Resource Management in Wireless Heterogeneous Networks: an Optimization Perspective

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Maziar Sanjabi Boroujeni

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Dedication

To my wife, Pardees

Abstract

In this dissertation we consider the central task of resource management in wireless Heterogeneous Networks (HetNets). Resource management plays an important role in satisfying the increasing need for wireless data in HetNets. Our emphasis is mainly on cross layer strategies. Various aspects of cross layer resource management can be formulated as optimization problems. Throughout this dissertation, we use advanced optimization techniques to develop algorithms that are capable of efficiently solving these optimization problems. First, we consider the joint base station assignment and linear transceiver design problem. In order to gain a better understanding of resource management problems, we analyze the complexity of solving the resulting optimization problem. We establish the NP-hardness of this problem for a wide range of system-wide utility functions. Due to the fundamental difficulty of globally solving these problems, our emphasis in the rest of this dissertation is on devising efficient algorithms that can approximately solve these problems under different practical limitations. One major practical limitation of current resource management strategies is the need for the channel state information at the transmitter side. In this thesis we consider transceiver design in wireless HetNet when the channel state information is incomplete/inexact. We propose a general stochastic successive upper-bound minimization approach to optimize the average/ergodic utility of the system. We specialize our method to obtain an efficient stochastic sum-rate maximization algorithm. The proposed algorithm can use the statistical knowledge instead of actual channel values and is guaranteed to converge to the set of stationary points of the stochastic sum-rate maximization problem. We further generalize our stochastic method to a cross layer framework for jointly optimizing the base station clustering and the downlink beamformers in a partial coordinated transmission scenario. The partial coordination is crucial in improving the overall system performance by reducing backhaul overhead. We validate the effectiveness of our methods via numerical experiments.

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Chapter 1

Introduction

1.1 Heterogeneous Networks

The insatiable demand for high speed mobile communication has led to escalating growth of data transmission over wireless networks. One effective means to cope with the explosive data growth in current and future 5G networks is to increase the existing spectral efficiency by reducing the cell size, adding more base stations (BS) and increasing frequency reuse. These techniques have led to the deployment of pico base stations (pico BS) or relays within a large macro-cell, and have resulted in the use of femto BS (also known as home BS) which are low power, short range transmitters in residential houses or crowded business areas [1, 2]. The overall mesh network resulted by massive dense deployment of such low powered access points with wireless or wired backhaul support is called Heterogeneous Network (HetNet); see Figure 1.1. It is clear that the traditional high powered single-hop access mode between serving BS and its users (Figure 1.2) is no longer applicable for this new network architecture.

With the increase in the number of transmitters, simultaneously operating within the same frequency band, interference has become a major performance limiting factor for HetNets in the physical layer. Meanwhile, the growth in high speed data usage by mobile users not only affects the physical layer performance, but also stretches the backhaul infrastructure of cellular network to its current limits. Note that the possibility of load sharing with Wi-Fi/DSL networks [2] as well as having access nodes with wireless backhaul support (e.g. relays) results in a very complicated backbone. Thus, the management of such a complicated backbone network, affects the physical layer strategies and vice versa. Therefore, the classical cellular network management strategies that are based on partitioning the management into disjoint tasks across different layers (e.g., physical layer, MAC layer and network layer) are not effective anymore. Thus, new management strategies are needed in order to jointly manage all resources in the network.

In this thesis we use advanced signal processing and optimization techniques to propose a cross layer optimization framework for management of wireless HetNets. Our approach integrates cross-layer techniques such as interference mitigation in the physical layer as well as MAC layer algorithms to handle user scheduling, BS assignment and BS clustering.

It is clear that such cross-layer optimization framework involves many more design variables compared to the traditional cellular network management problem. Moreover, practical issues such as backhaul capacity limitations or channel state information overhead can complicate the cross-layer design problem. Despite the complexity of the arising problems, we devise efficient algorithms to tackle the cross-layer design. As we will see, these algorithm can lead to high performance gains compared to the traditional management strategies. We believe that, with the help of high computational power of massive scale cloud architecture for future 5G networks [3, 4], these methods are able to manage large scale HetNets efficiently.

1.2 Background

Management of wireless communication networks has extensive history in the literature. It has been studied in various contexts and by different tools. In order to obtain a general view of different strategies used to tackle the resource management problem in wireless communication networks, we briefly introduce such approaches and their contexts.

Information Theoretic Approaches

The goal of wireless communication networks is to carry information around and information theory deals with the theoretical limitations of transmitting information. Therefore, multiuser communication systems have been extensively studied in the information

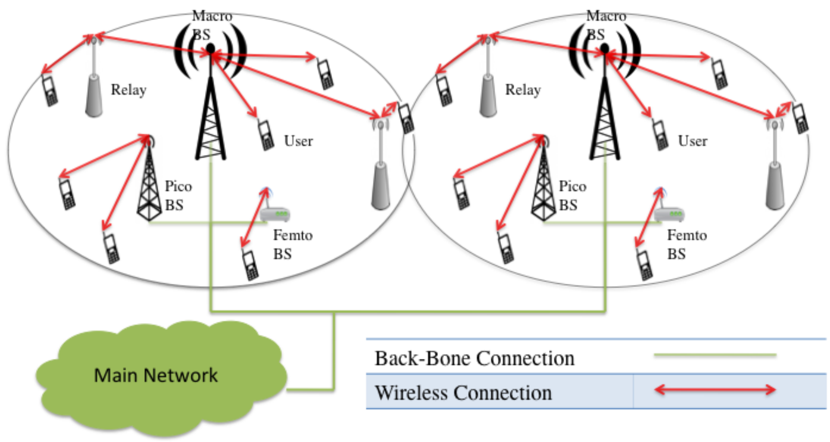


Figure 1.1: Heterogeneous Network (HetNet)

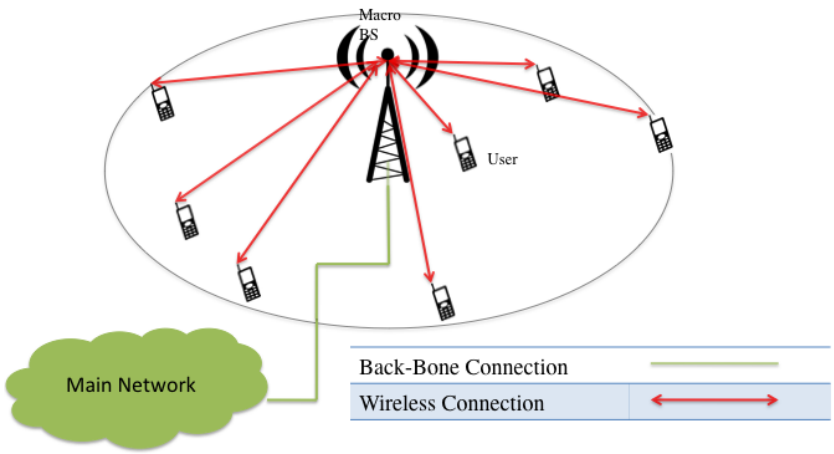


Figure 1.2: Traditional model for wireless communication networks

theory context [5, 6, 7]. Due to such close relationship between wireless communication and information theory, the ideas and strategies evolved in the latter have been used as guide lines for devising management protocols in wireless communication networks.

For most practical network setups, finding the fundamental limits of data transmission and the strategies to achieve them is not an easy task. In an attempt to make the problem more tractable, much simpler representations have been used to model the original complicated network. For example, it is usually assumed that the association variables such as user-BS assignment/clustering are pre-fixed. It is clear that such assumption is not true in practice; but it helps simplifying the problem of network management into finding resource allocation strategies in the physical layer. In the physical layer of a multiuser wireless communication network, the most important performance limiting factor is multiuser interference. Therefore, resource management in this scenario and with the aforementioned simplifying assumption boils down to handling interference. Dirty Paper Coding (DPC) and Successive Interference Cancelation (SIC) are two commonly used methods in information theory to deal with multiuser interference at the transmitter and receiver, respectively. The DPC result [6] states that the known interference could be precanceled at the transmitter side without sacrificing the achievable rate or using extra power. For example, in the case of broadcast channel, as the transmitter knows the signal of all receivers it can use this knowledge to pre-cancel the effect that signals of users $1, \cdots, k-1$ can have on user k. Thus, user k sees no interference from the signals of users $1, 2, \ldots, k-1$. The combination of DPC and TDMA techniques can achieve any point in the capacity region of the broadcast channel [8, 9, 10]. However, due to its high complexity, DPC technique is impractical. SIC is another technique to iteratively decode the interference and subtract it from the received signal. By subtracting the interference from the received signal, the SINR value of the signal increases and therefore, higher rates can be provided. The combination of SIC and TDMA can achieve any tuple of rates in the multi-access channel [7, 11, 12]. However, the complexity and the security issues, arising from decoding the interference, made this approach far from being practical.

In recent years more effective ways for managing multiuser interference have been proposed. For example, Jafar and Cadambe [5] have shown that the throughput of the interference channel (a network of transmitter-receiver pairs) can be much higher than the previous beliefs. In fact, it is shown that the total throughput of the interference channel (IC) increases linearly with the number of users. The technique they used is so called *interference alignment* where transmitters cooperatively transmit to *align* their interferences at different receivers. However, this technique in its current form is impractical due to the implementation/practical issues.

As these information theoretic approaches are proven to be far from practical implementation, in this thesis we focus more on coordinated beamforming (CB) and joint processing (JP) as interference mitigation strategies. In contrast with information theoretic approaches, CB and JP can be extended into cross-layer frameworks.

Coordinated Beamforming

As we mentioned, multiuser interference is one of the major performance limiting factors in wireless communication networks. A practical and somehow indirect way to overcome the problem of interference in multiuser wireless communication networks is to look at the system as a whole and try to optimize the performance of the network. In contrast with the information theoretic methods, in this approach we limit ourselves to simple and practical strategies such as linear beamforming at the transmitter and receiver side to control the interference in the physical layer. In order to fulfill this goal, we can consider a system-wide utility function and maximize the utility subject to the existing constraints that represent the resources or requirements in the network. It is clear that this system wide utility implicitly takes into account the effect of interference. One typical optimization problem is to maximize the total system throughput [13, 14, 15, 16], i.e.,

$$\begin{aligned} \max \quad & \sum_{i=1}^{I} R_i \\ \text{s.t.} \quad & \text{constraints} \end{aligned}$$

where R_i is the rate of user i. The constraints of the problem could vary from power budget constraint, to quality of service requirements. Unfortunately, in most of the cases, this problem becomes non-convex and computationally difficult to solve. In addition, although such objective denotes the throughput of the system, it is known that maximizing it might lead to unfair resource allocation among users in some scenarios.

Hence, people also consider different objective functions [17, 18, 19, 20, 21, 22] in the above optimization problem. Some commonly used utility functions $U(R_1, \cdots, R_I)$ are summarized in Table 1.1.

Table 1.1: Most common utility functions

Sum rate	$\sum_{i=1}^{I} R_i$
Harmonic mean	$\left(\sum_{i=1}^{I} R_i^{-1}\right)^{-1}$
Geometric mean	$\left(\prod_{i=1}^{I} R_i\right)^{1/I}$
Min rate	$\min_i R_i$

Such resource management strategies have been extensively studied in the literature by different names and for different means. For example, when the channels are diagonal and no correlated signaling is allowed across different antennas, we are basically led to the dynamic spectrum management problem. The dynamic spectrum management problem, which is a key core in the performance of DSL systems, has been a topic of intensive research in the signal processing community. The authors in [23] have studied this problem for different well-known utility functions and characterized the efficient solvability of the problems in different cases. Several algorithms have been proposed which provide varied performance in different channel conditions. These include: Iterative Water-filling Algorithm (IWFA) [24], Successive Convex Approximation Low complExity (SCALE) algorithm [25], Autonomous Spectrum Balancing (ASB)[26], Optimal Spectrum Balancing (OSB) [27]. Furthermore, different algorithms are proposed for the case when the channel matrices are non-diagonal. Authors in [28, 29, 30, 31, 32] proposed IWFA based algorithms for power allocation. However, these selfish approaches work well only in low SNR cases or when the interference is low.

A well-known CB approach that has been proven to be effective in dealing with multiuser interference is Weighted Mean Squared Error (WMMSE) algorithm [33, 34, 19]. This method is applicable to the multiple input multiple output (MIMO) networks with possibly non-diagonal channel matrices. It is also capable of maximizing utility functions other than simple sum rate [19]. In spite of its effectiveness WMMSE algorithm is limited to design of physical layer variables (beamformers). The key idea of WMMSE algorithm is to exploit the famous relationship between rate and mean squared error [33, 34, 19]. In this thesis we will frequently use this technique to extend the WMMSE type algorithms to cross-layer resource management methods.

Joint Processing

Joint Processing coordinated multi-point (JP-CoMP) transmission is another new strategy where base stations cooperate to transmit and receive the signals in order to mitigate the inter-user interference. In the JP-CoMP strategy, data to a single user is simultaneously transmitted from different base stations and the user jointly process the received signals from different base stations. Theoretically, this cooperative strategy can improve the performance of cell-edge users in cellular networks [35, 36, 37, 38, 39]. Most of the CoMP proposed techniques in the literature require each base station to have full/partial channel state information (CSI) as well as the knowledge of actual independent data streams to all remote terminals. With the complete sharing of data streams and CSI, the multi-cell scenario is effectively reduced to a single cell interference management problem with either total [40] or per-group-of-antennas power constraints [41, 42]. While these techniques can offer significant improvement on the raw data throughput, they also have several drawbacks including stringent requirement on base station coordination, the large demand on the communication bandwidth of backhaul links and the heavy computational load associated with the increasing number of cells [43, 44].

1.3 Challenges

In this section we briefly introduce two of the main challenges of resource management in HetNets that we will try to address in the rest of this dissertation.

1.3.1 Joint Design of Physical Layer Techniques and Higher Level Protocols

The joint design of physical layer techniques and higher level protocols has been studied in the literature in various contexts. For example, [45, 46] study the joint design of precoders and user admission. In addition to these references that focus on user admission, there are only a few prior studies that deal with the joint association and precoder design problem [47, 48, 49, 50]. For example, the authors of [47] proposed an algorithm for joint user assignment and power control in order to minimize the total power consumption in the downlink direction of a Code Division Multiple Access (CDMA) system, subject to some quality of service constraints. In addition, the references [48, 49] considered the same problem in the uplink direction of a CDMA network. The algorithms that they proposed are closely related to the simple distributed algorithm introduced in [51]. Moreover, [50] deals with the joint access point assignment and beamforming in the downlink direction of a congested system. Note that all these works [47, 48, 49, 50] aim at minimizing the total power rather than trying to maximize the throughput of the system. As a result, the direct application of the existing algorithms for multi-cell interference management cannot yield sufficient spectrum efficiency improvement.

It is worth noting that due to the growing interest in HetNets, there are a few recent studies considering joint base station assignment, beamforming and power allocation in uplink and downlink directions [52, 53, 54, 55, 56, 57].

In chapters 3 and 5 of this thesis we will show how the joint design of physical layer techniques (e.g., the transceiver structure) and higher level protocols (e.g. base station assignment/clustering) in the downlink direction of HetNets can provide more gains in practice. Moreover, in chapter 3, we theoretically examine the complexity of joint BS assignment and beamforming problem.

1.3.2 Imperfect/Incomplete Channel State Information at Transmitters (CSIT)

Most of the proposed methods for management of wireless networks require the perfect and full channel state information (CSI) of all links [58, 59, 60, 61, 17, 19, 18, 62, 63, 64, 65, 66]–an assumption that is clearly impractical due to channel aging and channel estimation errors. Obtaining the full CSI for all links would inevitably require a prohibitively large amount of training overhead and is therefore practically infeasible.

One approach to deal with the channel aging and the full CSI problem is to use the robust optimization methodology. To date, various robust optimization algorithms have been proposed to address this issue [67, 68, 69, 70, 71]. However, these methods are typically rather complex compared to their non-robust counterparts. Moreover, they are mostly designed for the worst case scenarios and therefore, due to their nature, are suboptimal when the worst cases happen with small probability. An alternative approach is to design the transceivers by optimizing the *average performance* using a stochastic optimization framework which requires only the *statistical channel knowledge* rather than the full instantaneous CSI. In contrast with the perfect CSIT case, the *ergodic/stochastic* transceiver design problem has not been studied throughly in the literature. In spite of its importance, there are only a few previous works that tackle the problem of imperfect CSIT using an *ergodic/stochastic* objective [72]. The approach in [72] is limited as the authors maximize a lower bound of expected sum rate that is only valid for Gaussian estimation error in the CSIT. Note that such Gaussian estimation error cannot model the cases where no estimate of the channel state is available (incomplete CSIT). In chapters 4 and 5 we discuss the possibility of resource management in presence of incomplete/inexact CSIT without confining our framework to Gaussian or any other channel estimation error model.

1.4 Thesis Structure

The rest of the thesis is structured as follows. In chapter 2 we introduce a generic model for downlink direction of HetNets which is the main focus of this thesis. In addition, we briefly present some technical preliminaries and mathematical definitions that are used in the rest of this dissertation.

In chapter 3 we focus on a coordinated beamforming strategy in which the assignments of users to BSs are not fixed. We will formulate the problem as a joint optimization of physical layer variables (i.e. beamformers) and user-BS assignment. Then we prove that such problem is computationally difficult to solve (NP-hard). In spite of its difficulty, we provide an efficient algorithm that produces high quality sub-optimal solutions for this problem. We perform extensive numerical experiments to provide evidence for benefits of our proposed method.

In chapter 4 we go back to designing physical layer variables, i.e. beamformers, in the presence of uncertainty in the channel values at the transmitter side. In this scenario, instead of maximizing the instantaneous throughput, we set the average/ergodic throughput as our objective. Using the stochastic optimization techniques we provide an algorithm for solving this problem. Furthermore, we prove that such algorithm converges to a stationary solution of the average throughput maximization problem. Our numerical experiments validate the efficacy of our approach.

In chapter 5 we consider a JP-CoMP scenario with uncertainty in channel values. In order to reduce the load on the backhaul network, our goal would be to maximize the average/ergodic throughput while keeping the number of BSs serving each user as low as possible. We formulate this problem into a regularized stochastic optimization problem. We provide an efficient algorithm that solves this problem to a stationary solution. Furthermore, we perform comprehensive numerical experiments to show the effectiveness of this method.

Finally, chapter 6 is dedicated to a summary of results as well as conclusions and further discussions.

For the sake of readability most of the technical details and proofs are relegated to the appendices. Note that in the appendices there is a section, section B, that is dedicated to Stochastic Successive Upper-bound Minimization (SSUM). SSUM framework is the base for the algorithms proposed in chapters 4 and 5. In appendix B we discuss the SSUM framework in detail and prove the convergence of SSUM algorithm under some assumptions. Furthermore, we introduce a few other applications of SSUM framework that are not necessarily related to resource management in HetNets.

Chapter 2

Models, Notations and Technical Preliminaries

2.1 Technical Preliminaries and Notations

In order to facilitate the presentation of the thesis we adopt the following notations. The set of real numbers is denoted by $\mathbb{R}$. Moreover, we denote the set of complex numbers by $\mathbb{C}$. We typify the expectation operator by $\mathbb{E}(\cdot)$. Note that the relations between random variables are almost surely, unless stated otherwise. In addition, We use $\mathrm{Tr}(\cdot)$ and $\det(\cdot)$ to denote the trace and determinant of a matrix respectively. Likewise, the Hermitian (conjugate transpose) of a complex matrix is denoted by $(\cdot)^H$, while $(\cdot)^T$ stands for transpose of a matrix with real entries. Additionally, we use $\mathbf{I}$ to denote the identity matrix of an appropriate size.

The rest of the adopted definitions and notations are presented below.

- **Complex Gaussian distribution:** For any vector $\mu \in \mathbb{C}^n$ and positive semi-definite hermitian matrix $\Sigma \in \mathbb{C}^{n\times n}$, $\mathcal{CN}(\mu, \Sigma)$ represents complex Gaussian distribution with mean μ and covariance Σ.

- **Real Gaussian distribution:** Likewise for any vector $\mu \in \mathbb{R}^n$ and positive semi-definite symmetric matrix $\Sigma \in \mathbb{R}^{n\times n}$, $\mathcal{N}(\mu, \Sigma)$ denotes real Gaussian distribution with mean μ and covariance Σ.

- **Distance of a point to a set:** The distance of a point x to a given non-empty set $S \subset \mathbb{R}^n$ or $\mathbb{C}^n$ is defined as

$$d(x, S) \triangleq \inf_{s \in S} \|x - s\|,$$

where $\|\cdot\|$ denotes the 2-norm in $\mathbb{R}^n$ or $\mathbb{C}^n$.

- **Directional derivative:** Let $h: D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$ is a convex set. The directional derivative of the function h at a point $x \in D$ in the direction $d \in R^n$ is defined as

$$h'(x; d) \triangleq \liminf_{t \downarrow 0} \frac{h(x + td) - h(x)}{t}.$$

Moreover, we define $h'(x; d) \triangleq +\infty$, if $x + td \notin D$, $\forall t > 0$.

Note that if function h is differentiable and set $D = \mathbb{R}^n$, i.e. h is defined on the whole space, then the directional derivative of h at any point x and in any direction d will simply be

$$\nabla h(x)^T d,$$

where $\nabla h(x)$ is the gradient of h at point x.

- **Stationary points of a function:** Let $h: D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$ is a convex set. The point $x \in \mathbb{R}^n$ is a stationary point of $h(\cdot)$ if

$$h'(x; d) \geq 0, \qquad \forall\ d \in \mathbb{R}^n. \tag{2.1}$$

Note that due to the definition of directional derivative $h'(x; d) = +\infty$, when d is not a feasible direction at point x.

Note that when function $h : D \to \mathbb{R}$ is convex, with closed convex D, then any stationary point of $h(\cdot)$ is an optimal solution of

$$\min_{x \in D}\ h(x) \tag{2.2}$$

- **Strongly Convex Function**: A function $h : D \to \mathbb{R}$ is called strongly convex with constant $\gamma > 0$ if

$$h(y) - h(x) \geq h'(x; y - x) + \frac{\gamma}{2}\|x - y\|^2,\ \forall\ x,\ y \in D.$$

The following definition is a key to establishing our inexact method in chapter 5.

Definition 1 *For any strongly convex function $h : D \to \mathbb{R}$, with closed convex D, we define the set of inexact stationary points (minimizers) of* (2.2)*, which are within ϵ accuracy of the stationary (optimal) solution as*

$$\mathcal{I}_\epsilon(h) \;=\; \{x \mid x \in D,\; h'(x;d) \geq -\epsilon\|d\|,\; \forall\; d\}. \tag{2.3}$$

It is easy to see that if h is differentiable and D is the whole space, then the set of ϵ accuracy solutions of (2.2), defined in (2.3), reduces to the set of points that satisfy the famous gradient condition $\|\nabla h(x)\| \leq \epsilon$.

- **Polynomial time complexity:** If the amount of time that it takes for an algorithm to solve a problem is a polynomial of the size of input string that is enough for specifying the problem instances, then the algorithm is called polynomial time.

- **Polynomial time reduction:** Polynomial time reduction from problem A to problem B is an algorithm that solves instances of problem A by a polynomial time transformation of the them to instances of problem B plus a polynomial number of calls to a subroutine for solving problem B.

- **Natural history of a stochastic process:** Consider a real valued stochastic process $\{Z^r\}_{r=1}^{\infty}$. For each r, we define the natural history of the stochastic process up to time r as

  $$\mathcal{F}^r \;=\; \sigma(Z^1, \ldots, Z^r),$$

  where $\sigma(Z^1, \ldots, Z^r)$ denotes the σ-algebra generated by the random variables $Z^1, \ldots, Z^r$.

- **Infinity norm of a function:** Let $h : \; D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$. The infinity norm of the function $h(\cdot)$ is defined as

  $$\|h\|_\infty \triangleq \sup_{x \in D} |h(x)|.$$

In the following section we present a generic model for the downlink direction of a HetNet. Our focus here will be on the generic notations and definitions that are common among all the scenarios considered in the forthcoming chapters.

2.2 Generic Model for a Heterogeneous Wireless Communication Network

Consider a downlink multi-cell HetNet consisting of K cells. Within each cell k there is a set of Q_k distributed BSs that provide service to users located within cell k. We denote the set of BSs in cell k by $\mathcal{Q}_k$. These BSs could be Macro, Pico or Femto BSs. Assume that in each cell k, there is a backhaul network connecting the set of BSs in that cell to the network.

Let us denote the set of users in cell k by $\mathcal{I}_k$, consisting of I_k users. We let a subset of BSs in cell k to serve user $i_k \in \mathcal{I}_k$. Note that the size of this subset could vary depending on the practical limitations. For example, in coordinated beamforming strategy, this subset only contains one BS. In contrast, a user could be served by all the BSs in its cell in the case of JP-CoMP. Observe that the load of overhead communication imposed by this user on the backhaul network highly depends on the size of this subset.

Let $\mathcal{I}$ denote the set of all the users. For simplicity, let us assume that each BS has M transmit antennas, and each user has N receive antennas. Let $\mathbf{H}_{i_k}^{q_l} \in \mathbb{C}^{N\times M}$ be the channel between the q-th transmitter of the l-th cell to the i-th user in the k-th cell. Moreover, let $\mathbf{H}_{i_k}^{l} = [\mathbf{H}_{i_k}^{1}, \cdots, \mathbf{H}_{i_k}^{Q_l}] \in \mathbb{C}^{N\times MQ_l}$ be the channel matrix between all the BSs in the l-th cell and the user i_k.

Let $\mathbf{x}^{q_k}$ denote the transmitted vector of BS q_k and the set of all transmitted signal in cell k as $\mathbf{x}^k = [(\mathbf{x}^1)^H, \cdots, (\mathbf{x}^{Q_k})^H]^H$. Therefore, when this signal is transmitted through the channel, the received signal $\mathbf{y}_{i_k} \in \mathbb{C}^N$ of user i_k is

$$\mathbf{y}_{i_k} = \sum_l \mathbf{H}_{i_k}^{l} \mathbf{x}^l + \mathbf{n}_{i_k}, \tag{2.4}$$

where $\mathbf{n}_{i_k} \in \mathbb{C}^{N\times 1}$ is additive white zero mean Gaussian noise with covariance $\sigma_{i_k}^2 \mathbf{I}$.

In literature it is most common to assume that the intended message for each user $i_k \in \mathcal{I}_k$ can be modeled as a random vector $\mathbf{s}_{i_k} \in \mathbb{C}^{d_{i_k}}$. For the sake of simplicity we set $d_{i_k} = d$ for all users i_k. Moreover, it is commonly presumed that $\mathbf{s}_{i_k}$ comprises of independent zero mean real or complex valued standard normal random variables. As a result $\mathbb{E}(\mathbf{s}_{i_k}) = \mathbf{0}$ and $\mathbb{E}(\mathbf{s}_{i_k}^H \mathbf{s}_{i_k}) = \mathbf{I}$.

In this thesis our focus is on the linear beamforming strategies in the physical layer due to their practical simplicity. In other words, we assume that the transmitted signal

$\mathbf{x}^k$ at cell k is a linear function of the messages $\mathbf{s}_{i_k}$, $i_k \in \mathcal{I}_k$, i.e.

$$\mathbf{x}^k = \mathfrak{V}_k(\{\mathbf{s}_{i_k} \mid i_k \in \mathcal{I}_k\}), \tag{2.5}$$

where $\mathfrak{V}_k : \mathbb{C}^{I_k d} \to \mathbb{C}^{MQ_l}$ is a linear mapping. In each of the following chapters we will provide more explicit forms for the linear mappings $\mathfrak{V}_k$ based on the corresponding scenario.

On the receiver side linear beamforming means that each receiver i_k uses a linear mapping of its received signal $\mathbf{y}_{i_k}$ to estimate the message $\mathbf{s}_{i_k}$, i.e.

$$\hat{\mathbf{s}}_{i_k} = \mathfrak{U}_{i_k}(\mathbf{y}_{i_k}), \tag{2.6}$$

where $\hat{\mathbf{s}}_{i_k} \in \mathbb{C}^d$ is the estimate of $\mathbf{s}_{i_k}$ and $\mathfrak{U}_{i_k} : \mathbb{C}^N \to \mathbb{C}^d$ is a linear mapping. For each user $i_k \in \mathcal{I}_k$, the mean squared error (MSE) matrix of this estimation can be defined as

$$\mathbf{E}_{i_k} = \mathbb{E}\Big((\hat{\mathbf{s}}_{i_k} - \mathbf{s}_{i_k})(\hat{\mathbf{s}}_{i_k} - \mathbf{s}_{i_k})^H\Big) \tag{2.7}$$

One major constraint on the design of physical layer precoders, i.e. transmit beamformers, is the power budget. It means that the power of transmitted signal by each BS cannot exceed a certain threshold, i.e. its power budget. Note that the transmitted signal of BS q_k, $\mathbf{x}^{q_k}$, is a function of random messages for users and therefore is random too. Thus, instead of imposing power constraint on the actual power we limit the amount of average power that is used in each BS. In other words, each BS q_k has a transmit power budget P_{q_k} such that

$$\mathbb{E}\big((\mathbf{x}^{q_k})^H \mathbf{x}^{q_k}\big) \leq P_{q_k}. \tag{2.8}$$

In each of the following chapters, we will further discuss this constraint and how it relates to the corresponding scenario of that chapter.

Now that we have introduced a generic model for downlink of HetNets, we are ready to present our results on the management of heterogeneous wireless communication networks.

Chapter 3

Joint BS Assignment and Downlink Beamforming for Wireless Heterogeneous Networks

3.1 Problem Formulation

Consider a wireless HetNet in the downlink direction. In this chapter we use the generic model introduced in chapter 2. For the sake of simplicity in presentation we assume that there is only one cell, i.e. $K = 1$. Therefore, we can drop the index k in our notations. As it is clear from our models, the transmitters are sharing the same frequency band for communication. We denote transmitter q by $\mathcal{B}_q$ and user i by $\mathcal{U}_i$.

The scenario that we consider in this chapter is coordinated baemforming. In other words, we assume that each user is served by a *unique* transmitter (rather than *multiple* transmitters as in [73], which entails extra communication/coordination overhead). In order to capture the idea of user Base Station (BS) assignment, we define a set of binary variable $a_{iq} \in \{0, 1\}$, $(i, q) \in \mathcal{I} \times \mathcal{Q}$ to denote the assignments, i.e., $a_{iq} = 1$ means user i associates to base station q. Each user can only be served by one base station; hence, for each user i,

$$\sum_{q \in \mathcal{Q}} a_{iq} \leq 1. \tag{3.1}$$

As we assume that the users are not pre-assigned to any BS, each base station can potentially transmit to all the users. Thus, we let $\mathbf{V}_{iq}$ denote the beamforming matrix used by base station q when it transmits to user i. In other words, the transmitted signal of BS q would be

$$\mathbf{x}^q = \sum_{i \in \mathcal{I}} \mathbf{V}_{iq} \mathbf{s}_i, \tag{3.2}$$

where $\mathbf{s}_i \in \mathbb{C}^d$ contains d independent Gaussian messages for user i and the matrix $\mathbf{V}_{iq} \in \mathbb{C}^{M \times d}$ is the linear transmit precoder applied to the data streams of user $\mathcal{U}_i$ by base station $\mathcal{B}_q$. Clearly $\mathbf{V}_{iq}$ should be zero if user $\mathcal{U}_i$ is not served by BS $\mathcal{B}_q$ (i.e., $\mathbf{V}_{iq} = \mathbf{0}$ if $a_{iq} = 0$). As we mentioned in the generic model for HetNets the power of the transmitted signal of $\mathcal{B}_q$ should be less than the power budget at $\mathcal{B}_q$ denoted by P_k. Note that, we made the assumption $\mathbb{E}(\mathbf{s}_i^H \mathbf{s}_i) = \mathbf{I}$. Moreover, we presume that the messages of different users are statistically independent. With these assumptions the power budget constraint at base station $\mathcal{B}_q$ can be written as

$$\mathbb{E}((\mathbf{x}^q)^H \mathbf{x}^q) = \sum_{i=1}^{I} \mathrm{Tr}(\mathbf{V}_{iq}^H \mathbf{V}_{iq}) \leq P_q. \tag{3.3}$$

Assuming that each user treats interference as noise, the rate for user i is given by

$$R_i(\mathbf{V}; \mathbf{a}) = \sum_{q=1}^{Q} a_{iq} \log\det\left(\mathbf{I} + \left(\sigma_i^2 \mathbf{I} + \sum_{p=1}^{Q} \sum_{m \neq i} \mathbf{H}_i^p \mathbf{V}_{mp} \mathbf{V}_{mp}^H (\mathbf{H}_i^p)^H\right)^{-1} \mathbf{H}_i^q \mathbf{V}_{iq} \mathbf{V}_{iq}^H (\mathbf{H}_i^q)^H\right), \tag{3.4}$$

where in the above summation at most one of the terms is nonzero (due to constraint (3.1)). In contrast to the prior works [47, 48, 49, 50], our goal is not to minimize the total transmit power. Instead, we assign base stations to users and design the precoders jointly to maximize a system wide utility function. In other words, our objective is to maximize a system wide utility function $U(R_1, \cdots, R_I)$ by choosing the assignment variables a_{iq} and the precoders $\mathbf{V}_{iq}$. Throughout, we assume that U preserves the partial order of $\mathbb{R}^I$, i.e. $\hat{R}_i \geq R_i, \ \forall\ i$ implies that $U(\hat{R}_1, \cdots, \hat{R}_I) \geq U(R_1, \cdots, R_I)$. Under these assumptions, the problem of joint BS assignment and precoder can be formulated as follows

$$\max_{\{\mathbf{a},\mathbf{V}\}} \quad U(R_1,\cdots,R_I) \tag{3.5}$$

$$\text{s.t.} \quad \sum_{q\in\mathcal{Q}} a_{iq} \le 1,\ \forall\, i\in\mathcal{I},\ a_{iq}\in\{0,1\},(i,k)\in\mathcal{I}\times\mathcal{Q} \tag{C1}$$

$$\sum_{i=1}^{I} \mathrm{Tr}(\mathbf{V}_{iq}\mathbf{V}_{iq}^H) \le P_q,\ \forall\, q\in\mathcal{Q}, \tag{C2}$$

with R_i given by (3.4) for all i. In the above formulation, we do not explicitly enforce $\mathbf{V}_{iq}=0$ when $a_{iq}=0$ because this constraint will be satisfied automatically at the optimality. To see this, consider an optimal solution $\{a^*,\mathbf{V}^*\}$ of (3.5) for which some user $\mathcal{U}_i$ is not assigned to BS $\mathcal{B}_q$ ($a_{iq}^*=0$) but $\mathbf{V}_{iq}^*$ is nonzero. Since the utility function $U(\cdot)$ is non-decreasing in each argument, by setting $\mathbf{V}_{iq}^*=\mathbf{0}$, we obtain a feasible solution with a higher (or at least the same) objective value. Hence, we can always refine the solutions of (3.5) such that $\mathbf{V}_{iq}=\mathbf{0}$ whenever $a_{iq}=0$.

The optimization problem (3.5) is non-convex and involves discrete optimization variables. In the remainder of this paper, we will first analyze the intrinsic complexity of this problem and establish its NP-hardness. Then, we develop an efficient iterative algorithm to compute a local optimal solution of (3.5).

3.2 Complexity Analysis

In this section we analyze the theoretical complexity status of problem (3.5) for different utility functions.

One important family of utility functions, denoted by $\mathcal{F}$, is the so called α-fairness utility function [74] defined as

$$U_\alpha(R_1,\cdots,R_I)=\begin{cases}\sum_{i=1}^{I}\frac{R_i^{1-\alpha}}{1-\alpha} & \text{if } \alpha\ge 0,\ \alpha\ne 1;\\ \sum_{i=1}^{I}\log(R_i) & \text{if } \alpha=1.\end{cases} \tag{3.6}$$

As it is shown in the following table, many popular utility functions belong to $\mathcal{F}$ [74, 23].

α	***Utility***	***Expression***
0	*Sum-Rate*	$\sum_{i=1}^{I} R_i$
1	*Proportional Fairness*	$\sum_{i=1}^{I} \log(R_i)$
2	*Harmonic-Rate*	$(\sum_{i=1}^{I} R_i^{-1})^{-1}$
∞	*Max-Min*	$\min_{1\leq i\leq I} R_i$

The following result characterizes the complexity status of the problem (3.5) for different utility functions as the number of base stations and users increases.

Theorem 1 *The joint base station assignment and precoder design problem* (3.5) *is NP-hard for any utility function $U(\cdot)$ in $\mathcal{F}$ if the number of receive antennas is at least 3 ($N \geq 3$). Moreover, for the sum-rate utility function, the problem remains NP-hard regardless of the number of transmit/receive antennas (i.e., for any M and N).*

Proof The NP-hardness proof for the sum-rate utility is based on a polynomial time reduction from the MAX 2-SAT problem [75]. The latter is the problem of determining the maximum number of clauses among a set of 2-literal Boolean clauses that can be satisfied simultaneously using Boolean variable assignments. Since the MAX 2-SAT problem is NP-hard, the problem of optimal base station assignment and precoder design is NP-hard as well. The details of the proof are given in Appendix A.1.

To prove the NP-hardness in the case of a general utility function $U(\cdot) \in \mathcal{F}$, we need to assume $N \geq 3$. In this case, we use a polynomial time reduction from the graph 3-colorability problem which is known to be NP-complete. The graph 3-colorability problem is the problem of determining if the vertices of a graph G can be colored with 3 colors such that none of the adjacent nodes are colored the same. To construct a polynomial time reduction from the graph 3-colorability problem, consider a graph $G = (V, E)$, where $|V| = I$. Let there be I mobile users in the system, each corresponding to a node in the graph G. Assume that there are $Q = 3I$ base stations, each having a power budget of $P = 1$. Let us also set the noise power at any user to $\sigma^2 = 1$. Furthermore, we assume that all users are equipped with 3 antennas and the BSs are equipped with only one antenna. For any user $\mathcal{U}_i$, there are 3 corresponding BSs denoted by $\mathcal{B}_{i_1}$, $\mathcal{B}_{i_2}$, and $\mathcal{B}_{i_3}$. The channels are constructed as follows:

1. The channel between $\mathcal{B}_{i_\ell}$ and $\mathcal{U}_i$ is $\mathbf{e}_\ell$, for all $i = 1, \cdots, N$, and all $\ell = 1, 2, 3$,

where $\mathbf{e}_\ell$ is the unit vector of size 3 with all elements equal to zero except the ℓ-th one.

2. If $(i,j) \in E$, the channel between $\mathcal{B}_{i_\ell}$ and $\mathcal{U}_j$ is $\frac{1}{2}\mathbf{e}_\ell$, for all $\ell = 1, 2, 3$, and otherwise it is zero.

The channel construction is depicted in Figure 3.1 for the case where graph $G = (V, E)$ and $V = \{i, j, k\}$ and $E = \{(i, j)\}$.

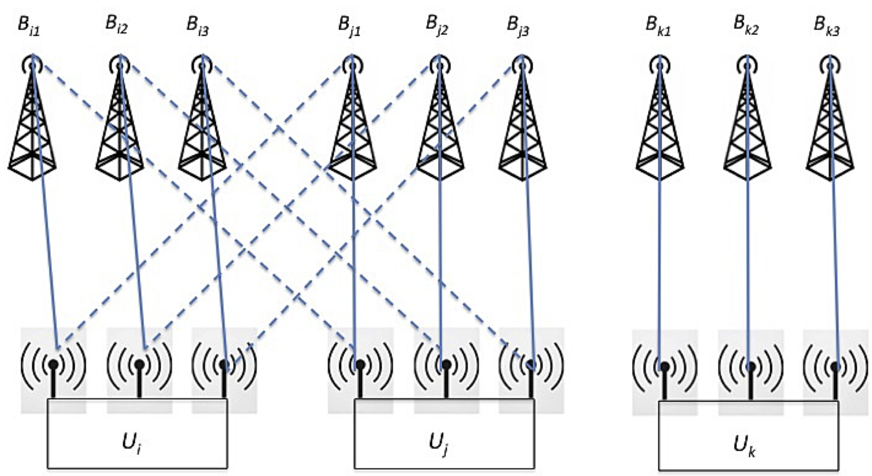


Figure 3.1: The channel construction for $G = (V, E)$, where $V = \{i, j, k\}$ and $E = \{(i, j)\}$. A solid line means that the channel is 1, a dashed line means that the channel is $\frac{1}{2}$ and no line means that the channel is 0.

Then, in order to show the NP-hardness of (3.5), it suffices to prove the following claim.

Claim 1 *In the above constructed network, the optimization problem* (3.5) *has an optimal value greater than or equal to* $U(\log(2), \cdots, \log(2))$ *if and only if the graph* G *is 3-colorable.*

The proof of Claim 1 is given in Appendix A.1.1. This completes the proof of Theorem 1.

A couple of remarks are in order. First, although the scenario considered here is MIMO, the proofs can be easily extended to the multi-carrier OFDM combined with Single Input Single Output (SISO) setup when at least 3 tones are utilized in the system. Note that in the multi-carrier SISO OFDM scenario, the channels, as well as the resulting precoders, are diagonal (see [23]). A detailed description of the modification needed in the proof can be found in Appendix A.1.1. Moreover, as SISO OFDM is a special case of MIMO OFDM, the NP-hardness analysis extends naturally to the MIMO OFDM scenario as well. Second, it should be noted that Theorem 1 is not a direct consequence of the results in [23] . The proofs in [23] are based on the difficult scenarios with strong cross links and weak direct links (i.e., high interference), whereas in our case there are no preassigned direct links. If transmitter-receiver associations can be re-assigned, the high interference scenarios considered in [23] would be easy.

3.3 Utility Maximization Using Matrix-Weighted-Sum-MSE

In this section we develop an algorithm for the joint base station assignment and precoder design problem (3.5). We first need to define some new notations and variables. Let $\mathbf{U}_{iq} \in \mathbb{C}^{N\times d}$ denote the linear receive beamformer that the receiver $\mathcal{U}_q$ uses to decode the data coming from the BS $\mathcal{B}_q$. For such estimation, we can define the mean squared error matrix

$$\mathbf{E}_{iq} \triangleq (\mathbf{I} - \mathbf{U}_{iq}^H \mathbf{H}_i^q \mathbf{V}_{iq})(\mathbf{I} - \mathbf{U}_{iq}^H \mathbf{H}_i^q \mathbf{V}_{iq})^H + \sum_{\ell=1}^{Q} \sum_{j\neq i} \mathbf{U}_{iq}^H \mathbf{H}_i^\ell \mathbf{V}_{j\ell} \mathbf{V}_{j\ell}^H (\mathbf{H}_i^\ell)^H \mathbf{U}_{iq} + \sigma_i^2 \mathbf{U}_{iq}^H \mathbf{U}_{iq}, \tag{3.7}$$

As a result, the achievable data rate between the BS q and the user i is

$$r_{iq} \triangleq \log\det\left(\mathbf{I} + \left(\sigma_i^2 \mathbf{I} + \sum_{p=1}^{Q} \sum_{j\neq i} \mathbf{H}_i^p \mathbf{V}_{jp} \mathbf{V}_{jp}^H (\mathbf{H}_i^p)^H\right)^{-1} \mathbf{H}_i^q \mathbf{V}_{iq}^H \mathbf{V}_{iq}^H (\mathbf{H}_i^q)^H\right). \tag{3.8}$$

Summing the rates across all BSs with a_{iq}'s as the corresponding weights, we get the total rate expression (3.4) for user i:

$$R_i = \sum_{q=1}^{Q} a_{iq} r_{iq}. \tag{3.9}$$

Next we will show that

$$r_{iq} = \max_{\substack{\mathbf{W}_{iq}\succeq 0\\ \mathbf{U}_{iq}}} \quad \log\det(\mathbf{W}_{iq}) - \mathrm{Tr}(\mathbf{W}_{iq}\mathbf{E}_{iq}) + d. \tag{3.10}$$

Checking the first order optimality condition of (3.10), the optimum value of $\mathbf{U}$ and $\mathbf{W}$ is given by [19]:

$$\mathbf{W}_{iq}^{\mathrm{opt}} = \mathbf{E}_{iq}^{-1} \tag{3.11}$$

$$\mathbf{U}_{iq}^{\mathrm{opt}} = \mathbf{U}_{iq}^{\mathrm{mmse}} = \left(\sum_{\ell,m} \mathbf{H}_i^{\ell}\mathbf{V}_{m\ell}\mathbf{V}_{m\ell}^{H}(\mathbf{H}_i^{\ell})^{H} + \sigma_i^2\mathbf{I}\right)^{-1} \mathbf{H}_i^{q}\mathbf{V}_{iq},\ \forall\ i,\ q. \tag{3.12}$$

It is easy to see that by plugging in (3.11) and (3.12) in (3.10) we obtain (3.8). Combining (3.9) and (3.10), we get

$$R_i = \max_{\{\mathbf{W}_{iq}\succeq 0,\ \mathbf{U}_{iq}\}_{q=1}^{Q}} \quad \sum_{q=1}^{Q} a_{iq}\bigg(\log\det(\mathbf{W}_{iq}) - \mathrm{Tr}(\mathbf{W}_{iq}\mathbf{E}_{iq}) + d\bigg). \tag{3.13}$$

In the sequel, we use this relation to propose an efficient algorithm for finding a local optimal solution of (3.5).

3.3.1 Weighted Sum-Rate Maximization

In this subsection we devise an algorithm for maximizing the weighted sum rate utility $U(R_1,\cdots,R_I) = \sum_{i=1}^{I}\gamma_i R_i$, where $\gamma_i \geq 0$ is the weight of user i. Using (3.13), problem (3.5) with weighted sum rate as its objective can be equivalently written as

$$\begin{aligned} \max_{\{a,\mathbf{V},\mathbf{U},\mathbf{W}\}} \quad & \sum_{i=1}^{I}\gamma_i\bigg(\sum_{q=1}^{Q} a_{iq}\left(\log\det(\mathbf{W}_{iq}) - \mathrm{Tr}(\mathbf{W}_{iq}\mathbf{E}_{iq}) + d\right)\bigg) \\ \text{s.t.} \quad & \text{(C1), and (C2),} \\ & \mathbf{W}_{iq} \succeq \mathbf{0},\ \forall\ i,\ q. \end{aligned} \tag{3.14}$$

As a consequence of Theorem 1 this problem is NP-hard. It is non-convex, and moreover it contains binary variables a_{iq}. In the first step we relax the binary variable constraint $a_{iq} \in \{0,1\}$ into $a_{iq} \in [0,1]$, i.e., we solve

$$\begin{aligned} \max_{\{a,\mathbf{V},\mathbf{U},\mathbf{W}\}} \quad & \sum_{i=1}^{I} \gamma_n \left(\sum_{q=1}^{Q} a_{iq} \left(\log\det(\mathbf{W}_{iq}) - \mathrm{Tr}(\mathbf{W}_{iq}\mathbf{E}_{iq}) + d \right) \right) \\ \text{s.t.} \quad & a_{iq} \geq 0, \; \forall\, i,\; q \;\text{ and }\; \sum_{q} a_{iq} \leq 1, \; \forall\, i \\ & \sum_{i} \mathrm{Tr}(\mathbf{V}_{iq}\mathbf{V}_{iq}^H) \leq P_q, \; \forall\, q \\ & \mathbf{W}_{iq} \succeq \mathbf{0}, \; \forall\, i,\; q. \end{aligned} \tag{3.15}$$

Clearly, the relaxed optimization problem (3.15) is linear in $\mathbf{a}$ and hence there exists a solution on the vertices of the constraint set. In other words, there exists an optimal solution of (3.15) for which the assignment variables $\{a_{iq}\}$ are binary; therefore the relaxed optimization problem (3.15) is equivalent to the original optimization problem and achieves the same optimal objective value.

There are four sets (blocks) of variables $\{\mathbf{a}, \mathbf{V}, \mathbf{U}, \mathbf{W}\}$ in problem (3.14). One strategy to solve (3.14) is the coordinate descent type of update rule [76], where at each iteration of the algorithm we update only one block of the variables while the rest of the variables are kept fixed. In particular, we update the four blocks of variables in the following manner:

- Fix $\mathbf{V}, \mathbf{W}, \mathbf{a}$. Update $\mathbf{U}$ using (3.12).
- Fix $\mathbf{U}, \mathbf{V}, \mathbf{a}$. Update $\mathbf{W}$ according to (3.11).
- Fix $\mathbf{U}, \mathbf{V}, \mathbf{W}$. Update $\mathbf{a}$ using one gradient projection step; see step 5 in Table 3.1.
- Fix $\mathbf{U}, \mathbf{W}, \mathbf{a}$. Update $\mathbf{V}$ by solving (3.15) with respect to $\mathbf{V}$ only; see [19] and step 6 of Table 3.1.

It is also worth noting that in the update of the variable $\mathbf{a}$, we only do one simple gradient descent step instead of solving to the global optimality. The reason is that if we solve it to the global optimality at the initial steps, the $\{a_{iq}\}$ values will be fixed at zero/one and will not change during the algorithm any more. Therefore, there would be no chance for the algorithm to further optimize the assignment variables. The overall algorithm is presented in Table 3.1 in details.

Joint Base Station Assignment and Precoder Design for Weighted Sum-Rate Maximization

1 Initialize $\mathbf{V}_{iq}$'s randomly and set $a_{iq} = \frac{1}{Q}$

2 **repeat**

3 *Update* $\mathbf{U}$ using the MMSE receiver (3.12)
$\mathbf{U}_{iq} \leftarrow \left(\sum_{\ell,m} \mathbf{H}_i^{\ell}\mathbf{V}_{m\ell}\mathbf{V}_{m\ell}^H(\mathbf{H}_i^{\ell})^H + \sigma_n^2\mathbf{I}\right)^{-1}\mathbf{H}_i^q\mathbf{V}_{iq},\ \forall\, i,\ q$

4 *Calculate* $\mathbf{E}_{iq}, \forall\, i,\ q$ according to (3.7) and *update* $\mathbf{W}$: $\mathbf{W}_{iq} \leftarrow \mathbf{E}_{iq}^{-1},\ \forall\, i,\ q$

5 *Update* $\mathbf{a}$: Apply a gradient projection step to the relaxed optimization problem

$$\max_{a} \quad \sum_{i=1}^{I}\gamma_i\Big(\sum_{q=1}^{Q} a_{iq}r_{iq}\Big), \text{ s.t. } \quad \sum_{q} a_{iq} \leq 1,\ \ a_{iq} \geq 0,\ \ \forall\, i,\ q \tag{3.16}$$

according to $\mathbf{a}_i \leftarrow P_{\Omega}(\mathbf{a}_i + \lambda\gamma_i[r_{i1}, \cdots, r_{iQ}]^T),\ \forall\, i$, where $\mathbf{a}_i = [a_{i1}, \cdots, a_{iQ}]^T$, λ is the step-size, and $P_{\Omega}(\cdot)$ is the projection to the set $\Omega = \{\mathbf{a} \in \mathbb{R}^Q : \mathbf{a} \geq 0,\ \sum_i \mathbf{a}_i \leq 1\}$.

6 *Update* $\mathbf{V}$: $\mathbf{V}_{iq} \leftarrow a_{iq}\gamma_i\left(\sum_{\ell,m} a_{m\ell}\gamma_m(\mathbf{H}_m^q)^H\mathbf{U}_{m\ell}\mathbf{W}_{m\ell}\mathbf{U}_{m\ell}^H\mathbf{H}_m^q + \mu_q^*\mathbf{I}\right)^{-1}(\mathbf{H}_i^q)^H\mathbf{U}_{iq}\mathbf{W}_{iq}$, $\forall\ i,\ \forall\ q$, where $\mu_q^* \geq 0$ is the optimal Lagrange multiplier (for the constraint $\sum_j \mathrm{Tr}(\mathbf{V}_{jq}\mathbf{V}_{jq}^H) \leq P_q$) which can be found using bisection (see [19]).

7 **until** convergence

Table 3.1: Pseudocode of the proposed algorithm for the weighted sum rate maximization

The following theorem shows a desirable convergence behavior of the proposed algorithm.

Theorem 2 *Assume $\sigma_i^2 > 0, \forall i$. Then every limit point of the iterates generated by the proposed algorithm is a stationary point of* (3.15).

Proof To prove this theorem, we use the convergence result of the block successive upper-bound minimization (BSUM) algorithm [77]. In order to use the results in [77], we need to check that the assumptions of [77, Theorem 2] are satisfied. In particular, we need to show that for updating each variable, we solve a locally tight upper-bound of the objective function which satisfies [77, Assumption 2]. Clearly, in the steps of updating the variables $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$, the original objective function is minimized while the rest of the variables are held fixed; and therefore the assumptions in [77, Assumption 2] are satisfied. On the other hand, the step of updating the variable $\{a_{iq}\}$ is equivalent to solving

$$\begin{aligned} \min_{a} \quad & -\sum_{i=1}^{I} \gamma_i \Big(\sum_{q=1}^{Q} a_{iq} r_{iq} \Big) + \frac{1}{2\lambda} \sum_{i,q} \| a_{iq} - \hat{a}_{iq} \|^2, \\ \text{s.t.} \quad & \sum_{q} a_{iq} \le 1, \quad a_{iq} \ge 0, \quad \forall \; i, \; q, \end{aligned} \tag{3.17}$$

where $\hat{a}_{iq}$ is the value of the variable a_{iq} at the current step. It can be easily checked that the objective function in (3.17) satisfies the conditions in [77, Assumption 2] and therefore [77, Theorem 2] implies that every limit point of the iterates generated by the algorithm is a stationary point of the original objective function in (3.15).

3.3.2 Maximizing U_α Utility

In this section, we modify the proposed joint base station association and precoder design algorithm in order to maximize the general α-fairness utility functions in (3.5). The resulting optimization problem is given by

$$\begin{aligned} \max_{\{\mathbf{a},\mathbf{V},\mathbf{U},\mathbf{W}\}} \quad & U_\alpha \left(R_1, ..., R_I \right) \\ \text{s.t.} \quad & \text{(C1)}, \; \text{(C2) and } \mathbf{W}_{iq} \succeq \mathbf{0}, \; \forall \; i, \; q, \end{aligned} \tag{3.18}$$

where $R_i = \sum_{q=1}^{Q} a_{iq} \left(\log\det(\mathbf{W}_{iq}) - \text{Tr}(\mathbf{W}_{iq}\mathbf{E}_{iq}) + d \right), \; \forall \; i$. Similar to the case of weighted sum rate utility, we apply the coordinate-wise update strategy to each set of the variables alternately. First note that the use of α-utility function does not change the maximization subproblems defining the update of $\mathbf{U}$ or $\mathbf{W}$. Hence the updates for $\mathbf{U}$ and $\mathbf{W}$ are the same as before. To update the association variables $\{a_{iq}\}$, we need to find the gradient $\nabla_{\mathbf{a}} U_\alpha$ using chain rule. Using the definition (3.6), the partial derivative of $\mathbf{U}_\alpha$ with respect to a_{iq} is given by

$$\frac{\partial U_\alpha}{\partial a_{iq}} = \frac{r_{iq}}{(R_i)^\alpha}, \tag{3.19}$$

and the gradient projection update for association vector of user $\mathcal{U}_i$, $\mathbf{a}_i$, can be written as

$$\mathbf{a}_i \leftarrow P_\Omega \left(\mathbf{a}_i + \frac{1}{R_i^\alpha} \lambda [r_{i1}, \cdots, r_{iQ}]^T \right).$$

To update $\mathbf{V}$, we fix all the other variables and maximize the objective function with respect to $\mathbf{V}$. Unfortunately it is difficult to find a simple closed form solution for updating $\mathbf{V}$ due to the complex nature of U_α utility. Therefore, we propose to locally linearize the objective around the current MSE matrix $\mathbf{E}$. In particular, suppose that the current point is $\{\hat{\mathbf{W}}, \hat{\mathbf{U}}, \hat{\mathbf{V}}, \hat{\mathbf{E}}, \hat{a}\}$ and define $\hat{R}_i = \sum_q \hat{a}_{iq}(\log\det(\hat{\mathbf{W}}_{iq}) - \mathrm{Tr}(\hat{\mathbf{W}}_{iq}\hat{\mathbf{E}}_{iq}) + d)$ as the rate at current point. Then the locally linearized version of the problem is

$$\begin{aligned} \max_{\mathbf{V}} \quad & \sum_{i=1}^{I} \frac{1}{\hat{R}_i^\alpha} \sum_{q=1}^{Q} a_{iq} \left(\log\det(\mathbf{W}_{iq}) - \mathrm{Tr}(\mathbf{W}_{iq}\mathbf{E}_{iq}) \right) \\ \text{s.t.} \quad & \sum_{i} \mathrm{Tr}(\mathbf{V}_{iq}\mathbf{V}_{iq}^H) \le P_q, \ \forall q. \end{aligned}$$

This problem is exactly the same as weighted sum-rate maximization by setting $\gamma_i = 1/\hat{R}_i^\alpha$. Hence we can apply the same update as the step 6 of the algorithm described in the previous section. The overall proposed algorithm is described in Table 3.2.

3.3.3 Complexity Analysis of the Proposed Algorithms

To implement the WMMSE algorithm with greedy base station assignment, we assume that each user can sense the strongest channel. For example, the mobile users are usually able to sense the nearby transmitters' signal in practice and pick the strongest one. With the greedy base station assignment, we can implement the WMMSE [19] whose computational complexity per iteration is $\mathcal{O}(I^2NM^2 + I^2MN^2 + I^2M^3 + IN^3)$

(for more details see [19]). Note that in this iteration complexity analysis we have assumed that $d = M$.

In comparison, proposed joint base station assignment and beamforming approach treats base station assignment as a part of the optimization process. In practical situations, the user can only be served by the nearby base stations. We assume that for each mobile user there are at most L candidate base stations that can serve that user. In practice it is usually reasonable to further assume that $1 < L \ll Q$. Note that this assumption is not necessary for our algorithm to work.
The beamforming at each iteration of the proposed algorithm is equivalent to an iteration of WMMSE with LI users (because we have to do beamforming for all possible direct connections). Moreover, in each gradient update of the assignment variables we need to calculate a total of $\mathcal{O}(N^3)$ rates for each of the LI connections, thus giving a total of $\mathcal{O}(LIN^3)$ for assignment variables update. As a result, the per-iteration complexity of the proposed algorithm is $\mathcal{O}(L^2I^2NM^2 + L^2I^2MN^2 + L^2I^2M^3 + LIN^3)$.

3.4 Simulation Results

In this section we present the simulation results evaluating the performance of the proposed algorithm. We compare our method with WMMSE algorithm with fixed user association [19], where each user is assigned to the BS with the strongest channel matrix (in terms of Frobenius norm). Our simulation setup is similar to the one presented in [57], which considers the joint BS assignment and beamforming in the up-link direction in a HetNet. We consider a single dense macro cell in a HetNet consisting of 7 pico BSs where the distance between the adjacent pico Base Stations is 200 meters (see Figures 3.5(b)-3.5(a)). Each entry of the channel between BS $\mathcal{B}_q$ and user $\mathcal{U}_i$ is drawn from a Gaussian distribution with mean zero and standard deviation parameter $\nu_{iq} = (200/D_{iq})^{3.5}S_{iq}$, where D_{iq} denotes the distance between $\mathcal{B}_q$ and $\mathcal{U}_i$ and $10\log_{10} S_{iq} \sim \mathcal{N}(0, 8)$ captures the shadowing effect. In our simulations, the power of noise is normalized to one at all receivers. Moreover, we set $L = Q = 7$ when implementing the proposed algorithms.

First, we consider a scenario with $I = 16$ users, among which half of them are located randomly and uniformly at the cell edge of the first pico BS ($D_{i1} \in [90, 100]$), and the

Joint Base Station Assignment and Precoder Design for General α-Fairness Utility Maximization

1 Initialize $\mathbf{V}_{iq}$'s randomly and set $a_{iq} = \frac{1}{Q}$

2 **Repeat**

3 $\mathbf{U}_{iq} \leftarrow \left(\sum_{\ell,m} \mathbf{H}_i^{\ell}\mathbf{V}_{m\ell}\mathbf{V}_{m\ell}^H(\mathbf{H}_i^{\ell})^H + \sigma_i^2\mathbf{I}\right)^{-1}\mathbf{H}_{iq}\mathbf{V}_{iq},\ \forall\, i,\ \forall\, q$

4 *Calculate* $\mathbf{E}_{iq}, \forall\, i,\ q$ according to (3.7) and *update* $\mathbf{W}_{iq} \leftarrow \mathbf{E}_{iq}^{-1},\ \forall\, i,\ \forall\, q$

5 $\hat{R}_i \leftarrow \sum_k a_{iq}(\log\det(\mathbf{W}_{iq}) - \mathrm{Tr}(\mathbf{W}_{iq}\mathbf{E}_{iq})), \forall\, n$

6 $\mathbf{a}_i \leftarrow P_\Omega\left(\mathbf{a}_i + \frac{1}{\hat{R}_i^{\alpha}}\lambda[r_{i1},\cdots,r_{iQ}]^T\right), \forall\, i$

7 $\hat{R}_i \leftarrow \sum_k a_{iq}(\log\det(\mathbf{W}_{iq}) - \mathrm{Tr}(\mathbf{W}_{iq}\mathbf{E}_{iq})),\ \gamma_i = 1/\hat{R}_i^{\alpha}, \forall\, i$

8 $\mathbf{V}_{iq} \leftarrow \gamma_i a_{iq}\big(\sum_{\ell,m}\gamma_m a_{m\ell}(\mathbf{H}_m^q)^H\mathbf{U}_{m\ell}\mathbf{W}_{m\ell}\mathbf{U}_{m\ell}^H\mathbf{H}_m^q + \mu_{iq}^*\mathbf{I}\big)^{-1}(\mathbf{H}_i^q)^H\mathbf{U}_{iq}\mathbf{W}_{iq}, \forall\, i,\ \forall\, q$

9 **until** convergence

Table 3.2: Pseudocode of the proposed algorithm for the α-fairness utility

rest of the users are randomly located at the cell edge of the other pico base stations. Furthermore we assume that all the users are equipped with $N = 4$ antennas and all the base stations have the same power budget $P_q = P$. In the first two plots we compare the proposed algorithm for Sum-Rate Maximization and WMMSE algorithm when all the base stations are equipped with $M = 4$. The results are averaged over 500 random initializations of users' positions and channels.

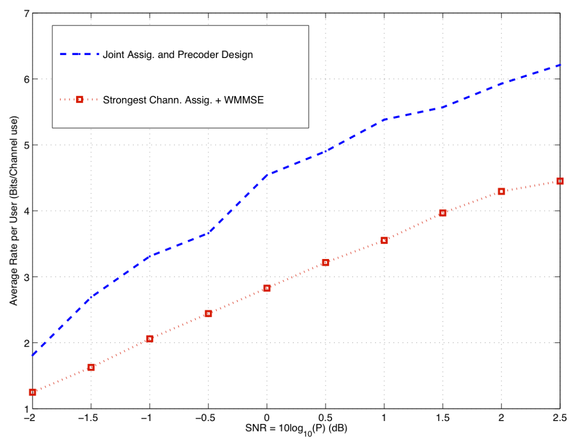


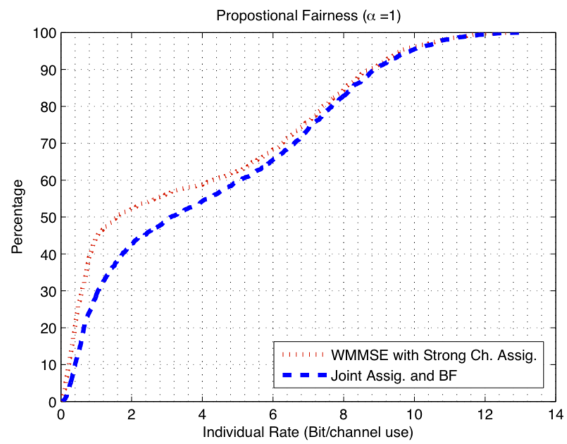
Figure 3.2: Average rate per user versus power in (dB) for sum-rate maximization.

As can be seen in Figure 3.2, the proposed algorithm for Sum Rate maximization outperforms WMMSE with fixed greedy assignment in terms of achievable sum rates. Note that in case of Sum-Rate utility ($\alpha = 0$), the algorithm in Table 3.2 reduces to the same algorithm in Table 3.1.

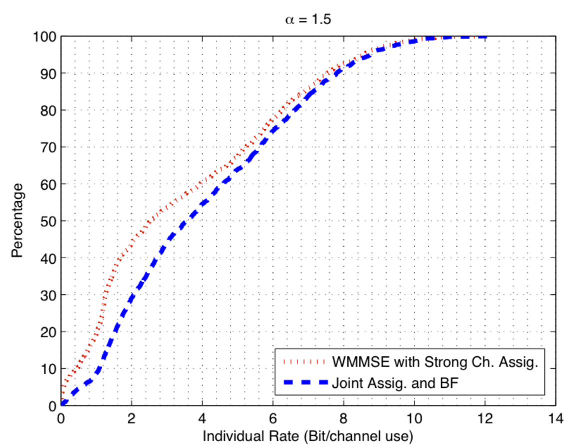
In order to compare our algorithm with WMMSE algorithm in terms of fairness among the users, we fix the power budget $P = 0.1$ and apply our methods as well as WMMSE to the Proportional Fairness utility and U_α for $\alpha = 1.5$ and 2 (Harmonic Mean utility). We have plotted the cumulative distribution function (CDF) of individual rate that is obtained with and without fixing the channel assignments in Figures 3.3(a)-3.3(c). The results are plotted by averaging over 100 different random realizations of users' positions and channels.

To see the effect of different number of antennas at different transmitters, we assume that the congested BS along with 3 other base stations have only 4 antennas, but the remaining base stations are equipped with 8 antennas. All the other conditions are kept the same as before. We apply our algorithm for the case of proportional fairness utility as well as U_α for $\alpha = 1.5$. It can be seen in Figures 3.4(a)-3.4(b) that our algorithm outperforms the WMMSE in terms of the rate CDF.

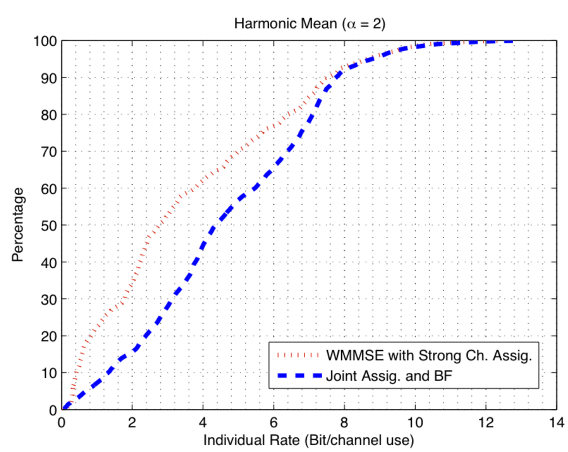
Finally we present figures to show how the change of assignment can balance the load in the network. As can be seen in Figures 3.5(a) and 3.5(b), the BS in the center of the picture is congested when the assignments are done based on strongest channel only (Figure 3.5(b)). However, the joint assignment and beamforming algorithm achieves a balanced load among the BSs (Figure 3.5(a)). It is interesting that the "*cell breathing*" phenomenon [49], in which the cell sizes dynamically change with the level of congestion, can be observed in Figure 3.5(a).



(a)

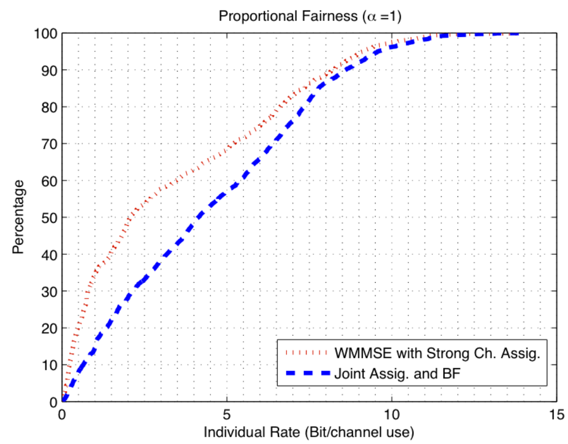


(b)

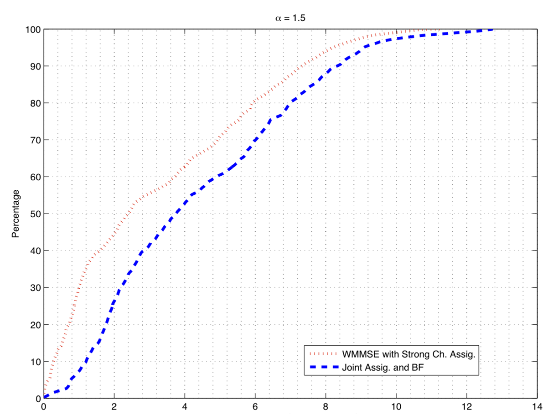


(c)

Figure 3.3: Rate CDF when $P = 0.1$, $M = 4$, $N = 4$, and $I = 16$ for different α-Fairness utilities: (a) Proportional Fairness utility ($\alpha = 1$), (b) $\alpha = 1.5$ and (c) Harmonic Mean utility ($\alpha = 2$).

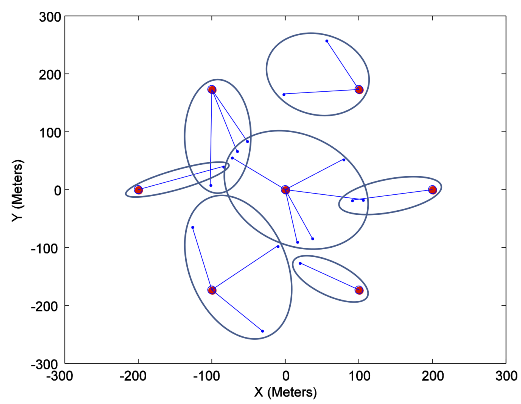


(a)

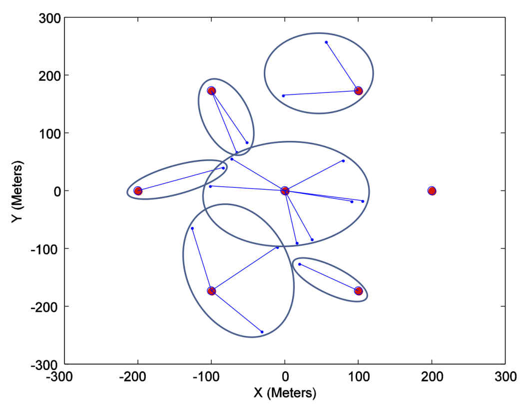


(b)

Figure 3.4: Rate CDF when $P = 0.1$, and some BS have 8 antennas, and the other base stations (including the congested BS) have 4 antennas, $N = 4$, and $I = 16$ for different α-Fairness utilities: (a) Proportional Fairness utility ($\alpha = 1$), (b) $\alpha = 1.5$.



(a)



(b)

Figure 3.5: The assignment of users to BSs by joint assignment and beamforming (a), versus the case where the users are assigned to the BS with strongest channel (b).

Chapter 4

Beamforming with Inexact and Incomplete CSIT Using Stochastic Optimization

In the previous chapter we proposed an algorithm to do joint beamforming and BS assignment in wireless HetNets. One of the main assumptions of the proposed algorithm, along with many other management methods for HetNets [60, 61, 17, 19, 18, 62, 64, 65, 66], is the availability of a perfect and full CSIT.

In order to prepare CSIT the channel state should first be estimated at the receiver using pilot messages. The allowance of co-channel transmission results in interference which in turn requires the need for estimating interfering channels as well as direct channels. Therefore, the number of channels to estimate increases quadratically with the size of the network. Thus, performing channel estimation in a large network is time consuming which alongside with the fast changing environment can lead to channel aging effect. In order to have CSIT, the estimated channel matrices should be sent to the transmitter. Consequently, the estimated channel states are rounded to a finite accuracy in order to be fed back to the transmitter through the wireless link. This finite precision feedback introduces rounding error to the already inexact channel values. The feedback process is also time consuming given the huge number of estimated channels. The situation only gets worse as the number of antennas in the transmitter and/or receiver side increases. Therefore, assuming complete and perfect CSIT is clearly impractical.

The key idea to deal with incomplete or imperfect CSIT is to model the channel states as a random variable with known distribution. Then, the distributions can be calibrated using experimental data (e.g. channel estimation). The distribution by which we model the channel state is usually defined by long term physical parameters such as path-loss coefficients, propagation distance and channel training time/power. Such parameters are usually much easier to be sent through feed back links compared to the actual channel matrices.

One approach to deal with the complete but inexact CSIT is to use the robust optimization techniques. Various robust optimization algorithms have been proposed to address this issue [67, 68, 69, 70, 71]. In these methods after modeling the channel state as random variables, high probability quality of service constraints are formed based on the distribution of those random variables. As a result, they can guarantee a low probability of outage (measured based on the channel distribution). Note that these methods, by nature, are mainly designed for the worst case scenario. As a result they might be sub-optimal in the cases where the worst case happens with low probability. Moreover, as they need to work with outage probability type of constraints, these methods are typically rather complex compared to their non-robust counterparts and require more computational effort. In addition, they have a limited scope and cannot deal with realistic channel distributions due to analytical intractability. In fact, most of them use a Gaussian channel distribution with estimated channels as the means for the Gaussian distributions. Therefore, utilizing these methods still requires estimating the channel for all the links in the network (including all the interfering links), albeit the channel estimation need not be very accurate. For a large HetNet, this approach is still not practical since estimating all the channel states requires excessive training overhead.

In this chapter, we propose a simple stochastic iterative optimization algorithm for solving the ergodic sum rate maximization problem. Our approach is based on the correspondence between the well-known weighted sum rate maximization and weighted mean squared error (MSE) minimization. Unlike the previous approach of [72] which maximizes a lower bound of the expected weighted sum rate problem, our work directly maximizes the ergodic sum rate and is guaranteed to converge to the set of stationary points of the ergodic sum rate maximization problem. Moreover, our approach can adapt easily to situations when the channel statistics change over time. Although presented for sum rate maximization, our algorithm and its convergence can be easily extended to other system utilities.

4.1 Problem Formulation

Let us assume a wireless HetNet in the downlink direction with the generic model introduced in chapter 2. For the sake of simplicity, we assume that there is only one BS in each cell, i.e. $Q_k = 1$. With this assumption, we can drop the index q in our notations. It is easy to generalize our approach to the case where there are more BSs in each cell.

In the transmitter side, we assume that the BS k uses a beamforming matrix $\mathbf{V}_{i_k} \in \mathbb{C}^{M\times d}$ to transmit the message of user i_k. Therefore, the transmitted message of BS k is

$$\mathbf{x}^k = \sum_{i\in\mathcal{I}_k} \mathbf{V}_{i_k}\mathbf{s}_{i_k}. \tag{4.1}$$

Using this equation, it is easy to see that the power budget constraint at BS k would be

$$\sum_{i\in\mathcal{I}_k} \mathrm{Tr}(\mathbf{V}_{i_k}\mathbf{V}_{i_k}^H) \leq P_k. \tag{4.2}$$

Moreover, at the receiver side we assume that each user i_k, utilizes the linear beamforming matrix $\mathbf{U}_{i_k} \in \mathbb{C}^{N\times d}$ to estimate its corresponding message using the following relationship

$$\hat{\mathbf{s}}_{i_k} = \mathbf{U}_{i_k}^H \mathbf{y}_{i_k}, \tag{4.3}$$

where $\mathbf{y}_{i_k}$ is its received signal and $\hat{\mathbf{s}}_{i_k}$ is the estimate of $\mathbf{s}_{i_k}$.

Under these assumptions, it is known that the instantaneous achievable rate of user i_k is given by [19, 78, 79]:

$$R_{i_k}(\mathbf{U}_{i_k}, \mathbf{V}, \mathbf{H}) \;=\; \log\det(\mathbf{E}_{i_k}^{-1}(\mathbf{V}, \mathbf{U}_{i_k})), \tag{4.4}$$

where

$$\mathbf{E}_{i_k}(\mathbf{V},\mathbf{U}_{i_k}) \triangleq (\mathbf{I}-\mathbf{U}_{i_k}^H\mathbf{H}_{i_k}^k\mathbf{V}_{i_k})(\mathbf{I}-\mathbf{U}_{i_k}^H\mathbf{H}_{i_k}^k\mathbf{V}_{i_k})^H + \sum_{(j,\ell)\neq(k,i)} \mathbf{U}_{i_k}^H\mathbf{H}_{i_k}^j\mathbf{V}_{\ell_j}\mathbf{V}_{\ell_j}^H(\mathbf{H}_{i_k}^j)^H\mathbf{U}_{i_k} + \sigma_{i_k}^2\mathbf{U}_{i_k}^H\mathbf{U}_{i_k} \tag{4.5}$$

denotes the mean square error matrix. Furthermore, it is not hard to check that the optimum receive beamformer $\mathbf{U}_{i_k}$ which maximizes (4.4) is the MMSE receiver [19, 78, 79]:

$$\mathbf{U}_{i_k}^{\mathrm{mmse}} = \mathbf{J}_{i_k}^{-1}\mathbf{H}_{i_k}^k\mathbf{V}_{i_k}, \tag{4.6}$$

where $\mathbf{J}_{i_k} \triangleq \sum_{j=1}^{K}\sum_{\ell=1}^{I_j}\mathbf{H}_{i_k}^j\mathbf{V}_{\ell_j}\mathbf{V}_{\ell_j}^H(\mathbf{H}_{i_k}^j)^H + \sigma_{i_k}^2\mathbf{I}$ is the covariance matrix of the total received signal at receiver i_k.

To maximize the sum of the rates of all users in the network, we need to solve

$$\begin{aligned} \max_{\mathbf{V},\mathbf{U}_{i_k}} \quad & \sum_{k=1}^{K}\sum_{i=1}^{L_k} R_{i_k}(\mathbf{U}_{i_k},\mathbf{V},\mathbf{H}) \\ \text{s.t.} \quad & \sum_{i=1}^{L_k}\mathrm{Tr}(\mathbf{V}_{i_k}\mathbf{V}_{i_k}^H) \le P_k,\ \forall\ k=1,\cdots,K, \end{aligned}$$

which requires the knowledge of all instantaneous channel matrices $\mathbf{H}$ at the transmitters. Due to practical limitations, the exact channel state information (CSI) of all channels, which changes rapidly in time, is typically not available at the base stations. A more realistic assumption is to assume that an approximate CSI is know for a few links, while a statistical model of the CSI is known for the rest of the links. The latter changes more slowly in time and is easier to track. In such situations, we are naturally led to maximize the expected sum rate of all users, where the expectation is taken over the channel statistics.

Notice that, in practice, the optimum receive beamformer in (4.6) can be updated by measuring the received signal covariance matrix. Hence even though the complete channel knowledge is not available at the transmitters, the receive beamformers can be optimized according to the instantaneous channel values by measuring the received signal covariance matrices. Therefore, the expected sum rate maximization problem can be written as

$$\begin{aligned} \max_{\mathbf{V}} \ & \mathbb{E}_{\mathbf{H}}\left\{\sum_{k=1}^{K}\sum_{i=1}^{L_k}\max_{\mathbf{U}_{i_k}}\{R_{i_k}(\mathbf{U}_{i_k},\mathbf{V},\mathbf{H})\}\right\} \\ \text{s.t.} \ & \sum_{i=1}^{L_k}\mathrm{Tr}(\mathbf{V}_{i_k}\mathbf{V}_{i_k}^H) \leq P_k, \ \forall\ k=1,\cdots,K, \end{aligned} \tag{4.7}$$

To be consistent with the optimization literature, let us rewrite (4.7) as a minimization problem:

$$\begin{aligned} \min_{\mathbf{V}} \ & \mathbb{E}_{\mathbf{H}}\{g_1(\mathbf{V},\mathbf{H})\} \\ \text{s.t.} \ & \sum_{i=1}^{L_k}\mathrm{Tr}(\mathbf{V}_{i_k}\mathbf{V}_{i_k}^H) \leq P_k, \ \forall\ k=1,\cdots,K, \end{aligned} \tag{4.8}$$

where

$$g(\mathbf{V},\mathbf{H}) \ = \ \sum_{k=1}^{K}\sum_{i=1}^{L_k}\min_{\mathbf{U}_{i_k}}\{-R_{i_k}(\mathbf{U}_{i_k},\mathbf{V},\mathbf{H})\}\,. \tag{4.9}$$

It can be checked that g is smooth but non-convex in $\mathbf{V}$ [19]. In practice, due to other design requirements, one might be interested in adding some convex non-smooth regularizer to the above objective function. In the next chapter we will see how such a regularizer can help us enforce more favorable designs in our formulation.

4.2 Stochastic Successive Upper-Bound Minimization

In this section we briefly introduce the idea of Stochastic Successive Upper-Bound Minimizations (SSUM) that was developed in [80]. The SSUM method serves as a base for devising our iterative stochastic algorithm to solve (4.8) . For a more general version of SSUM as well as its convergence analysis see Appendix B.

Expected value optimization is an important problem that arises in many contexts; for different examples of that see [80] and the references therein. In stochastic optimization, the problem is usually formulated as

$$\begin{aligned} \min \quad & f(x) \triangleq \mathbb{E}_{\xi}[g(x,\xi)] \\ \text{s.t.} \quad & x \in \mathcal{X}, \end{aligned} \tag{4.10}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is a bounded closed convex set; ξ is a random vector drawn from a set $\Xi \in \mathbb{R}^m$, and $g : \mathcal{X} \times \Xi \mapsto \mathbb{R}$ is a real-valued function. A classical approach for solving the above optimization problem is the sample average approximation (SAA) method. At each iteration of the SAA method, a new realization of the random vector ξ is obtained and the optimization variable x is updated by solving

$$\begin{aligned} x^r \in \; & \arg\min \; \frac{1}{r}\sum_{i=1}^{r} g(x,\xi^i) \\ & \text{s.t.} \quad x \in \mathcal{X}. \end{aligned}$$

Here $\xi^1, \xi^2, \ldots$ are some independent, identically distributed realizations of the random vector ξ. We refer the readers to [81, 82, 83, 84, 85] for the roots of the SAA method and [86, 87, 88] for several surveys on SAA.

One of the major drawbacks of the SAA method is the complexity of each step. In general, solving (B.2) may not be easy due to the non-convexity and/or non-smoothness of $g(\cdot,\xi)$. To overcome the difficulties in solving the subproblem (B.2), we propose an inexact SAA method whereby at each step a well-chosen approximation of the function $g(\cdot,\xi)$ in (B.2) is minimized. Specifically, at each iteration r, we update the optimization variable according to

$$x^r \leftarrow \arg\min_{x\in\mathcal{X}} \; \frac{1}{r}\sum_{i=1}^{r} \hat{g}(x, x^{i-1}, \xi^i), \tag{4.11}$$

where $\hat{g}(\cdot, x^{i-1}, \xi^i)$ is an approximation of the function $g(\cdot,\xi^i)$ around the point x^{i-1}. To ensure convergence of this method, we require the approximation function $\hat{g}(\cdot, x^{i-1}, \xi^i)$ to be a *locally tight*, *strongly convex upper bound* of the original function $g(\cdot,\xi^i)$ around the point x^{i-1}, for each $i = 0, \ldots, r-1$. For this reason, we call the above algorithm (B.3) a stochastic successive upper-bound minimization method (SSUM). The proposed SSUM method is described in Table B.1.

Under some assumptions (assumptions A-B in Appendix B) the SSUM method converges almost surely to set of stationary solutions of (4.10). For a detailed proof of this convergence analysis see Appendix B. In the next section we will see how to specialize the SSUM method to solve problem (4.8).

Table 4.1: The SSUM algorithm

Find a feasible point $x^0 \in \mathcal{X}$ and set $r = 0$.
repeat
$r \leftarrow r + 1$
$x^r \leftarrow \arg\min_{x \in \mathcal{X}} \frac{1}{r} \sum_{i=1}^{r} \hat{g}_(x, x^{i-1}, \xi^i)$
until some convergence criterion is met.

4.3 Stochastic Weighted Minimum Mean Squared Error Algorithm

In order to utilize the SSUM algorithm for solving (4.8), we need to find a convex tight upper-bound approximation of $g(\mathbf{V}, \mathbf{H})$. To do so, let us introduce a set of variables $\mathbf{P} \triangleq (\mathbf{W}, \mathbf{U}, \mathbf{Z})$, where $\mathbf{W}_{i_k} \in \mathbb{C}^{d \times d}$ (with $\mathbf{W}_{i_k} \succeq \mathbf{0}$) and $\mathbf{Z}_{i_k} \in \mathbb{C}^{M \times d}$ for any $i \in \mathcal{I}_k$ and for all $k = 1, \cdots, K$. Furthermore, define

$$\hat{R}_{i_k}(\mathbf{W}_{i_k}, \mathbf{Z}_{i_k}, \mathbf{U}_{i_k}, \mathbf{V}, \mathbf{H}) \triangleq -\log\det(\mathbf{W}_{i_k}) + \mathrm{Tr}(\mathbf{W}_{i_k}\mathbf{E}_{i_k}(\mathbf{U}_{i_k}, \mathbf{V})) + \frac{\rho}{2}\|\mathbf{V}_{i_k} - \mathbf{Z}_{i_k}\|^2 - d, \tag{4.12}$$

for some fixed $\rho > 0$ and

$$\mathcal{G}(\mathbf{V}, \mathbf{P}, \mathbf{H}) \triangleq \sum_{k=1}^{K} \sum_{i=1}^{L_k} \hat{R}_{i_k}(\mathbf{W}_{i_k}, \mathbf{Z}_{i_k}, \mathbf{U}_{i_k}, \mathbf{V}, \mathbf{H}). \tag{4.13}$$

Using the first order optimality condition, we can check that

$$g(\mathbf{V}, \mathbf{H}) = \min_{\mathbf{P}} \mathcal{G}(\mathbf{V}, \mathbf{P}, \mathbf{H}).$$

Now, let us define

$$\hat{g}(\mathbf{V}, \bar{\mathbf{V}}, \mathbf{H}) = \mathcal{G}(\mathbf{V}, \mathcal{P}(\bar{\mathbf{V}}, \mathbf{H}), \mathbf{H}),$$

where

$$\mathcal{P}(\bar{\mathbf{V}}, \mathbf{H}) = \arg\min_{\mathbf{P}} \mathcal{G}(\bar{\mathbf{V}}, \mathbf{P}, \mathbf{H}).$$

Clearly, we have

$$g(\bar{\mathbf{V}},\mathbf{H}) = \min_{\mathbf{P}} \ \mathcal{G}(\bar{\mathbf{V}},\mathbf{P},\mathbf{H}) = \mathcal{G}(\bar{\mathbf{V}},\mathcal{P}(\bar{\mathbf{V}},\mathbf{H}),\mathbf{H}) = \hat{g}(\bar{\mathbf{V}},\bar{\mathbf{V}},\mathbf{H}),$$

and

$$g(\mathbf{V},\mathbf{H}) = \min_{\mathbf{P}} \ \mathcal{G}(\mathbf{V},\mathbf{P},\mathbf{H}) \leq \mathcal{G}(\mathbf{V},\mathcal{P}(\bar{\mathbf{V}},\mathbf{H}),\mathbf{H}) = \hat{g}(\mathbf{V},\bar{\mathbf{V}},\mathbf{H}).$$

Furthermore, $\hat{g}_1(\mathbf{V},\bar{\mathbf{V}},\mathbf{H})$ is strongly convex in $\mathbf{V}$ with parameter ρ due to the quadratic term in (4.12). Hence $\hat{g}(\mathbf{V},\bar{\mathbf{V}},\mathbf{H})$ is *locally tight*, *strongly convex upper bound* of the function $g(\bar{\mathbf{V}},\mathbf{H})$. Therefore, we can apply the SSUM algorithm.

Define $\mathbf{H}^r$ to be the r-th channel realization. Let us further define

$$\mathbf{P}^r \triangleq \arg\min_{\mathbf{P}} \ \mathcal{G}(\mathbf{V}^{r-1},\mathbf{P},\mathbf{H}^r), \tag{4.14}$$

where $\mathbf{V}^{r-1}$ denotes the transmit beamformer at iteration $r-1$. Notice that $\mathbf{P}^r$ is well defined since the optimizer of (4.14) is unique. With these definitions, the update rule of the SSUM algorithm becomes

$$\begin{aligned} \mathbf{V}^r \leftarrow \arg\min_{\mathbf{V}} \ & \frac{1}{r}\sum_{i=1}^{r} \hat{g}(\mathbf{V},\mathbf{V}^{i-1},\mathbf{H}^i) \\ \text{s.t.} \quad & \sum_{i\in\mathcal{I}_k} \mathrm{Tr}(\mathbf{V}_{i_k}\mathbf{V}_{i_k}^H) \leq P_k, \quad \forall\ k \end{aligned}$$

or equivalently

$$\begin{aligned} \mathbf{V}^r \leftarrow \arg\min \ & \frac{1}{r}\sum_{i=1}^{r} \mathcal{G}(\mathbf{V},\mathbf{P}^i,\mathbf{H}^i) \\ \text{s.t.} \quad & \sum_{i\in\mathcal{I}_k} \mathrm{Tr}(\mathbf{V}_{i_k}\mathbf{V}_{i_k}^H) \leq P_k, \quad \forall\ k. \end{aligned} \tag{4.15}$$

In order to make sure that the SSUM algorithm can efficiently solve (4.8), we need to confirm that the update rules of the variables $\mathbf{V}$ and $\mathbf{P}$ can be performed in a computationally efficient manner in (4.14) and (4.15). Checking the first order optimality condition of (4.14), it can be shown that the updates of the variable $\mathbf{P} = (\mathbf{W},\mathbf{U},\mathbf{Z})$ can be done in closed form; see Table 4.2. Moreover, for updating the variable $\mathbf{V}$, we need to solve a simple quadratic problem in (4.15). Using the Lagrange multipliers, the update rule of the variable $\mathbf{V}$ can be performed using a one dimensional search method over the Lagrange multiplier [19]. Table 4.2 summarizes the SSUM algorithm applied to the expected sum rate maximization problem; we name this algorithm as stochastic weighted mean square error minimizations (stochastic WMMSE) algorithm. Notice that although in the SSUM algorithm the update of the precoder $\mathbf{V}_{i_k}$ depends on all the past realizations, Table 4.2 shows that all the required information (for updating $\mathbf{V}_{i_k}$) can be encoded into two matrices $\mathbf{A}_{i_k}$ and $\mathbf{B}_{i_k}$, which are updated recursively.

The following theorem which is a simple corollary of Theorem 5 (in Appendix B) guarantees the convergence of Stochastic WMMSE algorithm (Table 4.2).

Theorem 3 *If the channel realizations/samples are bounded almost surely and the noise power $\sigma_{i_k} > 0$, for all k and $i \in \mathcal{I}_k$, then the iterates generated by the Stochastic WMMSE algorithm will converge almost surely to the set of stationary solutions of* (4.8).

Remark 1 *Similar to the deterministic WMMSE algorithm [19] which works for the general α-fairness utility functions, the Stochastic WMMSE algorithm can also be extended to maximize the expected sum of such utility functions; see [19] for more details on the derivations of the respective update rules.*

Distributed Computation

Note that statistical channel state information such as path loss coefficients are easy to share with all the BSs. Moreover, they are less prone to changes over time and the performance of the network is less dependent on their accurate values. Therefore, they do not need to be updated very frequently. As a result, it is reasonable to assume that all the BSs have information about the statistical channel state information. In this case, if each BS is equipped with one pseudo random number generator, they can solve a copy of the problem without any further need to cooperate with other BSs. In this scenario, each BS can use a higher accuracy level for finding its own beamforming vector, while using a low accuracy level for finding the other precoders.

Another scenario in which a distributed online implementation can be performed is when the channels are half duplex and uplink and downlink connections share the same frequency band. In this case, each user i_k at each iteration calculates $\mathbf{U}_{i_k}$ and $\mathbf{W}_{i_k}$ based on the direct channel values that it has estimated as well as the covariance matrix

Table 4.2: SSUM algorithm applied to expected sum rate maximization (Stochastic WMMSE)

Initialize $\mathbf{V}$ randomly such that $\sum_{i\in\mathcal{I}_k} \mathrm{Tr}\left(\mathbf{V}_{i_k}\mathbf{V}_{i_k}^H\right) = P_k,\ \forall\, k$ and set $r=0$.

repeat

$r \leftarrow r+1$

Obtain the new channel estimate/realization $\mathbf{H}^r$

$\mathbf{U}_{i_k} \leftarrow \left(\sum_{j=1}^{K}\sum_{l\in\mathcal{I}_j}(\mathbf{H}_{i_k}^j)^r\mathbf{V}_{l_j}\mathbf{V}_{l_j}^H((\mathbf{H}_{i_k}^j)^r)^H + \sigma_{i_k}^2\mathbf{I}\right)^{-1}(\mathbf{H}_{i_k}^k)^r\mathbf{V}_{i_k},\ \forall\, k,\ i=1,\cdots,L_k$

$\mathbf{W}_{i_k} \leftarrow \left(\mathbf{I}-\mathbf{U}_{i_k}^H(\mathbf{H}_{i_k}^k)^r\mathbf{V}_{i_k}\right)^{-1},\ \forall\, k,\ i\in\mathcal{I}_k$

$\mathbf{Z}_{i_k} \leftarrow \mathbf{V}_{i_k},\ \forall\, k,\ i\in\mathcal{I}_k$

$\mathbf{A}_{i_k} \leftarrow \mathbf{A}_{i_k} + \rho\mathbf{I} + \sum_{j=1}^{K}\sum_{l=1}^{L_j}((\mathbf{H}_{l_j}^k)^r)^H\mathbf{U}_{l_j}\mathbf{W}_{l_j}\mathbf{U}_{l_j}^H(\mathbf{H}_{l_j}^k)^r,\ \forall\, k,\ i\in\mathcal{I}_k$

$\mathbf{B}_{i_k} \leftarrow \mathbf{B}_{i_k} + \rho\mathbf{Z}_{i_k} + ((\mathbf{H}_{i_k}^k)^r)^H\mathbf{U}_{i_k}\mathbf{W}_{i_k},\ \forall\, k,\ i\in\mathcal{I}_k$

$\mathbf{V}_{i_k} \leftarrow (\mathbf{A}_{i_k}+\mu_k^*\mathbf{I})^{-1}\mathbf{B}_{i_k},\ \forall\, k,\ i\in\mathcal{I}_k$, where μ_k^* is the optimal Lagrange multiplier for the constraint $\sum_{i=1}^{L_k}\mathrm{Tr}(\mathbf{V}_k\mathbf{V}_k^H)\leq P_k$ which can be found using bisection.

until some convergence criterion is met.

of the received signal. Then it feeds back $\mathbf{H}_{i_k}^H\mathbf{U}_{i_k}\mathbf{W}_{i_k}$, that is needed for computing matrix $\mathbf{B}_{i_k}$. To feedback this message in the uplink direction it uses a beamformer such that the covariance of its transmitted signal would be $\mathbf{U}_{i_k}\mathbf{W}_{i_k}\mathbf{U}_{i_k}^H$. Then, BS k only needs to estimate the covariance of its received signal to obtain the information that is needed to update matrix $\mathbf{A}_{i_k}$. Note that the progress of the algorithm in this scenario would be over time, while the channels are changing. In this scenario there is no need for statistical channel state information and the algorithm could be viewed as an online method. In other words, there are no assumptions on the statistical distribution of the channels.

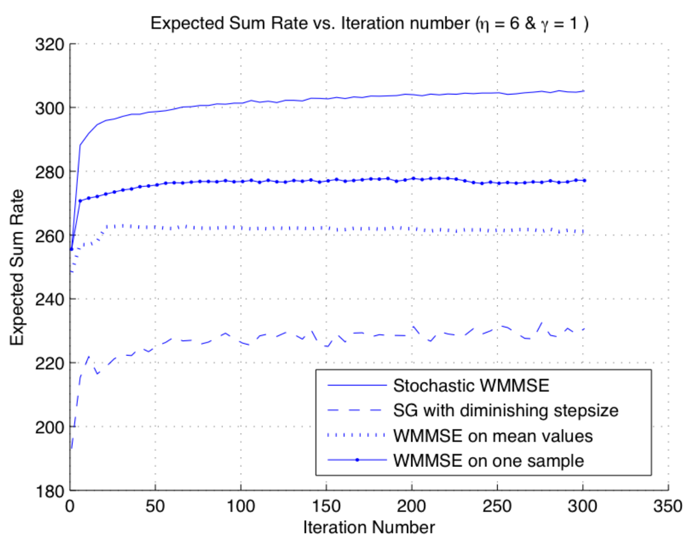
4.4 Numerical Experiments

In this section we numerically evaluate the performance of the SSUM algorithm for maximizing the expected sum-rate in a wireless network. In our simulations, we consider $K = 57$ base stations each equipped with $M = 4$ antennas and serve a two antenna user in its own cell. The path loss and the power budget of the transmitters are generated using the 3GPP (TR 36.814) evaluation methodology [89]. We assume that partial channel state information is available for some of the links. In particular, each user estimates only its direct link, plus the interfering links whose powers are at most η (dB) below its direct channel power. For these estimated links, we assume a channel estimation error model in the form of $\hat{h} = h + z$, where h is the actual channel; $\hat{h}$ is the estimated channel, and z is the estimation error. Given a MMSE channel estimate $\hat{h}$, we can determine the distribution of h as $\mathcal{CN}(\hat{h}, \frac{\sigma_l^2}{1+\gamma \mathrm{SNR}})$ where γ is the effective signal to noise ratio (SNR) coefficient depending on the system parameters (e.g. the number of pilot symbols used for channel estimation) and σ_l is the path loss. Moreover, for the channels which are not estimated, we assume the availability of estimates of the path loss σ_l and use them to construct statistical models (Rayleigh fading is considered on top of the path loss).

In our first set of simulations, we compare the performance of four different algorithms: *one sample WMMSE*, *mean WMMSE*, *stochastic gradient*, and *Stochastic WMMSE*. In "one sample WMMSE", we apply the WMMSE algorithm [19] on one realization of all channels, i.e., we fix $\xi = \hat{\xi}$ and apply the deterministic WMMSE algorithm to it. On the other hand, in the "mean WMMSE", the realization is assumed to be the mean channel matrices, i.e., $\hat{\xi} = \mathbb{E}[\xi]$ is used in the deterministic WMMSE algorithm. In the SG method, we apply the stochastic gradient method with the diminishing step size rule of $\gamma^r = 10^{12}/r$ (which is the best candidate based on our extensive numerical experiments) to the ergodic sum rate maximization problem; see Section B.4.2 for more details. In the Stochastic WMMSE method, it is observed that, when the number of users is large, the approximation function $\hat{g}(\cdot)$ is strongly convex by itself in most of the cases and therefore adding the regularizer is not necessary. However, to be consistent with our theoretical part, we add the proximal regularizer with $\rho = 10^{-12}$. It is further observed that smaller values of ρ, which result in tighter upper-bound, perform better in our numerical experiments. We have also considered a forgetting factor, similar to [90], in the first 100 iterations to reduce the initialization effects on the algorithm. Figure 4.1 shows our simulation results when each user only estimates about 3% of its channels, while the others are generated synthetically according to the channel distributions. The expected sum rate in each iteration is approximated in this figure by a Monte-Carlo averaging over 500 independent channel realizations. As can be seen from Figure 4.1, the Stochastic WMMSE algorithm significantly outperforms the rest of the algorithms. Although the stochastic gradient algorithm with diminishing step size (of order $1/r$) is guaranteed to converge to a stationary solution, its convergence speed is sensitive to the step size selection and is usually slow. We have also experimented the SG method with different constant step sizes in our numerical simulations, but they typically led to divergence.

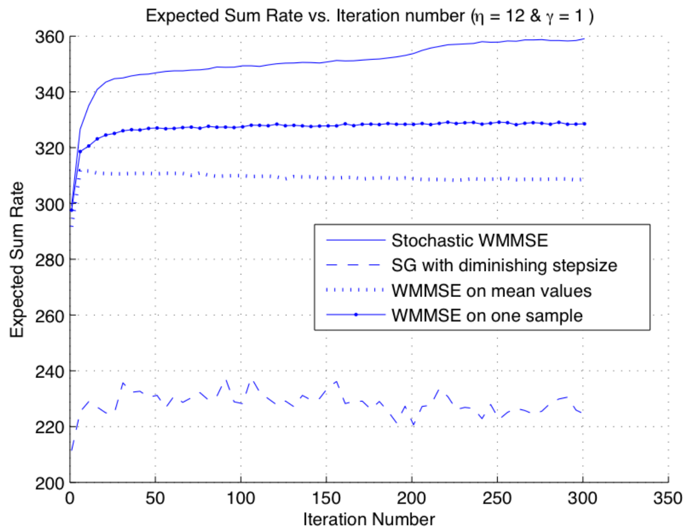
Figure 4.2 illustrates the performance of the algorithms for $\eta = 12$ whereby about 6% of the channels are estimated.

Figure 4.1: Expected sum rate vs. iteration number. We set $\eta = 6$, $\gamma = 1$ and consequently only 3% of the channel matrices are estimated, while the rest are generated by their path loss coefficients plus Rayleigh fading. The signal to noise ratio is set SNR = 15 (dB).



In our second set of numerical experiments, we compare the performance of the stochastic WMMSE algorithm with the sample average approximation (SAA) method.

Figure 4.2: Expected sum rate vs. iteration number. We set $\eta = 12$, $\gamma = 1$ and consequently only 6% of the channel matrices are estimated, while the rest are represented by their path loss coefficients plus Rayleigh fading. The signal to noise ratio is set SNR = 15 (dB).



In order to apply the SAA method, we need to solve (B.2) for the sum rate maximization problem. In other words, one needs to solve the following optimization problem at each step:

$$\begin{aligned} \max_{\mathbf{V}} \quad & \frac{1}{r} \sum_{t=1}^{r} \sum_{k=1}^{K} \sum_{i \in \mathcal{I}_k} \max_{\mathbf{U}_{i_k}} R_{i_k}(\mathbf{U}_{i_k}, \mathbf{V}, \mathbf{H}^t) \\ \text{s.t.} \quad & \sum_{i \in \mathcal{I}_k} \text{Tr}(\mathbf{V}_{i_k} \mathbf{V}_{i_k}^H) \leq P_k, \ \forall \ k = 1, \cdots, K. \end{aligned} \tag{4.16}$$

Clearly, the above problem is NP-hard since it covers the instantaneous sum rate maximization as a special case [23]. Moreover, most of the off-the-shelf solvers which are not customized to this problem perform poorly in practice and the global solvers could only solve problems of small sizes; see [91, 64]. Therefore, we utilize the idea of the WMMSE algorithm, which is one of the most efficient algorithms for the instantaneous sum rate maximization [33, 19, 34, 92, 93, 22], to solve the above SAA subproblem.

More precisely, the optimization problem (4.17) can be rewritten as

$$\begin{aligned} \max_{\mathbf{V}} \ & \frac{1}{r}\sum_{t=1}^{r}\sum_{k=1}^{K}\sum_{i\in\mathcal{I}_k}\max_{\mathbf{U}_{i_k},\mathbf{W}_{i_k}} \left[\log\det(\mathbf{W}_{i_k}) - \mathrm{Tr}\left(\mathbf{W}_{i_k}\mathbf{E}_{i_k}(\mathbf{U}_{i_k},\mathbf{V},\mathbf{H}^t)\right)\right] \\ \text{s.t.} \ & \sum_{i=1}^{L_k}\mathrm{Tr}(\mathbf{V}_{i_k}\mathbf{V}_{i_k}^H) \le P_k, \ \forall\, k=1,\cdots,K. \end{aligned} \tag{4.17}$$

By duplicating the auxiliary variables $\mathbf{U}$ and $\mathbf{W}$, (4.18) can be equivalently written as

$$\begin{aligned} \max_{\mathbf{V},\{\mathbf{U}^t,\mathbf{W}^t\}_{t=1}^r} \ & \frac{1}{r}\sum_{t=1}^{r}\sum_{k=1}^{K}\sum_{i\in\mathcal{I}_k}\log\det(\mathbf{W}_{i_k}^t) - \mathrm{Tr}\left(\mathbf{W}_{i_k}^t\mathbf{E}_{i_k}(\mathbf{U}_{i_k}^t,\mathbf{V},\mathbf{H}^t)\right) \\ \text{s.t.} \ & \sum_{i=1}^{L_k}\mathrm{Tr}(\mathbf{V}_{i_k}\mathbf{V}_{i_k}^H) \le P_k, \ \forall\, k=1,\cdots,K. \end{aligned} \tag{4.18}$$

The above problem can be solved using the block coordinate descent method on the variables $\mathbf{U},\mathbf{V},\mathbf{W}$, and every step is closed form [19]. The resulting algorithm is called *"SAA – WMMSE"* and compared with stochastic WMMSE method. Another approach tried in our simulation results is *"SAA – Gradient Descent"* method, which is the result of solving (4.17) via gradient descent method. Figure 4.3 compares these three methods. Notice that in the SAA – WMMSE and SAA – Gradient Descent methods, we set $r = 300$ and input all the 300 channel realizations to the algorithm at the first iteration and solve (4.17); while in the stochastic WMMSE method, only one channel realization is used at each iteration. Therefore, the SAA method have access to more information in the initial stages. As can be seen from this figure, the SAA algorithm with gradient descent method results in a local optimum point with a poor objective. However, the SAA method solved with the WMMSE algorithm can reach almost the same objective value as the stochastic WMMSE method. On the other hand, each step of the SAA – WMMSE method is much more expensive than the stochastic WMMSE method since the dimension of the problem is much larger. This fact can be seen in Figure 4.4. As can be seen in this figure, even one iteration of the SAA – WMMSE method takes more time than the Stochastic WMMSE method.

Figure 4.3: SSUM vs. SAA for $\eta = 6$, $\gamma = 1$.

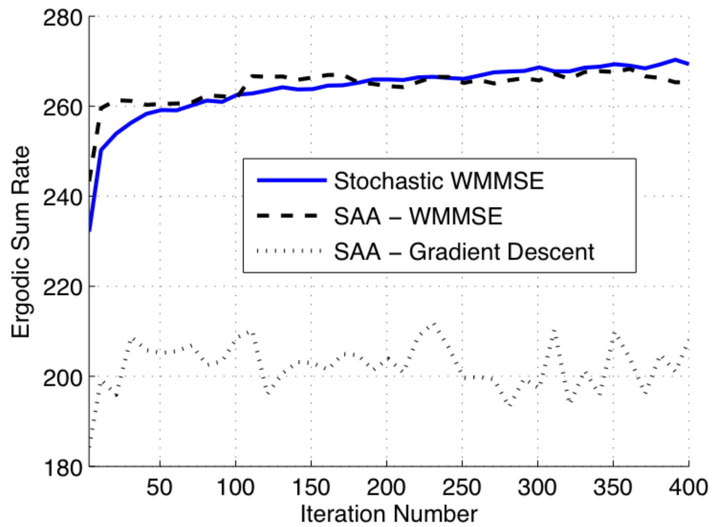
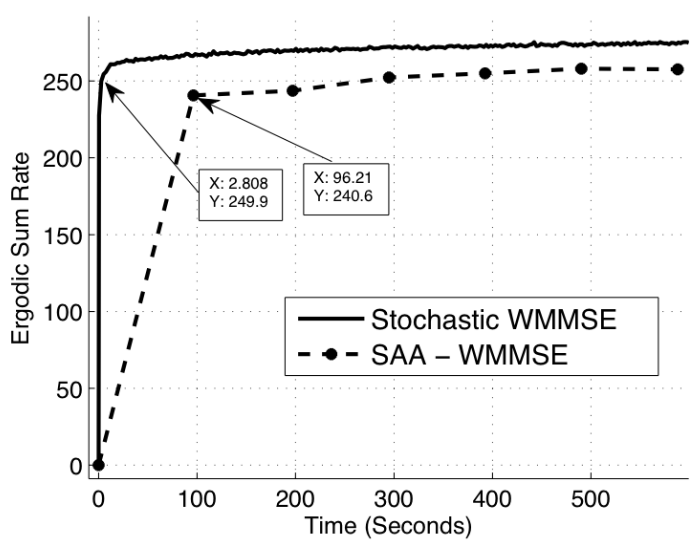


Figure 4.4: SSUM vs. SAA for $\eta = 6$, $\gamma = 1$.



Chapter 5

Joint Base Station Clustering and Beamformer Design for Partial Coordinated Transmission Using Incomplete and Inexact CSIT

It is widely accepted that combining physical layer signal processing techniques such as beamforming (BF) with multi-cell coordination can effectively mitigate the multiuser interference. As we mentioned earlier, there are two main approaches for the coordinated transmission and reception in a multi-cell MIMO network: joint processing (JP) and coordinated beamforming (CB) [94]. In CB the transmitters are only allowed to share information about channel states and/or design variables such as precoders [18, 19, 95], and each user is served by a single BS. In contrast, JP allows for further sharing of the users' data messages among the BSs [96], which makes it possible to have multiple BSs jointly serve a single user. Clearly JP methods achieve higher rates compared to CB methods, but at the same time consume much higher backhaul bandwidth as well [14]. The latter fact makes the full JP scheme, where all the BSs jointly serve each user, practically infeasible. A key question then is how to properly combine the JP and CB to yield an effective *partial* coordinated transmission scheme [73]. The ultimate goal in partial coordinated transmission is to have high throughput (similar to JP), while keeping the amount of communication overhead as low as possible (similar to CB).

One interesting approach to partial coordinated transmission is to group the BSs into coordination clusters of small sizes, within which they perform JP for one user. In this case, each user's data signals are only shared among a small number of its serving BSs, thus greatly reducing the overall backhaul signaling cost. Many recent works have developed various BS clustering strategies and heuristics for such purpose; see [96, 97, 98, 99, 100, 101, 73, 102]. Among all such works, references [73, 102] propose to dynamically construct (possibly overlapping) coordination clusters while at the same time optimizing the downlink transmit beamformers. Such dynamic joint clustering and BF is made possible by recognizing certain group sparsity structures in the collection of the beamformers from all the BSs. However, one major limitation of this approach is that such method requires perfect and full CSIT, just as many other approaches for interference management in HetNets [60, 61, 17, 19, 18, 62, 64].

In this chapter we consider joint BS clustering and beamformer design when with partial/incomplete CSIT. In other words, we assume that only channel distribution information (CDI) is available to the transmitter. Our approach is similar to the one proposed in previous chapter, i.e. we consider maximizing the *averaged* performance of the system. An advantage of our method is its suitability for practical scenarios where having perfect and complete CSIT is infeasible. Moreover, our method does not require any additional computational effort compared to its non-stochastic counterpart [73]. It is also worth noting that the recent work [103] considers a related problem, which is to perform long-term user-BS association using imperfect channel state information. Unlike [103], this work focuses on designing partial coordinated BF strategies. Moreover, our proposed method does not require extensive memory, compared to the method in [103]. In addition, our algorithm is guaranteed to converge to a stationary solution of the stochastic optimization problem.

5.1 Problem Formulation

Let us consider a wireless HetNet in downlink direction with the model we introduced in chapter 2. Due to the heterogeneity of the network the BSs capabilities and access to users' data differ. We assume that there is a backhaul network connecting the set of BSs in cell k to a central controller (usually the macro BS), who has access to the users' data signals and makes the resource allocation decisions for cell k. In other words, the macro BS is the only BS in each cell that has access to the data of the users. Therefore, conveying the messages to users, even if it is done by other BSs in the cell, requires passing information from macro BS to those BSs through the backhaul network. For simplicity of notations, let us assume that all the cells have the same number of transmitters, i.e., $Q_k = Q,\ \forall k$. Moreover, we assume that the number of data streams for each user is $d = 1$. Note that the analysis and algorithms do not depend on these simplifying assumptions and hold true in general.

Let $\mathbf{v}_{i_k}^{q_k} \in \mathbb{C}^{M\times 1}$ be the transmit beamformer that BS q_k uses to transmit the Gaussian data stream $s_{i_k} \sim \mathcal{CN}(0,1)$ to user i_k. Let $\mathbf{x}^{q_k} = \sum_{i_k\in\mathcal{I}_k} \mathbf{v}_{i_k}^{q_k} s_{i_k} \in \mathbb{C}^M$ denote the transmitted vector of BS q_k. Assume that each BS q_k, $q \in \mathcal{Q}_k$, $k \in \mathcal{K}$ has its own power budget P_{q_k} that cannot be exceeded. As the messages for different users are assumed to be independent, the power budget constraint for each BS q_k can be expressed as

$$\mathbb{E}[(\mathbf{x}^{q_k})^H \mathbf{x}^{q_k}] = \sum_{i_k\in\mathcal{I}_k} \|\mathbf{v}_{i_k}^{q_k}\|^2 \le P_{q_k}. \tag{5.1}$$

Collect all the beamforming vectors used for user i_k into $\mathbf{v}_{i_k} = [(\mathbf{v}_{i_k}^{1_k})^H, \cdots, (\mathbf{v}_{i_k}^{Q_k})^H]^H \in \mathbb{C}^{MQ\times 1}$. Similarly, we collect the transmitted signal in cell k as

$$\mathbf{x}^k = [(\mathbf{x}^{1_k})^H, \cdots, (\mathbf{x}^{Q_k})^H]^H = \sum_{i_k\in\mathcal{I}_k} \mathbf{v}_{i_k} s_{i_k} \in \mathbb{C}^{MQ\times 1}, \tag{5.2}$$

where the latter equality is based on the definitions of $\mathbf{v}_{i_k}$ and $\mathbf{x}^{q_k}$.

For a given channel realization, the maximum achievable rate for user i_k when treating the interference as noise is given by

$$\begin{aligned} R_{i_k}(\mathbf{v},\mathbf{H}) = \log\det\Bigg(&\mathbf{I} + \mathbf{H}_{i_k}^k \mathbf{v}_{i_k} \mathbf{v}_{i_k}^H (\mathbf{H}_{i_k}^k)^H \\ \times\Big(&\sum_{(l,j)\neq(k,i)} \mathbf{H}_{i_k}^l \mathbf{v}_{j_l} \mathbf{v}_{j_l}^H (\mathbf{H}_{i_k}^l)^H \ +\ \sigma_{i_k}^2 \mathbf{I}\Big)^{-1}\Bigg), \end{aligned} \tag{5.3}$$

where we use $\mathbf{v}$ and $\mathbf{H}$ to denote the set of all transmit beamformers and channels respectively.

As we discussed in the previous chapter, obtaining instantaneous channel values is a difficult task. Thus, optimizing the instantaneous achievable rates (5.3) that depends on the actual channel values is impractical. As an alternative, we prefer to model channels as random variables and optimize the following *averaged* system performance

$$\mathbb{E}_{\mathbf{H}}\left\{\sum_{k=1}^{K}\sum_{i_k\in\mathcal{I}_k}\mathcal{U}_{i_k}(R_{i_k}(\mathbf{v},\mathbf{H}))\right\}, \tag{5.4}$$

where the expectation is taken over the statistical model of the channel, and $\mathcal{U}_{i_k}(R_{i_k}(\mathbf{v},\mathbf{H}))$ denotes user i_k's utility function. It is worth emphasizing that such measure is not dependent on the instantaneous realization of the channel, but rather its long term distribution. So any scheme that optimizes such measure should be relatively independent of the instantaneous channel realizations as well.

Notice that the rate given in (5.3) is achievable if all the BSs $\mathcal{Q}_k$ share the data messages intended for the users $\mathcal{I}_k$ and perform JP. To reduce the resulting overhead in the backhaul network, we adopt the partial coordinated transmission scheme, where each user $i_k\in\mathcal{I}_k$ is served by only a subset of the BSs in $\mathcal{Q}_k$. Clearly the clustering structure needs to be decided at the same time with the BF strategies to yield the best performance.

A popular strategy for doing so is to induce certain sparsity pattern into the beamformer $\mathbf{v}$ [73, 102]. The main idea is that when a given BS q_k does not serve a user i_k, then $\mathbf{v}_{i_k}^{q_k}$ should be set to zero. Therefore, requiring only a few BSs in cell k serving user i_k is equivalent to having only a few non-zero blocks in $\mathbf{v}_{i_k}$. That is, $\mathbf{v}_{i_k}$ should be group sparse [104]. In order to optimize the system level performance while keeping the beamforming vector structure group sparse, we penalize the system performance objective with a group sparse encouraging penalty such as ℓ_2/ℓ_1 norm. Therefore, the problem can be formulated as

$$\begin{aligned}\max_{\mathbf{v}}\ &\mathbb{E}_{\mathbf{H}}\left\{\sum_{k=1}^{K}\sum_{i_k\in\mathcal{I}_k}\mathcal{U}_{i_k}(R_{i_k})\right\}-\sum_{k=1}^{K}\sum_{i_k\in\mathcal{I}_k}\lambda_k\sum_{q_k\in\mathcal{Q}_k}\|\mathbf{v}_{i_k}^{q_k}\|,\\ \text{s.t.}\ &\sum_{i_k\in\mathcal{I}_k}\|\mathbf{v}_{i_k}^{q_k}\|^2\le P_{q_k},\ q_k\in\mathcal{Q}_k,\ \forall k,\end{aligned} \tag{P}$$

where $\lambda_k\ge 0$ is the parameter that controls the level of sparsity within each cell.

Clearly the resulting beamforming and BS clustering structure only depends on the channel distribution information, which changes slowly over time.

This problem is non-convex, non-smooth and also incorporates an expected value. In the rest of this chapter, we focus on the problem (P) with sum rate utility. In the next section, we will devise an efficient algorithm that solves this problem to a stationary solution.

5.2 Sparse Stochastic Weighted Minimum Mean Squared Error Algorithm

In this section, we focus on problem (P), when $\mathcal{U}_{i_k}(R_{i_k}) = R_{i_k}$ (Sum-Rate objective). The algorithm we present here generalizes the well-known Weighted Minimum Mean Squared Error (WMMSE) algorithm [19] to sparse and stochastic setup. It is an interesting inexact variation of the proposed Stochastic Successive Upper-bound Minimization (SSUM) method; see Appendix B.

We assume that user i_k uses the linear receive beamformer $\mathbf{u}_{i_k}$ to decode its signal, i.e., it estimates s_{i_k} as $\hat{s}_{i_k} = \mathbf{u}_{i_k}^H \mathbf{y}_{i_k}$. The mean squared error (MSE) of such estimation would be $e_{i_k} = \mathbb{E}_{\mathbf{n}_{i_k}}\{(s_{i_k} - \hat{s}_{i_k})(\overline{s_{i_k} - \hat{s}_{i_k}})\}$, where "—" denotes the complex conjugate. Assuming Gaussian noise, e_{i_k} can be expressed as

$$\begin{aligned} e_{i_k}(\mathbf{u}_{i_k}, \mathbf{v}, \mathbf{H}) &= (1 - \mathbf{u}_{i_k}^H \mathbf{H}_{i_k}^k \mathbf{v}_{i_k})(\overline{1 - \mathbf{u}_{i_k}^H \mathbf{H}_{i_k}^k \mathbf{v}_{i_k}}) \\ &+ \sum_{(l,j)\neq(k,i)} \mathbf{u}_{i_k}^H \mathbf{H}_{i_k}^l \mathbf{v}_{j_l} \mathbf{v}_{j_l}^H (\mathbf{H}_{i_k}^l)^H \mathbf{u}_{i_k} + \sigma_{i_k}^2 \mathbf{u}_{i_k}^H \mathbf{u}_{i_k}. \end{aligned} \tag{5.5}$$

The following Lemma [105] establishes the well-known connection between MSE and the achievable rate.

Lemma 1 *Given* (5.5)*, we have*

$$\begin{aligned} R_{i_k}(\mathbf{v}, \mathbf{H}) - 1 = \max_{\mathbf{u}_{i_k}, w_{i_k}, \mathbf{z}_{i_k}} \; & \log(w_{i_k}) - w_{i_k} e_{i_k}(\mathbf{u}_{i_k}, \mathbf{v}, \mathbf{H}) \\ & - \beta \|\mathbf{z}_{i_k} - \mathbf{v}_{i_k}\|^2, \end{aligned} \tag{5.6}$$

for any $\beta > 0$*. Moreover, the optimal solution* $(\mathbf{u}_{i_k}^*, w_{i_k}^*, \mathbf{z}_{i_k}^*)$ *of* (5.6) *is given by*

$$\begin{aligned}\mathbf{u}_{i_k}^* &= \left(\sum_{(l,j)} \mathbf{H}_{i_k}^l \mathbf{v}_{j_l} \mathbf{v}_{j_l}^H (\mathbf{H}_{i_k}^l)^H + \sigma_{i_k}^2 \mathbf{I}\right)^{-1} \mathbf{H}_{i_k}^k \mathbf{v}_{i_k} \\ &= \mathbf{J}_{i_k}^{-1} \mathbf{H}_{i_k}^k \mathbf{v}_{i_k}, \qquad (5.7)\\ w_{i_k}^* &= e_{i_k}^{-1}(\mathbf{u}_{i_k}^*, \mathbf{v}, \mathbf{H}) = (1 - \mathbf{u}_{i_k}^H \mathbf{H}_{i_k}^k \mathbf{v}_{i_k})^{-1}, \qquad (5.8)\\ \mathbf{z}_{i_k}^* &= \mathbf{v}_{i_k}, \qquad (5.9)\end{aligned}$$

where $\mathbf{J}_{i_k}$ is the covariance matrix of user i_k's received signal $\mathbf{y}_{i_k}$ and $\mathbf{u}_{i_k}^*$ is called the Minimum MSE receiver.

Note that receiver i_k only needs to estimate the covariance matrix $\mathbf{C}_{i_k}$ and the direct channel $\mathbf{H}_{i_k}$, in order to find $\mathbf{u}_{i_k}^*$. Let us define the new auxiliary variable $\mathbf{P} = (\mathbf{u}, \mathbf{w}, \mathbf{z})$ where $\mathbf{u} = \{\mathbf{u}_{i_k} \mid i_k \in \mathcal{I}\}$, $\mathbf{w} = \{w_{i_k} \mid i_k \in \mathcal{I}\}$, and $\mathbf{z} = \{\mathbf{z}_{i_k} \mid i_k \in \mathcal{I}\}$ are defined as the collection of the corresponding variables across all users. Using Lemma 1, we can rewrite problem (P), when $\mathcal{U}_{i_k}(R_{i_k}) = R_{i_k}$, in the following equivalent form

$$\begin{aligned}\min_{\mathbf{v}} \ & \mathbb{E}_{\mathbf{H}} \left\{ \min_{\mathbf{P}} \sum_{k=1}^{K} \sum_{i_k \in \mathcal{I}_k} w_{i_k} e_{i_k} - \log(w_{i_k}) + \beta \|\mathbf{z}_{i_k} - \mathbf{v}_{i_k}\|^2 \right\} \\ & + \sum_{k=1}^{K} \sum_{i_k \in \mathcal{I}_k} \lambda_{i_k} \sum_{q_k \in \mathcal{Q}_k} \|\mathbf{v}_{i_k}^{q_k}\|, \qquad \text{(Q)} \\ \text{s.t.} \ & \sum_{i_k \in \mathcal{I}_k} \|\mathbf{v}_{i_k}^{q_k}\|^2 \le P_{q_k}, \ q_k \in \mathcal{Q}_k, \ \forall k.\end{aligned}$$

From this reformulation, it is clear that variables $(\mathbf{u}, \mathbf{w}, \mathbf{z})$ should be updated based on the channel realizations, while the precoder $\mathbf{v}$ could be adjusted based on the expected value (long term objective). Motivated by the sample average approximation (SAA) method, we propose to update the variables $(\mathbf{u}, \mathbf{w}, \mathbf{z})$ based on (5.7), (5.8) and (5.9) respectively, and update the variables $\mathbf{v}$ using the ensemble average of the objective function of (Q). More specifically, at iteration r after generating/obtaining a channel realization $\mathbf{H}^r = \{(\mathbf{H}_{i_k}^l)^r | i_k \in \mathcal{I}_k, k, l = 1, \cdots, K\}$, we find the auxiliary variables

$(\mathbf{u}^r, \mathbf{w}^r, \mathbf{z}^r)$ by

$$(\mathbf{u}^r, \mathbf{w}^r, \mathbf{z}^r) = \arg\min_{(\mathbf{u},\mathbf{w},\mathbf{z})} \sum_{k=1}^{K} \sum_{i_k \in \mathcal{I}_k} w_{i_k} e_{i_k}(u_{i_k}, \mathbf{v}^{r-1}, \mathbf{H}^r) \\ - \log(w_{i_k}) + \beta \|\mathbf{z}_{i_k} - \mathbf{v}_{i_k}^{r-1}\|^2. \tag{5.10}$$

Moreover, the precoders $\mathbf{v}$ is updated using

$$\mathbf{v}^r = \arg\min_{\mathbf{v}} \ \frac{1}{r} \sum_{t=1}^{r} \sum_{k=1}^{K} \sum_{i_k \in \mathcal{I}_k} w_{i_k}^t e_{i_k}(\mathbf{u}_{i_k}^t, \mathbf{v}, \mathbf{H}^t) + \beta \|\mathbf{z}_{i_k}^t - \mathbf{v}_{i_k}\|^2 \\ + \sum_{k=1}^{K} \sum_{i_k \in \mathcal{I}_k} \lambda_{i_k} \sum_{q_k \in \mathcal{Q}_k} \|\mathbf{v}_{i_k}^{q_k}\|, \tag{5.11} \\ \text{s.t.} \ \sum_{i_k \in \mathcal{I}_k} \|\mathbf{v}_{i_k}^{q_k}\|^2 \le P_{q_k}, \ q_k \in \mathcal{Q}_k, \ \forall k$$

Note that fixing variables $\mathbf{u}$ and channel realization $\mathbf{H}$, e_{i_k} is a convex quadratic function of variables $\mathbf{v}$. In other words, the smooth part of the objective function of (5.11) can be summarized as

$$\sum_{k=1}^{K} \sum_{i_k \in \mathcal{I}_k} \mathbf{v}_{i_k}^H \mathbf{A}_{i_k}^r \mathbf{v}_{i_k} + (\mathbf{b}_{i_k}^r)^H \mathbf{v}_{i_k} + \mathbf{v}_{i_k}^H \mathbf{b}_{i_k}^r, \tag{5.12}$$

where

$$\mathbf{A}_{i_k}^r = \frac{1}{r} \sum_{t=1}^{r} \big(\beta \mathbf{I} + \sum_{j_l \in \mathcal{I}} w_{j_l} (\mathbf{H}_{j_l}^k)^H \mathbf{u}_{j_l}^t (\mathbf{u}_{j_l}^t)^H \mathbf{H}_{j_l}^k\big) \succ \mathbf{0}, \tag{5.13}$$

and

$$\mathbf{b}_{i_k}^r = \frac{1}{r} \sum_{t=1}^{r} \big(\beta \mathbf{z}_{i_k}^t + w_{i_k} (\mathbf{H}_{i_k}^k)^H \mathbf{u}_{i_k}\big). \tag{5.14}$$

As a result in each cell k the beamformer vectors should be updated by solving the following problem

$$\min_{\{\mathbf{v}_{i_k},\ i_k \in \mathcal{I}_k\}} \ \sum_{i_k \in \mathcal{I}_k} \mathbf{v}_{i_k} \mathbf{A}_{i_k}^r \mathbf{v}_{i_k} + (\mathbf{b}_{i_k}^r)^H \mathbf{v}_{i_k} + \mathbf{v}_{i_k}^H \mathbf{b}_{i_k}^r \\ + \sum_{i_k \in \mathcal{I}_k} \lambda_{i_k} \sum_{q_k \in \mathcal{Q}_k} \|\mathbf{v}_{i_k}^{q_k}\|, \\ \text{s.t.} \ \sum_{i_k \in \mathcal{I}_k} \|\mathbf{v}_{i_k}^{q_k}\|^2 \le P_{q_k}, \ \forall \ q_k \in \mathcal{Q}_k. \tag{5.15}$$

Therefore, the update of $\mathbf{V}$ requires solving convex quadratic problems with a mixed ℓ_2/ℓ_1-penalty term over block constraint sets $\sum_{i_k \in \mathcal{I}_k} \|\mathbf{v}_{i_k}^{q_k}\|^2 \leq P_{q_k},\ \forall\ q_k \in \mathcal{Q}_k,\ \forall\ k$. Unfortunately, such problem does not have closed-form solution. One needs to resort to standard techniques such as the Block Coordinate Descent (BCD) method to solve it to the desired accuracy, say ϵ_r. We refer the interested readers to [73] for a detailed account on how to use BCD method to solve (5.15). In addition, note that BCD is proved [106] to be an efficient method to solve convex quadratic problems with mixed ℓ_2/ℓ_1 penalty term (R-linearly convergent). It is important to bear in mind that the authors in [73] need to solve the same sub-problem as in (5.11), even though their algorithm uses perfect CSIT. Therefore, our method is computationally the same as that of [73].

The overall algorithm is summarized in Table 5.1.

Remark 2 *In general, the SAA type update rules in the stochastic programming require the storage of all the realizations of the random parameters in the stochastic optimization (e.g. algorithm proposed in [103]). However, in our problem, all the required information is summarized in the matrices $\mathbf{A}_{i_k}$ and $\mathbf{b}_{i_k}$, which can be updated recursively (see the updates in Algorithm 5.1). Therefore, Algorithm 5.1 does not require extensive amount of memory for storing all the channel realizations.*

Remark 3 *Note that problem* (5.15) *decomposes over the cells. Therefore, update of* $\mathbf{V}$ *could be done for different cells in parallel.*

The nature of Sparse Stochastic WMMSE (SSWMMSE) algorithm (table 5.1) is a double loop algorithm where an iterative method is used to update the transmit beamformer at each outer iteration. Therefore, one needs to know the desired accuracy level to terminate the inner loop at each outer iteration. In other words, SSWMMSE algorithm relies on an iterative method, i.e. BCD, to solve (5.15) as well as a subroutine to identify if a generated solution satisfies the accuracy criterion required in each outer loop. In the following subsection, we will provide a simple method to check the accuracy of any given solution of (5.15).

Table 5.1: Stochastic Sparse WMMSE with Inexact Updates

Initialize $\mathbf{v}$ randomly such that $\sum_{i_k\in\mathcal{I}_k}\|\mathbf{v}_{i_k}^{q_k}\|^2 \le P_{q_k},\ q_k = 1,\cdots,Q_k,\ \forall k$. Set $r = 0$.

repeat

$r \leftarrow r + 1$
Obtain the new channel estimate/realization $\mathbf{H}^r$
Update $\mathbf{u}_{i_k}$ using (5.7) with channels $\mathbf{H}^r$, $\forall\ i_k \in \mathcal{I}$.
Update w_{i_k} using (5.8) with channels $\mathbf{H}^r$, $\forall\ i_k \in \mathcal{I}$.
$\mathbf{z}_{i_k} \leftarrow \mathbf{v}_{i_k}$, $\forall\ i_k \in \mathcal{I}$.
$\mathbf{A}_{i_k} \leftarrow \frac{1}{r}\sum_{j_l\in\mathcal{I}} w_{j_l}((\mathbf{H}_{j_l}^k)^r)^H\mathbf{u}_{j_l}(\mathbf{u}j_l)^H(\mathbf{H}_{j_l}^k)^r + \frac{r-1}{r}\mathbf{A}_{i_k} + \frac{1}{r}\beta\mathbf{I}$, $\forall\ i_k \in \mathcal{I}$
$\mathbf{b}_{i_k} \leftarrow \frac{r-1}{r}\mathbf{b}_{i_k} + \frac{1}{r}(\beta\mathbf{z}_{i_k} + w_{i_k}((\mathbf{H}_{i_k}^k)^r)^H\mathbf{u}_{i_k})$, $\forall\ i_k \in \mathcal{I}$.
Update $\mathbf{v}$ solving (5.11), using BCD method, within ϵ_r accuracy; the accuracy is measured using Definition 1.

until some convergence criterion is met.

5.2.1 Estimating Accuracy of Solutions to (5.15)

As problem (5.15) is solved for any cell k separately, for the sake of simplicity we can drop all the sub-indices k without loss of generality. Moreover, for the simplicity of presentation, let us assume real-valued beamforming vectors and channels. It is easy to see that the results can be generalized for the complex case. With these simplifying assumptions, problem (5.15) can be rewritten as

$$\begin{aligned} \min_{\{\mathbf{v}_i,\ i\in\mathcal{I}\}}\quad & \sum_{i\in\mathcal{I}}\mathbf{v}_i\mathbf{A}_i^r\mathbf{v}_i + 2(\mathbf{b}_i^r)^T\mathbf{v}_i \\ & + \sum_{i\in\mathcal{I}}\lambda_i\sum_{q\in\mathcal{Q}}\|\mathbf{v}_i^q\|, \\ \text{s.t.}\quad & \sum_{i\in\mathcal{I}}\|\mathbf{v}_i^q\|^2 \le P_q,\ \forall\ q\in\mathcal{Q}. \end{aligned} \tag{5.16}$$

We assume that we are given a feasible solution $\mathbf{V}$ to problem 5.16. Our goal is to estimate the inexactness of the this solution in the sense that is presented in Definition

1. In other words, we would like to find an $\epsilon \geq 0$ such that for any feasible direction $\mathbf{D}$ at this point $\mathbf{V}$, the directional derivative of the objective in (5.16) is greater than $-\epsilon\|\mathbf{D}\|$.

Our first step towards such a goal is to identify the feasible directions. Let us define the set $\mathcal{T}$ to be the set of all q such that the constraint $\sum_{i\in\mathcal{I}} \|\mathbf{v}_i^q\|^2 \leq P_q$ is tight. We also denote the complement of $\mathcal{T}$ as $\mathcal{T}^c$. Moreover, Let us define the set of all (i,q) such that $\mathbf{v}_i^q = \mathbf{0}$ as $\mathcal{S}$ and its complement as $\mathcal{S}^c$. Now for every $q \in \mathcal{T}$, and any direction $\mathbf{D} = \{\mathbf{d}_i^q\}_{(i,q)}$ the necessary condition for the solutions of the form $\mathbf{V} + \alpha\mathbf{D}$ to remain feasible for small enough $\alpha > 0$, is that the derivative of the left hand side with respect to α remain non-positive. In contrary, any constraints in (5.16) that corresponds to a $q \in \mathcal{T}^c$ does not restrict the choice of direction $\mathbf{d}$. Therefore, we can write the constraints on the feasible directions as follows

$$\sum_{i\in\mathcal{I}} (\mathbf{v}_i^q)^T \mathbf{d}_i^q \leq 0, \quad \forall\, q \in \mathcal{T}. \tag{5.17}$$

Note that for the terms with $\mathbf{v}_i^q = \mathbf{0}$, i.e. $(i,q) \in \mathcal{S}$, the direction $\mathbf{d}_i^q$ does not contribute the constraint above. Moreover, each vector $\mathbf{d}_i^q$ can appear only in one constraint.

Now that we have categorized the possible directions, we need to find the directional derivative of the objective in (5.16) with respect to any direction $\mathbf{D}$. To do so, let us define the derivative of the smooth part of the objective in (5.16) (the quadratic part) with respect to $\mathbf{v}_i^q$ as $\mathbf{g}_i^q$. It is clear that the directional derivative of the smooth part with respect to any direction $\mathbf{D}$ corresponds to the inner product of the direction $\mathbf{D}$ with the gradient $\mathbf{G} = \{\mathbf{g}_i^q\}_{(i,q)}$. In addition, the directional derivative of the ℓ_2/ℓ_1 part can be easily characterized as

$$\lim_{\alpha \downarrow 0} \frac{\|\mathbf{v}_i^q + \alpha\mathbf{d}_i^q\| - \|\mathbf{v}_i^q\|}{\alpha} = \begin{cases} (\mathbf{v}_i^q)^T \mathbf{d}_i^q / \|\mathbf{v}_i^q\| & (i,q) \in \mathcal{S}^c \\ \|\mathbf{d}_i^q\| & (i,q) \in \mathcal{S} \end{cases} \tag{5.18}$$

Now our goal would be to minimize the directional derivative of the (5.16) with respect to the feasible direction $\mathbf{D}$ at the feasible solution $\mathbf{V}$. Note that the directional derivative scales linearly with $\|\mathbf{D}\|$. Therefore, we need to impose a constraint on $\mathbf{D}$. Recalling the definition of inaccuracy in (2.3), we are interested to find ϵ by solving

$$\begin{aligned} -\epsilon \ = \ \min_{\mathbf{D}} \quad & \sum_i \sum_q (\mathbf{g}_i^q)^T \mathbf{d}_i^q + \sum_i \lambda_i \left(\sum_{(i,q)\in\mathcal{S}} \|\mathbf{d}_i^q\| + \sum_{(i,q)\in\mathcal{S}^c} (\mathbf{v}_i^q)^T \mathbf{d}_i^q / \|\mathbf{v}_i^q\| \right) \\ \text{s.t.} \quad & \sum_{i\in\mathcal{I}} (\mathbf{v}_i^q)^T \mathbf{d}_i^q \leq 0, \quad \forall\, q \in \mathcal{T}, \qquad (5.19) \\ & \|\mathbf{D}\|^2 \leq 1. \end{aligned}$$

Unfortunately, solving (5.19) might not be easy due to the coupling constraint on the norm of $\mathbf{D}$. One strategy to mitigate this problem is to relax this constraint and find an upper bound for the accuracy measure ϵ. To do so, let use define $\tilde{\mathbf{g}} = \{\tilde{\mathbf{g}}_i^q\}_{(i,q)}$ as

$$\tilde{\mathbf{g}}_i^q = \begin{cases} \mathbf{g}_i^q & (i,q) \in \mathcal{S} \\ \mathbf{g}_i^q + \frac{\mathbf{v}_i^q}{\|\mathbf{v}_i^q\|} & (i,q) \in \mathcal{S}^c \end{cases}$$

Moreover, for any q we define $\hat{\mathbf{d}}^q = \{\mathbf{d}_i^q\}_{(i,q)\in\mathcal{S}^c}$, $\hat{\mathbf{g}}^q = \{\tilde{\mathbf{g}}_i^q\}_{(i,q)\in\mathcal{S}^c}$ and $\hat{\mathbf{v}}^q = \{\mathbf{v}_i^q\}_{(i,q)\in\mathcal{S}^c}$. Now we are ready to formulate a relaxation of problem (5.19):

$$\begin{aligned} -\bar{\epsilon} = \min_{\mathbf{D}} \quad & \sum_q (\hat{\mathbf{g}}^q)^T \hat{\mathbf{d}}^q \ + \sum_{(i,q)\in\mathcal{S}} \left(\lambda_i \|\mathbf{d}_i^q\| + (\tilde{\mathbf{g}}_i^q)^T \mathbf{d}_i^q \right) \\ \text{s.t.} \quad & (\hat{\mathbf{v}}^q)^T \hat{\mathbf{d}}^q \leq 0, \quad \forall\, q \in \mathcal{T}, \\ & \|\hat{\mathbf{d}}^q\|^2 \leq 1, \ \forall\, q \in \mathcal{Q}, \qquad (5.20) \\ & \|\mathbf{d}_i^q\|^2 \leq 1, \ \forall (i,q) \in \mathcal{S}. \end{aligned}$$

With such relaxation method, the problem decomposes over different variables. The following Lemma characterizes the optimal solution $\mathbf{D}^*$ of (5.20).

Lemma 2 *For any $q \in \mathcal{T}^c$ the optimal $(\hat{\mathbf{d}}^q)^*$ can be characterized as*

$$(\hat{\mathbf{d}}^q)^* = -\hat{\mathbf{g}}^q / \|\hat{\mathbf{g}}^q\|, \ \forall q \in \mathcal{T}^c. \qquad (5.21)$$

Moreover, for any $q \in \mathcal{T}$ the optimal $(\hat{\mathbf{d}}^q)^*$ can be computed easily using the following rules.

- *If $(\hat{\mathbf{g}}^q)^T \hat{\mathbf{v}}^q \geq 0$, then*

$$(\hat{\mathbf{d}}^q)^* = -\frac{\hat{\mathbf{g}}^q}{\|\hat{\mathbf{g}}^q\|}. \qquad (5.22)$$

- *If* $(\hat{\mathbf{g}}^q)^T\hat{\mathbf{v}}^q < 0$*, then*

$$(\hat{\mathbf{d}}^q)^* = -\frac{\hat{\mathbf{g}}^q + \nu^q\hat{\mathbf{v}}^q}{\|\hat{\mathbf{g}}^q + \nu^q\hat{\mathbf{v}}^q\|}, \tag{5.23}$$

  where $\nu^q = -\frac{(\hat{\mathbf{g}}^q)^T\hat{\mathbf{v}}^q}{\|\hat{\mathbf{v}}^q\|^2}$.

In addition, for any $(i,q) \in \mathcal{S}$*, the optimal solution* $(\mathbf{d}_i^q)^*$ *can be easily found as follows.*

- *If* $\|\tilde{\mathbf{g}}_i^q\| \leq \lambda_i$*, then*

$$(\mathbf{d}_i^q)^* = \mathbf{0}. \tag{5.24}$$

- *If* $\|\tilde{\mathbf{g}}_i^q\| > \lambda_i$*, then*

$$(\mathbf{d}_i^q)^* = -\frac{\tilde{\mathbf{g}}_i^q}{\|\tilde{\mathbf{g}}_i^q\|}. \tag{5.25}$$

Proof Proof of this Lemma is relegated to Appendix D.

Using the optimal solution $\mathbf{D}^*$ that is obtained above, it is easy to compute $\bar{\epsilon}$, which is an upper bound on ϵ. Note that the directional derivative value is a linear function of $\|\mathbf{D}^*\|$. Therefore, if we plug in $\mathbf{D}^*/\|\mathbf{D}^*\|$, which is feasible solution of (5.19), in the objective we get $-\bar{\epsilon}/\|\mathbf{D}^*\|$ that would give us a lower bound on the desired accuracy ϵ. In other words

$$\bar{\epsilon}/\|\mathbf{D}^*\| \leq \epsilon \leq \bar{\epsilon} \tag{5.26}$$

Therefore, we can use $\bar{\epsilon}$ to estimate the accuracy of $\mathbf{D}^*$ as a solution to the sub-problem (5.16).

Remark 4 *As we mentioned in Remark 3, Problem* (5.15) *decomposes over the cells. Therefore, we can measure the accuracy of a solution to the decomposed problem in each cell without considering the rest of the cells using* (5.26)*.*

5.3 Convergence Analysis and Remarks

The following theorem states the convergence of Sparse Stochastic WMMSE algorithm and provides a guideline for choosing the accuracy level of the inner loop at each iteration.

Theorem 4 *Assume that the channel realizations* $\{\mathbf{H}^r\}$ *are drawn independently from a bounded distribution. Also assume that the channel noise variances are positive at all users. If the error satisfies* $\epsilon_r = \mathcal{O}(\frac{1}{r})$, *then the iterates generated by Sparse Stochastic WMMSE Algorithm (Table 5.1) converge to the set of stationary points of* (P) *with sum rate objective almost surely, i.e.,*

$$\lim_{r\to\infty} \; d(\mathbf{V}^r, \mathbb{V}) = 0 \;\text{ almost surely},$$

where $\mathbb{V}$ *is the set of stationary points of Problem* (P) *with sum rate utility .*

The Sparse Stochastic WMMSE algorithm is a special case of a more general SSUM framework, introduced in Appendix B. In Theorem 5 of Appendix B, we state a general convergence analysis. Theorem 4 could be easily proved as a corollary of Theorem 5; see Corollary 1 in Appendix B.

Remark 5 *Obtaining a channel realization* $\mathbf{H}^r$ *in each iteration of Algorithm 5.1 can be done by either generating virtual channel realizations based on the known channel distribution, or simply using the inaccurate measured CSI if it is available.*

Remark 6 *In practice, the channel statistics may vary over time. Assuming that the channel realizations are coming from a time varying distribution, one can modify the update rule of* $\mathbf{A}_{i_k}$ *and* $\mathbf{b}_{i_k}$ *by using a forgetting factor* δ, $0 < \delta < 1$*:*

$$\mathbf{A}_{i_k} \leftarrow \delta \mathbf{A}_{i_k} + \beta \mathbf{I} + \sum_{j_l \in \mathcal{I}} w_{j_l} ((\mathbf{H}_{j_l}^k)^r)^H \mathbf{u}_{j_l} (\mathbf{u} j_l)^H (\mathbf{H}_{j_l}^k)^r,$$
$$\mathbf{b}_{i_k} \leftarrow \delta \mathbf{b}_{i_k} + \beta \mathbf{z}_{i_k} + w_{i_k} ((\mathbf{H}_{i_k}^k)^r)^H \mathbf{u}_{i_k}.$$

Using such forgetting factor means that there is no need to restart algorithm once the distribution changes. In this way, the algorithm can be viewed as implementing an online joint clustering and BF strategy. In this scenario, the algorithm might run indefinitely, and the generated clusters and beamforming vectors can be used in real time.

Remark 7 *Similar to the original WMMSE algorithm [19], Algorithm 5.1 and its convergence can be easily generalized to other utility functions satisfying the conditions of [19, Theorem 2].*

5.4 Numerical Experiments

In this section we present some numerical experiments to evaluate the performance of Sparse Stochastic WMMSE Algorithm, Table 5.1. Our goal is to show that this algorithm can effectively reduce the size of BS clusters formed to serve each user, while maintaining the high average throughput. In our simulations, we consider $K = 7$ cells each containing $Q = 10$ BSs and $I = 20$ users. Each BS is equipped with $M = 4$ antennas and the receivers have $N = 2$ antennas. Moreover, we assume $\lambda_{i_k} = \lambda = 0.2$, for all $i_k \in \mathcal{I}$. The path loss of the channels are generated using the 3GPP (TR 36.814) evaluation methodology [89]. We assume that partial channel state information is available for some of the links. In particular, each user estimates only its strongest direct channel (to a BS) in addition to the links whose powers are at most η (dB) below this strongest channel power. For these estimated channels, we assume a channel estimation error model in the form of $\hat{h} = h + z$, where h is the actual channel; $\hat{h}$ is the estimated channel, and z is the estimation error. Given an MMSE channel estimate $\hat{h}$, we can determine the distribution of h as $\mathcal{CN}(\hat{h}, \frac{\sigma_l^2}{1+\gamma \text{SNR}})$ where γ is the effective signal to noise ratio (SNR) coefficient depending on the system parameters (e.g. the number of pilot symbols used for channel estimation) and σ_l is the path loss. Moreover, for the channels which are not estimated, we assume the availability of estimates of the path loss σ_l and use them to construct statistical models (Rayleigh fading is considered on top of the path loss). For the sake of simplicity we assume that all the BSs have the same power budget.

The methods that are compared throughout this section are as follows.

- **Partial JP:** In this method we apply Algorithm 5.1 to find joint clustering and beamforming in the stochastic setup. If not mentioned the error in computation of each iterate is of the order $\frac{I}{r}$, where I is the number of users.

- **Complete JP:** In this method we apply the stochastic WMMSE algorithm [80] to find the beamforming without any clustering. Therefore, the solution that is returned by this method might overload the backhaul network.

- **De-biased Partial JP:** In this method after applying SSWMMSE algorithm and obtaining the clustering and beamforming solution, we fix the clustering and apply the Stochastic WMMSE algorithm [80] to de-bias the beamforming solution and obtain better expected rates using the same sparse clusters.

- **One Sample Sparse WMMSE:** In this heuristic method we apply the Sparse WMMSE algorithm [73] on one channel sample.

- **Average Channel Sparse WMMSE**: In this heuristic method we apply the Sparse WMMSE algorithm [73] on average channel values.

- **Nearest Neighbor (NN) Stochastic WMMSE:** In this heuristic method each user is first assigned to the BS with strongest channel. Then, Stochastic WMMSE [80] is applied to find the beamforming.

In our simulations, we approximate the expected sum rate by a Monte-Carlo averaging over 100 independent channel realizations.

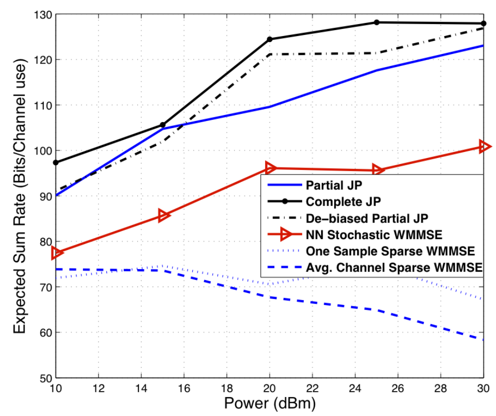
Figures 5.1(a) and 5.1(b) compare the aforementioned methods in terms of achievable average throughput and average size of clusters respectively.

As it can be seen from Figure 5.1, the performance of partial JP method is close to the complete JP strategy in terms of throughput, but with much smaller cluster sizes (and hence the message exchange overhead). Moreover, the solution provided by Partial JP can be further refined to give higher average throughput specially in higher power regimes. In such refinement step (also called de-biasing), we fix the clusters created by Algorithm Partial JP, and use the stochastic WMMSE algorithm to further adjust the beamformer vectors that are chosen to be non-zero (active). Such de-biasing step is beneficial due to the large amount of bias caused by ℓ_2/ℓ_1 penalty term in high power regime. As it can be seen from Figure 5.1(a) Partial JP combined with de-biasing step (de-biased partial JP) gives an average throughput that is almost identical to the complete JP.

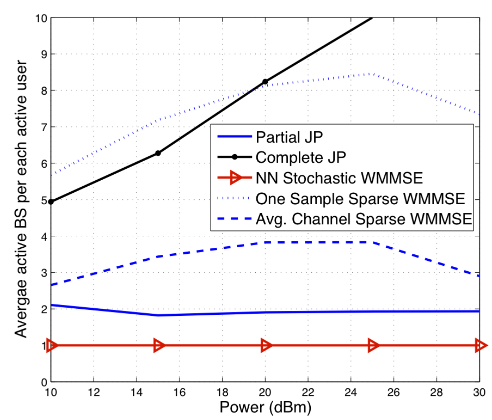
In order to compare the convergence speed of different algorithms, we plot the average throughput of the system versus the iteration number for each algorithm in Figure 5.2 (a) when the power budget is fixed at 30 dBm. Note that De-biased Partial JP is not included in this figure as its results depend on the clustering generated by Partial JP method.

Moreover, we compare those methods in terms of power consumption in Figure 5.2(b). In Figure 5.2(b) we plot the relative power consumption of each method at fixed power budget in proportion to the power consumption of the complete JP method at the same power budget. Note that using ℓ_2/ℓ_1 would result in solutions with lower power consumption. It can be seen that our Partial JP method consumes significantly less power compared to Complete JP.

Figure 5.1: $\lambda = 0.2$, $\gamma = 1$, $\eta = 8$; only 10% of the channels are estimated.



(a)



(b)

Finally, we perform numerical experiments to show the effectiveness of our inexactness measure in reducing computation time. Unfortunately, it is practically impossible to perform the exact version, i.e. $\epsilon_r = 0$, of Sparse Stochastic WMMSE method, table

5.1. Therefore, instead of the exact version, we compare our method versus an algorithm that performs a fixed number (in this case 20 rounds) of BCD updates in each iteration. We denote such algorithm in our simulation plots and tables by Exact Method. In contrast to the Exact Method, we allow our inexact algorithm (Partial JP Method) to terminate this process when the error is below the required threshold. In order to estimate the error we use the technique described in Section 5.2.1 and specifically the inequalities in (5.26). Note that based on Remarks 3 we can solve the problem (5.15) in each cell separately. Therefore, as it is mentioned in Remark 4 we estimate the accuracy of the solution to the problem in each cell individually. The accuracy level at iteration r, ϵ_r, for updating $\mathbf{v}$ at cell k is set to be $\frac{10I_k}{r}$, where I_k is the number of users in cell k. Note that in our simulations $I_k = I = 20$. In order to show that measuring the inexactness would be beneficial compared to simple heuristics, we apply an algorithm that performs a few (in our case 5) rounds of BCD method in each iteration. In our simulation plots and tables we denote this algorithm by Heuristic Inexact Method. In addition, as an alternative to $\frac{1}{r}$ decreasing error in finding the iterates in Partial JP, we implement another algorithm that calculates the errors in the iterates and stops when the error is below a fixed threshold, $\epsilon_r = \frac{I_k}{\sqrt{50}}$. We denote this algorithm in the simulations by Fixed Error Method. We conduct our simulations for different power budgets in the network. Table 5.2 shows the value of the objective in (P), calculated by a Monte-Carlo, after 50 iterations of each of the above three methods. Moreover, it depicts the average number of BCD cycles executed per cell per iteration by each of the three algorithms. As it can be seen from the table the partial JP method can provide the solutions with objective almost identical to the exact method with much less computational load.

In addition to Tables 5.2 and 5.3, in Figures 5.3(a)-(c) we depict the progress of the objective of (P) vs. the iteration number for each of the three methods in all the aforementioned power budget regimes. As it is clear from the Figures 5.3(a)-(c) alongside with Tables 5.2-5.3, our Partial JP provides a good balance between the computation time and convergence speed. Moreover, it can be seen in the Figures 5.3(a)-(c) that having an exact solution to the sub-problem does not necessarily lead to a higher objective value in the first few iterations where only a few samples have been used.

Table 5.2: Objective of (P) for each method when there are $K = 7$ cells

Power (dBm)	10 dBm	20 dBm	30 dBm
Exact Method	72.40	91.61	94.54
Partial JP	72.94	88.68	95.18
Heuristic Inexact Method	67.87	83.81	86.18
Fixed Error Method	63.46	82.09	84.13

Table 5.3: Average BCD cycles per cell per iteration for each method when there are $K = 7$ cells

Power (dBm)	10 dBm	20 dBm	30 dBm
Exact Method	20	20	20
Partial JP	4.37	4.93	5.08
Heuristic Inexact Method	5	5	5
Fixed Error Method	3.13	3.25	3.51

We further compare the exact and inexact methods over networks with different sizes to show the effectiveness of our constants. For this purpose, we use networks with $K = 4$ and $K = 10$ cells.

Next we pick a smaller size network with $K = 3$ cells, $Q = 3$ BSs in each cell, and $I = 5$ users in each cell . This time our goal is to check the stationarity condition of the original stochastic optimization. We use Monte Carlo channel samples of size 100 to approximate the gradient of the expected rate with respect to the transmit beamformers. Having the approximate gradient of the non-convex stochastic part of the objective, it is easy to calculate the first order stationarity condition of the solution using the technique introduced in section 5.2.1. Note that a point is stationary if it satisfies the condition (2.1). Moreover, a point could be viewed as an approximate stationary point if it satisfies the condition in Definition 1. In other words, we approximately calculate the first order stationarity gap for the original problem using the relation in (5.26). In Table 5.5 we

Table 5.4: Comparing the exact and inexact methods for different network sizes

Power (dBm)	10 dBm	20 dBm	30 dBm
Exact Method objective $K = 10$	92.90	116.45	116.61
Partial JP objective $K = 10$	99.46	115.48	117.05
Exact Method BCD iterations $K = 10$	20	20	20
Partial JP BCD iterations $K = 10$	5.51	5.96	5.69
Exact Method objective $K = 3$	33.09	51.18	54.41
Partial JP objective $K = 3$	33.77	52.07	57.89
Exact Method BCD iterations $K = 3$	20	20	20
Partial JP BCD iterations $K = 3$	4.06	5.17	4.30

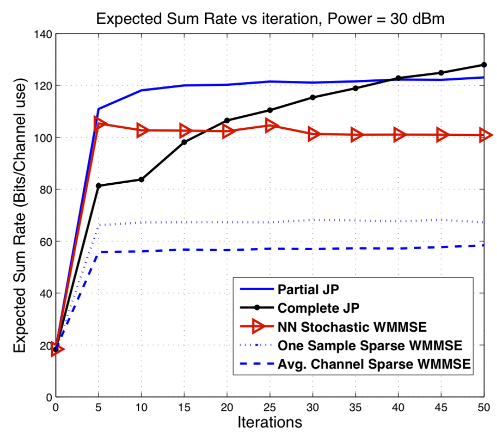
report the stationarity gap normalized by the total number of users. The row indicated by the "Initial Solution" shows the stationarity gap for the random initialization.

As it can be seen from Table 5.5 the Partial JP method is capable of generating solutions with stationarity gap that is close to the exact method, while the other two heuristics tend to generate solutions that are farther from being a stationary solution of the original problem.

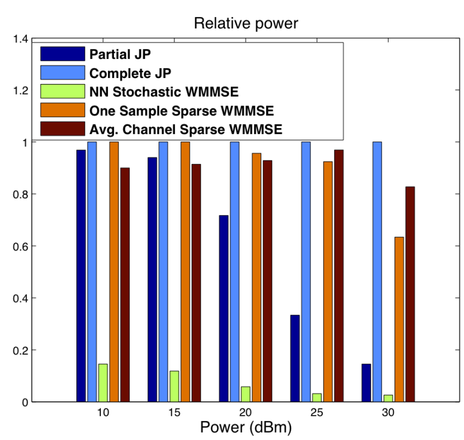
Table 5.5: Stationarity gap of problem (P) for each method after 50 iterations

Power (dBm)	10 dBm	20 dBm	30 dBm
Initial Solution	1.38	1.43	1.37
Exact Method	0.15	0.09	0.14
Partial JP	0.15	0.13	0.15
Heuristic Inexact Method	0.21	0.29	0.18
Fixed Error Method	0.37	0.42	0.38

Figure 5.2: $\lambda = 0.2$, $\gamma = 1$, $\eta = 8$; only 10% of the channels are estimated. Power budget is 30 dBm.

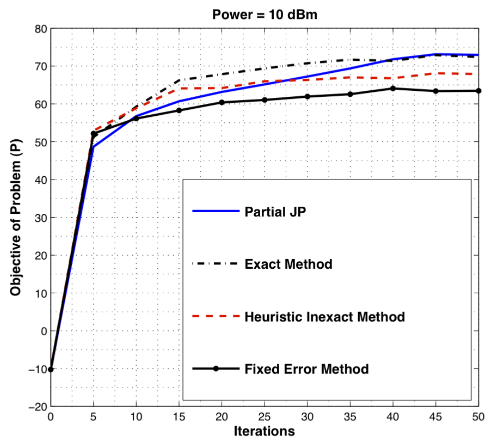


(a)

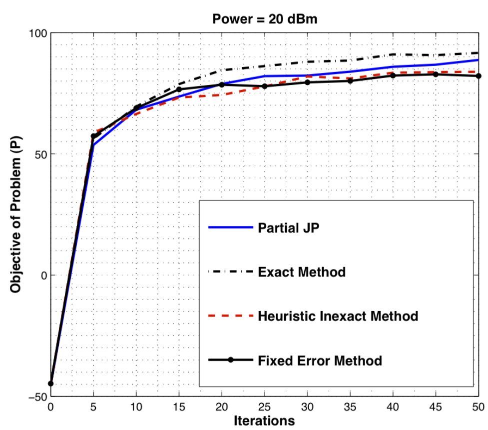


(b)

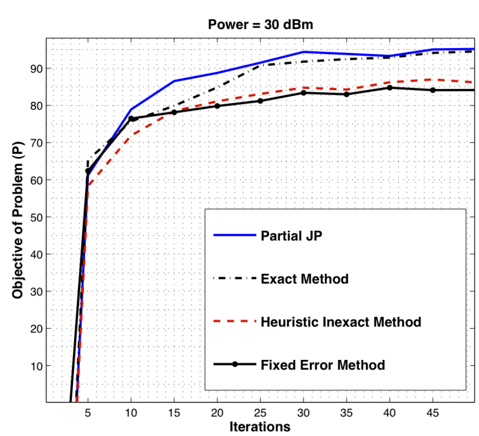
Figure 5.3: $K = 7$, $\lambda = 0.2$, $\gamma = 1$, $\eta = 8$; only 10% of the channels are estimated. Objective vs. Iteration



(a)



(b)



(c)

Chapter 6

Conclusion and Discussion

In this dissertation we considered the problem of resource management in wireless HetNets. We discussed this problem in multiple scenarios and under practical limitations and assumptions.

In chapter 3 we considered the downlink joint base station assignment and beamforming problem for a wireless heterogeneous network, with the goal of maximizing a system wide utility function. Our complexity analysis showed that this optimization problem is NP-hard for a wide range of utility functions. This suggests that designing an efficient algorithm to globally optimize the joint base station assignment and beamforming is an unrealistic goal. As a practical alternative, we proposed an algorithm that can not only compute a suboptimal (stationary) point of the joint optimization problem efficiently, but also achieve a better network performance than the existing approaches in the literature.

In the formulation of joint BS assignment and beamforming problem, we assumed the knowledge of channel state information at the transmitters. As we discussed in details, such assumption can be impractical in some scenarios. In chapter 4 we proposed a framework to relax this assumption by considering a stochastic optimization method. We proposed an algorithm that could efficiently optimize the *average/ergodic* throughput of the system with almost surely convergence to a stationary solution.

Our stochastic beamforming strategy in chapter 4 is focused on the physical layer precoder design. In practice it is always favorable to merge these physical layer strategies with higher level protocols such as BS clustering. Thus, in chapter 5 we combined the idea of *average/ergodic* throughput maximization of chapter 4 with dynamic BS clustering in partial coordinated transmission. We proposed an efficient method to solve the joint clustering and beamforming problem to a stationary solution almost surely. It is worth noting that our proposed stochastic algorithms in chapters 4 and 5 (i.e. Stochastic WMMSE and Sparse Stochastic WMMSE) do not require more computational effort or memory compared to their non-stochastic counterparts.

In the sequel we discuss a few possible directions to expand the work in this dissertation.

- Notice that our formulation considers only the downlink transmission (which is typically the main bottleneck). In practice, it may be desirable to have the same user association in uplink and downlink directions. How to jointly design user association and precoders in this scenario is an interesting direction for future research.
- As a future direction it is interesting to combine the stochastic optimization techniques with strategies such as user grouping [22] that are proven to have substantial gains.
- The algorithms that are proposed in this dissertation are limited to smooth system-wide utilities. It is interesting to extend the scope of these methods to non-smooth system-wide utilities such as min-rate utility. Unfortunately, there are only a few prior works that deal with such utilities [21].
- Combining the stochastic beamforming strategy with traffic engineering in the network as well as user scheduling is another possible extension of our work. Note that there are only a few works that consider such joint optimization [107] in deterministic setup.
- In our work, we do not consider the cases where the users join or leave the network. As a future research direction it is interesting to devise algorithms that can consider such scenario and react appropriately.
- In practice the channel samples that are obtained using estimation methods might not be independent. In this scenario, it would be more reasonable to model them

using a Markov chain. Performing a convergence analysis of SSUM algorithm when using dependent samples is another future extension of our work.

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Appendix A

Proof of Theorem 1

A.1 NP-hardness of Sum-Rate Utility Maximization

Channel Construction

Consider an instance of the MAX 2-SAT problem with V variables and C clauses. To relate the MAX 2-SAT problem and problem (3.5), we consider a wireless system with $I = 2V + C$ BSs, each with a single antenna. The set of BSs is partitioned into two subsets. The first subset consists of $2V$ BSs named as *variable BSs*. Each of these BSs corresponds to a variable or its negation in the MAX 2-SAT problem. More specifically, for any variable x in the MAX 2-SAT problem, there are two *variable* base stations, denoted by $\mathcal{B}_x$ and $\mathcal{B}_{\bar{x}}$, corresponding to x and $\bar{x}$ respectively.

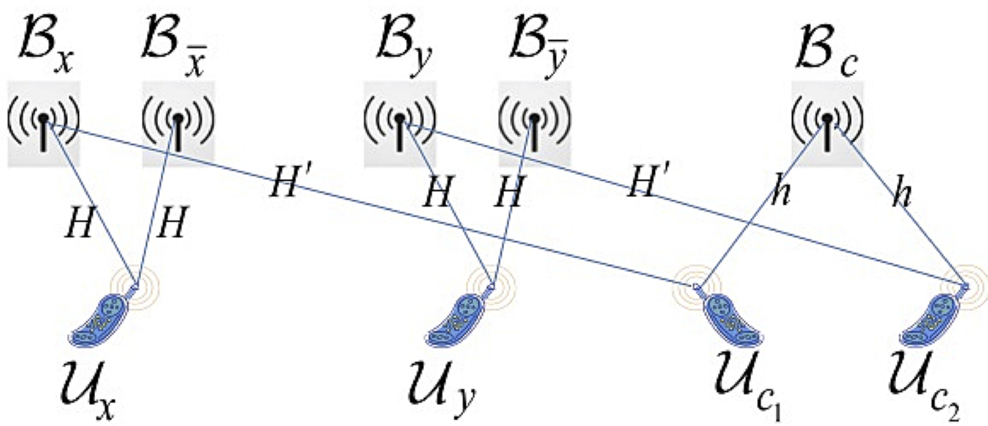
The rest of the BSs are called *clause BSs* since each of them will correspond to a clause in the MAX 2-SAT problem. In particular, for any clause c, we denote its corresponding BS by $\mathcal{B}_c$. Let us also assume that all the BSs have equal power budget P, that is $P_n^{\max} = P$, for all n.

Let there be $K = V + 2C$ users, each with a single antenna. The first V users will be called the *variable users* and the other $2C$ users will be called the *clause users*. Each variable user corresponds to a variable in the MAX 2-SAT problem. Let $\mathcal{U}_x$ be the user corresponding to the variable x. In addition, for any clause c in the MAX 2-SAT problem, there are two corresponding clause users denoted by $\mathcal{U}_{c_1}$ and $\mathcal{U}_{c_2}$. We assume that the noise power at all the users is equal to σ^2, i.e., $\sigma_i^2 = \sigma^2$, $\forall i$.

In our construction, the base stations $\mathcal{B}_x$ and $\mathcal{B}_{\bar{x}}$ are the only base stations that are connected to $\mathcal{U}_x$ with nonzero channel gain H. Furthermore, for any clause c, the channel gains between $\mathcal{B}_c$ and the users $\mathcal{U}_{c_1}$ and $\mathcal{U}_{c_2}$ are both equal to $h > 0$; while the channels between $\mathcal{B}_c$ and the rest of the users are assumed to be zero.

To construct the channels between the *variable* base stations and the clause users, let us consider a sample clause $c : x + y$. Since x is the first variable appeared in the clause[1] c, the BS $\mathcal{B}_x$ is connected to user $\mathcal{U}_{c_1}$ by a nonzero channel gain H'. Similarly, the channel gain between $\mathcal{B}_y$ and $\mathcal{U}_{c_2}$ is H'. The channel demonstration is depicted in Fig. A.1.

Figure A.1: Channels corresponding to the clause $c : x + y$



Now, it suffices to show that by setting $P = 1$, $h = 1/C^4$, $H = 3C^4$, $H' = C^2$, and $\sigma = 1$, the following claim is true.

Claim 2 *The MAX 2-SAT problem (with $C \geq 3$) has an optimal value of m iff the BS assignment and beamforming problem* (3.5) *has an optimal objective value no less than $V \log(1 + HP) + m \log(1 + hP)$.*

In order to prove Claim 2 we need the following lemmas.

[1] Note that our channel construction depends on the order of appearance of the variables in each clause, which can be fixed in advance.

Lemma 3 *In any single input, single output broadcast channel the optimal beamforming (power allocation) strategy for maximizing the Sum-Rate is to transmit with maximum power to the user with the highest channel gain* [2] *.*

Proof of Lemma 3 We prove the lemma for a two user broadcast channel. Extending the result to more than two users is straightforward. Let us denote the channel gains between the BS and the two users as h_1 and h_2. Moreover the noise power at the receivers is assumed to be 1. Then the Sum-Rate maximization would be

$$\max_{0\leq p\leq P} \ \log\left(1+\frac{h_1 p}{1+h_1(P-p)}\right)+\log\left(1+\frac{h_2(P-p)}{1+h_2 p}\right), \tag{A.1}$$

where P is the total power budget of the BS. The second order derivative of the objective with respect to p is

$$\frac{h_1^2}{(1+h_1(P-p))^2}+\frac{h_2^2}{(1+h_2 p)^2}>0. \tag{A.2}$$

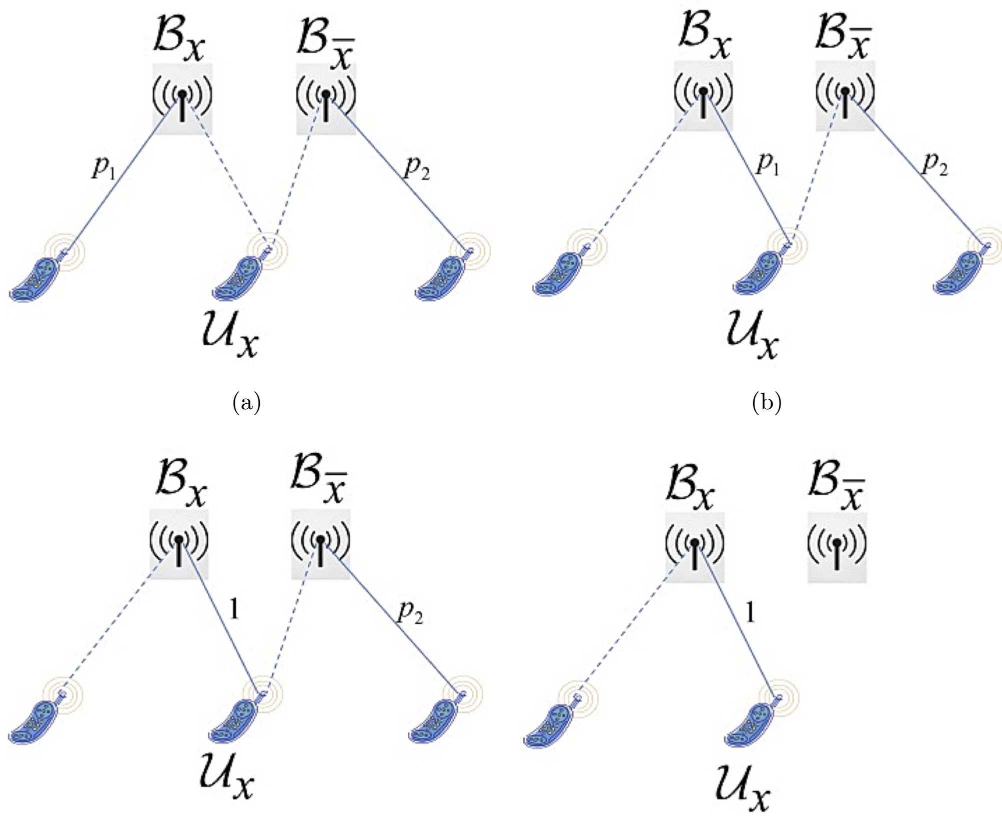
Hence the objective is strictly convex and the solution lies on the boundary ($p=0$ or $p=P$).

Lemma 4 *Let $C\geq 3$. At the optimal solution of problem* (3.5) *for any binary variable x, exactly one of the following cases can happen*
Case 1: Neither $\mathcal{B}_x$ nor $\mathcal{B}_{\bar{x}}$ transmits any signal.
Case 2: Exactly one of the two base stations $\mathcal{B}_x$, $\mathcal{B}_{\bar{x}}$ serves $\mathcal{U}_x$ with full power, while the other one does not transmit any signal.

Proof of Lemma 4 We prove this lemma in three steps. First we prove that if case 1 does not happen, then either $\mathcal{B}_x$ or $\mathcal{B}_{\bar{x}}$ serves $\mathcal{U}_x$. We use a contradiction argument. Assume that at an optimal solution there exists a variable x such that $\mathcal{B}_x$ and $\mathcal{B}_{\bar{x}}$ are both serving users other than $\mathcal{U}_x$, with powers p_1 and p_2 respectively. Without loss of generality let us assume that $p_1\geq p_2$ and $p_1>0$. The situation is depicted in the Fig. A.2(a).

Now consider another feasible solution in which $\mathcal{B}_x$ transmits with power p_1 to user $\mathcal{U}_x$ and $\mathcal{B}_{\bar{x}}$ transmits no signal, while the rest of the power allocations and BS

[2] *We assume that the noise power is normalized to 1 at all the receivers and the interference is treated as noise.*



(a) (b) (c) (d)

Figure A.2: The solid lines represent the direct links and the dashed lines represent the interference links. The transmitted powers are specified for the direct links.

assignments are kept fixed. We prove this new feasible solution results in a higher objective value than the optimal solution which is a contradiction. To show this, let us calculate the difference between the original objective value and the objective value achieved by the new feasible solution. In the new feasible solution, the user $\mathcal{U}_x$ does not receive any interference, because it is connected to $\mathcal{B}_x$ and $\mathcal{B}_{\bar{x}}$ only (by the construction) and $\mathcal{B}_{\bar{x}}$ transmits no signal. Thus, the new feasible solution achieves a rate of $\log(1 + Hp_1)$ by $\mathcal{B}_x$ and $\mathcal{B}_{\bar{x}}$. In contrast, the original solution achieves a sum rate of at most $\log(1 + H'p_1) + \log(1 + H'p_2)$ (ignoring the interference) by $\mathcal{B}_x$ and $\mathcal{B}_{\bar{x}}$. We subtract the original sum rate from the new sum rate to get the difference

$$\log(1+Hp_1) - \log(1+H'p_1) - \log(1+H'p_2) = \log\left(\frac{1+Hp_1}{(1+H'p_1)(1+H'p_2)}\right). \quad \text{(A.3)}$$

Notice that

$$\begin{aligned} 1+Hp_1-(1+H'p_1)(1+H'p_2) &\geq 1+Hp_1-(1+H'p_1)^2 \\ &= (H-2H')p_1-(H')^2p_1^2. \end{aligned} \quad \text{(A.4)}$$

Note that (A.4) is a concave function of $0 < p_1 \leq 1$, and hence to check its sign over $(0,1]$ we just need to check the end points of the interval. By plugging in $p_1 = 0$ we obtain 0 in (A.4). Furthermore, if we plug in $p_1 = 1$ we get $H - 2H' - (H')^2 = 3C^4 - 2C^2 - C^4 > 0$ (for any $C \geq 2$). Therefore, we can conclude that (A.4) is always greater than zero for $p_1 \in (0,1]$. As a result, (A.3) is always positive which is a contradiction.

So far we have proved that if case 1 in Lemma 4 does not happen, then either $\mathcal{B}_x$ or $\mathcal{B}_{\bar{x}}$ (in our case $\mathcal{B}_x$) should serve $\mathcal{U}_x$. In the next step we prove that the BS serving $\mathcal{U}_x$ (in our case $\mathcal{B}_x$) serves user $\mathcal{U}_x$ with full power.

Assume the contrary that $\mathcal{B}_x$ is serving $\mathcal{U}_x$ using power p_1, and $\mathcal{B}_{\bar{x}}$ is serving another user with power p_2 as it is depicted in Fig. A.2(b). We can separate the terms in the sum rate objective in which p_1 appears:

$$r(p_1) = \log\left(1+\frac{Hp_1}{1+Hp_2}\right) + \sum_{i\in S_x}\log\left(1+\frac{X_i}{1+I_i+H'p_1}\right), \quad \text{(A.5)}$$

where the set S_x, is the set of *clause* users that are connected to $\mathcal{B}_x$, X_i is the received power of the message intended for user $i \in S_x$, and I_i is the interference caused by other base stations to user $i \in S_x$. Clearly, $0 \leq X_i \leq h$ and $I_i \geq 0$ for every $i \in S_x$. Taking the derivative of r with respect to p_1, we have

$$\begin{aligned} \frac{\partial r}{\partial p_1} =& \frac{H}{1+Hp_2+Hp_1} + \sum_{i\in S_x}\frac{-H'X_i}{(1+I_i+H'p_1)(1+I_i+X_i+H'p_1)} && \text{(A.6)} \\ \geq& \frac{H}{1+2H} - \sum_{i\in S_x}\frac{H'h}{(1+H'p_1)(1+H'p_1)} && \text{(A.7)} \\ \geq& \frac{H}{1+2H} - CH'h = \frac{3C^4}{1+6C^4} - \frac{1}{C} > 0, && \text{(A.8)} \end{aligned}$$

where the last inequality holds for $C \geq 3$. Hence r is an increasing function of p_1 and therefore at the optimality, p_1 should be equal to 1.

Now we just need to prove that if case 1 in Lemma 4 does not hold, then among the *variable* base stations $\mathcal{B}_x$ and $\mathcal{B}_{\bar{x}}$ the one which is not serving $\mathcal{U}_x$ (in our case $\mathcal{B}_{\bar{x}}$) should not transmit. Assume the contrary that $\mathcal{U}_x$ is served by $\mathcal{B}_x$. As proved before, the power of $\mathcal{B}_x$ should be 1. In addition, let us assume that $\mathcal{B}_{\bar{x}}$ transmits a message with power $p_2 > 0$ to some other user (see Fig. A.2(c)). Now we prove that if $\mathcal{B}_{\bar{x}}$ transmits no power, the objective value will increase. The difference between the objective values that can be achieved in these two scenarios is lower bounded by

$$\begin{aligned}&\log(1+H) - \left[\log(1+\frac{H}{1+Hp_2}) + (\log(1+H'p_2))\right] = \\ &\log\left(\frac{(1+H)(1+Hp_2)}{(1+H+Hp_2)(1+H'p_2)}\right). \end{aligned} \tag{A.9}$$

In order to prove that (A.9) is greater than zero and get a contradiction, we only need to prove that $(1+H)(1+Hp_2) > (1+H+Hp_2)(1+H'p_2)$ for any value of $0 < p_2 \leq 1$. In other words, we need to show that

$$HH'p_2^2 + (H' + HH' - H^2)p_2 < 0, \tag{A.10}$$

for any $0 < p_2 \leq 1$. But (A.10) is a strictly convex function of p_2 and hence we only need to check the extreme points of the interval $(0, 1]$ in order to make sure that the inequality holds true. At $p_2 = 0$ it is zero, and at $p_2 = 1$ it is:

$$2HH' + H' - H^2 = 6C^6 + C^2 - 9C^8 < 0, \text{ for } C \geq 1. \tag{A.11}$$

Hence, the inequality (A.10) holds true $C \geq 1$. This completes the proof of lemma 4.

Lemma 4 implies that at the optimality if a variable user is served, then the allocated power and the channel association is of the form depicted in Fig. A.2(d).

Lemma 5 *For $C \geq 2$, if the optimal value of problem* (3.5) *is greater than or equal to $V\log(1+H)$, then all the variable users are served at the optimal solution.*

Proof of Lemma 5 According to Lemma 4, none of the clause users is served by a variable BS at the optimality. Hence, the best rate that a clause user can achieve is

$\log(1+h)$. Therefore, the highest achievable sum rate over all clause users is $C\log(1+h) = C\log(1+\frac{1}{C^4}) \leq \frac{1}{C^3}$. On the other hand, due to Lemma 4, at the optimality when one variable user is served, then it should be served with full power. Consequently, the rate of a variable user is $\log(1+H) = \log(1+3C^4) \geq 1$ which is clearly greater than $\frac{1}{C^3}$. Consequently, in order to achieve the sum rate of $V\log(1+H)$, all the variable users have to be served.

Proof of Claim 2 If there exists an assignment of binary variables such that m clauses are satisfied in the MAX 2-SAT problem, then we can achieve the objective value of $V\log(1+H) + m\log(1+h)$ in problem (3.5) by choosing the following solution:

1. For any variable x if $x = 1$, then $\mathcal{B}_{\bar{x}}$ serves $\mathcal{U}_x$ with power $P = 1$ and $\mathcal{B}_x$ does not transmit. Otherwise, if $x = 0$, $\mathcal{B}_x$ transmits with full power $P = 1$ to user $\mathcal{U}_x$ and $\mathcal{B}_{\bar{x}}$ does not transmit any signal.

2. For each clause c, there should exist at least one of its corresponding users, say $\mathcal{U}_{c_1}$, which does not receive any interference from *variable* base stations. Hence, $\mathcal{B}_c$ can transmit with full power to serve the user with no interference.

Clearly, the above scheme will result in total sum rate of $V\log(1+H) + m\log(1+h)$. Hence, the optimal value is greater than or equal to $V\log(1+HP) + m\log(1+hP)$.

Conversely, assume that there exists an optimal solution of problem (3.5) with objective value greater than or equal to $V\log(1+HP) + m\log(1+hP)$. Due to Lemma 3 we can assume that at the optimality each BS is serving only one user. This is due to the fact that for any BS, if we fix all the other base stations' power allocation and assignments, then this BS can choose its best user and only serve that user. Now we can construct a binary truth assignment for the MAX 2-SAT problem based on the optimal solution of problem (3.5) as follows:

- Set the variable $x = 1$, if $\mathcal{B}_{\bar{x}}$ is transmitting;

- Set $x = 0$ otherwise.

From Lemmas 4 and 5, at most one of the BSs $\mathcal{B}_x$ and $\mathcal{B}_{\bar{x}}$ is transmitting at the optimality. Thus, the above truth assignment is legitimate. According to Lemma 5, the total achievable rate by variable users will be $V\log(1+H)$. Now we prove that in order to achieve the extra term $m\log(1+h)$, there should exist at least m *clause* users that do not receive interference from the variable base stations.

Due to Lemma 3, the clause BSs only serve one of their users at optimality. Hence, if one of their users does not receive any interference, then they can easily serve that user and achieve the rate of $\log(1+h)$. On the other hand, if both of their users have interference, then they can achieve at most

$$\log\left(1+\frac{h}{1+H'}\right)=\log\left(1+\frac{1/C^4}{1+C^2}\right).$$

So the total achievable rate for those *clause* base stations whose users all receive interference is at most $C\log(1+\frac{1/C^4}{1+C^2})$. It can be easily checked that

$$C\log(1+\frac{1/C^4}{1+C^2})\leq\frac{1}{C^3+C^5}\leq\frac{1}{C^4}-\frac{1}{C^8}\qquad(\text{for } C\geq 2)\tag{A.12}$$

and

$$\frac{1}{C^4}-\frac{1}{C^8}\leq\log(1+\frac{1}{C^4})=\log(1+h).\tag{A.13}$$

Consequently, in order to get the extra term $m\log(1+h)$ at the optimality, there should be at least m *clause* base stations each serving at least one interference-free clause user. This is exactly equivalent to having at least m clauses satisfied by the aforementioned binary truth assignment in the MAX 2-SAT problem. This completes the proof of claim 2.

A.1.1 Proof of Claim 1

Proof First assume that graph G is 3-colorable. Now we will show that there exists a beamforming and BS assignment scheme which can achieve an objective value equal to $U(\log(2),\cdots,\log(2))$. To this end, we choose the transmission strategy based on the solution of the graph 3-colorability problem:

- If node i in graph G is colored by color ℓ, then $\mathcal{B}_{i_\ell}$ will transmit to user $\mathcal{U}_i$ with full power $P=1$; and the rest of the transmitters $\mathcal{B}_{i_j}$, $j\neq l$ will not transmit any signal.

As the coloring is a proper 3-coloring of the graph, no two adjacent nodes are colored the same, implying that there will be no interference for any user. Hence, each user can obtain the rate of $\log(2)$. This yields the system-wide utility of $U(\log(2), \cdots, \log(2))$.

To prove the converse direction, let us assume that there exists a transmission strategy which results in an objective value greater than or equal to $U(\log(2), \cdots, \log(2))$. On the one hand, note that the maximum rate that each user can achieve is $\log(2)$, since each user can be supported by at most one BS. On the other hand, since $U(\cdot)$ is a monotonically increasing function of its arguments, each user should achieve at least the rate of $\log(2)$. This means that for each i, $i = 1, \cdots, N$, user $\mathcal{U}_i$ should be served by exactly one of the base stations $\mathcal{B}_{i_1}$, $\mathcal{B}_{i_2}$, or $\mathcal{B}_{i_3}$. In addition, it should not receive any interference from other users. Now consider the coloring of the graph G, where each node i is colored by color ℓ if $\mathcal{U}_i$ is served by $\mathcal{B}_{i_\ell}$. Clearly, this coloring is well defined because each user $\mathcal{U}_i$ is served by exactly one of the base stations $\mathcal{B}_{i_\ell}$, $\ell = 1, 2, 3$. In addition, it is a proper coloring since no two transmissions would cause interference to each other. This completes the proof of Claim 1.

Note that we can modify the above proof to work for a SISO OFDM network when there are at least 3 tones in use. In particular, we just need to do the following modifications in the definition of the channels[3] .

1. The channel between $\mathcal{B}_{i_\ell}$ and $\mathcal{U}_i$ is $\mathbf{e}_\ell \mathbf{e}_\ell^H$, for all $i = 1, \cdots, N$, and all $\ell = 1, 2, 3$ (instead of being just $\mathbf{e}_\ell$).

2. If $\{i, j\} \in E$, the channel between $\mathcal{B}_{i_\ell}$ and $\mathcal{U}_j$ is $\frac{1}{2}\mathbf{e}_\ell \mathbf{e}_\ell^H$, for all $\ell = 1, 2, 3$ (instead of being $\frac{1}{2}\mathbf{e}_\ell$), and otherwise a zero matrix.

This proof extends naturally to the MIMO OFDM, because the diagonal channels in SISO OFDM could be considered as special cases of block diagonal channels in MIMO OFDM.

[3] Note that the channel gains in the SISO OFDM setup can be viewed as diagonal matrices.

Appendix B

A Stochastic Successive Minimization Method for Nonsmooth Nonconvex Optimization

Consider the problem of minimizing the expected value of a cost function parameterized by a random variable. The classical sample average approximation (SAA) method for solving this problem requires minimization of an ensemble average of the objective at each step, which can be expensive. In this chapter, we propose a stochastic successive upper-bound minimization method (SSUM) which minimizes an *approximate* ensemble average at each iteration. To ensure convergence and to facilitate computation, we require the approximate ensemble average to be a locally tight upper-bound of the expected cost function and be easily optimized. As we will see shortly our convergence analysis will readily prove the convergence of Stochastic WMMSE (Table 4.2) and Sparse Stochastic WMMSE (Table 5.1) as its corollaries. Moreover, using the SSUM framework, we extend the classical stochastic (sub-)gradient (SG) method to the case of minimizing a nonsmooth nonconvex objective function and establish its convergence.

B.1 Introduction

Consider the optimization problem

$$\begin{aligned} \min \quad & f(x) \triangleq \mathbb{E}_{\xi}[g(x,\xi)] \\ \text{s.t.} \quad & x \in \mathcal{X}, \end{aligned} \tag{B.1}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is a bounded closed convex set; ξ is a random vector drawn from a set $\Xi \in \mathbb{R}^m$, and $g : \mathcal{X} \times \Xi \mapsto \mathbb{R}$ is a real-valued function. A classical approach for solving the above optimization problem is the sample average approximation (SAA) method. At each iteration of the SAA method, a new realization of the random vector ξ is obtained and the optimization variable x is updated by solving

$$\begin{aligned} x^r \in \; \arg\min \; & \frac{1}{r}\sum_{i=1}^{r} g(x,\xi^i) \\ \text{s.t.} \quad & x \in \mathcal{X}. \end{aligned} \tag{B.2}$$

Here $\xi^1, \xi^2, \ldots$ are some independent, identically distributed realizations of the random vector ξ. We refer the readers to [81, 82, 83, 84, 85] for the roots of the SAA method and [86, 87, 88] for several surveys on SAA.

A main drawback of the SAA method is the complexity of each step. In general, solving (B.2) may not be easy due to the non-convexity and/or non-smoothness of $g(\cdot,\xi)$. Even when (B.2) is convex, finding a solution of it might require the use of an iterative procedure which may be inefficient. To overcome the difficulties in solving the subproblem (B.2), we propose an inexact SAA method whereby at each step a well-chosen approximation of the function $g(\cdot,\xi)$ in (B.2) is minimized. Specifically, at each iteration r, we update the optimization variable according to

$$x^r \leftarrow \arg\min_{x\in\mathcal{X}} \; \frac{1}{r}\sum_{i=1}^{r} \hat{g}(x, x^{i-1}, \xi^i), \tag{B.3}$$

where $\hat{g}(\cdot, x^{i-1}, \xi^i)$ is an approximation of the function $g(\cdot,\xi^i)$ around the point x^{i-1}. To ensure the convergence of this method, we require the approximation function $\hat{g}(\cdot, x^{i-1}, \xi^i)$ to be a locally tight upper bound of the original function $g(\cdot,\xi^i)$ around the point x^{i-1}, for each $i = 0, \ldots, r-1$. For this reason, we call the above algorithm (B.3) a stochastic successive upper-bound minimization method (SSUM).

The idea of successive upper-bound minimization (also known as majorization minimization or successive convex optimization) has been widely studied in the literature for deterministic optimization problems; see [77] and the references therein. In the successive upper-bound minimization (SUM) framework, a locally tight approximation of the function is minimized at each step of the algorithm. This technique is key to many important practical algorithms such as the concave-convex procedure [108] and the expectation maximization algorithm [109, 110].

While the successive upper-bound minimization idea is well studied and widely used in deterministic settings, very little is known about its use in the stochastic setup. The main contributions of this paper are to extend the technique of successive upper-bound minimization to the stochastic setup (B.1) and to illustrate its use in applications. In particular, we first establish the convergence of SSUM defined by (B.3), and then describe two important applications of the SSUM framework: the sum rate maximization problem for wireless communication networks and the online dictionary learning problem. For the stochastic wireless beamforming problem, our numerical experiments indicate that the SSUM approach significantly outperforms the other existing algorithms in terms of the achievable ergodic sum rate in the network. In addition, we show that the traditional stochastic gradient (SG) algorithm for unconstrained smooth minimization is a special case of the SSUM method. Moreover, using the SSUM framework, we extend the SG algorithm to the problem of minimizing a nonsmooth nonconvex objective function and establish its convergence.

B.2 Stochastic Successive Upper-bound Minimization

To be more specific, consider the optimization problem

$$\begin{aligned} \min \quad & \left\{ f(x) \triangleq \mathbb{E}_\xi \left[g_1(x,\xi) + g_2(x,\xi) \right] \right\} \\ \text{s.t.} \quad & x \in \mathcal{X}, \end{aligned} \tag{B.4}$$

where $\mathcal{X}$ is a bounded closed convex set and ξ is a random vector drawn from a set $\Xi \in \mathbb{R}^m$. We assume that the function $g_1 : \ \mathcal{X} \times \Xi \mapsto \mathbb{R}$ is a twice continuously differentiable (and possibly non-convex) function in x, while $g_2 : \ \mathcal{X} \times \Xi \mapsto \mathbb{R}$ is a convex continuous (and possibly non-smooth) function in x. Due to the non-convexity and non-smoothness of the objective function, it may be difficult to solve the subproblems (B.2) in the SAA method. This motivates us to consider an inexact SAA method by using an approximation of the function $g(\cdot,\xi)$ in the SAA method (B.2) as follows:

$$\begin{aligned} x^r \;\leftarrow\; & \arg\min_x \; \frac{1}{r}\sum_{i=1}^{r}\left(\hat{g}_1(x,x^{i-1},\xi^i)+g_2(x,\xi^i)\right) \\ \text{s.t.} \quad & x \in \mathcal{X}, \end{aligned} \tag{B.5}$$

where $\hat{g}_1(x,x^{i-1},\xi^i)$ is an approximation of the function $g_1(x,\xi^i)$ around the point x^{i-1}. Table B.1 summarizes the SSUM algorithm [80]. Note that the Stochastic WMMSE Algorithm in Table 4.2 is a special case of the SSUM algorithm when the function $g_2 \equiv 0$ and g_1 is the sum rate function.

Table B.1: The SSUM algorithm

Find a feasible point $x^0 \in \mathcal{X}$ and set $r = 0$.
repeat
$r \leftarrow r+1$
$x^r \leftarrow \arg\min_{x\in\mathcal{X}} \frac{1}{r}\sum_{i=1}^{r}\left(\hat{g}_1(x,x^{i-1},\xi^i)+g_2(x,\xi^i)\right)$
until some convergence criterion is met.

Clearly, the function $\hat{g}_1(x,y,\xi)$ should be related to the original function $g_1(x,\xi)$. In this paper, we assume that the approximation function $\hat{g}_1(x,y,\xi)$ satisfies the following conditions.

Assumption A:
Let $\mathcal{X}'$ be an open set containing the set $\mathcal{X}$. Suppose the approximation function $\hat{g}(x,y,\xi)$ satisfies the following

A1- $\hat{g}_1(y,y,\xi) = g_1(y,\xi), \quad \forall\, y \in \mathcal{X},\ \forall\, \xi \in \Xi$

A2- $\hat{g}_1(x,y,\xi) \geq g_1(x,\xi), \quad \forall\, x \in \mathcal{X}',\ \forall\, y \in \mathcal{X},\ \forall\, \xi \in \Xi$

A3- $\hat{g}(x, y, \xi) \triangleq \hat{g}_1(x, y, \xi) + g_2(x, \xi)$ is uniformly strongly convex in x, i.e., for all $(x, y, \xi) \in \mathcal{X} \times \mathcal{X} \times \Xi$,

$$\hat{g}(x + d, y, \xi) - \hat{g}(x, y, \xi) \geq \hat{g}'(x, y, \xi; d) + \frac{\gamma}{2}\|d\|^2, \quad \forall\ d \in \mathbb{R}^n,$$

where $\gamma > 0$ is a constant.

The assumptions A1-A2 imply that the approximation function $\hat{g}_1(\cdot, y, \xi)$ should be a locally tight approximation of the original function $g_1(\cdot, \xi)$. We point out that the above assumptions can be satisfied in many cases by the right choice of the approximation function and hence are not restrictive. For example, when the gradient of the function $g_1(\cdot, \xi)$ is Lipschitz continuous with a given constant L, then it is not hard to check that the function

$$\hat{g}_1(x, y, \xi) = g_1(y, \xi) + \langle \nabla g_1(y, \xi), x - y \rangle + \frac{\alpha}{2}\|x - y\|^2,$$

is a valid approximation and satisfies A1-A3 with α being a large enough positive constant. Note that in this example the approximation function $\hat{g}_1(\cdot, y, \xi)$ is strongly convex even though the function $g_1(\cdot, y)$ itself may not even be convex; see Section 3 and Section 4 for other examples.

To ensure the convergence of the SSUM algorithm, we further make the following assumptions.

Assumption B:

B1- The functions $g_1(x, \xi)$ and $\hat{g}_1(x, y, \xi)$ are continuous in x for every fixed $y \in \mathcal{X}$ and $\xi \in \Xi$

B2- The feasible set $\mathcal{X}$ is bounded

B3- The functions $g_1(\cdot, \xi)$ and $\hat{g}_1(\cdot, y, \xi)$, and their derivatives are uniformly bounded. More precisely, there exists a constant $K > 0$ such that for all $(x, y, \xi) \in \mathcal{X} \times \mathcal{X} \times \Xi$ we have

$$\begin{aligned} &|g_1(x, \xi)| \leq K, \quad \|\nabla_x g_1(x, \xi)\| \leq K, \\ &|\hat{g}_1(x, y, \xi)| \leq K, \quad \|\nabla_x \hat{g}_1(x, y, \xi)\| \leq K, \quad \|\nabla_x^2 \hat{g}_1(x, y, \xi)\| \leq K, \end{aligned}$$

B4- The function $g_2(x, \xi)$ is convex in x for every fixed $\xi \in \Xi$

B5- The function $g_2(x, \xi)$ and its directional derivative are uniformly bounded. In other words, there exists $K' > 0$ such that for all $(x, \xi) \in \mathcal{X} \times \Xi$, we have $|g_2(x, \xi)| \leq K'$ and

$$|g_2'(x, \xi; d)| \leq K'\|d\|, \quad \forall\, d \in \mathbb{R}^n \text{ with } x + d \in \mathcal{X}.$$

B6- Let $\hat{g}(x, y, \xi) = \hat{g}_1(x, y, \xi) + g_2(x, y, \xi)$. There exists $\bar{g} \in \mathbb{R}$ such that

$$|\hat{g}(x, y, \xi)| \leq \bar{g}, \quad \forall\, (x, y, \xi) \in \mathcal{X} \times \mathcal{X} \times \Xi.$$

Notice that in the assumptions B3 and B5, the derivatives are taken with respect to the x variable only. Furthermore, one can easily check that the assumption B3 is automatically satisfied if the mappings $g_1(x, \xi)$, $\nabla_x g_1(x, \xi)$, $\hat{g}_1(x, y, \xi)$, $\nabla_x \hat{g}_1(x, y, \xi)$, $\nabla_x^2 \hat{g}_1(x, y, \xi)$ are continuous in (x, y, ξ) and the set Ξ is bounded; or when the above mappings are continuous in (x, y) and Ξ is finite. As will be seen later, this assumption can be easily satisfied in various practical problems. It is also worth mentioning that since the function $g_2(x, \xi)$ is assumed to be convex in x in B4, its directional derivative with respect to x in B5 can be written as

$$\begin{aligned} g_2'(x, \xi; d) &= \liminf_{t \downarrow 0} \ \frac{g_2(x + td, \xi) - g_2(x, \xi)}{t} \\ &= \inf_{t > 0} \ \frac{g_2(x + td, \xi) - g_2(x, \xi)}{t} \\ &= \lim_{t \downarrow 0} \ \frac{g_2(x + td, \xi) - g_2(x, \xi)}{t}. \end{aligned} \tag{B.6}$$

In addition to these assumptions, we would like to define the following random functions to facilitate the presentation of the algorithms and proofs.

$$\begin{aligned} f_1^r(x) &\triangleq \frac{1}{r} \sum_{i=1}^{r} g_1(x, \xi^i), \\ f_2^r(x) &\triangleq \frac{1}{r} \sum_{i=1}^{r} g_2(x, \xi^i), \\ \hat{f}_1^r(x) &\triangleq \frac{1}{r} \sum_{i=1}^{r} \hat{g}_1(x, x^{i-1}, \xi^i), \\ f^r(x) &\triangleq f_1^r(x) + f_2^r(x), \\ \hat{f}^r(x) &\triangleq \hat{f}_1^r(x) + f_2^r(x), \end{aligned}$$

for $r = 1, 2, \ldots$. Clearly, the above random functions depend on the realization $\xi^1, \xi^2, \ldots$ and the choice of the initial point x^0.

B.2.1 Stochastic Successive Inexact Upper-Bound Minimization

Due to non-smoothness of the function g_2 and the existence of constraint $x \in \mathcal{X}$, there might be no closed form solution for (B.5) in many practical problems (e.g. problem (P) in chapter 5). Therefore, in each step of an SSUM method, we may have to solve another convex program using an iterative method. Unfortunately, such requirement can result in unnecessarily long running time for solving sub-problems. Our approach is to solve each sub-problem (B.5) to an accuracy which is enough for the overall algorithm to converge. This way we can avoid applying too many iterations of the iterative method when solving (B.5). In other words we allow some level of error in computing the iterate x^r, so that we gain speed in solving the sub-problem (B.5) in each iteration. In order to capture the idea of inexactness in computing the iterates we use Definition 1. We denote this inexact algorithm as Stochastic Successive Inexact Upper-bound Minimization (SSIUM). The overall SSIUM algorithm is summarized in Table B.2. It is worth noting that the Sparse Stochastic Algorithm 5.1 is a special case of SSIUM Algorithm. In addition, it is easy to see that when error $\epsilon_r = 0$ at each iterate r, then the SSIUM algorithm boils down to the original SSUM method.

Table B.2: The SSIUM algorithm

Find a feasible point $x^0 \in \mathcal{X}$ and set $r = 0$.
repeat
  $r \leftarrow r + 1$
  Choose x^r such that
    (1) $x^r \in \mathcal{I}_{\epsilon_r}\left(\hat{f}^r\right)$
    (2) $\hat{f}^r(x^r) \leq \hat{f}^r(x^{r-1})$
until some convergence criterion is met.

B.3 Convergence Analysis

As we mentioned earlier, the SSUM algorithm is a special case of SSIUM algorithm with no errors in computation of iterates, i.e. $\epsilon_r = 0, \ \forall r$. Therefore, in this section we only provide the proof for convergence of SSIUM method. The convergence of SSUM algorithm would follow as a corollary.

Theorem 5 *Suppose that Assumptions A and B are satisfied and $\epsilon_r = O(\frac{1}{r})$. Then the iterates generated by the SSIUM algorithm converge to the set of stationary points of* (B.4) *almost surely, i.e.,*

$$\lim_{r\to\infty} d(x^r, \mathcal{X}^*) = 0,$$

where $\mathcal{X}^*$ is the set of stationary points of (B.4).

Before proving Theorem 5, we need to prove the following lemma that characterizes the properties of the points within ϵ accuracy set $\mathcal{I}_\epsilon(h)$ (see Definition 1).

Lemma 6 *For any strongly convex function h defined on a compact convex set $\mathcal{X}$ with constant γ, if x^* is the exact minimizer of h on $\mathcal{X}$, then any point $x \in \mathcal{I}_\epsilon(h)$ satisfies*

$$\|x - x^*\| \leq \frac{2}{\gamma}\epsilon. \tag{B.7}$$

Moreover, if there exists a constant K such that

$$|h'(x,d)| \leq K\|d\|, \quad \forall \ d \text{ with } x + d \in \mathcal{X}, \tag{B.8}$$

then

$$h(x) \leq h(y) + \frac{2K}{\gamma}\epsilon, \ \forall \ y \in \mathcal{X} \tag{B.9}$$

Proof of Lemma 6 Writing the strong convexity assumption A3 at point $x \in \mathcal{I}_\epsilon(h)$ gives us

$$\begin{aligned} h(x^*) - h(x) &\geq h'(x, x^* - x) + \frac{\gamma}{2}\|x - x^*\|^2 \\ &\geq -\epsilon\|x^* - x\| + \frac{\gamma}{2}\|x - x^*\|^2, \end{aligned} \tag{B.10}$$

where the last inequality is due to inexactness condition (2.3). On the other hand, note that $h(x^*) - h(x) \leq 0$. Therefore, we can easily see that

$$\|x - x^*\| \leq \frac{2\epsilon}{\gamma}. \tag{B.11}$$

Using the convexity of h,

$$h(x) + h'(x, x^* - x) \leq h(x^*), \tag{B.12}$$

it is easy to see that

$$0 \leq h(x) - h(x^*) \leq -h'(x, x^* - x) \leq K\|x - x^*\|, \tag{B.13}$$

where the last inequality is due to assumption (B.8). Combining (B.11) with (B.13) and noting the fact that x^* is the minimizer of h yields the desired result.

Now we are ready to prove Theorem 5.

Proof of Theorem 5 First of all, since the iterates $\{x^r\}$ lie in a compact set, it suffices to show that every limit point of the iterates is a stationary point. To show this, let us consider a subsequence $\{x^{r_j}\}_{j=1}^{\infty}$ converging to a limit point $\bar{x}$. Note that since $\mathcal{X}$ is closed, $\bar{x} \in \mathcal{X}$ and therefore $\bar{x}$ is a feasible point. Moreover, since $|g_1(x, \xi)| < K$, $|g_2(x, \xi)| < K'$ for all $\xi \in \Xi$ (due to B3 and B5), using the strong law of large numbers [111], one can write

$$\lim_{r \to \infty} f_1^r(x) = \mathbb{E}\left[g_1(x, \xi)\right] \triangleq f_1(x), \quad \forall\, x \in \mathcal{X}, \tag{B.14}$$

$$\lim_{r \to \infty} f_2^r(x) = \mathbb{E}\left[g_2(x, \xi)\right] \triangleq f_2(x), \quad \forall\, x \in \mathcal{X}. \tag{B.15}$$

Furthermore, due to the assumptions B3, B5, and (B.6), the family of functions $\{f_1^{r_j}(\cdot)\}_{j=1}^{\infty}$ and $\{f_2^{r_j}(\cdot)\}_{j=1}^{\infty}$ are equicontinuous and therefore by restricting to a subsequence, we have

$$\lim_{j \to \infty} f_1^{r_j}(x^{r_j}) = \mathbb{E}_{\xi}\left[g_1(\bar{x}, \xi)\right], \tag{B.16}$$

$$\lim_{j \to \infty} f_2^{r_j}(x^{r_j}) = \mathbb{E}_{\xi}\left[g_2(\bar{x}, \xi)\right]. \tag{B.17}$$

On the other hand, $\|\nabla_x \hat{g}(x, y, \xi)\| < K$, $\forall\, x, y, \xi$ due to the assumption B3 and therefore the family of functions $\{\hat{f}_1^r(\cdot)\}$ is equicontinuous. Moreover, they are bounded and defined over a compact set; see B2 and B4. Hence the Arzelà–Ascoli theorem [112] implies that, by restricting to a subsequence, there exists a uniformly continuous function $\hat{f}_1(x)$ such that

$$\lim_{j\to\infty} \hat{f}_1^{r_j}(x) = \hat{f}_1(x), \quad \forall\, x \in \mathcal{X}, \tag{B.18}$$

and

$$\lim_{j\to\infty} \hat{f}_1^{r_j}(x^{r_j}) = \hat{f}_1(\bar{x}), \quad \forall\, x \in \mathcal{X}. \tag{B.19}$$

Furthermore, it follows from assumption A2 that

$$\hat{f}_1^{r_j}(x) \geq f_1^{r_j}(x), \quad \forall\, x \in \mathcal{X}'.$$

Letting $j \to \infty$ and using (B.14) and (B.18), we obtain

$$\hat{f}_1(x) \geq f_1(x), \quad \forall\, x \in \mathcal{X}'. \tag{B.20}$$

On the other hand, using the update rule of the SSIUM algorithm, one can show the following lemma.

Lemma 7 $\lim_{r\to\infty} \hat{f}_1^r(x^r) - f_1^r(x^r) = 0,$ almost surely.

The proof of Lemma 7 is rather technical. Thus, it is relegated to the Appendix C for the sake of coherence in the presentation.

Combining Lemma 7 with (B.16) and (B.19) yields

$$\hat{f}_1(\bar{x}) = f_1(\bar{x}). \tag{B.21}$$

It follows from (B.20) and (B.21) that the function $\hat{f}_1(x) - f_1(x)$ takes its minimum value at the point $\bar{x}$ over the open set $\mathcal{X}'$. Therefore, the first order optimality condition implies that

$$\nabla \hat{f}_1(\bar{x}) - \nabla f_1(\bar{x}) = 0,$$

or equivalently

$$\nabla \hat{f}_1(\bar{x}) = \nabla f_1(\bar{x}). \tag{B.22}$$

On the other hand, using Lemma 6, we have

$$\hat{f}_1^{r_j}(x^{r_j}) + f_2^{r_j}(x^{r_j}) \leq \hat{f}_1^{r_j}(x) + f_2^{r_j}(x) + \frac{2(K+K')}{\gamma}\epsilon_{r_j}, \quad \forall\, x \in \mathcal{X}.$$

Letting $j \to \infty$ and using (B.17) and (B.19) and the fact that $\epsilon_r = O(\frac{1}{r})$ yield

$$\hat{f}_1(\bar{x}) + f_2(\bar{x}) \leq \hat{f}_1(x) + f_2(x), \quad \forall\, x \in \mathcal{X}. \tag{B.23}$$

Moreover, the directional derivative of $f_2(\cdot)$ exists due to the bounded convergence theorem [111]. Therefore, (B.23) implies that

$$\langle \nabla \hat{f}_1(\bar{x}), d\rangle + f_2'(\bar{x}; d) \geq 0, \quad \forall\, d.$$

Combining this with (B.22), we get

$$\langle \nabla f_1(\bar{x}), d\rangle + f_2'(\bar{x}; d) \geq 0, \quad \forall\, d,$$

or equivalently

$$f'(\bar{x}; d) \geq 0, \quad \forall\, d,$$

which means that $\bar{x}$ is a stationary point of $f(\cdot)$. This completes the proof of Theorem 5.

Corollary 1 *[Theorem 4] Convergence of Stochastic Sparse WMMSE with Inexact Updates is guaranteed as a direct result of Theorem 5.*

Proof Using Lemma 1, it is easy to see that the weighted MMSE function (5.6) is a tight upper-bound for the rate function. As the feasible set is compact, it is obvious that the functions satisfy the assumptions A and B. The non-smooth part of the objective is also convex. As a result, the convergence follows directly.

B.3.1 A Few Remarks and Discussions

A few remarks are in the sequel.

Remark 8 *In Theorem 5, we assume that the set $\mathcal{X}$ is bounded. It is not hard to see that the result of the theorem still holds even if $\mathcal{X}$ is unbounded, so long as the iterates lie in a bounded set.*

Remark 9 *The boundedness of the Hessian of $\hat{g}_1(\cdot)$ in assumption B3 is used to show the existence of the gradient of $\hat{f}_1(\cdot)$ at the point $\bar{x}$. It is worth noticing that this assumption can be relaxed by just assuming that the gradient of $\hat{g}_1(\cdot)$ is uniformly locally Lipschitz continuous around the limit point $\bar{x}$.*

Remark 10 *It is worth noticing to the special cases: $g_1(x,\xi) = 0$ and $g_2(x,\xi) = 0$. When $g_1(x,\xi) = 0$, then the SSUM algorithm reduces to the traditional SAA algorithm. On the other hand, when $g_2(x,\xi) = 0$, we successively approximate the whole objective function at each iteration. An example of this case is given in subsections B.4.1 and B.4.2.*

Remark 11 *In contrast with many existing results (see [113, 114, 115]) on the convergence of similar methods with inexact computation of the iterates under the assumption $\sum_r \epsilon_r < \infty$, our method only requires $\epsilon_r = \mathcal{O}(\frac{1}{r})$ which is less restrictive.*

In this part we give some intuition on why $\epsilon_r = \mathcal{O}(\frac{1}{r})$ is achievable by means of an example. Let us consider the simplest situation where $g_2 \equiv 0$ and set $\mathcal{X}$ is the whole space.

First of all note that if we stack the samples $\xi_1, \cdots, \xi_S$ and make a batch out of them and run the SSUM algorithm on this batch instead of running it on one sample at a time, the same convergence result holds. This is due to the independence of the samples and additive nature of the objective. Now we will show that this batching technique is a special case of our SSIUM method, when considering one sample at a time. Let us assume that at an iteration r_0 which is an integer multiple of S,

$$x^{r_0} = \arg\min_x \ \hat{f}^{r_0}(x). \tag{B.24}$$

From the first order optimality condition, it is obvious that $\nabla \hat{f}^{r_0}(x^{r_0}) = 0$. Now we will prove that if $\epsilon_r = \frac{S \cdot K}{r}$, then for any $r_0 \leq r \leq r_0 + S - 1$, $x^{r_0} \in \mathcal{I}_{\epsilon_r}(\hat{f})$. We do so by a simple induction.

Let us assume that $\|\nabla \hat{f}^r x^{r_0}\| \leq \frac{(r-r_0).K}{r}$ that obviously holds for $r = r_0$. Now we prove that $\|\nabla \hat{f}^{r+1} x^{r_0}\| \leq \frac{(r+1-r_0)K}{r+1}$ and therefore $x^{r_0} \in \mathcal{I}_{\epsilon_{r+1}}(\hat{f}^{r+1})$. As a result x^{r_0} could be chosen as a candidate solution at the $(r+1)$-th step of SSIUM algorithm. To prove this, we compute $\nabla \hat{f}^{r+1}(x^{r_0})$, for any $r_0 \leq r \leq r_0 + S - 1$.

$$\|\nabla \hat{f}^{r+1}(x^{r_0})\| \leq \frac{r}{r+1}\|\nabla \hat{f}^r(x)\| + \frac{1}{r+1}\|\nabla \hat{g}_1(x^{r_0}, x^{r_0}, \xi_{r+1})\| \tag{B.25}$$

$$\leq \frac{r}{r+1}\frac{(r-r_0)K}{r} + \frac{1}{r+1}K \tag{B.26}$$

$$\leq \frac{(r+1-r_0)K}{r+1}, \tag{B.27}$$

where (B.25) is due to triangle inequality and (B.26) is due to the induction assumption and the Lipschitz condition B3. As a result (B.27), in SSIUM $x^r = x^{r_0}$, $r_0 \leq r \leq r_0 + S - 1$, meaning that x^r only needs to be updated exactly every S iterations. This coincides with the iterations of SSUM algorithm with batches of size S.

Note that although this example gives an intuition why the not absolutely summable error $\epsilon_r = \mathcal{O}(\frac{1}{r})$ still leads to convergent sequence, but in many cases SSUM algorithm is impossible to implement, even in batches, due to complexity of the sub-problems. In those cases, there might not be a direct connection between SSUM algorithm on batches and its inexact counterpart, SSIUM.

B.4 Other Applications of SSUM Method

In this section we discuss some other applications and special cases of SSUM method.

B.4.1 Online Dictionary Learning

Consider the classical dictionary learning problem: Given a random signal $y \in \mathbb{R}^n$ drawn from a distribution $P_Y(y)$, we are interested in finding a dictionary $D \in \mathbb{R}^{n \times k}$ so that the empirical cost function

$$f(D) \triangleq \mathbb{E}_y\left[g(D, y)\right]$$

is minimized over the feasible set $\mathcal{D}$; see [116, 117, 118]. The loss function $g(D, y)$ measures the fitting error of the dictionary D to the signal y. Most of the classical and modern loss functions can be represented in the form of

$$g(D, y) \triangleq \min_{\alpha \in \mathcal{A}} \; h(\alpha, D, y), \tag{B.28}$$

where $\mathcal{A} \subseteq \mathbb{R}^k$ and $h(\alpha, D, y)$ is a convex function in α and D separately. For example, by choosing $h(\alpha, D, y) = \frac{1}{2}\|y - D\alpha\|_2^2 + \lambda\|\alpha\|_1$, we obtain the sparse dictionary learning problem; see [118].

In order to apply the SSUM framework to the online dictionary learning problem, we need to choose an appropriate approximation function $\hat{g}(\cdot)$. To this end, let us define

$$\hat{g}(D, \bar{D}, y) = h(\bar{\alpha}, D, y) + \frac{\gamma}{2}\|D - \bar{D}\|_2^2,$$

where

$$\bar{\alpha} \triangleq \arg\min_{\alpha \in \mathcal{A}} h(\alpha, \bar{D}, y).$$

Clearly, we have

$$\hat{g}(\bar{D}, \bar{D}, y) = h(\bar{\alpha}, \bar{D}, y) = \min_{\alpha \in \mathcal{A}} h(\alpha, \bar{D}, y) = g(\bar{D}, y),$$

and

$$\hat{g}(D, \bar{D}, y) \geq h(\bar{\alpha}, D, y) \geq g(D, y).$$

Furthermore, if we assume that the solution of (B.28) is unique, the function $g(\cdot)$ is smooth due to Danskin's Theorem [76]. Moreover, the function $\hat{g}(D, \bar{D}, y)$ is strongly convex in D. Therefore, the assumptions A1-A3 are satisfied. In addition, if we assume that the feasible set $\mathcal{D}$ is bounded and the signal vector y lies in a bounded set $\mathcal{Y}$, the assumptions B1-B6 are satisfied as well. Hence the SSUM algorithm is applicable to the online dictionary learning problem.

Remark 12 *Choosing* $h(\alpha, D, y) = \frac{1}{2}\|y - D\alpha\|_2^2 + \lambda\|\alpha\|_1$ *and* $\gamma = 0$ *leads to the online sparse dictionary learning algorithm in [118]. Notice that the authors of [118] had to assume the uniform strong convxity of* $\frac{1}{2}\|y - D\alpha\|_2^2$ *for all* $\alpha \in \mathcal{A}$ *since they did not consider the quadratic proximal term* $\gamma\|D - \bar{D}\|^2$*.*

B.4.2 Stochastic Gradient Method and its Extensions

In this section, we show that the classical SG method and the incremental gradient method are special cases of the SSUM method. We also present an extension of these classical methods using the SSUM framework.

To describe the SG method, let us consider a special (unconstrained smooth) case of the optimization problem (B.1), where $g_2 \equiv 0$ and $\mathcal{X} = \mathbb{R}^n$. One of the popular algorithms for solving this problem is the stochastic gradient (also known as stochastic approximation) method. At each iteration r of the stochastic gradient (SG) algorithm, a new realization ξ^r is obtained and x is updated based on the following simple rule [87, 119, 120, 121, 122, 123, 124, 125]:

$$x^r \;\leftarrow\; x^{r-1} - \gamma^r \nabla_x g_1(x^{r-1}, \xi^r). \tag{B.29}$$

Here γ^r is the step size at iteration r. Due to its simple update rule, the SG algorithm has been widely used in various applications such as data classification [126, 127], training multi-layer neural networks [128, 129, 130, 131], the expected risk minimization [132], solving least squares in statistics [133], and distributed inference in sensor networks [134, 135, 136]. Also the convergence of the SG algorithm is well-studied in the literature; see, e.g., [137, 121, 87, 138].

The popular incremental gradient method [130, 131, 139, 133, 132] can be viewed as a special case of the SG method where the set Ξ is finite. In the incremental gradient methods, a large but finite set of samples Ξ is available and the objective is to minimize the empirical expectation

$$\hat{\mathbb{E}}\{g(x,\xi)\} \;=\; \frac{1}{|\Xi|}\sum_{\xi\in\Xi} g(x,\xi). \tag{B.30}$$

At each iteration r of the incremental gradient method (with random updating order), a new realization $\xi^r \in \Xi$ is chosen randomly and uniformly, and then (B.29) is used to update x. This is precisely the SG algorithm applied to the minimization of (B.30). In contrast to the batch gradient algorithm which requires computing $\sum_{\xi\in\Xi} \nabla_x g(x,\xi)$, the updates of the incremental gradient algorithm are computationally cheaper, especially if $|\Xi|$ is very large.

In general, the convergence of the SG method depends on the proper choice of the step size γ^r. It is known that for the constant step size rule, the SG algorithm might diverge even for a convex objective function; see [130] for an example. There are many variants of the SG algorithm with different step size rules [140, 141] and even different inexact versions [142]. In the following, we introduce a special form of the SSUM algorithm that can be interpreted as the SG algorithm with diminishing step sizes. Let us define

$$\hat{g}_1(x,y,\xi) = g_1(y,\xi) + \langle \nabla g_1(y,\xi), x-y\rangle + \frac{\alpha}{2}\|x-y\|^2, \tag{B.31}$$

where α is a function of y and is chosen so that $\hat{g}_1(x, y, \xi) \geq g_1(x, \xi)$. One simple choice is $\alpha^r = L$, where L is the Lipschitz constant of $\nabla_x g_1(x, \xi)$. Choosing $\hat{g}_1$ in this way, the assumptions A1-A3 are clearly satisfied. Moreover, the update rule of the SSUM algorithm becomes

$$x^r \leftarrow \arg\min_x \frac{1}{r} \sum_{i=1}^{r} \hat{g}_1(x, x^{i-1}, \xi^i). \tag{B.32}$$

Checking the first order optimality condition of (B.32), we obtain

$$x^r \leftarrow \frac{1}{\sum_{i=1}^r \alpha^i} \left(\sum_{i=1}^{r} (\alpha^i x^{i-1} - \nabla_x g_1(x^{i-1}, \xi^i)) \right). \tag{B.33}$$

Rewriting (B.33) in a recursive form yields

$$x^r \leftarrow x^{r-1} - \frac{1}{\sum_{i=1}^r \alpha^i} \nabla_x g_1(x^{r-1}, \xi^r), \tag{B.34}$$

which can be interpreted as the stochastic gradient method (B.29) with $\gamma^r = \frac{1}{\sum_{i=1}^r \alpha_i}$. Notice that the simple constant choice of $\alpha^i = L$ yields $\gamma^r = \frac{1}{rL}$, which gives the most popular diminishing step size rule of the SG method.

Remark 13 *When $\mathcal{X}$ is bounded and using the approximation function in* (B.31)*, we see that the SSUM algorithm steps become*

$$z^r = \frac{1}{\sum_{i=1}^r \alpha^i} \left(\sum_{i=1}^{r-1} \alpha^i z^{r-1} + \alpha^r x^{r-1} - \nabla_x g_1(x^{r-1}, \xi^r) \right),$$
$$x^r = \Pi_{\mathcal{X}}(z^r),$$

where $\Pi_{\mathcal{X}}(\cdot)$ signifies the projection operator to the constraint set $\mathcal{X}$. Notice that this update rule is different from the classical SG method as it requires generating the auxiliary iterates $\{z^r\}$ which may not lie in the feasible set $\mathcal{X}$.

It is also worth noting that in the presence of the non-smooth part of the objective function, the SSUM algorithm becomes different from the classical stochastic subgradient method [87, 119, 120, 121]. To illustrate the ideas, let us consider a simple deterministic nonsmooth function $g_2(x)$ to be added to the objective function. The resulting optimization problem becomes

$$\min_x \quad \mathbb{E}\left[g_1(x, \xi)\right] + g_2(x).$$

Using the approximation introduced in (B.31), the SSUM update rule can be written as

$$x^r \leftarrow \arg\min_x \quad \frac{1}{r}\sum_{i=1}^{r} \hat{g}_1(x, x^{i-1}, \xi^i) + g_2(x). \tag{B.35}$$

Although this update rule is similar to the (regularized) dual averaging method [143, 144] for convex problems, its convergence is guaranteed even for the nonconvex nonsmooth objective function under the assumptions of Theorem 5. Moreover, similar to the (regularized) dual averaging method, the steps of the SSUM algorithm are computationally cheap for some special nonsmooth functions. As an example, let us consider the special non-smooth function $g_2(x) \triangleq \lambda\|x\|_1$. Setting $\alpha^r = L$, the first order optimality condition of (B.35) yields the following update rule:

$$\begin{aligned} z^{r+1} &\leftarrow \frac{rz^r + x^r - \frac{1}{L}\nabla g_1(x^r, \xi^{r+1})}{r+1}, \\ x^{r+1} &\leftarrow \text{shrink}_{\frac{\lambda}{L}}(z^{r+1}), \end{aligned} \tag{B.36}$$

where $\{z^{r+1}\}_{r=1}^{\infty}$ is an auxiliary variable sequence and $\text{shrink}_\tau(z)$ is the soft shrinkage operator defined as

$$\text{shrink}_\tau(z) = \begin{cases} z - \tau & z \geq \tau \\ 0 & \tau \geq z \geq -\tau \\ z + \tau & z \leq -\tau \end{cases}.$$

Notice that the algorithm obtained in (B.36) is different from the existing stochastic sub-gradient algorithm and the stochastic proximal gradient algorithm [145, 139]; furthermore, if the conditions in Theorem 5 is satisfied, its convergence is guaranteed even for nonconvex objective functions.

Appendix C

Proof of Lemma 7

The proof requires the use of quasi martingale convergence theorem [146], much like the convergence proof of online learning algorithms [118, Proposition 3]. In particular, we will show that the sequence $\{\hat{f}^r(x^r)\}_{r=1}^{\infty}$ converges almost surely. Notice that

$$
\begin{aligned}
&\hat{f}^{r+1}(x^{r+1}) - \hat{f}^r(x^r) \\
&\quad = \hat{f}^{r+1}(x^{r+1}) - \hat{f}^{r+1}(x^r) + \hat{f}^{r+1}(x^r) - \hat{f}^r(x^r) \\
&\quad = \hat{f}^{r+1}(x^{r+1}) - \hat{f}^{r+1}(x^r) + \frac{1}{r+1}\sum_{i=1}^{r+1}\hat{g}(x^r, x^{i-1}, \xi^i) - \frac{1}{r}\sum_{i=1}^{r}\hat{g}(x^r, x^{i-1}, \xi^i) \\
&\quad = \hat{f}^{r+1}(x^{r+1}) - \hat{f}^{r+1}(x^r) - \frac{1}{r(r+1)}\sum_{i=1}^{r}\hat{g}(x^r, x^{i-1}, \xi^i) + \frac{1}{r+1}\hat{g}(x^r, x^r, \xi^{r+1}) \\
&\quad = \hat{f}^{r+1}(x^{r+1}) - \hat{f}^{r+1}(x^r) - \frac{\hat{f}^r(x^r)}{r+1} + \frac{1}{r+1}g(x^r, \xi^{r+1}) \\
&\quad \leq \frac{-\hat{f}^r(x^r) + g(x^r, \xi^{r+1})}{r+1},
\end{aligned}
$$

where the last equality is due to the assumption A1 and the inequality is due to the update rule of the SSIUM algorithm. Taking the expectation with respect to the natural history yields

$$\mathbb{E}\left[\hat{f}^{r+1}(x^{r+1}) - \hat{f}^r(x^r)\middle|\mathcal{F}^r\right] \leq \mathbb{E}\left[\frac{-\hat{f}^r(x^r) + g(x^r,\xi^{r+1})}{r+1}\middle|\mathcal{F}^r\right]$$
$$= \frac{-\hat{f}^r(x^r)}{r+1} + \frac{f(x^r)}{r+1}$$
$$= \frac{-\hat{f}^r(x^r) + f^r(x^r)}{r+1} + \frac{f(x^r) - f^r(x^r)}{r+1} \tag{C.1}$$
$$\leq \frac{f(x^r) - f^r(x^r)}{r+1} \tag{C.2}$$
$$\leq \frac{\|f - f^r\|_\infty}{r+1}, \tag{C.3}$$

where (C.2) is due to the assumption A2 and (C.3) follows from the definition of $\|\cdot\|_\infty$. On the other hand, the Donsker theorem (see [118, Lemma 7] and [147, Chapter 19]) implies that there exists a constant k such that

$$\mathbb{E}\left[\|f - f^r\|_\infty\right] \leq \frac{k}{\sqrt{r}}. \tag{C.4}$$

Combining (C.3) and (C.4) yields

$$\mathbb{E}\left[\left(\mathbb{E}\left[\hat{f}^{r+1}(x^{r+1}) - \hat{f}^r(x^r)\middle|\mathcal{F}^r\right]\right)_+\right] \leq \frac{k}{r^{3/2}}, \tag{C.5}$$

where $(a)_+ \triangleq \max\{0, a\}$ is the projection to the non-negative orthant. Summing (C.5) over r, we obtain

$$\sum_{r=1}^{\infty}\mathbb{E}\left[\left(\mathbb{E}\left[\hat{f}^{r+1}(x^{r+1}) - \hat{f}^r(x^r)\middle|\mathcal{F}^r\right]\right)_+\right] \leq M < \infty, \tag{C.6}$$

where $M \triangleq \sum_{r=1}^{\infty}\frac{k}{r^{3/2}}$. The equation (C.6) combined with the quasi-martingale convergence theorem (see [146] and [118, Theorem 6]) implies that the stochastic process$\{\hat{f}^r(x^r) + \bar{g}\}_{r=1}^{\infty}$ is a quasi-martingale with respect to the natural history $\{\mathcal{F}^r\}_{r=1}^{\infty}$ and $\hat{f}^r(x^r)$ converges. Moreover, we have

$$\sum_{r=1}^{\infty}\left|\mathbb{E}\left[\hat{f}^{r+1}(x^{r+1}) - \hat{f}^r(x^r)\middle|\mathcal{F}^r\right]\right| < \infty, \quad \text{almost surely.} \tag{C.7}$$

Next we use (C.7) to show that $\sum_{r=1}^{\infty}\frac{\hat{f}^r(x^r) - f^r(x^r)}{r+1} < \infty$, almost surely. To this end, let us rewrite (C.1) as

$$\frac{\hat{f}^r(x^r) - f^r(x^r)}{r+1} \leq \mathbb{E}\left[-\hat{f}^{r+1}(x^{r+1}) + \hat{f}^r(x^r)\middle|\mathcal{F}^r\right] + \frac{f(x^r) - f^r(x^r)}{r+1}. \tag{C.8}$$

Using the fact that $\hat{f}^r(x^r) \geq f^r(x^r),\ \forall\ r$ and summing (C.8) over all values of r, we have

$$\begin{aligned}0 &\leq \sum_{r=1}^{\infty} \frac{\hat{f}^r(x^r) - f^r(x^r)}{r+1} \\ &\leq \sum_{r=1}^{\infty} \left| \mathbb{E}\left[-\hat{f}^{r+1}(x^{r+1}) + \hat{f}^r(x^r) \middle| \mathcal{F}^r \right] \right| + \sum_{r=1}^{\infty} \frac{\|f - f^r\|_\infty}{r+1}.\end{aligned} \tag{C.9}$$

Notice that the first term in the right hand side is finite due to (C.7). Hence in order to show $\sum_{r=1}^{\infty} \frac{\hat{f}^r(x^r) - f^r(x^r)}{r+1} < \infty$, almost surely, it suffices to show that $\sum_{r=1}^{\infty} \frac{\|f - f^r\|_\infty}{r+1} < \infty$, almost surely. To show this, we use the Hewitt-Savage zero-one law; see [148, Theorem 11.3] and [111, Chapter 12, Theorem 19]. Let us define the event

$$\mathcal{A} \triangleq \left\{ (\xi^1, \xi^2, \ldots) \mid \sum_{r=1}^{\infty} \frac{\|f^r - f\|_\infty}{r+1} < \infty \right\}.$$

It can be checked that the event $\mathcal{A}$ is permutable, i.e., any finite permutation of each element of $\mathcal{A}$ is inside $\mathcal{A}$; see [148, Theorem 11.3] and [111, Chapter 12, Theorem 19]. Therefore, due to the Hewitt-Savage zero-one law [148], probability of the event $\mathcal{A}$ is either zero or one. On the other hand, it follows from (C.4) that there exists $M' > 0$ such that

$$\mathbb{E}\left[\sum_{r=1}^{\infty} \frac{\|f^r - f\|_\infty}{r+1} \right] \leq M' < \infty. \tag{C.10}$$

Using Markov's inequality, (C.10) implies that

$$Pr\left(\sum_{r=1}^{\infty} \frac{\|f^r - f\|_\infty}{r+1} > 2M' \right) \leq \frac{1}{2}.$$

Hence combining this result with the result of the Hewitt-Savage zero-one law, we obtain $Pr(\mathcal{A}) = 1$; or equivalently

$$\sum_{r=1}^{\infty} \frac{\|f^r - f\|_\infty}{r+1} < \infty, \quad \text{almost surely.} \tag{C.11}$$

As a result of (C.9) and (C.11), we have

$$0 \leq \sum_{r=1}^{\infty} \frac{\hat{f}^r(x^r) - f^r(x^r)}{r+1} < \infty, \quad \text{almost surely.} \tag{C.12}$$

On the other hand, it follows from the triangle inequality that

$$\begin{aligned}
&\left|\hat{f}^{r+1}(x^{r+1}) - f^{r+1}(x^{r+1}) - \hat{f}^{r}(x^{r}) + f^{r}(x^{r})\right| \\
&\quad\leq \left|\hat{f}^{r+1}(x^{r+1}) - \hat{f}^{r}(x^{r})\right| + \left|f^{r+1}(x^{r+1}) - f^{r}(x^{r})\right| \qquad \text{(C.13)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\hat{f}^{r+1}(x^{r+1}) - \hat{f}^{r}(x^{r})\right| \\
&\leq \left|\hat{f}^{r+1}(x^{r+1}) - \hat{f}^{r+1}(x^{r})\right| + \left|\hat{f}^{r+1}(x^{r}) - \hat{f}^{r}(x^{r})\right| \\
&\leq \kappa\|x^{r+1} - x^{r}\| + \left|\frac{1}{r+1}\sum_{i=1}^{r+1}\hat{g}(x^{r}, x^{i-1}, \xi^{i}) - \frac{1}{r}\sum_{i=1}^{r}\hat{g}(x^{r}, x^{i-1}, \xi^{i})\right| \qquad \text{(C.14)} \\
&\leq \kappa\|x^{r+1} - x^{r}\| + \left|\frac{1}{r(r+1)}\sum_{i=1}^{r}\hat{g}(x^{r}, x^{i-1}, \xi^{i}) + \frac{\hat{g}(x^{r}, x^{r}, \xi^{r+1})}{r+1}\right| \\
&\leq \kappa\|x^{r+1} - x^{r}\| + \frac{2\bar{g}}{r+1} \qquad \text{(C.15)} \\
&= \mathcal{O}\left(\frac{1}{r}\right), \qquad \text{(C.16)}
\end{aligned}$$

where (C.14) is due to the assumption B3 (with $\kappa = (K + K')$); (C.15) follows from the assumption B6, and (C.16) will be shown in Lemma 8. Similarly, one can show that

$$|f^{r+1}(x^{r+1}) - f^{r}(x^{r})| = \mathcal{O}\left(\frac{1}{r}\right). \qquad \text{(C.17)}$$

It follows from (C.13), (C.16), and (C.17) that

$$\left|\hat{f}^{r+1}(x^{r+1}) - f^{r+1}(x^{r+1}) - \hat{f}^{r}(x^{r}) + f^{r}(x^{r})\right| = \mathcal{O}\left(\frac{1}{r}\right). \qquad \text{(C.18)}$$

Let us fix a random realization $\{\xi^r\}_{r=1}^{\infty}$ in the set of probability one for which (C.12) and (C.18) hold. Define

$$\varpi^{r} \triangleq \hat{f}^{r}(x^{r}) - f^{r}(x^{r}).$$

Clearly, $\varpi^r \geq 0$ and $\sum_r \frac{\varpi^r}{r} < \infty$ due to (C.12). Moreover, it follows from (C.18) that $|\varpi^{r+1} - \varpi^{r}| < \frac{\tau}{r}$ for some constant $\tau > 0$. Hence Lemma 9 implies that

$$\lim_{r\to\infty} \varpi^{r} = 0,$$

which is the desired result.

Lemma 8 $\|x^{r+1} - x^r\| = \mathcal{O}(\frac{1}{r})$.

Proof The proof of this lemma is similar to the proof of [80, Lemma 2]; see also [149, Proposition 4.32]. First of all, due to the update of SSIUM method

$$\hat{f}^r(x^r; d) \geq -\epsilon_r \|d\|, \quad \forall\, d \in \mathbb{R}^n.$$

Hence, it follows from the assumption A3 that

$$\hat{f}^r(x^{r+1}) - \hat{f}^r(x^r) \geq -\epsilon_r \|x^{r+1} - x^r\| + \frac{\gamma}{2}\|x^{r+1} - x^r\|^2. \tag{C.19}$$

On the other hand,

$$\begin{aligned}
\hat{f}^r(x^{r+1}) - \hat{f}^r(x^r) &\leq \hat{f}^r(x^{r+1}) - \hat{f}^{r+1}(x^{r+1}) + \hat{f}^{r+1}(x^r) - \hat{f}^r(x^r) \qquad \text{(C.20)}\\
&\leq \frac{1}{r(r+1)} \sum_{i=1}^{r} |\hat{g}(x^{r+1}, x^{i-1}, \xi^i) - \hat{g}(x^r, x^{i-1}, \xi^i)| \\
&\quad + \frac{1}{r+1} |\hat{g}(x^{r+1}, x^r, \xi^{r+1}) - \hat{g}(x^r, x^r, \xi^{r+1})| \\
&\leq \frac{\theta}{r+1} \|x^{r+1} - x^r\|, \qquad \text{(C.21)}
\end{aligned}$$

where (C.20) follows from the update of SSIUM algorithm, the second inequality is due to the definitions of $\hat{f}^r$ and $\hat{f}^{r+1}$, while (C.21) is the result of the assumptions B3 and B5. Combining (C.19) and (C.21) we get

$$\frac{\gamma}{2}\|x^{r+1} - x^r\| \leq \frac{\theta}{r+1} + \epsilon_r.$$

Using the fact that $\epsilon_r = \mathcal{O}(\frac{1}{r})$, we can readily see that $\|x^{r+1} - x^r\| = \mathcal{O}(\frac{1}{r})$.

Lemma 9 *Assume* $\varpi^r > 0$ *and* $\sum_{r=1}^{\infty} \frac{\varpi^r}{r} < \infty$. *Furthermore, suppose that* $|\varpi^{r+1} - \varpi^r| \leq \tau/r$ *for all* r. *Then* $\lim_{r\to\infty} \varpi^r = 0$.

Proof Since $\sum_{r=1}^{\infty} \frac{\varpi^r}{r} < \infty$, we have $\liminf_{r\to\infty} \varpi^r = 0$. Now, we prove the result using contradiction. Assume the contrary so that

$$\limsup_{r\to\infty} \varpi^r > \epsilon, \tag{C.22}$$

for some $\epsilon > 0$. Hence there should exist subsequences $\{m_j\}$ and $\{n_j\}$ with $m_j \leq n_j < m_{j+1}, \forall\ j$ so that

$$\frac{\epsilon}{3} < \varpi^r \qquad m_j \leq r < n_j, \tag{C.23}$$

$$\varpi^r \leq \frac{\epsilon}{3} \qquad n_j \leq r < m_{j+1}. \tag{C.24}$$

On the other hand, since $\sum_{r=1}^{\infty} \frac{\varpi^r}{r} < \infty$, there exists an index $\bar{r}$ such that

$$\sum_{r=\bar{r}}^{\infty} \frac{\varpi^r}{r} < \frac{\epsilon^2}{9\tau}. \tag{C.25}$$

Therefore, for every $r_0 \geq \bar{r}$ with $m_j \leq r_0 \leq n_j - 1$, we have

$$\begin{aligned}
|\varpi^{n_j} - \varpi^{r_0}| &\leq \sum_{r=r_0}^{n_j-1} |\varpi^{r+1} - \varpi^r| \\
&\leq \sum_{r=r_0}^{n_j-1} \frac{\tau}{r} && \text{(C.26)} \\
&\leq \frac{3}{\epsilon} \sum_{r=r_0}^{n_j-1} \frac{\tau}{r} \varpi^r && \text{(C.27)} \\
&\leq \frac{3\tau\epsilon^2}{9\epsilon\tau} = \frac{\epsilon}{3}, && \text{(C.28)}
\end{aligned}$$

where the equation (C.27) follows from (C.23), and (C.28) is the direct consequence of (C.25). Hence the triangle inequality implies

$$\alpha^{r_0} \leq \varpi^{n_j} + |\varpi^{n_j} - \alpha^{r_0}| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3},$$

for any $r_0 \geq \bar{r}$, which contradicts (C.22), implying that

$$\limsup_{r \to \infty} \varpi^r = 0.$$

Appendix D

Proof of Lemma 2

Proof For any $q \in \mathcal{T}^c$, Finding $(\hat{\mathbf{d}}^q)^*$ reduces to

$$\begin{aligned} \min_{\hat{\mathbf{d}}^q} \quad & (\hat{\mathbf{g}}^q)^T \hat{\mathbf{d}}^q \\ \text{s.t.} \quad & \|\hat{\mathbf{d}}^q\|^2 \leq 1, \end{aligned}$$

which leads to optimal solution $(\hat{\mathbf{d}}^q)^* = \frac{-\hat{\mathbf{g}}^q}{\|\hat{\mathbf{g}}^q\|}$.

For any $q \in \mathcal{T}$, finding $(\hat{\mathbf{d}}^q)^*$ can be reformulated as

$$\begin{aligned} \min_{\hat{\mathbf{d}}^q} \quad & (\hat{\mathbf{g}}^q)^T \hat{\mathbf{d}}^q \\ \text{s.t.} \quad & (\hat{\mathbf{v}}^q)^T \hat{\mathbf{d}}^q \leq 0, \\ & \|\hat{\mathbf{d}}^q\|^2 \leq 1. \end{aligned}$$

Introducing the Lagrangian multipliers $\nu^q \geq 0$ and $\omega^q \geq 0$ for first and second constraint respectively, we can write the first order optimality condition as

$$(\hat{\mathbf{g}}^q) + \nu^q \hat{\mathbf{v}}^q + 2\omega^q (\hat{\mathbf{d}}^q)^* = 0. \tag{D.1}$$

Therefore, $(\hat{\mathbf{d}}^q)^* = -\frac{(\hat{\mathbf{g}}^q) + \nu^q \hat{\mathbf{v}}^q}{2\omega^q}$. Now using the complementary slackness conditions

$$\begin{aligned} \nu^q (\hat{\mathbf{v}}^q)^T (\hat{\mathbf{d}}^q)^* = 0 \\ \omega^q (\|(\hat{\mathbf{d}}^q)^*\|^2 - 1) = 0, \end{aligned}$$

alongside with the non-negativity of multipliers, we can see that

- If $(\hat{\mathbf{g}}^q)^T\hat{\mathbf{v}}^q \geq 0$, then $\nu^q = 0$. Moreover, if $\hat{\mathbf{g}}^q \neq 0$, then $\omega^q > 0$. This means that $\|(\hat{\mathbf{d}}^q)^*\|^2 = 1$. As a result $(\hat{\mathbf{d}}^q)^* = -\frac{\hat{\mathbf{g}}^q}{\|\hat{\mathbf{g}}^q\|}$.

- If $(\hat{\mathbf{g}}^q)^T\hat{\mathbf{v}}^q < 0$, then ν^q should be chosen such that $(\hat{\mathbf{v}}^q)^T\hat{\mathbf{d}}^q = 0$. Doing the simple calculations we find $\nu^q = -\frac{(\hat{\mathbf{g}}^q)^T\hat{\mathbf{v}}^q}{\|\hat{\mathbf{v}}^q\|^2} \geq 0$. Similar to the previous case, $\omega^q > 0$ is chosen so that $\|(\hat{\mathbf{d}}^q)^*\|^2 = 1$. Thus,

$$(\hat{\mathbf{d}}^q)^* = -\frac{\hat{\mathbf{g}}^q + \nu^q\hat{\mathbf{v}}^q}{\|\hat{\mathbf{g}}^q + \nu^q\hat{\mathbf{v}}^q\|}.$$

For any $(i, q) \in \mathcal{S}$, the problem of finding $(\mathbf{d}_i^q)^*$ is equivalent to

$$\begin{aligned} \min_{\mathbf{d}_i^q} \quad & \lambda_i\|\mathbf{d}_i^q\| + (\tilde{\mathbf{g}}_i^q)^T\mathbf{d}_i^q \\ \text{s.t.} \quad & \|\mathbf{d}_i^q\|^2 \leq 1. \end{aligned}$$

Introducing the Lagrange multiplier $\pi_i^q \geq 0$ corresponding to the constraint, the first order optimality condition is

$$0 \in \lambda_i\partial\|(\mathbf{d}_i^q)^*\| + \tilde{\mathbf{g}}_i^q + 2\pi_i^q(\mathbf{d}_i^q)^*, \tag{D.2}$$

where $\partial\|(\mathbf{d}_i^q)^*\|$ is the sub-differential set of function $\|\cdot\|$ at point $\mathbf{d}_i^q$. Note that $\partial\|\mathbf{d}_i^q\| = \{\frac{\mathbf{d}_i^q}{\|\mathbf{d}_i^q\|}\}$ when $\mathbf{d}_i^q \neq 0$. Moreover, $\partial\|\mathbf{d}_i^q\| = \{\mathbf{e} \mid \|\mathbf{e}\| \leq 1\}$. Let us consider two different situations.

- If $\|\tilde{\mathbf{g}}_i^q\| \leq \lambda_i$, then choosing $\pi_i^q = 0$ as well as $(\mathbf{d}_i^q)^* = 0$ will satisfy the optimality condition (D.2).

- If $\|\tilde{\mathbf{g}}_i^q\| > \lambda_i$, then $(\mathbf{d}_i^q)^* \neq 0$. Therefore the optimality condition boils down to

$$\lambda_i\frac{(\mathbf{d}_i^q)^*}{\|(\mathbf{d}_i^q)^*\|} + \tilde{\mathbf{g}}_i^q + 2\pi_i^q(\mathbf{d}_i^q)^* = 0. \tag{D.3}$$

  Now if we assume $\|(\mathbf{d}_i^q)^*\| < 1$, then complementary slackness condition $2\pi_i^q.(\|(\mathbf{d}_i^q)^*\| - 1) = 0$ means that $\pi_i^q = 0$. Substituting this in (D.3) leads to $\|\tilde{\mathbf{g}}_i^q\| = \lambda_i$ which is a contradiction. Therefore, $\|(\mathbf{d}_i^q)^*\| = 1$. Using this fact alongside with (D.3) means $(\mathbf{d}_i^q)^* = -\frac{\tilde{\mathbf{g}}_i^q}{\|\tilde{\mathbf{g}}_i^q\|}$.