 1$, the case for $0 < \epsilon \le \beta_0$ is derived readily from maximum principle.

The following proposition improves the estimate obtained in [2].

Proposition 2.3 *Let $u^{\pm}(x,t)$ be the solution of (1.1)-(1.4) obtained in [2]. Then there exists $\beta > 0$ such that for any ϵ, $0 < \epsilon \le \beta$, the set*

$$\mathsf{S}_{\pm\epsilon} = \{(x,t) : u^{\pm}(x,t) = \pm\epsilon,\ 0 \le t \le T\}$$

can be written as

$$x = s_{\pm\epsilon}(t),\ \ s_{\pm\epsilon}(t) \in C^{\infty}(0,T] \cap C^{1}[0,T]. \tag{2.2}$$

In addition, there exists $\delta > 0$ such that for the derivatives $u_x^{\pm}(x,t)$ in R_{β} where they exist, we have

$$\frac{1}{\delta} \ge u_x^{\pm}(x,t) \ge \delta > 0, \text{ in } R_{\beta}. \tag{2.3}$$

PROOF: First of all, we can choose $\beta \ll 1$ so that $\beta \le \beta_0$ in the Proposition 2.2 and $\pm\beta$ are not critical values for $u^{\pm}(x,t)$. Consequently by Sard lemma and following the same argument as in [2],

$$x = s_{\pm\beta}(t) \in C^{\infty}(0,T] \cap C^{1}[0,T],\ \ u^{\pm}(s_{\pm\beta}(t),t) = \pm\beta. \tag{2.4}$$

Besides, by assumption (H2), we can also choose β such that

$$\delta_0^{-1} \ge \partial_x u_0^{\pm}(x) \ge \delta_0 > 0,\ \ -\beta \le x \le \beta. \tag{2.5}$$

For the above chosen β, we need only to show (2.3), because (2.3) implies that none of the values $\epsilon \in [-\beta, 0) \cup (0, \beta]$ is critical value of $u^{\pm}(x,t)$ and (2.2) follows by Sard lemma.

(2.3) can be shown by similar argument in [2] as follows. Let $c(u)$ be defined by

$$c(u) = \begin{cases} \int_0^u \alpha^+(\xi)d\xi \equiv c^+(u), \quad u > 0, \\ [-1, 0], \quad u = 0, \\ -1 + \int_0^u \alpha^-(\xi)d\xi \equiv c^-(u), \quad u < 0. \end{cases}$$

Construct smooth approximate sequences

$$c_n(u) \in C^\infty(R^1), \quad U_{0n}(x) \in C^2[-\beta, \beta]$$

such that

- $\lim_{n\to\infty} c_n(u) = c(u)$ in $L^2(R^1)$;
- $c_n'(u) \geq 1/n$ and $c_n'(u) = c'(u)$ when $|u| \geq \delta_n$ for small $\delta_n > 0$ and $\delta_n \to 0$ as $n \to \infty$;
- $U_{0n}(x) = u_0^\pm(x)$ in $[x_{-0}, x_{-0}/2]$, $[x_{+0}/2, x_{+0}]$ with $x_{\pm 0} = s_{\pm\beta}(0)$;
- $|U_{0n}(x)| \leq \beta$, $U_{0n}(x) \to u_0^\pm(x)$ in $[x_{-0}, x_{+0}]$ uniformly and

$$\frac{2}{\delta_0} \geq \partial_x U_{0n}(x) \geq \frac{\delta_0}{2}. \tag{2.6}$$

Consider the approximation problem as [2] in $R_\beta \equiv [s_{-\beta}, s_{+\beta}] \times [0, T]$:

$$(P_n) \begin{cases} \partial_t c_n(U_n) = \partial_x^2 U_n \ \text{in}(s_{-\beta}(t), s_{+\beta}(t)) \times (0, T), \\ U_n(s_{\pm\beta}(t), t) = \pm\beta, \quad 0 \leq t \leq T, \\ U_n(x, 0) = U_{0n}(x), \quad x_{-0} \leq x \leq x_{+0}. \end{cases}$$

The problem (P_n) has a unique classical solution $U_n(x,t) \in C^{2,1}(R_\beta)$ and there is a subsequence of $\{U_n(x,t)\}$ converging uniformly to the solution $u^\pm(x,t)$ of (1.1)-(1.4) in R_β. It was established in [2, Lemma 3, 4] that

$$0 < N_n \leq U_{nx}(x,t) \leq M \quad \text{in} \quad R_\beta, \tag{2.7}$$

with M independent of n, but N_n may not be uniform with respect of n. However, with our choice of the domain R_β, we are going to show that actually N_n can be chosen independent of n

$$N_n \geq \delta > 0. \tag{2.8}$$

Once (2.8) is proven, we derive for the limit function $u^{\pm}(x,t)$ the estimate (2.3) by the well-known theorem in functional analysis that the strong closure and weak closure for convex set coincide.

To prove (2.8), first we notice that on the boundaries $x = s_{\pm\beta}(t)$, $U_n(x,t)$ takes its maximum and minimum value $\pm\beta$. The boundaries $x = s_{\pm\beta}(t)$ are smooth so that we can apply the strong maximum principle to derive

$$U_{nx}(s_{\pm\beta}(t), t) \geq \delta'. \tag{2.9}$$

We show that δ' can be chosen independent of n and t. Since $[0,T]$ is compact, we need only to show that for each fixed t ($0 \leq t \leq T$), δ' is independently of n. Since at $t = 0$, $U_{nx}(s_{\pm\beta}(t),t) = U_{0x}(x_{\pm 0}) \geq \delta_0/2$, we may assume $t > 0$. In the following we consider the point on $x = s_{+\beta}(t)$, the case $x = s_{-\beta}(t)$ can be discussed in the same way.

For any fixed point $P(x^*,t^*) \equiv (s_{+\beta}(t^*),t^*)$, ($t^* > 0$), we can construct a sphere B contained in R_β, centered at $(\bar{x},\bar{t})$ ($\bar{x} \neq x^*$) and tangent to $x = s_{+\beta}(t)$ only at $P(x^*,t^*)$. As in the standard proof of the strong maximum principle [5], divide the plane into two parts π^- and π^+ by a straight line π such that

$$(\bar{x},\bar{t}) \in \pi^-, \quad (x^*,t^*) \in \pi^+; \quad B^+ \equiv \pi^+ \cap B \neq \emptyset;$$

$$|\bar{x} - x| \geq \text{const} > 0, \quad (x,t) \in B^+.$$

Denote the boundary of B^+ as $\partial B^+ = S_1 \cup S_2$,

$$S_1 = \partial B^+ \cap \partial B, \quad S_2 = \partial B^+ \cap \pi.$$

Construct barrier function

$$h(x,t) = e^{-\kappa[|x-\bar{x}|^2 + |t-\bar{t}|^2]} - e^{-\kappa r^2}$$

where r is the radius of B. Then $h = 0$ on S_1, $h \geq 0$ on $\bar{B}^+$. For $\kappa \gg 1$, we have

$$(\partial_x^2 - c_n'(U_n)\partial_t)h > 0, \quad \text{in} \quad B^+$$

independent of n. Taking $\epsilon \ll 1$, the function $V_n(x,t) = U_n(x,t) + \epsilon h(x,t) < \beta$ on S_2 uniformly in n for large n because

$$u^+(x,t) \leq \beta - \epsilon_1, \quad \text{for} \quad \epsilon \ll 1 \quad \text{on} \quad S_2$$

and U_n converges uniformly to u^+. On S_1, $V_n(x,t) = U_n(x,t) < \beta$ for $(x,t) \neq (x^*,t^*)$ and $V_n(x^*,t^*) = U_n(x^*,t^*) = \beta$. By the strong maximum principle, $V_n(x,t) < \beta$ in B^+. Therefore $\partial_x V_n(x^*,t^*) \geq 0$. But $h_x(x^*,t^*) < 0$, so

$$\partial_x U_n(x^*,t^*) > -h_x(x^*,t^*) > 0$$

which is uniformly true for all n. Applying ∂_x to the problem (P_n) and employing the strong maximum principle as in [2] to $\partial_x U_n$ in R_β, we obtain (2.8). This concludes the proof of Proposition 2.3.

2.2 Hodograph transformation

Since $u^{\pm}(x,t)$ are known to be C^{∞} smooth away from $x = s(t), \pm 1$, and near the fixed boundaries $x = \pm 1$, the regularity results of the solutions are classical. In the following we will consider the solution $u^{\pm}(x,t)$ only in the domain R_{β}.

We perform the hodograph transformation [6,9] as in (2.1) to reduce the problem (1.1)-(1.4) inside R_{β} into a problem in $[-\beta, \beta] \times [0,T]$. Applying hodograph transformation in the discussion of multi-dimensional nondegenerate Stefan problem was previously found in [10]. Introduce new variables

$$\bar{x} = u(x,t), \quad \bar{t} = t. \tag{2.10}$$

By Proposition 2.3, the transformation (2.10) is invertible in $R_{\beta}^{-} = [s_{-\beta}(t), s(t))$ and $R_{\beta}^{+} = (s(t), s_{+\beta}(t)]$ respectively. Denote the inverse transformation as:

$$x = v(\bar{x}, \bar{t}), \quad t = \bar{t}. \tag{2.11}$$

Taking derivative of $x = v(u(x,t),t)$ with respect to x and t, we have

$$u_t = -\frac{v_{\bar{t}}}{v_{\bar{x}}}, \quad u_x = \frac{1}{v_{\bar{x}}}. \tag{2.12}$$

Hence

$$u_{xx} = \frac{1}{v_{\bar{x}}} \partial_{\bar{x}} u_x = \frac{1}{v_{\bar{x}}} \partial_{\bar{x}} \left(\frac{1}{v_{\bar{x}}} \right) = -\frac{v_{\bar{x}\bar{x}}}{v_{\bar{x}}^3}. \tag{2.13}$$

From (2.11), obviously

$$s'(t) = v_t(0,t). \tag{2.14}$$

Omitting bar in new coordinates, we obtain a new transformed problem in $[-\beta, \beta] \times [0,T]$:

1. Interior equations:

$$\begin{cases} \alpha_-(x) v_t^- + \partial_x \left(\dfrac{1}{v_x^-} \right) = 0, & -\beta < x < 0, \; 0 < t \le T, \\ \alpha_+(x) v_t^+ + \partial_x \left(\dfrac{1}{v_x^+} \right) = 0, & 0 < x < \beta, \; 0 < t \le T. \end{cases} \tag{2.15}$$

2. Initial conditions:

$$v^{\pm}(x,0) = v_0^{\pm}(x), \quad 0 \le \pm x \le \beta; \quad v_{0x}^{\pm}(0) > 0. \tag{2.16}$$

3. Boundary conditions:

$$v^{\pm}(\pm\beta, t) = s_{\pm\beta}(t) \in C^{\infty}(0, T]; \tag{2.17}$$

$$v^{+}(0,t) - v^{-}(0,t) = 0, \ \ 0 < t \leq T; \tag{2.18}$$

$$v_t(0,t) = \frac{1}{v_x^{-}(0,t)} - \frac{1}{v_x^{+}(0,t)}, \ \ 0 < t \leq T. \tag{2.19}$$

According to the Proposition 2.3, we see that

$$0 < \delta \leq v_x^{\pm}(x,t) \leq \delta^{-1}, \ \text{uniformly in} \ [\pm\beta, 0) \times [0, T]. \tag{2.20}$$

Therefore the smooth solutions $v^{\pm}$ for the problem (2.15)-(2.19) in $\{[-\beta, 0] \cup [0, \beta]\} \times [0, T]$ are equivalent to the smooth solutions $(u^{\pm}, s)$ of the original problem (1.1)-(1.4). Consequently the proof of Theorem 1.2 is reduced to the proof of the following equivalent

Theorem 2.1 *In the problem (2.15)-(2.19),*

- *assume that $v_0^{\pm}(x) \in C^4[\pm\beta, 0]$, $s_{\pm\beta}(t) \in C^2[0,T]$ satisfy the 2-order compatibility conditions, there is a unique solution $v^{\pm}(x,t) \in C^{\infty}((\pm\beta, 0] \times (0, T])$.*
- *If the data $v_0^{\pm}(x) \in C^{\infty}[\pm\beta, 0]$ and $s_{\pm\beta}(t) \in C^{\infty}[0,T]$ satisfy the C^{∞} compatibility conditions, then the unique solution is also C^{∞} smooth up to the boundaries $t = 0$ and $x = \pm\beta$.*

Here the k-order and C^{∞} compatibility conditions are similarly defined as in (1.5)-(1.7) of the Definition 1.1.

We prove the theorem 2.1 in section 3 and 4.

3 A Priori Estimate for Nonlinear Problem

3.1 A priori estimate

For simplicity, we will denote briefly $v(x,t) = v^{\pm}(x,t)$ and $\alpha(x) = \alpha^{\pm}(x)$ in $\pm x \geq 0$. We denote $(\cdot, \cdot)$ the inner product of $L^2[-\beta, \beta]$ and $\|\cdot\|$ the corresponding norm. Therefore

$$\|\sqrt{\alpha} v(t)\|^2 \equiv \int_{-\beta}^{0} \alpha_{-}(x)|v^{-}(x,t)|^2 dx + \int_{0}^{\beta} \alpha_{+}(x)|v^{+}(x,t)|^2 dx$$

and so on.

First we prove the standard energy estimate for solutions $v^{\pm}(x,t) \in C^{\infty}([\pm\beta, 0] \times [0, T])$ of (2.15)-(2.19) when the C^{∞} compatibility conditions are satisfied. The main result of this section is the following theorem.

Theorem 3.1 *Let $v^{\pm}(x,t) \in C^{\infty}([\pm\beta, 0] \times [0,T]$ be a solution of (2.15)-(2.19). Then for any nonnegative integer k and $t \in [0,T]$, it satisfies the following estimate*

$$\sum_{j=0}^{k} \partial_t \left(\|\sqrt{\alpha}\partial_t^j v(t)\|^2 + |\partial_t^j v(0,t)|^2 \right) + \sum_{2k_1+k_2 \leq 2k} \|\partial_t^{k_1}\partial_x^{k_2+1} v(t)\|^2 \leq C_k. \tag{3.1}$$

Here the constant C_k depends upon the known initial and boundary values at $t = 0$ and $x = \pm\beta$ up to the order related to k and is independent of the time $t \in [0,T]$.

PROOF: The proof of the theorem is carried out by induction on k.

1. $k = 0$:

   Taking inner product of (2.15) with $v(x,t)$ and integrating by parts over $-\beta \leq x \leq \beta$, we have
$$\frac{1}{2}\partial_t\|\sqrt{\alpha}v(t)\|^2 - \frac{v^+(0,t)}{v_x^+(0,t)} + \frac{v^-(0,t)}{v_x^-(0,t)} - 2 \leq C_0.$$
   Making use of the boundary conditions (2.18)(2.19), we obtain
$$\partial_t \left(\|\sqrt{\alpha}v(t)\|^2 + |v(0,t)|^2 \right) \leq C. \tag{3.2}$$
   The estimate for $\partial_x v$ follows directly from (2.20).

2. $k = 1$:

   Applying ∂_t to (2.15), taking its inner product with $v_t(x,t)$ and integrating by parts over $-\beta \leq x \leq \beta$, we have
$$\frac{1}{2}\partial_t\|\sqrt{\alpha}v_t(t)\|^2 - (\partial_t v_x^{-1}, v_{tx}) - v_t^+(0,t)\left(\frac{1}{v_x^+}\right)_t(0,t) + v_t^-(0,t)\left(\frac{1}{v_x^-}\right)_t(0,t) \leq C_1.$$
   Making use of the boundary conditions (2.18)(2.19) and noticing (2.20), we obtain
$$\partial_t \left(\|\sqrt{\alpha}v_t(t)\|^2 + |v_t(0,t)|^2 \right) + \|v_{tx}(t)\|^2 \leq C_1. \tag{3.3}$$
   From (2.15) and (2.20), we obtain the estimate for $\|v_{xx}(t)\|$ and $\|v_{xxx}\|$ since
$$\|v_{xxx}(t)\| = \|\partial_x(v_x^2\alpha(x)v_t)(t)\| \leq C\left(\|v_t(t)\| + \|v_{xx}(t)\| + \|v_{tx}(t)\|\right).$$

3. $k = 2$:

Applying ∂_t^2 to (2.9), taking its inner product with $v_{tt}(x,t)$ and integrating by parts over $-1 \le x \le 1$, we have

$$\frac{1}{2}\partial_t\|\sqrt{\alpha}v_{tt}(t)\|^2 - (\partial_t^2 v_x^{-1}, v_{ttx}) - v_{tt}^+(0,t)\left(\frac{1}{v_x^+}\right)_{tt} + v_{tt}^-(0,t)\left(\frac{1}{v_x^-}\right)_{tt} \le C_2.$$

Making use of the boundary conditions (2.18)(2.19) and noticing (2.20), we obtain

$$\partial_t\left(\|\sqrt{\alpha}v_{tt}(t)\|^2 + |v_{tt}(0,t)|^2\right) + \|v_{ttx}(t)\|^2 \le C_2 + 2|(v_{tx}^2 v_x^{-3}, v_{ttx})|. \tag{3.4}$$

Since

$$2|(v_{tx}^2 v_x^{-3}, v_{ttx})| \le \frac{1}{2}\|v_{ttx}(t)\|^2 + 4\|v_{tx}^2(t)\|^2 \le \frac{1}{2}\|v_{ttx}(t)\|^2 + 4|v_{tx}(t)|_{L^\infty}^2\|v_{tx}(t)\|^2$$

and

$$\begin{aligned}|v_{tx}(t)|_{L^\infty} &\le C(1 + \|v_{txx}(t)\|) = C(1 + \|\partial_t(v_x^2\alpha v_t)(t)\|) \\ &\le C(1 + \|v_{tx}\| + \|\sqrt{\alpha}v_{tt}(t)\|) \le C_1(1 + \|\sqrt{\alpha}v_{tt}(t)\|),\end{aligned} \tag{3.5}$$

Noticing (3.3) we obtain the following

$$\partial_t\left(\|\sqrt{\alpha}v_{tt}(t)\|^2 + |v_{tt}(0,t)|^2\right) + \|v_{ttx}\|^2 \le C_2(1 + \|\sqrt{\alpha}v_{tt}(t)\|^2).$$

Employing Gronwall inequality, we have

$$\partial_t\left(\|\sqrt{\alpha}v_{tt}(t)\|^2 + |v_{tt}(0,t)|^2\right) + \|v_{ttx}\|^2 \le C_2. \tag{3.6}$$

Using (2.20) and the equations (2.15), we derive readily from (3.6) the estimates

$$\sum_{2k_1+k_2\le 4}\|\partial_t^{k_1}\partial_x^{k_2+1}v(t)\|^2 \le C_2.$$

Therefore (3.1) is proved for $k = 2$.

4. Induction on k:

Assume (3.1) true for all $j \le k$, $k \ge 2$. Applying ∂_t^{k+1} to (2.15), taking its inner product with $\partial_t^{k+1}v(x,t)$ and integrating by parts over $-\beta \le x \le \beta$, we have

$$\begin{aligned}&\frac{1}{2}\partial_t\|\sqrt{\alpha}\partial_t^{k+1}v(t)\|^2 - (\partial_t^{k+1}v_x^{-1}, \partial_t^{k+1}v_x) \\ &\quad -\partial_t^{k+1}v^+(0,t)\partial_t^{k+1}\left(\frac{1}{v_x^+}\right)(0,t) + \partial_t^{k+1}v^-(0,t)\partial_t^{k+1}\left(\frac{1}{v_x^-}\right)(0,t) \le C_{k+1}.\end{aligned} \tag{3.7}$$

Making use of the boundary conditions (2.11)(2.12) and noticing v_x is bounded, we obtain

$$\partial_t\left(\|\sqrt{\alpha}\partial_t^{k+1}v(t)\|^2+|\partial_t^{k+1}v(0,t)|^2\right)+\|\partial_t^{k+1}v_x(t)\|^2\le C_{k+1}(1+|(F_k,\partial_t^{k+1}v_x)|) \quad (3.8)$$

where F_k has the form

$$F_k=\partial_t^k\left(\frac{v_{xt}}{v_x^2}\right)-\frac{\partial_t^k v_{xt}}{v_x^2}=\sum_{k_1+k_2=k-1}A_{k_1,k_2}\left(\partial_t^{k_1+1}v_x\right)\partial_t^{k_2+1}\left(\frac{1}{v_x^2}\right). \quad (3.9)$$

(a) To estimate F_k.

- For $k_1\le k-2$ and $k_2\le k-2$, by induction we have

$$\begin{aligned}\|\partial_t^{k_1+1}v_x\partial_t^{k_2+1}v_x^{-2}\| &\le C_{k+1}|\partial_t^{k_1+1}v_x|_{L^\infty}|\partial_t^{k_2+1}v_x|_{L^\infty}\\ &\le C_{k+1}\left(1+\|\partial_t^{k_1+1}v_{xx}\|\right)\left(1+\|\partial_t^{k_2+1}v_{xx}\|\right)\le C_{k+1}.\end{aligned}$$

- For $k_1=k-1$, $k_2=0$ or $k_1=0$, $k_2=k-1$, we have

$$\begin{aligned}\|\partial_t^{k_1+1}v_x\partial_t^{k_2+1}v_x^{-2}\| &\le C_{k+1}|\partial_t v_x|_{L^\infty}\|\partial_t^k v_x\|\\ &\le C_{k+1}\left(1+\|\partial_t^k v_x\|\right)\le C_{k+1}.\end{aligned}$$

Therefore $\|F_k\|\le C_{k+1}$. Combining with (3.8), we have

$$\partial_t\left(\|\sqrt{\alpha}\partial_t^{k+1}v(t)\|^2+|\partial_t^{k+1}v(0,t)|^2\right)+\|\partial_t^{k+1}v_x(t)\|^2\le C_{k+1}. \quad (3.11)$$

(b) To prove the Theorem 3.1, we need to estimate all the terms of

$$\partial_t^{k_1}\partial_x^{k_2+1}v,\ \ 2k_1+k_2\le 2(k+1). \quad (3.12)$$

These terms can be estimated by induction on k_2. The case for $k_2=0$ is (3.11). We will assume (3.12) is estimated for $k_2\le 2m$, $m>0$ and derive the estimate for $k_2=2m+1$ and $k_2=2m+2$.

- For $k_2=2m+1$, then $k_1\le k-m$.
  If $k_1<k-m$, then $2k_1+k_2\le 2k$. So the terms in (3.12) are estimated in (3.1) by the induction assumption on k.
  In $k_1=k-m$, $k_2=2m+1$,

$$\begin{aligned}\partial_t^{k-m}\partial_x^{2m+2}v &=\partial_x^{2m}\partial_t^{k-m}(v_x^2\alpha v_t)\\ &=\partial_x^{2m}\sum_{j_1+j_2=k-m}B_{j_1,j_2}\left(\partial_t^{j_1}(v_x^2)\right)\left(\partial_t^{j_2}(\alpha v_t)\right).\end{aligned} \quad (3.13)$$

By the Nirenberg inequality [11], we have

$$
\begin{aligned}
&\|\partial_t^{k-m}\partial_x^{2m+2}v\| \\
&\le C_{k+1}\sum_{j_1+j_2=k-m}\Big(|\partial_t^{j_2}(\alpha v_t)|_{L^\infty}\|\partial_t^{j_1}\partial_x^{2m}v_x^2\| + |\partial_t^{j_1}v_x^2|_{L^\infty}\|\partial_t^{j_2}\partial_x^{2m}(\alpha v_t)\|\Big)
\end{aligned}
\tag{3.14}
$$

By the induction assumption on k_2 we derive from (3.14) the estimate

$$
\|\partial_t^{k-m}\partial_x^{2m+2}v\| \le C_{k+1}. \tag{3.15}
$$

- For $k_2 = 2m+2$, then $k_1 \le k-m$. If $k_1 < k-m$, the terms are estimated by the induction assumption on k. Consider $k_2 = 2m+2$ and $k_1 = k-m$. Similar as (3.13)-(3.15), we have

$$
\begin{aligned}
&\|\partial_t^{k-m}\partial_x^{2m+3}v\| = \|\partial_x^{2m+1}\partial_t^{k-m}(v_x^2\alpha v_t)\| \\
&\le C_{k+1}\sum_{j_1+j_2=k-m}\Big(|\partial_t^{j_2}(\alpha v_t)|_{L^\infty}\|\partial_t^{j_1}\partial_x^{2m+1}v_x^2\| + |\partial_t^{j_1}v_x^2|_{L^\infty}\|\partial_t^{j_2}\partial_x^{2m+1}(\alpha v_t)\|\Big)
\end{aligned}
$$

  which can be estimated by (3.15) and the induction assumption on $k_2 = 2m$. This finished the induction proof on k_2.

3.2 Truncated estimate

Now we assume that the solutions $v^{\pm}(x,t)$ be C^∞ only in $t > 0$ and be C^2 up to $t = 0$. Then we have the following theorem of the truncated estimate for (2.15)-(2.19).

Theorem 3.2 *Let $v^{\pm}(x,t) \in C^\infty([\pm\beta,\beta]\times(0,T]$ be a solution of (2.15)-(2.19) and being C^2 up to $t = 0$. Then for any nonnegative integer k and $0 \le t \le T$, the solution satisfies the following estimate*

$$
\begin{aligned}
&\sum_{j=0}^{k}\partial_t\left(\|\sqrt{\alpha}D_t^j\partial_t^2 v(t)\|^2 + |D_t^j\partial_t^2 v(0,t)|^2\right) \\
&+ \sum_{\substack{2k_1+k_2\le 2k \\ 2j_1+j_2\le 4}} \|D_t^{k_1}D_x^{k_2}\partial_t^{j_1}\partial_x^{j_2+1}v(t)\|^2 \le C_k',
\end{aligned}
\tag{3.16}
$$

where

$$
D_t \equiv t\partial_t, \quad D_x \equiv t\partial_x.
$$

Here in (3.16) the constant C_k' depends upon the C^4 norms of the initial data at $t = 0$ and C^{k+2} truncated norms of the boundary value on $x = \pm\beta$.

PROOF: The case $k = 0$ is included in (3.1). We derive (3.16) by induction on k as in the proof of Theorem 3.1.

1. Applying $D_t^{k+1}\partial_t^2$ to (2.15) and integrating by parts its inner product with $D_t^{k+1}\partial_t^2 v$ over $[-\beta, \beta]$, we have

$$\frac{1}{2}\partial_t \left(\|\sqrt{\alpha} D_t^{k+1}\partial_t^2 v(t)\|^2 + |D_t^{k+1}\partial_t^2 v(0,t)|^2 \right) + \|D_t^{k+1}\partial_t^2 v_x(t)\|^2 \leq G_k \tag{3.17}$$

where

$$\begin{aligned} G_k &\equiv G_k' + G_k'' + C_{k+1}' \\ &= (\alpha[\partial_t, D_t^{k+1}]\partial_t^2 v, D_t^{k+1}\partial_t^2 v) + \langle [\partial_t, D_t^{k+1}]\partial_t^2 v, D_t^{k+1}\partial_t^2 v\rangle_{x=0} \\ &\quad -([D_t^{k+1}\partial_t, v_x^{-2}]v_{tx}, D_t^{k+1}\partial_t^2 v_x) + C_{k+1}'. \end{aligned} \tag{3.18}$$

- Estimate G_k'. Using the equation (2.15), integrating by parts and using the boundary conditions (2.18)(2.19), we have

$$\begin{aligned} G_k' &= (\alpha[\partial_t, D_t^{k+1}]\partial_t^2 v, D_t^{k+1}\partial_t^2 v) + \langle [\partial_t, D_t^{k+1}]\partial_t^2 v, D_t^{k+1}\partial_t^2 v\rangle_{x=0} \\ &= -([\partial_t, D_t^{k+1}]\partial_t\partial_x v_x^{-1}, D_t^{k+1}\partial_t^2 v) + \langle [\partial_t, D_t^{k+1}]\partial_t^2 v, D_t^{k+1}\partial_t^2 v\rangle_{x=0} \\ &= ([\partial_t, D_t^{k+1}]\partial_t v_x^{-1}, D_t^{k+1}\partial_t^2 v_x). \end{aligned}$$

Therefore , we obtain

$$\|G_k'\|^2 \leq \frac{1}{4}\|D_t^{k+1}\partial_t^2 v\|^2 + 4\|[\partial_t, D_t^{k+1}]\partial_t v_x^{-1}\|^2 \leq \frac{1}{4}\|D_t^{k+1}\partial_t^2 v\|^2 + C_k' \tag{3.19}$$

which follows from

$$\|[\partial_t, D_t^{k+1}]\partial_t v_x^{-1}\| \leq C_k' \sum_{j\leq k} \left\| D_t^j \partial_t \left(\frac{v_{tx}}{v_x^2}\right)\right\| \leq C_k' \sum_{j_1, j_2 \leq k} |D_t^{j_2} v_x|_{L^\infty} \|D_t^{j_1} v_{ttx}\| \leq C_k'.$$

- Estimate G_k''. From (3.18)

$$\|G_k''\|^2 \leq \frac{1}{4}\|D_t^{k+1}\partial_t^2 v_x\|^2 + 4\|[D_t^{k+1}\partial_t, v_x^{-2}]v_{tx}\|^2. \tag{3.20}$$

It is easy to derive

$$\begin{aligned} &\|[D_t^{k+1}\partial_t, v_x^{-2}]v_{tx}\|^2 \leq C\left(\|D_t^{k+1}(v_{tx}^2 v_x^{-3})\| + \sum_{k_1+k_2=k} \|D_t^{k_1+1} v_x^{-2} D_t^{k_2} v_{ttx}\| \right) \\ &\leq C_k' \|D_t^k v_{ttx}\|^2 \leq C_k'. \end{aligned} \tag{3.21}$$

Combining (3.17)-(3.21), we have

$$\partial_t \left(\|\sqrt{\alpha} D_t^{k+1} \partial_t^2 v(t)\|^2 + |D_t^{k+1} \partial_t^2 v(0,t)|^2 \right) + \|D_t^{k+1} \partial_t^2 v_x(t)\|^2 \leq C_k'. \tag{3.22}$$

2. It remains to estimate the following terms to finish the proof of the Theorem 3.2

$$\sum_{\substack{2k_1 + k_2 \leq 2k+2 \\ 2j_1 + j_2 \leq 4}} \|D_t^{k_1} D_x^{k_2} \partial_t^{j_1} \partial_x^{j_2+1} v(t)\|^2 \leq C_k'. \tag{3.23}$$

(3.23) can proven similarly as in the proof of the Theorem 3.1 by induction on k_2. The operations involve using the equation (2.15) and employing (3.22). We omit the straightforward details.

This finishes the proof of the Theorem 3.2.

4 Existence of C^∞ Solution

Once the a priori estimates in the Theorem 3.1, Theorem 3.2 are obtained, there are a number of ways to establish the existence of the solution.

1. We need only to show the existence for the case when C^∞ compatibility conditions are satisfied because we can take an approximating sequence of C^∞ compatible initial data and the estimate in the Theorem 3.2 guarantees the convergence of the solution sequence to the desired solution.

2. For the C^∞ compatible data, the existence of smooth solution for (2.15)-(2.19) can be reduced to the existence of smooth solutions for a nondegenerate problem since (3.1) is also valid for $\alpha^\pm(x) > 0$.

   Let $\chi(x) \in C_0^\infty(R^1)$ such that

$$\chi(x) = \begin{cases} 1, & |x| < \beta/4, \\ 0, & |x| > \beta/2. \end{cases}$$

   Choose a sequence of functions $\alpha_\epsilon^\pm(x) \equiv \alpha^\pm(x) + \epsilon\chi(x)$ so that $\alpha_\epsilon^\pm(x) > 0$ and $\alpha_\epsilon^\pm(x) \to \alpha^\pm(x)$ in $C^\infty[\pm\beta, 0]$ as $\epsilon \to 0$.

   Then choose a sequence of initial data $v_{0\epsilon}^\pm(x)$ such that $v_{0\epsilon}^\pm(x) \to v_0^\pm(x)$ in $C^\infty[\pm\beta, 0]$ and $v_{0\epsilon}^\pm(x)$ is C^∞ compatible with respect to $\alpha_\epsilon^\pm(x)$.

The initial data $v_{0\epsilon}^{\pm}(x)$ can be chosen as follows.

$$v_{0\epsilon}^{\pm}(x) = v_0^{\pm}(x) + \epsilon\chi(x)\sigma^{\pm}(x),$$

where $\sigma^{\pm}(x) \in C^{\infty}[\pm\beta, 0]$ are constructed from there traces at $x = 0$. Actually, we choose $\sigma^{\pm}(0) = \sigma_x^{\pm}(0) = 0$ and determine all the higher order traces from

$$v_{xx}^{\pm} = (v_x^{\pm})^2(\alpha(x) + \epsilon)\left(\frac{1}{v_x^-} - \frac{1}{v_x^+}\right).$$

Consider the following perturbed nondegenerate problem of (2.15)-(2.19):

(a) Interior equations:

$$\begin{cases} \alpha_\epsilon^-(x)v_t^- + \partial_x\left(\dfrac{1}{v_x^-}\right) = 0, & -\beta < x < 0,\ 0 \le t \le T, \\ \alpha_\epsilon^+(x)v_t^+ + \partial_x\left(\dfrac{1}{v_x^+}\right) = 0, & 0 < x < \beta,\ 0 \le t \le T. \end{cases} \tag{4.1}$$

(b) Initial conditions:

$$v^{\pm}(x, 0) = v_{0\epsilon}^{\pm}(x), \quad 0 \le \pm x \le \beta. \tag{4.2}$$

(c) Boundary conditions:

$$v^+(0, t) - v^-(0, t) = 0, \tag{4.3}$$

$$v_t(0, t) = \frac{1}{v_x^-(0, t)} - \frac{1}{v_x^+(0, t)}. \tag{4.4}$$

If we can find C^{∞} smooth solutions $v_\epsilon^{\pm}$ for (4.1)-(4.4), then let $\epsilon \to 0$, There is a subsequence converging to a smooth solution $v^{\pm}$ for (2.9)-(2.12), by the Theorem 4.

3. To prove the existence of C^{∞} smooth solutions for (4.1)-(4.4), one can transform it back to the original free boundary problem and apply the result of the C^{∞} smooth solutions for nondegenerate Stefan problem [1,4,12]. We can also prove directly the existence for (4.1)-(4.4) by linear iteration. The proof can be sketched as follows.

   Let E_T^k be the space of functions $v(x, t)$ in $(-\beta, \beta) \times (0, T)$ such that

$$|||v|||_{k,T} \equiv \sup_{0 \le t \le T} \sum_{2k_1 + k_2 \le 2k+1} \left(\|\partial_t^{k_1}\partial_x^{k_2} v(t)\|^2\right)^{\frac{1}{2}} \le \infty. \tag{4.5}$$

We need to show that for any given integer k, the problem (4.1)-(4.4) has the solution $v(x, t) \in E_T^k$.

By the estimate of Theorem 3.1, we need only to show that for any given k, there is a $t_0 > 0$ such that in $(0, t_0)$, (4.1)-(4.4) has a solution $v(x,t) \in E^k_{t_0}$. Because once this is proved, the solution at $t = t_0$ has finite norm in $E^k_{t_0}$ and obviously the $(k-1)$-order compatibility conditions are satisfied at $x = 0,\ t = t_0$. So one can solve (4.1)-(4.4) again from the time $t = t_0$ and extend the solution $v(x,t)$ to the time $t_1 > t_0$. This procedure can be repeated until $t = T$.

The existence of solution in $[0, t_0]$ for (4.1)-(4.4) can be proved by linear iteration. Because the nonlinear terms in the boundary condition (4.4) contain $v^{\pm}_x(0,t)$ and we are unable to obtain their estimate for the linearized problem, the Nash-Moser iteration can be employed. Let $H^k(0,T)$ be the space defined by the norm $\|v\|_{k,T}$:

$$\|v\|^2_{k,T} \equiv \sum_{2k_1+k_2 \le 2k+1} \int_0^T \|\partial_t^{k_1} \partial_x^{k_2} v(t)\|^2 dt \le \infty.$$

Denote briefly

$$L(v)v \equiv \left(\alpha(x)\partial_t - \frac{1}{v_x^2}\partial_{xx}\right) v, \quad B(v)v \equiv \partial_t v - \frac{1}{v_x^-} + \frac{1}{v_x^+}. \tag{4.6}$$

Then the linearized operators of (4.6) at v are

$$\ell(v)\dot{v} \equiv L(v)\dot{v} + \frac{2v_{xx}}{v_x^3}\partial_x \dot{v} \tag{4.7}$$

and

$$b(v)\dot{v} \equiv \partial_t \dot{v} + \frac{1}{(v_x^-)^2}\partial_x \dot{v}^- - \frac{1}{(v_x^+)^2}\partial_x \dot{v}^+. \tag{4.8}$$

Consider the linearized problem

$$\begin{cases} \ell(v^-)\dot{v}^- = \dot{F}^-, & \text{in } -\beta < x < 0, \\ \ell(v^+)\dot{v}^+ = \dot{F}^+, & \text{in } 0 < x < \beta. \end{cases} \tag{4.9}$$

$$\begin{cases} b(v)\dot{v} = \dot{G}, \\ \dot{v}^+ - \dot{v}^- = 0 \end{cases} \quad \text{on } x = 0, \quad \dot{v}^{\pm}(\pm\beta, t) = 0. \tag{4.10}$$

$$\dot{v}^{\pm}(x, 0) = 0. \tag{4.11}$$

For the linearized problem (4.9)-(4.11), we can easily derive the following energy estimate

$$\|\dot{v}\|^2_{k,T} \le C_k \left(\|\dot{F}\|^2_{k,T} + |\dot{G}|^2_{k,T}\right). \tag{4.12}$$

by the usual integration by parts and the Gronwall inequality. This is the "tame" estimate [7] required in applying the Nash-Moser inverse function theorem. We omit the details here and the reader is referred to [8]. Thus the existence of smooth solutions is established for (4.1)-(4.4). This finishes the existence proof for the Theorem 3.1.

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