A Mixture with Local Kelvin-Voigt Damping

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Dedication

This dissertation is dedicated to my father.

Contents

Chapter 1

Introduction

In this thesis, we will further explore a theory of viscoelastic mixtures proposed by [Iesan, 2004] and analyzed by [Quintanilla, 2005]. In the first of these papers, the author Iesan introduced a theory for a mixture between a viscoelastic material and an elastic material. In this theory, the dissipation effects are determined by the viscosity of rate type of a constituent and the relative velocity. [Quintanilla, 2005] states the linear equations of the thermomechanical deformations. Then, he studies several qualitative properties such as well-posedness and exponential stability. An important result of the paper of [Quintanilla, 2005] is a proof of exponential decay of solution when only one of the dissipative mechanisms is present in the case of anti-shear deformations. We concentrate our attention on a similar case where only one of the dissipative mechanisms is present and is localized. This means that the dissipation can be zero in a part of the domain, but have positive measure support. The formulation of the viscoelastic mixture in [Quintanilla, 2005] results in the following system of equations.

$$\begin{cases} u_{tt} &= \alpha u_{xx} + \beta w_{xx} - \xi(u-w) \\ w_{tt} &= \beta u_{xx} + \gamma w_{xx} + \xi(u-w) + (a(x)w_{tx})_x \\ & u(0) = u(\pi) = 0 \qquad w(0) = w(\pi) = 0 \end{cases}$$

under the following conditions:

i) $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ is positive definite

ii) $\inf(a(x)) > 0$ on $(x_1, x_2) \subset [0, \pi] \qquad a(x) = 0$ on $[0, \pi] \sim (x_1, x_2)$

As in [Quintanilla, 2005], we also place the following restriction on β, ξ:

$$\beta < 0 \quad \xi \geq 0$$

In order to provide preliminaries for our proof of polynomial stability, we begin by introducing the theory of semigroups. This introduction of semigroup theory is largely paraphrased in part by [Engel and Nagel, 2000] and also from [Kreyszig, 1989]. We also state an important theorem proved by Borichev and Tomilov to prove polynomial decay of solutions for the case discussed in [Quintanilla, 2005] with localized damping. Then, the final section introduces the equations that govern the problem of viscoelastic mixtures. It is in this section that we include proof of polynomial stability of such a system.

Chapter 2

Preliminaries

The theory of one parameter semigroups was developed over the course of the past fifty years. It can be said that the theory has its roots in the Hille-Yosida generation theorem, and attained its first milestone with the publication of the book Semigroups and Functional Analysis by [Hille and Phillips, 1996]. Proceeding this, the theory became highly developed as a result of numerous mathematicians. Many deterministic systems can be described by maps $S(t)$, $t \geq 0$ satisfying the functional equation

$$S(t+s) = S(t)S(s) \tag{2.1}$$

where we consider t as the time parameter, and each $S(t)$ maps the "state" of the system therefore determining the time evolution of the space. Most importantly, if one is given an initial state x_0 at time $t_0 = 0$, then at time t the space is in state $S(t)x_0$.

Definition 1 *[Liu and Zheng, 1999] A family $S(t)(0 \leq t < \infty)$ of bounded linear operators in a Banach space H is called a one-parameter semigroup (in short, a semigroup), if*

1. $S(t_1 + t_2) = S(t_1)S(t_2)\ \forall t_1, t_2 \geq 0$

2. $S(0) = I$

We now introduce typical examples of semigroups on a Banach space which become increasingly abstract.

2.1 Matrix operator semigroups.

Definition 2 *[Engel and Nagel, 2000] For any matrix $A \in M_n(\mathbb{C})$ and $t \in \mathbb{R}$, the matrix exponential e^{tA} is defined by*

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

For any $A \in M_n(\mathbb{C})$, the map $S(\cdot) : t \in \mathbb{R} \mapsto e^{tA} \in M_n(\mathbb{C})$ is continuous and satisfies the definition of a semigroup. Thus, we call (e^{tA}) the one-parameter semigroup generated by the matrix $A \in M_n(\mathbb{C})$. In fact, any such map is differentiable and satisfies the differential equation

$$\frac{d}{dt}S(t) = AS(t) \quad t \geq 0 \tag{2.2}$$

$$S(0) = I \tag{2.3}$$

While every matrix A generates a semigroup, we also have the following theorem

Theorem 2.1.1 *[Engel and Nagel, 2000] Every continuous semigroup $(S(t))_{t\geq 0}$ on $\mathbb{C}^n$ is of the form*

$$S(t) = e^{tA}, \quad t \geq 0 \tag{2.4}$$

for some complex valued matrix $A \in M_n(\mathbb{C})$.

Thus not only does every $A \in M_n(\mathbb{C})$ generate a semigroup through the map $t \in \mathbb{R} \mapsto$

$e^{tA} \in M_n(\mathbb{C})$, but also, conversely, every semigroup is generated by such a mapping. This relationship is vital in describing the asymptotic behavior of such a family of operators.

Definition 3 *[Engel and Nagel, 2000] A continuous one-parameter semigroup is stable if*

$$\lim_{t\to\infty} \|e^{tA}\| = 0,$$

The classical Liapunov stability theorem characterizes stability in terms of the location of the eigenvalues of A.

Theorem 2.1.2 *[Engel and Nagel, 2000] Let $(e^{tA})_{t\geq 0}$ be the one-parameter semigroup generated by $A \in M_n(\mathbb{C})$. Then the following statements are equivalent.*

- *The semigroup is stable, i.e., $\lim_{t\to\infty} \|e^{tA}\| = 0$.*
- *All eigenvalues of A have negative real part.*

Such a conclusive result cannot be so directly found when the semigroup is a family of more abstract operators.

2.2 Uniformly Continuous Operator Semigroups

We now wish to present the same results for semigroups $(T(t))_t \geq 0$ on an infinite dimensional Banach space X. We briefly state some of the necessary notions. For an operator $A \in \mathcal{L}(X)$ where $\mathcal{L}(X)$ denotes the set of all bounded linear operators of some Banach space X, we denote by $\sigma(A)$ its spectrum. For A, its spectrum $\sigma(A)$ is defined to be the set of all $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not bijective. The resolvent set of an operator $\rho(A)$ is the set $\{\lambda \in \mathbb{C} : \lambda \notin \sigma(A)\}$. From the fact that

$\sigma(A)$ is a compact subset of $\mathbb{C}$, $\rho(A)$ is open and it can be shown that the resolvent map

$$R(\lambda, A) = (\lambda - A)^{-1} \in \mathcal{L}(X) \tag{2.5}$$

yields an analytic map from $\rho(A)$ into $\mathcal{L}(X)$. We construct a similar argument as for matrix semigroups under the condition that the operator is uniformly continuous.

Definition 4 *[Engel and Nagel, 2000] For any bounded operator $A \in \mathcal{L}(X)$, e^{tA} is defined by*

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

for each $t \geq 0$.

Definition 5 *[Engel and Nagel, 2000] A family $S(t)(0 \leq t < \infty)$ of bounded linear operators in a Banach space X is called a uniformly continuous semigroup, if*

1. $S(t_1 + t_2) = S(t_1)S(t_2)\ \forall t_1, t_2 \geq 0$
2. $S(0) = I$
3. $t \mapsto S(t)$ *is continuous on* $t \in [0, \infty)$

With these definitions in place, we can obtain results akin to those for matrix semigroups.

Theorem 2.2.1 *[Engel and Nagel, 2000] For $A \in \mathcal{L}(X)$ and (e^{tA}) as previously defined, the following properties hold:*

1. $S(t) = (e^{tA})$ *is a semigroup on* X *such that*

$$S(\cdot) : t \in \mathbb{R} \mapsto e^{tA} \in \mathcal{L}(X) \tag{2.6}$$

is continuous.

2. *The map* $S(\cdot) : t \in \mathbb{R} \mapsto e^{tA} \in \mathcal{L}(X)$ *is differentiable and satisfies the differential equation*

$$\frac{d}{dt}S(t) = AS(t) \quad t \geq 0 \tag{2.7}$$

$$S(0) = I \tag{2.8}$$

As we were able to classify matrix semigroups, so we are also able to classify uniformily continuous semigroups.

Theorem 2.2.2 *[Engel and Nagel, 2000] Every uniformly continuous semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* X *is of the form*

$$S(t) = e^{tA}, \quad t \geq 0 \tag{2.9}$$

for some linear bounded operator $A \in \mathcal{L}(X)$.

With this apt description, we can begin to consider the asymptotic properties of the semigroup.

Definition 6 *[Engel and Nagel, 2000]* e^{tA} *is said to be uniformly exponentially stable if there exist positive constants* α *and* $M \geq 1$ *such that*

$$\|e^{tA}\| \leq Me^{-\alpha t}, \quad \forall t \geq 0. \tag{2.10}$$

Proposition 2.2.3 *[Engel and Nagel, 2000] For a uniformly continuous semigroup* $(S(t))_{t\geq 0}$, *the following assertions are equivalent.*

- $(S(t))_{t\geq 0}$ *is uniformily exponentially stable.*

- $\lim_{t\to\infty} \|S(t)\| = 0$.
- *There exists* $t_0 > 0$ *such that* $\|S(t_0)\| < 1$.
- *There exists* $t_1 > 0$ *such that* $r(S(t_1)) < 1$, *where* $r(S(t))$ *denotes the spectral radius of* $S(t)$.

Lemma 2.2.4 *[Engel and Nagel, 2000] (Spectral Mapping Theorem) For every uniformly continuous semigroup* $(e^{tA})_{t\geq 0}$ *and its generator* A *one has*

$$\sigma(e^{tA}) = e^{t\sigma(A)} := \{e^{t\lambda} : \lambda \in \sigma(A)\} \tag{2.11}$$

Theorem 2.2.5 *[Engel and Nagel, 2000] For a uniformly continuous semigroup with generator* A, *the statements in Proposition 3.2 are equivalent to*

$$\Re(\lambda) < 0 \quad \forall\lambda \in \sigma(A). \tag{2.12}$$

While this result is useful and is in a more abstract setting, many "natural" one-parameter semigroups exist which are not uniformly continuous. Hence, we are required to study an even more general class of semigroups, namely C_0 semigroups.

2.3 Strongly Continuous Operator Semigroups.

Definition 7 *[Engel and Nagel, 2000] A family* $S(t)(0 \leq t < \infty)$ *of bounded linear operators in a Banach space* H *is called a strongly continuous semigroup (in short, a* C_0*-semigroup), if*

1. $S(t_1 + t_2) = S(t_1)S(t_2)\ \forall t_1, t_2 \geq 0$
2. $S(0) = I$

3. *For each $x \in H$, $\xi_x : t \mapsto \xi_x(t) := S(t)x$ is continuous on $t \in [0, \infty)$*

Now we wish to obtain what we obtained for uniformly continuous semigroups for strongly continuous semigroups. For every uniformly continuous semigroup $(S(t))_{t \geq 0}$, we found an operator A such that $(S(t))_{t \geq 0} = e^{tA}$. There is an analogue of A called the *generator* of the semigroup. It will be a linear operator defined only on a dense subspace of the Banach space X. It can be shown that generators of semigroups are necessarily closed, have dense domain, and have their spectrum contained in some proper left half-plane. These conditions do not suffice for the operator to generate a strongly continuous semigroup. For such a semigroup $S(t)$, we define an operator A with domain $D(A)$ consisting of points x such that the limit

$$Ax = \lim_{h \to 0} \frac{S(h)x - x}{h}, \quad x \in D(A) \tag{2.13}$$

exists. Then A is called the infinitesimal generator of the semigroup $S(t)$.

Given an operator A, if A is the infinitesimal generator of $S(t)$, then we say that it generates a strongly continuous semigroup $S(t)$, $t \geq 0$. Sometimes we also denote $S(t)$ by e^{At}. For a linear operator A which generates a C_0-semigroup on a Hilbert space H, we have that

Theorem 2.3.1 *[Hille and Phillips, 1996] A linear operator A is the infinitesimal generator of a C_0 semigroup of contractions $S(t)$, $t \geq 0$, if and only if*

- *A is closed and $D(A)$ is dense in H.*
- *the resolvent set $\rho(A)$ contains $\mathbb{R}^+$ and for every $\lambda > 0$,*

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda} \tag{2.14}$$

Additionally,

Theorem 2.3.2 *[Liu and Zheng, 1999] Let A be a linear operator with dense domain in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0 semigroups of contractions.*

The necessary and sufficient conditions of exponential stability of a C_0 semigroup on a Hilbert space are stated below:

Theorem 2.3.3 *[Liu and Zheng, 1999] Let $S(t) = e^{tA}$ be a C_0 semigroup on a Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\sup\{\Re\lambda : \lambda \in \sigma(A)\} < 0 \tag{2.15}$$

and

$$\sup_{\Re\lambda \geq 0} \|(\lambda I - A)^{-1}\| < \infty \tag{2.16}$$

hold.

2.3.1 Exponential Stability of a C_0 Semigroup

As a simple example of an application of such theorems, consider the equation

$$u_{tt} - u_{xx} + au_t = 0, \quad x \in (0, \pi), \forall t > 0 \tag{2.17}$$

where $a > 0$ and $a \in \mathbb{R}$. We consider the initial boundary value problem subject to the following boundary conditions and initial conditions:

$$u|_{x=0} = u|_{x=\pi} = 0 \tag{2.18}$$

and

$$u|_{t=0} = u_0(x), u_t|_{t=0} = u_1(x), \quad x \in [0, \pi] \tag{2.19}$$

Semigroup Formulation

With the stated initial conditions and boundary conditions, we now transform the problem into an abstract problem on a suitable Hilbert space. We introduce the variable

$$v = u_t$$

and the energy space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$$

equipped with the inner product

$$\langle y_1, y_2 \rangle_{\mathcal{H}} = \int_\Omega v_1 \bar{v}_2 + u_{1,x} \bar{u}_{2,x} \, dx.$$

where

$$y_1 = (u_1, v_1)^T, \quad y_2 = (u_2, v_2)^T. \tag{2.20}$$

With this, the problem can be reduced to the following initial value problem for a first-order evolution equation on the Hilbert space $\mathcal{H}$.

$$\begin{aligned} \frac{dy}{dt} &= \mathcal{A}y, \quad t > 0 \\ y(0) &= (u_0, u_1)^T, \end{aligned} \tag{2.21}$$

where the operator $\mathcal{A}$ is defined as

$$\mathcal{A}y = \begin{bmatrix} v \\ u_{xx} - av \end{bmatrix}.$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$ and independent of time $t > 0$. Next, we prove that the operator $\mathcal{A}$ generates a semigroup of contractions in $\mathcal{H}$. From Liu and Zheng [Liu and Zheng, 1999], it suffices to prove that $0 \in \rho(\mathcal{A})$ and $\mathcal{A}$ is dissipative.

Well-posedness

Lemma 2.3.4 *For $y = (u, v)^T \in D(\mathcal{A})$ we have*

$$\Re\langle \mathcal{A}y, y\rangle_{\mathcal{H}} \leq 0. \tag{2.22}$$

Proof. A straightforward calculation gives that

$$\begin{aligned} \Re\langle \mathcal{A}y, y\rangle_{\mathcal{H}} &= \Re \int_\Omega v_x \bar{u}_x + (u_{xx} - av)\bar{v}\, dx \\ &= \Re \int_\Omega v_x \bar{u}_x - u_x \bar{v}_x - av\bar{v}\, dx \\ &= \Re \int_\Omega -a|v|^2\, dx < 0 \end{aligned}$$

In view of the conditions established for a, the lemma is proved. □

Lemma 2.3.5 *The operator $\mathcal{A}$ satisfies the condition*

$$0 \in \rho(\mathcal{A}) \tag{2.23}$$

Proof. Let $F = (f, g)^T \in \mathcal{H}$. We must show the equation

$$\mathcal{A}y = F \tag{2.24}$$

has a solution $y = (u, v)^T \in \mathcal{H}$. If we examine the operator $\mathcal{A}$, then we find the system

$$f = v \in H_0^1 \tag{2.25}$$

$$g = u_{xx} - av \in L^2 \tag{2.26}$$

Substituting $v = f$, we obtain the following system:

$$u_{xx} = af + g \in L^2 \tag{2.27}$$

It easily follows from the standard result on elliptic systems that the system (2.9)-(2.10) has a unique solution y. Therefore, $0 \in \rho(\mathcal{A})$. $\square$

As consequence of the Hille-Yosida theorem [Liu and Zheng, 1999], we have that $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}}$ on $\mathcal{H}$. From semigroup theory, $y(t) = e^{t\mathcal{A}}y_0$ is the unique solution of the system described. The proof is complete.

Exponential Stability

Theorem 2.3.6 *Let $\rho(\mathcal{A})$ be the resolvent set of the operator $\mathcal{A}$ and $S(t) = e^{t\mathcal{A}}$ be the C_0-semigroup of contractions generated by $\mathcal{A}$. Then, $S(t)$ is exponentially stable if and only if and only if*

$$i\mathbb{R} = \{i\omega : \omega \in \mathbb{R}\} \subset \rho(\mathcal{A}), \tag{2.28}$$

$$\limsup_{|\omega|\to\infty} \|(i\omega I - \mathcal{A})^{-1}\| < \infty. \tag{2.29}$$

Lemma 2.3.7 *The operator $\mathcal{A}$ satisfies (2.12).*

Proof. If (2.12) is not true, it means that there is an $\omega \in \mathbb{R}$ such that $\omega \neq 0$ and ω is in the spectrum of $\mathcal{A}$. Since $\mathcal{A}^{-1}$ is compact, there is a vector function

$$y = (u, v) \in D(\mathcal{A}), \text{ with } \|y\|_{\mathcal{H}} = 1$$

such that $\mathcal{A}y = i\omega y$, which is equivalent to

$$i\omega u - v = 0, \tag{2.30}$$

$$i\omega v - u_{xx} + av = 0 \tag{2.31}$$

Taking the inner product of $(i\omega\mathcal{ID} - A)y$ with y in $\mathcal{H}$ and then taking its real part yields

$$\int_\Omega a|v|^2\,dx = 0 \tag{2.32}$$

This implies

$$\|v\|_{L^2}^2 = 0 \tag{2.33}$$

On the other hand, taking the inner product with u in L^2 gives us that

$$\langle i\omega v - u_{xx} + av, u\rangle = 0 \implies -\int_\Omega u_{xx}\bar{u}\,dx = 0 \tag{2.34}$$

Therefore, from integration by parts, it follows that

$$0 = \|u\|_{H_0^1}^2 \tag{2.35}$$

which contradicts the assumptions on the norm $0 = \|v\|_{L^2}^2 + \|u\|_{H_0^1}^2 = \|y\|_{\mathcal{H}} = 1.$ □

Lemma 2.3.8 *The operator $\mathcal{A}$ satisfies (2.13).*

Proof. To prove (2.13) we use contradiction argument again. If (2.13) is not true, there exists a real sequence ω_n, with $\omega_n \to \infty$ and a sequence of vector functions $y_n \in D$ with unit norm in $\mathcal{H}$ that satisfies

$$\limsup_{|\omega_n|\to\infty} \|(i\omega_n I - \mathcal{A})^{-1} y_n\| = \infty. \tag{2.36}$$

This can be reformulated as

$$\|(i\omega_n I - \mathcal{A}) y_n\|_{\mathcal{H}} \to 0. \tag{2.37}$$

Hence,

$$f_n = i\omega_n u_n - v_n \to 0, \tag{2.38}$$

$$g_n = i\omega_n v_n - u_{n,xx} - a v_n \to 0 \tag{2.39}$$

Taking the inner product of $(i\omega_n I - \mathcal{A}) y_n$ with y_n in $\mathcal{H}$ and then taking its real part yields

$$v_n \to 0 \text{ in } L^2 \tag{2.40}$$

By assumption, the vector functions y_n have unit norm and so the sequence y_n is bounded. Since the sequence ω_n tends to infinity, we see that,

$$u_n \to 0 \text{ in } L^2 \tag{2.41}$$

On the other hand, taking the inner product with u_n in L^2 gives us that

$$\langle i\omega_n v_n - u_{n,xx} + av_n, u_n\rangle = \langle g_n, u_n\rangle \tag{2.42}$$

Therefore, from integration by parts, it follows that

$$0 = \|u\|^2_{H^1_0} \tag{2.43}$$

which contradicts the assumptions on the norm $0 = \|v\|^2_{L^2} + \|u\|^2_{H^1_0} = \|y\|_{\mathcal{H}} = 1$. Thus, the proof is complete. □

We have proved the following result:

Theorem 2.3.9 *Let $y(t) = (u(t), v(t))$ be a solution of the problem determined by the system (2.1), the boundary conditions (2.2) and the initial conditions (2.3). Then there exists two positive constants M, c such that*

$$\|y(t)\| \leq Me^{-ct}\|y_0\| \tag{2.44}$$

where $y_0 = (u_0, u_1)$

2.3.2 Polynomial Stability

Our result in this paper is not a result of exponential stability, but polynomial stability. Thus it is necessary to state the theorems which would allows us to make such a conclusion. As a preliminary, recall the linear evolution equation on Hilbert Space $\mathcal{H}$:

$$\frac{dy}{dt} = Ay$$

$$y(0) = y_0$$

Assume that

1. A generates a bounded C_0-semigroup $S(t) = e^{tA}$ on $\mathcal{H}$

2. $i\mathbb{R} \cap \sigma(A) = \emptyset$

We say that for a solution $y(t) = e^{tA}y_0$, $y(t)$ is strongly stable if

$$\lim_{t\to\infty} \|y(t)\|_{\mathcal{H}} = 0$$

for all $y_0 \in \mathcal{H}$. We also say that $y(t)$ is exponentially stable if there exists constants $M, k > 0$ such that

$$\|y(t)\|_{\mathcal{H}} \leq Me^{-kt}\|y_0\|_{\mathcal{H}}, \quad t > 0$$

There exist many systems which are strongly stable, but not exponentially stable. As an example, it was shown in [Liu and Liu, 1998] that for a one dimensional wave equation with localized Kelvin-Voigt damping, the solutions are exponentially stable only under certain restrictions of the local function. For that purpose, other decay rates have been introduced to provide a more specific classification. If there is a positive function $f(t)$ with $\lim_{t\to\infty} f(t) = 0$ such that

$$\|y(t)\|_{\mathcal{H}} \leq f(t)\|y_0\|_{D(A)}, \quad t > 0$$

we say the solution decays at a rate of $f(t)$ for all $y_0 \in D(A)$. In this paper we examine the condition of polynomial stability. A solution $y(t)$ is polynomial stable if there exists constants $M, k > 0$ such that

$$\|y(t)\|_{\mathcal{H}} \leq Mt^{-k}\|y_0\|_{D(A)}, \quad t > 0$$

It was proven in [Borichev and Tomilov, 2009], that this is equivalent to the order of boundedness of the resolvent operator on the imaginary axis.

Theorem 2.3.10 ([Borichev and Tomilov, 2009]) *Let $(S(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $l > 0$ the following conditions are equivalent:*

1. $\|(i\lambda - A)^{-1}\|_{\mathcal{H}} = O(\lambda^l)$
2. *There exists constant $M > 0$ such that $\|y(t)\|_{\mathcal{H}} \leq Mt^{-1/l}\|y_0\|_{\mathcal{D}(\mathcal{A})}, t > 0$*

With this theorem, we can prove polynomial decay of solutions for the case discussed in [Quintanilla, 2005] with localized damping.

2.4 Lax-Milgram Theorem

This section will state the Lax-Milgram theorem which will be a condition used to obtain unique solvability in the proceeding chapter.

Theorem 2.4.1 ([Szepessy, 2020]) *Let B be a bilinear functional and L a linear functional defined on a Hilbert space $\mathcal{H}$ where the following conditions hold:*

1. *Coerciveness: $\exists\alpha > 0$ such that $B(u,u) \geq \alpha\|u\|^2_{\mathcal{H}}$ for all $u \in \mathcal{H}$.*
2. *B is continuous: $\exists C \in \mathbb{R}$ such that $|B(u,v)| \leq C\|u\|_{\mathcal{H}}\|w\|_{\mathcal{H}}$ for all $u, w \in \mathcal{H}$.*
3. *L is continuous: $\exists C \in \mathbb{R}$ such that $|L(u)| \leq C\|u\|_{\mathcal{H}}$ for all $u \in \mathcal{H}$.*

Then there is a unique function $u \in \mathcal{H}$ such that $B(u,v) = L(v)$

Now, we will continue to the proof of polynomial stability for the stated system of differential equations.

Chapter 3

Mixture with Localized Kelvin Voigt Damping

In this chapter, we shall consider the stability of the system for the mixture with localized Kelvin-Voigt damping. Recall that the formulation of the viscoelastic mixture in [Quintanilla, 2005] results in the following system of equations.

$$\begin{cases} u_{tt} &= \alpha u_{xx} + \beta w_{xx} - \xi(u - w) \\ w_{tt} &= \beta u_{xx} + \gamma w_{xx} + \xi(u - w) + (a(x)w_{tx})_x \end{cases}$$

under the following conditions:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \text{ positive definite}$$

$$u(0) = u(\pi) = 0 \qquad w(0) = w(\pi) = 0$$

$$\inf(a(x)) > 0 \text{ on } (x_1, x_2) \subset [0, \pi] \qquad a(x) = 0 \text{ on } [0, \pi] \sim (x_1, x_2)$$

As in [Quintanilla, 2005], we also place the following restriction on β, ξ:

$$\beta < 0 \quad \xi \geq 0$$

3.1 Semigroup Formulation

With the stated initial conditions and boundary conditions, we now transform the problem into an abstract problem on a suitable Hilbert space. We introduce the variables,

$$v = u_t, \quad y = w_t.$$

and the energy space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$$

equipped with the inner product

$$\begin{aligned} \langle z_1, z_2 \rangle_{\mathcal{H}} =& \frac{1}{2} \int_\Omega v_1 \bar{v}_2 + y_1 \bar{y}_2 + \alpha u_{1,x} \bar{u}_{2,x} + \beta (u_{1,x} \bar{w}_{2,x} + w_{1,x} \bar{u}_{2,x}) \\ &+ \gamma w_{1,x} \bar{w}_{2,x} + \xi (u_1 - w_1)(\bar{u}_2 - \bar{w}_2)\, dx. \end{aligned}$$

where

$$z_1 = (u_1, v_1, w_1, y_1)^T, \quad z_2 = (u_2, v_2, w_2, y_2)^T.$$

With this, the problem can be reduced to the following initial value problem for a first-order evolution equation on the Hilbert space $\mathcal{H}$.

$$\begin{aligned} z_t &= \mathcal{A} z, \quad t > 0 \\ z_0 &= (u_0, v_0, w_0, y_0)^T, \end{aligned} \tag{3.1}$$

where the operator $\mathcal{A}$ is defined as

$$\mathcal{A}z = \begin{bmatrix} v \\ \alpha u_{xx} + \beta w_{xx} - \xi(u-w) \\ z \\ \beta u_{xx} + \gamma w_{xx} + \xi(u-w) + (a(x)y_x)_x \end{bmatrix}. \tag{3.2}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega). \tag{3.3}$$

Next, we prove that the operator $\mathcal{A}$ generates a C_0 semigroup of contractions in $\mathcal{H}$. From [Lumer and Phillips, 1961], it suffices to prove that $I - \mathcal{A}$ is surjective and $\mathcal{A}$ is dissipative.

3.2 Well-posedness

We now wish to establish that our operator $\mathcal{A}$ is the generator of a C_0 semigroup of contractions.

Theorem 3.2.1 *An operator $\mathcal{A}$ is defined to be the generator of a semigroup of contractions when*

$$\text{For } z = (u, v, w, y)^T \in D(\mathcal{A}) \text{ we have } \Re\langle \mathcal{A}z, z\rangle_{\mathcal{H}} \leq 0 \tag{3.4}$$

$$Ran(I - \mathcal{A}) = \mathcal{H} \tag{3.5}$$

In what follows, we show that $\mathcal{A}$ as defined in (3.2) satisfies both of the previously mentioned properties and is thus the generator of a semigroup of contractions.

Proposition 3.2.2 *The operator $\mathcal{A}$ satisfies* (3.4).

Proof. A straightforward calculation gives that

$$\begin{aligned}\Re\langle \mathcal{A}z, z\rangle_{\mathcal{H}} = \Re \int_\Omega \Big\{ &\alpha v_x \bar{u}_x + \beta\Big[v_x \bar{w}_x + y_x \bar{u}_x\Big] + \gamma y_x \bar{w}_x \\ &+ (\alpha u_{xx} + \beta w_{xx} - \xi(u-w))\bar{v} \\ &+ (\beta u_{xx} + \gamma w_{xx} + \xi(u-w) + (a(x)y_x)_x)\bar{y} \\ &+ \xi(v_1 - y_1)(\bar{u}_2 - \bar{w}_2)\Big\}\, dx \\ = -\int_\Omega a(x)|y_x|^2\, dx\end{aligned}$$

In view of the conditions established for $a(x)$, the lemma is proved. □

Proposition 3.2.3 *The operator* $\mathcal{A}$ *satisfies* (3.5).

Proof. Let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$. We must show the equation

$$(I - \mathcal{A})z = F \tag{3.6}$$

has a solution $z = (u, v, w, y)^T \in D(A)$. If we examine the operator $\mathcal{A}$ by its individual components, then from (3.2) we find the system

$$\begin{aligned} f_1 &= u - v \\ f_2 &= v - \alpha u_{xx} - \beta w_{xx} + \xi(u - w) \\ f_3 &= w - y \\ f_4 &= y - \beta u_{xx} - \gamma w_{xx} - \xi(u - w) - (a(x)y_x)_x \end{aligned}$$

Since $v = u - f_1$, $y = w - f_3$, it is enough to solve

$$
\begin{aligned}
g_1 &= u - \alpha u_{xx} - \beta w_{xx} + \xi(u - w) \\
g_2 &= w - \beta u_{xx} - \gamma w_{xx} - \xi(u - w) - (a(x)w_x)_x
\end{aligned}
$$

where $g_1 = f_2 + f_1$ and $g_2 = f_4 + f_3 - (a(x)(f_3)_x)_x$. In order to obtain unique solvability we will apply the Lax-Milgram theorem (2.4.1). The associated bilinear form will be defined as

$$
\begin{aligned}
B(\{u_1, w_1\}, \{u_2, w_2\}) =& \alpha\langle u_1, u_2\rangle_{H_0^1} + \beta(\langle u_1, w_2\rangle_{H_0^1} + \langle w_1, u_2\rangle_{H_0^1}) \\
&+ \gamma\langle w_1, w_2\rangle_{H_0^1} + \xi\langle u_1 - w_1, u_2 - w_2\rangle_{L^2}
\end{aligned}
$$

It is easy to see that this bilinear form is coercive. This is because

$$
\begin{aligned}
B(\{u, w\}, \{u, w\}) =& \alpha\|u\|_{H_0^1}^2 + \beta(\langle u, w\rangle_{H_0^1} + \langle w, u\rangle_{H_0^1}) + \gamma\|w\|_{H_0^1}^2 + \xi\|u - w\|_{L^2}^2 \\
=& \|\{u, w\}\|_{\mathcal{H}}^2
\end{aligned}
$$

The bilinear form is also bounded:

$$\begin{aligned}|B(\{u_1,w_1\},\{u_2,w_2\})| = {} & |\alpha\langle u_1,u_2\rangle_{H_0^1} + \beta(\langle u_1,w_2\rangle_{H_0^1} + \langle w_1,u_2\rangle_{H_0^1}) \\ & + \gamma\langle w_1,w_2\rangle_{H_0^1} + \xi\langle u_1-w_1,u_2-w_2\rangle_{L^2}| \\ \leq {} & |\alpha\langle u_1,u_2\rangle_{H_0^1}| + |\beta(\langle u_1,w_2\rangle_{H_0^1} + \langle w_1,u_2\rangle_{H_0^1})| \\ & + |\gamma\langle w_1,w_2\rangle_{H_0^1}| + |\xi\langle u_1-w_1,u_2-w_2\rangle_{L^2}| \\ \leq {} & \alpha\|u_1\|_{H_0^1}\|u_2\|_{H_0^1} + \beta(\|u_1\|_{H_0^1}\|w_2\|_{H_0^1} + \|w_1\|_{H_0^1}\|u_2\|_{H_0^1}) \\ & + \gamma\|w_1\|_{H_0^1}\|w_2\|_{H_0^1} + \xi\|u_1-w_1\|_{L^2}\|u_2-w_2\|_{L^2} \\ \leq {} & C\|\{u_1,w_1\}\|_{\mathcal{H}}\|\{u_2,w_2\}\|_{\mathcal{H}}\end{aligned}$$

Because the bilinear form is bounded, it is also continuous and so the result follows by using Lax-Milgrams' lemma. Therefore the equation

$$(I-\mathcal{A})z = F \tag{3.7}$$

has a solution $z = (u,v,w,y)^T \in \mathcal{H}$. □

As consequence of the Lumer-Phillips theorem [Liu and Zheng, 1999], we have that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions on $\mathcal{H}$ satisfying (3.4) and (3.5). The proof is complete.

3.3 Polynomial stability

Recall the sufficient and necessary conditions for a C_0-semigroup to be polynomially stable:

Theorem 3.3.1 *Let $\rho(\mathcal{A})$ be the resolvent set of the operator $\mathcal{A}$ and $S(t) = e^{t\mathcal{A}}$ be the C_0-semigroup of contractions generated by $\mathcal{A}$. Then, $S(t)$ is polynomial stable of* *order l if and only if*

$$i\mathbb{R} = \{i\lambda : \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A}), \tag{3.8}$$

$$\limsup_{|\lambda|\to\infty} \frac{1}{\lambda^l} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty. \tag{3.9}$$

Proposition 3.3.2 *The operator $\mathcal{A}$ satisfies* (3.8)*.*

Proof. If (3.8) is not true, it means that there is an $\lambda \in \mathbb{R}$ such that $\lambda \neq 0$ and $i\lambda$ is in $\sigma(\mathcal{A})$. Let z be a vector function with

$$z = (u, v, w, y) \in D(\mathcal{A}), \text{ with } \|z\|_{\mathcal{H}} = 1$$

such that $(i\lambda I - \mathcal{A})z = 0$. This is equivalent to the following system of equations:

$$i\lambda u - v = 0, \tag{3.10}$$

$$i\lambda v - \alpha u_{xx} - \beta w_{xx} + \xi(u - w) = 0, \tag{3.11}$$

$$i\lambda w - y = 0, \tag{3.12}$$

$$i\lambda y - \beta u_{xx} - \gamma w_{xx} - \xi(u - w) - (a(x)y_x)_x = 0, \tag{3.13}$$

Taking the inner product of $(i\lambda I - \mathcal{A})z$ with z in $\mathcal{H}$ and then taking its real part yields

$$\int_\Omega a(x)|y_x|^2 \, dx = 0 \tag{3.14}$$

From (3.22), this implies

$$-\int_\Omega \lambda^2 a(x)|w_x|^2\,dx = 0 \tag{3.15}$$

Because the measure of the support of $a(x)$ is positive, it must be that w_x is locally zero. We will find that this is in fact a contradiction. The system of equations can be simplified to be

$$-\lambda^2 u - \alpha u_{xx} - \beta w_{xx} + \xi(u - w) = 0 \tag{3.16}$$

$$-\lambda^2 w - \beta u_{xx} - \gamma w_{xx} - \xi(u - w) = 0 \tag{3.17}$$

The solutions of this system would be of the type $u = A\sin(nx), w = B\sin(nx)$. Therefore, it follows that

$$0 = a(x)w_x = a(x)Bn\cos(nx) \tag{3.18}$$

which contradicts the assumptions on $a(x)$ that the measure of the support of $a(x)$ is positive. Thus it must be that (3.8) is true. □

Proposition 3.3.3 *The operator $\mathcal{A}$ satisfies (3.9) for $l = 2$.*

Proof. To prove (3.9) we use contradiction argument again. If (3.9) is not true, there exists a real sequence λ_n, with $\lambda_n \to \infty$ and a sequence of vector functions $z_n \in D(\mathcal{A})$ with unit norm in $\mathcal{H}$ that satisfies

$$\lambda_n^l \|(i\lambda_n I - \mathcal{A})z_n\|_{\mathcal{H}} \to 0. \tag{3.19}$$

Hence we define the components of the sequence of vector functions componentwisely,

$$f_n = \lambda_n^l(i\lambda_n u_n - v_n) \tag{3.20}$$

$$g_n = \lambda_n^l(i\lambda_n v_n - \alpha u_{n,xx} - \beta w_{n,xx} + \xi(u_n - w_n)) \tag{3.21}$$

$$h_n = \lambda_n^l(i\lambda_n w_n - y_n) \tag{3.22}$$

$$k_n = \lambda_n^l(i\lambda_n y_n - \beta u_{n,xx} - \gamma w_{n,xx} - \xi(u_n - w_n) - (a(x)y_{n,x})_x), \tag{3.23}$$

From (3.19), each of these sequences tend to zero in their respective spaces.

$$f_n \to 0 \text{ in } H_0^1$$

$$g_n \to 0 \text{ in } L^2$$

$$h_n \to 0 \text{ in } H_0^1$$

$$k_n \to 0 \text{ in } L^2$$

We first wish to establish some estimates on the bounds of the different parts of our vector function. These estimates will be used later on in the proof.

Lemma 3.3.4

$$u_n \to 0 \textit{ in } L^2 \qquad w_n \to 0 \textit{ in } L^2 \tag{3.24}$$

Proof. The sequence z_n has unit norm and so each component of its norm is thus bounded. Recall the definition of $\langle z_n, z_n\rangle_{\mathcal{H}}$,

$$\begin{aligned}\langle z_n, z_n\rangle_{\mathcal{H}} =& \langle v_n, v_n\rangle_{L^2} + \langle y_n, y_n\rangle_{L^2} + \langle \alpha u_{n,x}, u_{n,x}\rangle_{L^2} + \langle \gamma w_{n,x}, w_{n,x}\rangle_{L^2} \\ &+ \langle \xi(u_n - w_n), (u_n - w_n)\rangle_{L^2} + 2\beta\Re\langle u_{n,x}, w_{n,x}\rangle_{L^2}\end{aligned}$$

Thus v_n is bounded in L^2 and so we see from (3.20) that $i\lambda_n u_n$ is also bounded in L^2.

From the fact that the sequence λ_n tends to infinity while $i\lambda_n u_n$ remains bounded, it must be that

$$u_n \to 0 \text{ in } L^2$$

An analogous result can be seen in (3.22) that

$$w_n \to 0 \text{ in } L^2$$

□

Lemma 3.3.5

$$\sqrt{\lambda_n^l a(x)}y_{n,x} \to 0 \textit{ in } L^2 \qquad \sqrt{\lambda_n^{l+2} a(x)}w_{n,x} \to 0 \textit{ in } L^2 \tag{3.25}$$

Proof. Taking the inner product of $\lambda_n^l(i\lambda_n I - \mathcal{A})z_n$ with z_n in $\mathcal{H}$ and then taking its real part yields

$$\sqrt{\lambda_n^l a(x)}y_{n,x} \to 0 \text{ in } L^2$$

From (3.22), we will have that $i\lambda_n w_n - y_n \to 0$. Thus

$$\sqrt{\lambda_n^{l+2} a(x)}w_{n,x} \to 0 \text{ in } L^2$$

□

Lemma 3.3.6

$$\frac{-\alpha u_{n,xx} - \beta w_{n,xx}}{\lambda_n} \quad \textit{is bounded in } L^2 \tag{3.26}$$

Proof. In (3.21), we divide by λ_n^{l+1} to obtain that

$$iv_n + \frac{-\alpha u_{n,xx} - \beta w_{n,xx}}{\lambda_n} + \frac{\xi(u_n - w_n)}{\lambda_n} \to 0 \text{ in } L^2$$

From (3.24), this simplifies to be

$$iv_n + \frac{-\alpha u_{n,xx} - \beta w_{n,xx}}{\lambda_n} \to 0 \text{ in } L^2$$

Since v_n is bounded, it follows that

$$\frac{-\alpha u_{n,xx} - \beta w_{n,xx}}{\lambda_n} = O(1) \text{is also bounded in } L^2$$

□

With these properties, we will now examine how the sequence of vector functions z_n behaves.

Lemma 3.3.7

$$\langle u_{n,x}, a(x)u_{n,x}\rangle_{L^2} \to 0 \tag{3.27}$$

Proof. Taking inner product of (3.23) with $a(x)u_n$ in L^2,

$$\langle i\lambda_n y_n - \beta u_{n,xx} - \gamma w_{n,xx} + \xi(u_n - w_n) - (a(x)y_{n,x})_x, a(x)u_n\rangle_{L^2} \to 0 \tag{3.28}$$

We will examine each term individually. For $\langle i\lambda_n y_n, a(x)u_n\rangle_{L^2}$, we have

$$\langle i\lambda_n y_n, a(x)u_n\rangle_{L^2} = \langle \sqrt{a(x)}y_n, -i\lambda_n \sqrt{a(x)}u_n\rangle_{L^2}$$

From (3.20), we know $-i\lambda_n\sqrt{a(x)}u_n$ is bounded. From (3.25), we know that $\sqrt{a(x)}y_n \to 0$ in L^2. Thus,

$$\langle i\lambda_n y_n, a(x)u_n\rangle_{L^2} \to 0$$

For $\langle -\beta u_{n,xx}, a(x)u_n\rangle_{L^2}$, by integration by parts, we have

$$\begin{aligned}\langle -\beta u_{n,xx}, a(x)u_n\rangle_{L^2} &= \langle \beta u_{n,x}, (a(x)u_n)_x\rangle_{L^2} \\ &= \langle \beta u_{n,x}, a(x)u_{n,x}\rangle_{L^2} + \langle \beta u_{n,x}, a_x(x)u_n\rangle_{L^2}\end{aligned}$$

From the fact that z_n is a sequence of unit norm, $u_{n,x}$ is bounded. From (3.24) and that $a(x) \in C^1$, we know that $\langle \beta u_{n,x}, a_x(x)u_n\rangle_{L^2} \to 0$. Thus we are left with

$$\langle -\beta u_{n,xx}, a(x)u_n\rangle_{L^2} - \langle \beta u_{n,x}, a(x)u_{n,x}\rangle_{L^2} \to 0 \tag{3.29}$$

For $\langle -\gamma w_{n,xx}, a(x)u_n\rangle_{L^2}$, by integration by parts, we have

$$\begin{aligned}\langle -\gamma w_{n,xx}, a(x)u_n\rangle_{L^2} &= \langle \gamma w_{n,x}, (a(x)u_n)_x\rangle_{L^2} \\ &= \langle \gamma w_{n,x}, a(x)u_{n,x}\rangle_{L^2} + \langle \gamma w_{n,x}, a_x(x)u_n\rangle_{L^2}\end{aligned}$$

From the fact that z_n is a sequence of unit norm, $w_{n,x}$ is bounded. From (3.24) we know that $\langle \gamma w_{n,x}, a_x(x)u_n\rangle_{L^2} \to 0$. Thus we are left with

$$\langle -\gamma w_{n,xx}, a(x)u_n\rangle_{L^2} - \langle \gamma w_{n,x}, a(x)u_{n,x}\rangle_{L^2} \to 0$$

This can be written as $\langle \gamma\sqrt{a(x)}w_{n,x}, \sqrt{a(x)}u_{n,x}\rangle_{L^2}$. From the fact that z_n is a sequence of unit norm, $\sqrt{a(x)}u_{n,x}$ is bounded. From (3.25), we know that $\sqrt{a(x)}w_n \to 0$ in L^2. So it is that $\langle \gamma\sqrt{a(x)}w_{n,x}, \sqrt{a(x)}u_{n,x}\rangle_{L^2} \to 0$. Thus

$$\langle -\gamma w_{n,xx}, a(x)u_n\rangle_{L^2} \to 0$$

For $\langle \xi(u_n - w_n), a(x)u_n\rangle_{L^2}$, by (3.24) this term tends to zero. Finally, for $\langle (a(x)y_{n,x})_x, a(x)u_n\rangle_{L^2}$, we have, using integration by parts, that,

$$-\langle (a(x)y_{n,x})_x, a(x)u_n\rangle_{L^2} = \langle \sqrt{a(x)}y_{n,x}, (a(x))^{\frac{3}{2}}u_{n,x}\rangle_{L^2} + \langle \sqrt{a(x)}y_{n,x}, \sqrt{a(x)}a_x(x)u_{n,x}\rangle_{L^2}$$

From the fact that z_n is a sequence of unit norm, we know both $\sqrt{a(x)}a_x(x)u_{n,x}$, $(a(x))^{\frac{3}{2}}u_{n,x}$ are bounded. From (3.25), we know that $\sqrt{a(x)}y_n \to 0$. Thus

$$\langle \sqrt{a(x)}y_{n,x}, (a(x))^{\frac{3}{2}}u_{n,x}\rangle_{L^2} + \langle \sqrt{a(x)}y_{n,x}, \sqrt{a(x)}a_x(x)u_{n,x}\rangle_{L^2} \to 0$$

and

$$-\langle (a(x)y_{n,x})_x, a(x)u_n\rangle_{L^2} \to 0$$

Thus (3.28) becomes:

$$\langle -\beta u_{n,xx}, a(x)u_n\rangle_{L^2} \to 0$$

And with our knowledge from (3.29), we have the desired result:

$$\langle u_{n,x}, a(x)u_{n,x}\rangle_{L^2} \to 0$$

□

Lemma 3.3.8

$$\langle v_n, a(x)v_n\rangle_{L^2} \to 0 \tag{3.30}$$

Proof. Taking inner product of (3.21) with $a(x)u_n$ in L^2,

$$\langle i\lambda_n v_n - \alpha u_{n,xx} - \beta w_{n,xx} - \xi(u_n - w_n), a(x)u_n\rangle_{L^2} \to 0 \tag{3.31}$$

We will examine each term individually. For $\langle i\lambda_n v_n, a(x)u_n\rangle_{L^2}$, we have

$$\langle i\lambda_n v_n, a(x)u_n\rangle_{L^2} = -\langle v_n, i\lambda_n a(x)u_n\rangle_{L^2}$$

and thus

$$\langle i\lambda_n v_n, a(x)u_n\rangle_{L^2} + \langle v_n, i\lambda_n a(x)u_n\rangle_{L^2} \to 0$$

From (3.20), we have that $v_n = i\lambda_n u_n - f_n$, and because $f_n \to 0$,

$$\langle i\lambda_n v_n, a(x)u_n\rangle_{L^2} + \langle v_n, a(x)v_n\rangle_{L^2} \to 0$$

For $\langle -\alpha u_{n,xx}, a(x)u_n\rangle_{L^2}$, by integration by parts, we have

$$\begin{aligned}\langle -\alpha u_{n,xx}, a(x)u_n\rangle_{L^2} &= \langle \alpha u_{n,x}, (a(x)u_n)_x\rangle_{L^2}\\ &= \langle \alpha u_{n,x}, a(x)u_{n,x}\rangle_{L^2} + \langle \alpha u_{n,x}, a_x(x)u_n\rangle_{L^2}\end{aligned}$$

From the fact that z_n is a sequence of unit norm, $u_{n,x}$ is bounded. From (3.24) we know that $\langle \alpha u_{n,x}, a_x(x)u_n\rangle_{L^2} \to 0$. From (3.27), $\langle \alpha u_{n,x}, a(x)u_{n,x}\rangle_{L^2} \to 0$. Thus

$$\langle -\alpha u_{n,xx}, a(x)u_n\rangle_{L^2} \to 0$$

For $\langle -\beta w_{n,xx}, a(x)u_n\rangle_{L^2}$, by integration by parts, we have

$$\begin{aligned}\langle -\beta w_{n,xx}, a(x)u_n\rangle_{L^2} &= \langle \beta w_{n,x}, (a(x)u_n)_x\rangle_{L^2} \\ &= \langle \beta w_{n,x}, a(x)u_{n,x}\rangle_{L^2} + \langle \beta w_{n,x}, a_x(x)u_n\rangle_{L^2}\end{aligned}$$

From the fact that z_n is a sequence of unit norm, $w_{n,x}$ is bounded. From (3.24) we know that $\langle \beta w_{n,x}, a_x(x)u_n\rangle_{L^2} \to 0$. Thus we are left with

$$\langle -\beta w_{n,xx}, a(x)u_n\rangle_{L^2} - \langle \beta w_{n,x}, a(x)u_{n,x}\rangle_{L^2} \to 0$$

This can be written as $\langle \beta\sqrt{a(x)}w_{n,x}, \sqrt{a(x)}u_{n,x}\rangle_{L^2}$. From the fact that z_n is a sequence of unit norm, $\sqrt{a(x)}u_{n,x}$ is bounded. From (3.25), we know that $\sqrt{a(x)}w_n \to 0$. So it is that $\langle \beta\sqrt{a(x)}w_{n,x}, \sqrt{a(x)}u_{n,x}\rangle_{L^2} \to 0$. Thus

$$\langle -\beta w_{n,xx}, a(x)u_n\rangle_{L^2} \to 0$$

For $\langle \xi(u_n - w_n), a(x)u_n\rangle_{L^2}$, by (3.24) this term tends to zero. Thus (3.31) becomes:

$$\langle v_n, a(x)v_n\rangle_{L^2} \to 0$$

$\square$

Lemma 3.3.9

$$\langle y_n, a(x)y_n\rangle_{\mathcal{H}} \to 0 \tag{3.32}$$

Proof. Take the inner product of (3.23) with $a(x)y_n$ and apply integration by parts. □

With these previous lemmas, we can show that the norm of the sequence of vector functions tends to zero on the domain of the dissipation.

Lemma 3.3.10

$$\langle z_n, a(x)z_n\rangle_{\mathcal{H}} \to 0 \tag{3.33}$$

Proof.

For $\langle z_n, a(x)z_n\rangle_{\mathcal{H}}$, we have

$$\begin{aligned}\langle z_n, a(x)z_n\rangle_{\mathcal{H}} &= \langle v_n, a(x)v_n\rangle_{L^2} + \langle y_n, a(x)y_n\rangle_{L^2} + \langle \alpha u_{n,x}, a(x)u_{n,x}\rangle_{L^2}\\ &\quad + \langle \gamma w_{n,x}, a(x)w_{n,x}\rangle_{L^2} + \langle \xi(u_n - w_n), a(x)(u_n - w_n)\rangle_{L^2} + 2\beta\Re\langle u_{n,x}, a(x)w_{n,x}\rangle_{L^2}\end{aligned}$$

Thus it is necessary to prove that each of the terms individually tend to zero in L^2. From (3.30), $\langle v_n, a(x)v_n\rangle_{L^2} \to 0$. From (3.27), $\alpha\langle u_{n,x}, a(x)u_{n,x}\rangle_{L^2} \to 0$. From (3.32), $\langle y_n, a(x)y_n\rangle_{L^2} \to 0$. From (3.25), $\gamma\langle w_{n,x}, a(x)w_{n,x}\rangle_{L^2} \to 0$. From (3.24), $\langle \xi(u_n - w_n), a(x)(u_n - w_n)\rangle_{L^2} \to 0$. Finally, from the fact that $\sqrt{a(x)}u_{n,x}$ is bounded and (3.25), $2\beta\Re\langle w_{n,x}, a(x)u_{n,x}\rangle_{L^2} \to 0$. Thus

$$\langle z_n, a(x)z_n\rangle_{\mathcal{H}} \to 0$$

□

Now we can construct the setting for the contradiction argument. The contradiction will be that the norm of the sequence of vectors functions on the local domain tends to a constant multiple of the global norm which was presumed to be unit norm. This contradicts the previous lemma that the norm of the sequence of vector functions tends to 0 on the local domain. From the system (3.20)-(3.23), we can obtain the equalities

$$\lambda_n^l(-\lambda_n^2 u_n - \alpha u_{n,xx} - \beta w_{n,xx} - \xi(u_n - w_n)) = g_n + i\lambda_n f_n \tag{3.34}$$

$$\begin{aligned}\lambda_n^l(-\lambda_n^2 w_n - \beta u_{n,xx} - \frac{\beta_1^2}{\alpha} w_{n,xx} - (\gamma - \frac{\beta_1^2}{\alpha}) w_{n,xx}\\ + \xi(u_n - w_n) - (a(x) y_{n,x})_x) = h_n + i\lambda_n k_n\end{aligned} \tag{3.35}$$

Let $p(x)$ be an arbitrary real function in C^1 which will be chosen later. Also set $d = \gamma - \frac{\beta_1^2}{\alpha}$. Taking the inner product of (3.36) with $2p(x)du_{n,x}$, (3.37) with $2p(x)(dw_{n,x} + a(x)y_{n,x})$ in L^2, taking the real part, and then summing together, we obtain the following result. Hereafter, we omit the subscript and as an example set $u'' = u_{n,xx}$. We also omit the space for which the inner product is taken unless it is not L^2. Thus we obtain

$$\lambda^l \Re\langle -\lambda^2 u - \alpha u'' - \beta w'' - \xi(u-w), 2pdu'\rangle \tag{3.36}$$

$$+ \lambda^l \Re\langle -\lambda^2 w - \beta u'' - \frac{\beta^2}{\alpha} w'' - (\gamma - \frac{\beta^2}{\alpha}) w'' + \xi(u-w) - (ay')', 2p(dw' + ay')\rangle \to 0 \tag{3.37}$$

Lemma 3.3.11 *It can be deduced from* (3.36),(3.37) *that*

$$\begin{aligned}\lambda^{\frac{l}{2}-1}d\Big(&\langle v,p'v\rangle+\langle y,p'y\rangle+\langle\alpha p'u',u'\rangle+\langle\xi(u-w),p'(u-w)\rangle\\&+2\beta\Re\langle u',p'w'\rangle+\langle\gamma w',p'w'\rangle\\&-\alpha p|u'|^2(\cdot)-\gamma p|w'|^2(\cdot)-2\Re\beta pw'u'(\cdot)\Big)\to 0.\end{aligned}\tag{3.38}$$

where for a given function f: $f(\cdot)=f(\pi)-f(0)$.

Proof. We can simplify both of (3.36),(3.37). First, (3.36) can be written as $\Re\big\{\langle-\lambda^2u,2pdu'\rangle+\langle-\alpha u'',2pdu'\rangle+\langle-\beta w'',2pdu'\rangle+\langle-\xi(u-w),2pdu'\rangle\big\}$. Each of these terms can be individually decomposed using integration by parts as follows.

$$\begin{aligned}&\Re\langle-\lambda^2u,2pdu'\rangle=d\langle\lambda^2p'u,u\rangle\\&\Re\langle-\alpha u'',2pdu'\rangle=-\alpha dp|u'|^2(\cdot)+d\langle\alpha p'u',u'\rangle\end{aligned}$$

All together, (3.36) can be written as

$$d\langle\lambda^2p'u,u\rangle+d\langle\alpha p'u',u'\rangle-\alpha dp|u'|^2(\cdot)+\Re\big\{\langle-\beta w'',2pdu'\rangle+\langle-\xi(u-w),2pdu'\rangle\big\}\tag{3.39}$$

For (3.37), this can be written as

$$\begin{aligned}\Re\Big\{&\langle-\lambda^2w,2pdw'\rangle+\langle-\beta u'',2pdw'\rangle+\langle-\frac{\beta^2}{\alpha}w'',2pdw'\rangle+\langle\xi(u-w),2pdw'\rangle\\&+\langle-(dw'+ay')',2p(dw'+ay')\rangle+\langle-\lambda^2w-\beta(u''+\frac{\beta}{\alpha}w'')+\xi(u-w),2pay'\rangle\Big\}\end{aligned}$$

. For each term we have

$$
\begin{aligned}
&\Re\langle -\lambda^2 w, 2pdw'\rangle = d\langle \lambda^2 p' w, w\rangle \\
&\Re\langle -\beta u'', 2pdw'\rangle = -2\Re\beta dpw'u'(\cdot) + \Re\langle \beta u', (2pdw')'\rangle = \\
&\qquad -2\Re\beta dpw'u'(\cdot) + 2d\beta\Re\langle u', p'w'\rangle + 2d\beta\Re\langle u', pw''\rangle \\
&\Re\langle -\frac{\beta^2}{\alpha} w'', 2pdw'\rangle = -d\frac{\beta^2}{\alpha} p|w'|^2(\cdot) + d\frac{\beta^2}{\alpha}\langle w', p'w'\rangle \\
&\Re\langle -(dw' + ay')', 2p(dw' + ay')\rangle = \\
&\qquad -p|(dw' + ay')|^2(\cdot) + \langle (dw' + ay'), p'(dw' + ay')\rangle
\end{aligned}
$$

Thus (3.37) can be written as

$$
\begin{aligned}
&d\langle \lambda^2 p' w, w\rangle + 2d\beta\Re\langle u', p'w'\rangle + d\frac{\beta^2}{\alpha}\langle w', p'w'\rangle + \langle (dw' + ay'), p'(dw' + ay')\rangle \\
&\qquad - d\frac{\beta^2}{\alpha} p|w'|^2(\cdot) - p|(dw' + ay')|^2(\cdot) - 2\Re\beta dpw'u'(\cdot) \\
&\qquad + \Re\big\{2d\beta\langle u', pw''\rangle + \langle \xi(u - w), 2pdw'\rangle + \langle -\lambda^2 w - \beta(u'' + \frac{\beta}{\alpha} w'') + \xi(u - w), 2pay'\rangle\big\}
\end{aligned}
$$

Finally, the sum of (3.36), (3.37) can be expressed as

$$
\begin{aligned}
&\lambda^l d\Big(\langle \lambda^2 p' u, u\rangle + \langle \alpha p' u', u'\rangle + \langle \xi(u - w), p'(u - w)\rangle + \langle \lambda^2 p' w, w\rangle \\
&\qquad + 2\beta\Re\langle u', p'w'\rangle + \frac{\beta^2}{\alpha}\langle w', p'w'\rangle + \frac{1}{d}\langle (dw' + ay'), p'(dw' + ay')\rangle \\
&\qquad - \alpha p|u'|^2(\cdot) - \frac{\beta^2}{\alpha} p|w'|^2(\cdot) - \frac{1}{d} p|(dw' + ay')|^2(\cdot) - 2\Re\beta pw'u'(\cdot)\Big) \\
&\qquad + \lambda^l \Re\big\{\langle -\lambda^2 w - \beta(u'' + \frac{\beta}{\alpha} w'') + \xi(u - w), 2pay'\rangle\big\} \to 0.
\end{aligned}
$$

We divide by $\lambda^{l/2+1}$ to get:

$$\begin{aligned}
&\lambda^{\frac{l}{2}-1}d\Big(\langle\lambda^2p'u,u\rangle+\langle\alpha p'u',u'\rangle+\langle\xi(u-w),p'(u-w)\rangle+\langle\lambda^2p'w,w\rangle\\
&\quad+2\beta\Re\langle u',p'w'\rangle+\frac{\beta^2}{\alpha}\langle w',p'w'\rangle+\frac{1}{d}\langle(dw'+ay'),p'(dw'+ay')\rangle\\
&\quad-\alpha p|u'|^2(\cdot)-\frac{\beta^2}{\alpha}p|w'|^2(\cdot)-\frac{1}{d}p|(dw'+ay')|^2(\cdot)-2\Re\beta pw'u'(\cdot)\Big)\\
&\quad+\lambda^{\frac{l}{2}-1}\Re\big\{\langle-\lambda^2w-\beta(u''+\frac{\beta}{\alpha}w'')+\xi(u-w),2pay'\rangle\big\}\to 0.
\end{aligned}$$

Examining the last line, we have:

$$\begin{aligned}
&\lambda^{\frac{l}{2}-1}\Re\big\{\langle-\lambda^2w-\beta(u''+\frac{\beta}{\alpha}w'')+\xi(u-w),2pay'\rangle\big\}\\
&\quad=\Re\big\{\langle-\lambda w\sqrt{a}p,2\lambda^{\frac{l}{2}}\sqrt{a}y'\rangle+\langle-\beta(\frac{u''+\frac{\beta}{\alpha}w''}{\lambda})p,\omega^{\frac{l}{2}}ay'\rangle+\langle\frac{2}{\lambda}\xi(u-w)p,\lambda^{\frac{l}{2}}ay'\rangle\big\}.
\end{aligned}$$

From (3.22), $-\lambda w\sqrt{a}p$ is bounded. From (3.25), $2\lambda^{\frac{l}{2}}\sqrt{a}y'\to 0$. Thus the first term tends to zero in L^2. From (3.26), $\frac{u''+\frac{\beta}{\alpha}w''}{\lambda}$ is bounded and so the second term tends to zero in L^2. Finally, the last term also tends to zero as a result of (3.24). Ultimately, it is that the last line tends to zero in L^2. Thus the equation becomes:

$$\begin{aligned}
&\lambda^{\frac{l}{2}-1}d\Big(\langle\lambda^2p'u,u\rangle+\langle\alpha p'u',u'\rangle+\langle\xi(u-w),p'(u-w)\rangle+\langle\lambda^2p'w,w\rangle\\
&\quad+2\beta\Re\langle u',p'w'\rangle+\frac{\beta^2}{\alpha}\langle w',p'w'\rangle+\frac{1}{d}\langle(dw'+ay'),p'(dw'+ay')\rangle\\
&\quad-\alpha p|u'|^2(\cdot)-\frac{\beta^2}{\alpha}p|w'|^2(\cdot)-\frac{1}{d}p|(dw'+ay')|^2(\cdot)-2\Re\beta pw'u'(\cdot)\Big)\to 0.
\end{aligned}$$

For the terms $\frac{\beta^2}{\alpha}\langle w', p'w'\rangle + \frac{1}{d}\langle (dw' + ay'), p'(dw' + ay')\rangle$, using (3.25), we have that

$$\begin{aligned}
&\frac{\beta^2}{\alpha}\langle w', p'w'\rangle + \frac{1}{d}\langle (dw' + ay'), p'(dw' + ay')\rangle \\
&= \frac{\beta^2}{\alpha}\langle w', p'w'\rangle + d\langle w', p'w'\rangle \\
&= \langle \gamma w', p'w'\rangle
\end{aligned}$$

Thus the equation becomes:

$$\begin{aligned}
\lambda^{\frac{l}{2}-1} d\Big(&\langle \lambda^2 p'u, u\rangle + \langle \alpha p'u', u'\rangle + \langle \xi(u-w), p'(u-w)\rangle + \langle \lambda^2 p'w, w\rangle \\
&+ 2\beta\Re\langle u', p'w'\rangle + \langle \gamma w', p'w'\rangle \\
&- \alpha p|u'|^2(\cdot) - \frac{\beta^2}{\alpha} p|w'|^2(\cdot) - \frac{1}{d} p|(dw' + ay')|^2(\cdot) - 2\Re\beta p w' u'(\cdot)\Big) \to 0.
\end{aligned}$$

The expression can be simplified as $\frac{1}{d}p|(dw' + ay')|^2(\cdot) = dp|w'|^2(\cdot)$ from the fact that $a(0), a(\pi) = 0$. Then we have $-(d + \frac{\beta^2}{\alpha})p|w'|^2(\cdot)$ which can be written as $-\gamma p|w'|^2(\cdot)$. Now the equation has been simplified to

$$\begin{aligned}
\lambda^{\frac{l}{2}-1} d\Big(&\langle \lambda^2 p'u, u\rangle + \langle \alpha p'u', u'\rangle + \langle \xi(u-w), p'(u-w)\rangle + \langle \lambda^2 p'w, w\rangle \\
&+ 2\beta\Re\langle u', p'w'\rangle + \langle \gamma w', p'w'\rangle \\
&- \alpha p|u'|^2(\cdot) - \gamma p|w'|^2(\cdot) - 2\Re\beta p w' u'(\cdot)\Big) \to 0.
\end{aligned}$$

An additional simplification can be made using $(3.20), (3.22)$ to express $\langle \lambda^2 p'u, u\rangle$ as $\langle p'v, v\rangle$ and $\langle \lambda^2 p'w, w\rangle$ as $\langle p'y, y\rangle$. Thus our final form of $(3.36), (3.37)$ is

$$\begin{aligned}
\lambda^{\frac{l}{2}-1} d\Big(&\langle v, p'v\rangle + \langle \alpha u', p'u'\rangle + \langle \xi(u-w), p'(u-w)\rangle + \langle y, p'y\rangle \\
&+ 2\beta\Re\langle u', p'w'\rangle + \langle \gamma w', p'w'\rangle \\
&- \alpha p|u'|^2(\cdot) - \gamma p|w'|^2(\cdot) - 2\Re\beta p w' u'(\cdot)\Big) \to 0.
\end{aligned}$$

□

Lemma 3.3.12 *Let* $a_0 = \int_0^\pi a(x)\,dx$*. Then*

$$\lambda^{\frac{l}{2}-1} d\Big(-\langle z, az\rangle + \frac{a_0}{\pi}\|z\|^2\Big) \to 0. \tag{3.40}$$

Proof. Taking $p = x$ in (3.38), we will have that

$$\begin{aligned}\lambda^{\frac{l}{2}-1} d\Big(&\langle v, v\rangle + \langle \alpha u', u'\rangle + \langle \xi(u-w), (u-w)\rangle + \langle y, y\rangle \\ &+ 2\beta\Re\langle u', w'\rangle + \langle \gamma w', w'\rangle \\ &- \alpha\pi|u'|^2(\pi) - \gamma\pi|w'|^2(\pi) - 2\Re\beta\pi w'u'(\pi)\Big) \to 0.\end{aligned}$$

Recall that

$$\langle z, z\rangle_{\mathcal{H}} = \langle v, v\rangle + \langle y, y\rangle + \langle \alpha u', u'\rangle + \langle \gamma w', w'\rangle + \langle \xi(u-w), (u-w)\rangle + 2\beta\Re\langle u', w'\rangle$$

It follows that

$$\|z\|^2_{\mathcal{H}} - \alpha\pi|u'|^2(\pi) - \gamma\pi|w'|^2(\pi) - 2\Re\beta\pi w'u'(\pi) \to 0.$$

Next, taking $p = \int_0^x a(s)\,ds$ in (3.38), this becomes

$$\begin{aligned}\lambda^{\frac{l}{2}-1} d\Big(&-\langle v, av\rangle - \langle \alpha u', au'\rangle - \langle \xi(u-w), a(u-w)\rangle - \langle y, ay\rangle \\ &- 2\beta\Re\langle u', aw'\rangle - \langle \gamma w', aw'\rangle \\ &+ \alpha a_0|u'|^2(\pi) + \gamma a_0|w'|^2(\pi) + 2\Re\beta a_0 w'u'(\pi)\Big) \to 0.\end{aligned}$$

Thus

$$\lambda^{\frac{l}{2}-1} d\Big(- \langle z, az \rangle_{\mathcal{H}} + \frac{a_0}{\pi} \|z\|^2_{\mathcal{H}} \Big) \to 0.$$

□

With this information and (3.33),

$$\frac{a_0}{\pi} \|z\|^2_{\mathcal{H}} = 0. \tag{3.41}$$

This contradicts the assumptions on $a(x)$ that $\int_0^\pi a(x)\, dx > 0$ and that $\|z\|^2_{\mathcal{H}} = 1$. Thus the proof is complete. □

We have proved the following result:

Theorem 3.3.13 *Let $z(t) = (u(t), v(t), w(t), y(t))$ be a solution of the problem determined by the system, boundary conditions and the initial conditions described in section 2. Then there exists two positive constants $M, l = 2$ such that*

$$\|z(t)\| \leq M t^{-1/2} \|z_0\|$$

where $z_0 = (u_0, v_0, w_0, y_0)$

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