ANALYSIS OF A SINGULARLY PERTURBED TRAVELING WAVE PROBLEM

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ANALYSIS OF A SINGULARLY PERTURBED TRAVELING WAVE PROBLEM*

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Abstract. We analyse a singularly perturbed traveling wave problem which is derived from the drift diffusion model describing electron transport in two-valley semiconductor materials. We show that the reduced problem, defined on a submanifold of the phase space, has homoclinic orbits for certain parameters. By using methods from invariant manifold theory and methods from homoclinic bifurcation theory we prove that the homoclinic orbits of the reduced problem persist in a global center manifold near the manifold of the reduced problem.

Key words. Singular Perturbations, Traveling Waves, Invariant Manifolds, Homoclinic Orbits, Semiconductors.

1. Introduction. In this paper we analyse the dynamics of the following four-dimensional nonlinear system of singularly perturbed ordinary differential equations

$$\begin{aligned}
\dot{E} &= (n-1)/\lambda^2 \\
\dot{n} &= v_1(E)u + v_2(E)(n-u) - cn + \gamma \\
\varepsilon\,\dot{u} &= w \\
\varepsilon\,\dot{w} &= \varepsilon(v_1(E)-c)\,w + \varepsilon^2 v_1'(E)u(n-1)/\lambda^2 + (1+\alpha(E))u - n\,.
\end{aligned} \tag{1.1}$$

System (1.1) describes traveling wave solutions of a system of partial differential equations which models electron transport in certain semiconductor materials. The dependent variables in system (1.1) are the electric field E, the total electron concentration n which is the sum of the concentrations of two different species of electrons one of which is denoted by u, the other one is then given by $n-u$. The variable w is defined as the derivative of u by the usual reduction of a second order equation to two first order equations. The physical problem and the mathematical model which leads to the traveling wave problem (1.1) are described in detail in Section 2.

At this point we only mention that the physical problem has dynamics on two very different time scales (described by the small parameter $\varepsilon \ll 1$) which causes the traveling wave problem (1.1) to be singularly perturbed, i.e. the solution has slow components varying on the time scale t and fast components varying on the time scale $\tau := t/\varepsilon$. By

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transforming the slow system (1.1) to the independent variable τ we obtain the equivalent fast system

$$
\begin{aligned}
E' &= \varepsilon(n-1)/\lambda^2 \\
n' &= \varepsilon(v_1(E)u + v_2(E)(n-u) \; - c\,n + \gamma) \\
u' &= w \\
w' &= \varepsilon(v_1(E)-c)\,w + \varepsilon^2 v_1'(E)u(n-1)/\lambda^2 + (1+\alpha(E))u - n\,.
\end{aligned} \tag{1.2}
$$

Physically significant traveling wave solutions correspond to homoclinic or heteroclinic orbits connecting fixed points of equation (1.1). The proof of existence of homoclinic and heteroclinic orbits in finite-dimensional dynamical systems in dimension greater than two is a difficult task. In many cases the proof of the existence of homoclinic and heteroclinic orbits is based on perturbation methods by identifying the system of interest as a perturbation of a simpler or lower-dimensional system which is easier to analyse and has the desired orbits (see [5], [20]).

For this application the simpler, lower-dimensional system which captures the essential dynamics is given by the reduced problem obtained by setting the singular perturbation parameter zero. By setting $\varepsilon = 0$ in the slow system (1.1) we obtain the reduced problem

$$
\begin{aligned}
\dot{E} &= (n-1)/\lambda^2 \\
\dot{n} &= (v(E)-c)\,n + \gamma \\
0 &= w \\
0 &= (1+\alpha(E))\,u - n
\end{aligned} \tag{1.3}
$$

where the function $v(E)$ is defined by

$$
v(E) := \frac{v_1(E) + \alpha(E)v_2(E)}{1+\alpha(E)}\,. \tag{1.4}
$$

The reduced problem defines a dynamical system on a two-dimensional manifold $\mathcal{C}$ which is defined by the last two equations in system (1.3). By setting $\varepsilon = 0$ in the fast system (1.2) we obtain the layer problem

$$
\begin{aligned}
E' &= 0 \\
n' &= 0 \\
u' &= w \\
w' &= (1+\alpha(E))\,u - n\,.
\end{aligned} \tag{1.5}
$$

which approximates the fast dynamics away from the manifold $\mathcal{C}$.

In Section 3 we summarize the necessary results from [4] and [13] on invariant manifold theory for singularly perturbed problems. In Section 4 we apply these methods to system (1.1) and show the existence of a global center manifold close the manifold of the reduced problem. This allows us to conclude that hyperbolic fixed points of the reduced problem (1.3) persist as hyperbolic fixed points of the singularly perturbed problem (1.1). The existence of homoclinic orbits of the reduced problem (1.3) is analysed in Section 5. We show that for $\gamma = 0$ the reduced problem (1.3) can be transformed to a Hamiltonian System which has homoclinic orbits in an interval of possible wavespeeds, and that for $\gamma \neq 0$ no closed orbits of the reduced problem exist. An application of the Melnikov Integral proves the persistence of the homoclinic orbits of the reduced problem (1.3) in the global center manifold of the singularly perturbed problem (1.1) for small ε. For a generalization of these methods to construct transversal heteroclinic and homoclinic orbits in singularly perturbed problems we refer to [19].

2. Physical Problem. We consider semiconductor materials, the most important of which is Gallium Arsenide, which possess two different species of electrons. The existence of these different species of electrons is due to quantum theoretic effects in the bandstructure of the semiconductor material, i.e. there are electrons in a lower and in a higher energetic state. These states are called valleys due to the form of the bandstructure of the conduction band. The most important effect of this configuration is that electrons in the lower valley have a much smaller effective mass than electrons in the upper valley. Therefore electrons in the lower resp. upper valley are called light resp. heavy electrons. We denote their concentration densities by n_1 resp. n_2. As a consequence their drift velocities $v_1(E)$ resp. $v_2(E)$ due to the acceleration by an electric field E are of different orders of magnitude, i.e. $v_1(E) \gg v_2(E)$. Therefore, the terms light and fast resp. heavy and slow are used simultaneously for the electrons in the lower resp. upper valley. The functions v_1 and v_2 are odd, monotone increasing functions of the electric field. For small fields they are essentially linear and they saturate at values v_1^s resp. v_2^s as $E \to \infty$ (see Figure 1).

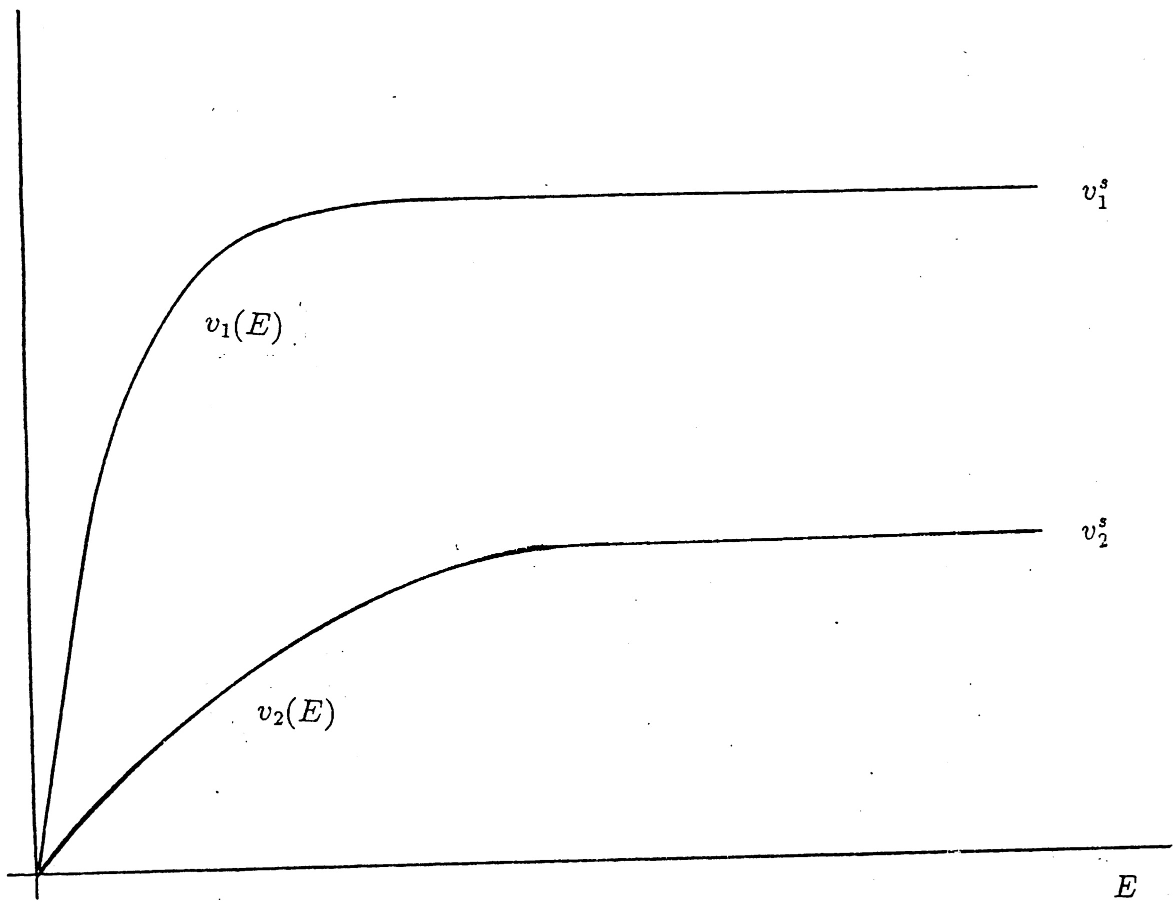

Figure 1.

The intervalley scattering of the electrons between these two states at a certain point within the semiconductor depends mainly on the strength of the electric field at this point (see [12]). We use a simple phenomenological description of this process suitable for the macroscopic drift-diffusion model

$$
\begin{aligned}
&\lambda^2 \frac{\partial^2 \psi}{\partial x^2} = n_1 + n_2 - 1 \\
&\frac{\partial n_1}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial n_1}{\partial x} - v_1(E) n_1\right) - \frac{\alpha(E) n_1 - n_2}{\varepsilon^2} \\
&\frac{\partial n_2}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial n_2}{\partial x} - v_2(E) n_2\right) + \frac{\alpha(E) n_1 - n_2}{\varepsilon^2}
\end{aligned}
\tag{2.1}
$$

subject to Dirichlet boundary conditions for ψ, n_1, and n_2 and nonnegative initial values for n_1 and n_2. The equations are written in scaled dimensionless quantities, for details of the scaling we refer to [10], a detailed discussion of the drift-diffusion model is given in [14] and [16]. Time is denoted by t, the spatial variable x takes values in the interval $(0,1)$. The first equation in (2.1) is Poisson's equation which determines the electrostatic potential ψ for a given space charge $\rho := n_1 + n_2 - 1$. The (negative) electric field E is given by

$$E := \frac{\partial \psi}{\partial x}\ .$$

The constant one in the expression for ρ represents a uniform background charge density called doping profile. The constant λ^2 multiplying the second spatial derivative of ψ is due to the scaling. The other two equations are continuity equations for the concentration densities of the fast and slow electrons. The corresponding currents are the sum of a diffusion current and of a drift current.

We would like to make the following comments on the form of the source term $S := (\alpha(E)n_1 - n_2)/\varepsilon^2$ which models the intervalley transfer: (i) the source term S appears with different signs in the continuity equations since the total electron concentration $n := n_1 + n_2$ remains unchanged by the scattering process; (ii) the parameter ε^2 is very small, because the intervalley transfer occurs on a much faster time scale than the other dynamics; (iii) by transforming the system (2.1) to the fast time scale $\tau := t/\varepsilon^2$ and by keeping only the leading order terms we obtain the following system of linear differential equations for n_1 and n_2 with the variables x and $E(x)$ acting as parameters

$$
\begin{aligned}
\frac{\partial n_1}{\partial \tau} &= -\alpha(E) n_1 + n_2 \\
\frac{\partial n_2}{\partial \tau} &= \alpha(E) n_1 - n_2 .
\end{aligned}
\tag{2.2}
$$

All solutions of the linear equation (2.2) converge exponentially to the stationary solution given by

$$n_1 = \frac{n}{1+\alpha(E)}\ , \quad n_2 = \frac{n\,\alpha(E)}{1+\alpha(E)} \tag{2.3}$$

where $n := n_1 + n_2$ denotes the total electron concentration which does not change during this simplified fast time scale process. This motivates the assumptions : (A_1) α is a monotone increasing function of the field strength, i.e. $\alpha(E) = \alpha(|E|)$, and (A_2) $\alpha(0) = 0$. A particular simple choice for the function $\alpha(E)$ is an even power of E which has been used in [8]. We expect from the simple dynamics of the fast time scale process (2.2) that the distribution of the electrons between the two valleys for the full singularly perturbed problem is – at least approximately – described by equations (2.3), i.e. for low fields the majority of the electrons is in the lower valley whereas for high fields most of the electrons are in the upper valley.

By eliminating the fast time scale process, i.e. by assuming the validity of equations (2.3) for $(x,t) \in (0,1) \times \mathbf{R}^+$, we obtain formally the reduced problem corresponding to system (2.1). By adding the two continuity equations and by using equations (2.1) we obtain an equation for the total electron concentration $n := n_1 + n_2$

$$\begin{aligned} \lambda^2 \frac{\partial^2 \psi}{\partial x^2} &= n - 1 \\ \frac{\partial n}{\partial t} &= \frac{\partial}{\partial x} \left(\frac{\partial n}{\partial x} n - v(E) n \right) . \end{aligned} \tag{2.4}$$

Thus we are left with a single continuity equation where the averaged electron velocity $v(E)$ is given by equation (1.4). Due to our assumptions on the electron velocities v_1, v_2 and the function α we expect the graph of the velocity coefficient v to have the form shown in Figure 2. For small values of the field the velocity is almost linear $v(E) \sim v_1(E)$, as the field increases the proportion of electrons in the upper (slower) valley increases causing a decrease of the velocity for large electric field. For even larger fields, i.e. if most of the electrons are in the upper valley, we have $v(E) \sim v_2(E)$. We do not bother to give precise conditions on the coefficients v_1, v_2, and α which guarantee this behavior we simply assume that we have a configuration with this qualitative behavior.

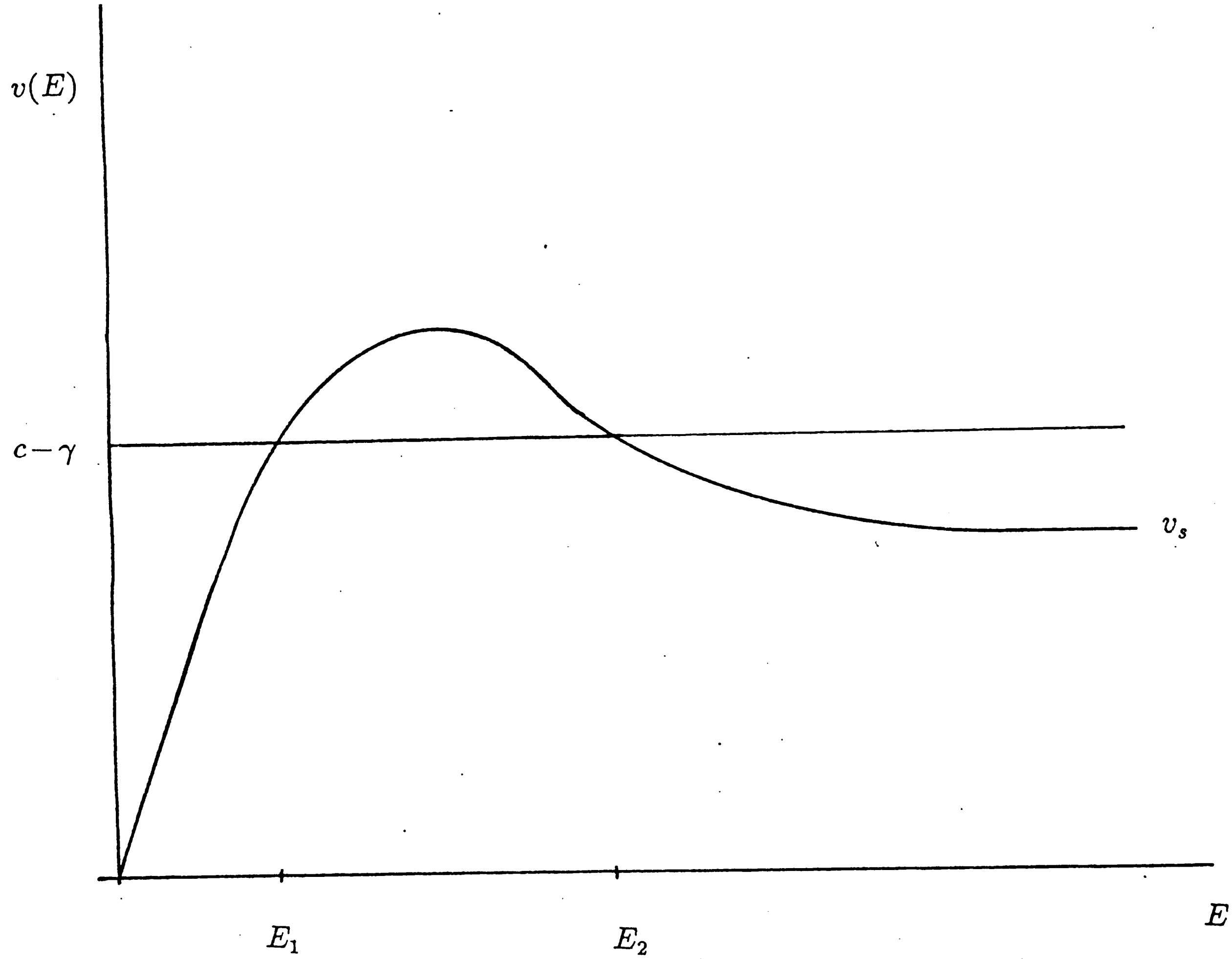

Figure 2.

The surprising phenomenon that the electron velocity decreases with increasing field in a certain range of the field is called negative differential resistance and causes interesting nonlinear behavior of solutions of equations (2.1) resp. (2.4). It was first discovered experimentally by *Gunn* [6] that semiconductor materials which exhibit negative differential resistance develop spontaneous oscillations and high field pulses traveling through the device, which is now known as the 'Gunn effect'. The importance of the sign of $v'(E)$ on the stability of the trivial solution $n \equiv 1$ becomes apparent by writing the continuity equation for n in differentiated form

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - v(E)\frac{\partial n}{\partial x} - v'(E)\frac{n(n-1)}{\lambda^2} \, .$$

So far the mathematical analysis of the 'Gunn effect' has been based mostly on the reduced system (2.4). We refer to [8], [15], and [16] for the engineering literature where also results from numerical simulations of more complicated models like (2.1) can be found. Recently a more rigorous discussion of the existence and stability of various types of traveling wave solutions of system (2.4) has been given in [17]. In [18] the (spatial) singular perturbation nature of (2.4), i.e. $\lambda^2 \ll 1$ is very small too, has been used to show the significance of these traveling wave solutions for the boundary value problem on the finite interval $(0,1)$.

This suggests to rewrite system (2.1) in terms of the slow variable $n := n_1 + n_2$ and one of the fast variables n_1 or n_2, we choose n_1. In these variables the equations have the form

$$
\begin{aligned}
\lambda^2 \frac{\partial^2 \psi}{\partial x^2} &= n - 1 \\
\frac{\partial n}{\partial t} &= \frac{\partial}{\partial x}\left(\frac{\partial n}{\partial x} - v_1(E)n_1 - v_2(E)(n - n_1)\right) \\
\frac{\partial n_1}{\partial t} &= \frac{\partial}{\partial x}\left(\frac{\partial n_1}{\partial x} - v_1(E)n_1\right) + \frac{n - (1+\alpha(E))n_1}{\varepsilon^2}
\end{aligned}
\tag{2.5}
$$

One expects - at least heuristically - that for ε sufficiently small the solution of system (2.5) is close to the solution of the reduced problem (2.4). We shall show that this is true in the finite-dimensional case of traveling wave solutions. Furthermore we would like to make the remark that equations (2.5) could be the starting point of a similar analysis for the infinite-dimensional initial boundary value problem (2.5).

Traveling wave solutions are solutions depending on the single variable $\xi := x - ct$. By transforming system (2.5) to the variable ξ and by integrating the equation for n once we obtain

$$
\begin{aligned}
\dot{E} &= (n-1)/\lambda^2 \\
\dot{n} &= v_1(E)u + v_2(E)(n-u) - cn + \gamma \\
\varepsilon\, \dot{u} &= w \\
\varepsilon\, \dot{w} &= \varepsilon(v_1(E) - c)\, w + \varepsilon^2 v_1'(E)(n-1)u/\lambda^2 + (1+\alpha(E))u - n
\end{aligned}
\tag{2.6}
$$

where we have used the substitution

$$
u := n_1 \quad , \quad w := \varepsilon\, \dot{n}_1
\tag{2.7}
$$

and γ is an arbitrary integration constant. Thus, we obtain the singularly perturbed traveling wave problem (1.1). Since we consider only the traveling wave problem from now on we use the traditional independent variable t instead of ξ, i.e. the superscript '$\cdot$' denotes differentiation with respect to t.

The phase space $\mathcal{V}$ for this problem is $\mathcal{V} := \mathbf{R}\times\mathbf{R}_0^+\times\mathbf{R}_0^+\times\mathbf{R}$ since the electron concentrations have to be nonnegative, however, mathematically the equations are well defined on $\mathbf{R}^4$ and define a smooth flow on this space. This will cause no confusion because the dynamics we are interested in take place in a submanifold of $\mathcal{V}$. The smoothness of the nonlinear right hand side of (2.6) is determined by the smoothness of $\alpha(E)$ which we assume to be smooth.

Since λ^2 is small the equation for E is also singularly perturbed, which is important to show the significance of the traveling wave solutions for the system (2.1) of partial differential equations on the finite interval $(0,1)$. In [18] asymptotic expansions (in λ) of solutions of system (2.4) have been constructed, which consist essentially of moving internal layer terms which are the leading order terms of an expansion (in λ) of the traveling waves given by Theorem 4. The most important impact of the parameter λ^2 is that $n \sim 1$ ($n \equiv 1$ is the solution of the reduced problem $\lambda^2 = 0$) because even small deviations of n from one cause large variations of E. However, in this paper we are only concerned with the asymptotic behavior of solutions for small ε and small but fixed λ^2.

3. Geometric Singular Perturbation Theory. We briefly review some of the results on invariant manifold theory for singularly perturbed problems. The subject goes back to the the work of Fenichel [4] but has only recently become more popular, see e.g. [13] for details on the invariant manifold theory and [2], [9],and [19] for applications of these geometric methods. We consider singularly perturbed systems of differential equations

$$\begin{aligned} \dot{x} &= f(x,\, y,\, \varepsilon) \\ \varepsilon\dot{y} &= g(x,\, y,\, \varepsilon) \end{aligned} \tag{3.1}$$

with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 > 0$ small, and $(x,y) \in \mathbf{R}^{m+k}$. We assume that $f \in \mathbf{R}^m$ and $g \in \mathbf{R}^k$ are C^r functions of (x, y, ε). For $\varepsilon \neq 0$ these equations define a smooth dynamical system. We consider x and y as functions of the variable t. By transforming the slow system (3.1) to the fast variable τ we obtain the equivalent fast system

$$\begin{aligned} x' &= \varepsilon\, f(x,\, y,\, \varepsilon) \\ y' &= \quad g(x,\, y,\, \varepsilon)\,. \end{aligned} \tag{3.2}$$

By setting $\varepsilon = 0$ in equation (3.1) and (3.2) we obtain two essentially different problems, the reduced problem

$$\begin{aligned} \dot{x} &= f(x,\, y,\, 0) \\ 0 &= g(x,\, y,\, 0) \end{aligned} \tag{3.3}$$

and the layer problem

$$\begin{aligned} x' &= 0 \\ y' &= g(x,\, y,\, 0) \end{aligned} \tag{3.4}$$

We make the assumptions :

(*i*) The equation $g(x,y,0)=0$ has a smooth manifold $\mathcal{C}$ of solutions which is given as a graph of a C^r function $h:\mathbf{R}^m\to\mathbf{R}^k$

$$\mathcal{C}=\{(x,h(x)),\ x\in\mathbf{R}^m\}\ , \tag{3.5}$$

i.e. the dynamics of the reduced problem (3.3) are governed by the equation

$$\dot{x}=f(x,\,h(x),\,0)\,. \tag{3.6}$$

(*ii*) There exists an integer j such that the matrix

$$\frac{\partial g(x,h(x),0)}{\partial y}$$

has j eigenvalues with negative real part and $k-j$ eigenvalues with positive real part.

The significance of these assumptions is that compact subsets of the manifold $\mathcal{C}$ satisfy the hyperbolicity assumptions for normally hyperbolic invariant manifolds of the layer problem (3.4). The following theorem which is contained in Theorem 9.1 in [4] asserts that the invariant manifold $\mathcal{C}$ perturbs into an invariant manifold $\mathcal{C}_\varepsilon$ for small ε. We call $\mathcal{C}_\varepsilon$ a center-like manifold since it is the ε section of an $m+1$ dimensional manifold which has the role of a center-manifold for system (3.2) extended by the trivial equation $\varepsilon'=0$. The linearization of this extended system at points in $\mathcal{C}\times\{0\}$ has j eigenvalues with negative real parts, an eigenvalue zero of algebraic multiplicity $m+1$, and $k-j$ eigenvalues with positive real parts. The version of the theorem we use can be found in [13].

THEOREM 1. *Under the assumptions made in this section let K be a compact subset of $\mathbf{R}^m$. Then there exists $\varepsilon_0>0$ and a C^{r-1}-function $h:R^m\times[0,\varepsilon_0)\to R^k$ such that the manifold $\mathcal{C}_\varepsilon$ defined by*

$$\mathcal{C}_\varepsilon:=\{(x,h(x,\varepsilon),\ x\in K\},\quad \varepsilon\in(0,\varepsilon_0)$$

is locally invariant under the flow generated by equations (3.1) resp. (3.2). Furthermore $h(x,0)=h(x),\ x\in K$ holds.

For details we refer to [4], [13] where additionally the existence of stable and unstable manifolds for $\mathcal{C}_\varepsilon$ is shown and to the literature on invariant manifolds, e.g. [3], [7]. It is obvious that Theorem 1 remains valid if the functions f and g in system (3.1) depend smoothly on additional parameters

$$f=f(x,y,\varepsilon,\eta),\quad g=g(x,y,\varepsilon,\eta),\quad \eta\in\mathbf{R}^l\,.$$

In this case the invariant manifold $\mathcal{C}_\varepsilon$ depends smoothly on η.

Theorem 1 implies that the slow system (3.1) restricted to the invariant manifold $\mathcal{C}_\varepsilon$ can be extended smoothly to $\varepsilon = 0$ near $\mathcal{C}$. Thus, a singular perturbation problem (3.1) which satisfies the assumptions of Theorem 1 becomes a regular perturbation problem of the reduced problem (3.6) by restricting it to the center-like manifold $\mathcal{C}_\varepsilon$. The dynamics in $\mathcal{C}_\varepsilon$ are determined by the equation

$$\dot{x} = f(x, h(x,\varepsilon), \varepsilon) .$$

The importance of this observation is that ([4], pp. 91) *'any structure in $\mathcal{C}$ which persists under regular perturbations persists under singular perturbations'* by restricting the flow generated by the singularly perturbed problem (3.1) to the center-like manifold $\mathcal{C}_\varepsilon$. Normally hyperbolic invariant manifolds of the reduced problem, e.g. hyperbolic fixed points and hyperbolic periodic orbits, persist as normally hyperbolic invariant manifolds of the singularly perturbed problem (see Theorems 12.2, 13.2 in [4]). Another immediate consequence of Theorem 1 is that transversal heteroclinic orbits of the reduced problem persist as heteroclinic orbits of the singularly perturbed problem. However, the existence of homoclinic orbits can not be shown by this argument, since it is well known that stable and unstable manifolds of a hyperbolic fixed point can not intersect transversally in a homoclinic orbit.

In the following we outline how transversality can be regained in singularly perturbed problems which depend on additional parameters. We consider systems of the form

$$\begin{aligned} \dot{x} &= f(x,\, y,\, \varepsilon,\, \mu) \\ \varepsilon\dot{y} &= g(x,\, y,\, \varepsilon,\, \mu) . \end{aligned} \tag{3.7}$$

which satisfy the assumptions of Theorem 1 for $\mu \in U$, where U is an open subset of $\mathbf{R}^l$. Suppose there exists $\mu_0 \in U$ such that the reduced problem

$$\begin{aligned} \dot{x} &= f(x,\, y,\, 0,\, \mu) \\ 0 &= g(x,\, y,\, 0,\, \mu) . \end{aligned} \tag{3.8}$$

has an orbit $\omega_0(\mu_0) \subset \mathcal{C}$ parametrized by $(x_0(t), h(x_0(t), \mu_0))$, $t \in \mathbf{R}$ homoclinic to the hyperbolic fixed point p_0, i.e. the unstable manifold W^u intersects W^s the stable manifold of p_0. We assume that the nontransversality is of the least degenerate type, i.e. the the intersection

$$T_q\, W^u \cap T_q\, W^s, \quad q \in \omega_0$$

is one-dimensional. The last assumption is equivalent to the assumption that $\dot{x}_0$ is the only bounded solution of the linearized equation

$$\dot{x} = A(t)\, x \tag{3.9}$$

where the matrix A is defined by

$$A(t) := D_x\, f(x_0(t), h(x_0(t), \mu_0), 0, \mu_0) .$$

It is important to notice that the derivative D_x in this formula has to be computed by the chain rule because $h = h(x, \mu)$ holds. The same remark holds for the derivative D_μ in equation (3.10) below.

THEOREM 2. *Under the above assumptions define the vector $M \in \mathbf{R}^l$ by*

$$M := \int_{-\infty}^{\infty} (\psi(t) \cdot D_\mu f(x_0(t), h(x_0(t), \mu_0), 0, \mu_0))\, dt \tag{3.10}$$

where the function ψ is the (up to a scalar multiple) unique bounded solution of the adjoint equation of (3.9)

$$\dot{\psi} = -A^T \psi\, . \tag{3.11}$$

If $M \neq \mathbf{0}$ then there exists $\varepsilon_1 > 0$ and a smooth family

$$\{\mathcal{H}_\varepsilon,\ \varepsilon \in [0, \varepsilon_1)\}$$

of codimension one hypersurfaces $\mathcal{H}_\varepsilon \subset \mathbf{R}^l$ with $\mu_0 \in \mathcal{H}_0$ such that the singularly perturbed problem (3.7) has an orbit $\omega_\varepsilon(\mu)$ homoclinic to the hyperbolic fixed point $p_\varepsilon(\mu)$ for $\mu \in \mathcal{H}_\varepsilon$, $\varepsilon \in (0, \varepsilon_1)$. Each of the homoclinic orbits $\omega_\varepsilon(\mu)$ lies in the center-like manifold $\mathcal{C}_\varepsilon$ uniformly close to the orbit ω_0.

If the i-th component of the vector $M = (\mu_1, \ldots, \mu_l)$ is different from zero then the hypersurface $\mathcal{H}_\varepsilon$ can be represented as the graph of a smooth function

$$\mu_i = \mu_i(\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_l, \varepsilon)\, .$$

Proof. We choose a compact set $K \subset \mathbf{R}^m$ such that the orbit ω_0 is contained in the interior of the set $\{(x, h(x, \mu_0)), x \in K\}$. We conclude from Theorem 3 the existence of a center-like manifold $\mathcal{C}_\varepsilon$ for system (3.7) on which the dynamics are governed by the equation

$$\dot{x} = f(x, h(x, \mu, \varepsilon), \varepsilon, \mu)\, . \tag{3.12}$$

Now the theorem is a straightforward application of homoclinic bifurcation theory [5], [11] for the regular perturbation problem (3.12). Therefore we only sketch the main idea, details can be found in [19]. Our assumptions imply that equation (3.12) has an orbit homoclinic to the hyperbolic fixed point p_0 for $\mu = \mu_0$, $\varepsilon = 0$. The hyperbolicity of p_0 implies that there exists a neighbourhood $V \subset \mathbf{R}^l$ of μ_0, $\varepsilon_1 > 0$ and points $p_\varepsilon(\mu)$ which are hyperbolic fixed points of equation (3.12) for $\mu \in V$, $\varepsilon \in (0, \varepsilon_1)$. There exists a smooth function $Q(\varepsilon, \mu)$ which measures the distance between the unstable manifold $W^u(p_\varepsilon(\mu))$ and the stable manifold $W^s(p_\varepsilon(\mu))$ in a hyperplane transversal to the unperturbed homoclinic orbit. The function $Q(\varepsilon, \mu)$ is usually called the Melnikov function. Obviously, solutions of the equation $Q(\varepsilon, \mu) = 0$ correspond to homoclinic orbits of equation (3.12). In particular $Q(0, \mu_0) = 0$ holds. It is well known that

$$\frac{\partial Q(0, \mu_0)}{\partial \mu} = M$$

holds (see [5], [11]). Thus the assumption $M \neq 0$ implies the theorem since we can solve the equation $Q(\varepsilon, \mu) = 0$ according to the implicit function theorem. □

We would like to make the remark that homoclinic orbits may exist for parameter values which are not solutions of the equation $Q(\varepsilon, \mu)$ due to the possibility of homoclinic doubling at resonant eigenvalues (see the analysis in [1]).

4. Singularly Perturbed Problem. Now we apply the methods from the previous section to equations (1.1) which are of the form (3.1) with $m = 2$ and $k = 2$. System (1.2) is defined on $\mathbf{R}^4$, where we keep in mind that only nonnegative values of n and u have a physical meaning.

THEOREM 3. *Assume that $\alpha : \mathbf{R} \to \mathbf{R}$ is C^r, $r > 2$. Let K be a compact subset of $\mathbf{R}^2$. Then there exists $\varepsilon_0 > 0$ such that system (1.1) has an invariant center-like manifold for $\varepsilon \in (0, \varepsilon_0)$ which is given by*

$$\mathcal{C}_\varepsilon = \{(E, n, u(E, n, \varepsilon), w(E, n, \varepsilon)),\ (E, n) \in K\}$$

The functions $u, w : K \times [0, \varepsilon_0) \to \mathbf{R}$ are C^{r-1} and have the asymptotic expansion

$$\begin{aligned} u(E, n, \varepsilon) &= \frac{n}{1+\alpha} + O(\varepsilon^2) \\ w(E, n, \varepsilon) &= \varepsilon \left(\frac{v(E) - c)n + \gamma}{1 + \alpha} - \frac{\alpha' n(n-1)}{(1+\alpha)^2 \lambda^2} \right) + O(\varepsilon^3). \end{aligned} \tag{4.1}$$

Proof. The theorem is a direct application of Theorem 1. The manifold $\mathcal{C}$ of stationary solutions of (1.5) is the two-dimensional graph $\mathcal{C} := \{(E, n, \frac{n}{1+\alpha(E)}, 0),\ (E, n) \in \mathbf{R}^2\}$, i.e. assumption (i) is satisfied. The assumption (ii) is satisfied, the relevant eigenvalues of the linearization of equation (1.5) are

$$\lambda^s(E, n) = -\sqrt{1 + \alpha(E)}\,, \quad \lambda^u(E, n) = \sqrt{1 + \alpha(E)}\,.$$

Thus, Theorem 1 implies the existence assertions of the theorem. We know that

$$u(E, n, 0) = \frac{n}{1 + \alpha(E)}\quad , \quad w(E, n, 0) = 0$$

holds for $(E, n) \in K$. The functions u and w are C^{r-1} functions of their arguments which allows to approximate them by expansions in ε. Differentiating the equations for u and w with respect to τ gives

$$u' = \frac{\partial u}{\partial E} E' + \frac{\partial u}{\partial n} n'\,.$$

The invariance of $\mathcal{C}_\varepsilon$ implies that we can use the equations (1.2) to eliminate the derivatives with respect to τ which gives

$$w = \frac{\partial u}{\partial E} \varepsilon (n-1)/\lambda^2 + \frac{\partial u}{\partial n} \varepsilon (v_1(E) u + v_2(E)(n - u) - cn + \gamma).$$

By differentiating this equation with respect to ε and by setting $\varepsilon = 0$ we obtain

$$\frac{\partial w}{\partial \varepsilon}(E,n,0) = \frac{(v(E)-c)n+\gamma}{1+\alpha} - \frac{\alpha' n(n-1)}{(1+\alpha)^2\lambda^2} \tag{4.2}$$

where we have used that $\frac{\partial u}{\partial E}$ and $\frac{\partial u}{\partial n}$ are known on $K \times \{0\}$. Repeating this procedure once more gives

$$\frac{\partial^2 w}{\partial \varepsilon^2}(E,n,0) = 0 \tag{4.3}$$

A similar calculation starting with w shows that

$$\frac{\partial u}{\partial \varepsilon}(E,n,0) = 0 \tag{4.4}$$

holds which proves the theorem. □

By inserting expansion (4.1) into system (1.1) we obtain equations which approximate the flow in the center-like manifold $\mathcal{C}_\varepsilon$.

$$\begin{aligned}
\dot{E} &= (n-1)/\lambda^2 \\
\dot{n} &= (v(E)-c)n+\gamma+O(\varepsilon^2) \\
\dot{u} &= \frac{(v(E)-c)n+\gamma}{1+\alpha} - \frac{\alpha' n(n-1)}{(1+\alpha)^2\lambda^2} \;+\; O(\varepsilon^2) \\
\dot{w} &= O(\varepsilon)
\end{aligned} \tag{4.5}$$

Note that we could divide the equations for u and w by ε which corresponds to the smoothness of the vector field (1.1) near the manifold $\mathcal{C}$ even for $\varepsilon = 0$. By writing the equations in the form (4.5) it is apparent that the flow in the center-like manifold $\mathcal{C}_\varepsilon$ is a regular perturbation of the reduced problem

$$\begin{aligned}
\dot{E} &= (n-1)/\lambda^2 \\
\dot{n} &= (v(E)-c)n+\gamma \\
\dot{u} &= \frac{(v(E)-c)n+\gamma}{1+\alpha} - \frac{\alpha' n(n-1)}{(1+\alpha)^2\lambda^2} \\
\dot{w} &= 0\,.
\end{aligned} \tag{4.6}$$

System (4.6) is simply the projection of the planar two-dimensional reduced problem (1.3) in E, n coordinates on the two-dimensional manifold $\mathcal{C}$. Clearly, we knew this a priori from Theorem 1, however, we find it instructive to go through the construction in some detail.

5. Homoclinic Orbits. We now turn to the discussion of the fixed points of (1.1) and to the analysis of the existence of homoclinic orbits which correspond to traveling wave solutions of system (2.1). We restrict ourselves to the case of waves traveling to the right, i.e. $c > 0$. However, all the traveling waves exist for $c < 0$ as well because our analysis is based on the dynamics of the reduced problem (1.3) which is invariant under the transformation

$$(E, c, \gamma, t) \to (-E, -c, -\gamma, -t) \,. \tag{5.1}$$

The fast dynamics are basically uneffected by the transformation (5.1) which corresponds to a reversal of time in the layer problem (1.5).

The fixed points of (1.1) are given by

$$p_i := (E_i, 1, \frac{1}{1+\alpha(E_i)}, 0), \quad i = 1, 2 \tag{5.2}$$

where the values E_i are the solutions of the equation $v(E) = c - \gamma$. Let $v_{\max}$ denote the local maximum of $v(E)$ and let v_s denote the saturation velocity. For $c - \gamma \in (v_s, v_{\max})$ there exist two fixed points which we assume to be ordered as in Figure 2. Due to the special structure of the equations the fixed points (5.2) of the singularly perturbed problem (1.1) are identical with the fixed points of the reduced problem (1.3).

We will use Theorem 2 to prove the existence of orbits of system (1.1) homoclinic to the fixed point p_1. Thus, the starting point is to find values of the parameters c and γ such that the reduced problem

$$\begin{aligned} \dot{E} &= (n-1)/\lambda^2 \\ \dot{n} &= (v(E) - c)\,n + \gamma \end{aligned} \tag{5.3}$$

has homoclinic orbits. We use the notation p_i, $i = 1, 2$ for the fixed points of the planar system (5.3) as well as for those of the reduced problem (1.3). An easy calculation shows that the fixed point p_1 is a saddle for all choices of the parameters. Thus, Theorem 3 and Theorem 12.2 in [4] imply

LEMMA 1. *There exists $\varepsilon_1 > 0$ such that the point p_1 is a hyperbolic fixed point of system (1.1) with two-dimensional stable and unstable manifolds for $\varepsilon \in (0, \varepsilon_1)$.*

The fixed point p_2 is a center for $\gamma = 0$ and a stable or unstable node or focus depending on the parameters c, γ and λ^2. We are not more specific on that since p_2 is not important in the following.

The substitution $n = e^z$ transforms (5.3) to

$$\begin{aligned} \dot{E} &= (e^z - 1)/\lambda^2 \\ \dot{z} &= v(E) - c + \gamma e^{-z} \end{aligned} \tag{5.4}$$

which is a Hamiltonian system for $\gamma = 0$. Since γ is an arbitrary integration constant we start by analysing the case $\gamma = 0$. In this case the integrability of equations (1.3) resp. (5.3) makes it straightforward to prove

THEOREM 4. *For $\gamma = 0$ equation (1.3) has an an orbit $\omega_0(c)$ homoclinic to the point $p_1(c)$ for all $c \in (v_s, v_{\max})$. There exists a family of periodic orbits around the center p_2. There exists $\lambda_0 > 0$ such that n is strictly positive along these orbits for $0 < \lambda \leq \lambda_0$. In the case $\gamma \neq 0$ no closed orbits exist.*

Proof. By using E as independent variable we obtain the differential equation

$$\frac{d}{dE} n = \frac{\lambda^2 (v(E) - c) n}{n - 1} \tag{5.5}$$

By integrating this equation starting at the point p_1 we obtain the equation

$$n(E) - \ln(n(E)) - 1 = \lambda^2 \int_{E_1}^{E} (v(e) - c) \, de \tag{5.6}$$

which describes the stable and unstable manifolds of p_1. The value $n = 1$ is the only zero of the function $n - \ln n - 1$. The unstable manifold of p_1 crosses the line $n = 1$ at a maximal value of the field $E_{\max}$ which is determined by the 'equal area rule'

$$\int_{E_1}^{E_{\max}} (v(e) - c) \, de \; = \; 0$$

and returns to the fixed point forming the homoclinic orbit. For $c \to v_{\max}-$ the homoclinic loop $\omega_0(c)$ shrinks to the corresponding fixed point p_1, for $c \to v_s+$ the homoclinic loop $\omega_0(c)$ becomes unbounded, i.e. $E_{\max} \to \infty$ holds. The existence of the family of periodic orbits inside the homoclinic loop is proved in the same way. By expanding $\ln(n(E))$ in (5.6) in a power series around $n = 1$ we obtain that $n(E)$ is given asymptotically by $n(E) = 1 + O(\lambda)$ which proves the positivity of n along the homoclinic and periodic orbits for small λ. In the case $\gamma \neq 0$ no closed orbits exist because the divergence of the vectorfield (5.4) is strictly positive resp. negative for $\gamma < 0$ resp. $\gamma > 0$. $\square$

Now we use Theorem 2 to prove that the homoclinic orbits of the reduced problem (1.3) given by Theorem 4 persist for the singularly perturbed problem (1.1) for small ε.

THEOREM 5. *Assume that $\alpha : \mathbf{R} \to \mathbf{R}$ is C^r, $r > 2$. Then for $[c_-, c_+] \subset (v_s, v_{\max})$ there exist $\varepsilon_1 > 0$ and a C^{r-1}-function $\gamma : [c_-, c_+] \times [0, \varepsilon_1) \to \mathbf{R}$ such that the singularly perturbed problem (1.1) has an orbit $\omega_\varepsilon(c)$ homoclinic to the fixed point p_1 for $\gamma = \gamma(c, \varepsilon)$, $c \in [c_-, c_+]$, and $\varepsilon \in (0, \varepsilon_1)$. The orbit $\omega_\varepsilon(c)$ lies in the center-like manifold $\mathcal{C}_\varepsilon$ given by Theorem 3. There exists $\lambda_0 > 0$ such that n, u are strictly positive along all the homoclinic orbits $\omega_\varepsilon(c)$ for $0 < \lambda < \lambda_0$.*

Proof. We choose the compact set $K \subset \mathbf{R} \times \mathbf{R}^+$ in Theorem 3 such that all homoclinic orbits given by Theorem 4 for $c \in [c_-, c_+]$ are contained in its interior. Let $c \in [c_-, c_+]$

be fixed. We know from Theorem 4 that for the parameter values c, $\gamma = 0$ the reduced problem (5.3) resp. (5.4) has an orbit ω_0 homoclinic to p_1. Since the reduced problem is two-dimensional the assumptions of Theorem 2 are satisfied, i.e. we are left with computing the vector M for which we use equations (5.4). Let $(E_0(t), z_0(t))$, $t \in \mathbf{R}$ be a parametrization of the orbit ω_0. For the two-dimensional Hamiltonian system (5.4) the solution ψ of the adjoint equation of the linearization (3.9) is given by $(-\dot{z}_0, \dot{E}_0)$, see e.g. [5]. The computation

$$M = \int_{-\infty}^{\infty} (-\dot{z}_0, \dot{E}) \begin{pmatrix} 0 & 0 \\ -1 & e^{-z_0} \end{pmatrix} dt = \int_{-\infty}^{\infty} (-\dot{z}_0, \dot{E}_0\, e^{-z_0})\, dt$$

shows that the first component of M vanishes. The second component of M is positive because

$$\int_{-\infty}^{\infty} \dot{E}_0\, e^{-z_0}\, dt = \int_{\omega_0} e^{-z}\, dE = \int_{\Omega_0} e^{-z}\, dE dz$$

holds, where Ω_0 denotes the interior of the homoclinic loop ω_0. Thus, Theorem 2 implies the existence of the smooth function $\gamma(c, \varepsilon)$ locally near the point $(c, 0)$ which can uniquely be extended to a smooth function $\gamma : [c_-, c_+] \times [0, \varepsilon_1) \to \mathbf{R}$ since $c \in [c_-, c_+]$ was arbitrary. The positivity of the components n, u along all orbits $\omega_\varepsilon(c)$ follows from Theorem 2 and Theorem 4. □

The first component of M has to be zero because we know from Theorem 4 that homoclinic orbits exist for $c \in (v_s, v_{\max})$, $\gamma = 0$. The nonvanishing second component of M corresponds to the breaking of the homoclinic loop of problem (5.3) resp. (5.4) for $\gamma \neq 0$.

We conclude from Theorem 5 that all homoclinic orbits of the reduced problem (1.3) persist as homoclinic orbits of the singularly perturbed problem (1.1) for small ε. Thus, we have shown the existence of traveling wave solutions of system (2.1) resp. (2.5). Our analysis shows that - in the context of the traveling wave description of the 'Gunn-effect' - the reduced problem (2.4) is a valid zeroth order approximation of the more sophisticated two-valley model (2.1) resp. (2.5).

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