SOME FINITELY ADDITIVE PROBABILITY

by

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Abstract

In their book <u>How to Gamble if You Must</u>, Lester E. Dubins and Leonard J. Savage showed how to define a large family of finitely additive probability measures on infinite product spaces. The probabilities they considered were defined on an algebra of sets rich enough for their purposes but not including many of the sets which occur in the usual statements of such probabilistic limit laws as the martingale convergence theorem. In some unpublished notes Dubins and Savage also conjectured that there might be a natural way to extend their measures and proceeded to carry out one step of an extension. We show here it is possible to make a further natural extension to the sigma-field generated by the original algebra of Dubins and Savage. It is then possible to state and prove many of the classical countably additive limit theorems in this finitely additive setting. If assumptions of countable additivity are imposed, the extension studied here, when restricted to the usual product sigma-field, agrees with the conventional extension.

1. Introduction.

Let γ be a finitely additive probability defined on all subsets of the set N of positive integers. (The main body of the paper will consider probabilities on an arbitrary set.) If γ is countably additive, it is well known that there exists a unique countably additive probability which assigns to each subset of $N^N (= N \times N \times \ldots)$ of the form

$$A^1 \times A^2 \times \ldots \times A^j \times N \times N \times \ldots, \qquad j \in N,\ A^i \subseteq N,$$

the probability $\prod_{i=1}^{j} \gamma(A_i)$, and whose domain is the sigma-algebra generated by these sets. Is there a counterpart to this product measure theorem in the case that γ is not countably additive? In the first place, it is easy to see what probabilities should be assigned to the subsets of N^N which depend on only finitely many coordinates. However, once there, the conventional methods, relying as they do on the countable additivity of γ, give no indication as to what the values of the measure should be on a wider class of sets. In much greater generality this problem has already been considered by Lester Dubins and the late Leonard Savage in their book <u>How to Gamble if You Must</u>, (1965). In order to surmount the apparent arbitrariness involved in the extension, Dubins and Savage require that a certain natural condition be satisfied, which for the special case being considered here, reduces to the following:

(1) $$\pi(D) = \int_N \pi(Dx)d\gamma(x) .$$

Here π is the extension-to-be, $D \subseteq N^N$, $Dx = \{z \in N^N | (x, z_1, z_2, \ldots) \in D\}$. For reasons given in Dubins and Savage (1965, pp. 12 - 20), but too lengthy to present here, it is natural to ask that (1) hold for all sets D which are clopen (simultaneously closed and open) in the product topology on N^N determined by assigning N the discrete topology. Then, although they do not do this directly, their method can easily be adapted to show that there is exactly one finitely additive probability π which is defined on the clopen subsets of N^N and which satisfies (1) for all clopen D.

To compare this situation with the countably additive one described in the opening paragraph, note that the collection of clopen sets in N^N includes properly the collection of sets which depend on finitely many co-ordinates. However, the clopen sets form a much smaller class than the domain of the conventional product measure. In fact, the latter coincides with the sigma-field generated by the former. Is it possible to extend π in some natural way to a larger collection of sets? Dubins and Savage posed this question, again in greater generality, in some unpublished notes written in the fall of 1962 (For a relevant quotation from these notes, see Dubins (1973a).) In the same notes they proceeded to begin to answer it by assigning to each open set the supremum of the measures of the clopen sets contained within it, and then showing that the resulting extension, which in this particular instance might be called π_*, satisfied $\pi_*(O \cup P) + \pi_*(O \cap P) = \pi_*(O) + \pi_*(P)$ for all open sets O, P.

This is the point of departure of our efforts. We were priviliged to see these notes and were immediately tempted by the possibility, suggested in the notes by Dubins and Savage, of even further extension. A time honored first step in such a situation is to form the collection $\mathcal{G}$ of all sets which can be approximated from without by an open set and within by a closed set in such a way that the measure of their set-theoretic difference, which is open, can be made arbitrarily small. It is clear how the extension should be defined on these sets.

It is not difficult, in fact only a matter of verifying that typical measure-theoretic arguments suffice, to show that $\mathcal{G}$ is an algebra of sets which contains the open sets, and the proposed extension is finitely additive on $\mathcal{G}$. This is part of the content of Theorem 2.1 below. The next question is: how large is this algebra? For example, does it contain the sigma-field generated by the open sets? We found this question difficult, even in such a special case, and the results to follow come from our attempt to answer it.

Our answer, given in somewhat greater generality in Theorem 5.1 below, is yes. This theorem makes it possible to state finitely additive counterparts of such classical limit theorems of probability as the strong law of large numbers and the martingale convergence theorem. (Incidentally, a recent paper of Lester Dubins (1974) contains some pointed remarks concerning the merits of the various ways of formulating a limit law). Many of these finitely additive limit theorems (see Theorems 7.3 and 10.2 for example) can then be proved by applying well-known arguments in conjunction with the stop rule methods devised to establish Theorem 5.1. However, some results (e.g., the Levy 0 - 1 law which is Theorem 8.1) do seem to require additional effort. Finally, if γ is countably additive, the extension of π considered here coincides with the familiar countably additive product measure (see section 11 for a more comprehensive result) and so the methods of extension described here are consistent with the conventional ones.

2. Basic Framework.

Throughout, probabilities and probability measures are finitely additive unless explicitly stated otherwise. Let N be the positive integers and X an arbitrary non-empty set. Let $H = X^N = X \times X \times \ldots$ and give H the product topology determined by assigning X the discrete topology. Subsets of H which are simultaneously closed and open in this topolgy will be referred to as clopen. (Strictly speaking, any of the topology in what follows is logically dispensable, and perhaps even a little distracting. But it does offer the convenience of familiarity.)

The following theorem defines the particular extension of a finitely additive probability which will be studied in this paper. There are two novel ideas in Theorem 1, both due to Dubins and Savage. The first is the notion of extending from the clopen sets to the open sets by assigning to each open set the supremum of the measures of the clopen sets contained within. The second is the invoking of a special relation between these two collections of sets in order to show that the extension remains a probability. See Corollary 3.1 for a statement of this relation. A thorough account of its significance is given by Lester Dubins in Dubins (1973a). In particular Theorem 1 is more or less rendered redundant by Dubins (1973a). First, Dubins shows there how to extend to open sets, and then, as he points out, a general extension theorem applies to yield a unique extension from the lattice of open sets to the algebra it generates. All that remains, then, is to make the completion to obtain the extension we will be considering. However, the extension theorem is not as easy in general as it is in this special case, so it seemed worthwhile to us to present a self-contained account of the extension in the form of Theorem 1. We will not verify here that the extension of Theorem 1 coincides with that suggested by Dubins.

Theorem 1.

Let μ be a finitely additive probability defined on the clopen subsets of H. Then there is a unique finitely additive probability λ such that

(i) the domain of λ is an algebra $\mathcal{A}$ of sets containing the open sets;

(ii) λ extends μ;

(iii) If O is open and $\delta > 0$, there is a clopen set K such that $K \subseteq O$ and $\lambda(K) > \lambda(O) - \delta$;

(iv) $A \in \mathcal{A}$ if and only if, for every positive ϵ, there are O open, C closed such that $C \subseteq A \subseteq O$ and $\lambda(O - C) < \epsilon$.

The proof is given in Section 3. The only preliminaries needed to read Section 3 are the definitions of stop rule, and incomplete stop rule, given below.

At this point it is reasonable to inquire, for example, whether or not the $\mathcal{A}$ of Theorem 1 contains the Borel sets (sigma-field generated by the open sets). This is not so in general and an example to this effect is given in the next section. The next three paragraphs introduce a class of probabilities, the "probabilities determined by strategies," for which we have shown it does. This class, first considered by Dubins and Savage, is essential to our proof, which typically involves working with all of its members simultaneously.

Let X^* be the set of all finite sequences of members of X, including the empty one. A strategy σ is a function which assigns to each $p \in X^*$ a probability measure $\sigma(p)$, defined on all subsets of X. The probability assigned by σ to the empty sequence will be denoted σ_0. Informally, a strategy generates a chance sequence $(x_1, x_2, x_3, \ldots)$ of members of X in the following manner: let x_1 be chosen at random according to σ_0, let x_2 be chosen according to $\sigma(x_1)$, x_3 to be chosen according to $\sigma(x_1, x_2)$, and so on. For the special case considered in the introduction, $X = N$ and the corresponding strategy is the (constant) function which assigns γ to all members of N^*.

Before stating the precise sense in which a strategy determines a probability on the clopen subsets of H, a few preliminaries will be required. Let $p, q \in X^*$ and $h \in H$. Then pq is the member of X^* whose terms consist of the terms of p followed by the terms of q, and ph is the member of H whose terms consist of the terms of p followed by the terms of h. If $A \subseteq H$, $Ap = \{h \in H | ph \in A\}$. If w is a function defined on H, wp is defined by $wp: h \to w(ph)$, $h \in H$. If p consists of a single term $x \in X$, Ap will be written Ax and wp will be written wx. If σ is a strategy and $p \in X^*$, $\sigma[p]$ is the <u>conditional strategy</u> defined by $\sigma[p](q) = \sigma(pq)$, all $q \in X^*$. If $p = (x)$, $x \in X$, $\sigma[p]$ will be written $\sigma[x]$. If S is any set 1_S, the indicator of S, is the function which is 1 on S and 0 off S.

For the next few paragraphs only, let $\mathbb{S}$ be the set of all strategies and $\mathbb{C}$ be the set of all bounded functions on H to the real line which are continuous when the latter is endowed with the discrete topology.

(A characterization of $\mathcal{C}$ as the bounded "finitary" functions is presented in Dubins and Savage (1965).) Then there exists a unique real-valued function E, defined on $\mathcal{S} \times \mathcal{C}$, such that for every $(\sigma, x) \in \mathcal{S} \times \mathcal{C}$,

$$(1) \qquad \begin{aligned} &E(\sigma, c) = c, \quad \text{for every constant } c, \\ &E(\sigma, w) = \int E(\sigma[x], wx)\, d\sigma_0(x) . \end{aligned}$$

Further, for each $\sigma \in \mathcal{S}$, the function $w \to E(\sigma, w)$, $w \in \mathcal{C}$, is a positive linear functional on $\mathcal{C}$. Then the <u>probability determined by a strategy</u> σ is defined to be the set function $K \to E(\sigma, 1_K)$, K clopen.

The proof of the claims of the preceding paragraph, given in Section 2.8 of Dubins and Savage (1965), is a transfinite recursion argument which turns on the fact that $\mathcal{C}$ can be arranged in an ordinally indexed hierarchy in such a way that wx, for $x \in X$, is always "below" w in the hierarchy, for all nonconstant $w \in \mathcal{C}$. The idea of the inductive step, is that once E has been defined for all pairs of the form (σ, wx) it can be extended up the hierarchy to (σ, w) by using (1). A precise definition of this hierarchy will not be presented here. It appears in Section 2.7 of Dubins and Savage. An understanding of the contents of this section of their book, especially the notion of the structure of a function in $\mathcal{C}$, will be required in order to follow some of the main arguments of this paper. In particular, "structure of a clopen set K" will be used here to refer to the structure of 1_K.

To digress briefly, the process of defining E amounts, loosely, to extending all strategies simultaneously to linear functionals on $\mathcal{C}$. However, it is not necessary, only convenient, to extend this far in order to define probabilities uniquely on the clopen subsets of H. Nor

is it necessary to work with the set of all strategies. The class $\mathfrak{S}$ may be replaced by any $\mathfrak{R} \subseteq \mathfrak{S}$ provided it has the property that $\sigma \in \mathfrak{R}$ and $x \in X$ implies $\sigma[x] \in \mathfrak{R}$. The existence of E then takes the following more modest form. There is a unique function $m_{\mathfrak{R}}$ defined for all (σ, K) with $\sigma \in \mathfrak{R}$, K clopen such that

$$m_{\mathfrak{R}}(\sigma, \phi) = 0, \qquad m_{\mathfrak{R}}(\sigma, H) = 1,$$

$$m_{\mathfrak{R}}(\sigma, K) = \int m_{\mathfrak{R}}(\sigma[x], Kx)d\sigma_0(x),$$

for all $\sigma \in \mathfrak{R}$, K clopen. Further, for each fixed σ, the function $K \to m_{\mathfrak{R}}(\sigma, K)$, K clopen, is a probability. The proof is essentially the same as that for E. For an example, let $X = N$, γ be a probability on N, and $\mathfrak{R}$ have as its sole member the strategy which assigns γ to all members of N^*. Then $m_{\mathfrak{R}}$ gives the extension π claimed to exist in the introduction. Finally, the $m_{\mathfrak{R}}$ are all consistent with E: $m_{\mathfrak{R}}(\sigma, K) = E(\sigma, 1_K)$ for $\sigma \in \mathfrak{R}$, K clopen, and any $\mathfrak{R}$ (such that $\sigma \in \mathfrak{R}$, $x \in X$ implies $\sigma[x] \in \mathfrak{R}$).

If σ is a strategy, it determines, as just indicated the probability μ: $K \to E(\sigma, 1_K)$, K clopen. For this μ, let $\mathfrak{A}(\sigma)$ be the algebra determined by Theorem 1, and with some harmless ambiguity, let σ be the probability λ determined by the same theorem. This convention will be in force from now on.

A <u>stop rule</u> is a function $r: H \to N$ such that if h, h' belong to H and $h_i = h'_i$ $i = 1,\dots, r(h)$, then $r(h) = r(h')$. A set $K \subseteq H$ is said to be <u>determined by time</u> r, provided that $h \in K$, $h' \in H$ and $h_i = h'_i$, $i = 1,\dots, r(h)$ implies $h' \in K$. It is shown in Dubins and

Savage (1965) that the clopen sets are exactly those sets which are determined by time r for some stop rule r. A sequence of stop rules $r_1, r_2, \ldots$ is said to be <u>strictly increasing</u> provided $r_1(h) < r_2(h) < \ldots$, for all $h \in H$.

An <u>incomplete stop rule</u> is a function $t: H \to N \cup \{\infty\}$ such that if $t(h) < \infty$ and $h_i = h'_i$, $i = 1,\ldots, t(h)$, then $t(h) = t(h')$. If t is an incomplete stop rule, the set $[t < \infty](= \{h \in H | t(h) < \infty\})$ is open. Conversely, if O is open there is an incomplete stop rule t such that $O = [t < \infty]$. One such t, the <u>minimal incomplete stop rule associated with</u> O, is defined by taking $t(h)$ to be the least k (if any) such that if $h' \in H$ and $h'_i = h_i$, $i = 1,\ldots, k$, then $h' \in O$; if no such k exists, $t(h) = \infty$.

There is a basic integration formula involving stop rules which will often be called upon. To render it more readable, the value of E at (σ, w) will be denoted $\int w(h)d\sigma(h)$ or σw. Let $p \in X^*$. We will use the suggestive notation $\sigma(g|p)$ in place of $\sigma[p](gp)$ and $\sigma(K|p)$ for $\sigma[p](Kp)$. Also, set $p_n(h) = (h_1,\ldots, h_n)$ for $h \in H$, $n \in N$; and, if s is a stop rule set $p_s(h) = p_n(h)$ where $n = s(h)$. Then if σ is a strategy

$$\begin{aligned} \sigma w &= \int \sigma(w | p_s(h)) d\sigma(h) \\ \sigma(K) &= \int \sigma(K | p_s(h)) d\sigma(h) \end{aligned} \tag{2}$$

for all w in $\mathcal{C}$ and clopen K. The special case of (2) obtained by taking $s \equiv 1$, after a standard change of variable, is just the condition (1). The formula (2) is proved from this special case by induction on the structure of p_s. A slightly more general version of (2) is formula 3.7.1 in Dubins and Savage (1965).

Savage (1965) that the player acts one at a time. These acts which are determined by time t. [illegible] A sequence of stop rules $t_1, t_2, \ldots$ is said to be strictly increasing provided $t_1(h) < t_2(h) < \ldots$ for all $h \in H$.

An incomplete stop rule is a function $t: H \to N \cup \{\infty\}$ such that if $t(h) < \infty$ and $h'_k = h_k$, $k = 1, \ldots, t(h)$, then $t(h) = t(h')$. If t is an incomplete stop rule, the set $[t < \infty] (= \{h \in H: t(h) < \infty\})$ is open. Conversely, if O is open there is an incomplete stop rule t such that $O = [t < \infty]$. One such t is the minimal incomplete stop rule associated with O. It is defined by taking $t(h)$ to be the least n (if any) such that if $h' \in H$ and $h'_k = h_k$, $k = 1, \ldots, n$, then $h' \in O$; if no such n exists, $t(h) = \infty$.

There is a basic theorem in [illegible] involving stop rules which will often be called upon. To render it more readable, the value of [illegible] at (t, w) will be denoted $\int [illegible](h) d\sigma(h)$ or [illegible]. [illegible] We will use the strategy notation $u(x, p)$ in place of $u[p](x)$; and [illegible] for $u[p](x)$.

Also, set $p_n(h) = (h_1, \ldots, h_n)$ for $n \in N$, [illegible]; and, if t is a stop rule set $p_t(h) = p_n(h)$ where $n = t(h)$. Then if σ is a strategy

$$(2) \qquad \begin{aligned} \sigma u &= \int \sigma(u \mid p_t(h))\, d\sigma(h) \\ \sigma(A) &= \int \sigma(A \mid p_t(h))\, d\sigma(h) \end{aligned}$$

for all u in [illegible] and stop rules t. The trivial case of (2) obtained by taking $t \equiv 1$, [illegible] is just the condition (1). The formula (2) is proved [illegible] this special case by induction on the structure of t. A slightly more general version of (2) is formula [illegible] in Dubins and Savage (1965):

Finally, the following notation will appear on occasion. If $p \in X^*$, $|p|$ is the number of terms of p; if p is empty, $|p| = 0$. If $p \in X^*$, and $A \subseteq H$, pA is the set of all $h \in H$ with $h = ph'$, for some $h' \in A$.

3. A Proof of Theorem 2.1 and an example.

Before proceeding to the proof, the following lemma and corollaries are required. These are well known in some circles (e.g., logicians) but included here for completeness.

Lemma 1.

Let $O, \check{O}$ be open sets in H. Then there exist $P, \check{P}$ open such that

$$P \subseteq O, \quad \check{P} \subseteq \check{O},$$

$$P \cup \check{P} = O \cup \check{O},$$

and $P, \check{P}$ are disjoint.

Proof:

Let $t, \check{t}$ be the minimal incomplete stop rules associated with $O, \check{O}$ respectively. Set

$$P = [t < \infty,\ t \leq \check{t}], \quad \check{P} = [\check{t} < \infty,\ t > \check{t}].$$

The claimed properties of $P, \check{P}$ are easily verified. □

Corollary 1.

Let $K \subseteq O \cup \check{O}$ where K is clopen and $O, \check{O}$ are open. Then there exist $L, \check{L}$ clopen such that

$$L \subseteq O, \ \check{L} \subseteq \check{O}$$

$$L \cup \check{L} = K .$$

Proof:

Using the $P, \check{P}$ of the preceding lemma, set $L = K \cap P$, $\check{L} = K \cap \check{P}$. □

Corollary 2:

If $C, \check{C}$ are closed and disjoint there exists K clopen with $K \supseteq C$ and K disjoint from $\check{C}$.

Proof:

In Lemma 1, let $O, \check{O}$ be the complements of $C, \check{C}$ respectively. Then take K to be $\check{P}$. □

Proof of Theorem 1:

The uniqueness of λ is easily verified. The proof of existence proceeds in two stages, the first due to Dubins and Savage, the second being conventional measure theory.

$$\eta(O) = \sup\{\mu(K) \mid K \text{ clopen}, K \subseteq O\}$$

for each open set O. Plainly,

$$\eta(O) + \eta(\check{O}) \leq \eta(O \cup \check{O})$$

for $O, \check{O}$ open disjoint. For the other inequality, which does not require that $O, \check{O}$ be disjoint, let $K \subseteq O \cup \check{O}$. Then, using the $L, \check{L}$ of Corollary 1,

$$\mu(K) \leq \mu(L) + \mu(\check{L}) \leq \eta(O) + \eta(\check{O})$$

It follows that η is finitely additive and subadditive on the open subsets of H. This completes the first stage of the extension.

For the second stage, set

$$\eta^*(A) = \inf\{\eta(O) \mid O \text{ open}, O \supseteq A\}$$

for every $A \subseteq H$; and $\mathcal{A}$ to be the collection of all $A \subseteq H$ satisfying:

For every $\epsilon > 0$, there exist O open, C closed with

$$C \subseteq A \subseteq O \quad \text{and} \quad \eta^*(O - C) < \epsilon .$$

Then verify the following, where A, B are arbitrary subsets of H.

(a) If $A \subseteq B$, $\eta^*(A) \leq \eta^*(B)$.

(b) $\eta^*(A \cup B) \leq \eta^*(A) + \eta^*(B)$. This requires the subadditivity of η.

(c) $\mathcal{A}$ is an algebra of sets.

(d) If $A \subseteq B$, $\eta^*(B) - \eta^*(A) \leq \eta^*(B-A)$.

(e) $\eta^*(A) = \sup\{\eta^*(C) \mid C \text{ closed}, \ C \subseteq A\}$.

This follows easily from the definition of $\mathcal{A}$, (d), and (a).

(f) If C, D are closed, disjoint

$$\eta^*(C \cup D) = \eta^*(C) + \eta^*(D) .$$

Given $\epsilon > 0$, there is an O open such that $O \supseteq C \cup D$ and $\eta(O) \leq \eta^*(C \cup D) + \epsilon$. Using Corollary 2 (or even the normality of H) there are open disjoint $Q, \check{Q}$ such that $Q \supseteq C$, $\check{Q} \supseteq D$. Then

$$\eta(O) \geq \eta(O \cap (Q \cup \check{Q})) = \eta(O \cap Q) + \eta(O \cap \check{Q}) .$$

It follows that $\eta^*(C \cup D) \geq \eta^*(C) + \eta^*(D)$.

(g) If A, B belong to $\mathcal{A}$ and are disjoint,

$$\eta^*(A \cup B) = \eta^*(A) + \eta^*(B) .$$

Use (b), (e), (f).

(h) The open sets belong to $\mathcal{A}$. To see this, let $\epsilon > 0$ and O be open. There is a K clopen such that $K \subseteq O$ and $\mu(K) > \eta(O) - \epsilon$. Then verify that $\eta^*(O - K) < \epsilon$.

The theorem now follows by taking λ to be the restriction of η^* to $\mathcal{G}$.

Definition.

If λ is as in Theorem 1,

$$\lambda^*(A) = \inf\{\lambda(O) \mid O \text{ open}, \ O \supseteq A\}$$

$$\lambda_*(A) = \sup\{\lambda(C) \mid C \text{ closed}, \ C \subseteq A\}$$

for all $A \subseteq H$.

Corollary 3.

If λ, $\mathcal{G}$ are as in Theorem 1, $\mathcal{G}$ coincides with the collection of all $A \subseteq H$ such that $\lambda^*(A) = \lambda_*(A)$.

Suppose for this paragraph that X is an infinite set. Then, as will be shown in the remainder of this section, there is a finitely additive probability μ defined on the clopen subsets of H such that μ takes only the values 0 and 1, and if D is any closed nowhere dense set in H, there is a clopen K with $K \supseteq D$ and $\mu(K) = 0$. Such a μ is of interest for two reasons. The first is that the extension λ, as defined by Theorem 2.1, does not include all G_δ's in its domain; the second is that μ cannot be approximated (in a sense to be specified later) by any probability which is determined by a strategy. L. Dubins (1973b) has already given an example of this latter phenomenon. The reasoning will be given following the proof of existence of μ. The latter begins immediately with the statement of two very easy lemmas. In the first, pQ, where $p \in X^*$ and $Q \subseteq H$, is the set of all $h \in H$ with $h = ph'$, for some $h' \in Q$.

Lemma 2. Let $p \in X^*$ and $Q \subseteq H$.

(i) If $A \subseteq Qp$, then $pA \subseteq Q$

(ii) If Q is dense in H, Qp is dense in H

(iii) If Q is open in H, Qp is open in H

(iv) If Q is clopen in H, pQ is clopen in H.

Lemma 3. Let Z be a topological space.

Suppose $(Z_i, i \in I)$ is a family of pairwise disjoint clopen subsets of Z whose union is a clopen subset of Z. Then if $(C_i, i \in I)$ is a family of closed subsets of Z and $C_i \subseteq Z_i$, all $i \in I$, the set $C = \bigcup_{i \in I} C_i$ is a closed subset of Z.

The next two definitions and lemma establish the existence of a function which inserts a clopen set in each open dense set in such a way that finite intersections of the inserted sets are never empty. The notation $J(p) = \{h \in H \mid h_i = p_i, 1 \leq i \leq |p|\}$ will be used. (If p is the empty sequence $J(p) = H$).

Definition. Let O be open dense in H. Let q be any member of X^* such that $J(q) \subseteq O$ and if $J(p) \subseteq O$, $p \in X^*$, then $|p| \geq |q|$. Set $\beta(O) = J(q)$ and $d(O) = |q|$.

There is more freedom in the choice of the function β than is apparent from the above definition. Many other β's will work just as well in what follows.

Definition.

The equations below define inductively, for each positive integer n, a function β_n, which has its domain as the collection of all open dense

Lemma 2. Let $p \in X^*$, and $Q \subseteq H$.

(i) If $A \subseteq Q_p$, then $pA \subseteq Q$.

(ii) If Q is dense in H, Q_p is dense in H.

(iii) If Q is open in H, Q_p is open in H.

(iv) If Q is clopen in H, Q_p is clopen in H.

Lemma _. Let Y be a topological space.

Suppose $(G_i, i \in I)$ is a family of pairwise disjoint clopen subsets of Y whose union is a clopen subset of Y. Then if $(C_i : i \in I)$ is a family of closed subsets of Y and $C_i \subseteq G_i$ for all $i \in I$, the set $C = \bigcup_{i \in I} C_i$ is a closed subset of Y.

The next two definitions and lemma establish the existence of a function which inserts a clopen set in each open dense set in such a way that finite intersections of the inserted sets are never empty. The notation $\bar{d}(p) = \{n \in N : n_i = p_i, 1 \le i \le |p|\}$ will be used. (If p is the empty sequence $\bar{d}(p) = N$.)

Definition. Let O be open dense in H. Let q be any member of X^* such that $f(q) \subseteq O$ and if $f(p) \subseteq O$, $p \in X^*$, then $|p| \ge |q|$. Let $f(O) = f(q)$ and $d(O) = |q|$.

There is more freedom in the choice of the function f than is apparent from the above definition. Many other f's will work just as well in [illegible] what follows.

Definition.

The functions below define inductively, for each positive integer n, a function f_n which has its domain as the collection of all open dense subsets of H and which takes as values subsets of H. The inductive step requires Lemma 2(ii)

$$\beta_1 = \beta$$

$$\beta_{n+1}(O) = \beta(O) \cup \left(\bigcup_{|p| \leq k} p\beta_n(Op) \right) ,$$

where O is open dense, $k = d(O)$, $p \in X^*$.

<u>Lemma 4</u>. Let n be a positive integer.

Then, if $O, O^1, \ldots, O^n$ are any open dense sets in H,

(a) $\beta_n(O)$ is a clopen set in H,

(b) $\beta_n(O) \subseteq O$,

(c) $\beta_n(O^1) \cap \ldots \cap \beta_n(O^n)$ is non-empty.

<u>Proof</u>:

This is done by induction on n. If $n = 1$, the lemma is immediate from the definition of β. Now suppose it holds for a positive integer n. To verify (a) for β_{n+1}, let O be open dense and set

$$Q_j = \bigcup_{|p|=j} p\beta_n(Op)$$

for each $j = 0, \ldots, d(O)$. Now $\beta_n(Op)$ is clopen by lemma 2(ii) and the inductive hypothesis. Next $p\beta_n(Op)$ is clopen by lemma 2 (iv) and since $p\beta_n(Op) \subseteq J(p)$ for all $p \in X^*$, lemma 3 applies with $I = \{p \in X^* | \ |p| = j\}$ and $Z_p = J(p)$ all $p \in I$, to show that Q_j is closed. It is also open, so Q_j is clopen, $j = 0, \ldots, d(O)$. Since $\beta_{n+1}(O)$ is the union of $\beta(O)$ together with the Q_j's, $\beta_{n+1}(O)$ is clopen.

subsets of N and which takes as values subsets of N. The inductive step [illegible] Lemma 2(i)

$$p_n = \emptyset$$

$$f_{n+1}(O) = f(O) \cup \left(\bigcup_{|p|=n} pf_n(O_p)\right),$$

where O is open dense, $k = d(O)$, $p \in K^*$.

Lemma 3. Let n be a positive integer.

Then if $O, O^1, \ldots, O^n$ are any open dense sets in N,

(a) $f_n(O)$ is a closed set in N.

(b) $f_n(O) \subseteq O$.

(c) $f_n(O^1) \cap \ldots \cap f_n(O^n)$ is non-empty.

Proof:

This is done by induction on n. If $n = 1$, the result is immediate from the definition of f_1. Now suppose it holds for a positive integer n. We verify (a) for f_{n+1}. Let O be open dense and set

$$Q_i = \bigcup_{|p|=i} pf_n(O_p)$$

for each $i = 0, \ldots, d(O)$. Now $f_n(O_p)$ is closed by Lemma 2(ii) and the inductive hypothesis. Hence $pf_n(O_p)$ is closed by Lemma 2 (iv) and since $pf_n(O_p) \subseteq N(p)$ for all $p \in K^*$, Lemma [illegible] applies with $I = \{p \in K^* : |p| = i\}$ and $A_p = N(p)$ all $p \in I$, to show that Q_i is closed. It is also open, so Q_i is clopen, $i = 0, \ldots, d(O)$. Thus $f_{n+1}(O)$ is the union of $f(O)$ together with the Q_i's, $f_{n+1}(O)$ is clopen.

To verify (b), note that the inductive hypothesis and lemma 2(ii) together imply that $\beta_n(Op) \subseteq Op$, so that, by lemma 2(i), $p\beta_n(Op) \subseteq O$. This is so for any $p \in X^*$, so $\beta_{n+1}(O) \subseteq O$.

To verify (c), let $O^1, \ldots, O^{n+1}$ be open dense in H. Assume that $d(O^1) \leq d(O^i)$ for $i = 2, \ldots, n+1$, and set $\beta(O^1) = J(q)$, where $q \in X^*$. Since $|q| = d(O^1) \leq d(O^i)$, the set $\beta_{n+1}(O^i)$ includes $q\beta_n(O^i q)$ as a subset, $i = 2, \ldots, n+1$. By the induction hypothesis, $\beta_n(O^2 q), \ldots, \beta_n(O^{n+1} q)$ have a non-empty intersection. Therefore the sets $q\beta_n(O^2 q), \ldots, q\beta_n(O^{n+1} q)$ have a non-empty intersection, which is also a subset of $J(q)$. As the intersection of $\beta_{n+1}(O^2), \ldots, \beta_{n+1}(O^{n+1})$ includes this subset, and $\beta(O^1) = J(q)$, the proof of the inductive step is complete. □

<u>Corollary 4</u>:

If X is an infinite set, there is a function β which assigns to each open dense set in H a clopen set in H in such a way that if O is any open dense set, $\beta(O) \subseteq O$; and if n is any positive integer and $O^1, \ldots, O^n$ are any open dense sets, the sets $\beta(O^1), \ldots, \beta(O^n)$ have a non-empty intersection.

<u>Proof</u>:

Let $w_1, w_2, \ldots$ be an infinite sequence of distinct members of X^* with $|w_i| = 1$ all $i \in N$. If O is open dense, set

$$\beta(O) = \bigcup_{i \in N} w_i \beta_i(O w_i)$$

where β_i is defined in lemma 4. The set Ow_i is in the domain of β_i by lemma 2(ii).

It is now easy to check that β has the desired properties. For example, if $O^1, \ldots, O^n$ are open dense, the set $\beta(O^1) \cap \ldots \cap \beta(O^n)$ is non-empty essentially because $\beta_n(O^1 w_n) \cap \ldots \cap \beta_n(O^n w_n)$ is. For if h is a member of the latter, $w_n h$ is a member of $w_n \beta_n(O' w_n) \cap \cdots \cap w_n \beta_n(O^n w_n)$, which in turn is a subset of $\beta(O') \cap \ldots \cap \beta(O^n)$. The distinctness of the w_i's has not been used. It is required to quarantee that $\beta(O)$ be closed (as well as open). In lemma 3 let $I = N$ and $Z_i = J(w_i)$, $i \in N$. Then the Z_i's are pairwise disjoint by virtue of the distinctness of the w_i's. $\square$

Corollary 5:

Let X be an infinite set. There is a finitely additive probability μ defined on the clopen subsets of H such that μ only takes on the values 0 and 1, and if C is any closed nowhere dense set in H, there is a clopen set $K \supseteq C$ for which $\mu(K) = 0$.

Proof:

Let $\mathfrak{F}$ be the collection of all clopen sets L such that $L = \beta(O)$ for some open dense set O. Then, since every finite intersection of members of $\mathfrak{F}$ is non-empty, there is an ultrafilter of subsets of H which includes $\mathfrak{F}$ as a subcollection. This ultrafilter determines a finitely additive probability μ on the clopen subsets of H in the usual manner: assign probability one to all clopen sets which belong to the ultrafilter and probability zero to all other clopen sets. Then, from the definition of μ, every open dense set contains a clopen set L with $\mu(L) = 1$. As a set is closed nowhere dense if and only if its complement is open dense, the proof is completed by taking complements.

Returning now to the setting of Theorem 2.1, let λ be the extension of the μ of corollary 5 and let $\mathcal{G}$ be the domain of λ. Fix a member x of X and let $S = \{h \in H \mid h_j = x,$ for all $j \in N$ sufficiently large $\}$. Then S cannot belong to $\mathcal{G}$. First, S is dense in H so that $O \supseteq S$ and O open imply O dense and $\lambda(O) = 1$. Second S has an empty interior so that $C \subseteq S$ and C closed imply C nowhere dense and $\lambda(C) = 0$. Thus $S \notin \mathcal{G}$, and consequently the complement, $H - S$ does not belong to $\mathcal{G}$. The set S is a countable union of closed sets, so $H - S$ is a countable intersection of open sets which does not belong to $\mathcal{G}$. □

The proabability μ cannot be approximated by a strategy: for $\epsilon = 1/2$, say, there is no strategy σ such that $|\mu(K) - \sigma(K)| < \epsilon$ for all clopen K. For, as will be shown in the remark at the end of section 6, given any strategy σ, there is a closed nowhere dense set D such that $K \supseteq D$ and K clopen implies $\sigma(K) \geq 3/4$. But there is a clopen set K such that $K \supseteq D$ and $\mu(K) = 0$. For that K, $|\mu(K) - \sigma(K)| \geq 3/4$.

4. An extension of the Basic Integration Formula.

The result of this section is a straightforward extension of formula (2) of Section 2. It will often be used in the sequel without reference.

Theorem 1.

Let σ be a strategy. For every $A \subseteq H$,

$$\sigma^*(A) = \int \sigma[x]^*(Ax)d\sigma_0(x), \text{ and}$$

$$\sigma_*(A) = \int \sigma[x]_*(Ax)d\sigma_0(x).$$

Proof:

The conclusion holds for clopen sets. It is next established for open sets.

If $K \subseteq O$, $Kx \subseteq Ox$; so that

$$\sigma(O) = \sup \sigma(K) = \sup \int \sigma[x](Kx)d\sigma_0(x) \le \int \sigma[x](Ox)d\sigma_0(x),$$

where the sup is taken over all clopen sets $K \subseteq O$.

For the opposite inequality, let $\epsilon > 0$. If O is open, Ox is open and so, for each $x \in X$ there is a $K(x) \subseteq Ox$ with $K(x)$ clopen and $\sigma[x]K(x) \ge \sigma[x](Ox) - \epsilon$. Set $K = \bigcup_{x \in X} xK(x)$ and check that K is clopen and $Kx = K(x)$, all $x \in X$. Then $K \subseteq O$, and

$$\sigma(O) \ge \sigma(K) = \int \sigma[x](Kx)d\sigma_0(x) \ge \int \sigma[x](Ox)d\sigma_0(x) - \epsilon.$$

The argument just given can be easily adapted now to give the first equation in Theorem 1. The second equation follows from the first together with the fact that $\sigma_*(A) = 1 - \sigma^*(A^c)$. □

4. Extension of the Basic Integration Formula.

The result of this section is a straightforward extension of formula (1) of Section 2. It will often be used in the sequel without reference.

Theorem 1.

Let σ be a strategy. For every $A \subseteq H$,

$$\sigma^*(A) = \int \sigma[x]^*(Ax)\,d\sigma_0(x), \text{ and}$$

$$\sigma_*(A) = \int \sigma[x]_*(Ax)\,d\sigma_0(x).$$

Proof:

The conclusion holds for clopen sets. It is next established for open sets.

If $K \subseteq O$, $Kx \subseteq Ox$, so that

$$\sigma(O) = \sup \sigma(K) = \sup \int \sigma[x](Kx)\,d\sigma_0(x) \leq \int \sigma[x](Ox)\,d\sigma_0(x),$$

where the sup is taken over all clopen sets $K \subseteq O$.

For the opposite inequality, let $\varepsilon > 0$. If O is open, Ox is open, and so, for each $x \in X$, there is a $K(x) \subseteq Ox$ with $K(x)$ clopen and $\sigma[x]K(x) \geq \sigma[x](Ox) - \varepsilon$. Set $K = \bigcup_x xK(x)$ and check that K is clopen and $Kx = K(x)$, all $x \in X$. Then K [illegible] clopen, $K \subseteq O$, and

$$\sigma(O) \geq \sigma(K) = \int \sigma[x](Kx)\,d\sigma_0(x) \geq \int \sigma[x](Ox)\,d\sigma_0(x) - \varepsilon.$$

The argument just given can be easily adapted now to give the first equation in Theorem 1. The second equation follows from the first together with the fact that $\sigma_*(A) = 1 - \sigma^*(A^c)$. □

Corollary 1.

Let σ be a strategy and s a stop rule. Then, for every $A \subseteq H$,

$$\sigma^*(A) = \int \sigma[p_s(h)]^*(Ap_s(h))d\sigma(h), \quad \text{and}$$

$$\sigma_*(A) = \int \sigma[p_s(h)]_*(Ap_s(H))d\sigma(h).$$

This corollary extends Theorem 1, and is proved from it by induction on the structure of p_s.

Notice that, if $A \in \mathcal{G}(\sigma)$, then, by Theorem 1, $\int \sigma[x]^*(Ax)d\sigma_0(x) = \int \sigma[x]_*(Ax)d\sigma_0(x)$. Nevertheless, it can happen that $Ax \notin \mathcal{G}(\sigma[x])$ for all x even when $A \in \mathcal{G}(\sigma)$. For an example, let $X = N$ and γ be a probability on N which gives measure zero to all finite sets. Let β be the probability on N which assings measure 1/2 to each of the points 1, 2. Set $\sigma_0 = \gamma$ and $\sigma(p) = \beta$ if p is a non-empty partial history. The next step will not be executed precisely here. However, it can be justified by the contents of section 11. Namely, $\sigma[x]$ is essentially the coin tossing measure for every x. For that reason, there is a set $B(x)$ such that $\sigma[x]^*(B(x)) = 1/x$ and $\sigma[x]_*(B(x)) = 0$. Define A by $Ax = B(x)$.

5. $\mathcal{G}(\sigma)$ contains the Borel sets.

A critical ingredient in the theorems of this section is a very weak Heine-Borel property of H, presented in Lemma 1 below. To state it, the following notation will be required. Let $A^1, A^2, \ldots$ be subsets of H. If s is a stop rule, $A^s = \{h \in H | h \in A^{s(h)}\}$. It is easy to check that if the A^i are all open (closed), then A^s is open (closed).

Lemma 1.

Let $O^1, O^2, \ldots$ be open sets in H. If $O^1 \subseteq O^2 \subseteq \ldots$ and $H = \bigcup_i O^i$ then there is a stop rule s such that $H = O^s$.

Proof.

For each $h \in H$, there is a least $n \geq 1$ such that the basic neighborhood $\{h' \in H | h'_i = h_i,\ i = 1, \ldots, n\}$ is a subset of at least one of the O^i; and for that n there is a least k such that $O^k \supseteq \{h' \in H | h'_i = h_i,\ i = 1, \ldots, n\}$. If $k \leq n$, set $s(h) = n$; if $k > n$, set $s(h) = k$. Then s is a stop rule and $H = O^s$. (This argument, which was suggested to us by David Blackwell, is more perspicuous than our original proof.) □

Corollary 1. Let $O^1, O^2, \ldots$ be open sets in H. If $O^1 \subseteq O^2 \subseteq \ldots$, C is closed, and $C \subseteq \bigcup_i O^i$, there is a stop rule s such that $C \subseteq O^s$.

Proof.

Set $Q^i = O^i \cup (H - C)$, $i \in N$, and apply the preceding lemma to the sets $Q^1, Q^2, \ldots$. □

Corollary 2.

Let $C^1, C^2, \ldots$ be closed sets in H. If $C^1 \supseteq C^2 \supseteq \ldots$, O is open, and $O \supseteq \bigcap_i C^i$, there is a stop rule s such that $O \supseteq C^s$.

Proof.

Take complements in the preceding corollary. □

Corollary 3.

Let μ be a finitely additive probability on the algebra of clopen sets and λ be the extension of μ defined by Theorem 2.1.

(i) If $C^1 \supseteq C^2 \supseteq \ldots$ are closed, and $C = \bigcap_i C^i$, then $\lambda(C) = \inf_s \lambda(C^s)$.

(ii) If $O^1 \subseteq O^2 \subseteq \ldots$ are open, and $O = \bigcup_i O^i$ then $\lambda(O) = \sup_s \lambda(O^s)$.

(The infimum and supremum are taken over all stop rules s.)

Proof.

Use corollaries 1, 2, and Theorem 2.1.

The next lemma gives a crude sufficient condition for a countable union of open sets to have small probability.

Lemma 2.

Let $R^1, R^2, \ldots$ be a sequence of open sets in H. Let σ be a strategy. If δ is a mon-negative quantity and

$$\sigma(R^{|q|}|q) \le \delta/2^{|q|},$$

for all non-empty $q \in X^*$, then

$$\sigma(R^1 \cup \ldots \cup R^s) \le \delta ,$$

for all stop rules s. Further $\sigma(R^1 \cup R^2 \cup \ldots) \le \delta$.

Proof.

The second assertion follows from the first by using Corollary 3.

The first assertion is a consequence of showing the following statement holds for every ordinal β:

Let s be a stop rule of structure β. If $R^1, R^2, \ldots$ is any sequence of open sets, σ any strategy, δ any non-negative quantity such that

$$\sigma(R^{|q|}|q) \le \delta/2^{|q|}$$

for all non-empty $q \in X^*$, then $\sigma(R^1 \cup \ldots \cup R^s) \le \delta$.

This assertion holds if $\beta = 0$. For then there is a positive integer m such that $s(h) = m$, all $h \in H$. Hence,

$$\begin{aligned}\sigma(R^1 \cup \ldots \cup R^s) &= \sigma(R^1 \cup \ldots \cup R^m) \\ &\le \sum_{i=1}^{m} \sigma(R^i) \\ &= \sum_{i=1}^{m} \int \sigma(R^i|p_i(h))d\sigma(h) \\ &\le \sum_{i=1}^{m} \delta/2^i \le \delta.\end{aligned}$$

For the inductive step, assume the assertion holds for all $\beta < \alpha$, when $\alpha > 0$ is an ordinal. Let s be a stop rule with structure exactly α. Let $R^1, R^2, \ldots$ be open, σ a strategy, $\delta \ge 0$, all satisfying the hypothesis of the assertion. Set $M^i = R^{i+1}$, $i = 1, 2, \ldots$. Then

$$R^1 \cup \ldots \cup R^s \subseteq R^1 \cup (M^1 \cup \ldots \cup M^s)$$

Now $\sigma(R^1) \le \delta/2$, so it suffices to show $\sigma(M^1 \cup \ldots \cup M^s) \le \delta/2$. First,

$$\sigma(M^1 \cup \ldots \cup M^s) = \int \sigma[x](M^1 \cup \ldots \cup M^s)x d\sigma_0(x)$$

The second assertion is a consequence of showing the following statement holds for every ordinal α:

Let s be a stop rule of structure α. If $E^1, E^2, \ldots$ is any sequence of open sets, σ any strategy, δ any non-negative quantity such that

$$\sigma(E^{i}_{p_i}|x) \le \frac{\delta}{2^{i}}$$

for all non-empty $\{p_i\}$, then $\sigma(E^1 \cup \ldots \cup E^s) \le \delta$.

This assertion holds if $\alpha = 0$. For then there is a positive integer n such that $s(x) = n$, all $x \in H$. Hence

$$\sigma(E^1 \cup \ldots \cup E^s) = \sigma(E^1 \cup \ldots \cup E^n)$$

$$\le \sum_{i=1}^{n} \sigma(E^i)$$

$$= \sum_{i=1}^{n} \int \sigma(E^i_{p_i}(x))\, d\sigma(x)$$

$$\le \sum_{i=1}^{n} \delta/2^i \le \delta .$$

For the inductive step, assume the assertion holds for all $\beta < \alpha$, when $\alpha > 0$ is an ordinal. Let s be a stop rule of structure exactly α. Let $E^1, E^2, \ldots$ be open, σ a strategy, $\delta \ge 0$, all satisfying the hypothesis of the assertion. Set $M^i = E^{i+1}$, $i = 1, 2, \ldots$. Then

$$E^1 \cup \ldots \cup E^s \subseteq E^1 \cup (M^1 \cup \ldots \cup M^s)$$

Now $\sigma(E^1) \le \delta/2$, so it suffices to show $\sigma(M^1 \cup \ldots \cup M^s) \le \delta/2$. Thus,

$$\sigma(M^1 \cup \ldots \cup M^s) = \int \sigma[x](M^1_x \cup \ldots \cup M^s_x)\, d\sigma_1(x)$$

Fix an $x \in X$ temporarily. Set

$$L^i = M^i x \qquad i = 1, 2, \ldots ,$$

$$r(h) = s(xh), \qquad h \in H ,$$

$$\rho = \sigma[x] .$$

Then r is a stop rule with structure less than α. Further, the sets $L^1, L^2, \ldots,$ the strategy ρ, and the quantity $\delta/2$ all satisfy the hypothesis of the assertion. Therefore the inductive hypothesis implies that $\rho(L^1 \cup \ldots \cup L^r) \leq \delta/2$. But, as is easily checked,

$$L^1 \cup \ldots \cup L^r = (M^1 \cup \ldots \cup M^s)x .$$

It now follows that the integrand in the above expression for $\sigma(M^1 \cup \ldots \cup M^r)$ cannot exceed $\delta/2$. □

As will be shown shortly, $\mathfrak{G}(\sigma)$ need not be a sigma-field. The following subcollection of sets is a sigma-field however, this being the content of Theorem 1.

Definition.

If σ is a strategy,

$$\mathfrak{F}(\sigma) = \{A \subseteq H \mid Ap \in \mathfrak{G}(\sigma[p]) \text{ for all } p \in X^*\}$$

Theorem 1.

If σ is a strategy, then $\mathfrak{F}(\sigma)$ is a sigma-field which contains the Borel sigma-field and is contained in $\mathfrak{G}(\sigma)$.

Proof.

First, $\mathfrak{F}(\sigma)$ contains the open sets, as A open implies Ap open, and $\mathcal{G}(\sigma[p])$ contains all the open sets. Next, $\mathfrak{F}(\sigma)$ is closed under complementation, as $A^c p = (Ap)^c$ and $\mathcal{G}(\sigma[p])$ is closed under complementation. The remainder of the proof is devoted to showing that $\mathfrak{F}(\sigma)$ is closed under countable intersection.

Let $A^1, A^2, \ldots$ be members of $\mathfrak{F}(\sigma)$ and $A = \bigcap_i A^i$. Fix $p \in X^*$ and set

$$B = Ap,$$

$$B^i = A^i p, \qquad i = 1, 2, \ldots$$

$$\tau = \sigma[p] .$$

The aim is to show $B \in \mathcal{G}(\tau)$. In other words, given $\varepsilon > 0$, to show there exists D closed, P open such that $D \subseteq B \subseteq P$ and $\tau(P-D) < \varepsilon$. Let δ be a small positive quantity, to be chosen later.

Step 1.

For each $q \neq 0$ in X^*, there is a closed set C^q and an open set O^q such that

$$C^q \subseteq B^{|q|}_q \subseteq O^q ,$$

and

$$\tau[q](O^q - C^q) \leq \delta/2^{|q|} .$$

To see this, set $i = |q|$ and observe that since $A^i \in \mathfrak{F}(\sigma)$, $A^i pq \in \mathcal{G}(\sigma[pq])$. But $A^i pq = B^i q$ and $\sigma[pq] = \tau[q]$.

Step 2.

Choose (axiom of choice) for each $q \neq 0$ in X^* a C^q, O^q satisfying the conditions of step 1. Set, for each $n \in N$,

$$C^n = \bigcup_{|q|=n} q\, C^q, \qquad O^n = \bigcup_{|q|=n} q\, O^q .$$

Then C^n is closed (see Lemma 3.3 for example), O^n is open, $C^n \subseteq B^n \subseteq O^n$, and $C^n q = C^q$, $O^n q = O^q$ for all $q \in X^*$ with $|q| = n$.

Step 3.

Set, for $n \in N$,

$$P^n = O^1 \cap \ldots \cap O^n$$

$$D^n = C^1 \cap \ldots \cap C^n.$$

Then, for all stop rules s, $D^s \subseteq P^s$ and

$$\tau(P^s - D^s) \leq \delta .$$

The inclusion $D^s \subseteq P^s$ follows from the fact (step 2) that $C^i \subseteq O^i$, all $i \in N$. To establish the inequality, set $R^i = O^i - C^i$, $i \in N$. First,

$$P^s - D^s \subseteq R^1 \cup \ldots \cup R^s ,$$

which follows from

$$P^m - D^m = (P^m - C^1) \cup \ldots \cup (P^m - C^m) ,$$

$$\subseteq (O^1 - C^1) \cup \ldots \cup (O^m - C^m)$$

where $m \in N$, and the i-th term of the second union is $O^i - C^i$. Now, if $q \neq 0$ in X^*,

$$\tau[q]R^{|q|}\ q \le \delta/2^{|q|},$$

using steps 1 and 2. By lemma 2, this suffices to show $\tau(R^1 \cup \ldots \cup R^s) \le \delta$.

Step 4.

Set

$$a = \inf_s \tau(D^s)$$

$$b = \inf_s \tau\,(P^s)$$

where the infimum is over all stop rules s. Then $0 \le b - a \le \delta$.

If not, $b - a = \delta + \delta_1$ where $\delta_1 > 0$. Now choose s such that $\tau(D^s) < a + \delta_1$. Then $\delta < \tau(P^s - D^s)$ contradicting the preceding step.

Step 5.

Set $D = \bigcap_i D^i$. Then

$$D \subseteq \bigcap_i B^i \subseteq P^s$$

for each stop rule s.

Step 6.

There is a stop rule r such that

$$\tau(P^r) < a + 2\delta.$$

Let r be such a stop rule and set $P = P^r$. Then $\tau(P-D) < 2\delta$.

The first assertion follows from step 4. To show the second, observe that $a = \tau(D)$ by Corollary 3.

Step 7.

By taking $\delta = \epsilon/2$ and noting that $B = \bigcap_i B^i$, the proof is completed. □

It is pleasing that $\mathcal{A}(\sigma)$ is the completion, in the usual sense, of σ restricted to the Borel sets. That is, $\mathcal{A}(\sigma)$ is exactly equal to the collection of all sets A for which there exists Borel E, F with $E \subseteq A \subseteq F$ and $\sigma(F-E) = 0$. For suppose $A \in \mathcal{A}(\sigma)$. For each $n \in N$, there is a closed set C_n, an open set O_n such that $C_n \subseteq A \subseteq O_n$ and $\sigma(O_n - C_n) < 1/n$. Set $E = \bigcup_n C_n$, $F = \bigcap_n O_n$. Then E, F are Borel and $F - E \subseteq O_n - C_n$ for every $n \in N$. It follows that $\sigma(F-E) = 0$. The other direction is a consequence of Theorem 1. Perhaps suprisingly, $\mathcal{A}(\sigma)$ may not be a sigma-field. The example is not difficult. Let X and σ be as in the example of section 4. Then there is a set B of infinite sequences of 1's and 2's such that $\sigma[x]^*B = 1$, $\sigma[x]_*B = 0$. Set $A^n = nB$, $n \in N$. Then $A^n \in \mathcal{A}(\sigma)$, all $n \in N$, but $\bigcup_n A^n$ is not in $\mathcal{A}(\sigma)$. The set A^n is in $\mathcal{A}(\sigma)$ because

$$\sigma^*(A^n) = \int \sigma[x]^* A^n x \, d\sigma_0(x) = 0$$

as $A^n x = \emptyset$ unless $x = n$. However, since $(\bigcup A^n)x = B$, $\sigma^*(\bigcup_n A^n) = 1$ and $\sigma_*(\bigcup_n A^n) = 0$.

<u>Theorem 2</u>. Let $A^1, A^2, \ldots$ be sets in $\mathcal{F}(\sigma)$.

(i) If s is a stop rule, $A^s \in \mathcal{F}(\sigma)$.

(ii) If $A^1 \supseteq A^2 \supseteq \ldots$, and $A = \bigcap_i A^i$, then $\sigma(A) = \inf_s \sigma(A^s)$.

(iii) If $A^1 \subseteq A^2 \subseteq \ldots$, and $A = \bigcup_i A^i$, then $\sigma(A) = \sup_s \sigma(A^s)$.

(The infinum and supremum are taken over all stop rules s)

<u>Proof:</u> (i) $A^s \in \mathcal{F}(\sigma)$, since $\mathcal{F}(\sigma)$ is a sigma-field and $A^s = \bigcup_i A^i \cap \{h \in H \mid s(h) = i\}$

(ii) Let $\epsilon > 0$ and $\delta = \epsilon/2$. Follow the argument given in steps 1-6 of the proof of Theorem 1, but take $p = \emptyset$ Since $A^1 \supseteq A^2 \supseteq \ldots$, A^s is a subset of P^s. Then

$$D \subseteq A^s \subseteq P^s ,$$

and

$$\sigma(D) \le \sigma(A^s) \le \sigma(P^s) ,$$

for each stop rule s. Using the stop rule r of step 6 allows us to conclude that

$$\sigma(A^s) - \sigma(A) \le 2\delta = \epsilon$$

(iii) This follows from (ii) by taking complements. □

We will want to use lemma 2 for sets R^1, R^2, ... which belong to $\mathfrak{F}(\sigma)$. In the light of theorems 1 and 2, the same proof will apply.

6. The Measure of Countable Intersections.

It is easy to verify that

(1) $$P(\bigcap_1^\infty A_n) = P(A_1) \prod_2^\infty P(A_n | A_1 \cap \dots \cap A_{n-1})$$

when P is a countably additive probability and both sides are well-defined. For a finitely additive P, the left side of (1) can be smaller than the right. However, the theorems of this section give formulas analogous to (1) which are appropriate even in a finitely additive setting. In particular, Theorem 4 states that (1) holds for "independent events."

Let $\{K_n\}$ be a sequence of clopen sets and let $\{r_n\}$ be a strictly increasing sequence of stop rules such that, for every positive integer n, K_n is determined by time r_n. Define, for every $n \in N$ and $h \in H$, $q_n(h) = p_{r_n}(h)$. Finally, let $\{\alpha_n\}$ be a sequence of numbers satisfying $0 \le \alpha_n \le 1$ for all n and let σ be a strategy.

Theorem 1.

If $\sigma(K_1) \ge \alpha_1$ and if, for all $n = 1,2,\dots$ and all $h \in \bigcap_1^n K_i$, $\sigma(K_{n+1} | q_n(h)) \ge \alpha_{n+1}$, then $\sigma(\bigcap_1^\infty K_i) \ge \prod_1^\infty \alpha_i$.

Proof:

The set $\bigcap_1^\infty K_i$ is closed. Let K be clopen and $K \supseteq \bigcap_1^\infty K_i$. By Theorem 2.1, it suffices to show $\sigma(K) \ge \prod_1^\infty \alpha_i$.

The argument is by induction on the structure of K. We can and do assume $\alpha_i > 0$ for all i.

Suppose K has structure 0. Then either $K = H$ or $K = \emptyset$. If $K = H$, then $\sigma(K) = 1 \ge \prod_1^\infty \alpha_i$. We show K cannot be empty by constructing

7. <u>The Measure of Countable Intersections.</u>

It is easy to verify that

$$(1) \qquad P\left(\bigcup_{i} A_i\right) = P(A_1) + \sum_{n} P(A_n \cap A_1' \cap \cdots \cap A_{n-1}')$$

when P is a countably additive probability and both sides are well-defined. For a finitely additive P, the left side of (1) can be smaller than the right. However, the theorems of this section give formulas analogous to (5.1) which are appropriate even in a finitely additive setting. In particular, Theorem 2 states that (1) holds for independent events.

Let $\{K_n\}$ be a sequence of clopen sets and let $\{r_i\}$ be a strictly decreasing sequence of [illegible] such that for every positive integer n, K_n is determined by [illegible]. Define, for every [illegible] and $h \in H$, $a_n(h) = p_n(h)$. Finally, let $\{q_n\}$ be a sequence of numbers satisfying $0 \le q_n \le 1$ for all n and let [illegible] be [illegible].

<u>Theorem 1:</u>

If $\sigma(K_1) = q_1$ and [illegible] for $n = 1, 2, \ldots$ and all $h \in \bigcap_{1}^{n} K_i$, $\sigma(K_{n+1} \mid a_n(h)) [illegible] q_{n+1}$, then $\sigma\left(\bigcap_{1}^{\infty} K_i\right) = \prod q_i$.

<u>Proof:</u>

The set $\bigcap_{1}^{\infty} K_i$ is clopen. Let K be clopen and $K \supseteq \bigcap_{1}^{\infty} K_i$. By Theorem 5.1, it suffices to show $\sigma(K) \ge \prod_{1}^{\infty} q_i$.

The argument is by induction on the structure of K. We can and do assume $q_i > 0$ for all i.

Suppose K has structure 0. Then either $K = H$ or $K = \emptyset$. If $K = H$, then $\sigma(K) = 1 \ge \prod_{1}^{\infty} q_i$. [illegible] K cannot be empty by construction a history $h \in \bigcap_1^\infty K_i$. Since $\sigma(K_1) \geq \alpha_1 > 0$, there exists $h^1 \in K_1$. Since $\sigma(K_2 | q_1(h^1)) \geq \alpha_2 > 0$, there exists $h^2 \in K_2$ such that h^2 agrees with h^1 up to time $r_1(h^1)$. Continue in this fashion to define $h^n \in K_n$ such that h^n agrees with h^{n-1} up to time $r_{n-1}(h^{n-1})$. Then let h be that history which agrees with h^n up to time $r_n(h^n)$ for all n. Since K_n is determined by time r_n and $h^n \in K_n$, we have $h \in K_n$ for all n.

For the inductive step, assume the desired result for sets of structure less than α and suppose K has structure $\alpha > 0$. Then, for all h, $Kq_1(h) \supseteq (\bigcap_1^\infty K_n)q_1(h) = \bigcap_1^\infty (K_n q_1(h))$ and $Kq_1(h)$ has structure less than α.

Fix $h \in K_1$. Set $q = q_1(h)$ and define $\sigma' = \sigma[q]$, $K_n' = K_{n+1}q$, and $r_n'(h') = r_{n+1}(q\,h') - r_1(h)$ for $h' \in H$. Then $\sigma'(K_1') = \sigma(K_2 | q_1(h)) \geq \alpha_2$. Also, if $h' \in \bigcap_1^n K_i'$, then $q\,h' \in \bigcap_1^{n+1} K_i$ and $\sigma'(K_{n+1}' | q_n'(h')) = \sigma(K_{n+2} | q_{n+1}(qh')) \geq \alpha_{n+2}$.

By the inductive assumption, if $h \in K_1$, then $\sigma(K | q_1(h)) \geq \prod_2^\infty \alpha_n$. Hence, by equation 2.2,

$$\begin{aligned}\sigma(K) &= \int \sigma(K|q_1(h))\, d\,\sigma(h) \\ &\geq \int_{K_1} \sigma(K|q_1(h))\, d\,\sigma(h) \\ &\geq \sigma(K_1) \prod_2^\infty \alpha_n \\ &\geq \prod_1^\infty \alpha_n . \quad \square\end{aligned}$$

Let $B_n = \{h \in \bigcap_1^n K_i | \sigma(K_{n+1}|q_n(h)) \geq \alpha_n\}$ for $n = 1,2,\ldots$ It would be convenient to replace the assumption in Theorem 1 that $B_n = \bigcap_1^n K_i$ by the milder one that $\bigcap_1^n K_i - B_n$ has small probability. An example shows there history $h \in \cap K_i$. Since $\nu(K_1) \geq \varepsilon_1 > 0$, there exists $h^1 \in K_1$. Since $\nu(K_2|q_1(h^1)) \geq \varepsilon_2 > 0$ there exists $h^2 \in K_2$ such that h^2 agrees with h^1 up to time $t_1(h^1)$. Continue in this fashion to define $h^n \in K_n$ such that h^n agrees with h^{n-1} up to time $t_{n-1}(h^{n-1})$. Then let h be that history which agrees with h^n up to time $t_n(h^n)$ for all n. Since K_n is determined by time t_n and $h^n \in K_n$ we have $h \in K_n$ for all n.

For the inductive step, assume the desired result for sets of structure less than α and suppose K has structure $\alpha > 0$. Then for all h, $K_{q_1}(h) \supseteq (\cap K_n)_{q_1}(h) = \cap_1 (K_n q_1(h))$ and $K_{q_1}(h)$ has structure less than α.

Fix $h \in K_1$. Set $q = q_1(h)$ and define $\omega' = \omega[\,]$, $K'_i = K_{i+1}$ and $t'_i(h') = t_{i+1}(q h') - t_1(h)$. [illegible] Then $\nu'(K') = \nu(K|q_1(h))$ [illegible]. Also, if $h' \in \cap_1^n K'_i$ then $q h' \in \cap_1^{n+1} K_i$ [illegible] $\nu'(K'_n|q'_n(h')) = \nu(K_{n+1}|q_{n+1}(qh')) \geq \varepsilon_{n+1}$.

By the inductive assumption [illegible] Then $\nu(K|q_1(h))$ [illegible]. Hence, by equation 2.1,

$$\nu(K) = \int \nu(K|q_1(h))\, d\nu(h)$$

$$\geq \int_{K_1} \nu(K|q_1(h))\, d\nu(h)$$

$$\geq \nu(K_1) \prod \varepsilon_n$$

$$\geq \prod_1 \varepsilon_n .$$

Let $B_n = \{h \in \cap_1^n K_i : \nu(K_{i+1}|q_i(h)) \geq \varepsilon_n\}$ for $n = 1, \ldots$. It would be convenient to replace the assumption in Theorem 1 that $B_n = \cap_1^n K_i$ by the milder one that $\nu(\cap_1^n K_i - B_n)$ has small probabilities. An example [illegible] that is no hope for such a generalization.

Example: Let $X = N$; let σ_0 be a finitely additive probability on all subsets of N such that $\sigma_0(\{n\}) = 0$ for all n ; let $\sigma[n]$ assign mass one to the history $(n,n,\ldots)$ for all n; let $r_n = n$, $K_n = \{h | h_n \geq n\}$ and $\alpha_n = 1$. Then $\sigma(B_n) = 1$ for all n but $\sigma(\bigcap_1^\infty K_n) = 0$.

The next result is a simple inequality which goes in the opposite direction from Theorem 1.

Theorem 2.

Assume $\sigma(K_1) \leq \alpha_1$ and let $C_n = \{h \in \bigcap_1^n K_i | \sigma(K_{n+1} | q_n(h)) > \alpha_{n+1}\}$ for $n = 1,2,\ldots$ Then

$$\sigma(\bigcap_1^n K_i) \leq \prod_1^n \alpha_i + \sum_1^{n-1} \sigma(C_i), \text{ for } n = 2,3,\ldots, \tag{2}$$

and, hence,

$$\sigma(\bigcap_1^\infty K_i) \leq \prod_1^\infty \alpha_i + \sum_1^\infty \sigma(C_i).$$

Proof:

Assume (2) is true for n. Then

$$\sigma(\bigcap_1^{n+1} K_i) = \int_{\bigcap_1^n K_i} \sigma(K_{n+1} | q_n(h)) d\sigma(h)$$

$$= \int_{C_n} \sigma(K_{n+1} | q_n(h)) d\sigma(h) + \int_{\bigcap_1^n K_i - C_n} \sigma(K_{n+1} | q_n(h)) d\sigma(h)$$

$$\leq \sigma(C_n) + \alpha_{n+1}(\prod_1^n \alpha_i + \sum_1^{n-1} \sigma(C_i))$$

$$\leq \prod_1^{n+1} \alpha_i + \sum_1^n \sigma(C_i). \;\square$$

The next result is immediate from Theorems 1 and 2.

Theorem 3.

If $\sigma(K_1) = \alpha_1$ and if, for all $n = 1,2,\ldots$ and all $h \in \bigcap_1^n K_i$, $\sigma(K_{n+1} | q_n(h)) = \alpha_{n+1}$, then $\sigma(\bigcap_1^\infty K_i) = \prod_1^\infty \alpha_i$.

Now let $\{\gamma_n\}$ be a sequence of probabilities defined on all subsets of X. Define the strategy $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ by $\sigma_0 = \gamma_1$ and, for all p of length n, $\sigma(p) = \gamma_{n+1}$. Such a strategy is said to be independent. Notice that $\sigma[p] = \gamma_{n+1} \times \gamma_{n+2} \times \cdots$ for every p of length n.

Theorem 4.

Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy and let $A_n \subseteq X$ for $n = 1,2,\ldots.$ Then

$$\sigma(A_1 \times A_2 \times \cdots) = \gamma_1(A_1)\, \gamma_2(A_2)\cdots.$$

Proof:

Let $K_n = \{h: h_n \in A_n\}$, and let $r_n = n$. Then $\bigcap_1^\infty K_n = A_1 \times A_2 \times \cdots$ and $\sigma(K_{n+1} | q_n(h)) = \gamma_{n+1}(A_{n+1})$ for all n and h. □

Remark. Theorem 1 implies the following fact, which is well known in the case that X is a finite set, where probabilities on the clopen sets are automatically countably additive. Let σ be a strategy and $0 < \epsilon < 1$. If X is infinite, there is a closed nowhere dense subset D of H such that $\sigma(D) \geq 1-\epsilon$. The proof is not difficult. Let $\epsilon_1, \epsilon_2, \ldots$ be a sequence of positive real numbers such that $\sum_{i=1}^\infty \epsilon_i \leq \epsilon$. For each nonempty $p \in X^*$, let $X(p)$ be a proper subset of X such that $\sigma(p)(X(p)) \geq 1 - \epsilon_i$, The next result is immediate from Theorems 1 and 2.

Theorem 3.

If $\sigma(K_1) = \alpha_1$ and if, for all $n = 1, 2, \ldots$ and all $h \in \bigcap_1^n K_i$,

$\sigma(K_{n+1} | h_n(h)) = \alpha_{n+1}$, then $\sigma(\bigcap_1^\infty K_i) = \prod_1^\infty \alpha_i$.

Now let γ_n be a sequence of probabilities defined on all subsets of X. Define the strategy $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ by $\sigma_0 = \gamma_1$ and, for all p of length n, $\sigma(p) = \gamma_{n+1}$. Such a strategy is said to be independent. Notice that $\sigma[p] = \gamma_{n+1} \times \gamma_{n+2} \times \ldots$ for every p of length n.

Theorem 4.

Let $\sigma = \gamma_1 \times \gamma_2 \times \ldots$ be an independent strategy and let $A_n \subseteq X$ for $n = 1, 2, \ldots$. Then

$$\sigma(A_1 \times A_2 \times \ldots) = \gamma_1(A_1)\gamma_2(A_2)\ldots$$

Proof:

Let $K_n = \{h: h_n \in A_n\}$ and let $\alpha_n = \gamma_n(A_n)$. Then $\bigcap_1^\infty K_i = A_1 \times A_2 \times \ldots$

and $\sigma(K_{n+1} | h_n(h)) = \gamma_{n+1}(A_{n+1})$ for all n and h.

Remark: Theorem 1 implies a ... which is well known in the case X is a finite set: there probabilities on the Borel sets are automatically countably additive. Let σ be a strategy and $0 < \varepsilon < 1$. If X is infinite, there is a closed nowhere dense subset U of X such that $\sigma(U) > 1-\varepsilon$. The proof is not difficult. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive real numbers such that $\sum_{i=1}^\infty \varepsilon_i < \varepsilon$. For each ... $p \in X_*$, let $K(p)$ be a compact subset of X such that $\sigma(p)(K(p)) > 1 - \varepsilon_i$.

where i is the number of terms of p. Finally, set

$$K_1 = H,$$

$$K_2 = \{h \in H | h_n \in X(p_{n-1}(h))\}, \qquad n \geq 2, \quad n \in N.$$

Then using Theorem 1 and the elementary inequality $\prod_{i=1}^{\infty} (1 - \varepsilon_i) \geq 1 - \sum_{i=1}^{\infty} \varepsilon_i$,

it can be shown that the set $D = \bigcap_1^{\infty} K_i$ has σ-measure at least $1 - \varepsilon$.

From the definition of the sets K_i, D must be closed nowhere dense.

7. <u>The Borel-Cantelli Lemmas and a Strong Law of Large Numbers</u>.

The question which is tentatively raised in this section and again in sections 8 and 10 is whether the conventional strong limit theorems of probability continue to hold in a finitely additive theory and, if so, in what form. Theorems 1 and 2 below are finitely additive versions of the Borel-Cantelli lemmas. A slight modification of a standard Borel-Cantelli argument together with the result of Theorem 1 enable us to establish Theorem 3, which corresponds to the strong law for independent, uniformly bounded variables.

For the first two theorems, assume the same setting as in paragraph 2 of section 6.

<u>Theorem 1</u>: Suppose that, for $n = 1,2,\ldots$ and $h \in H$, $\sigma(K_{n+1} | q_n(h)) \leq \alpha_{n+2}$. If $\sum \alpha_n < \infty$, then $\sigma\,[K_n \text{ i.o}] = 0$. (The symbol "i.o." is short for "infinitely often".)

<u>Proof</u>: Let $n \in N$. Notice that $\sigma(K_{n+1}^c) = \int \sigma(K_{n+1}^c | q_n(h))d\sigma(h) \geq 1 - \alpha_{n+1}$, and, for $k \in N$ and $h \in H$, $\sigma(K_{n+k+1}^c | q_{n+k}(h)) \geq 1 - \alpha_{n+k+1}$. By Theorem 6.1, $\sigma(\bigcap_{i>n} K_i^c) \geq \prod_{i>n} (1 - \alpha_i) \geq 1 - \sum_{i>n} \alpha_i \to 1$ as $n \to \infty$. (The last inequality uses the elementary fact that $\prod p_i \geq 1 - \sum p_i$ for numbers p_i such that $0 \leq p_i \leq 1$.) Since $[K_m \text{ i.o.}] \subseteq \bigcup_{i>n} K_i = (\bigcap_{i>n} K_i^c)^c$ for all n, the proof is complete. □

In the conventional theory, the result corresponding to the previous theorem states that, for arbitrary events A_n, if $\sum P(A_n) < +\infty$, then $P[A_n \text{ i.o.}] = 0$. The same is not true here as the following example shows.

7. <u>The Borel-Cantelli Lemma and the Strong Law of Large Numbers.</u>

The question which is tentatively raised in this section [illegible] sections 2 and [illegible] whether the conventional strong limit theorems of probability continue to hold in a finitely additive theory and, if so, in what form. Theorems 1 and 2 below are finitely additive versions of the Borel-Cantelli Lemma. A slight modification of a standard Borel-Cantelli argument together with the results of Theorem 1 enable us to establish Theorem 3, which corresponds to the strong law for independent uniformly bounded variables.

For the first two theorems, assume the case σ being as in paragraph 2 of section 6.

<u>Theorem 1</u>: Suppose that, for $n = 1, 2, \ldots$ and $h \in H$, $\sigma(K_{n+1} | [illegible](h)) \le a_{n+1}$. If $\sum a_n < \infty$, then $\sigma[K_n \text{ i.o.}] = 0$. (The symbol "i.o." stands for "infinitely often".)

<u>Proof</u>: Let [illegible]. Then $\sigma(K^c_{n+1}) = \int \sigma(K^c_{n+1} | [illegible](h))\, d\sigma(h) \ge 1 - a_{n+1}$, and, for $k \in N$ and $h \in H$, $\sigma(K^c_{n+k+1} | [illegible](h)) \ge 1 - a_{n+k+1}$. By Theorem 6.1, $\sigma(\bigcap_{k \ge n} K^c_k) \ge \prod_{k \ge n} (1 - a_k) \ge 1 - [illegible]$. (The last inequality is the elementary result that $\prod (1 - a_k) \ge 1 - \sum a_k$ for numbers such that $0 \le a_k \le 1$.) Since $[K_n \text{ i.o.}] \subseteq \bigcup_{k \ge n} K_k = (\bigcap_{k \ge n} K^c_k)^c$ for all n, the proof is complete.

In the conventional theory, the result corresponding to the previous theorem states that, for arbitrary events A_n, if $\sum P(A_n) < +\infty$, then $P[A_n \text{ i.o.}] = 0$. This statement is not true as the following example shows.

Example: Let X and σ be as in the example of section 6. Let $K_n = \{h | h_n \le n\}$ for $n \in N$. Then $\sigma(K_n) = 0$ for all n, but $\sigma[K_n \text{ i.o}] = 1$ as can be seen by applying Theorem 4.1.

Theorem 2: Suppose that, for $n = 1,2,\ldots$ and $h \in H$, $\sigma(K_{n+1} | q_n(h)) \ge \alpha_{n+1}$. If $\sum \alpha_n = \infty$, then $\sigma[K_n \text{ i.o.}] = 0$.

Proof: For $k \in N$, let $A^k = \bigcup_{n \ge k} K_n$. Then the A^k decrease to $[K_n \text{ i.o}]$ as $k \to \infty$. So, by Theorem 5.2, $\sigma(A^s)$ decreases to $\sigma[K_n \text{ i.o.}]$ as $s \to \infty$ through the stop rules.

Let s be a stop rule. For $h \in H$, define $\hat{s}(h) = r_n(h)$ if $s(h) = n$. Then $\hat{s} \ge s$ and it is easy to check that $\hat{s}$ is a stop rule. Thus it suffices to show $\sigma(A^{\hat{s}}) = 1$.

Let $h \in H$ and suppose $s(h) = n$. Then

$$\begin{aligned}\sigma(A^{\hat{s}} | p_{\hat{s}}(h)) &= \sigma[p_{\hat{s}}(h)](A^{\hat{s}} p_{\hat{s}}(h)) \\ &= \sigma[q_n(h)](\bigcup_{i \ge n} K_i q_n(h)) \\ &= 1 - \sigma[q_n(h)](\bigcap_{i \ge n} K_i^c q_n(h)) \\ &\ge 1 - \prod_{i \ge n} (1 - \alpha_i).\end{aligned}$$

The last inequality is obtained by applying Theorem 5.1 to the strategy $\sigma[q_n(h)]$ and the clopen sets $\{K_i^c q_n(h)\}_{i \ge n}$. Since

$$\begin{aligned}\prod_{i \ge n} (1 - \alpha_i) &\le \prod_{i=n}^{m} (1 - \alpha_i) \\ &< \exp(-\sum_{i=n}^{m} \alpha_i) \to 0 \text{ as } m \to \infty,\end{aligned}$$

it follows that $\sigma(A^{\hat{s}} | p_{\hat{s}}(h)) = 1$. Now apply Corollary 4.1 to get $\sigma(A^{\hat{s}}) = 1$. □

Corollary 1.

Let $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ be an independent strategy on H as defined in Section 6. Let $i_1, i_2, \ldots$ be a sequence of positive integers and suppose

$$A_n \subseteq X^{i_n} \text{ for all } n.$$

Set

$$r_1 = 1;\ r_n = i_1 + \ldots + i_{n-1} + 1,\ n \geq 2;$$

$$s_n = i_1 + \ldots + i_n,\ n \geq 1; \text{ and}$$

$$K_n = \{h: (h_{r_n}, \ldots, h_{s_n}) \in A_n\},\ n \geq 1.$$

If $\Sigma\, \sigma(K_n) < \infty$, then $\sigma[K_n \text{ i.o}] = 0$.

Proof:

For each n, K_n is determined by time r_n and, for all h, $\sigma(K_{n+1} | q_n(h)) = \sigma(K_{n+1})$. Now use Theorem 1. □

Corollary 2.

Let σ and the K_n be as in Corollary 1. If $\Sigma\, \sigma(K_n) = \infty$, then $\sigma[K_n \text{ i.o.}] = 1$.

Proof:

Similar to that of Corollary 1. □

We are grateful to David Freedman for pointing out that conventional methods now suffice to prove the next result. The particular proof given is similar to that of Theorem 5.1.2 in Chung (1968).

Theorem 3.

Let σ be an independent strategy on H and, for $n = 1,2,\ldots$, let Y_n be a real-valued function on H which depends only on the n^{th} coordinate. Assume that $|Y_n(h)| \leq 1$ for all n and h, and that $\sigma Y_n = 0$ for all n. Then the set $\{h: \frac{1}{n} \Sigma_{i=1}^{n} Y_i(h) \to 0\}$ has σ-measure one.

Proof:

Let $S_0 = 0$, $S_n = Y_1 + \ldots + Y_n$, and $T_n = S_{n!} - S_{(n-1)!}$ for $n = 1,2,\ldots$ Notice that $\sigma(S_n^2) \leq n$ and $\sigma(T_n^2) \leq \sigma(S_{n!}^2) \leq n!$ for all n.

The proof of Theorem 3 is in four steps.

Step 1. $\sigma[\frac{T_n}{n!} \to 0] = 1$.

To see this, let $K_n = [\frac{|T_n|}{n!} \geq \frac{1}{n}]$. Then apply Chebyshev's inequality, which clearly holds for finitely additive measures, to get

$$\sigma(K_n) = \sigma[\frac{T_n^2}{(n!)^2} \geq \frac{1}{n^2}] \leq \frac{n^2}{n!} .$$

Thus $\Sigma\, \sigma(K_n) < \infty$ and, since the K_n are defined in terms of disjoint sets of coordinates, Corollary 1 applies to give $\sigma[K_n \text{ i.o.}] = 0$. Since

$$[\frac{T_n}{n!} \to 0] \supseteq [K_n \text{ i.o.}]^c ,$$

step 1 is complete.

Step 2. $\sigma[\frac{S_{n!}}{n!} \to 0] = 1$.

Notice that

$$\frac{|S_{n!}|}{n!} \leq \frac{|S_{n!}-T_n|}{n!} + \frac{|T_n|}{n!} \leq \frac{(n-1)!}{n!} + \frac{|T_n|}{n!}$$

and, hence

Theorem 3.

Let [illegible] be an independent [illegible] on Ω and, for n = 1,2,...., let Y_n be a real-valued function on Ω which depends only on the n[illegible] coordinate. Assume that $|Y_n(\omega)| \le 1$ for all ω and n, and that [illegible] $Y_n = 0$ for all n. Then the set $[\omega: \frac{1}{n}\sum_{i=1}^{n} Y_i(\omega) \to 0]$ has σ-measure one.

Proof:

Let $S_0 = 0$, $S_n = Y_1 + \cdots + Y_n$ and $T_n = S_n^4 - S_{(n-1)}^4$ for n = 1,2,.... Notice that [illegible]

Proof of Theorem 3 in four steps.

Step 1. $\sigma[\frac{T_n}{n^3} \to 0] = 1.$

To see this, let $K_n = [\frac{|T_n|}{n^3} > \frac{1}{n}]$. Then [illegible] inequality [illegible] clearly holds [illegible]

$$\sigma(K_n) \le [\frac{1}{n} [illegible]] [illegible]$$

Then $\sum \sigma(K_n) < \infty$ and since the K_n are defined in terms of disjoint sets of coordinates, Corollary 1 applies to give $\sigma[K_n \text{ i.o.}] = 0$. Since

$$[\frac{T_n}{n^3} \to 0] \supseteq [K_n \text{ i.o.}]^c$$

step 1 is complete.

Step 2. $\sigma[\frac{S_n^4}{n^4} \to 0] = 1.$

Notice that

$$[illegible]$$

and hence

$$[\frac{S_{n!}}{n!} \to 0] \supseteq [\frac{T_n}{n!} \to 0].$$

__Step 3.__ For n = 1, 2,..., set $D_n = \max\{|S_k - S_{n!}| : n! < 1 \le (n+1)!\}$. Then $\sigma[\frac{D_n}{n!} \to 0] = 1$.

To check thisstep, first notice that, for all n,

$$\sigma(S_n^4) = \sum_{i=1}^{n} \sigma(Y_i^4) + 6\sum_{1\le i<j\le n} \sigma(Y_i^2)\sigma(Y_j^2) \le 3n^2.$$

Clearly, the same inequality holds if S_n is replaced by the sum of any n distinct Y_i's. Furthermore,

$$D_n^4 \le \sum_{k=n!+1}^{(n+1)!} |S_k - S_{n!}|^4$$

and, hence,

$$\sigma(D_n^4) \le 3 \sum_{k=1}^{(n+1)!-n!} k^2 \le 3[(n+1)! - n!]^3 \le 3[(n+1)!]^3.$$

The completion of this step is similar to step 1. For each n, let $L_n = [\frac{D_n}{n!} \ge \frac{1}{\sqrt[4]{n}}]$. By Chebyshev,

$$\sigma(L_n) = \sigma[\frac{D_n^4}{(n!)^4} \ge \frac{1}{n}] \le \frac{3n(n+1)^3}{n!} .$$

Since $\Sigma\, \sigma(L_n) < \infty$, $\sigma[L_n \text{ i.o.}] = 0$. The step is finished because

$$[\frac{D_n}{n!} \to 0] \supseteq [L_n \text{ i.o.}]^c .$$

__Step 4.__ $\sigma[\frac{S_n}{n} \to 0] = 1$.

For each n, let m = m(n) be that integer such that $m! < n \le (m+1)!$.

$$[\frac{S_{m!}}{m!} \to 0] \subseteq [\frac{T_n}{n} \to 0].$$

Note 5a For n = 1, 2, ..., let $D_n = \max\{|S_k - S_{n!}| : n! \le k < (n+1)!\}$.

Then $P[\frac{D_n}{n!} \to 0] = 1$.

To check this assertion, first notice that, for all n,

$$E(S_n^4) = \sum_{k=1}^{n} E(Y_k^4) + 6 \sum_{1 \le i < j \le n} E(Y_i^2)\,E(Y_j^2) \le Kn^2.$$

Clearly the same inequality holds if S_n is replaced by the sum of any n distinct Y_k's. Furthermore,

$$D_n^4 \le \sum_{k=n!+1}^{(n+1)!} |S_k - S_{n!}|^4$$

and, hence,

$$E(D_n^4) \le \sum_{k=1}^{(n+1)!-n!} Kk^2 \le K[(n+1)! - n!]^3 \le K[(n+1)!]^3.$$

The completion of this step is similar to step 1. For each n, let $T_n = [\frac{D_n}{n!} > \frac{1}{n^{1/4}}]$. By Chebyshev,

$$P(T_n) = P[\frac{D_n^4}{(n!)^4} > \frac{1}{n}] \le \frac{n(n+1)^3}{n!}.$$

Since $\sum P(T_n) < \infty$, $P[T_n \text{ i.o.}] = 0$ [illegible]

$$[\frac{D_n}{n!} \to 0] \supseteq [T_n \text{ i.o.}]^c.$$

Step 4. $P[\frac{S_n}{n^2} \to 0] = 1$.

For each n, let $m = m(n)$ be the integer such that $m! \le n < (m+1)!$.

Then

$$\frac{|S_n|}{n} \leq \frac{D_m + |S_{m!}|}{m!}.$$

Thus

$$[\frac{S_n}{n} \to 0] \supseteq [\frac{D_n}{n!} \to 0] \cap [\frac{S_{n!}}{n!} \to 0].$$

Step 4 and the proof of the theorem are now complete. □

Robert Chen has informed us that the assumption in Theorem 3 that the Y_n's are uniformly bounded can be replaced, as in the conventional theory, by the weaker assumption that

$$\sum_{n=1}^{\infty} \frac{\sigma(Y_n^2)}{n^2} < \infty.$$

8. Zero-One Laws.

The main result of this section is a finitely additive version of the Lévy 0-1 law. It becomes the conventional version if measurability assumptions are imposed as in section 11.

Theorem 1: Let σ be a strategy and $A \in \mathfrak{F}(\sigma)$. Then $\sigma\{h|\sigma(A|p_n(h)) \to 1_A(h)\} = 1$ (The notation $\sigma(A|p)$ is short for $\sigma[p](Ap)$.)

Proof: It suffices to show

(1) $\sigma(A \cap [\sigma(A|p_n) \to 1]) \geq \sigma(A)$,

since (1) together with the same result for A^c imply the theorem.

The proof of (1) is an application of Theorem 6.1. The idea is to construct a sequence of stop rules r_n so that with appropriately high probability all the numbers $\sigma(A|p_k)$, $k \geq r_n$, are larger than $1 - \alpha_n$ where $\alpha_n \to 0$.

The basic tool for the construction is the following lemma, which is adapted from Lévy's original argument (Section 41 of Lévy (1937)).

Notice that, for any $B \in \mathcal{a}(\sigma)$ and $\epsilon > 0$, there is a clopen set K such that $\sigma(K \Delta B) < \epsilon$, where $K \Delta B = (K \cap B^c) \cup (K^c \cap B)$. Such a K is obtained by first choosing O open such that $O \supseteq B$ and $\sigma(O - B) < \epsilon/2$ and then K clopen such that $K \subseteq O$ and $\sigma(O - K) < \epsilon/2$.

Lemma.

Let α, β, and γ be numbers between 0 and 1 with $\gamma = \alpha\beta$. Let $B \in \mathfrak{F}(\sigma)$, K be clopen, and $\sigma(K \Delta B) < \gamma$. Suppose K is determined by time s and define the incomplete stop rule t by

$$t(h) = \text{first } n \text{ (if any) such that } n \geq s(h) \text{ and } \sigma(B|p_n(h)) < 1-\alpha,$$
$$= +\infty \text{ if there is no such } n, \text{ for } h \in H.$$

Then

$$\sigma(K \cap [t < \infty]) < \beta$$

and, hence,

$$\sigma(K \cap [t = \infty]) > \sigma(K) - \beta > \sigma(B) - (\beta + \gamma).$$

<u>Proof of lemma</u>.

Let $\delta > 0$. By Corollary 5.3, $\sigma[t \le r] \to \sigma[t < \infty]$. Hence, there is a stop rule r_0 such that $r_0 \geq s$ and $\sigma[t \le r_0] \geq \sigma[t < \infty] - \delta$. Set $r = r_0 \wedge t$. Then $r \geq s$ and $[t \le r_0] = [t \le r] = [t = r]$. Notice that $K \cap [t \le r]$ is determined by time r. Let $\eta = \sigma(K \cap [t < \infty])$ and compute

$$\sigma(B^c \cap K \cap [t < \infty]) \geq \sigma(B^c \cap K \cap [t = r])$$

$$= \int \sigma[p_r]\,((B^c \cap K \cap [t = r])p_r)d\sigma$$

$$= \int_{K \cap [t = r]} \sigma[p_r]\,(B^c p_r)d\sigma \geq \alpha\, \sigma(K \cap [t = r])$$

$$= \alpha\, \sigma(K \cap [t \le r_0]) \geq \alpha(\eta - \delta).$$

Since δ was arbitrary, $\sigma(B^c \cap K \cap [t < \infty]) \geq \alpha\eta$. Hence, $\alpha\eta \le \sigma(B^c \cap K) < \gamma = \alpha\beta$. So $\eta < \beta$. □

Return now to the proof of the theorem.

Let $\epsilon > 0$. Let $\alpha_n, \beta_n, \gamma_n, \delta_n$ be numbers between 0 and 1 defined for each $n \in N$ and such that

(i) $\gamma_n = \alpha_n \beta_n$

(ii) $\beta_n < \delta_n^2$

(iii) $\alpha_n \to 0$

(iv) $(\sigma(A) - (\beta_1 + \gamma_1 + \delta_1)) \prod_{k=1}^{\infty} (1 - (\alpha_k + \beta_{k+1} + \gamma_{k+1} + \delta_k + \delta_{k+1})) > \sigma(A) - \epsilon.$

(14) [illegible]

(15) [illegible]

(16) [illegible]

(17) [illegible]

defined for each n ∈ K and each [illegible]

Let ε > 0. [illegible] be numbers as before [illegible]

Return now to the proof of the theorem.

So [illegible]

Since [illegible]

[illegible]

Proof of Lemma.

[illegible]

and hence

[illegible]

Thus [illegible]

It is not difficult to see that such numbers exist. One way to get started is to choose ϵ_k so that $(\sigma(A)-\epsilon_1)\prod_2^{\infty}(1-\epsilon_k)>\sigma(A)-\epsilon$. Then split up the ϵ_k in some manner to obtain the desired quantities.

Shortly a sequence $\{L_n\}$ of clopen sets and a pointwise strictly increasing sequence of stop rules $\{r_n\}$ will be constructed so as to have the following properties: For all $n = 1,2,\ldots$

(a) L_n is determined by time r_n,

(b) $\sigma(L_n - A) \le \gamma_n$,

(c) if $n > 1$, $L_n \subseteq \{h: \sigma(A|p_k(h)) \geq 1 - \alpha_{n-1}$ for $r_{n-1}(h) \le k \le r_n(h)\}$,

(d) if $q_n(h) = p_{r_n}(h)$ and

$$t_n(h) = \text{first } k \text{ (if any) such that } k \geq r_n(h) \text{ and } \sigma(A|p_k(h)) < 1 - \alpha_n$$

$$= +\infty \text{ if there is no such } k,$$

then

$$L_n \subseteq \{h: \sigma([t_n = \infty]|q_n(h)) > 1 - \delta_n\},$$

(e) $L_n \subseteq [t_n > r_n]$ (here t_n is as in (d)),

(f) $\sigma(L_1) \geq \sigma(A) - (\beta_1 + \gamma_1 + \delta_1)$, and, for $n > 1$, and $h \in L_{n-1}$,

$$\sigma(L_n|q_{n-1}(h)) \geq 1 - (\alpha_{n-1} + \beta_n + \gamma_n + \delta_{n-1} + \delta_n).$$

(Here, q_{n-1} is as in (d).)

Suppose temporarily that the construction has been carried out. Then the proof is easily completed.

It is not difficult to see that such numbers exist. One way to get them is to choose q_n so that $(r(A)-s_1) \cdot (1-q_n) > c(A)-\epsilon$. Then split [illegible]

the q_n in some manner to obtain the desired properties.

Shortly a sequence $\{L_n\}$ of closed sets and a nonnegative, [illegible] decreasing sequence of stop rules $\{r_n\}$ will be constructed so as to have the following properties: For all $n = 1, 2, \ldots$

(a) L_n is determined by the r_n,

(b) $c(L_n - A) \le \eta_n$,

(c) if $n > 1$, $L_n \subseteq \{h: c(A|s_n(h)) \ge 1 - q_{n-1}$ for $r_{n-1}(h) \le n \le r_n(h)\}$,

(d) if $c_n(h) = r_n(h)$ and

$$r_n(h) = \text{first } k \text{ (if any) such that } [illegible] \text{ and } c_1(A|s_k(h)) < 1 - q_n$$

$$= +\infty \text{ if there is no such } k,$$

then

$$L_n \subseteq \{h: c(I_n = \infty | s_n(h)) \ge 1 - q_n\}$$

(e) $L_n \subseteq [r_n > r_{n-1}]$ (here q_n [illegible] as in (d)).

(f) $c(L_1) \ge c(A) - (\epsilon_1 + \epsilon_1' + \ldots)$ and, for $n > 1$ and $h \in L_{n-1}$,

$$c(L_n | s_{r_{n-1}}(h)) \ge 1 - (\epsilon_{n-1} + \epsilon_n + \ldots + q_{n-1} + \ldots).$$

(Here, q_{n-1} is as in (d).)

Suppose temporarily that the construction has been carried out. Then the proof is easily completed.

Set $L = \cap L_n$. Then $\sigma(L - A) \le \sigma(L_n - A) \le \gamma_n$ (by (b)) $\le \alpha_n$ (by (i)) $\to 0$ (by (iii)). So $\sigma(L - A) = 0$. By (c) and (iii), $L \subseteq \{h\colon \sigma(A|p_k(h)) \to 1\}$. Thus, $\sigma(A \cap [\sigma(A|p_k) \to 1]) \ge \sigma(L)$. Finally, by Theorem 6.1 (a), (f), and (iv), $\sigma(L) \ge \sigma(A) - \epsilon$. Since ϵ is arbitrary, (1) follows.

It remains to show that the L_n and r_n exist. The proof will be by induction.

Choose K_1 clopen such that $\sigma(K_1 \Delta A) < \gamma_1$. Let r_1 be a stop rule such that K_1 is determined by time r_1. Let t_1 and q_1 be as in (d). Define

$$T_1 = [\sigma[q_1]([t_1 = \infty]q_1) > 1 - \delta_1].$$

$$L_1 = K_1 \cap T_1 \cap [t_1 > r_1].$$

Notice that L_1 satisfies (a) through (e) and so it is enough to check (f) to complete the case when $n = 1$.

Let $g(h) = \sigma[q_1(h)]([t_1 = \infty]q_1(h)) = \sigma([t_1 = \infty]|q_1(h))$, $h \in H$. Then $0 \le g \le 1$ and

$$\int_{K_1} g\,d\sigma = \sigma(K_1 \cap [t_1 = \infty]) \qquad \text{(Corollary 4.1)}$$

$$> \sigma(K_1) - \beta_1 \qquad \text{(by the Lemma)}$$

$$> \sigma(K_1) - \delta_1^2 \qquad \text{(by (ii))}.$$

It follows that

(2) $$\sigma(K_1 \cap T_1) = \sigma(K_1 \cap [g > 1 - \delta_1]) > \sigma(K_1) - \delta_1.$$

Also, by the lemma,

(3) $$\sigma(K_1 \cap [t_1 > r_1]) \ge \sigma(K_1 \cap [t_1 = \infty]) > \sigma(K_1) - \beta_1.$$

Hence, by (2) and (3),

Set $B = B_n$. Then $\mu(B - A) \le \mu(B_n - A) \le \varepsilon_n$ (by (6)) $\le \varepsilon_n$ (by (i)) $\to 0$ (by (ii')). So $\mu(B - A) = 0$. By (·) and (iii), $B \subseteq [f: \mu(A|p_n(f)) \to 1]$. Thus $\mu(A \cap [f: \mu(A|p_n) \to 1]) \ge \mu(B)$. Finally, by Theorem 6.1 (a), (b), and (iv), $\mu(B) \ge \mu(A) - \varepsilon$. Since ε is arbitrary (R) follows.

It remains to show that the B_n and r_n exist. The proof will be by induction.

Choose K_1 closed such that $\mu(K_1 - A) < \varepsilon_1$. Let r_1 be a rule such that K_1 is determined by time r_1. Let c_1 and d_1 be as in (4). Define

$$T_1 = [\omega[d_1]([c_1 = \omega]K_1) > 1 - \delta_1].$$

$$B_1 = K_1 \cap T_1 \cap [c_1 \le r_1].$$

Notice that B_1 satisfies (i) through (5) and so it is enough to check (6) to complete the case for $n = 1$.

Let $g(h) = \omega[d_1(h)]([c_1 = \omega]d_1(h)) = ([c_1 = \omega]|d_1(h))$, $h \in H$. Then $0 \le g \le 1$ and

$$\int_{K_1} g\,d\mu = \mu(K_1 \cap [c_1 = \infty]) \qquad \text{(corollary 4.1)}$$

$$\ge \mu(K_1) - \varepsilon_1' \qquad \text{(by the lemma)}$$

$$\ge \mu(K_1) - \varepsilon_1 \qquad \text{(by (ii))}.$$

It follows that

(7) $\mu(K_1 \cap T_1) = \mu(K_1 \cap [g > 1 - \delta_1]) \ge \mu(K_1) - \varepsilon_1'$.

Also, by the lemma,

(8) $\mu(K_1 \cap [c_1 \le r_1]) \ge \mu(K_1 \cap [c_1 = \infty]) \ge \mu(K_1) - \varepsilon_1'$.

Hence, by (7) and (8).

$$\sigma(L_1) = \sigma(K_1 \cap T_1 \cap [t_1 > r_1]) > \sigma(K_1) - (\beta_1 + \delta_1) > \sigma(A) - (\beta_1 + \gamma_1 + \delta_1),$$

which proves (f).

Assume now that $n > 1$ and that L_k and r_k are defined and satisfy (a) - (f) for $1 \leq k \leq n - 1$. It remains to find L_n and r_n with the desired properties.

For each $p \in X^*$, choose a clopen set $S(p)$ such that $\sigma[p](S(p) \Delta Ap) < \gamma_n$. Let $K_n = \bigcup_h q_{n-1}(h)S(q_{n-1}(h))$. Check that $K_n q_{n-1}(h) = S(q_{n-1}(h))$ for all $h \in H$ and K_n is clopen. Thus $\sigma[p]\ (K_n p \Delta Ap) < \gamma_n$ for p of the form $q_{n-1}(h)$. Suppose K_n is determined by time s_n and let $r_n = s_n V(r_{n-1} + 1)$ so that K_n is also determined by time r_n, a stop rule which is strictly larger than r_{n-1}. Let t_n and q_n be as in (d). Set

$$T_n = [\sigma([t_n = \infty] | q_n) > 1 - \delta_n],$$

$$L_n = K_n \cap T_n \cap [t_n > r_n] \cap [t_{n-1} > r_n].$$

We need to check (a) through (f) again.

Since each set occuring in the definition of L_n is determined by time r_n, (a) is true. To check (b), compute

$$\begin{aligned} \sigma(L_n - A) \leq \sigma(K_n - A) &= \int \sigma(K_n - A_n | q_{n-1}) d\sigma \\ &\leq \int \sigma(K_n \Delta A_n | q_{n-1}) d\sigma \\ &\leq \gamma_n . \end{aligned}$$

The final inequality holds because the integrand is uniformly bounded by γ_n. Since $L_n \subseteq [t_{n-1} > r_n]$, (c) is true. Also, (d) and (e) are clear from the definition of L_n. The proof will be complete once (f) is verified.

To do this, fix an $h \in H$ for this paragraph and set $p = q_{n-1}(h)$.

Define a stop rule s and an incomplete stop rule v by the formulas

$$s(h') = r_n(ph') - |p|$$

$$v(h') = t_n(ph') - |p|$$

for each $h' \in H$. These quantities are positive since $|p| = r_{n-1}(h)$. Then $[t_n = \infty]p = [v = \infty]$. Furthermore, the strategy $\sigma[p]$, the stop rule s, the incomplete stop rule v, and the sets $K_n p$ and A_p stand in the same relation as the σ, s, t, K, and B of the lemma when α, β, γ are replaced there by α_n, β_n, γ_n. It thus follows from the lemma that

(4) $$\sigma(K_n \cap [t_n > r_n] | p) \geq \sigma(K_n \cap [t_n = \infty] | p) > \sigma(K_n | p) - \beta_n.$$

An argument similar to that already given for the case $n = 1$ shows that

(5) $$\sigma(K_n \cap T_n | p) \geq \sigma(K_n | p) - \delta_n.$$

From this point on, assume $h \in L_{n-1}$. Then part (d) of the inductive hypothesis gives

(6) $$\sigma([t_{n-1} > r_n] | p) \geq \sigma([t_{n-1} = \infty] | p) > 1 - \delta_{n-1}.$$

By (4), (5), and (6),

$$\begin{aligned} \sigma(L_n | p) &\geq \sigma(K_n | p) - (\beta_n + \delta_{n-1} + \delta_n) \\ &\geq \sigma(A | p) - (\beta_n + \gamma_n + \delta_{n-1} + \delta_n). \\ &\geq 1 - (\alpha_{n-1} + \beta_n + \gamma_n + \delta_{n-1} + \delta_n). \end{aligned}$$

The last inequality uses part (e) of the inductive hypothesis. So (f) is verified.

This completes the inductive step. □

Another form of the Lévy martingale convergence theorem is proved as Theorem 3 in section 10.

Under the hypotheses of Theorem 1, the net $\sigma(A|p_r)$; r a stop rule, converges to 1_A in σ-probability as $r \to \infty$ through the stop rules (see the lemma in section 9). However, the sequence $\sigma(A|p_n)$; $n \in N$ need not do so.

Example: Let $X = N$. Let α and β be strategies which give probability 1 to the histories $(1,1,\ldots)$ and $(2,2,\ldots)$ respectively. Take σ to be a strategy such that $\sigma_0(\{n\}) = 0$ for all n, $\sigma(n_1,\ldots,n_k) = \delta(n_1)$ if $k < n_1$, and $\sigma[(n_1,\ldots,n_k)] = \frac{1}{2}\alpha + \frac{1}{2}\beta$ if $k = n_1$. Let $A = \{(n_1,n_2,\ldots)$: for some k, $n_k = 1\}$. Then A is open and, for all n, $\sigma\{h: \sigma(A|p_n(h)) = \frac{1}{2}\} = 1$.

By a tail set is meant a subset A of H such that $Ap = Ap'$ whenever p and p' are of the same length. Also, recall the definition of an independent strategy given in section 6.

In a measurable, countably additive setting such as that in section 11, an independent, measurable strategy would be one with respect to which the coordinate process on H is a sequence of independent random variables and a measurable tail set would be an element in the tail σ-field of the coordinate process. In such a setting, the next theorem would be the usual Kolmogorov 0-1 law.

Theorem 2: If σ is an independent strategy and A is a tail set in $\mathcal{C}(\sigma)$, then $\sigma(A)$ is 0 or 1.

Proof: First notice that, for all p, $\sigma^*(A|p) = \sigma(A)$. (Here $\sigma^*(A|p) = \sigma[p]^*(Ap)$.) To see this, fix $n \in N$. Then $\sigma[p_n(h)]$ is the same for all h and so is $Ap_n(h)$. But, by Corollary 4.1,

$$\sigma(A) = \int \sigma^*(A|p_n(h))\, d\sigma(h) = \sigma^*(A|p_n(h)).$$

Now let $\epsilon > 0$. Choose an open set O containing A such that $\sigma(O) < \sigma(A) + \epsilon$. Choose K clopen contained in O such that $\sigma(K) > \sigma(O) - \epsilon$.

Then
$$\begin{aligned}\sigma(A) &= \sigma(A \cap K) + \sigma(A \cap K^c) \\ &\leq \sigma(A \cap K) + \sigma(O \cap K^c) \\ &\leq \sigma(A \cap K) + \epsilon.\end{aligned}$$

Suppose K is determined by time r, where r is a stop rule. Then, by Corollary 4.1,

$$\begin{aligned}\sigma(A \cap K) &= \int \sigma^*(A \cap K \mid p_r)d\sigma \\ &= \int_K \sigma^*(A \mid p_r)d\sigma \\ &= \int_K \sigma(A)d\sigma \\ &= \sigma(A)\,\sigma(K).\end{aligned}$$

Hence,
$$\begin{aligned}\sigma(A) &\leq \sigma(A)\,\sigma(K) + \epsilon \\ &\leq \sigma(A)\,\sigma(O) + \epsilon \\ &\leq (\sigma(A))^2 + 2\epsilon .\end{aligned}$$

It follows that $\sigma(A) = 0$ or 1. □

Another proof of Theorem 2 is based on Theorem 1 and the remark about $\mathfrak{a}(\sigma)$ which follows Theorem 5.1.

The next result is rather curious.

<u>Theorem 3:</u> An independent strategy σ is countably additive when restricted to the collection of tail sets in $\mathfrak{F}(\sigma)$.

<u>Proof</u>: Let $A^1, A^2, \ldots$ be tail sets and in $\mathfrak{F}(\sigma)$. Assume $\sigma(A^n)=0$ for all n. By Theorem 2, it suffices to show $\sigma(\bigcup_1^\infty A^n) = 0$.

Let p be a partial history of length $n > 0$. Then, for all $h \in H$, $\sigma(A^n | p_n(h)) = \sigma(A^n | p)$. So $\sigma(A^n | p) = \int \sigma(A^n | p_n(h))d\sigma(h) = \sigma(A^n) = 0$. The theorem now follows from the remark after Theorem 5.2. □

Now let $\varepsilon > 0$. Choose an open set O containing A such that $\phi(O) \le \phi(A) + \varepsilon$. Then K is open contained in O such that $\phi(K) \ge \phi(O) - \varepsilon$.

$$\text{Then} \quad \phi(A) = \phi(A \cap K) + \phi(A \cap K^c)$$
$$\le \phi(A \cap K) + \phi(O \cap K^c)$$
$$\le \phi(A \cap K) + \varepsilon.$$

Suppose K is determined by time τ, where τ is a stop rule. Then, by Corollary 4.1,

$$\phi(A \cap K) = \int \phi^*(A \cap K \mid p_\tau)\,d\mu$$
$$= \int_K \phi^*(A \mid p_\tau)\,d\mu$$
$$= \int_K \phi(A)\,d\mu$$
$$= \phi(A)\,\phi(K).$$

$$\text{Hence,} \quad \phi(A) \le \phi(A)\,\phi(K) + \varepsilon$$
$$\le \phi(A)\,\phi(O) + \varepsilon$$
$$\le (\phi(A))^2 + 2\varepsilon.$$

It follows that $\phi(A) = 0$ or 1. □

Another proof of Theorem 9 is based on Theorem 4 and the remark about (c) which follows Theorem 5.1.

The next result is rather curious.

Theorem 11: An independent strategy σ is countably additive when restricted to the collection of tail sets in $B(\cdot)$. Assume $\phi(A^n) = 0$ for all n. By Theorem 7, it suffices to show $\phi(\bigcup_m A^n) = 0$. [illegible] $(A^n) = 0$ for all n. Therefore, it suffices to show $\phi(\bigcup A^n) = 0$.

Let p be a partial history of length $n \ge 1$. Then, for all $h \in H$, $\phi(A^n \mid \sigma_p^n(h)) = \phi(A^n \mid p)$. So $\phi(A^n \mid h) = \int \phi(A^n \mid \sigma_p^n(s))\,d\mu(s) = \phi(A^n) = 0$. The theorem now follows from the remark after Theorem 5.1. □

9. Some Integration Theory for Strategies

There is already available in the literature (cf. Chapter III of Dunford and Schwartz (1958)) a standard theory of integration with respect to a bounded, finitely additive measure on an algebra. Here we sketch the usual definition of integration via simple functions which simplifies a bit in the special case of strategies and point out some useful convergence theorems which apply to strategies. Another definition of integration, based on the inductive technique of Dubins and Savage (1965), is also useful and so is briefly introduced below. A few comments are made about the relationship between the two types of integration.

Let σ be a strategy, $n \in N$, and let $g = \sum_{i=1}^{n} a_i 1_{A_i}$ where $A_i \in \mathcal{A}(\sigma)$ and a_i is a real number for $i=1,\ldots,n$. Such a g is called a <u>σ-simple function</u> and we define

$$\sigma g = \int g d\sigma = \sum_{i=1}^{n} a_i \sigma(A_i).$$

Let $g, g_1, g_2, \ldots$ be real-valued functions defined on H. We say <u>g_n converges to g in σ-probability</u> and write $g_n \overset{\sigma}{\to} g$ if, for all $\epsilon > 0$, $\sigma^*[|g_n - g| > \epsilon] \to 0$ as $n \to \infty$. The definition for a generalized sequence $\{g_\alpha\}$ is similar.

Also, <u>g_n converges to g σ-almost surely</u> if $\{h | g_n(h) \to g(h)\}$ has σ-probability one.

A real-valued function g on H is <u>σ-integrable</u> if there is a sequence $\{g_n\}$ of σ-simple functions such that $g_n \overset{\sigma}{\to} g$ and $\sigma(|g_n - g_m|) \to 0$ as $n,$

$m \to \infty$. Set $\sigma g = \int g d\sigma = \lim_n \sigma g_n$. (It can be shown that the limit exists and the definition of σg is unambiguous.) This definition of the integral is consistent with that given in Dunford and Schwartz (1958) and the integral has the usual properties which are detailed there.

If $g \geq 0$ and $g \wedge n$ is σ-integrable for $n = 1,2,\ldots$, define $\sigma g = \lim_n \sigma(g \wedge n)$.

For a general real-valued g, write $g = g^+ - g^-$ and set $\sigma g = \sigma(g^+) - \sigma(g^-)$ whenever the right hand side makes sense.

A function g from H to the reals is, as usual, <u>Borel measurable</u> if $\{h | g(h) > r\}$ is a Borel subset of H for every real number r. It is straightforward to check that, if g is bounded and Borel measurable, then g is σ-integrable for every strategy σ and also that, if g is nonnegative and Borel measurable, then σg is well-defined for all σ. For simplicity, we will restrict ourselves to Borel measurable functions in what follows.

The standard convergence theorems such as the dominated convergence theorem ordinarily require in a finitely additive theory the hypothesis of convergence in probability rather than almost sure convergence. However, the following lemma provides us with a way around this obstacle and an easy path to variations on the dominated and monotone convergence theorems and Fatou's inequality. For the lemma and the next three theorems assume g, $g_1, \ldots$ are Borel measurable and σ is a strategy. Note that g_r is Borel measurable for every stop rule r, where $g_r(h) = g_{r(h)}(h)$.

<u>Lemma</u>: If $g_n \to g$ σ-almost surely as $n \to \infty$, then g_r converges to g

m[illegible]. Set $\nu g = \int g\,d\nu = \lim_n \nu(g \wedge n)$. (It can be shown that the limit exists and the definition of νg is unambiguous.) This definition of the integral is consistent with that given in Dunford and Schwartz (1958) and the integral has the usual properties which are detailed there.

If $g \geq 0$ and $g \wedge n$ is ν-integrable for $n = 1, 2, \ldots$, define
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A function g from H to the reals is, as usual, Borel measurable if $\{h: g(h) \leq r\}$ is a Borel subset of H for every real number r. It is straightforward to check that if g is bounded and Borel measurable, then g is ν-integrable [illegible]. If g is nonnegative and Borel measurable, then νg is well-defined for all ν. For simplicity we will restrict ourselves to Borel measurable functions in what follows.

The standard convergence theorems such as the dominated convergence theorem ordinarily require [illegible] finitely additive [illegible] the hypothesis of convergence in probability rather than almost sure convergence. However the following lemma provides [illegible] a way around this obstacle and an easy path to versions of the dominated and monotone convergence theorems and Fatou's inequality. For the lemma and the next three theorems assume g, $g_1, \ldots$ are Borel measurable and ν is a [illegible]. Note that g_n is Borel measurable for every [illegible], where $g_n(h) = g_{(n)}(h)$.

<u>Lemma</u>: If $g_n \to g$ ν-almost surely as $n \to \infty$, then νg_n converges to νg

in σ-probability as $r \to \infty$ through the directed set of stop rules.

Proof: Suppose $\sigma[\lim_n g_n = g] = 1$. Let $\epsilon > 0$, and set $A = [|g_n - g| \geq \epsilon \text{ i.o.}]$. then $[\lim_n g_n = g] \subseteq A^c$, so $\sigma(A) = 0$.

Also, if $A^n = \bigcup_{k \geq n} [|g_k - g| \geq \epsilon]$, then A^n decreases to A as $n \to \infty$.

Hence, by Theorem 5.2, $\sigma(A^r)$ converges to 0 as $r \to \infty$ through the stop rules. Since $[|g_r - g| \geq \epsilon] \subseteq A^r$, the proof is complete. □

The next theorem is immediate from the lemma and Theorem III. 3.7, p. 124 in Dunford and Schwartz (1958).

Theorem 1: Let $1 \leq \alpha < \infty$. Assume φ maps H to the reals, $|\varphi|^\alpha$ is σ-integrable, $|g_n(h)| \leq |\varphi(h)|$ for all n and h, and g_n converges to g σ-almost surely. Then $\sigma(|g|^\alpha) < \infty$, $\sigma(|g_r|^\alpha) < \infty$ for every stop rule r, and $\sigma(|g - g_r|^\alpha) \to 0$ as $r \to \infty$ through the stop rules.

Theorem 2: Assume $g_n \geq 0$ for all n and $g_n \uparrow g$ as $n \to \infty$. Then σg_r converges to σg as $r \to \infty$ through the stop rules.

Proof: If $\sigma g < \infty$, use Theorem 1. If $\sigma g = \infty$, then, for $k > 0$,

$$\lim_r \sigma g_r \geq \lim_r \sigma(g_r \wedge k)$$

$$= \sigma(g \wedge k) \to \infty \text{ as } k \to \infty. \quad \square$$

Theorem 3: Assume $g_n \leq 0$. Then $\sigma(\limsup_n g_n) \geq \limsup_r \sigma g_r$.

Proof: Let $W_n = \sup_{k \geq n} g_k$. Then $W_n \leq 0$ and $W_n \downarrow \limsup_k g_k$ as $k \to \infty$. By Theorem 2, $\sigma(W_r) \to \sigma(\limsup_n g_n)$. But $g_s \leq W_r$ if $s \geq r$ so that $\limsup_s \sigma g_s = \lim_r \sup_{s \geq r} \sigma g_s \leq \lim_r \sigma W_r$. □

Suppose g is a bounded, finitary function. Then, as sketched in section 2 of this paper, Dubins and Savage (1965) have given another definition of σg inductively on the structure. An inductive argument shows that the integral defined above is consistent with that of Dubins and Savage when restricted to the bounded, finitary functions.

Now consider a finitary function g which is not necessarily bounded and a strategy σ. Imitating Dubins and Savage, if g is a constant function c, define $\hat{\sigma}g = \int g d\hat{\sigma} = c$. Also, if $\varphi(x) = \widehat{\sigma[x]}(gx)$ has been defined on a set of x's having σ_0-probability one and if $\int \varphi d\sigma_0$ exists as a finite or infinite number, define $\hat{\sigma}g = \int g d\hat{\sigma} = \int \varphi d\sigma_0$. Let $I(\sigma)$ be the collection of g for which $\hat{\sigma}g$ is thus defined.

Theorem 4: Let σ be a strategy.

(a) If $g \in I(\sigma)$ and a is a real number, then $a\, g \in I(\sigma)$ and $\hat{\sigma}(ag) = a\, \hat{\sigma}(g)$.

(b) If $g, g' \in I(\sigma)$ and $\hat{\sigma}g + \hat{\sigma}g'$ is a well-defined number, then $g + g' \in I(\sigma)$ and $\hat{\sigma}(g + g') = \hat{\sigma}g + \hat{\sigma}(g')$.

(c) If $g \in I(\sigma)$, $\hat{\sigma}g < \infty$, g' is finitary and $g' \leq g$, then $g' \in I(\sigma)$ and $\hat{\sigma}\, g' \leq \hat{\sigma}g$.

(In particular, $I(\sigma)$ contains all finitary functions which are bounded from above or below.)

Proof: Straightforward using induction. □

We have had several useful conversations with Robert Chen about the contents of this section and, in particular, he showed us an example where σg and $\hat{\sigma}g$ are both well-defined and finite but unequal. Such an example could not occur in a measurable, countably additive theory as follows from

Proposition III. 2.1, p. 74 of Neveu (1965) and an inductive argument. However, in the general case, the following theorem is sometimes useful.

Theorem 6: Let σ be a strategy and g be a finitary, real-valued function on H.

(a) If g is bounded, $\sigma g = \hat{\sigma} g$.

(b) If g is nonnegative, $\sigma g \leq \hat{\sigma} g$.

Proof: Both (a) and (b) are proved by induction on the structure of g.

To prove (b), notice it is true if g has structure zero. Next assume it true for functions of smaller structure than g and let $g_n = g \wedge n$ for $n = 1, 2, \ldots$. Then $\sigma g = \lim_n \sigma g_n = \lim_n \hat{\sigma}\, g_n$ (by (a)) $= \lim_n \int \widehat{\sigma[x]}\, (g_n x)\, d\sigma_0(x) \leq \int \widehat{\sigma[x]}\, (gx)\, d\sigma_0(x) = \hat{\sigma} g$. □

10. Martingales

Let σ be a strategy. Let $Y = (Y^0, Y^1, Y^2, \ldots)$ be a sequence of real-valued functions on H such that for all n, Y^n is of structure at most n. So, for $n \geq 1$, Y^n depends only on the first n coordinates and we can write $Y^n(h) = Y^n(h_1, \ldots, h_n)$. If

$$\sigma_0 \, Y^1 \geq Y^0 ,$$

and, for all $n \geq 1$ and $(x_1, \ldots, x_n) \in X^n$,

$$\sigma(x_1, \ldots, x_n) \, Y^{n+1} (x_1, \ldots, x_n, \cdot) \geq Y^n(x_1, \ldots, x_n) ,$$

then Y is a submartingale with respect to σ. If the inequalities are reversed, Y is a supermartingale and if they are replaced by equations, Y is a martingale.

The major theorems about martingales in the conventional theory are the optional sampling theorem and the martingale convergence theorem (Theorems VII. 2.2 and VII. 4.1s of Doob (1953)). This section is mainly devoted to proving finitely additive versions of each of these. The simple proof of our optional sampling theorem (Theorem 1) is due to Dubins and Savage. They used it to prove Lemma 2.12.1 of [3] which is an optional sampling theorem for bounded martingales. The proof given here of the convergence theorem (Theorem 2) is an adaptation of Doob's original proof.

The results below are stated for submartingales. It is, of course, obvious how to change them so as to be appropriate for supermartingales or martingales.

If $Y = (Y^0, Y^1, \ldots)$ is a sequence of real-valued functions on H,

$p \in X^*$, and $n = |p|$, let

$$Yp = (Y_p^n, Y_p^{n+1}, \ldots).$$

In other terms, $(Yp)^k = (Y^{k+n})p$ for $k \geq 0$.

Lemma 1: If Y is a submartingale with respect to σ and $p \in X^*$, then Yp is a submartingale with respect to $\sigma[p]$.

Proof: Obvious. □

Theorem 1: Suppose Y is a submartingale with respect to σ, r is a stop rule, and $Y^r \in I(\sigma)$. Then $\hat{\sigma}\, Y^r \geq Y^0$. (Here, as usual, $Y^r(h) = Y^{r(h)}(h)$.)

Proof: Observe that $(Y^r)x = (Yx)^{r[x]}$, where $r[x](h) = r(xh) - 1$ for $h \in H$. Thus

$$\hat{\sigma}\, Y^r = \int \widehat{\sigma[x]}\, (Yx)^{r[x]}\, d\sigma_0(x).$$

Now use Lemma 1 and induction on the structure of r. □

This theorem gives a bit of new information even in the countably additive case where one usually considers incomplete stop rules t such that $\sigma[t < \infty] = 1$. Additional conditions on t or Y are then needed to guarantee that $\sigma\, Y^t \geq Y^0$. Such conditions can be used to ensure that $\sigma\, Y^t$ can be approximated by $\sigma\, Y^{t \wedge r}$ for large stop rules r. Even in the finitely additive case, the following is true.

Corollary 1: Suppose Y is a uniformly bounded submartingale with respect to σ and t is an incomplete stop rule for which $\sigma[t < +\infty] = 1$. Then $\sigma\, Y^t \geq Y^0$.

Proof: Since t is finite σ-almost surely, $t \wedge r \to r$ in σ-probability as $r \to \infty$ and, hence, $Y^{t \wedge r} \to Y^t$ in σ-probability. By Theorem III 3.8 of

Dunford and Schwartz (1958),

$$\sigma\, Y^t = \lim_r \sigma\, Y^{t \wedge r} = \lim_r \hat{\sigma}\, Y^{t \wedge r} \geq Y^0. \quad \Box$$

Theorem 2: Let Y be a submartingale with respect to σ. Assume that, for all $p \in X^*$ and all stop rules r, $|(Yp)^r| \in I(\sigma[p])$ and $\sup_r \widehat{\sigma[p]}\, |(Yp)^r| < +\infty$. Then $\sigma\{h | Y^n(h) \text{ converges}\} = 1$. Also, $\sigma |\liminf_n Y^n| < +\infty$.

Notice that the hypotheses of Theorem 2 are satisfied by bounded submartingales or even by submartingales which are dominated in an appropriate sense. Presumably the results of Dubins (1962) can be used to show also that submartingales which are uniformly bounded above converge with probability one.

The proof of Theorem 2 will be presented in several lemmas.

Lemma 2: Let $A^1, A^2, \ldots$ be subsets of H and suppose A^n has structure at most n for every n. Let J^n be the indicator function of A^n for every n and let r be a stop rule. Then

$$\hat{\sigma}\left(\sum_{k=1}^{r-1} J^k (Y^{k+1} - Y^k)\right) \geq 0.$$ (Here the sum is defined to be equal to zero where $r = 1$.)

Proof: Set $Z^0 = Z^1 = 0$ and, for $n \geq 2$, $Z^n = \sum_{k=1}^{n-1} J^k (Y^{k+1} - Y^k)$.

Check that Z is a submartingale with respect to σ. Use the hypotheses of Theorem 2, induction, and Theorem 1 to show $Z^r \in I(\sigma)$ and $\hat{\sigma}(Z^r) \geq 0$. □

Lemma 3 (Upcrossings inequality): Let $a < b$, r be a stopping time, and, for $h \in H$, $\beta_r(h)$ = the number of upcrossings of $[a, b]$ by $(Y^1(h),\ldots,Y^r(h))$. Then

$$\hat{\sigma}\, \beta_r \leq \frac{\hat{\sigma}\,(Y^r-a)^+}{b-a} \leq \frac{\hat{\sigma}\,(Y^r)^+ + |a|}{b-a}.$$

Proof: Set

$$S_1(h) = \inf\{j \mid j \leq r(h),\ Y^j(h) \leq a\}$$

$$T_1(h) = \inf\{j \mid S_1(h) \leq j \leq r(h),\ Y^j(h) \geq b\}$$

$$S_2(h) = \inf\{j \mid T_1(h) \leq j \leq r(h),\ Y^j(h) \leq a\}$$

et cetera. The convention here is that the empty set has infimum $= +\infty$.

Then $\beta_r = \max\{j \mid T^j < +\infty\}$.

Set

$$J^k(h) = 1 \quad \text{if } T^j(h) \leq k < S^{j+1}(h) \text{ for some } j,$$
$$= 0, \text{ otherwise.}$$

Then, for $k = 1,2,\ldots$, J^k has structure at most k. Also, a little thought shows

$$\sum_{k=1}^{r-1} J^k(Y^{k+1} - Y^k) \leq (a-b)\,\beta_r + (Y^r-a)^+.$$

By Lemma 2, $0 \leq (a-b)\,\hat{\sigma}(\beta_r) + \hat{\sigma}(Y^r-a)^+$, which proves the first inequality of the lemma. The second is a triviality. □

In the countably additive theory, the proof of Theorem 2 would now be

Proof: [illegible] Set $X_0 = X_1 = 0$ and, for [illegible], $= \sum_{i=1}^{j-1} c_i (X_{i+1} - X_i)$.

By the previous Lemma, [illegible] with [illegible].
Use the hypothesis of Theorem [illegible] Theorem 1 to show
$f^* = \Sigma(f)$ and $\hat{f}(\pi^*)$ [illegible].

Lemma 1 (Upcrossings inequality): Let $a < b$, [illegible] be a stopping time, and, for $n \in N$, $U_n(h) =$ the number of upcrossings of $[a, b]$ by $(Y_1(h), \ldots, Y_n(h))$.

Then

$$E\,U_n \le \frac{E(Y_n - a)^+}{b-a} \le \frac{E(Y_n)^+ + |a|}{b-a}.$$

Proof: Let

$$S_1(h) = \inf\{n : Y_n(h) \le a\}$$

$$T_1(h) = \inf\{n > S_1(h) : Y_n(h) \ge b\}$$

$$S_2(h) = \inf\{n > T_1(h) : Y_n(h) \le a\}$$

[illegible]. The [illegible] that the [illegible] infimum $= +\infty$.

Then $U_n = \max\{k : T_k \le n\}$.

Set

$$c_r(h) = 1 \text{ if } S_k(h) < r \le T_k(h) \text{ for some } k$$
$$= 0 \text{ otherwise.}$$

Then for $k = 1, \ldots, U_n$ [illegible] at least $b - a$. Also, a little thought shows

$$\sum_{r=1}^{n-1} c_r (Y_{r+1} - Y_r) \ge (b-a)\,U_n + (Y_n - a)^+$$

By Lemma [illegible], $(b-a)\,E\,U_n + E(Y_n - a)^+$ [illegible], which we may use since the [illegible] of the Lemma. The result then follows.

In the [illegible] this proof of Theorem 1 would [illegible] essentially complete. Here a little effort is still required.

Lemma 4: Let $a < b$ and, for $h \in H$, let $\beta(h)$ be the number of upcrossings of $[a, b]$ by the infinite sequence $Y(h) = (Y^0, Y^1(h), \ldots)$. Then $\sigma[\beta = +\infty] = 0$.

Proof: By the previous lemma and the hypotheses of Theorem 2, there is a constant B such that, for all stop rules r and positive integers k,

$$\sigma[\beta_r \geq k] \leq k^{-1}\, \sigma\beta_r \leq k^{-1}\, B.$$

But the sets $[\beta_n \geq k]$ increase up to $[\beta \geq k]$ as $n \to \infty$. So, by Theorem 5.2, $\sigma[\beta_r \geq k] \uparrow \sigma[\beta \geq k]$ as $r \to \infty$. Therefore, $\sigma[\beta = +\infty] \leq \sigma[\beta \geq k]$ $\leq k^{-1}B \to 0$ as $k \to +\infty$. □

Now let $I_1, I_2, \ldots$ be an enumeration of all closed intervals with rational endpoints and, for each n, let α_n be the number of upcrossings of I_n by Y. Let $A^n = [\alpha_n = +\infty]$ and $A = \bigcup_n A^n$.

Lemma 6: $\sigma(A) = 0$.

Proof: Let $p \in X^*$ and $n \in N$. Then Ap^n is the event that the submartingale Yp upcrosses I_n infinitely often. By the previous lemma, $\sigma[p](Ap^n) = 0$. Now use the remark at the end of Section 5. □

Since $A^c \subseteq [Y^n$ converges$]$, the first assertion of Theorem 2 is proved. The second follows from Theorems 9.6(b) and 9.3.

The next result is a version of the Lévy martingale convergence theorem. The proof imitates the usual one, but it yields somewhat less information, at least in the case of indicator functions, then did the proof of Theorem 8.1.

Theorem 3: Let σ be a strategy, g be a bounded, Borel measurable function from H to the reals, and define $Y^0 = \sigma g$, $Y^n = \sigma(g|p_n)$ for $n \in N$. Then $Y = (Y^0, Y^1, \ldots)$ in a bounded martingale with respect to σ and, hence, converges σ-almost surely. Moreover, if $\varphi = \lim_n \sup Y^n$, then, for every $\epsilon > 0$, $\sigma\{h| \ |\varphi(h) - g(h)| > \epsilon\} = 0$.

Proof: One can easily check that Y is a bounded martingale with respect to σ and thus converges σ-almost surely by Theorem 2. Since $\sigma(g|p_n) \to \varphi$ with σ-probability one and the $\sigma(g|p_n)$ are uniformly bounded, we have by Theorem 9.1 $\int_A \sigma(g|p_r)d\sigma \to \int_A \varphi d\sigma$ as $r \to \infty$ through the stop rules for every Borel set A.

Suppose A is clopen. Then, if A is determined by time r, $\sigma(1_A g|p_r) = 1_A \sigma(g|p_r)$ and

$$\begin{aligned}\int_A \sigma(g|p_r)d\sigma &= \int \sigma(1_A g|p_r)d\sigma \\ &= \int 1_A g d\sigma \\ &= \int_A g d\sigma \ .\end{aligned}$$

Hence, $\int_A g d\sigma = \int_A \varphi d\sigma$ for A clopen.

If A is an arbitrary Borel set, the same equation holds since one can take a sequence $\{A_n\}$ of clopen sets such that $\sigma(A_n \Delta A) \to 0$ and use dominated convergence (Theorem 9.1).

Now take $A = \{h: g(h) - \varphi(h) > \epsilon\}$. Since $\int_A (g - \varphi)d\sigma = 0$, $\sigma(A) = 0$. Likewise, $\sigma\{h: \varphi(h) - g(h) > \epsilon\} = 0$. □

In an important special case, the inequality of Theorem 9.3 becomes a

equation. The corresponding result in the conventional theory is in Sudderth (1971a). The proof here is essentially the same.

Theorem 4: Let $g_1, g_2, \ldots$ be a uniformly bounded sequence of functions from H to the reals and suppose that g_n is of structure at most n for every n. Then, for every strategy σ,

$$\limsup_r \sigma g_r = \sigma(\limsup_n g_n),$$

where the first lim sup is taken over the stop rules and the second over the positive integers.

Proof: Let $g = \limsup_n g_n$. By Theorem 9.3, $\limsup_r \sigma g_r \leq \sigma g$.

To prove the opposite inequality, let $\epsilon > 0$ and s be a stop rule. It suffices to find a stop rule r such that $r \geq s$ and $\sigma g_r \geq \sigma g - \epsilon$.

Let $\delta > 0$. Define, for $h \in H$, $t(h) = \inf\{k | k \geq s(h)$ and $\sigma(g|p_k)(h) < g_k(h) + \delta\}$. Then t is an incomplete stop rule and it follows from Theorem 3 that $\sigma[t < \infty] = 1$.

Choose a stop rule r^1 such that $r^1 \geq s$ and $\sigma[t \leq r^1] > 1 - \delta$ as is possible by Corollary 5.3. Set $r = t \wedge r^1$. So $\sigma[t = r] > 1 - \delta$ and $\sigma[\sigma(g|p_r) < g_r + \delta] > 1 - \delta$. Hence, if B is a bound on the absolute values of the g_n,

$$\begin{aligned} \sigma g - \sigma g_r &= \int\{\sigma(g|p_r) - g_r\}\, d\sigma \\ &\leq \delta(1 + 2B) \\ &< \epsilon \quad \text{for } \delta \text{ sufficiently small.} \quad \square \end{aligned}$$

This theorem has an interesting interpretation for gambling problems. Namely, let u be a bounded map from X to the reals and let σ be a strategy. Then Dubins and Savage (1965) define the utility under u of σ to be

$$u(\sigma) = \limsup_{r \to \infty} \int u\,(h_r)\,d\sigma .$$

By the theorem,

$$u(\sigma) = \int \limsup_{n \to \infty} u(h_n)\,d\sigma .$$

11. Relation to Countably Additive Theory.

If a strategy σ satisfies conventional measurability and countable additivity assumptions, then standard countably additive extension theorems can be applied. It is shown below that the present finitely additive extension is consistent with the conventional one and assigns measure to as many sets. Thus the finitely additive probability theorems of previous sections are, in a sense, extensions of the conventional theory.

Let $\mathcal{B}$ be a sigma-field of subsets of X and let $\mathcal{B}^\infty = \mathcal{B} \times \mathcal{B} \times \cdots$ be the product sigma-field of subsets of H. (If $\mathcal{B}$ is the set of all subsets of X, then $\mathcal{B}^\infty$ is just the collection of all Borel subsets of H.) It is assumed in this section that σ is a measurable strategy with respect to $\mathcal{B}$. That is, σ is assumed to satisfy

(i) σ_0 is countably additive when restricted to $\mathcal{B}$ and, for every finite sequence $(x_1, \ldots, x_n)$, $\sigma(x_1, \ldots, x_n)$ is countably additive when restricted to $\mathcal{B}$,

(ii) for every n and every $B \in \mathcal{B}$, $\sigma(x_1, \ldots, x_n)(B)$ is a jointly measurable function of $(x_1, \ldots, x_n)$.

Then, as is well-known, there is a unique countably additive probability measure ν on $\mathcal{B}^\infty$ such that $\nu(A) = \sigma(A)$ for every cylinder set A; that is, for every set A of the form $B_1 \times B_2 \times \cdots$ where each $B_i \in \mathcal{B}$ and $B_i = X$ for all but finitely many i. Let $\mathcal{C}$ be the completion of $\mathcal{B}^\infty$ under ν.

Theorem 1.

If σ is a measurable strategy with respect to $\mathcal{B}$, then $\mathcal{A}(\sigma)$ contains $\mathcal{C}$ and σ agrees with ν on $\mathcal{C}$.

The proof is given in several rather technical lemmas. The heart of the argument is Lemma 2.

Lemma 1.

Let K be a clopen set and let $K \in \mathcal{B}^{\infty}$. Then $\sigma(K) = \nu(K)$.

Proof:

The proof is by induction on the structure of K and is presented in detail in Section 2 of Sudderth (1971b). □

Lemma 2.

Let t be a $\mathcal{B}^{\infty}$-measurable incomplete stop rule. Then $\sigma[t < \infty] = \nu[t < \infty]$.

Proof:

Notice that

$$\begin{aligned}\sigma[t < \infty] &= \sup\{\sigma[t \leq s]: s \text{ a stop rule}\} \text{ (by Corollary 5.3)}\\ &\geq \sup\{\sigma[t \leq n]: n \text{ a positive integer}\}\\ &= \sup\{\nu[t \leq n]: n \text{ a positive integer}\} \text{ (by the previous lemma)}\\ &= \nu[t < \infty].\end{aligned}$$

The final equation above uses the countable additivity of ν on $\mathcal{B}^{\infty}$.

To complete the proof it suffices to show that, for every stop rule s,

$$(1) \qquad \sigma[t \leq s] \leq \sup_n \sigma[t \leq n] .$$

The proof of (1) is by induction on the structure of s. If s is constant, (1) is clear. It remains to check the inductive step.

Recall that

$$s[x](h) = s(xh) - 1,$$

and set

$$t[x](h) = t(xh) - 1.$$

Notice that, for each x, $s[x]$ is either a stop rule or identically equal to zero. Also, $s[x]$ has smaller structure than that of s if the structure of s is larger than zero. Similarly, $t[x]$ is either a $\mathcal{B}^{\infty}$-measurable incomplete stop rule or identically zero. Finally, the conditional

Lemma 1.

Let K be a clopen set and let $K \in \mathcal{C}^{\infty}$. Then $\gamma(K) = \nu(K)$.

Proof:

The proof is by induction on the structure of K and is presented in detail in Section 2 of [4]. (1.11). □

Lemma 2.

Let t be a $\mathcal{C}^{\infty}$-measurable incomplete stop rule. Then $\gamma[t < \infty] = \nu[t < \infty]$.

Proof:

Notice that

$$\gamma[t < \infty] = \sup\{\gamma[t \le s]: s \text{ a stop rule}\} \text{ (by Corollary 3.2)}$$
$$\le \sup\{\gamma[t \le n]: n \text{ a positive integer}\}$$
$$= \sup\{\nu[t \le n]: n \text{ a positive integer}\} \text{ (by the previous lemma)}$$
$$= \nu[t < \infty].$$

The final equation above uses the countable additivity of ν on $\mathcal{C}^{\infty}$.

To complete the proof it suffices to show that, for every stop rule s,

$$(10.1) \qquad \gamma[t \le s] \le \sup_n \gamma[t \le n].$$

The proof of (10.1) is by induction on the structure of s. If s is constant, (10.1) is clear. It remains to check the inductive step.

Recall that

$$s[x](h) = s(xh) - 1,$$

and set

$$t[x](h) = t(xh) - 1.$$

Notice that, for each x, $s[x]$ is either a stop rule or identically equal to zero. Also, $s[x]$ has smaller structure than that of s if the structure of s is larger than zero. Similarly, $t[x]$ is either a $\mathcal{C}^{\infty}$-measurable incomplete stop rule or identically zero. Finally, the conditional strategy $\sigma[x]$ is measurable, for each x, because σ is. Now compute

(2) $$\sigma[t \le s] = \int \sigma[x]([t \le s]x)d\sigma_0(x)$$
$$= \int \sigma[x][t[x] \le s[x]]d\sigma_0(x)$$
$$\le \int \sup_n \sigma[x][t[x] \le n]d\sigma_0(x).$$

The inequality follows from the inductive assumption.

Let $\epsilon > 0$. For $x \in X$, define

$$N(x) = \min\{k:(\sigma[x][t[x] \le k]) \ge (\sup_n \sigma[x][t[x] \le n]) - \epsilon\},$$

and let $M(h) = N(h_1) + 1$ for $h \in H$. Then, by (2),

(3) $$\sigma[t \le s] \le \int \sigma[x][t[x] \le N(x)]d\sigma_0(x) + \epsilon$$
$$= \int \sigma[x]([t \le M]x)d\sigma_0(x) + \epsilon$$
$$= \sigma[t \le M] + \epsilon$$
$$= \nu[t \le M] + \epsilon .$$

The last step, which follows from Lemma 1, requires that M be $\mathcal{B}^\infty$-measurable. This will follow easily from the $\mathcal{B}$-measurability of the function

$$x \to \sigma[x][t[x] \le n], \quad x \in X.$$

This has the form $x \to \sigma[x]Ax$, where A is $\mathcal{B}^\infty$-measurable and has finite structure. The quantity $\sigma[x]Ax$ can be evaluated in a natural way as an iterated integral (see, for example, formula 2.6.1 in Dubins and Savage (1965) involving finitely additive extensions of the countably additive $\sigma(p)$'s. A little reflection shows that the iterated integral has the same value as the usual Lebesgue integral. The $\mathcal{B}^\infty$-measurability of $x \to \sigma[x]Ax$ then follows by the standard arguments.

strictly $\varphi[x]$ is measurable, for each x, because φ is. Now compute

$$(17.2)\qquad \varphi[t \leq s] = \int \varphi[x]([t \leq s]x)d\nu_0(x)$$

$$= \int \varphi[x][t[x] \leq s[x]]d\nu_0(x)$$

$$\leq \int \sup_n \varphi[x][t[x] \leq n]d\nu_0(x).$$

The inequality follows from the inductive assumption.

Let $\varepsilon > 0$. For $x \in X$, define

$$N(x) = \min\{n:(\varphi[x][t[x] \leq n]) \geq (\sup_m \varphi[x][t[x] \leq m]) - \varepsilon\},$$

and let $M(x) = N(x) + 1$ for $x \in X$. Then, by (17.2),

$$(17.3)\qquad \varphi[t \leq s] \leq \int \varphi[x][t[x] \leq M(x)]d\nu_0(x) + \varepsilon$$

$$= \int \varphi[x]([t \leq M]x)d\nu_0(x) + \varepsilon$$

$$= \varphi[t \leq M] + \varepsilon$$

$$= \psi[t \leq M] + \varepsilon .$$

The last step, which follows from Lemma 1, requires that M be ν-measurable. This will follow easily from the ν-measurability of the function

$$x \to \varphi[x][t[x] \leq n]; \quad x \in X.$$

This has the form $x \to \varphi[x]_{A_x}$ where A is ν-measurable and has finite structure. The quantity $\varphi[x]_{A_x}$ can be evaluated in a natural way as an iterated integral (see, for example, formula 2.1.1 in [2]) involving [illegible] finitely additive extensions of the countably additive $\nu(\cdot)$'s. [illegible] reflection shows that the iterated integral has the same value as the usual Lebesgue integral. The ν-measurability of $x \to \varphi[x]_{A_x}$ then follows by the standard arguments.

Since M is $\mathcal{B}^\infty$-measurable and ν is countably additive, there exists an integer n such that $\nu[M \leq n] \geq 1 - \epsilon$. So, by (3), $\sigma[t \leq s] \leq \nu[t \leq M] + \epsilon \leq \nu[t \leq n] + 2\epsilon = \sigma[t \leq n] + 2\epsilon$. The last equation is by Lemma 1. Since ϵ was arbitrary, (1) is now proved. □

Let $\mathcal{T}$ be the collection of all $\mathcal{B}^\infty$-measurable incomplete stop rules t. For $A \subseteq H$, define

$$\nu^*(A) = \inf\{\nu[t < \infty]: t \in \mathcal{T}, A \subseteq [t < \infty]\},$$

and

$$\nu_*(A) = \sup\{\nu[t = \infty]: t \in \mathcal{T}, A \supseteq [t = \infty]\}.$$

Notice $\nu_*(A) = 1 - \nu^*(A^c)$.

Let

$$\mathcal{C}' = \{A \subseteq H: \nu^*(A) = \nu_*(A)\}.$$

<u>Lemma 3</u>.

The collections $\mathcal{C}$ and $\mathcal{C}'$ coincide. Also, ν^* restricted to $\mathcal{C}$ is the completion of ν and, in particular, ν^* is countably additive on $\mathcal{C}$.

<u>Proof</u>:

First notice that $\mathcal{C}'$ is a sigma-field. To see this check in order that $\mathcal{C}'$ is closed under the taking of complements, finite unions, and countable increasing unions.

Now let A be a cylinder set in $\mathcal{B}^\infty$. Then there is an $n \in N$ and a set $B \subseteq X^n$ such that $A = \{(x_1, \ldots, x_n, \ldots) : (x_1, \ldots, x_n) \in B\}$. Define $t(h) = \infty$ or n according as $h \notin A$ or $h \in A$; and $v(h) = \infty$ or n according as $h \in A$ or $h \notin A$. Then $t, v \in \mathcal{T}$ and $[t < \infty] = A = [v = \infty]$. Thus $A \in \mathcal{C}'$ and $\mathcal{C}' \supseteq \mathcal{B}^\infty$.

To see $\mathcal{C}' \subseteq \mathcal{C}$, let $A \in \mathcal{C}'$. Write O for sets of the form $[t < \infty]$ and C for sets of the form $[t = \infty]$ when $t \in \mathcal{T}$. Then there exist sets

Since M is $\mathscr{F}^\infty$-measurable and v is countably additive, there exists an integer n such that $v[M \le n] \ge 1 - \epsilon$. So, by (10.3), $c[t \le e] \le v[t \le M] + \epsilon \le v[t \le n] + 2\epsilon = c[t \le n] + 2\epsilon$. The last equation is by Lemma 1. Since ϵ was arbitrary, (11.1) is now proved. □

Let $\mathscr{T}$ be the collection of all $\mathscr{F}^\infty$-measurable incomplete stop rules t. For $A \subseteq H$, define

$$v^*(A) = \inf\{v[t < \infty]: t \in \mathscr{T}, A \subseteq [t < \infty]\},$$

and

$$v_*(A) = \sup\{v[t = \infty]: t \in \mathscr{T}, A \subseteq [t = \infty]\}.$$

Notice $v_*(A) = 1 - v^*(A^c)$.

Let

$$\mathscr{C}' = \{A \subseteq H: v^*(A) = v_*(A)\}.$$

Lemma 2.

The collections $\mathscr{C}$ and $\mathscr{C}'$ coincide. Also, v^* restricted to $\mathscr{C}$ is the completion of v and, in particular, v^* is countably additive on $\mathscr{C}$.

Proof:

First notice that $\mathscr{C}'$ is a sigma-field. To see this check in order that $\mathscr{C}'$ is closed under the taking of complements, finite unions, and countable increasing unions.

Now let A be a cylinder set in $\mathscr{C}$. Then there is an $n \in N$ and a set $B \subseteq X^n$ such that $A = \{(x_1, \ldots, x_n, \ldots) : (x_1, \ldots, x_n) \in B\}$. Define $t(h) \equiv \infty$ or n according as $h \notin A$ or $h \in A$; and $v(h) = \infty$ or n according as $h \in A$ or $h \notin A$. Then $t, v \in \mathscr{T}$ and $[t < \infty] = A = [v = \infty]$. Thus $A \in \mathscr{C}'$ and $\mathscr{F}^\infty \subseteq \mathscr{C}'$.

To see $\mathscr{C}' \subseteq \mathscr{C}$, let $A \in \mathscr{C}'$. Write O for sets of the form $[t < \infty]$ and C for sets of the form $[t = \infty]$ when $t \in \mathscr{T}$. Then there exist sets

O_n and C_n for $n \in N$, such that the O_n's are decreasing, the C_n's are increasing, $O_n \supseteq A \supseteq C_n$, $\nu(O_n) \to \nu^*(A)$, and $\nu(O_n - C_n) \to 0$. Then $\cup C_n \subseteq A$, $A - \cup C_n \subseteq \cap O_n - \cup C_n$, and $\nu(\cap O_n - \cup C_n) = 0$. Thus A differs from $\cup C_n$ by a subset of a $\mathcal{B}^\infty$ set which is ν-null and, hence, $A \in \mathcal{C}$. Notice also that $\nu^*(A) = \nu(\cup C_n)$. Hence, ν^* agrees with the completion of ν on $\mathcal{C}'$. But $\mathcal{C}'$ is clearly complete for ν^* and so is complete for ν. Therefore $\mathcal{C} = \mathcal{C}'$. □

The next lemma finishes the proof of Theorem 1.

Lemma 4.

For every $A \subseteq H$,

$$\nu^*(A) \geq \sigma^*(A) \geq \sigma_*(A) \geq \nu_*(A).$$

Hence, $\mathcal{A}(\sigma) \supseteq \mathcal{C}$ and σ^* agrees with ν^* on $\mathcal{C}$.

Proof:

Easy, but it requires Lemma 2. □

Two brief remarks conclude this section.

Suppose X is finite or countable and $\mathcal{B}$ is the set of all subsets of X. Then every incomplete stop rule is $\mathcal{B}^\infty$-measurable. Hence, $\sigma^* = \nu^*$ and $\mathcal{A}(\sigma)$ is just the usual completion of $\mathcal{B}^\infty$ under ν. In particular, the usual examples of non-measurable sets give examples of sets not in $\mathcal{A}(\sigma)$.

Finally, $\mathcal{A}(\sigma)$ is sometimes strictly larger than $\mathcal{C}$, since $\mathcal{A}(\sigma)$ always contains all clopen sets and it can easily happen that some clopen sets are not measurable.

O_n and C_n for $n \in N$, such that the O_n's are decreasing, the C_n's are increasing, $O_n \supseteq A \supseteq C_n$, $v(O_n) \to v^*(A)$, and $v(O_n - C_n) \to 0$. Then $\bigcap O_n \supseteq A$, $A - \bigcup C_n \subseteq \bigcap O_n - \bigcup C_n$, and $v(\bigcap O_n - \bigcup C_n) = 0$. Thus A differs from $\bigcup C_n$ by a subset of a [illegible] set which is v-null and, hence, $A \in$ [illegible]. Notice also that $v^*(A) = v(\bigcup C_n)$. Hence, v^* agrees with the completion of v on [illegible]'. But [illegible]' is clearly complete for v^* and so is complete for v. Therefore [illegible] = [illegible]'. □

The next lemma finishes the proof of Theorem 1.

<u>Lemma 4.</u>

For every $A \subseteq E$,

$$v^*(A) \geq \sigma^*(A) \geq \sigma_*(A) \geq v_*(A).$$

Hence, [illegible]$(\sigma) \supseteq$ [illegible] and σ^* agrees with v^* on [illegible].

<u>Proof:</u>

Easy, but it requires Lemma 2. □

Two brief remarks conclude this section.

Suppose E is finite or countable and [illegible] is the set of all subsets of E. Then every incomplete stop rule is [illegible]$^\infty$-measurable. Hence, $\sigma^* = v^*$ and [illegible](σ) is just the usual completion of [illegible]$^\infty$ under v. In particular, the usual examples of non-measurable sets give examples of sets not in [illegible](σ).

Finally, [illegible](σ) is sometimes strictly larger than [illegible], since [illegible](σ) always contains all clopen sets and it can easily happen that some clopen sets are not measurable.

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