

CONSTRUCTION OF SMOOTH ERGODIC COCYCLES FOR SYSTEMS  
WITH FAST PERIODIC APPROXIMATIONS

BY

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Construction of Smooth Ergodic Cocycles for Systems  
with Fast Periodic Approximations

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## Introduction:

In this paper we prove results about lifting various dynamical and ergodic properties of a given smooth dynamical system to its skew product extensions by smooth cocycles. The classical small divisor argument shows that in general such results are not possible. However we will show that if the dynamical system admits a "fast periodic approximation" then indeed certain qualitative behaviour which is prohibited by "small divisor type conditions" is now in fact generic. In Section I we give the basic definitions and state the main theorem, Section II is for proofs. In Section III we apply these results to study some question about the qualitative behaviour of solutions of linear differential equations with almost periodic coefficients. Here we show how the well approximability by rationals of the of the eigenfrequencies of the coefficient matrix yields generically a strange qualitative behaviour.

### SECTION (I.1) Basic Definitions and Notation:

Let  $\Omega$  be a compact, connected  $C^\infty$  manifold. Let  $T$  denote either the group of integers ( $\mathbb{Z}$ ) or reals ( $\mathbb{R}$ ). A triple  $(\Omega, T, \mu)$  is called a  $C^k$ -smooth dynamical system if  $T$  acts on  $\Omega$  with a jointly  $C^k$ , i.e.  $k$  times continuously differentiable  $k \in \mathbb{N} \cup \{\infty\}$  action  $(w, t) \rightarrow w \cdot t$ ,  $\forall w \in \Omega, t \in T$  and preserving a smooth Borel probability measure  $\mu$  on  $\Omega$ . If  $T = \mathbb{Z}$ , we will denote again by letter  $T$  the diffeomorphism generating the  $\mathbb{Z}$  action and if  $T = \mathbb{R}$  we will denote by  $(T_t)_{t \in \mathbb{R}}$  the one parameter group of diffeomorphisms of  $\Omega$ .

Given a connected Lie group  $G$ , a cocycle from  $(\Omega, T)$  into  $G$  is a continuous map  $\phi: \Omega \times T \rightarrow G$  such that  $\phi(w, t_1 + t_2) = \phi(w \cdot t_1, t_2) \phi(w, t_1)$ ,

$\forall w \in \Omega, t_1, t_2 \in T$ . A cocycle  $\phi$  is of class  $C^k$  if for each  $t \in T$  the map  $w \rightarrow \phi(w, t)$  is a  $C^k$  map. Let  $Z_k(\Omega, G)$  denote the set of all  $C^k$  cocycles into  $G$ .

If  $T = \mathbb{Z}$ , the cocycle  $\phi$  is completely determined by its values on  $\Omega \times \{1\}$  and hence will be identified with a function on  $\Omega$  into  $G$ , (i.e. cocycle  $\phi$  determined by a function  $\phi$  is given by  $\phi(w, n) = \phi(T^{n-1}w) \phi(T^{n-2}w) \cdots \phi(w), n \in \mathbb{Z}^+$ ) Thus for  $T = \mathbb{Z}$ , we will identify  $Z_k(\Omega, G)$  with  $C^k(\Omega, G)$  the space of all  $C^k$  functions from  $\Omega$  into  $G$ . Thus consequently the  $C^k$  norm  $\|\cdot\|_k$  on  $C^k(\Omega, G)$  induces a metric  $D_k$  on  $Z_k(\Omega, G)$ . Metrics on all other spaces will be denoted by letter  $d$ .

Suppose  $Z$  is another  $C^\infty$  manifold on which a Lie group  $G$  acts on the left with a jointly  $C^k$  action  $(g, z) \rightarrow gz$ . Then given a cocycle  $\phi \in Z_k(\Omega, G)$ , we define on  $Z \times \Omega$  the skew product  $T$ -action by setting  $(z, w) \cdot t = (\phi(w, t)z, w \cdot t)$ . In the case  $T = \mathbb{Z}$ , the diffeomorphism generating the skew product action will be denoted by  $T_\phi$ , i.e.  $T_\phi(z, w) = (\phi(w)z, Tw)$ . If  $\nu$  is a  $G$  invariant Borel

measure on  $Z$  then clearly  $\nu \times \mu$  is invariant under the skew product action. The following are two important cases to be kept in mind. (a) when  $Z = G$  and  $G$  acts on itself by left multiplication. Here  $\nu \equiv \eta$  - a left Haar measure on  $G$  is an invariant measure. (b) Take  $G = GL(n, \mathbb{R})$  - the general linear group and  $Z = \mathbb{P}^{n-1}(\mathbb{R})$  - the real projective  $n-1$  space. In this case we do not have any invariant measure on  $Z$ .

Given any  $f \in C^k(\Omega, G)$ , it generates a cocycle  $l^f \in Z_k(\Omega, G)$  by setting  $l^f(w, t) = f(w \cdot t) f(w)^{-1}$ . Cocycles of this form are called coboundaries and the set of all such coboundaries will be denoted  $B_k(\Omega, G)$ . The trivial cocycle is the cocycle  $l^1$ , generated by the map  $w \rightarrow e$  (- the identity element of  $G$ ) and will be denoted by  $1$ . Given a  $\phi \in Z_k(\Omega, G)$  and  $l^f \in B_k(\Omega, G)$  set,  $\phi \cdot l^f(w, t) = f(w \cdot t) \phi(w, t) f(w)^{-1}$ . It is easy to verify that  $\phi \cdot l^f \in Z_k(\Omega, G)$ . Given  $\phi_1, \phi_2 \in Z_k(\Omega, G)$  we call them cohomologous via a transfer function  $f$  if  $\phi_2 = \phi_1 \cdot l^f$  for some  $f \in C^k(\Omega, G)$ . Cohomologous cocycles gives rise to isomorphic skew-product actions.

Our main goal is to construct cocycles for which the corresponding skew products are ergodic. The following property of the action of  $G$  on  $Z$  is necessary for the ergodicity of the skew product actions, (see Herman [7]). Let  $\nu$  be a  $\sigma$ -finite  $G$  invariant measure on  $Z$ , then the triple  $(Z, G, \nu)$  is said to have a  $L^\infty$ -fixed point property if every  $G$  invariant weakly compact set  $K \subseteq L^\infty(Z, \nu)$  contains a fixed point for the  $G$  action. This property also helps in constructing cocycles with ergodic skew products. We list a few examples of actions having this property. (1) If  $\nu$  is a finite Borel measure then  $(Z, G, \nu)$  has this property. (2) Let  $G$  be amenable,  $Z = G$  and action of

$G$  be by left-multiplication. Take  $\nu = \eta$  - a left Haar measure. This system does leave  $L^\infty$  fixed point property. (3) Let  $Z = \mathbb{R}^2$  and  $G = \text{SL}(2, \mathbb{R})$  or the group  $\left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} / a \in \mathbb{R}^*, b \in \mathbb{R} \right\}$  with standard linear action on  $Z$  and  $\nu$  be the Lebesgue measure on  $\mathbb{R}^2$ . Then  $(Z, G, \nu)$  has a  $L^\infty$  fixed point property.

### SECTION (1.2) Lifting ergodicity by a smooth cocycle

One important question in the study of skew product dynamical systems has been lifting ergodicity i.e. if  $(\Omega, T, \mu)$  is ergodic, can one find a cocycle  $\phi$  such that the corresponding skew-product flow is ergodic with respect to the product measure? If in the above set up everything is just "continuous" (and not smooth) then the existence of a continuous cocycle  $\phi$  giving ergodic skew-products is known - even for more general dynamical systems, where  $T$  can be any reasonable amenable group, (see [17]). In fact such cocycles are residual in the class  $\overline{B}_0(\Omega, G)$ , (we will denote by  $\overline{B}_k$  the closure of  $B_k$  in  $Z_k$ ). However even for integer actions (i.e.  $T = \mathbb{Z}$ ) the problem of producing sufficiently smooth function  $\phi$  for which  $T_\phi$  is ergodic is delicate and difficult. In fact the classical "small divisor" argument shows that for some dynamical systems this is impossible. For example, let  $\Omega = S^1$  - the circle.  $T = R_\alpha$  - the irrational rotation by  $\alpha$  and  $\alpha$  be badly approximable by rationals. Then given any smooth enough (say  $C^k, k \geq 2$ ) function  $\phi: \Omega \rightarrow \mathbb{R}$  with  $\int_\Omega \phi d\mu = 0$  ( $\mu$  being the Lebesgue measure on  $S^1$ ), the functional equation  $\phi(w) = f(R_\alpha w) - f(w)$  has always a solution  $f$  of class - say  $C^{k-2}$ . This result of Kolmogorov - Siegel shows that  $\overline{B}_k(S^1, \mathbb{R}) \subseteq B_{k-2}(S^1, \mathbb{R})$  and hence  $\forall \phi \in \overline{B}_k(\Omega, \mathbb{R})$  the skew product diffeomorphism  $(R_\alpha)_\phi$  on  $\mathbb{R} \times S^1$  can not be ergodic.

As very different phenomenon occurs when  $T$  is a point transitive Anosov diffeomorphism. In this case A. Livšic has given (see [15]) a precise condition for the solvability of the cohomology equation  $f(Tw) - f(w) = \phi(w)$ , for a given smooth  $\phi$ . Using this, one can show that  $\overline{B}_1(\Omega, \mathbb{R}) = B_1(\Omega, \mathbb{R})$  and hence lifting ergodicity in the class  $\overline{B}_1(\Omega, \mathbb{R})$  is impossible. However using a Parry-Jones type argument along with A. Livšic's result one can show that cocycles lifting ergodicity in the class  $Z_1(\Omega, G)$  are residual when  $G = \mathbb{T}^n$  - the  $n$  torus. A similar result about density of cocycles lifting ergodicity in the class  $Z_1(\Omega, G)$  for any compact connected Lie group  $G$  is obtained by M. Brin (see [2]). We also remark that this result of A. Livšic has been generalized to the case when  $T$  is a point transitive diffeomorphism satisfying a "closing lemma", (see [11]). Also versions of this result for geodesic flows are known. Now a result of J. Hawkins, ([6]) based on slight modification of Parry-Jones argument shows that ergodicity can be generically lifted in the class  $Z_k(\Omega, G)$  for any diffeomorphism if  $G = \mathbb{T}^n$  the  $n$ -torus. This shows that lifting ergodicity in the class  $Z_k$  and in the class  $B_k$  are in some sense different problems. Also non compactness and non abelianness of the group  $G$  contribute substantially to the difficulties in constructing smooth cocycles lifting ergodicity.

In this paper we develop a smooth version of a technique of Glasner and Weiss, (see [5]). The small divisor argument shows that in general their technique has no smooth analog. The main obstacle being construction of a coboundary with "desired Properties" while keeping it close to the identity cocycle in  $C^k$  norm, (in  $C^0$  norm this is always possible.) However we will

show that if the transformation admits an "approximation by periodic transformation" with "sufficient speed", or is " $C^\infty$  rigid", then the above obstacle can be overcome. Thus our results are in some sense opposite of the consequences of small divisor type arguments.

### SECTION (I.3) Fast periodic approximation and statement of the main theorem

Definition (I.3.1) Consider a  $C^r$ -smooth discrete dynamical system  $(\Omega, T, \mu)$  on a  $C^\infty$  manifold  $\Omega$ . Let  $a(n) \geq 0$ ,  $n \in \mathbb{N}$  be a sequence such that  $\lim_{n \rightarrow \infty} a(n) = 0$ .

The diffeomorphism  $T$  is said to admit a  $C^r$ -fast-periodic approximation (or  $C^r$ -rigid) with speed  $a(n)$  if there exists a sequence  $(q_n)_n \in \mathbb{N}$  of positive integers and a constant  $K > 0$  such that  $q_n \rightarrow \infty$  and,

$$\|h \circ T^{q_n} - h\|_r \leq K \|h\|_{r+1} a(q_n), \quad \forall n \in \mathbb{N}, \quad \forall h \in C^{r+1}(\Omega, \mathbb{R}).$$

If  $T$  is  $C^\infty$  and above inequality holds  $\forall r \in \mathbb{N}$ , (with  $M$  depending on  $r$ ) then we say  $T$  is  $C^\infty$  rigid with speed  $a(n)$ .

The following is a typical example of such diffeomorphisms.

Example: Let  $\Omega = S^1$ , be the circle and  $T = R_\alpha$ , be the irrational rotation by  $\alpha$ . If  $\exists$  a sequence  $(p_n/q_n)_n \in \mathbb{N}$  of irreducible fractions with  $q_n \rightarrow \infty$

as  $n \rightarrow \infty$  and a constant  $K > 0$  such that  $|\alpha - p_n/q_n| < \frac{K}{(q_n)^{p+1} \epsilon}$  then  $(\Omega, R_\alpha)$

is  $C^\infty$  rigid with speed  $a(n) = \frac{1}{n^{p+1} \epsilon}$ . If  $\alpha$  is a Liouville number i.e.

$|\alpha - p_n/q_n| < \frac{K}{(q_n)^n}$  for some sequence  $p_n, q_n$  of integers with  $q_n \rightarrow \infty$  and

$K > 0$  is a constant then we can take  $a(n) = \frac{1}{n^n}$ .

An immediate generalization of this example is obtained by taking  $\Omega = \mathbb{T}^n$  - the  $n$  torus and  $R_\alpha$  - the irrational rotation by  $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ , where  $\bar{\alpha}$  is a "Liouville vector" i.e.  $\alpha_1, \dots, \alpha_n$  are rationally independent irrationals and for some sequence  $q_n$  of integers such that  $q_n \rightarrow \infty$  as  $n \rightarrow \infty$  one has  $||| \alpha_i q_n ||| \leq \frac{K}{(q_n)^n}, \forall i, n$ , where  $K > 0$  is a constant and  $||| |||$  denotes the distance from the nearest integer. Again  $(\Omega, R_\alpha)$  is  $C^\infty$  rigid with speed  $1/q_n$ .

We also mention that besides this prime example, examples constructed by Anosov and Katok in [1] also admit  $C^r$ -fast periodic approximations.

Now we state our main theorem.

Theorem (I.3.2) Let  $(\Omega, T, \mu)$  be a  $C^r$  dynamical system with a smooth invariant ergodic probability measure  $\mu$ . Let  $(\Omega, T)$  admit  $C^r$  fast-periodic approximation with speed  $a(n)$ . Let  $G$  be a connected Lie group acting  $C^r$ -smoothly on a connected  $C^\infty$  manifold  $Z$  and preserving a  $\sigma$ -finite ergodic measure  $\nu$ .

Assume that  $(Z, G, \nu)$  has the  $L^\infty$  fixed point property. If either  $r < \infty$  and

$$a(n) = \frac{1}{n^{2r+1+\rho}} (0 < \rho < 1) \text{ or } r = \infty \text{ and } a(n) = \frac{1}{a^n}, \text{ where } a > 1 \text{ is some}$$

constant, then the set,  $\{\phi \in \bar{B}_r(\Omega, G) \mid (Z \times \Omega, T_\phi, \nu \times \mu) \text{ is ergodic}\}$

is residual in  $\bar{B}_r(\Omega, G)$ .

Remark (I.3.3) (a) If  $(\Omega, T)$  is as in the example described before, namely the irrational rotation on  $n$ -torus one may only assume that  $a_n = \frac{1}{n^{2r+\rho}}$ ,

furthermore if  $G = \mathbb{R}$  then the assumption on the decay rate of  $a(n)$  can be made weaker by degree  $r$ , i.e.  $a(n) = \frac{1}{n^{2r+\rho}}$  ( $0 < \rho < 1$ ). The reason for this

is also explained later in Remark (II.3). This in particular for  $\Omega = S^1$  gives a generic version of a theorem of Krygin [14]. We also believe that this is the least speed we need to get a generic lifting of ergodicity in the class  $\overline{B}_r$ .

(b) If  $\mu$  is the unique ergodic measure on  $\Omega$  and either (i)  $G$  is amenable or (ii)  $G = SL(2, \mathbb{R})$  and  $Z = P^{n-1}(\mathbb{R})$  then in the conclusion of Theorem (1.3.2) ergodicity can be replaced by unique ergodicity.

(c) Although we do not discuss "affine cocycles" in this paper, using the techniques developed here we can prove an "affine extension" of Theorem (1.3.2). As a consequence we get the following theorem.

Theorem (I.3.4) Let  $(\Omega, T, \mu)$  be as in theorem (1.3.2). Let  $G = \mathbb{T}^n$  be the  $n$ -torus and  $\sigma$  be any automorphism of  $G$ . Then the set  $\{\phi \in C^r(\Omega, G) \mid \text{the map } T_\phi(g, w) = (\sigma(\phi(w)g), Tw) \text{ on } G \times \Omega \text{ is ergodic with respect to the product measure}\}$  is residual in  $C^r(\Omega, G)$ .

Now we turn to the proof of the main theorem.

SECTION II Proof of Theorem (I.3.2)

Let  $H = L^1(Z \times \Omega, \nu \times \mu)$  and  $H_0 = \{f \mid f \in H, \int f d\nu \times \mu = 0\}$ . Let  $\| \cdot \|_1$  be the  $L^1$  norm on  $H$ . Given a cocycle  $\phi \in Z_r(\Omega, G)$ , define operator  $U_\phi$  on  $H$  by setting  $U_\phi f(z, w) = f(\phi(w)z, Tw)$ . Let  $V_\phi^N = \frac{1}{N} \sum_{i=0}^{N-1} U_\phi^i$ . Also, given a function  $\psi \in C^r(\Omega, G)$ , define operator  $L_\psi$  on  $H$  by  $L_\psi f(z, w) = f(\psi(w)z, w)$ .

Now given  $f \in H_0$ ,  $\epsilon > 0$  and  $m \in \mathbb{N}$ , define  $W(f, \epsilon, m) = \{\phi \mid \phi \in \overline{B}_r(\Omega, G)$  such that  $M \in \mathbb{N}$ ,  $M > m$  and  $\|V_\phi^M f\|_1 < \epsilon\}$ . Now if

$\phi \in \bigcap_j \bigcap_n \bigcap_m W(f_j, 1/n, m)$ , (where  $(f_j)_{j \in \mathbb{N}} \in H_0$  is a dense subset) then

$(Z \times \Omega, T_\phi, \nu \times \mu)$  is ergodic, (see [7] for a proof). Thus once we prove that

each  $W(f, \epsilon, m)$  is open and dense in  $\overline{B}_r$  then Theorem (I.3.2) is a consequence of the Baire Category theorem. Openness is easy to verify. To prove density,

first observe that  $V_\phi^N L_\psi = L_\psi V_{\phi.1}^N$ . This shows that if  $1 \in \overline{W(f, \epsilon, m)}$ ,  $\forall f, \epsilon, m$

then  $1^\psi \in \overline{W(f, \epsilon, m)}$ ,  $\forall f, \epsilon, m, \psi$  i.e.  $W(f, \epsilon, m)$  is dense in  $\overline{B}_r$ . Thus it is

enough to prove that, given  $f, \epsilon, m$  and  $\delta > 0$  there exists a  $\psi \in C^r(\Omega, G)$  such that, (i)  $D_r(1^\psi, 1) < \delta$  and  $\|V_{1^\psi}^M f\|_1 < \epsilon$  for some  $M > m$ . But now ergodic theorem

implies that,  $\|V_{1^\psi}^n f\|_1 \rightarrow \|\int_\Omega f(\psi(w)z, w) d\mu(w)\|_1$  as  $n \rightarrow \infty$ . Hence if we can

choose  $\|\int_\Omega f(\psi(w)z, w) d\mu(w)\|_1 < \epsilon/2$ , then by choosing  $M$  large enough, we can make sure that  $\|V_{1^\psi}^M f\|_1 < \epsilon$ . Thus we have reduced the proof of the theorem

to the following lemma.

Lemma (II.1) Given  $f \in H_0$ ,  $\epsilon > 0$  and  $\delta > 0$ , there exist a  $\psi \in C^r(\Omega, G)$  such that (i)  $D_r(1^\psi) < \delta$  and (ii)  $\| \int_\Omega f(\psi(w)z, w) d\mu(w) \|_1 < \epsilon$

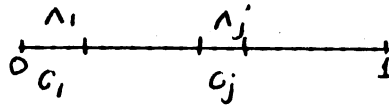
Proof: Let  $0 < \gamma < 1/2$  be a small number (it's relation to  $f, \epsilon, \delta$  will be determined later). Since  $C_0(Z \times \Omega)$  is dense in  $L^1(Z \times \Omega)$ , standard approximation arguments allow us to assume (without loss of generality) that the given  $f$  is continuous, has compact support and  $\| f \|_1 < \epsilon/8$ . We will first assume that  $f$  depends on  $z$  alone, i.e.  $f(z, w) \equiv f(z)$  and let  $M_1 = \sup\{|f(z)| / z \in Z\}$ .

We will fix a family  $(O_\alpha, \phi_\alpha)_{\alpha=1}^R$  of local charts so that (i)  $(O_\alpha)_{\alpha=1}^R$  is an open cover of  $\Omega$  (ii)  $\phi_\alpha: \rightarrow Q$  (where  $Q \equiv \{x = (x_1, \dots, x_n) \in \mathbb{R}^n / 0 < x_i < 1\}$ ,  $n = \dim \Omega$ ) a  $C^\infty$  diffeomorphism such that  $(\phi_\alpha)_*(\mu|_{O_\alpha}) = m|_Q$ , where  $\mu|_{O_\alpha}$  and  $m|_Q$  are the normalized restrictions of measure  $\mu$  and the Lebesgue measure  $m$  to  $O_\alpha$  and  $Q$  respectively, (such a chart can be chosen because  $\mu$  has a positive  $C^\infty$  density in each chart, see [1]). Let  $M_2 = \sup_{1 < i < R} \| \phi_i \|_{r+1}$ , note that  $M_2$  depends only on  $r$  and the manifold  $\Omega$ .

Since  $(Z, G, \nu)$  has  $L^\infty$  fixed point property, there are  $c_i \in [0, 1]$  and  $g_i \in G$ ,  $1 \leq i \leq s$  such that  $\sum_{i=1}^s c_i = 1$  and  $\| \sum_{i=1}^s c_i f_{g_i} \|_1 < \epsilon/4$ , where  $f_{g_i}(z) = f(g_i z)$ ,

(see [7] for details.)

Divide interval  $[0,1]$  into subintervals  $\Lambda_j$  of length  $C_j$  as shown in figure



and define a  $C^\infty$  map  $h: [0,1] \rightarrow G$  by setting it equal to  $g_i$  on "most of the interval  $\Lambda_i$ " and making sure that

$$\| \int_0^1 f_{h(t)}(z) dt \|_1 < \epsilon / 2 \quad (1)$$

Pick  $N \in \mathbb{N}$  such that,

$$(i) \quad \frac{LK}{(q_N)^p} < \delta \text{ and } (ii) \quad \frac{(M_2 C) K 2^{r+1}}{(q_N)^p} < \delta^* \quad (2)$$

where  $\delta^* > 0$  is such that if  $|t_1 - t_2| < \delta$  then  $|f(h(t_1)z) - f(h(t_2)z)| < \epsilon/4$

$Z$  and  $L, C$  are constants depending only on  $r$ , function  $h, f, \epsilon$  and manifold  $\Omega$ .

Using Rokhlin's lemma pick a Borel set  $E \subseteq \Omega$  such that (i)  $E, TE, \dots, T^{q_N-1}E$  are mutually disjoint and

$$(ii) \quad \mu \left( \bigcup_{i=0}^{q_N-1} T^i E \right) > 1 - (\gamma'/M_1) \quad (3)$$

Let  $(B_j)_{j=1}^p$  be a partition of  $E$  into disjoint Borel sets with positive measure such that for each  $1 \leq i \leq p, (p \leq R)$  there is some  $\alpha(i) \in \{1, 2, \dots, R\}$  such that  $B_i \subseteq T^{\alpha(i)} E$ . Then pick open sets  $V_j$  ( $1 \leq j \leq p$ ) such that  $B_j \subseteq V_j \subseteq T^{\alpha(j)} E$ ,

$$V_i \cap V_j = \emptyset \text{ if } i \neq j \text{ and } \mu(V_j - B_j) < \frac{\gamma' \mu(B_j)}{(q_N)^p M_1} \quad (4)$$

Such a choice of disjoint  $V_j$ 's can be made, because each  $B_j$  can be replaced by a compact subset of it with measure arbitrarily close to that of  $B_j$ . Thus if necessary replacing the original  $E$  by a slightly smaller set.

$$\text{Set } V = \bigcup_{j=1}^p V_j, \quad \lambda = \mu(V) \text{ and } \lambda_j = \mu(V_j).$$

The following sublemma (proof of which will be given later) is the first step in the construction of the desired function  $\psi$ .

Sublemma (II.2): Let  $Q \equiv \{(x_1, \dots, x_n) \in \mathbb{R}^n / 0 < x_i < 1\}$ ,  $W \subseteq Q$  be an open set,  $I \subseteq [0, 1]$  be an interval of length  $\ell$  and  $f_\beta: [0, 1] \rightarrow \mathbb{R}$ ,  $\beta \in I$  be an equicontinuous family of maps. Then given  $\epsilon > 0$ , and  $r \in \mathbb{N}$  there exists a  $C^\infty$  map  $\eta: Q \rightarrow I$  such that

(a)  $\text{supp } \eta \subseteq W$  and  $\eta \equiv 0$  in a small neighbourhood of  $\partial W$  and

$|\eta|_i \leq C \left[ \frac{\ell}{m(W)} \right]^i$ ,  $1 \leq i \leq r$  where  $C$  is a constant depending on  $r, \epsilon, f_\beta$  and the dimension of  $Q$ .

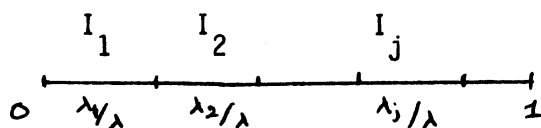
$$(b) \left| \frac{1}{m(W)} \int_W (f_\beta \circ \eta) dm - \frac{1}{|I|} \int_I f_\beta(t) dt \right| < \epsilon, \quad \forall \beta \in I. \quad (5)$$

With the help of above sublemma, we now construct maps  $\theta_j: V_j \rightarrow \mathbb{R}$  as follows:

Let  $\theta_j = \eta_j \circ \phi_{\alpha(j)}$  where  $\eta_j: \phi_{\alpha(j)}(V_j) \rightarrow I_j$  be as in the sublemma where

the equicontinuous family of maps is  $t \mapsto f(h(t)z)$ ,  $z \in Z$ ,  $\epsilon = \gamma'/R$  and  $I_j$  (6)

is an interval of length  $\lambda_j/\lambda$  as shown in the figure



Clearly  $\text{supp } \theta_j \subseteq V_j$  and  $\theta_j$  can be extended to all of  $\Omega$  by setting it equal to

0 outside  $V_j$ . Thus  $\theta_j: \Omega \rightarrow \mathbb{R}$  is  $C^{r+1}$  map and

$$\|\theta_j\|_{r+1} \leq \frac{M_2 C(\lambda_j/\lambda)^{r+1}}{m(\phi_{\alpha(j)}(V_j))^{r+1}} = \frac{M_2 C(\lambda_j/\lambda)^{r+1}}{[\lambda_j/m(0_{\alpha(j)})]^{r+1}} = \frac{M_3}{\lambda^{r+1}} \quad (7)$$

where  $M_3 = M_2 C \max_{1 \leq i \leq R} m(0_{\alpha(i)})$

Now set  $\tilde{\theta} = \sum_{j=1}^p \theta_j$ , then  $\tilde{\theta}$  is a  $C^{r+1}$  map with  $\text{supp } \tilde{\theta} \subseteq V$  and

since  $V_j$ 's are disjoint,  $\|\tilde{\theta}\|_{r+1} \leq \frac{M_3}{\lambda^{r+1}}$

Writing  $A \sim B$  to mean  $|A-B| < \gamma$ , we note that,

$$\int_E f([h \circ \tilde{\theta}(w)]z) d\mu(w) = \sum_{j=1}^p \int_{B_j} f([h \circ \tilde{\theta}](z)) d\mu(w)$$

$$\mu(V_j) \gamma' \sum_{j=1}^p \int_{V_j} f([h \circ \tilde{\theta}(w)]z) d\mu(w), \text{ by (4)}$$

Now  $\int_{V_j} f([h \circ \tilde{\theta}(w)]z) d\mu(w) = \frac{\mu(V_j)}{m(S_j)} \int_{S_j} f([h \circ \eta_j(r)]z) dm(r)$ , where  $S_j = \phi_{\alpha(j)}(V_j)$ .

$$\frac{(\mu(V_j) \gamma') / R}{\lambda_j} \int_{I_j} f(h(t)z) dt \quad \text{by (5)}$$

$$\text{Hence } \int_V f([h \circ \tilde{\theta}(w)]z) d\mu \stackrel{\mu(V)^{r+1}}{\sim} \mu(V) \int_0^1 f(h(t)z) dt \quad (8)$$

Now define  $\theta: \Omega \rightarrow \mathbb{R}$  by setting  $\theta(w) = \sum_{i=0}^{q_{N-1}} \tilde{\theta}(T^i w)$  and set  $\psi = h \circ \theta$ .

We will show that  $\psi$  is the required map. Clearly  $\psi$  is of class  $C^{r+1}$ . We

first show that  $D_r(1^\psi, 1) < \delta$ .

$$D_r(1^\psi, 1) = D_r[(\psi \circ T)\psi^{-1}, 1] = D_r[(\psi \circ T)\psi^{-1}, \psi \psi^{-1}]$$

$$\leq D_r(\psi \circ T, \psi) L_1 D_r(\psi^{-1}) \quad (\text{by "product rule", where } L_1$$

is a constant depending on  $r$ )

$$\leq L_1 D_r(h \circ \theta \circ T, h \circ \theta) D_r(h^{-1} \circ \theta)$$

$$\leq L_1 L_2 \|\theta \circ T - \theta\|_r L_3 \|\theta\|_r \quad (\text{by "chain rule", where } L_2$$

$L_3$  are constants depending only on  $r$  and  $h$ ).

$$\leq \tilde{L} \|\theta \circ T - \theta\|_r \|\theta\|_r, \quad \text{where } \tilde{L} = L_1 L_2 L_3.$$

$$\begin{aligned} \text{Now, } \|\theta \circ T - \theta\|_r &= \left\| \sum_{i=0}^{q_{N-1}} (\tilde{\theta} \circ T^{i+1} - \tilde{\theta} \circ T^i) \right\|_r \\ &= \|\tilde{\theta} \circ T^{q_N} - \tilde{\theta}\|_r \leq K \|\tilde{\theta}\|_{r+1} a(q_N) \end{aligned}$$

$$\begin{aligned} \text{Thus } D_r(1^\psi, 1) &\leq \tilde{L}K\|\tilde{\Theta}\|_{r+1}\|\tilde{\Theta}\|_r a(q_N) \\ &\leq K\tilde{L}(M_3)^2 \left[\frac{1}{\lambda^{2r+1}}\right] a(q_N) \text{ by (7)} \end{aligned}$$

$$\text{Now } \lambda = \mu(V) \geq \mu(E) \geq \frac{1-\gamma'}{q_N} \geq \frac{1}{2q_N} \text{ (Since } \gamma' < 1/2\text{).}$$

$$\text{Hence } D_r(1^\psi, 1) \leq LKa(q_N)q_N^{2r+1} = \frac{LK}{q_N^\rho} < \delta \text{ (by 3), where } L = 2^{r+1} \tilde{L}(M_3)^2 \text{ is}$$

a constant depending only on  $h, r$  and  $\omega$ .

Now we prove that  $\|\int_{\Omega} f(\psi(w)z)d\mu\|_1 < \epsilon$ . First consider  $\int_E f(\psi(w)z)d\mu$ .

$$\text{Set } E' = E - \bigcup_{i=1}^{q_{N-1}} T^i V. \text{ Note that, } \mu(E \cap T^i V) = \mu(E \cap T^i(V-E)) \leq \mu(V-E) < \frac{\gamma'}{q_N^{M_1}} \text{ by (4)}$$

$$\text{Thus } \mu(E \cap \bigcup_{1 \leq i} T^i V) < \sum_{i=1}^{q_{N-1}} \frac{\gamma'}{q_N^{M_1}} < \frac{\gamma'}{q_N^{M_1}}$$

$$\text{Hence } |\mu(E) - \mu(E')| < \gamma'/(q_N^{M_1}) \quad (9)$$

Furthermore, if  $w \in E'$  then  $\Theta(w) = \tilde{\Theta}(w)$ , thus (8) gives

$$\left| \int_E f(\psi(w)z)d\mu(w) - \mu(E) \int_0^1 f(h(t)z)dt \right| < \frac{4\gamma'}{q_N} \quad (10)$$

We now estimated  $\int_{T^i E} f(\psi(w)z)d\mu$ , for a fixed  $i \in \{1, 2, \dots, q_{N-1}\}$ .

Let  $\tilde{w} \in T^i E'$ , say  $\tilde{w} = T^i w$ ,  $w \in E'$ , then

$$\psi(\tilde{w}) = h \circ \Theta(T^i w) = h \circ \left( \sum_{j=0}^{q_{N-1}} \tilde{\Theta} \circ T^{i+j}(w) \right) = h \circ \left[ \sum_{j=i}^{q_{N-1}} \tilde{\Theta}(T^i w) + \sum_{j=0}^{i-1} (\tilde{\Theta} \circ T^{q_N})(T^j) \right]$$

Now note that,

$$\begin{aligned} & \sup_{w \in \Omega} \left| \sum_{j=0}^{i-1} \tilde{\theta}^{q_N+j}(T^j w) - \sum_{j=0}^{i-1} \tilde{\theta}(T^j w) \right| \\ & \leq K i \|\tilde{\theta}\|_{r+1} a(q_N) \\ & \leq \frac{K q_N^{2r+1} M_2 C q_N^{r+1}}{(q_N)^{2r+1+\rho}} < \delta^* \quad \text{by (2)} \end{aligned}$$

Thus if  $w \in E'$  then by (2) we have,

$$|f(\psi(T^i w)z) - f(\psi(w)z)| < \epsilon/4, \quad \forall z \in Z \quad (11)$$

$$\text{Hence, } \int_{T^i E} f(\psi(\tilde{w})z) d\mu \sim \frac{\gamma'}{q_N} \int_{T^i E'} f(\tilde{\gamma}(\tilde{w})z) d\mu \quad \text{by (9)}$$

$$\epsilon/4 \mu(E') \int_{TE'} f(\psi(w)z) d\mu \sim \frac{4\gamma'}{q_N} \mu(E) \int_0^1 f(h(t)z) dt \quad \text{by (10)}$$

and since  $\mu(E') < 1/q_N$  we get

$$\int_{\Omega} f(\psi(w)z) d\mu \sim \sum_{i=0}^{q_N-1} \int_{T^i E} f(\psi(w)z) d\mu$$

$$\text{and } \left| \sum_{i=0}^{q_N-1} \int_{T^i E} f(\psi(w)z) d\mu - \sum_{i=0}^{q_N-1} \mu(E) \int_0^1 f(h(t)z) dt \right| < \sum_{i=0}^{q_N-1} (\epsilon/4 + \gamma' + 4\gamma') \frac{1}{q_N}$$

$$\text{Thus } \left| \int_{\Omega} f(\psi(w)z) d\mu - \int_0^1 f(h(t)z) dt \right| < 2\gamma' + (\epsilon/4 + \gamma') + 4\gamma', \quad \forall z \in Z$$

Now using (1) it follows that,

$$\| \int_{\Omega} f(\psi(w)z) d\mu \|_1 < 7\gamma' + \epsilon/4 + \epsilon/4$$

Hence choosing  $\gamma'$  small enough we can make  $\| \int_{\Omega} f(\psi(w)z) d\mu \|_1 < \epsilon$ .

The general case: Now let  $f \in C_0(Z \times \Omega)$  be any function. Set

$g(z) = \int_{\Omega} f(z,w) d\mu(w)$ . Now given  $g, \epsilon, \delta$  we will apply the previous technique to construct the function  $\psi$  such that,

$$D_r(1^\psi, 1) < \delta \text{ and } \| \int_{\Omega} g(\psi(w)z) d\mu(w) \|_1 < \epsilon/4 \quad (a)$$

In doing so we will be little more careful in choosing  $N \in \mathbb{N}$  (see (2)).

Here we pick  $N \in \mathbb{N}$  so that in addition to conditions in (2) we also have,

$$(i) \quad \| \frac{1}{q_N} \sum_{i=0}^{q_N-1} f(z, T^i w) - g(z) \|_1 < \epsilon/4 \quad (b)$$

(This is possible by Ergodic theorem). and (ii) the choice of  $\delta^*$  be such that in addition to conditions in (2) we also have the following.

If  $|t_1 - t_2| < 5\delta^*$  then  $|f(h(t_1)z, w) - f(h(t_2)z, w)| < \epsilon/4, \forall (w, z)$

$$\text{Let } f_1(z, w) = \frac{1}{q_N} \sum_{i=0}^{q_N-1} f(z, T^i w).$$

$$\text{Claim: } \| \int_{\Omega} f_1(\psi(w)z, w) d\mu - \int_{\Omega} f(\psi(w)z, w) d\mu \|_1 < (\epsilon/4) + \gamma' \quad (c)$$

to see this note that,

$$\int_{\Omega} f_1(\psi(w)z, w) d\mu = \frac{1}{q_N} \sum_{i=0}^{q_N-1} \int_{\Omega} f(\psi(w), T^i w) d\mu.$$

Now we have shown before that

$|\theta(w) - \theta(T^i w)| < \delta^*$  if  $w \in E'$  and  $0 \leq i \leq q_N-1$ . Using the same technique one can easily verify that  $|\theta(w) - \theta(T^i w)| < 5\delta^*$  if  $w \in T^i E', 0 \leq i, j \leq q_N-1$

Now our choice of  $N$  implies the proof of the claim.

$$\text{Thus } \left\| \int_{\Omega} f(\psi(w)z, w) d\mu \right\|_1 \sim \epsilon/4 + \gamma' \left\| \int_{\Omega} f_1(\psi(w)z, w) d\mu \right\|_1 \sim \epsilon/4 \left\| \int_{\Omega} g(\psi(w)z) d\mu \right\|_1 \sim 0$$

Thus choosing  $\gamma' < \epsilon/4$  the proof is complete.

Remark (II.3) Note that in the previous proof the decay rate of  $a(n)$  was used at two places. The first being to estimate  $D_r(1^\psi, 1)$ . Clearly if  $G = \mathbb{R}$  then  $(\psi \circ T)\psi^{-1}$  is just  $\psi \circ T(w) - \psi(w)$  and we do not have to apply chain rule which in terms eliminate the additional factor  $\|\theta\|_r$  (see the computation) and hence one can lower the decay rate of  $a(n)$  by  $r$ -degrees.

Furthermore note that in defining the function  $\theta$  we did not translate  $\tilde{\theta}$  on  $T^i E$ 's and took  $\theta = \frac{1}{q_N} \sum_{i=0}^{q_N-1} \tilde{\theta} \circ T^i$ . If we were to do this, the estimates for  $D_r(1^\psi, 1)$  would involve  $C^r$  norms of powers of transformation  $T$  on which we have no control. But if  $C^r$  norms of arbitrary powers of  $T$  were uniformly bounded then averaging would have helped us in lowering the degree of decay rate by one, provided it was possible to choose open sets  $V$  such that  $T^i V$ 's are mutually disjoint. For the example of irrational rotation on  $n$ -torus this is possible. The approximating rational rotation provides an approximate but "very nice" Rokhlin tower of disjoint open sets triangulating the  $n$ -torus into parallelograms which enable one to define  $\theta$  periodically by translating along the approximating rational rotation, (see [18] for the details in the case of a 2-torus.)

The second place where we need the decay rate was in estimating  $|\theta(T^i w) - \theta(w)|$ ,  $0 \leq i \leq q_N - 1$ . But in the case of irrational rotations since  $\theta$  is defined periodically and the approximating rational rotation and the given irrational rotation are very close to each other up to their first  $q_n$  powers, one only needs a decay rate of the order  $\frac{1}{n^{r+p}}$

This explains why for irrational rotations and  $G = \mathbb{R}$  one can get results with lower decay rates. Finally we sketch the proof of the sublemma.

First let  $W$  be the cube  $\{(x_1 \dots x_n) \mid 0 < x_i < a\}$  and  $I = [0, \lambda]$ . Let  $\chi: W \rightarrow [0, 1]$  be a  $C^\infty$  map such that  $\chi \equiv 0$  outside  $\{\bar{x} \mid \frac{1}{2\epsilon} < x_i < a_i - \frac{1}{2\epsilon}\}$  and  $\chi \equiv 1$  inside  $\{\bar{x} \mid \frac{1}{\epsilon} < x_i < a_i - \frac{1}{\epsilon}\}$  and  $\|\chi\|_{r, \infty} \leq M(r) \epsilon^r$ ,  $1 \leq i \leq n$  where  $\epsilon = (a/N)$ ,  $N \in \mathbb{N}$  and  $M(r)$  is some constant depending on  $r$  alone. Explicit construction of such a function can be found in [21] and is based on the usual convolution technique. Take  $g(\bar{x}) = g(x_1 \dots x_n) = (\lambda x_n / a)$  and set  $n(\bar{x}) = \chi(\bar{x})g(\bar{x})$ . It is easy to verify that by choosing  $N \in \mathbb{N}$  large enough one can make  $\frac{1}{m(W)} \int_W f \circ n \, dm$  arbitrarily close to  $\frac{1}{\lambda} \int_0^\lambda f(t) dt$  and this can be done uniformly over the class of functions which are uniformly bounded (in particular for an equicontinuous family). If  $W$  is not a cube then write "most of  $W$ " as a disjoint union of such  $P$  cubes and apply above result replacing  $\lambda$  by  $\lambda/p$  and interval of length  $\lambda$  by intervals of length  $\lambda/p$  and take the sum of the corresponding functions.

SECTION III: Linear differential equations with almost periodic coefficients.

SECTION (III.1) Preliminaries:

Let  $(\Omega, T_t)$  be a  $C^k$  flow on a compact, connected  $C^\infty$  manifold  $\Omega$ . Given a continuous function  $A: \Omega \rightarrow M(n, \mathbb{R})$  - the set of all  $n \times n$  real matrices, we consider the family of linear differential equations given by,

$$\dot{x} = A(w \cdot t)x, \quad x \in \mathbb{R}^n \quad (I)$$

We will always assume that the flow  $(\Omega, T_t)$  is (1) minimal (i.e. the orbit  $\{T_t w | t \in \mathbb{R}\}$  of each  $w \in \Omega$  is dense in  $\Omega$ ) and (2)  $(\Omega, T_t)$  is almost periodic (i.e. the family  $(T_t)_{t \in \mathbb{R}}$  of maps is equicontinuous). Thus given a single linear differential equation  $\dot{x} = A(t)x$  with almost periodic coefficient matrix  $A(t)$ , taking  $\Omega$  to be the hull of  $A(t)$  transforms this problem into our general set up.

Let  $X_A(w, t): \Omega \times \mathbb{R} \rightarrow GL(n, \mathbb{R})$  be the fundamental matrix solution of (I) satisfying  $X_A(w, 0) = I, \forall w \in \Omega$ . Then  $X_A$  is a cocycle. Also  $A(w)$  can be recovered from  $X_A$  (i.e.  $A(w \cdot t) = \frac{d}{dt} X_A(w, t) X_A(w, t)^{-1}$ ). In this way we will here onwards identify the set  $Z_r(\Omega, GL(n, \mathbb{R}))$  with systems  $\dot{x} = A(w \cdot t)x$  where  $A \in C^r(\Omega, M(n, \mathbb{R}))$ . Furthermore note that  $X_A$  and  $X_B$  are cohomologous iff systems  $\dot{x} = A(w \cdot t)x$  and  $\dot{x} = B(w \cdot t)x$  are kinematically similar.

If  $A(w) \equiv A$  - a constant matrix, then one knows that the qualitative behaviour of the solutions of (1) is determined by the real parts of eigenvalues of  $A$ . If the flow on  $\Omega$  is periodic; then by the Floquet theorem (which states that the cocycle  $X_A$  is cohomologous to cocycle of the form

$(w,t) \rightarrow e^{tK}$  for some constant matrix  $K$ ). Again the real parts of eigenvalues of  $K$  determine the qualitative behaviour completely. However when  $(\Omega, T_t)$  is almost periodic (but not periodic) the overall qualitative behaviour becomes quite complex. We will use certain skew-product flows to formulate and study such qualitative behaviour.

Consider  $\Sigma = p^{n-1} \times \Omega$  and the skew-product flow on  $\Sigma$  defined by cocycle  $X_A$  i.e.  $([x], w) \cdot t = ([X_A(w,t)x], w \cdot t)$  where  $[x]$  denotes the ray containing  $x \in \mathbb{R}^n$  system (I) will be called recurrent if the skew product flow on  $\Sigma$  is minimal. System (I) is said to be proximal if the projection  $\pi: \Sigma \rightarrow \Omega$  ( $\pi([x], w) = w$ ) is a proximal extension, (i.e. given  $([x_1], w_0), ([x_2], w_0) \in \Sigma$  there exists a sequence  $t_n \in \mathbb{R}$  such that,

$$d([x_1], w_0) \cdot t_n, ([x_2], w_0) \cdot t_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Qualitatively this means that for any  $w \in \Omega$ , the angle between any two solutions of (I) tends to 0 for some sequence  $t_n$  of times. System (I) will be called uniquely ergodic if the skew product flow on  $\Sigma$  has only one invariant Borel probability measure.

If  $A(w) \equiv A$  - is a constant matrix then system (I) can not be recurrent and proximal. This is because recurrence requires that eigenvalues of  $A$  be purely imaginary (real eigenvectors would give rise to fixed points for the skew product flow), but then in this case the flow on  $\Sigma$  is "rigid" i.e. preserves distances between points, thus can not be proximal. This shows (by Floquet-theorem) that if the flow on  $\Omega$  is periodic, system (I) can not be recurrent and proximal. As a consequence of the techniques developed in the previous section, we will show that when the flow  $(\Omega, \mathbb{R})$  is almost periodic (but not periodic) and admits "fast periodic approximation", recurrent-proximal behaviour of the linear systems based on  $\Omega$  is generic.

### SECTION III.2. The Spectrum:

The (Sacker-Sell) spectrum of system (I) is defined to be the set  $\sigma(A) = \{\lambda \mid \lambda \in \mathbb{R} \text{ such that for some } w \in \Omega, \text{ the equation } \dot{x} = [A(w \cdot t) - \lambda I]x \text{ has a non trivial bounded solution.}\}$  The spectral theorem of Sacker and Sell says that  $\sigma(A) = \bigcup_{i=1}^k [a_i, b_i]$ , where  $k \leq n$  and the union is a disjoint union of  $k$ -intervals. If  $a_i = b_i$   $1 \leq i \leq k$  then system (I) is said to have a discrete spectrum else it is said to have a band spectrum. If the flow on  $\Omega$  is periodic then the spectrum  $\sigma(A)$ , for any  $A$  is discrete.

Given  $([x], w) \in \Sigma$ , the upper Lyapunov exponents at  $([x], w)$  are defined by,

$$\lambda_u^\pm([x], w) = \limsup_{t \rightarrow \pm\infty} \frac{1}{t} \log \|X_A(w, t)x\|$$

The lower Lyapunov exponents (denoted by  $\lambda_\ell^\pm([x], w)$ ) are defined similarly by taking  $\liminf$ . It is known that  $\lambda_{u(\ell)}^\pm([x], w) \in \sigma(A)$ , (see [9]). Lyapunov exponents can also be written down as ergodic averages (with respect to the skew product flow on  $\Sigma$ ) of a suitable function  $H$  on  $\Sigma$ . This function  $H$  is defined by,  $H([x], w) = \langle [A(w)x], [x] \rangle$ , where  $\langle \cdot \rangle$  is the standard inner product on  $\mathbb{R}^n$ . One can verify that,  $\lambda_u^\pm([x], w) = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t H \{([x], w) \cdot s\} ds$ .

We will call system (I) elliptic iff  $\int_\Sigma H dm = 0$  for all Borel probability measures  $m$  on  $\Sigma$  which are invariant under the skew product flow. In the case  $n = 2$ , if system (I) is not elliptic is is said to be hyperbolic, (see [9] for details.)

The following proposition is proved in [9] for  $n = 2$ , however it can be easily generalized for any  $n$ .

Proposition (III.2.1) (a) If system (I) is uniquely ergodic then it is elliptic.

(b) If System (I) is elliptic then  $\sigma(A) = \{ \int_{\Omega} \text{tr } A(w \cdot s) ds \}$ . Now we state an important result due to R. Johnson, (see [10]).

Theorem (III.2.2) Let  $n = 2$ . If system (I) is hyperbolic and proximal then it has a band spectrum.

### SECTION (III.3)

Now we state the main result and very briefly indicate the proof. Again the notion of fast periodic approximation is similar, the diffeomorphism

$T_n^q$  in definition (I.3.1) is replaced by  $T_n^q$  where  $(T_t)$  is the one parameter group of diffeomorphisms generated by the flow. Typical example of such a flow is again the flow on  $n$ -torus generated by irrational rotation by Liouville vectors. For simplicity we state the following theorem only for such systems.

parameter group of diffeomorphisms generated by the flow. Typical example of such a flow is again the flow on  $n$ -torus generated by irrational rotation by Liouville vectors. For simplicity we state the following theorem only for such systems.

Theorem (III.3.1) Let  $(\Omega, T_t)$  be the rotation flow on  $n$ -torus generated by Liouville vectors. Then the set  $\{X_A \in \overline{B}_{\infty}(\Omega, SL(n, \mathbb{R})) \mid \dot{x} = A(w \cdot t)x \text{ is recurrent and proximal}\}$  is residual in  $\overline{B}_{\infty}(\Omega, SL(n, \mathbb{R}))$ .

The techniques used are as before, we will only sketch the ideas involved in lifting proximality generically. Fix  $w_0 \in \Omega$ . Given open sets  $U, V \subseteq \mathbb{R}^n$

and  $\epsilon > 0$ , define  $P(U, V, \epsilon) = \{X_A \in \overline{B_\infty} \mid \text{diam} (H_t^{X_A}(U \times \{w_0\}) \cup H_t^{X_A}(V \times \{w_0\})) < \epsilon,$   
 for some  $t \in \mathbb{R}\}$ , where  $H_t^{X_A}$  is the homeomorphism given by  $H_t^{X_A}([x], w)$   
 $= ([X_A(w, t)x], w \cdot t)$ . If  $X_A \in P(U, V, 1/n)$ ,  $\forall n$  and  $\forall U, V$  belonging to a countable  
 base of  $\Sigma$ , then any pair of points in the fiber over  $w_0$  is proximal and since  
 $(\Omega, \mathbb{R})$  is minimal, this implies  $\dot{x} = A(w \cdot t)x$  is a proximal system. Thus as  
 before the main problem is to prove that  $P(U, V, \epsilon)$  is dense in  $\overline{B_\infty}$ . Using  
 the same arguments as in the proof of Theorem (I.3.2) we reduce this problem  
 to showing the  $1 \in \overline{P(U, V, \epsilon)}$  and this is equivalent to showing that given a  $g_0$   
 $\in \text{SL}(n, \mathbb{R})$  and  $\delta > 0$ ,  $\exists \psi \in C^\infty(\Omega, \text{SL}(n, \mathbb{R}))$  such that  $D_\infty(1^\psi, 1) < \delta$  and  
 $\psi(w_0) = g_0$ , (see [18] for details in the case of 2-torus). Construction of  
 such a function is essentially as before, in fact much easier, (see [18]).

In the same way, under the same assumptions one can show residuality of  
 uniquely ergodic systems in  $\overline{B_\infty}$ .

### Remarks (III.3.2)

Unfortunately the class  $\overline{B_r}$  of linear systems in which proximal systems  
 are generic is the class of elliptic systems. To get a generic class of  
 systems having band spectrum one needs to get proximal extensions in the class  
 of hyperbolic systems. This is a difficult task. In fact this problem is  
 somewhat similar to the problem of getting smooth minimal skew-products with  
 non zero exponents (i.e. positive entropy), (see [8] for details). To get non

zero exponents, one must look at the cohomology class of cocycles having non zero exponents and the simplest such cocycle is  $(w,t) \rightarrow \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix}$ ,  $a \neq 0$ , but then extending the previous technique to the closure of the cohomology class of this cocycle is not possible because of the highly noncommutative nature of  $SL(2, \mathbb{R})$ . (This means constructing a smooth function  $\psi$  with values in  $SL(2, \mathbb{R})$ , having some "desired properties" and furthermore satisfying the condition that  $\psi(w \cdot t) \begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix}$  be close to  $\begin{pmatrix} e^{at} & 0 \\ 0 & e^{-at} \end{pmatrix} \psi(w)$  in  $B^r$  norm is impossible). We have been able to get around this problem by considering (for  $n = 2$ ) the systems

$$\dot{x} = \left[ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} + p(w \cdot t)^{-1} \frac{d}{dt} p(w \cdot t) \right] x \quad (*)$$

where  $p: \Omega \rightarrow SO(2, \mathbb{R})$ , and  $\frac{d}{dt}$  is the derivative along the orbits of the flow on  $\Omega$ , and the flow on  $\Omega$  admits fast periodic approximations then in the class of systems given by the closure of systems of the form(\*) we can get a generic "band spectrum" behaviour. Due to space limitation and the non trivial modifications in the arguments for it's proof we will publish this result elsewhere.

Remark (III.3.3) In general the questions regarding proximality or band spectrum of linear systems  $\dot{x} = A(w \cdot t)x$  are related to the nature of the Mackey range and the recurrence and transience properties of  $X_A$ . However no concrete results of this nature are available. In this context one may also ask: under what conditions on the flow  $(\Omega, \mathbb{R})$ , a given smooth cocycle is cohomologous to a "constant cocycle"? i.e. given a  $\phi: \Omega \times \mathbb{R} \rightarrow SL(2, \mathbb{R})$  can one find a continuous function  $f: \Omega \rightarrow SL(2, \mathbb{R})$  and a constant matrix  $A$  such

that  $\phi(w,t) = f(w \cdot t) e^{tA} f(w)^{-1}$ . Even in the situation when  $\Omega = \mathbb{T}^2$  -the 2-torus and the flow on  $\Omega$  is the rotation flow generated by badly approximable irrational, the answer to this question is not easy, mainly because the desired analog of small divisor technique in this noncommutative situation do not seem to be simple. However if the cocycle  $\phi$  takes values in  $SO(2, \mathbb{R})$  or in the subgroup of upper triangular matrices then indeed above functional equation can be solved by essentially applying the Kolmogorov-Siegel theorem to each entry. For a general  $\phi$  however, this is impossible. An example of Millionschikov (see [16]) shows that for any given irrational rotation flow on  $\mathbb{T}^2$  one can always find a  $C^\infty$  cocycle into  $GL(2, \mathbb{R})$  which is not cohomologous to any constant cocycle. We feel that such examples for rotation flows generated by "badly approximable" irrationals if not impossible are of first category in the class of smooth cocycles. In passing we also mention that under the assumption of  $C^\infty$  rigidity with speed say  $a(n) = (1/n^n)$ , A. Katok has shown (see [11]) the existence of a cocycle not cohomologous to any constant cocycle.

Remark (III.3.4) We mentioned in above remark that one can always have a smooth hyperbolic, proximal cocycle on rotation flows generated by any irrational. Here we ask a similar question, but in the class of elliptic systems. Clearly our result shows that for flows admitting fast-periodic approximation smooth elliptic proximal cocycles are generic. Now for flows on  $\mathbb{T}^2$  generated by badly approximable irrationals, one can show that any smooth elliptic proximal cocycle taking values in the subgroup of upper triangular matrices is cohomologous to the cocycle  $\begin{pmatrix} e^t & bt \\ 0 & e^{-t} \end{pmatrix}$  and hence can not yield a minimal skew product. So a more appropriate question is: can one have

smooth recurrent - proximal elliptic cocycles based on such flows (i.e. flows generated by badly approximable irrationals.)

One can also ask these questions for linear systems of the form

$\dot{x} = \begin{pmatrix} 0 & 1 \\ q(w \cdot t) - \lambda & 0 \end{pmatrix} x$ . These systems arise from the 1-dimensional Schrödinger

equation  $\frac{d^2 y}{dt^2} + qy = \lambda y$  where  $q$  is an almost-periodic potential. Suppose  $q$  is

quasi-periodic with frequency module generated by irrationals  $(\alpha_1, \dots, \alpha_n)$  and

let  $(\alpha_1, \dots, \alpha_n)$  satisfy the "bad approximability" condition. Let  $\Omega = \mathbb{T}^n =$

hull of  $q$  and  $X_{q,\lambda}$  be the cocycle corresponding to this system. In [3],

Dinaburg and Sinai describe a set  $S \subseteq \mathbb{R}$  such that  $\lambda \in S \Rightarrow X_{q,\lambda}$  is elliptic.

However unfortunately these cocycles are not proximal. For linear systems arising from Schrödinger equation it is known (due to Pastur and Ishii) that the set of  $\lambda$ 's for which  $X_{q,\lambda}$  is elliptic is the support of the absolutely

continuous part of the spectral measure associated with the Schrödinger operator. Thus one may look at the known spectral results about Schrödinger

operators with almost periodic potentials to look for elliptic proximal cocycles supported on flows generated by badly approximable irrationals. We

end this remark by pointing out that in this respect results of P. Sarnak

([20]) needs to be studied more closely.

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