THE ENTROPY FORMULA FOR SRB-MEASURES OF LATTICE DYNAMICAL SYSTEMS

MIAOHUA JIANG

ABSTRACT. In this article we give the detailed proof of the entropy formula for SRB-measures of coupled hyperbolic map lattices. We show that the topological pressure for the potential function of the SRB-measure is zero.

1. INTRODUCTION

We proved in [2] that the thermodynamic limit of Sinai-Ruelle-Bowen measures for coupled hyperbolic maps over finite volumes of an integer lattice exists as the volume tends to infinity. The limiting measure, also called SRB-measure, is an equilibrium state satisfying the variational principle of statistical mechanics for a Hölder continuous function φ. The measure is invariant and exponentially mixing with respect to both temporal and spatial translations. The formula for computing the potential function φ is explicitly given. In this note, we give the detailed proof of a result in [2] that the topological pressure for this potential function is zero with respect to the group actions induced by both spatial and temporal translations. Thus, the entropy formula holds for the SRB-measure for the coupled hyperbolic map lattice. This result further justifies the name of the measure since topological pressure being zero is one of the characteristics of SRB-measure on hyperbolic attractors of finite dimension.

The proof is a straightforward computation using the definition of topological pressure for continuous functions on a compact metric space with respect to a $\mathbb{Z}^d$-action induced by d interchangeable homeomorphisms. First, we briefly describe the infinite-dimensional system:

Date: June, 1998.

1991 *Mathematics Subject Classification.* 58F15.

Key words and phrases. entropy formula, SRB measure, coupled map lattice, topological pressure .

weakly coupled identical systems with a uniformly hyperbolic attractor and the results concerning its SRB-measure. We then choose an appropriate cover of the space for computing the topological pressure. The most natural choice is the cover provided by the Markov partition. Finally, we use the properties of the potential function φ to show that the topological pressure is zero.

2. SRB-measures for coupled map lattices

Let M be a smooth compact Riemannian manifold and f a C^r-map of M, $r \geq 1$. We may assume that f is topologically mixing on the hyperbolic attractor Λ.

The direct product of identical copies of M over a d-dimensional integer lattice $\mathcal{M} = \otimes_{i\in\mathbb{Z}^d} M_i$ is an infinite-dimensional Banach manifold with the Finsler metric induced by the Riemannian metric on M.

The distance on $\mathcal{M}$ induced by the Finsler metric is

$$\rho(\bar{x}, \bar{y}) = \sup_{i\in\mathbb{Z}^d} d(x_i, y_i),$$

where $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$ are two points in $\mathcal{M}$ and d is the Riemannian distance on M.

The *direct product* map on $\mathcal{M}$ defined by $F = \otimes_{i\in\mathbb{Z}^d} f_i$ possesses an infinite-dimensional hyperbolic attractor $\Delta_F = \otimes_{i\in\mathbb{Z}^d}\Lambda_i$, where f_i and Λ_i are copies of f and Λ, respectively.

We recall the definitions of some objects discussed in [2].

Let S denote the spatial translation actions on $\mathcal{M}$ induced by the translations on the integer lattice $\mathbb{Z}^d$. Let the map G be a perturbation, at least C^2, of the identity map on $\mathcal{M}$. G is said to be spatially translation invariant if $G \circ S = S \circ G$. It is said to have *short range* property: if G is of the form $G = (G_i)_{i\in\mathbb{Z}^d}$, where $G_i : \mathcal{M} \to M_i$; for any fixed $k \in \mathbb{Z}^d$ and any points $\bar{x} = (x_j), \bar{y} = (y_j) \in \mathcal{M}$ with $x_j = y_j$ for all $j \in \mathbb{Z}^d, j \neq k$ we have

$$d(G_i(\bar{x}), G_i(\bar{y})) \leq C\theta^{|i-k|} d(x_k, y_k),$$

where $0 < \theta < 1$.

Define $\Phi = F \circ G$. The pair of actions (Φ, S) on $\mathcal{M}$ is called a *coupled map lattice*. If $G = id$, the lattice is called *uncoupled* .

The metric $\rho_q, 0 < q < 1$, is a family of metrics compatible with the Tychonov compact topology on $\mathcal{M}$, i.e., the direct product topology:

$$\rho_q(\bar{x}, \bar{y}) = \sup_{i \in \mathbb{Z}^d} q^{|i|} d(x_i, y_i)$$

where $|i| = |i_1| + |i_2| + \cdots + |i_d|$, $i = (i_1, i_2, \cdots, i_d) \in \mathbb{Z}^d$.

Fix a point $\bar{x}^* \in \Delta_\Phi$, and a finite volume $V \subset \mathbb{Z}^d$, the map Φ_V on $M_V = \otimes i \in V M_i$ defined by

$$\big(\Phi_V(x)\big)_i = \big(\Phi((x, x^*|_{\widehat{V}})\big)_i,$$

where $()_i$ denotes the coordinate at the lattice site i. It is a diffeomorphism of M_V when the perturbation is sufficiently small.

Since the diffeomorphism Φ_V is C^1-closed to the diffeomorphism F_V, by the structural stability theorem it possesses a hyperbolic attractor $\Delta_{\Phi,V}$. There exists a conjugating homeomorphism $h_V : \Delta_{F,V} \to \Delta_{\Phi,V}$, $\Phi_V \circ h_V = h_V \circ F_V$.

The maps Φ_V and h_V provide finite-dimensional approximations for the infinite-dimensional maps Φ and h, respectively.

We state the main results in [2] on the existence of SRB-measures for Φ and the properties of this measure.

(1) For any $\epsilon > 0$ there exists $0 < \delta < \delta_0$ such that, if $\text{dist}_{C^1}(\Phi, F) \leq \delta$, then there is a unique homeomorphism $h : \Delta_F \to \mathcal{M}$ satisfying $\Phi \circ h = h \circ F|_{\Delta_F}$ with $\text{dist}_{C^0}(h, id) \leq \epsilon$. In particular, the set $\Delta_\Phi = h(\Delta_F)$ is a topologically mixing hyperbolic attractor. The conjugating map h is spatial translation invariant whenever G is.

(2) For any $0 < \theta < 1$ there exists $\delta > 0$ such that if G is a C^2-spatial translation invariant short range map with a decay constant θ and $\text{dist}_{C^1}(G, id) \leq \delta$, then the conjugacy map h is Hölder continuous with respect to the metric ρ_q, $0 < q < 1$. Moreover,
$h = (h_i(\bar{x}))_{i \in \mathbb{Z}^d}$ satisfies the following property:

$$d(h_0(\bar{x}), h_0(\bar{y})) \leq C(\delta) d^\alpha(x_k, y_k)$$

for every $k \neq 0$ and any $\bar{x}, \bar{y} \in \mathcal{M}$ with $x_i = y_i, i \in \mathbb{Z}^d, i \neq k$, where $0 < \alpha < 1$ and $C(\delta) > 0$ is a constant. Furthermore, $C(\delta) \to 0$ as $\text{dist}_{C^1}(G, id) \to 0$.

(3) Let μ_V be the SRB-measure on the hyperbolic attractor Δ_{Φ_V} for the map Φ_V. Then, μ_V weakly converges to a measure μ on Δ_Φ. The measure μ is invariant and exponentially mixing under Φ and spatial translations S. It also satisfies the variational principle:

$$P_\tau(\varphi) = h_\mu(\tau) + \int \varphi d\mu,$$

where τ denotes the $\mathbb{Z}^{d+1}$ action on Δ_Φ induced by Φ and S, $P_\tau(\varphi)$ is the topological pressure for the potential function φ, and $h_\mu(\tau)$ is the measure theoretical entropy of μ with respect to τ.

(4) The construction of the potential function φ can be described in the following way. By assumptions that $\Phi = F \circ G$ are C^1 close to F and the interaction G has short range property, under an appropriately chosen local coordinate system, the restriction of the derivative operator of Φ_V to the unstable space at point $h_V(x)$ has the following matrix representation:

$$D\Phi|_{E^u_{\Phi_V}(h_V(x_V))} = (D^u f(x_i))(I + A_V(x_V)),$$

where $A_V(x_V) = (\mathbf{a}_{ij}(x_V))$ is a $|V| \times |V|$ matrix with submatrices $\mathbf{a}_{ij}(x_V)$ as entries, $(D^u f(x_i))$ is a diagonal matrix with $D^u f(x_i)$, $i \in V$ (the matrix representation of Df restricted to the unstable space) on its main diagonal, and $|V|$ is the cardinality of V. The norms of submatrices $\mathbf{a}_{ij}(x_V)$ are small and go to zero exponentially fast as $|i-j| \to \infty$. The entries $\mathbf{a}_{ij}(x_V)$ are also Hölder continuous with respect to the metric ρ_q. The determinant of $(I + A_V)$ is then calculated in the following way.

$$\det(I + A_V) = \exp(\text{trace}(\ln(I + A_V)) = \exp(-\sum_{i \in V} w_{Vi}),$$

where

$$w_{Vi}(x_V) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{trace}(\mathbf{a}^n_{ii}(x_V))$$

and $\mathbf{a}_{ii}^n(x_V)$ are submatrices on the main diagonal of $(A_V)^n$.

The functions $w_{Vi}(x_V)$ has the following properties. There exist constants $\epsilon_0 > 0, \beta > 0$ such that

$$|w_{Vi}(x_V) - w_{V'i}(y_{V'})| \leq \epsilon_0 e^{-d(i,V)}, \tag{1}$$

where $V \subset V'$, $x_V = y_{V'}|_V$, and $d(i,V)$ denotes the distance from the lattice site i to the boundary of V. The estimation (1) implies that the limit $\varphi_i(\bar{x}) = \lim_{V\to\mathbb{Z}^d} w_{Vi}(x_V)$ exists for each $i \in \mathbb{Z}^d$. This limit is also translation invariant in the following sense. Let $\psi(\bar{x}) = \lim_{V\to\mathbb{Z}^d} w_{V0}(x_V)$. Then, $\varphi_i(\bar{x}) = \psi(\sigma_s^i\bar{x})$. Moreover, $\psi(\bar{x})$ is Hölder continuous with respect to the metric ρ_q with a Hölder constant going to zero as the C^1-distance between Φ and F tends to zero.

The potential function φ for the SRB-measure for the coupled map lattice (Φ, S) composed with the conjugating map h is

$$\varphi(h(\bar{x})) = -\log J^u f(x_0) + \psi(\bar{x}). \tag{2}$$

This expression is slightly different from that in [2] since we have a hyperbolic attractor instead of an Anosov system.

3. Computing Topological Pressure

In this section we prove that the topological pressure of the potential function φ with respect to the $\mathbb{Z}^{d+1}$-action induced by the coupled map lattice (Φ, S) is zero on the hyperbolic attractor $\Delta_\Phi = h(\Delta_F)$. We first recall the definition of the topological pressure. It is directly taken from [5].

Let Ω be a compact metric space and τ a $\mathbb{Z}^{d+1}$-action on Ω induced by $d+1$ commuting homeomorphisms, $d \geq 0$. For two covers of Ω $\mathcal{U} = \{U_i\}$ and $\mathcal{B} = \{B_i\}$, $\mathcal{U} \vee \mathcal{B}$ denotes the cover of Ω consisting of all sets of the form $B_i \cap U_j$. For a finite volume $V \subset \mathbb{Z}^{d+1}$ define

$$\mathcal{U}^V = \vee_{i\in V}\tau^{-i}\mathcal{U}.$$

Let $\mathcal{U}$ be any cover of Ω, φ a continuous function on Ω, and V a finite subset of $\mathbb{Z}^{d+1}$. The partition function is defined by

$$Z_V(\varphi,\mathcal{U}) = \min_{\{B_j\}} \Big\{ \sum_j \exp\Big[\sup_{x\in B_j} \sum_{i\in V} \varphi(\tau^i x)\Big]\Big\}, \tag{3}$$

where the minimum is taken over all subcovers $\{B_j\}$ of $\mathcal{U}^V$. Because of the subadditivity of the partition function, the following limit exists and is call the topological pressure with respect to the cover $\mathcal{U}$.

$$P_\tau(\varphi,\mathcal{U}) = \lim_{a_1,\ldots,a_{d+1}\to\infty} \frac{1}{|V(a)|} \log Z_{V(a)}(\varphi,\mathcal{U}),$$

where $V(a)$ denotes the rectangular volume $[-a_1,a_1]\times\cdots\times[-a_{d+1},a_{d+1}] \subset \mathbb{Z}^{d+1}$. When $\mathcal{U}$ is an open cover, the quantity

$$P_\tau(\varphi) = \lim_{\operatorname{diam}\mathcal{U}\to 0} P_\tau(\varphi,\mathcal{U}) = \sup_{\mathcal{U}} P_\tau(\varphi,\mathcal{U})$$

is called the *topological pressure* of φ with respect to τ.

It is easy to see that for a fixed volume V_0

$$P_\tau(\varphi,\mathcal{U}) = P_\tau(\varphi,\mathcal{U}^{V_0}).$$

When τ is expansive and $\operatorname{diam}\mathcal{U}$ is smaller than the expansive constant, we have $P_\tau(\varphi) = P_\tau(\varphi,\mathcal{U})$.

Before we proceed to the actual computation, we first state the strategy:

First of all, we project every object onto the hyperbolic attractor Δ_F for F since $P_\tau(\varphi) = P_\tau(\varphi(h(\bar{x}))$. It is much easier to compute $P_\tau(\varphi(h(\bar{x}))$ since the $\mathbb{Z}^{d+1}$-action τ induced by (F,S) is now acting on the direct product space $\Delta_F = \otimes_{i\in\mathbb{Z}^d}\Lambda_i$.

The SRB-measure μ on Δ_Φ for the coupled map lattice (Φ,S) is the unique equilibrium measure μ_φ satisfying the variational principle. This is equivalent to saying that the measure $h^*(\mu)$ is an equilibrium state for function $\varphi\circ h$ on Δ_F.

We then show that it is not necessary to choose an open cover for Δ_F to compute the topological pressure. Instead we can choose the cover provided by the Markov partition. This transition makes it possible to compute the topological pressure.

When we actually compute the partition function for the potential function $\varphi(h(\bar{x}))$ with respect to the $\mathbb{Z}^{d+1}$-action τ, we compare it with the partition function of the potential function $-\log J^u\Phi_V(h_V(x_V))$ for the SRB-meassure of Φ_V projected onto the hyperbolic attractor $\Delta_{F,V}$. Note that this partition function is computed with respect to the $\mathbb{Z}$-action generated by F_V. Using the fact that the pressure is zero for $-\log J^u\Phi_V(h_V(x_V))$. We prove that the pressure for φ is also zero.

3.1. **Markov Partition.** Since we assume that f is topologically mixing on the hyperbolic attractor Λ, for any $\epsilon > 0$, there exists a Markov partition of Λ into *proper rectangles*

$$\Lambda = \cup_{i=1}^{p} R_i,$$

where $\{R_i\}$ satisfy the following properties [1] [4]:

(1) $\operatorname{diam}\mathcal{R} = \max_i diam(R_i) < \epsilon$;
(2) for each i, R_i is proper: $\overline{\operatorname{int} R_i} = R_i$;
(3) for any two points $x, y \in R_i$, there is a unique point in the intersection of the local stable manifold at x and the local unstable manifold at y; this point denoted by $[x, y]$ is also in R_i: $[x, y] = W^s_\epsilon(x) \cap W^u_\epsilon(y) \in R_i$, i.e. R_i is a rectangle;
(4) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ when $i \neq j$.

We assume that ϵ is sufficiently small so that the map f is well approximated by its derivative in the neighborhood containing each R_i. For each $x \in R_i$, we can also assume that the intersection $W^s_\epsilon(x) \cap R_i$ (denoted by $W^s(x, R_i)$) is proper and its boundary set $\partial W^s(x, R_i)$ is in ∂R_i, the boundary set of R_i. Similarly, one has $W^u(x, R_i)$. Note that $W^u(x, R_i)$ is a submanifold with boundary, but $W^s(x, R_i)$, in general, is not a submanifold. We use relative topologies on both objects.

(5) $f(W^s(x, R_i)) \subset W^s(f(x), R_j), W^u(x, R_i) \subset f^{-1}(W^u(f(x), R_j))$, when $x \in \operatorname{int} R_i$ and $f(x) \in \operatorname{int} R_j$.

One can show that for each R_i and any $x_0 \in R_i$,

$$R_i = \{z = [x, y], y \in W^u(x_0, R_i), x \in W^s(x_0, R_i)\}.$$

On the other hand, for any $x_0 \in \Lambda$ and any $\epsilon > 0$ sufficiently small, the set $\{z = [x, y], y \in W^u_\epsilon(x_0), x \in W^s(x_0) \cap \Lambda\}$ is a proper rectangle.

Next, we show that the cover $\mathcal{R} = \{R_i\}$ can be used to compute the topological pressure.

Lemma 1. *For any continuous function φ on Λ, there exist $\epsilon > 0$ and a Markov partition of Λ, $\{R_i\}$ with* $\operatorname{diam} \mathcal{R} < \epsilon$ *such that*

$$P_f(\varphi) = P_f(\varphi, \mathcal{R}).$$

Proof. We first look at the partition function computed with respect to the cover $\{R_i\}$ over the interval $[-n, n]$. We denote the cover $\mathcal{R}^{[-n,n]}$ by $\mathcal{R}^n$.

$$Z_n(\varphi, \mathcal{R}) = \min_{\{B_j\}} \Big(\sum_j \exp \sup_{x \in B_j} \sum_{i=-n}^{n} \varphi(f^i(x)) \Big),$$

where $\{B_j\}$ is a subcover from the cover $\mathcal{R}^n$, i.e., the collection of sets in the form of

$$f^{-n}(R_{i_{-n}}) \cap \cdots \cap f^{-1}(R_{i_{-1}}) \cap R_{i_0} \cap f(R_{i_1}) \cap \cdots \cap f^n(R_{i_n})$$

and the minimum is taken over all such subcovers.

One observes that the minimum is attained at the subcover denoted by $\mathcal{R}^n_o = \{B_j\}$ where each B_j has a non-empty interior since the interiors of R_i are disjoint. In fact, $\{B_j\}$ provides another Markov partition of proper rectangles with a smaller diameter. Therefore, we can use the following formula for the topological pressure relative to the cover $\mathcal{R}$.

$$Z_n(\varphi, \mathcal{R}) = \sum_j \exp \sup_{x \in B_j} \sum_{i=-n}^{n} \varphi(f^i(x)),$$

where $\{B_j\} = \mathcal{R}^n_o$.

Now, for each R_i, we can extend it a little to obtain an open rectangle Q_i so that Q_i is in the δ open neighborhood of R_i: $Q_i \subset O_\delta(R_i)$. This family of open sets forms an open cover of Λ: $\mathcal{Q} = \{Q_i\}$. Similarly, we define $\mathcal{Q}^n$. When both ϵ and δ are chosen small, we have that $\operatorname{diam} \mathcal{Q}$ is smaller than the expansive constant. Since f is a diffeomorphism, we can choose δ small enough such that every subcover of $\mathcal{Q}^n$ contains elements of the subcover $\mathcal{Q}_o^n$, where $\mathcal{Q}_o^n$ consists of open sets in the form of

$$f^{-n}(Q_{i_{-n}}) \cap \cdots \cap f^{-1}(Q_{i_{-1}}) \cap Q_{i_0} \cap f(Q_{i_1}) \cap \cdots \cap f^n(Q_{i_n})$$

with the corresponding set

$$f^{-n}(R_{i_{-n}}) \cap \cdots \cap f^{-1}(R_{i_{-1}}) \cap R_{i_0} \cap f(R_{i_1}) \cap \cdots \cap f^n(R_{i_n}) \in \mathcal{R}_o^n$$

This means that $\mathcal{Q}_o^n$ is the minimal subcover of $\mathcal{Q}^n$. Therefore, according to the definition, we have

$$P_f(\varphi) = P_f(\varphi, \mathcal{Q}) = \lim_{n\to\infty} \frac{1}{2n} \log Z_n(\varphi, \mathcal{Q})$$

and

$$Z_n(\varphi, \mathcal{Q}) = \sum_j \exp \sup_{x\in B_j} \sum_{i=-n}^{n} \varphi(f^i(x)),$$

where $\{B_j\} = \mathcal{Q}_o^n$.

Next, we compare $P_f(\varphi, \mathcal{R})$ and $P_f(\varphi, \mathcal{Q})$. We need only to compare the corresponding partition functions. Let $c(\delta) = \max\{|\varphi(x) - \varphi(y)|, x, y \in \Lambda, d(x,y) \leq \delta\}$. Since Λ is compact, $c(\delta)$ can be made arbitrarily small as long as δ is chosen small. Thus, we have

$$e^{-(2n+1)c(\delta)} Z_n(\varphi, \mathcal{Q}) \leq Z_n(\varphi, \mathcal{R}) \leq e^{(2n+1)c(\delta)} Z_n(\varphi, \mathcal{Q}). \tag{4}$$

Taking limit $n \to \infty$, we have

$$|P_f(\varphi, \mathcal{Q}) - P_f(\varphi, \mathcal{R})| \leq c(\delta),$$

which implies $P_f(\varphi) = P_f(\varphi, \mathcal{R})$. □

Remark. By the same type of inequality as (4), we can see that we can use the following expression as the definition of a partition function. The resulting topological pressure will be the same:

$$Z_n^*(\varphi, \mathcal{R}) = \sum_j \exp \operatorname*{eval}_{x\in B_j} \sum_{i=-n}^{n} \varphi(f^i(x)),$$

$$P_f(\varphi, \mathcal{R}) = \lim_{n\to\infty} \frac{1}{2n} \log Z_n^*(\varphi, \mathcal{R}),$$

where $\operatorname*{eval}_{x\in B_j}$ means evaluating the function at an arbitrary point in B_j. This observation will be used later in the proof of the main result.

The same arguments can be apply to the hyperbolic attractors $(\otimes_{i\in V}\Lambda, F_V)$ and $(\Delta, (F, S))$. Let $\mathcal{R}_V$ denote the Markov partition of $\otimes_{i\in V}\Lambda$ provided by the direct product of $\mathcal{R}$ over V. The Markov partition of Δ_F: $\mathcal{R} = \{\widetilde{R}_i\}$ is now understood in the following sense:

$$\widetilde{R}_i = \{\bar{x} \in \Delta_F, x_0 \in R_i\}.$$

Lemma 2.

$$P_{F_V}(\varphi) = P_{F_V}(\varphi, \mathcal{R}_V). \qquad P_\tau(\varphi) = P_\tau(\varphi, \mathcal{R}).$$

3.2. **Zero topological pressure.** In this section, we show that $P_\tau(\varphi, \mathcal{R}) = 0$ for the potential of the SRB-measure of coupled map lattice.

Theorem 1. *Let φ be the potential function for the SRB-measure μ defined in (2). Then,*

$$P_\tau(\varphi(h), \mathcal{R}) = 0.$$

Moreover, the entropy formula holds.

$$h_\tau(\mu) = -\int \varphi d\mu.$$

Proof. Since $\otimes_{i\in V}\Lambda$ is obviously a hyperbolic attractor for F_V, we have $P_{F_V}(\varphi, \mathcal{R}_V) = 0$, where $\varphi = -\log J^u\Phi_V(h_V)$. Using this fact, we will show that $P_\tau(\varphi, \mathcal{R}) = 0$ for the potential function corresponding to the SRB-measure for the coupled map lattice.

For simplicity, we shall assume $d = 1$ and V is an interval $[-m, m]$. We denote Φ_V by Φ_m, etc.

Let m be fixed. Let a_{nm} denote the following expression.

$$\frac{1}{2n}\log\sum_j \exp[\sup_{x\in B_j}\sum_{i=-n}^{n} -\log J^u\Phi_m(h_m(F_m^i(x)))],$$

where $\{B_j\}$ is the subcover from $\mathcal{R}_m^n$ such that it is the minimal. Then, $\lim_{n\to\infty} a_{nm} = 0$.

Now we compute the pressure with respect to the cover $\mathcal{R}$ over the volume $V_{nm} = [-m, m] \times [-n, n]$.

We let b_{nm} denote the expression

$$\frac{1}{4nm} \log \sum_j \exp[\sup_{\bar{x} \in B_j} \sum_{i,k=-n,-m}^{nm} \varphi(h_m(F_m^i S^k(\bar{x})))].$$

Then, $P_\tau(\varphi) = \lim_{n,m\to\infty} b_{nm}$.

Since $\lim_{n\to\infty} a_{nm} = 0$ for each m, we can find a sequence $\{n(m)\}$ such that

$$\lim_{m\to\infty} \frac{1}{2m} a_{n(m)m} = 0.$$

Note that $V_{nm} = [-m, m] \times [-n(m), n(m)] \to \mathbb{Z}^2$ in the sense of van Hove. We need only to show that

$$\lim_{m\to\infty} b_{nm} - \frac{1}{2m} a_{n(m)m} = 0.$$

We now use the decomposition of the Jacobian

$$J^u\Phi_m(h_m(x)) = \prod_{k=-m}^{m} J^u f_k(x_k) \exp(-\sum_{k=-m}^{m} w_{mk}(x)),$$

and the definition of the function φ:

$$\varphi(h(\bar{x})) = \lim_{m\to\infty} w_{m0}(x) - \log J^u f(x_0).$$

$$b_{nm} - \frac{1}{2m} a_{nm} =$$

$$\text{(5)} \qquad \frac{1}{4nm} \log \frac{\sum_j \exp[\sup_{\bar{x}\in B_j} \sum_{i,k=-n,-m}^{nm} \varphi(h(F^i S^k(\bar{x})))]}{\sum_j \exp[\sup_{y\in B_j} \sum_{i=-n}^{n} - \log J^u\Phi_m(h_m(F_m^i(y)))]}.$$

Since there are same number of terms in the numerator and the denominator in the logarithm in (5), we simply need to estimate the following expression.

$$w_{km}(F^i y) - \psi(F^i S^k \bar{x}).$$

From the Remark, we can, in fact, choose $y \in B_j$ in such a way that y is the restriction of $\bar{x}$ on to the volume $[-m, m]$, i.e., $y = \bar{x}|_{[-m,m]}$.

When we plug in the formulas for $J^u\Phi_m$ and $\varphi(h)$, the terms containing $J^u f_i$ are canceled out. Thus, by the estimation (1) we have

$$|w_{km}(F^i y) - \psi(F^i S^k \bar{x})| \leq \epsilon_0 e^{-\beta d(k,m)},$$

where $\beta > 0$ and $d(k,m) = \min\{m-k, k+m\}$.

Thus, we have

$$|\sum_{i,k=-n,-m}^{nm} (w_{km}(F^i y) - \psi(F^i S^k \bar{x}))| \leq (2n+1) \sum_{k=-m}^{m} \epsilon_0 e^{-\beta d(k,m)} \leq C(2n+1),$$

where $C > 0$ is a constant.

Therefore, we have

$$|b_{nm} - \frac{1}{2m} a_{nm}| \leq \frac{1}{4nm} |\log(e^{C(2n+1)})| = \frac{C}{2m} \cdot \frac{2n(m)+1}{2n(m)},$$

which goes to zero as $m \to \infty$.

□

4. Acknowledgment

The author thanks Y. B. Pesin and G. Keller for encouragements and helpful suggestions.

References

[1] R. Bowen 1975 Equilibrium State and the Ergodic Theory of Anosov Diffeomorphisms *Lecture Notes in Mathematics* No. 470 Springer-Verlag Berlin

[2] M. Jiang and Y. B. Pesin 1996 Equilibrium Measures for Coupled Map Lattices: Existence, Uniqueness, and Finite-Dimensional Approximation Preprint.

[3] A. Katok and B. Hasselblatt 1994 Introduction to the Modern Theory of Dynamical Systems Cambridge University Press.

[4] Ricardo Mañé 1987 Ergodic Theory and Differential Dynamics Springer-Verlag New York

[5] D. Ruelle 1978 Thermodynamic Formalism. *Encyclopedia of Mathematics and Its Applications* No.5 Addison Wesley New York

[6] I. G. Sinai 1994 Topics in ergodic theory Princeton University Press Princeton, N.J.

Institute for Mathematics and its Applications,, University of Minnesota, Minneapolis, MN 55455

E-mail address: `jiang@ima.umn.edu`