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FINITE-DIMENSIONAL DESCRIPTION OF DOUBLY DIFFUSIVE CONVECTION

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Abstract. Doubly diffusive convection in small aspect ratio systems exhibits complex temporal dynamics that has been attributed to the Shil'nikov mechanism. These results are reviewed and an asymptotic expansion suggested that leads in a systematic manner from the partial differential equations to a third order system of ordinary differential equations with Shil'nikov dynamics.

0. Introduction. The purpose of this article is to describe some attempts at providing a finite-dimensional description of the complex behavior that is known to occur in doubly diffusive systems. In section I we summarize the results of numerical experiments performed on the partial differential equations describing thermosolutal convection in small aspect ratio systems. In section II we describe an asymptotic régime in which these equations can be reduced systematically to a third order system of ordinary differential equations. This system contains the dynamics first described by Shil'nikov [22] and elaborated by Glendinning and Sparrow [23]. This dynamics bears a strong qualitative resemblance to the numerical results, suggesting that the Shil'nikov mechanism is responsible for the observed dynamics even for parameter values outside the asymptotic régime. Support for this conjecture is provided by truncations of (flat) Galerkin expansions which also exhibit the Shil'nikov mechanism. In section III we describe how our results are modified in large aspect ratio systems (modelled here by imposing periodic boundary conditions in the horizontal) and discuss prospects for improved finite-dimensional models of doubly diffusive systems.

I. Doubly diffusive convection. Doubly diffusive systems are characterized by a competition between a destabilizing force (typically thermal buoyancy) and a stabilizing force. The stabilization may be provided by an ambient concentration gradient as in thermosolutal convection; in binary mixtures the concentration gradient is set up in response to the thermal forcing by means of the Soret effect. In magnetoconvection or in rotating

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systems the restoring force is provided by the Lorentz and Coriolis forces, respectively. These systems differ from Rayleigh-Bénard convection in that the primary instability may be oscillatory. In thermosolutal convection such an instability occurs when during an oscillation the phase lag between the thermal and concentration fields extracts sufficient energy from the thermal stratification to overcome viscous dissipation. The resulting instability is diffusive and does not require the system to be dynamically unstable.

(a) The equations.

The basic equations describing thermosolutal convection are the Boussinesq equations

$$(1a) \quad \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \rho \mathbf{g} + \rho_0 \nu \nabla^2 \mathbf{u}$$

$$(1b) \quad \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa_T \nabla^2 T$$

$$(1c) \quad \frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S = \kappa_S \nabla^2 S$$

with

$$(1d) \quad \nabla \cdot \mathbf{u} = 0, \quad \rho = \rho_0(1 - \alpha T + \beta S).$$

Thus the Navier-Stokes equation is coupled to two advection-diffusion equations through the buoyancy term $\rho \mathbf{g}$. Here \mathbf{u}, p, T and S are the velocity, pressure, temperature and concentration fields, ρ is the density, and ν, κ_T and κ_S are the kinematic viscosity, thermal diffusivity and solutal diffusivity, respectively, assumed to be constant. The coefficients of expansion α, β are positive constants.

In the simplest situation we assume that the boundary conditions at the top and bottom of a plane layer are stress-free and that the temperature and concentration are fixed there:

$$(2a) \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0, \quad T = T_0 + \Delta T, \quad S = S_0 + \Delta S \quad \text{on } z = 0$$

$$(2b) \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0, \quad T = T_0, \quad S = S_0 \quad \text{on } z = Hh,$$

where $\mathbf{u} = (u, v, w)$, $\Delta T > 0$, $\Delta S > 0$, and H is dimensionless. It is convenient to define quantities θ, ϕ by

$$(3) \quad T = T_0 + \Delta T \left(1 - \frac{z}{Hh} + \theta \right), \quad S = S_0 + \Delta S \left(1 - \frac{z}{Hh} + \phi \right),$$

so that θ, ϕ denote departures of the temperature and concentration from the no-motion-state

$$(4) \quad \mathbf{u} = 0, \quad T = T_0 + \Delta T \left(1 - \frac{z}{Hh}\right), \quad S = S_0 + \Delta S \left(1 - \frac{z}{Hh}\right)$$

(hereafter the trivial solution). In two dimensions one may introduce the dimensionless streamfunction $\psi(x, z, t)$ such that $\mathbf{u} \equiv \kappa_T(-\partial_z\psi, 0, \partial_x\psi)$. In terms of the dimensionless variables $x/h, z/h, t/(h^2/\kappa_T)$ equations (1) become

$$(5a) \quad \sigma^{-1} [\partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi)] = R_T \partial_x \theta - R_S \partial_x \phi + \nabla^4 \psi$$

$$(5b) \quad \partial_t \theta + J(\psi, \theta) = \partial_x \psi + \nabla^2 \theta$$

$$(5c) \quad \partial_t \phi + J(\psi, \phi) = \partial_x \psi + \tau \nabla^2 \phi,$$

where $J(f, g) \equiv (\partial_x f)(\partial_z g) - (\partial_z f)(\partial_x g)$, and the dimensionless parameters are given by

$$(5d) \quad \sigma = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_S}{\kappa_T}, \quad R_T = \frac{g\alpha\Delta T h^3}{\kappa_T \nu}, \quad R_S = \frac{g\beta\Delta S h^3}{\kappa_T \nu}.$$

The corresponding boundary conditions are

$$(5e) \quad \psi = \psi_{zz} = \theta = \phi = 0 \quad \text{on} \quad z = 0, H.$$

A Hopf bifurcation from the trivial solution $\psi = \theta = \phi = 0$ occurs if $\tau < 1$ and $R_S > R_{S,CT}$ (see below).

(b) Numerical results.

Detailed numerical studies of equations (5) are available [7,13,17,18]. These calculations are carried out with the lateral boundary conditions

$$(5f) \quad \psi = \psi_{xx} = \theta_x = \phi_x = 0 \quad \text{on} \quad x = 0, \Lambda.$$

These boundary conditions force the planes $x = 0, x = \Lambda$ to be stress-free but impenetrable walls. The computations use $\Lambda = 2^{1/2}$ or 1.5 corresponding (approximately) to half the wavelength of the mode that first loses stability with increasing Rayleigh number R_T . The computations impose the symmetry

$$(6) \quad \psi(x, z) = \psi(\Lambda - x, H - z), \theta(x, z) = -\theta(\Lambda - x, H - z), \phi(x, z) = -\phi(\Lambda - x, H - z)$$

and employ a second order accurate finite difference scheme. The mesh typically used is equivalent to 594 independent variables. The dependence of the results on mesh size and time step has been extensively investigated [18]. In Figure 1 we summarize the results of these computations. The figure shows the Nusselt number, related to the square of

the amplitude of motion, as a function of R_T . The oscillations set in at $R_T^{(o)}$, increase in amplitude and undergo a sequence of bifurcations, the first of which breaks the *temporal* symmetry

$$(7) \quad \begin{aligned} \psi(x, z, t) &= -\psi\left(x, H - z, t + \frac{1}{2}P\right), & \theta(x, z, t) &= -\theta\left(x, H - z, t + \frac{1}{2}P\right), \\ \phi(x, z, t) &= -\phi\left(x, H - z, t + \frac{1}{2}P\right), \end{aligned}$$

where P is the oscillation period. We refer to the oscillations with the symmetry (7) as symmetric and those that break the symmetry (7) as asymmetric. Both types of solutions retain the imposed *spatial* symmetry (6). The asymmetric oscillations undergo a cascade of period-doubling bifurcations into chaos and back out of it. The chaotic region is interspersed with windows containing more complicated periodic oscillations, both symmetric and asymmetric, which themselves undergo period doubling bifurcations. For the symmetric oscillations these are again preceded by a bifurcation to asymmetry. The whole structure is repeated hysteretically in a sequence of bifurcation “bubbles”, four of which are shown in Figure 1. The results provide the most extensive study of chaotic behavior in a hydrodynamic system described by partial differential equations.

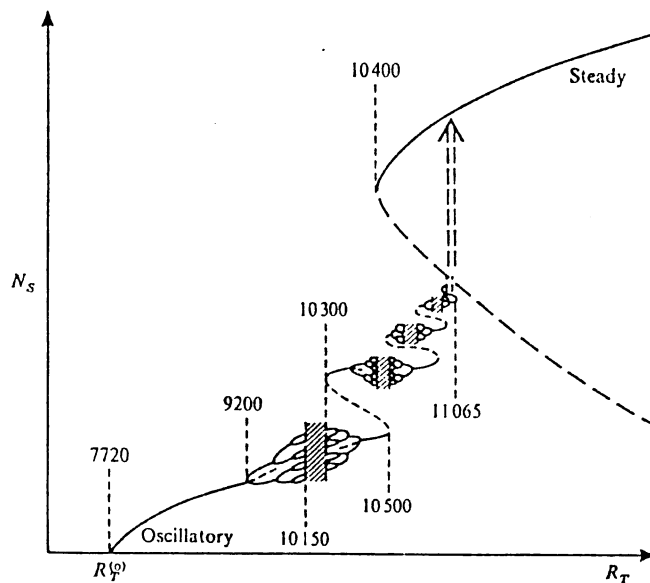


Figure 1. Schematic bifurcation diagram for numerical experiments on two-dimensional thermosolutal convection, showing both oscillatory and steady branches. Conjectured unstable solutions are denoted by broken lines. (From Knobloch et al [13].)

(c) The Shil'nikov mechanism.

Moore et al [17] suggested that the above behavior is related to the presence of a heteroclinic orbit connecting two saddle-foci to one another. These saddle-foci correspond to unstable steady (overturning) convection (hereafter SS) and are characterised by their eigenvalues. Suppose that the three dominant (i.e., least stable) eigenvalues are

$$(8a) \quad \gamma, -\alpha \pm i\beta, \quad \gamma > 0, \alpha > 0.$$

Since the saddle-foci are related by reflection in $x = \Lambda/2$ their eigenvalues are identical. We define

$$(8b) \quad \delta \equiv \alpha/\gamma$$

and let δ_h denote the value of δ at the parameter value for which a global connection is present. Shil'nikov [22] showed for a homoclinic connection that the condition $\delta_h < 1$ implies the presence of a countable number of horseshoes in the dynamics of the system. Motivated by the results of Figure 1 Glendinning and Sparrow [5] investigated in detail the sequence of bifurcations that gives rise to the complex orbits guaranteed by Shil'nikov's theorem as a bifurcation parameter μ passes through μ_h . Their results are summarized in Figure 2 showing the period $P(\mu)$ for the two cases $\delta > 1$ and $\frac{1}{2} < \delta < 1$.

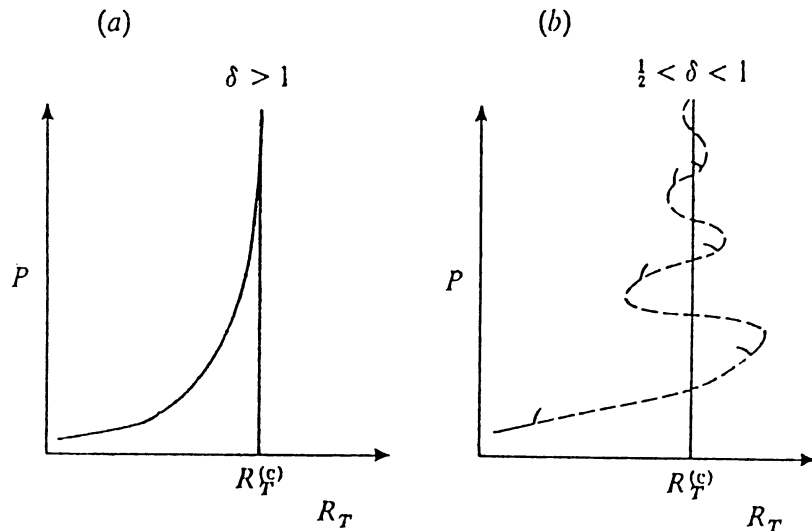


Figure 2. Period P as a function of the bifurcation parameter μ in the neighborhood of the homoclinic bifurcation when (a) $\delta > 1$, (b) $1/2 < \delta < 1$. Full and broken lines represent stable and non-stable solutions, respectively.

In the former the period increases monotonically as $\mu \rightarrow \mu_h$. In the latter the infinite period is approached by a series of wiggles containing period-doubling bubbles with chaotic intervals and hysteresis between successive bubbles. In addition their study revealed the existence of subsidiary homoclinic orbits with their own associated bubble structure. Since the period and amplitude are (typically) in 1-1 correspondence the analysis leads to results much like those in Figure 1. The above structure is present also for $\delta < \frac{1}{2}$ but is unstable.

The presence of a heteroclinic orbit in the partial differential equations is difficult to establish. Only in the limit where the two primary bifurcations coalesce ($R_T^{(o)} = R_T^{(e)}$), i.e.,

$$(9a) \quad R_T = R_{T,CT} \equiv \frac{\sigma + \tau}{\sigma(1 - \tau)} R_0$$

$$(9b) \quad R_S = R_{S,CT} \equiv \frac{\tau^2}{\sigma} \left(\frac{1 + \sigma}{1 - \tau} \right) R_0$$

$$(9c) \quad R_0 \equiv \pi^4 (H^2 + \Lambda^2)^3 / \Lambda^4 H^6,$$

and the dynamics of the p.d.e.'s reduce to the Takens-Bogdanov normal form

$$(10a) \quad \dot{v} = w$$

$$(10b) \quad \dot{w} = \mu v + \nu w + Av^3 + Bv^2w$$

with $A > 0, B < 0$, and $|\mu|, |\nu| \ll 1$ can one show the existence of a heteroclinic connection between the two saddle-points [6,14]. In this limit the saddle points are characterized by two dominant eigenvalues, one small and positive and one small and negative. All the other eigenvalues are $O(1)$ and negative. Consequently the vector field (10) is planar and no interesting dynamics occur near such a connection. Moore et al [17] and Knobloch et al [13] pointed out, however, that with increasing $R_T - R_{T,CT}$ and $R_S - R_{S,CT}$ the two least negative eigenvalues coalesce on the negative real axis, and thereafter move into the complex plane giving rise to the situation (8a). This scenario is supported by studying truncations of flat Galerkin expansions of which the simplest nontrivial one is

$$(11a) \quad \dot{a} = \sigma[-a + rb - sd]$$

$$(11b) \quad \dot{b} = -b + a(1 - c)$$

$$(11c) \quad \dot{c} = \varpi[-c + ab]$$

$$(11d) \quad \dot{d} = -\tau d + a(1 - c)$$

$$(11e) \quad \dot{e} = \varpi[-\tau e + ad],$$

where $r \equiv R_T/R_0, S \equiv R_S/R_0$ and $\varpi \equiv 4\Lambda^2/(H^2 + \Lambda^2)$. Numerical study of (11) indicates the presence of the Shil'nikov mechanism but in a régime where higher order modes can no longer be neglected [1,16]. In the following section we describe a systematic asymptotic procedure that enables us to derive a set of ordinary differential equations possessing

the Shil'nikov mechanism. The basic idea is to seek a codimension-three singularity that would add another equation to the Takens-Bogdanov normal form (10) and so describe the deformation of the planar heteroclinic orbit into one joining the saddle-foci. Since in the present problem there are no other primary bifurcations we look at the limit $\Lambda \rightarrow 0$, i.e., study convection in tall thin cells.

II. The asymptotic régime.

Before proceeding it is convenient to define horizontally averaged quantities denoted by an overbar, and fluctuating quantities denoted by a tilde. Thus, for example, $\bar{\theta} \equiv \frac{1}{\Lambda} \int_0^\Lambda \theta dx$ while $\tilde{\theta} \equiv \theta - \bar{\theta}$. To look at tall thin cells we set $\Lambda = \pi$ and let H be large. Following Proctor and Weiss [21] we use $\epsilon \equiv \pi H^{-1}$ as an expansion parameter and let

$$(12a) \quad \zeta = \epsilon z, \quad T = \epsilon^2 t,$$

$$\psi = \epsilon^2 \tilde{\psi}(x, \zeta, T; \epsilon), \quad \theta = \epsilon^2 \tilde{\theta}(x, \zeta, T; \epsilon) + \epsilon^3 \bar{\theta}(\zeta, T; \epsilon),$$

$$(12b) \quad \phi = \epsilon^2 \tilde{\phi}(x, \zeta, T; \epsilon) + \epsilon^3 \bar{\phi}(\zeta, T; \epsilon),$$

where

$$(12c) \quad \tilde{\psi} = \tilde{\psi}_0 + \epsilon^2 \tilde{\psi}_2 + \dots, \quad \tilde{\theta} = \tilde{\theta}_0 + \epsilon^2 \tilde{\theta}_2 + \dots, \quad \tilde{\phi} = \tilde{\phi}_0 + \epsilon^2 \tilde{\phi}_2 + \dots$$

$$(12d) \quad \bar{\theta} = \bar{\theta}_0 + \epsilon^2 \bar{\theta}_2 + \dots, \quad \bar{\phi} = \bar{\phi}_0 + \epsilon^2 \bar{\phi}_2 + \dots.$$

In addition we expand R_T and R_S around the codimension-two point:

$$(12e) \quad R_T = R_{T,CT} + \epsilon^2 R_{T,2} + \epsilon^4 R_{T,4} + \dots$$

$$(12f) \quad R_S = R_{S,CT} + \epsilon^2 R_{S,2} + \epsilon^4 R_{S,4} + \dots.$$

Here, as in (7), the overbar indicates a horizontal average and the tilde the fluctuating part. At $O(\epsilon^0)$ we obtain from (5) the solution

$$(13) \quad \tilde{\psi}_0 = a \sin x, \quad \tilde{\theta}_0 = \tau \tilde{S}_0 = a \cos x.$$

The ζ -dependence of $a(\zeta, T)$ follows from a solvability condition at $O(\epsilon^2)$:

$$(14) \quad a = \hat{a}(T) \sin \zeta, \quad R_{T,2} - R_{S,2}/\tau = 3.$$

The solution of the $O(\epsilon^2)$ problem is then given by

$$(15a) \quad \tilde{\psi}_2 = b_\psi \sin x, \quad \tilde{\theta}_2 = b_\theta \cos x, \quad \tilde{\phi}_2 = \frac{b_\phi}{\tau} \cos x,$$

where

$$(15b) \quad b_\psi \equiv b(\zeta, T), \quad b_\theta \equiv -\left(\frac{\partial \hat{a}}{\partial T} + \hat{a}\right) + b(\zeta, T), \quad b_\phi \equiv -\left(\frac{1}{\tau} \frac{\partial \hat{a}}{\partial T} + \hat{a}\right) + b(\zeta, T).$$

Here $b(\zeta, T)$ is as yet arbitrary. At $O(\epsilon^3)$ the mean fields enter:

$$(16a) \quad \frac{\partial \bar{\theta}_0}{\partial T} + \frac{1}{2} \frac{\partial}{\partial \zeta} (\hat{a}^2 \sin^2 \zeta) = \frac{\partial^2 \bar{\theta}_0}{\partial \zeta^2}$$

$$(16b) \quad \frac{\partial \bar{\phi}_0}{\partial T} + \frac{1}{2\tau} \frac{\partial}{\partial \zeta} (\hat{a}^2 \sin^2 \zeta) = \tau \frac{\partial^2 \bar{\phi}_0}{\partial \zeta^2}.$$

These equations have the solution

$$(17) \quad \bar{\theta}_0 = -d(T) \sin 2\zeta, \quad \bar{\phi}_0 = -c(T) \sin 2\zeta,$$

where

$$(18a) \quad c_T - \frac{1}{2\tau} \hat{a}^2 = -4\tau c$$

$$(18b) \quad d_T - \frac{1}{2} \hat{a}^2 = -4d.$$

Finally, at $O(\epsilon^4)$ the solvability condition for the fluctuating terms yields an expression for $b(\zeta, T)$ and the desired evolution equation for \hat{a} :

$$(19) \quad \hat{a}_{TT} \left[\frac{R_{S,CT}}{\tau^3} - R_{T,CT} \right] + \hat{a}_T \left[2 \left(\frac{R_{S,CT}}{\tau^2} - R_{T,CT} \right) + \frac{1}{\sigma} + R_{T,2} - \frac{R_{S,2}}{\tau^2} \right] \\ + \hat{a} \left[R_{T,2} - \frac{R_{S,2}}{\tau} - R_{T,4} + \frac{R_{S,4}}{\tau} \right] + R_{T,CT} \hat{a} d - \frac{R_{S,CT}}{\tau} \hat{a} c = 0.$$

Note that this is a second order equation. This is because of the proximity to the codimension-two point (9). Note also that even with the restriction $R_{T,2} - R_{S,2}/\tau = 3$ (see eq. (14)) one is able to vary the coefficients of \hat{a} and \hat{a}_T independently to unfold the bifurcation. Suitably scaled the final equations become

$$(20a) \quad a'' + \mu a' + \lambda a - ac + \kappa ad = 0$$

$$(20b) \quad c' = -\tau c + \frac{1}{\tau} a^2$$

$$(20c) \quad d' = -d + a^2,$$

where $\kappa \equiv (\sigma + \tau)/\tau(1 + \sigma)$ and the prime denotes differentiation with respect to T .

Equations (20) provide an asymptotic description of the p.d.e.'s in the sense that higher order terms vanish in the limit $\epsilon \downarrow 0$. It is possible to derive an even simpler system by

taking in addition the asymptotic limit $\tau \downarrow 0$. For most fluids this is in fact a good approximation. If we let

$$(21a) \quad a = \tau^2 \tilde{a}, \quad c = \tau^2 \tilde{c}, \quad d = \tau^4 \tilde{d},$$

$$(21b) \quad \frac{\partial}{\partial T} = \tau \frac{\partial}{\partial \tilde{T}}, \quad \mu = -\tau \tilde{\mu}, \quad \lambda = \tau^2 \tilde{\lambda},$$

we find that, as $\tau \downarrow 0$, the mode d decouples, and we are left with the system (dropping tildes):

$$(22a) \quad a_{TT} - \mu a_T + \lambda a = ac$$

$$(22b) \quad c_T + c = a^2.$$

We call this third order system the *canonical* system. It is exact in the limit $\varepsilon \downarrow 0, \tau \downarrow 0$ and is the smallest possible system that can describe Shil'nikov dynamics. Its properties are summarized in Figure 3.

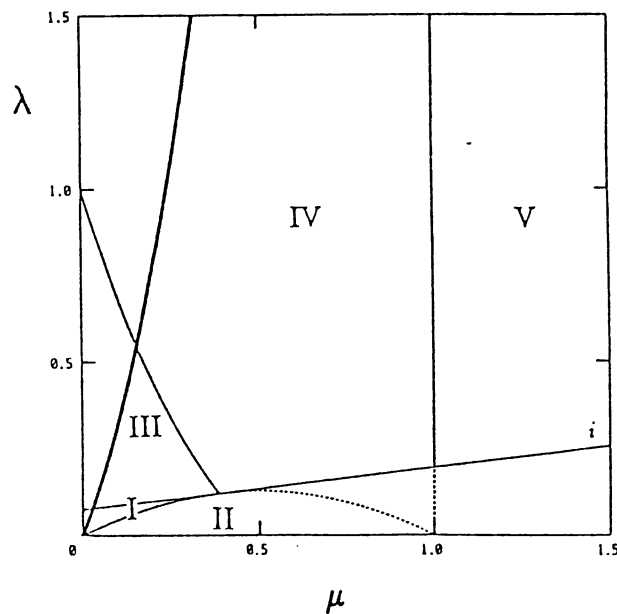


Figure 3. Eigenvalues at the non-stable fixed points of (22). Eigenvalues are real in regions I and II and complex elsewhere. The heavy curve shows the location of the heteroclinic bifurcation. Stable chaos is expected in region IV. (After Proctor & Weiss [21].)

The figure shows in the (μ, λ) plane the lines $\delta = \frac{1}{2}, 1$ as well as the line where the nontrivial fixed points have equal negative eigenvalues, all computed analytically. The locus of the heteroclinic connection was determined numerically and crosses the line $\delta = 1$ at $(\mu, \lambda) \simeq (0.15, 0.54)$. Consequently we expect the Shil'nikov dynamics to be present in region IV, and this is in accord with numerical integration of the system.

The canonical system can also be derived from the Galerkin expansion (11). These equations become exact at small amplitudes and in particular reduce to the Takens-Bogdanov normal form (10) in the neighborhood of the codimension-two point (9). For small τ these equations also simplify. Let $\hat{t} = \tau t$ be a slow time and

$$(23) \quad r = 1 + \mu\tau, \quad s = \tau^2\nu, \quad a = \tau\hat{a}, \quad b = \tau\hat{b}, \quad c = \tau^2\hat{c}.$$

Then $\hat{b} = \hat{a} - \tau\hat{a}' + O(\tau^2)$, and equations (11) reduce to [15]

$$(24a) \quad a' = ra - sd + O(\tau)$$

$$(24b) \quad d' = -d + a(1 - e) + O(\tau)$$

$$(24c) \quad e' = -\varpi e + \varpi ad + O(\tau),$$

where

$$r = \frac{\sigma\mu}{1 + \sigma}, \quad s = \frac{\sigma\nu}{1 + \sigma},$$

a replaces \hat{a} and the prime denotes differentiation with respect to \hat{t} . These equations can be transformed into the Lorenz equations

$$(25a) \quad x' = \tilde{\sigma}(y - x)$$

$$(25b) \quad y' = \tilde{r}x - y - xz$$

$$(25c) \quad z' = -\varpi z + xy,$$

where $\tilde{\sigma} = -r < 0$, and $\tilde{r} = s/r$. When $\tilde{\sigma} < 0$ these equations behave quite differently from the régime described by Sparrow [23], and in particular contain Shil'nikov dynamics [4]. However, like equations (11), equations (24) or (25) are truncations and not rational approximations to the partial differential equations. Only in the limit $\varpi \downarrow 0$, obtained by defining the superslow time $t' = \varpi\hat{t}$ and setting

$$(26) \quad r = 1 + \mu\varpi, \quad s - r = \lambda\varpi^2, \quad a = \varpi\tilde{a}, \quad d = \varpi b, \quad e = \varpi^2 c,$$

does one recover the canonical system (22) and hence obtains an asymptotic approximation to the p.d.e.'s. The third order system (24) does, however, provide a useful *model* of the p.d.e.'s when $\varpi = 0(1)$. Note that this argument shows that the limits $\varpi \downarrow 0$ and $\tau \downarrow 0$ commute.

Equation (26) shows that the canonical system is valid in an $O(\varpi^2)$ neighborhood of the line $r = s$, i.e., the locus of the pitchfork bifurcation from the trivial solution. Hence the Hopf bifurcation from the trivial solution occurring along $r = 1, s > 1$, is not described by the canonical system unless $s - 1 = O(\varpi^2)$. To describe what happens for

fixed $s - 1 = O(\varpi)$ as r is varied it is necessary to use a different scaling. In contrast to (26) let $\tilde{t} = \varpi^{1/2}\hat{t}$ and

$$(27) \quad r = 1 + \mu\varpi, \quad s = 1 + \nu\varpi, \quad a = \varpi^{1/2}\tilde{a}, \quad d = \varpi^{1/2}\tilde{d}, \quad e = \varpi\tilde{e}.$$

Dropping the tildes one now finds

$$(28a) \quad a_{tt} + (\nu - \mu - e)a = O(\varpi^{1/2})$$

$$(28b) \quad e_t = \varpi^{1/2}(-e + a^2) + O(\varpi).$$

Hence

$$(29) \quad a = A \sin \Omega t, \quad \Omega^2 = \nu - \mu - e,$$

with the amplitude A and frequency Ω evolving on the time scale $T = \varpi^{1/2}t = \varpi\hat{t}$ according to

$$(30a) \quad (A^2)_T = \left(\mu - \frac{1}{2}\right)A^2 + (\nu - \mu)(A^2/2\Omega^2) - (3A^4/8\Omega^4)$$

$$(30b) \quad (\Omega^2)_T = \nu - \mu - \Omega^2 - (A^2/2\Omega^2).$$

(see [15]). The periodic orbits are therefore given by

$$(31) \quad A^2 = \frac{8\mu(\nu - \mu)}{1 + 4\mu}, \quad \Omega^2 = \frac{\nu - \mu}{1 + 4\mu};$$

they appear in the primary Hopf bifurcation at $\mu = 0$ and persist until $\nu = \mu$ (i.e., $s - r = O(\varpi^2)$) where their frequency (and amplitude) vanishes. This description thus captures the behavior between the Hopf bifurcation and the Shil'nikov dynamics described by equation (22).

It is possible to go the other way as well. Consider the canonical system in the limit $\lambda = (\nu - \mu)/\varpi \rightarrow \infty$. Let $t' = t/\lambda^{1/2}$ be the slow time and

$$(32) \quad a = \lambda^{1/2}a', \quad c = \lambda c'.$$

After dropping primes equations (22) become

$$(33a) \quad a_{tt} + a(1 - e) = \lambda^{-1/2}\mu a_t$$

$$(33b) \quad e_t = \lambda^{-1/2}(-e + a^2).$$

Hence both $E \equiv a_t^2 + a^2(1 - e)$ and e vary slowly. With the superslow time $T = t/\lambda^{1/2} = t/\lambda$ we obtain the following averaged equations:

$$(34a) \quad E_T = 2\mu\langle a_t^2 \rangle - \langle a^4 \rangle + e\langle a^2 \rangle$$

$$(34b) \quad e_T = -e + \langle a^2 \rangle.$$

Hence

$$(35) \quad a = A \sin \omega t, \quad \omega^2 = 1 - e,$$

where

$$(36a) \quad E_T = \mu E - (3E^2/8\omega^4) + (1 - \omega^2)(E/2\omega^2)$$

$$(36b) \quad (\omega^2)_T = 1 - \omega^2 - (E/2\omega^2).$$

The limit cycle amplitude and frequency therefore satisfy

$$(37) \quad A^2 = \frac{8\mu}{1 + 4\mu}, \quad \omega^2 = \frac{1}{1 + 4\mu}.$$

It can now be checked that (31) and (37) agree if they are written in unscaled variables. Thus the outer limit of (22) matches onto the solution (29)-(31). We have therefore a complete description of the evolution of the oscillations with r for $s - 1 = O(\varpi)$. Figure 4 shows the variation of the period P with $r - 1$ for the two solutions, as given by (31) and (37), with $\varpi = 0.01$, and $s - 1 = 0.002$. The mismatch is $O(\varpi^{1/2})$, as expected.

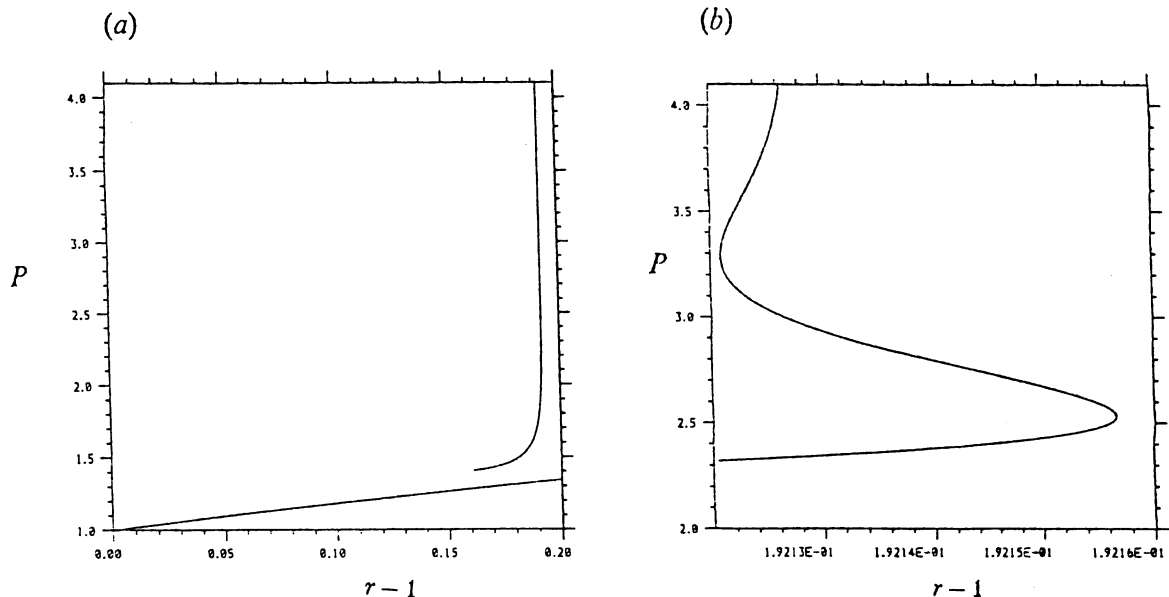


Figure 4. Period P as a function of $(r - 1)$ for $\omega = 0.01$, $s - 1 = 0.002$. (a) Inner and outer expansions given by (31) and (37). (b) Detail of behavior showing wiggly approach to heteroclinicity. (From Knobloch et al [15].)

III. Discussion. The analysis described above has been confined to oscillations in the form of standing waves that have the point symmetry (6). If this symmetry is relaxed one finds that typically the symmetric standing waves (hereafter SW) lose stability with increasing R_T to asymmetric SW. These secondary instabilities have been studied by Moore et al [19] for the p.d.e.'s and for a truncated Galerkin expansion describing magnetoconvection by Nagata et al [20]. Changing the boundary conditions from (5f) to the periodic boundary conditions

$$(38) \quad \psi(x + 2\Lambda) = \psi(x), \quad \theta(x + 2\Lambda) = \theta(x), \quad \phi(x + 2\Lambda) = \phi(x)$$

has more dramatic consequences. In particular it introduces the symmetry group $O(2)$ of rotations and reflections of a circle into the problem. Since the primary instability breaks this symmetry the multiplicity of the pure imaginary eigenvalues at $R_T^{(o)}$ is doubled. As a consequence *two* branches of nontrivial oscillations bifurcate simultaneously from the trivial solution. These are the SW already described together with a branch of spatially periodic travelling waves (hereafter TW). In addition the SW, like the steady states SS, are no longer isolated; instead there is a circle of both, obtained by applying translations (mod 2Λ) to any one solution. With the boundary conditions (5e) at top and bottom the SW are almost always unstable to travelling wave disturbances and evolve into either left- or right-travelling waves [3,10]. With increasing R_T these waves also undergo secondary bifurcations. Near the Takens-Bogdanov bifurcation the TW must lose stability at a secondary Hopf bifurcation to modulated travelling waves (hereafter MW) before terminating on the unstable SS branch [2]. These MW terminate in a homoclinic bifurcation by colliding with the TW 1-torus. This result holds for the p.d.e.'s since it is derived from the normal form for the Takens-Bogdanov bifurcation with $O(2)$ symmetry. Further away from this bifurcation a Galerkin truncation using 15 (real) modes carried out for binary fluid convection shows that the MW terminates by becoming heteroclinic to the circles of SS and SW solutions [11]. Under certain conditions the MW 2-torus may undergo a cascade of torus-doubling bifurcations [12] leading to chaotic travelling waves. Whether this scenario provides an explanation for the chaotic TW described by Deane et al [3] remains to be seen.

Three remarks are in order. In the presence of the $O(2)$ symmetry the bifurcations from SW that break the reflection (6) lead to drifting SW (i.e. to MW). The Galerkin expansions are not as good at describing the TW as they are for SW. This is because the reflection in a vertical plane takes a left-travelling wave into a right-travelling one and hence does not force the existence of a solution with reflection symmetry. This is in contrast to SW or SS for which a reflection-symmetric solution always exists. Consequently, Galerkin truncations that are suitable for SW or SS impose an unphysical symmetry on the TW and hence, for a given truncation, will describe the TW less accurately [9]. This difficulty is easily overcome by enlarging the space of basis functions to include both $\sin n\pi x/\Lambda$ and $\cos n\pi x/\Lambda$. An additional difficulty arises with the boundary conditions (5e) at top and

bottom. With these the analogue of the truncation (11) takes the form

$$\begin{aligned}
(39a) \quad & \dot{a} = \sigma[-a + r_T b - r_S d] \\
(39b) \quad & \dot{b} = -b + a(1 - c) \\
(39c) \quad & \dot{c} = \varpi[-c + \frac{1}{2}(a\bar{b} + \bar{a}b)] \\
(39d) \quad & \dot{d} = -\tau d + a(1 - e) \\
(39e) \quad & \dot{e} = \varpi[-\tau e + \frac{1}{2}(a\bar{d} + \bar{a}d)],
\end{aligned}$$

where a, b, d are complex and c, e are real. Within this system the TW take the form

$$(40) \quad (a, b, c, d, e) = (a_0 e^{i\omega t}, b_0 e^{i\omega t}, c_0, d_0 e^{i\omega t}, e_0),$$

but exist at $r_T = r_T^{(0)}$ only. This degeneracy can be anticipated from the small amplitude theory [10] and implies that a larger number of modes is required in order that the TW branch be nondegenerate. Note finally that the limit $\tau \downarrow 0$ of the system (39) yields a system analogous to (24) but one in which the TW remain degenerate. Neither this system nor the complex version of (22) serve as useful models of $O(2)$ -equivariant dynamics.

The procedure we have advocated for constructing Galerkin truncations utilizes all the modes that are generated to a particular order in perturbation theory around primary bifurcation points (e.g. [1,11]). The resulting truncation becomes *exact* at small amplitudes. The systems (11) and (39) provide simple examples. It is likely that the construction of approximate inertial manifolds will allow us to construct excellent (non-flat) Galerkin truncations of significantly lower order than those using the linear eigenfunctions. A particularly striking application of this technique can be found in Jolly et al [8] where a third order system (with six slaved modes) reproduces accurately the dynamics of the Kuramoto-Sivashinsky equation over a significant parameter range, including details of Shil'nikov dynamics. As mentioned above, care must be taken in any procedure not to force symmetries on solutions that do not possess them.

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