August, 1965

A NOTE ON THE TRIGAMMA FUNCTION

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Technical Report No. 60

University of Minnesota
Minneapolis, Minnesota

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The purpose of this note is to bring attention to the fact that the formula

$$(1)\qquad \frac{d^2}{dz^2}\log\Gamma(z+1) = \frac{1}{z+1} + \frac{1!}{2}\,\frac{1}{(z+1)_2} + \frac{2!}{3}\,\frac{1}{(z+1)_3} + \cdots, \quad \mathrm{Re}(z) > -1$$

where $(z+1)_j = (z+1)(z+2)\ldots(z+j)$, is a direct consequence of the well-known formula of Gauss for the value of the hypergeometric function with unit argument. The series in (1) converges absolutely and uniformly on the region $\mathrm{Re}(z) > -1 + \delta$ $(\delta > 0)$, and serves also as an asymptotic development for the trigamma function when $-\pi/2 < \arg(z+1) < \pi/2$. In fact, we have

$$F(a,b;c;1) = \sum_{0}^{\infty} \frac{(a)_j(b)_j}{j!(c)_j}$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}\,(c-a-b) > 0,\ c \neq 0, -1, -2, \ldots,$$

and therefore in particular

$$(2)\qquad \frac{\Gamma(c)\Gamma(c-2b)}{\Gamma^2(c-b)} = \sum_{0}^{\infty} \frac{\{(b)_j\}^2}{j!(c)_j}, \qquad \mathrm{Re}(c-2b) > 0,\ c \neq 0, -1, -2, \ldots\ .$$

(1) follows from (2) on letting $b \to 0$. It is actually slightly more convenient to reparametrize (2) by setting $z = c - b$, $h = b$ and taking logarithms before allowing h to tend to 0. Thus,

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*Research supported under a grant from the United States Air Force Office of Scientific Research - Office of Aerospace Research, Grant #AF-AFOSR-885-65.

$$\frac{\log \Gamma(z+h) - 2\log \Gamma(z) + \log \Gamma(z-h)}{h^2} = \frac{1}{h^2} \log \left(1 + \frac{\{(h)_1\}^2}{1!(z+h)_1} + \frac{\{(h)_2\}^2}{2!(z+h)_2} + \dots \right),$$

$$\mathrm{Re}(z-h) > 0, \quad z+h \neq 0, -1, -2, \dots, \quad h \neq 0,$$

which gives

$$\frac{d^2}{dz^2} \log \Gamma(z) = \sum_1^\infty \frac{\{(j-1)!\}^2}{j!(z)_j}, \qquad \mathrm{Re}(z) > 0,$$

as in (1), when $h \to 0$. [Here $\lim\limits_{h\to 0} \{(h)_j\}^2/\{(z+h)_j h^2\} = 1^2 \cdot 2^2 \dots (j-1)^2/(z)_j$ has been used. Justification of the term by term limiting process is elementary and is omitted.]. Absolute and uniform convergence of the series in (1) on the region $\mathrm{Re}(z) > -1 + \delta$ $(\delta > 0)$ follows from Raabe's test and

$$\left| \sum_1^\infty \frac{(j-1)!}{j\cdot(z+1)_j} \right| < \sum_1^\infty \frac{(j-1)!}{j\cdot\delta(\delta+1)\dots(\delta+j-1)} < \infty$$

(the ratio of the jth to the $(j+1)$th term in the dominating series being $(1 + 1/j)(1 + \delta/j)$), while the asymptotic nature of the series follows from

$$\left| \sum_{j=m+1}^\infty \frac{(j-1)!}{j\cdot(z+1)_j} \right| = \left| \frac{m!}{(m+1)\cdot(z+1)_{m+1}} \cdot \sum_{q=0}^\infty \frac{m+1}{m+1+q} \frac{(m+1)_q}{(z+m+2)_q} \right|$$

$$< \left| \frac{m!}{(m+1)\cdot(z+1)_{m+1}} \right| \cdot \sum_{q=0}^\infty \frac{m^q}{(m+1)_q}$$

$$< e^m \cdot \frac{m!}{m+1} \left| \frac{1}{(z+1)_{m+1}} \right| = o(|\text{mth term}|), \qquad |z| \to \infty.$$

(1) should be contrasted to the familiar result

$$\frac{d^2}{dz^2} \log \Gamma(z+1) = \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \dots .$$

(1) can, of course, be obtained independently. For example, it follows from Nörlund's series [1924, p. 261]

$$\psi(a - z) = \psi(a) = \left\{\frac{(z)_1}{(a)_1} + \frac{1}{2}\frac{(z)_2}{(a)_2} + \frac{1}{3}\frac{(z)_3}{(a)_3} + \dots\right\},$$

valid for $\mathrm{Re}(a - z) > 0$ and $a \neq 0, -1, -2, \dots$, in which $\psi(z) = (d/dz) \log \Gamma(z)$, by differentiation at $z = 0$, and can also be obtained by a slight modification of standard theory for the development of functions in series of inverse factorials [Whittaker and Watson 1946, pp.142-4]. (We remark in passing that the last method is here rather clumsy, and the coefficients of $1/(z + 1)_j$ are obtained as linear functions of the Bernouilli numbers.) Nevertheless, the proof of this note is simple and suggestive.

References

Nörlund, N. E. (1924). Vorlesungen über Differenzenrechnung. Springer.

Whittaker, E. T. and Watson, G. N. (1946). A Course of Modern Analysis. [4th Edn.] Cambridge.