UNIVERSALITY OF THE LOCAL EIGENVALUE STATISTICS FOR A CLASS OF UNITARY INVARIANT RANDOM MATRIX ENSEMBLES

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Abstract

The paper is devoted to the rigorous proof of the universality conjecture of the random matrix theory, according to which the limiting eigenvalue statistics of $n \times n$ random matrices within spectral intervals of the order $O(n^{-1})$ is determined by the type of matrices (real symmetric, Hermitian or quaternion real) and by the density of states. We prove this conjecture for a certain class of the Hermitian matrix ensembles that arose in the quantum field theory and have the unitary invariant distribution defined by a certain function (the potential in the quantum field theory) satisfying some regularity conditions.

Key words: random matrices, local asymptotic regime, universality conjecture, orthogonal polynomial technique.

1 Introduction. Problem and results.

The random matrix theory (RMT) has been extensively developed and used in a number of areas of theoretical and mathematical physics. In particular the theory provides quite satisfactory description of fluctuations in spectra of complex quantum systems such as heavy nuclei, small metallic particles, classically chaotic quantum models, etc. One of the important ingredients of this description is the universality conjecture of the RMT according to which the local eigenvalue statistics on $n \times n$ random matrices (probabilistic properties of their spectra within intervals of the order of $1/n$) do not depend on a particular ensemble in the limit $n = \infty$ and is completely determined by the invariance group of the ensemble probability distribution. There are three invariance groups (orthogonal, unitary and simplectic) and three respective classes of the random matrix ensembles that model quantum systems possessing respective invariance under the time reflection and the space rotations. The explicit form of the local eigenvalue statistics in the limit $n = \infty$ for each of these classes was found in sixties by Wigner,

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Mehta, Dyson and others who introduced and studied the explicitly solvable Gaussian and circular ensembles (see Ref.1 and references therein).

In this paper we consider the technically simplest case of the unitary invariant ensembles. Moreover we will study the class defined by the density

$$p_n(M)dM = Z_n^{-1}\exp\{-n\mathrm{Tr}V(M)\}dM \tag{1.1}$$

where M is a $n \times n$ Hermitian matrix,

$$dM = \prod_{j=1}^{n} dM_{jj} \prod_{j<k} d\Im M_{jk} d\Re M_{jk}$$

is the "Lebesgue" measure for Hermitian matrices, Z_n is the normalization factor and $V(\lambda)$ is a real valued function (see the Theorem below for explicit conditions).

The case $V(\lambda) = \frac{\lambda^2}{2}$ corresponds to the Gaussian unitary ensemble (GUE) which was introduced by Wigner in fifties. Ensembles with an arbitrary $V(\lambda)$ were introduced in sixties,[2-4] when some particular cases were studied . The new wave of interest to this class of unitary invariant ensembles was caused by quantum field theory, where they arise in large-n limit of quantum chromodynamics, 2-dimensional quantum gravity and bosonic string theory (see the review papers 5, 6). Analogous ensembles are used in condensed matter theory and statistical mechanics of random surfaces.[7,8]

Denote by $p_n(\lambda_1, ..., \lambda_n)$ the joint probability density of all eigenvalues which we assume to be symmetric without loss of generality. Let

$$p_l^{(n)}(\lambda_1, ..., \lambda_l) = \int p_n(\lambda_1, ..., \lambda_l, \lambda_{l+1}, ...\lambda_n)d\lambda_{l+1}...d\lambda_n \tag{1.2}$$

be its l-th marginal distribution density. The simplest case of $p_1^{(n)}(\lambda_1)$ is of particular interest. Indeed, if $\lambda_1^{(n)}, ..., \lambda_n^{(n)}$ are eigenvalues of a random Hermitian matrix M then

$$N_n(\Delta) = \frac{1}{n}\sum_{\lambda_i^{(n)}\in\Delta} 1, \qquad \Delta = (a, b) \tag{1.3}$$

is their normalized counting function (empirical eigenvalue distribution). Then

$$E\{N_n(\Delta)\} = \int_\Delta p_1^{(n)}(\lambda)d\lambda \equiv \int_\Delta \rho_n(\lambda)d\lambda \tag{1.4}$$

where $E\{...\}$ denotes the expectation with respect to density (1.1).

In the recent paper[9] it was proved that if $V(\lambda)$ is bounded below for all $\lambda \in R$ and satisfies the conditions

$$|V(\lambda)| \geq (2+\epsilon)\log|\lambda|, \quad |\lambda| \geq L_1 \tag{1.5a}$$

for some L_1 and

$$|V(\lambda_1) - V(\lambda_2)| \leq C(L_2)|\lambda_1 - \lambda_2|^{\gamma}, \quad |\lambda_{1,2}| \leq L_2 \tag{1.5b}$$

for any $0 < L_2 < \infty$ and some $\gamma > 0$, then $\rho_n(\lambda)$ converges to the limiting density $\rho(\lambda)$ (density of states) in the Hilbert space defined by the norm

$$\left(-\int \log|\lambda - \mu|\rho(\lambda)\rho(\mu)d\lambda d\mu\right)^{1/2} \tag{1.6}$$

and $\rho(\lambda)$ can be found from the certain variational procedure analogous to that known in the mean field theory of statistical mechanics. Moreover, there exist positive numbers L, A, and a such that

$$\rho_n(\lambda) \leq Ae^{-na(|\lambda|-L)}, \quad |\lambda| > L \tag{1.7}$$

and for any differentiable on $(-L, L)$ function $\phi(\mu)$ which grows not faster than $Be^{b\mu}$, $B, b > 0$ as $|\mu| \to \infty$

$$\left|\int \phi(\mu)\rho_n(\mu)d\mu - \int \phi(\mu)\rho(\mu)d\mu\right| \leq C||\phi'||_2^{1/2}||\phi||_2^{1/2}n^{-1/2}\log^{1/2} n. \tag{1.8}$$

where symbol $||...||_2$ denotes the L_2-norm on the $(-L, L)$.

Here and below the symbols C and C_i denote n-independ positive constants that may be different in different formulae.

Now we formulate the universality conjecture following Dyson.[4]

Universality conjecture. *For any n-independent integer l, λ_0 such that $\rho(\lambda_0) \neq 0$ and arbitrary fixed $(x_1, ...x_l) \in R^l$*

$$\lim_{n\to\infty}[n\rho_n(\lambda_0)]^{-l}p_l^{(n)}(\lambda_0 + \frac{x_1}{n\rho_n(\lambda_0)}, ..., \lambda_0 + \frac{x_l}{n\rho_n(\lambda_0)}) = \det ||S(x_1 - x_k)||_{j,k=1}^{l} \tag{1.9}$$

where

$$S(x) = \frac{\sin \pi x}{\pi x}. \tag{1.10}$$

In other words, the limit in the r.h.s. of (1.9) is the same for all $V(\lambda)$'s in (1.1) (modulo some weak conditions) and all λ_0 that belong to the "bulk" of the spectrum where $\rho(\lambda_0) \neq 0$. Thus the limit (1.9) for arbitrary V has to coincide with the same limit for the arhetype Gaussian case $V(\lambda) = \lambda^2/2$, whose form is given by the r.h.s. of (1.9) and is known since early sixties (see Ref.1 for respective results and discussions).

In this paper we prove the

Theorem. *Assume that the function $V(\lambda)$ satisfies the conditions (cf.(1.5))*

$$V(\lambda) \geq (2+\epsilon)\log|\lambda|, \qquad |\lambda| \geq L_1 \tag{1.11a}$$

for some $L_1 < \infty$ and

$$|V'''(\lambda)| \leq \infty, \qquad |\lambda| \leq L \tag{1.11b}$$

for $L < \infty$ defined by (1.7). Then the universality conjecture (1.9) is true uniformly in $(x_1, ...x_l)$ varying on compact sets of R^l.

Remarks.

1. The Theorem is valid without any restrictions on the growth of $V(\lambda)$ provided that (1.11) is satisfied. However our proofs are considerably simpler if $V(\lambda)$ grows at infinity not faster than $Be^{b|\lambda|}$ for some finite B and b and $V'(\lambda)$ exists for all $\lambda \in R$ and majorized at infinity by similar exponent (not necessary with the same B and b). For this reason we will assume below that $V(\lambda)$ satisfies these additional conditions.

2. Denote by $P_l^{(n)}(\lambda), l = 0, 1, ...$ orthogonal polynomials on R associated with the weight

$$w_n(\lambda) = e^{-nV(\lambda)}, \tag{1.12}$$

$$\int P_l^{(n)}(\lambda)P_m^{(n)}(\lambda)e^{-nV(\lambda)}d\lambda = \delta_{l,m} \tag{1.13}$$

and by

$$\psi_l^{(n)}(\lambda) = \exp\{-nV(\lambda)/2\}P_l^{(n)}(\lambda),\ l = 0, 1, ... \tag{1.14}$$

respective orthonormal system

$$\int \psi_l^{(n)}(\lambda)\psi_m^{(n)}(\lambda)d\lambda = \delta_{l,m}. \tag{1.15}$$

Then the joint probability density of all eigenvalues of ensemble (1.1) is[1]

$$p_n(\lambda_1, ...\lambda_n) = \hat{Z}_n^{-1} \prod_{1\leq j<k\leq n} (\lambda_i - \lambda_j)^2 \exp\{-n\sum_{j=1}^{n} V(\lambda_j)\} = \tag{1.16}$$

$$(n!)^{-1}\left(\det ||\psi_{j-1}(\lambda_k)||_{j,k=1}^{n}\right)^2, \tag{1.17}$$

and marginal densities (1.4) are

$$p_l^{(n)}(\lambda_1, ..., \lambda_l) = \frac{(n-l)!}{n!} \det ||k_n(\lambda_j, \lambda_k)||_{j,k=1}^l \tag{1.18}$$

where

$$k_n(\lambda, \mu) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu) \tag{1.19}$$

is known as the reproducing kernel of system (1.14). In particular

$$\rho_n(\lambda) \equiv p_1^{(n)}(\lambda) = K_n(\lambda, \lambda) \tag{1.20}$$

where

$$K_n(\lambda, \mu) = n^{-1} k_n(\lambda, \mu). \tag{1.21}$$

In view of (1.18) the proof of the universality conjecture (1.9) for the random matrix ensemble (1.1) reduces to the proof of the limiting relation

$$\lim_{n\to\infty} [\rho_n(\lambda_0)]^{-1} K_n(\lambda_0 + \frac{x}{n\rho_n(\lambda_0)}, \lambda_0 + \frac{y}{n\rho_n(\lambda_0)}) = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{1.22}$$

that being rewritten as

$$\lim_{n\to\infty} [k_n(\lambda_0, \lambda_0)]^{-1} k_n(\lambda_0 + \frac{x}{k_n(\lambda_0, \lambda_0)}, \lambda_0 + \frac{y}{k_n(\lambda_0, \lambda_0)}) = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{1.23}$$

can be regarded as a conjecture of purely analytic nature concerning the orthogonal polynomial (1.13). Since for a complete systems of orthonormal functions

$$\sum_{j=0}^{\infty} \psi_j^{(n)}(\lambda)\psi_j^{(n)}(\mu) = \delta(\lambda - \mu) \tag{1.24}$$

the relation (1.23) can be viewed as the one saying that the fine ("magnified") structure of the δ-function in (1.24) is universal and is given by the r.h.s. of (1.23).

The relation (1.23) can be readily proven if the precise enough asymptotic formula for respective orthogonal polynomials is known. Let us consider the simplest ("toy") case of an n-independent weight supported on a finite interval, say interval $[-1, 1]$. By using classical asymptotic formulae[10] we find that in this case $\rho(\lambda) = (\pi\sqrt{1-\lambda^2})^{-1}, |\lambda| \leq 1$ and relation (1.9) is valid for any $|\lambda| < 1$. Less trivial case corresponds to the weight (1.12) in which $V(\lambda) = |\lambda|^\alpha/\alpha$ with a positive α. In this case $P_l^{(n)}(\lambda) = n^{1/2\alpha}\pi_l(n^{1/\alpha}\lambda)$

where $\{\pi_l(x)\}_{l=0}^{\infty}$ are orthogonal polynomials associated with the n-independent weight $w(x) = \exp\{-|x|^\alpha/\alpha\}$. The case $\alpha = 2$ corresponds to the Gaussian unitary ensemble and the Hermite polynomials as $\pi_l(x)$. This case was studied in great details[1] on the basis of the Plancherel-Rotah asymptotic formula[10] describing the semiclassical regime of quantum oscillator. For the general case $\alpha > 1$ asymptotic formulae were recently obtained.[11,12] By using these formulae the limiting density $\rho(\lambda)$ can be found and relation (1.9) can be checked for $\lambda = 0$. Unfortunately asymptotic formulae[11,12] are not precise enough to prove (1.9) for$\lambda \neq 0$. This can be done only for $\alpha = 4, 6$ where more precise asymptotic formulae are known. We mention also physical papers[13,14] devoted to more general forms of $V(\lambda)$. Authors[13] considered the case of polynomials $V(\lambda)$. They formulated the semiclassical ansatz for the asymptotic form of respective orthogonal polynomials and established (1.9) and also the universal form of the correlation function $p_n^{(2)}(\lambda,\mu) - p_n^{(1)}(\lambda)p_n^{(1)}(\mu)$ on the much bigger scale $1 >> \lambda, \mu > n^{-1}$. In the recent physical paper[15] the universality conjecture was considered by studying the generated functional of densities (1.18) that was computed by applying the Laplace method to the Grassmam integral representation of the generating functional.

We will prove the Theorem by using the orthogonal polynomials technique that is rather powerful and widely used in in the random matrix theory and its numerous applications. However, since the asymptotic formulae for the general case treated in the Theorem are not known we combine the orthogonal polynomial technique with certain identities that were introduced in the random matrix theory in the seminal paper[5].

Our paper is organized as follows. In Section 2 we give the proof of the Theorem following the main line of the arguments. The important ingredient of our arguments is the pointwise convergence of $\rho_n(\lambda)$ to $\rho(\lambda)$ for the set $\{\lambda : \rho(\lambda) > 0\}$.

Proposition. *Under the conditions of the Theorem*

$$|\rho_n(\lambda) - \rho(\lambda)| \leq C\left(1 + \frac{1}{\rho(\lambda)}\right) n^{-1/4} \tag{1.25}$$

for some positive n-independent constant C.

Remark. The Proposition allows us to give another proof of the so-called approximation property of the weight $w(\lambda) = e^{-V(\lambda)}$. According to definition[18] weight $w(\lambda)$ has the approximation property on the open set O if for every continuous and vanishing outside of O function $f(\lambda)$ there exists a sequence of polynomilals $q_n(\lambda)$ such that $w^n(\lambda)q_n(\lambda)$ converges uniformly to $f(\lambda)$ on O.

The approximation theorem (see e.g. Ref.18 for the proof, references and discussion) says that $w(\lambda)$ has the approximation property on the set where $\rho(\lambda)$ is continuous and strictly positive. In our case $\rho(\lambda)$ is continuous and the Proposition allows us to construct the approximating polynomials by using sertain linear combinations of polynomials $\frac{1}{n}\sum_{j=0}^{k}[P_j^{(n)}(\lambda)]^2$ for $k = O(n)$. These polynomials possess in fact all properties of $\rho_n(\lambda)$ because they can be obtained, roughly speaking, by replacing $V(\lambda)$ by $\frac{k}{n}V(\lambda)$.

The Proposition is also proved in Section 2. Auxiliary facts which we need to establish the Theorem and the Proposition are proved in Section 3. We discuss some consequences of our results in Section 4.

2 Proofs of the Proposition and the Theorem.

Proof of the Proposition. Consider the Stieltjes transform of the normalized counting measure (1.3)

$$f_n(z) \equiv \int \frac{N_n(d\lambda)}{\lambda - z} = \frac{1}{n}\sum_{l=1}^{n}\frac{1}{\lambda_l - z}, \tag{2.1}$$

and denote

$$g_n(z) \equiv E\{f_n(z)\} = \int \frac{\rho_n(\lambda)d\lambda}{\lambda - z}. \tag{2.2}$$

According to the spectral theorem

$$f_n(z) = \frac{1}{n}\mathrm{Tr}G(z)$$

where $G(z) = (M - z)^{-1}$ is the resolvent of a Hermitian matrix M. By using Lemma 1 for $F(M) = G_{ik}(z)$ (a matrix element of the resolvent) and $B = B^{(ik)} = \{B_{jm}^{(ik)}\}_{j,m=1}^{n}$, $\quad B_{jm}^{(ik)} = \zeta\delta_{ij}\delta_{km} + \overline{\zeta}\delta_{im}\delta_{kj}$, where $\zeta \in C$ is a free parameter, it is easy to derive the identity

$$E\{\zeta G_{ii}G_{kk} + \overline{\zeta}G_{ik}^2 + nG_{ik}\left(\zeta(V'(M))_{ki} + \overline{\zeta}(V'(M))_{ik}\right) = 0$$

Since ζ is arbitrary we conclude that

$$E\{G_{ii}G_{kk} + nG_{ik}(V'(M))_{ki}\} = 0.$$

Now if we sum over $i, k = 1, ..., n$ this inequality and divide the result by n^2 we get

$$E\{f_n^2\} + E\{n^{-1}\mathrm{Tr}V'(M)G(z)\} = 0. \qquad (2.3)$$

By applying Lemma 3 to $f(\mu) = (\mu - z)^{-1}$, $z = \lambda + i\eta$, $\eta > 0$, we find that

$$E\{f_n^2\} = E^2\{f_n\} + O(n^{-2}\eta^{-4}). \qquad (2.4)$$

This bound, (2.1) and (2.2) yield the relation

$$g_n^2(z) + V'(\lambda)g_n(z) + Q_n(z) = O(n^{-2}\eta^{-4}) \qquad (2.5)$$

where

$$Q_n(z) = \int \frac{V'(\mu) - V'(\lambda)}{\mu - z}\, \rho_n(\mu)d\mu. \qquad (2.6)$$

is well defined due to (1.7) and our conditions on $V(\lambda)$ (see the Theorem and Remark 1). To proceed further we use the result (1.8) combining it with condition (1.11b). We obtain

$$Q_n(\lambda + i\eta) = Q(\lambda) + O(\eta^{-1/4}n^{-1/2}\log^{1/2} n) \qquad (2.7)$$

where

$$Q(\lambda) = \int \frac{V'(\mu) - V'(\lambda)}{\mu - \lambda}\rho(\mu)d\mu. \qquad (2.8)$$

Combining (2.5) and (2.7) with Lemma 4 we find that

$$\pi^{-1}\Im g_n(\lambda + in^{-1/3}) = \rho(\lambda) + O(n^{-1/3}). \qquad (2.9)$$

On the other hand it follows from Lemmas 5 and 6 that

$$|\pi^{-1}\Im g_n(\lambda + in^{-1/3}) - \rho_n(\lambda)| \leq C\left(1 + \frac{1}{\rho(\lambda)}\right) n^{-1/4}.$$

This bound and (2.9) imply (1.25).

Proof of the Theorem. According to (1.18) the proof of the Theorem reduces to the proof of the limiting relation (1.22) to the reproducing kernel (1.21) of orthonormal systems (1.14). We use the representation

$$K_n(\lambda, \mu) = Z_n^{-1} \int \prod_{j=2}^{n} d\lambda_j(\lambda - \lambda_j)(\mu - \lambda_j)\times$$

$$\prod_{2\le j<k\le n}(\lambda_j-\lambda_k)^2\exp\left\{-\frac{n}{2}V(\lambda)-\frac{n}{2}V(\mu)-\sum_{j=2}^{n}V(\lambda_j)\right\} \qquad (2.10)$$

which can be derived from the well-known in the RMT[1] identities

$$\prod_{1\le j<k\le n}(\lambda_j-\lambda_k)=\left(\prod_{l=1}^{n-1}\gamma_l^{(n)}\right)^{-1}\det||P_{j-1}^{(n)}(\lambda_k)||_{j,k=1}^{n},\quad Z_n=n!\prod_{l=0}^{n-1}(\gamma_l^{(n)})^{-2}$$

where $\gamma_l^{(n)}$ is the coefficient in front of λ^l in the polynomial $P_l^{(n)}$. If we substitute these identities into the r.h.s. of (2.10), set in the one of determinant $\lambda_1=\lambda$, in other $\lambda_1=\mu$ and then integrate the result with respect to $\lambda_2,...,\lambda_n$, using the orthogonality of polynomials $P_l^{(n)}$ we obtain the l.h.s. of (2.10).

We will consider the function $K_n(\lambda_0,\lambda_0+s/n)$. General case of function $K_n(\lambda_0+s/n,\lambda_0+t/n)$ can be reduced to $K_n(\lambda_0,\lambda_0+(s-t)/n)$ by using Lemma 7. Let us choose

$$\delta=\frac{\log n}{n}$$

and rewrite (2.10) in the form

$$K_n(\lambda_0,\lambda_0+\frac{s}{n})=\exp\{\frac{n}{2}V(\lambda_0)-\frac{n}{2}V(\lambda_0+\frac{s}{n})\}\left\langle\prod_{j=2}^{n}\left[\frac{s}{n}\cdot\frac{\chi_\delta(\lambda_0-\lambda_j)}{\lambda_0-\lambda_j}+e^{u(\lambda_j)}\right]\right\rangle. \qquad (2.11)$$

Here and below the symbol $<...>$ denotes the operation $E\{\delta(\lambda_0-\lambda_1)...\}$, $\chi_\delta(\lambda)$ is the indicator of the interval $|\lambda|\le\delta$, and

$$u(\lambda)=(1-\chi_\delta(\lambda_0-\lambda))\log\left(1+\frac{s}{n(\lambda_0-\lambda)}\right).$$

Rewrite (2.11) as

$$K_n(\lambda_0,\lambda_0+\frac{s}{n})=T_n(\lambda_0)\left[1+\sum_{l=1}^{n-1}C_{n-1}^{l}(\frac{s}{n})^l\left\langle\prod_{j=2}^{l+1}\frac{\chi_\delta(\lambda_0-\lambda_j)}{\lambda_0-\lambda_j}e^{U_n(\lambda_0)}\right\rangle Z_n^{-1}(\lambda_0)\right], \qquad (2.12)$$

where $C_n^l=\dfrac{n!}{l!(n-l)!}$ and

$$T_n(\lambda_0)=\exp\{-\frac{n}{2}V(\lambda_0)+\frac{n}{2}V(\lambda_0+\frac{s}{n})\}Z_n(\lambda_0), \qquad (2.13)$$

$$Z_n(\lambda_0)=\left\langle e^{U_n(\lambda_0)}\right\rangle, \qquad (2.14)$$

$$U_n(\lambda_0)=\sum_{j=2}^{n}u(\lambda_j). \qquad (2.15)$$

Introduce the probability density (cf. (1.16))

$$p_{tn}(\lambda_2,...,\lambda_n)=Z_{tn}^{-1}(\lambda_0)\prod_{j=2}^{n}(\lambda_0-\lambda_j)^2\prod_{2\le j<k\le n}(\lambda_j-\lambda_k)^2$$

$$\exp\left\{-n\sum_{j=2}^{n}V(\lambda_j)+t\sum_{j=2}^{n}u(\lambda_j)\right\}, \qquad (2.16)$$

where $Z_{tn}^{-1}(\lambda_0)$ is the normalization factor, and respective marginal densities

$$p_{tl}^{(n)}(\lambda_2,...,\lambda_{l+1})=\int p_{tn}(\lambda_2,...,\lambda_n)d\lambda_{l+2}...d\lambda_n. \qquad (2.17)$$

In particular for $t=0$

$$p_{0l}^{(n)}(\lambda_2,...,\lambda_{l+1})=\rho_n^{-1}(\lambda_0)p_{l+1}^{(n)}(\lambda_0,\lambda_2,...,\lambda_{l+1}). \qquad (2.18)$$

This allows us to rewrite (2.12) as follows

$$K_n(\lambda_0,\lambda_0+\frac{s}{n})=T_n(\lambda_0)\left[1+\sum_{l=1}^{n-1}C_{n-1}^{l}\left(\frac{s}{n}\right)^l\int\prod_{j=2}^{l+1}\frac{\chi_\delta(\lambda_0-\lambda_j)}{\lambda_0-\lambda_j}p_{1n}(\lambda_2,...,\lambda_{l+1})d\lambda_2...d\lambda_{l+1}\right]. \qquad (2.19)$$

Introduce

$$R_{tn}(\lambda,\mu)=\frac{1}{n-1}\sum_{k=0}^{n-2}\psi_{tk}(\lambda)\psi_{tk}(\mu) \qquad (2.20)$$

where

$$\psi_{tk}(\lambda)=(\lambda-\lambda_0)\exp\{-\frac{n}{2}V(\lambda)+\frac{t}{2}u(\lambda)\}P_{tl}(\lambda)$$

and $\{P_{tk}(\lambda)\}_{k=0}^{\infty}$ are polynomials that are orthogonal with respect to the weight
$(\lambda-\lambda_0)^2\exp\{-nV(\lambda+tu(\lambda)\}$:

$$\int P_{tl}(\lambda)P_{tm}(\lambda)(\lambda-\lambda_0)^2\exp\{-nV(\lambda+tu(\lambda)\}d\lambda=\delta_{lm}.$$

Then (cf.(1.18))

$$p_{tl}^{(n)}(\lambda_1,...,\lambda_{l+1})=\frac{(n-1)^l}{(n-1)...(n-l-1)}\det||R_{tn}(\lambda_{j+1},\lambda_{k+1})||_{j,k=1}^{l}$$

and after the change of variables $x_j = n(\lambda_{j+1} - \lambda_0)$ (2.19) can be rewritten in the form

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0)\,[1+$$

$$\sum_{l=1}^{n-1} \frac{s^l (n-1)^l}{n^l l!} \int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||R_{1n}(\lambda_0 + \frac{x_j}{n}, \lambda_0 + \frac{x_k}{n})||_{j,k=1}^{l} \Bigg]. \qquad (2.21)$$

We will prove that

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[1 + \sum_{l=1}^{n-1} \frac{s^l}{l!} \int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||R_{0n}(\lambda_0 + \frac{x_j}{n}, \lambda_0 + \frac{x_k}{n})||_{j,k=1}^{l} + O(\frac{1}{n\delta})\right]. \qquad (2.22)$$

To this end we use Lemma 9. Therefore we have to check conditions (3.41)-(3.45) of the Lemma for $A = R_{0n}$ and $B = R_{1n}$. Inequality (3.41) follows from (2.18) and Lemma 8, inequality (3.43) follows from (2.18) and Lemma 7, and inequality (3.42) follows from representation (2.20). To check (3.44) and (3.45) consider the derivative $R'_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n})$ of (2.20) with respect to t. By using arguments similar to those in the proof of Lemma 5 we obtain

$$R'_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) = \frac{1}{2}(u(\lambda_0 + \frac{x}{n}) + u(\lambda_0 + \frac{y}{n}))R_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) -$$

$$(n-1)\int R_{tn}(\lambda_0 + \frac{x}{n}, \mu) R_{tn}(\lambda_0 + \frac{y}{n}, \mu) u(\mu) d\mu. \qquad (2.23)$$

If $|x| < n\delta$ and $|y| < n\delta$, then the first term in the r.h.s. of (2.23) is zero. The second term can be estimated by using the Schwarz ineguality and the analogue of (3.8) for (2.20)

$$\left|(n-1)\int R_{tn}(\lambda_0 + \frac{x}{n}, \mu) R_{tn}(\lambda_0 + \frac{y}{n}, \mu) u(\mu) d\mu\right| \le$$

$$\left|(n-1)\int R_{tn}(\lambda_0 + \frac{x}{n}, \mu)^2 |u(\mu)| d\mu\right|^{1/2} \left|(n-1)\int R_{tn}(\lambda_0 + \frac{y}{n}, \mu)^2 |u(\mu)| d\mu\right|^{1/2} \le$$

$$\max_{\lambda} |u(\lambda)| \cdot \left[R_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) \cdot R_{tn}(\lambda_0 + \frac{y}{n}, \lambda_0 + \frac{y}{n})\right]^{1/2}. \qquad (2.24)$$

Hence

$$\left|R'_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n})\right| \le \frac{C}{n\delta}\left[R_{tn}(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{x}{n}) R_{tn}(\lambda_0 + \frac{y}{n}, \lambda_0 + \frac{y}{n})\right]^{1/2}. \qquad (2.25)$$

Besides

$$\max_t R_{tn}(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}) = R_{t^*n}(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}) =$$

$$R_{0n}(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}) + \int_0^{t^*} d\tau R'_{\tau n}(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}) \le$$

$$R_{0n}(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}) + \frac{C}{n\delta}R_{t^*n}(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}). \tag{2.26}$$

Thus it follows from (2.26) that for all x and t (cf.(3.44))

$$R_{tn}(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}) \le CR_{0n}(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}). \tag{2.27}$$

Combining (2.25) and (2.27) we obtain (cf.(3.44))

$$|R_{1n}(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n})-R_{0n}(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n})| \le \frac{C}{n\delta}R_{0n}^{1/2}(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n})R_{0n}^{1/2}(\lambda_0+\frac{y}{n},\lambda_0+\frac{y}{n}). \tag{2.28}$$

Inequality (2.28), identity (2.18) and Lemma 8 guarantee condition (3.44) of Lemma 9. Condition (3.45) can be proved by similar arguments. Thus we can apply Lemma 9 to the expression in the r.h.s. of (2.21) and obtain (2.22).

By using the analogue of representation (2.10) for $R_{0n}(\lambda,\mu)$ we get

$$R_{0n}(\lambda,\mu) = K_n(\lambda,\mu) - \frac{K_n(\lambda_0,\lambda)K_n(\lambda_0,\mu)}{K_n(\lambda_0,\lambda_0)}. \tag{2.29}$$

We will use this representation to prove that we can replace the function $R_{0n}(\lambda,\mu)$ in the r.h.s. of (2.22) by

$$R^*(x_j,x_k) = K_n(\lambda_0,\lambda_0+\frac{x_k-x_j}{n}) - \frac{K_n(\lambda_0,\lambda_0-x_j/n)K_n(\lambda_0,\lambda_0+x_k/n)}{K_n(\lambda_0,\lambda_0)}. \tag{2.30}$$

We use again Lemma 9 for $A = R_{0n}$, $B = R^*$. As it was explained above, conditions (3.41)-(3.43) for this A are true and thus we have to check (3.44) and (3.45). Since according to (2.29) and (2.30)

$$|R^*(x,y) - R_{0n}(x,y)| \le \left|K_n(\lambda_0,\lambda_0+\frac{y-x}{n}) - K_n(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n})\right| +$$

$$\left|\frac{K_n(\lambda_0,\lambda_0+y/n)}{K_n(\lambda_0,\lambda_0)}\right| |K_n(\lambda_0,\lambda_0+x/n) - K_n(\lambda_0,\lambda_0-x/n)|$$

it suffices to check that uniformly in $|y| \le n\delta$ and $n \to \infty$

$$|K_n(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) - K_n(\lambda_0, \lambda_0 + \frac{y-x}{n})|^2 \le \frac{Cx^2}{n^{1/4}}, \quad |x| \le 1, \qquad (2.31)$$

and

$$\int_{-n\delta}^{n\delta} |K_n(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}) - K_n(\lambda_0, \lambda_0 + \frac{x-y}{n})|^2 dx \le \frac{C(n\delta)^3}{n^{1/4}}. \qquad (2.32)$$

Estimate (2.31) follows from Lemma 7, because $|x|, |y| \le n\delta = \log n$. Estimate (2.32) can be obtained if we integrate (2.31) with respect to x. Thus we have proved that

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[1 + \sum_{l=1}^{n-1} \frac{s^l}{l!} \int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||R^*(x_j, x_k)||_{j,k=1}^{l} + O(\frac{1}{n\delta}) \right]. \qquad (2.33)$$

The next step is to prove that we can replace the integral over the interval $(-n\delta, n\delta)$ in the r.h.s. of (2.33) by the integral over the whole axis R. To this end let us notice first that since $R^*(x,x) = R^*(-x,-x)$

$$\int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||R^*(x_j, x_k)||_{j,k=1}^{l} = \int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||R^*(x_j, x_k)(1-\delta_{jk})||_{j,k=1}^{l}. \qquad (2.34)$$

Besides

$$\Delta_l \equiv \left| \int \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||R^*(x_j, x_k)(1-\delta_{jk})||_{j,k=1}^{l} - \int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||R^*(x_j, x_k)(1-\delta_{jk})||_{j,k=1}^{l} \right| \le$$

$$\sum_{p=1}^{l} C_l^p \int \prod_{j=1}^{p} \frac{(1-\chi_{n\delta}(x_j))dx_j}{|x_j|} \prod_{j=p+1}^{l} \frac{\chi_{n\delta}(x_j)dx_j}{|x_j|} \left| \det ||R^*(x_j, x_k)(1-\delta_{jk})||_{j,k=1}^{l} \right| \le$$

$$\sum_{p=1}^{l} C_l^p \sum_{m=0}^{l-p} C_{l-p}^m \int \prod_{j=1}^{p} \frac{(1-\chi_{n\delta}(x_j))dx_j}{|x_j|} \times$$

$$\prod_{i=p+1}^{p+m} \frac{(1-\chi_1(x_i))dx_i}{|x_j|} \prod_{k=p+m+1}^{l} \frac{\chi_1(x_k)dx_k}{|x_k|} \left| \det ||R^*(x_j, x_k)(1-\delta_{jk})||_{j,k=1}^{l} \right| \le$$

$$\sum_{p=1}^{l} C_l^p \sum_{m=0}^{l-p} C_{l-p}^m \left\{ \int \prod_{j=1}^{p} \frac{(1-\chi_{n\delta}(x_j))dx_j}{|x_j|^{5/4}} \prod_{i=p+1}^{p+m} \frac{(1-\chi_1(x_i))dx_i}{|x_j|^{5/4}} \prod_{k=p+m+1}^{l} dx_k \chi_1(x_k) \right\}^{1/2} \times$$

$$\left\{ \int \prod_{j=1}^{p} \frac{(1-\chi_{n\delta}(x_j))dx_j}{|x_j|^{3/4}} \prod_{i=p+1}^{p+m} \frac{(1-\chi_1(x_i))dx_i}{|x_j|^{3/4}} \times \right.$$

$$\left. \prod_{k=p+m+1}^{l} \frac{\chi_1(x_k)dx_k}{|x_k|^2} \left| \det ||R^*(x_j,x_k)(1-\delta_{jk})||_{j,k=1}^{l} \right|^2 \right\}^{1/2}.$$

The first factor in the r.h.s. of the last inequality can be estimated by $(n\delta)^{-p/4}C^{l-p}$. To estimate the second one we repeat almost literally the arguments of Lemma 9. We obtain

$$\Delta_l \le l^{(l+2)/2} C^l (n\delta)^{-1/4}.$$

Therefore

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[1 + \sum_{l=1}^{n-1} \frac{s^l}{l!} \int \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||R^*(x_j,x_k)||_{j,k=1}^{l} + O\left(\frac{1}{(n\delta)^{1/4}} \right) \right]. \tag{2.35}$$

Now by using the formula

$$\det ||a_{jk}||_{j,k=0}^{l} = a_{00} \det ||a_{jk} - \frac{a_{j0}a_{0k}}{a_{00}}||_{j,k=1}^{l}$$

we obtain from (2.35)

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[1 + \sum_{l=1}^{n-1} \frac{s^l}{l!\rho_n(\lambda_0)} \int \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||S_n(x_j - x_k)||_{j,k=0}^{l} + O\left(\frac{1}{(n\delta)^{1/4}} \right) \right]. \tag{2.36}$$

where $x_0 = 0$ and $S_n(x) = K_n(\lambda_0, \lambda_0 + \frac{x}{n})$. The integral in the r.h.s. of (2.36) can be computed by using the Fourier integral technique. This is done in Lemma 11 of Section 3. According to that Lemma

$$K_n(\lambda_0, \lambda_0 + \frac{s}{n}) = T_n(\lambda_0) \left[\frac{\sin \pi\rho_n(\lambda_0)s}{\pi\rho_n(\lambda_0)s} + O\left(\frac{1}{(n\delta)^{1/4}} \right) \right]. \tag{2.37}$$

Comparing this expression with (1.22) we see that to finish the proof of the Theorem we have to establish the relation

$$\lim_{n\to\infty} T_n(\lambda_0) = \rho(\lambda_0).$$

This relation follows from the Proposition and Lemma 10 of Section 3. Theorem is proved.

3 Auxiliary results.

In this section we prove a number of facts that we use in the proofs of the Theorem and the Proposition (Sec.2).

Lemma 1 *Let $F(t), t \in R$ be a continuously differentiable and polynomially bounded function, and B be an arbitrary Hermitian matrix. Then*

$$E\{F'_B(M)\} - nE\{F(M)TrV'(M)B\} = 0 \tag{3.1}$$

where $F'_B(M) = \lim_{\epsilon\to\infty} \epsilon^{-1}[F(M+\epsilon B) - F(M)]$.

Proof. We obtain the Lemma by differentiating with respect to t the identity

$$\int \exp\{-n\mathrm{Tr}V(M+tB)\}F(M+tB)dM = \int \exp\{-n\mathrm{Tr}V(M)\}F(M)dM$$

that follows from the invariance of the measure dM with respect to shift $M \to M+B$ by an arbitrary Hermitian matrix B.

Remark. This lemma was in fact proved by Bessis et al.[5]

Lemma 2 *Let $k_n(\lambda,\mu)$ be defined by (1.19). Then*

$$\int (\lambda-\mu)^2 k_n^2(\lambda,\mu)d\lambda d\mu \le C, \tag{3.2}$$

and for $\alpha = 1,2$

$$\int (\lambda-\mu)^\alpha k_n^2(\lambda,\mu)d\mu \le C\{(\psi_{n-1}^{(n)}(\lambda))^2 + (\psi_n^{(n)}(\lambda))^2\}. \tag{3.3}$$

Proof. It follows from the orthogonality relations (1.15) that for $j = 0,1,2,...$

$$r_j P_{j+1}(\lambda) + r_{j-1}P_{j-1}(\lambda) = \lambda P_j(\lambda) \quad (r_{-1} = 0) \tag{3.4}$$

where

$$r_j = \int \lambda P_j(\lambda)P_{j+1}(\lambda)e^{-nV(\lambda)}d\lambda \tag{3.5}$$

and we omit superscript n to simplify notations. Denote by $J = \{J_{jk}\}_{j,k=1}^\infty$ the Jacobi matrix defined by (3.4):

$$J_{jk} = r_j\delta_{j+1,k} + r_{j-1}\delta_{j-1,k}. \tag{3.6}$$

Then for any nonnegative integer p

$$(J^p)_{jk} = \int \lambda^p \psi_j^{(n)}(\lambda)\psi_k^{(n)}(\lambda)(\lambda)d\lambda. \tag{3.7}$$

By using the identity

$$\int k_n^2(\lambda,\mu)d\mu = k_n(\lambda,\lambda), \tag{3.8}$$

and (3.7) for $p = 1, 2$ we find that the l.h.s. of (3.2) is

$$2\left(\sum_{j=0}^{n-1}(J^2)_{jj} - \sum_{j,k=0}^{n-1} J_{jk}^2\right). \tag{3.9}$$

This relation and (3.6) yield

$$\int(\lambda-\mu)^2 k_n^2(\lambda,\mu)d\lambda d\mu = 2r_{n-1}^2. \tag{3.10}$$

Using (1.7) and (1.19)-(1.21) we obtain that for some n-independent $a,\ A,\ L > 0$

$$[\psi_l(\lambda)]^2 \le n\rho_n(\lambda) \le nAe^{-an(|\lambda|-L)}, \qquad |\lambda| \ge L. \tag{3.11}$$

and then (3.5) implies the bound

$$|r_l| \le C \tag{3.12}$$

for some C. This bound and (3.10) imply (3.2). Similar arguments and equation (3.4) yield

$$\int(\lambda-\mu)k_n^2(\lambda,\mu)d\mu = \psi_{n-1}(\lambda)\psi_n(\lambda)r_{n-1}.$$

Now (3.3) follows from this identity and (3.12). The case $\alpha = 2$ in the l.h.s. of (3.3) can be proved analogously. Lemma 2 is proved.

Lemma 3 *Let $f(\mu)$, $\mu \in R$, be a bounded and Holder continuous function:*

$$|f(\lambda) - f(\mu)| \le C|\lambda-\mu|^\alpha \tag{3.13}$$

for some $C > 0$ and $0 < \alpha \le 1$ and

$$f_n = \frac{1}{n}\sum_{j=1}^{n} f(\lambda_j^{(n)})$$

where $\{\lambda_j^{(n)}\}_{j=1}^n$ are eigenvalues of a random matrix. Then

$$D\{f_n\} \equiv E\{|f_n - E\{f_n\}|^2\} \le C_1 n^{-1-\alpha}. \tag{3.14}$$

Proof. By using (1.18) and (1.19) we can write (3.14) as follows

$$D\{f_n\} = \frac{1}{2n^2}\int |f(\lambda) - f(\mu)|^2 k_n^2(\lambda,\mu)d\lambda d\mu.$$

This representation, (3.13), the Holder inequality and the relation

$$\int k_n^2(\lambda,\mu)d\lambda d\mu = n \tag{3.15}$$

yield the bound

$$D\{f_n\} \le \frac{C^2}{2n^{1+\alpha}}\left[\int |\lambda - \mu|^2 k_n^2(\lambda,\mu)d\lambda d\mu\right]^\alpha$$

which implies (3.14) in view of Lemma 2. Lemma 3 is proved.

Lemma 4 *Assume that λ is a point of the spectral axis at which $\rho(\lambda) > 0$. Then*

$$\rho(\lambda) = \frac{1}{\pi}\sqrt{Q(\lambda) - (V'(\lambda))^2/4} \tag{3.16}$$

where $Q(\lambda)$ is defined by (2.8).

Proof. According to (1.8) $\rho_n(\lambda)$ converges weakly to $\rho(\lambda)$. This result allows us to perform the limiting transition in (2.5) and to obtain for nonreal z's the relation

$$g^2(z) + V'(\lambda)g(z) + Q(z) = 0 \tag{3.17}$$

where $g(z)$ is the Stietjes transform of the limiting density $\rho(\lambda)$. Condition (1.11b) of the Theorem and (2.8) imply that $Q(\lambda + i0)$ is a real valued, bounded function with a bounded derivative. Then by general principles

$$\rho(\lambda) = \frac{1}{\pi}\Im g(\lambda + i0) \tag{3.18}$$

is also bounded. Computing the real and the imaginary parts of (3.17) rewritten as

$$g = -\frac{Q}{V' + g} \tag{3.19}$$

we find (3.16). Lemma 4 is proved.

Lemma 5 *Under the conditions of the Theorem*

$$\sup_{n,\lambda} \rho_n(\lambda) \le C \tag{3.20}$$

and

$$\left|\frac{d\rho_n(\lambda)}{d\lambda}\right| \le C_1(\psi_{n-1}^2(\lambda) + \psi_n^2(\lambda)) + C_2. \tag{3.21}$$

Proof. According to (1.18) and (1.20)

$$\frac{d\rho_n(\lambda)}{d\lambda} = Z_n^{-1}\frac{d}{d\lambda}\int \exp\left\{-nV(\lambda) - n\sum_{j=2}^{n} V(\lambda+\mu_j)\right\}\prod_{j=2}^{n}(\lambda-\mu_j)^2 d\mu_j =$$

$$-nV'(\lambda)\rho_n(\lambda) - n(n-1)\int V'(\lambda_2)p_n(\lambda,\lambda_2,...\lambda_n)d\lambda_2...d\lambda_n =$$

$$= -nV'(\lambda)K_n(\lambda,\lambda) - n^2\int V'(\lambda_2)[K_n(\lambda,\lambda)K_n(\lambda_2,\lambda_2) - K_n^2(\lambda,\lambda_2)]d\lambda_2 \tag{3.22}$$

The identity (3.1) for $F(M)=1$ and $B=1$ yields

$$E\{\mathrm{Tr}V'(M)\} = n\int V'(\lambda)K_n(\lambda,\lambda)d\lambda = 0.$$

Hence by (3.22)

$$\rho_n'(\lambda) = n^2\int[V'(\mu) - V'(\lambda)]K_n^2(\lambda,\mu)d\mu. \tag{3.23}$$

Now we split this integral in two parts corresponding to the intervals $|\mu| > 2L$ and $|\mu| \le 2L$ where L is defined by (1.7). The former integral is bounded because of the inequality $K_n^2(\lambda,\mu) \le K_n(\lambda,\lambda)K_n(\mu,\mu)$ and bound (1.7) for $K_n(\lambda,\lambda) = \rho_n(\lambda)$. In the latter integral we write

$$V'(\mu) - V'(\lambda) = (\mu-\lambda)V''(\lambda) + \frac{(\mu-\lambda)^2}{2}V'''(\xi)$$

for some ξ depending on λ and μ and use Lemma 2 and condition (1.11b) of the Theorem. Combining the bounds for these two integrals we obtain (3.21). To obtain (3.20) we have to use (3.23) and (1.13). Lemma 5 is proved.

Lemma 6 *Assume that $\rho(\lambda) \ne 0$ and take $\epsilon = O(n^{-1/4})$. Then*

$$\int_{|\mu-\lambda|\le\epsilon}(\psi_{n-1}^2(\mu) + \psi_n^2(\mu))d\mu \le C_1\left(1+\frac{1}{\rho(\lambda)}\right)n^{-1/4}, \tag{3.24}$$

$$\frac{1}{n}(\psi_{n-1}^2(\mu) + \psi_n^2(\mu)) \le C_2\left(1+\frac{1}{\rho(\lambda)}\right)n^{-1/8}, \quad |\mu-\lambda| \le \epsilon. \tag{3.25}$$

Proof. Let us introduce the density

$$p_n^-(\lambda_1,...,\lambda_{n-1}) = \frac{1}{Z_n^-}\exp\left\{-n\sum_{j=1}^{n-1}V(\lambda_j)\right\}\prod_{1\le j<k\le n-1}(\lambda_j-\lambda_k)^2. \quad (3.26)$$

The difference of this density from density (1.2) written for $n-1$ variables $\lambda_1,...\lambda_{n-1}$ is that in the former we have the factor n in the exponent while in the latter we would have $n-1$. Set

$$\rho_n^-(\lambda) = \frac{n-1}{n}\int p_n^-(\lambda,\lambda_2,...,\lambda_{n-1})d\lambda_2...d\lambda_{n-1} = \frac{1}{n}\sum_{j=0}^{n-2}[\psi_j(\lambda)]^2. \quad (3.27)$$

Then

$$\psi_{n-1}(\lambda) = n[\rho_n(\lambda)-\rho_n^-(\lambda)]. \quad (3.28)$$

Furthermore, by using the analogue of identity (3.1) for the density p_n^- and the arguments similar to those proving (2.5) we obtain the relation

$$[g_n^-(z)]^2 + \int\frac{V'(\mu)\rho_n^-(\mu)}{\mu-z}d\mu = O(\frac{1}{n^2\eta^4}) \quad (3.29)$$

for the Stieltjes transform $g_n^-(z)$ of $\rho_n^-(\mu)$ and $z=\lambda+i\eta,\ \eta>0$. Denote

$$\Delta_n(z) = n(g_n(z)-g_n^-(z)) = \int\frac{\psi_{n-1}^2(\mu)}{\mu-z}d\mu, \quad (3.30)$$

subtract (3.29) from (2.5) and multiply the result by n. We obtain

$$\Delta_n(z)(g_n(z)+g_n^-(z)) + \int\frac{V'(\mu)\psi_{n-1}^2(\mu)}{\mu-z}d\mu = O(\frac{1}{n\eta^4}).$$

For $z=\lambda+in^{-1/4}$ this relation takes the form

$$\Delta_n(z)(2g_n(z)-V'(\lambda)) = \int\frac{(V'(\lambda)-V'(\mu))\psi_{n-1}^2(\mu)}{\mu-z}d\mu + O(1).$$

Then relation (3.16) and (3.18) imply that

$$\Im\Delta_n(\lambda+in^{-1/4}) \le C_3\left(1+\frac{1}{\rho(\lambda)}\right).$$

Integrating this inequality over the interval $|\lambda-\mu|\le\epsilon=O(n^{-1/4})$ and using (3.30) we obtain

$$\int_{|\mu-\lambda|\le\epsilon}\psi_{n-1}^2(\mu)d\mu \le C_4\left(1+\frac{1}{\rho(\lambda)}\right)n^{-1/4}. \quad (3.31)$$

Now we derive (3.25) from (3.31). Set

$$\Psi_{n-1} = \psi_{n-1}^2(\mu^*) = \max_{|\mu-\lambda|\le\epsilon}\{\psi_{n-1}^2(\mu)\}, \quad \mu_1 = \sup\{\mu : \mu \in (\lambda-\epsilon, \mu^*),\ \psi_{n-1}^2(\mu) \le \frac{\Psi_{n-1}}{2}\}.$$

Since $|\mu_1 - \mu^*| \le \epsilon$ we have from (3.31)

$$\frac{\Psi_{n-1}}{2}(\mu^* - \mu_1) \le \int_{\mu_1}^{\mu^*} \psi_{n-1}^2(\mu)d\mu \le C_4(1 + \frac{1}{\rho(\lambda)})n^{-1/4}. \tag{3.32}$$

On the other hand

$$\left(\frac{\Psi_{n-1}^{1/2} - (\frac{1}{2}\Psi_{n-1})^{1/2}}{\mu^* - \mu_1}\right)^2 \cdot (\mu^* - \mu_1) \le \int_{\mu_1}^{\mu^*} (\psi'_{n-1})^2(\mu)d\mu \le \int (\psi'_{n-1})^2(\mu)d\mu$$

and since

$$\int (\psi'_{n-1})^2(\mu)d\mu = \int \frac{n^2}{4}(V'(\mu))^2(\psi'_{n-1})^2(\mu)d\mu \le C_5 n^2$$

we obtain

$$\frac{\Psi_{n-1}}{\mu^* - \mu_1} \le C_5 n^2 \tag{3.33}$$

Now if we multiply (3.32) by (3.33) we get (3.25) for ψ_{n-1}.

To prove the analogous bounds for ψ_n^2 we have to repeat above arguments for the density (cf.(3.26))

$$p_n^+(\lambda_1, ..., \lambda_{n+1}) = \frac{1}{Z_n^+} \exp\left\{-n\sum_{j=1}^{n+1} V(\lambda_j)\right\} \prod_{1\le j<k\le n+1} (\lambda_j - \lambda_k)^2$$

and for

$$\rho_n^+(\lambda) = \frac{n+1}{n}\int p_n^+(\lambda, \lambda_2, ..., \lambda_{n+1})d\lambda_2...d\lambda_{n+1} = \frac{1}{n}\sum_{j=0}^{n}[\psi_j(\lambda)]^2,$$

so that $\psi_n^2(\lambda) = n(\rho_n^+(\lambda) - \rho_n(\lambda))$ (cf.(3.28)). Lemma 6 is proved.

Lemma 7 *If $\rho(\lambda_0) \ne 0$, then*

$$|K_n(\lambda_0+\frac{x}{n}, \lambda_0+\frac{y}{n}) - K_n(\lambda_0, \lambda_0+\frac{y-x}{n})| \le C|x|\left(\frac{1}{n^{1/8}} + \frac{|x-y|^2}{n^2} + e^{-naL/2}\right). \tag{3.34}$$

Proof. Repeating almost literally the derivation of (3.22) we get

$$\frac{d}{dt}K_n(\lambda_0+\frac{x-tx}{n},\lambda_0+\frac{y-tx}{n})=$$

$$xn\int K_n(\lambda_0+\frac{x-tx}{n},\lambda)K_n(\lambda_0+\frac{y-tx}{n},\lambda)\left(V'(\lambda)-\frac{1}{2}V'(\lambda_0+\frac{x-tx}{n})-\frac{1}{2}V'(\lambda_0+\frac{y-tx}{n})\right)d\lambda.$$

To estimate the r.h.s. of this relation we split this integral in two parts corresponding to the intervals $|\lambda|>2L$ and $|\lambda|\le 2L$ where L is defined by (1.7). The former integral is bounded by $C\exp\{-naL/2\}$ because of the inequality $K_n^2(\lambda,\mu)\le K_n(\lambda,\lambda)K_n(\mu,\mu)$ and bound (1.7) for $K_n(\lambda,\lambda)=\rho_n(\lambda)$. In the latter integral we write

$$V'(\lambda)-\frac{1}{2}V'(\lambda_0+\frac{x-tx}{n})-\frac{1}{2}V'(\lambda_0+\frac{y-tx}{n})=\frac{1}{2}(\lambda-\lambda_x)V''(\lambda_x))+\frac{1}{2}(\lambda-\lambda_y)V''(\lambda_y))+$$

$$O\left((\lambda-\lambda_x)^2+(\lambda-\lambda_y)^2\right)=\frac{1}{2}(\lambda-\lambda_x)V''(\lambda_x))+\frac{1}{2}(\lambda-\lambda_y)V''(\lambda_y))+O\left((\lambda-\lambda_x)(\lambda-\lambda_y)+\frac{|x-y|^2}{n^2}\right)$$

where

$$\lambda_x=\lambda_0+\frac{x-tx}{n},\quad \lambda_y=\lambda_0+\frac{y-tx}{n}.$$

According to (3.7)

$$n\int K_n(\lambda_x,\lambda)K_n(\lambda_y,\lambda)(\lambda-\lambda_{x,y})d\lambda=\frac{r_{n-1}}{n}\psi_n^{(n)}(\lambda_x)\psi_n^{(n)}(\lambda_y)$$

Besides, by the Schwartz inequality

$$n\left|\int K_n(\lambda_x,\lambda)K_n(\lambda_y,\lambda)(\lambda-\lambda_x)(\lambda-\lambda_y)d\lambda\right|\le$$

$$n\left[\int K_n(\lambda_x,\lambda)(\lambda-\lambda_x)^2d\lambda\int K_n(\lambda_y,\lambda)(\lambda-\lambda_y)^2d\lambda\right]^{1/2}.$$

Now the arguments similar to those used in the proof of Lemma 2 and Lemma 5 yield the estimate

$$\left|\frac{d}{dt}K(\lambda_0+\frac{x-tx}{n},\lambda_0+\frac{y-tx}{n})\right|\le$$

$$\frac{C}{n}|x|\left(\psi_n^2(\lambda_x)+\psi_{n-1}^2(\lambda_x)+\psi_n^2(\lambda_y)+\psi_{n-1}^2(\lambda_y)+\frac{|x-y|^2}{n}+e^{-naL}\right).$$

Combining this estimate with (3.25) we obtain (3.34).

Lemma 8 *Let $p_2^{(n)}(\lambda_1,\lambda_2)$ be specified by (1.18) for $l=2$. Then uniformly in n*

$$\int_{-1}^{1} \frac{p_2^{(n)}(\lambda_0+\frac{x}{n},\lambda_0)}{x^2}dx \le C. \tag{3.35}$$

Proof. Consider

$$W=\left\langle \prod_{i=2}^{n}\left|1-\frac{1}{n^2(\lambda_i-\lambda_0)^2}\right|\right\rangle$$

where symbol $< ... >$ was defined in (2.11). By the Schwarz inequality W^2 is bounded from above by the product of integrals

$$Z_n^{-1}\int \prod_{2\le j<k\le n}(\lambda_i-\lambda_j)^2\prod_{2\le j\le n}(\lambda_0+\sigma-\lambda_j)^2\exp\{-nV(\lambda_0)-n\sum_{j=2}^{n}V(\lambda_j)\}$$

for $\sigma=\pm 1/n$. Besides, $n(V(\lambda_0)-V(\lambda_0+\sigma))$ is bounded in n due to (1.11b). This allows us to write the bound

$$W\le C\cdot K_n^{1/2}(\lambda_0+\frac{1}{n},\lambda_0+\frac{1}{n})K_n^{1/2}(\lambda_0-\frac{1}{n},\lambda_0-\frac{1}{n})\le C_1. \tag{3.36}$$

On the other hand W can be written as

$$W=\left\langle\prod_{i=2}^{n}(\phi_1(\lambda_i)+\phi_2(\lambda_i))\right\rangle=$$

$$\left\langle\prod_{i=2}^{n}\phi_2(\lambda_i)\right\rangle+\sum_{k=1}^{n-1}C_{n-1}^{k}\left\langle\prod_{i=2}^{k+1}\phi_1(\lambda_i)\prod_{i=k+2}^{n}\phi_2(\lambda_i)\right\rangle$$

where

$$\phi_1(\lambda)=\begin{cases}\dfrac{(1-n^2(\lambda-\lambda_0)^2)^2}{n^2(\lambda-\lambda_0)^2}, & n|\lambda-\lambda_0|<1,\\ 0, & otherwise\end{cases}$$

and

$$\phi_2(\lambda)=\begin{cases}1-n^2(\lambda-\lambda_0)^2, & n|\lambda-\lambda_0|<1,\\ \dfrac{n^2(\lambda-\lambda_0)^2-1}{n^2(\lambda-\lambda_0)^2}, & otherwise.\end{cases}$$

Since $0\le\phi_2(\lambda)\le 1$, $\phi_1(\lambda)\ge 0$ and $<1>=\rho_n(\lambda_0)$, we get from this representation

$$W\ge-\rho_n(\lambda_0)+(n-1)\int d\lambda\phi_1(\lambda)\left\langle\delta(\lambda_2-\lambda)\exp\left\{\sum_{i=3}^{n}\log\phi_2(\lambda_i)\right\}\right\rangle. \tag{3.37}$$

By using the Jensen inequality and (1.2) we have

$$\frac{\langle \delta(\lambda_2-\lambda)\exp\{\sum_{i=3}^n \log\phi_2(\lambda_i)\}\rangle}{\langle\delta(\lambda_2-\lambda)\rangle} \ge$$

$$\exp\left\{\left\langle \delta(\lambda_2-\lambda)\sum_{i=3}^n \log\phi_2(\lambda_i)[p_2^{(n)}(\lambda_0,\lambda)]^{-1}\right\rangle\right\} =$$

$$\exp\left\{(n-2)\int \log\phi_2(\lambda')p_3^{(n)}(\lambda_0,\lambda,\lambda')d\lambda'[p_2^{(n)}(\lambda_0,\lambda)]^{-1}\right\}. \qquad (3.38)$$

According to (1.18)

$$p_3^{(n)}(\lambda_0,\lambda,\lambda') = \frac{n^2}{(n-1)(n-2)}\left(\frac{n-1}{n}\rho_n(\lambda')p_2^{(n)}(\lambda_0,\lambda) + 2K_n(\lambda_0,\lambda)K_n(\lambda_0,\lambda')K_n(\lambda,\lambda') - \right.$$

$$\left. \rho_n(\lambda_0)K_n^2(\lambda,\lambda') - \rho_n(\lambda)K_n^2(\lambda_0,\lambda')\right). \qquad (3.39)$$

Besides, since $\log\phi_2(\lambda')\le 0$ and

$$2K_n(\lambda_0,\lambda)K_n(\lambda_0,\lambda)K_n(\lambda',\lambda) \le 2K_n^{1/2}(\lambda_0,\lambda_0)K_n^{1/2}(\lambda,\lambda')|K_n(\lambda_0,\lambda')||K_n(\lambda',\lambda)| \le$$

$$\rho_n(\lambda_0)K_n^2(\lambda',\lambda) + \rho_n(\lambda)K_n^2(\lambda_0,\lambda')$$

we have

$$\int d\lambda' \log\phi_2(\lambda')\left(2K_n(\lambda_0,\lambda)K_n(\lambda_0,\lambda')K_n(\lambda',\lambda) - \rho_n(\lambda_0)K_n^2(\lambda,\lambda') - \rho_n(\lambda)K_n^2(\lambda_0,\lambda')\right) \ge 0.$$

Hence taking into account that $\rho_n(\lambda)$ is bounded from above uniformly in n
we get

$$W \ge -\rho_n(\lambda_0) + (n-1)\int d\lambda\phi_1(\lambda)p_2^{(n)}(\lambda_0,\lambda)\exp\left\{(n-1)\int\rho_n(\lambda')\log\phi_2(\lambda')d\lambda'\right\} \ge -\rho_n(\lambda_0) +$$

$$\int_{-1}^{1}\frac{(1-x^2)^2}{x^2}p_2^{(n)}(\lambda_0,\lambda_0+\frac{x}{n})dx\cdot\exp\left\{-C\left(\int_0^1|\log(1-y^2)|dy + \int_1^\infty \log(1-y^{-2})dy\right)\right\}. \qquad (3.40)$$

From (3.40) and (3.36) it is easy to derive (3.35).

Lemma 9 *Let the functions* $A(x,y)$, $B(x,y)$ *be defined for* $|x|,|y|\le n\delta$, $\delta = n^{-1}\log n$ *and satisfy the conditions:*

$$A(x,y)\le a(x)\le C_0,\ \int_{-1}^1 \frac{a^2(x)}{x^2}dx \le C_1^2, \qquad (3.41)$$

$$\int |A(x.y)|^2 dx \le C_2^2, \tag{3.42}$$

$$|A(x,x) - A(-x,-x)| \le \epsilon_1 x, \tag{3.43}$$

$$|A(x,y) - B(x,y)| \le \epsilon_2 b(x), \ \int_{-1}^{1} \frac{b^2(x)dx}{x^2} \le C_3^2, \qquad |b(x)| \le C_4, \tag{3.44}$$

$$\int_{-n\delta}^{n\delta} |A(x,y) - B(x,y)|^2 dx \le C_5^2 \epsilon_3^2. \tag{3.45}$$

Then

$$\int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||A(x_j,x_k)||_{j,k=1}^{l} \le (lC)^{l/2}, \tag{3.46}$$

and

$$\left| \int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} (\det ||A(x_j,x_k)||_{j,k=1}^{l} - \det ||B(x_j,x_k)||_{j,k=1}^{l}) \right| \le \epsilon l (lC)^{l/2} \tag{3.47}$$

where $\epsilon = n\delta(\epsilon_1 + \epsilon_2) + \epsilon_3$ *and* C *depend only on* $C_i, \ i = 1, ..., 5$

Proof. It suffices to prove estimates (3.46) and (3.47) for

$$A_0(x_j,x_k) = A(x_j,x_k)(1 - \delta_{jk}),$$

and

$$B_0(x_j,x_k) = B(x_j,x_k)(1 - \delta_{jk})$$

Indeed, due to conditions (3.43) and (3.44) the following inequalities are true

$$\left| \int_{-n\delta}^{n\delta} \frac{A(x,x)}{x} dx \right| \le 2n\delta\epsilon_1$$

and

$$\left| \int_{-n\delta}^{n\delta} \frac{B(x,x)}{x} dx \right| \le 2n\delta\epsilon_1 + 2\epsilon_2(C_3 + C_4 \log n\delta)$$

and we can easily obtain (3.46) and (3.47) for general A and B from the respective bounds for A_0 and B_0.

Consider

$$F(t) = \int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||A_t(x_j,x_k)||_{j,k=1}^{l}$$

where $A_t(x_j,x_k)=A_0(x_j,x_k)+t(B_0-A_0)(x_j,x_k)$. To obtain (3.46) we have to estimate $|F(1)-F(0)|$. Therefore it suffices to estimate $\frac{dF}{dt}$. Differentiating $F(t)$, making respective permutations of columns and rows and the same renumbering of variables we obtain

$$\frac{dF}{dt}=l\int_{-n\delta}^{n\delta}\prod_{j=1}^{l}\frac{dx_j}{x_j}\det||D_t(x_j,x_k)||_{j,k=1}^{l}$$

where

$$D_t(x_1,x_k)=(A_0-B_0)(x_1,x_k),\ \ D_t(x_j,x_k)=A_t(x_j,x_k),\ \ j\geq 2.$$

Thus

$$l^{-1}\left|\frac{dF}{dt}\right|\leq\int_{-1}^{1}\prod_{j=1}^{l}\frac{dx_j}{|x_j|}\left|\det||D_t(x_j,x_k)||_{j,k=1}^{l}\right|+\int_{-n\delta}^{n\delta}\prod_{j=1}^{l}\frac{dx_j}{|x_j|}(1-\chi_1(x_j))\left|\det||D_t(x_j,x_k)||_{j,k=1}^{l}\right|+$$

$$\sum_{m=1}^{l}C_l^m\left[\int_{-n\delta}^{n\delta}\prod_{j=1}^{m}\frac{dx_j}{|x_j|}\chi_1(x_j)\prod_{j=m+1}^{l}(1-\chi_1(x_j))\frac{dx_j}{|x_j|}\left|\det||D_t(x_j,x_k)||_{j,k=1}^{l}\right|+\right.$$

$$\left.\int_{-n\delta}^{n\delta}\prod_{j=1}^{m}\frac{dx_j}{|x_j|}(1-\chi_1(x_j))\prod_{j=m+1}^{l}\chi_1(x_j)\frac{dx_j}{|x_j|}\left|\det||D_t(x_j,x_k)||_{j,k=1}^{l}\right|\right]\qquad(3.48)$$

where $\chi_1(x)$ is the indicator of the interval $(-1,1)$. Let us estimate the last term

$$F_t^{(m)}=\int_{-n\delta}^{n\delta}\prod_{j=1}^{m}\frac{dx_j}{|x_j|}(1-\chi_1(x_j))\prod_{j=m+1}^{l}\chi_1(x_j)\frac{dx_j}{|x_j|}\left|\det||D_t(x_j,x_k)||_{j,k=1}^{l}\right|.$$

Other terms in the r.h.s. of (3.48) can be estimated similarly.

$$|F_t^{(m)}|\leq\left\{\int_{-n\delta}^{n\delta}\prod_{j=1}^{m}\frac{dx_j}{|x_j|^{3/4}}(1-\chi_1(x_j))\prod_{j=m+1}^{l}\chi_1(x_j)\frac{dx_j}{|x_j|^2}\left|\det||D_t(x_j,x_k)||_{j,k=1}^{l}\right|^2\right\}^{1/2}\times$$

$$\left\{\int_{-n\delta}^{n\delta}\prod_{j=1}^{m}\frac{dx_j}{|x_j|^{5/4}}(1-\chi_1(x_j))\prod_{j=m+1}^{l}\chi_1(x_j)dx_j\right\}^{1/2}\leq$$

$$\left\{\int_{-n\delta}^{n\delta}\prod_{j=1}^{m}\frac{dx_j}{|x_j|^{3/4}}(1-\chi_1(x_j))\prod_{j=m+1}^{l}\chi_1(x_j)\frac{dx_j}{|x_j|^2}\prod_{j=1}^{l}\sum_{k=1}^{l}D_t^2(x_j,x_k)\right\}^{1/2}\cdot 2^{m+l/2}. \tag{3.49}$$

Here we have used the Schwartz inequality and then the Hadamard estimate for determinants(16). Now the r.h.s. of (3.49) can be rewritten as the sum of the integrals

$$I_{j_1\dots j_l}=\int_{-n\delta}^{n\delta}\prod_{j=1}^{m}\frac{dx_j}{|x_j|^{3/4}}(1-\chi_1(x_j))\prod_{j=m+1}^{l}\chi_1(x_j)\frac{dx_j}{|x_j|^2}D_t^2(x_1,x_{j_1})\dots D_t^2(x_l,x_{j_l})\le$$

$$\int_{-n\delta}^{n\delta}\prod_{j=1}^{m}\frac{dx_j}{|x_j|^{3/4}}(1-\chi_1(x_j))\prod_{j=m+1}^{l}\chi_1(x_j)\frac{dx_j}{|x_j|^2}D_t^2(x_1,x_{j_1})\dots D_t^2(x_m,x_{j_m})a_t^2(x_{m+1})\dots a_t^2(x_l) \tag{3.50}$$

with $a_t(x)=a(x)+t\epsilon_3 b(x)$.

To estimate the last integral we start integrating with respect to the "free" variables, i.e. the variables that do not enter the set $(x_{j_1},\dots,x_{j_m})$. We use bound (3.42) for the integral with respect to x_1 and bounds (3.44) and (3.45) for integrals with respect to $x_2,\dots,x_m$. If there is no free variables, then we use the inequality $D_t^2(x_m,x_{j_m})\le(C_0+t\epsilon_2C_4)^2$ which makes the variable x_{j_m} free. Repeating this procedure we end up either with the estimate

$$I_{j_1\dots j_l}\le C^{l-1}\Delta_{1,j} \tag{3.51}$$

or with the estimate

$$I_{j_1\dots j_l}\le \epsilon_3^2C^{l-1}\Delta_{i,j} \tag{3.52}$$

where $C=\max\{2(C_1+\epsilon_2C_3),C_0+\epsilon_2C_4,2(C_2+\epsilon_3C_5)\}$ and

$$\Delta_{ij}=\int_{-n\delta}^{n\delta}D_t^2(x_i,x_j)(1-\chi_1(x_i))(1-\chi_1(x_j))\frac{dx_i}{|x_i|^{3/4}}\frac{dx_j}{|x_j|^{3/4}}$$

with some $i,j\le m$. Regarding $D_t^2(x_i,x_j)$ as the kernel of an integral operator acting in $L_2(-n\delta,n\delta)$ and using the bound $\sup_{x_i}\int_{-n\delta}^{n\delta}D_t^2(x_i,x_j)dx_j$ for the norm of this operator and bounds (3.42) and (3.45) we obtain that

$$I_{j_1\dots j_l}\le\epsilon_3^2C^l.$$

Repeating similar argument to estimate all other terms in (3.48) we obtain (3.46) and (3.47)

Lemma 10 *Let T_n be defined by (2.13). Then*

$$|T_n(\lambda_0) - \rho_n(\lambda_0)| \leq \frac{C}{\log n}. \tag{3.53}$$

Proof. We will prove the following bounds:

$$|E\{U_n(\lambda_0)\} - \frac{s}{2}V'(\lambda_0)| \leq Csn^{-1/4}\log n \tag{3.54}$$

$$\langle \exp\{2U_n(\lambda_0)\}\rangle \leq C \tag{3.55}$$

$$\langle |U_n(\lambda_0) - E\{U_n(\lambda_0)\}|^2\rangle \leq \frac{C}{n\delta} \tag{3.56}$$

where $U_n(\lambda_0)$ is specified by (2.15). Assuming that these bounds are true it is easy to prove (3.53) by using the Schwarz inequality and elementary inequality $|e^x - 1| \leq |x|(e^x + 1)$.

To prove (3.54) take $\delta_1 = n^{-1/4}$ and rewrite $E\{U_n(\lambda_0)\}$ as follows

$$E\{U_n(\lambda_0)\} = (n-1)\int u(\lambda)\rho_n(\lambda)d\lambda = \int ((n-1)u(\lambda) - \phi(\lambda))(\rho_n(\lambda) - \rho(\lambda))\,d\lambda +$$

$$\int \phi(\lambda)(\rho_n(\lambda) - \rho(\lambda))\,d\lambda + (n-1)\int u(\lambda)\rho(\lambda)d\lambda \tag{3.57}$$

where $\phi(\lambda)$ is a differantiable function of the form:

$$\phi(\lambda) = \begin{cases} (n-1)\dfrac{\lambda_0 + \delta_1 - \lambda}{2\delta_1}\ln\left(1 - \dfrac{s}{n\delta_1}\right) + (n-1)\dfrac{\lambda - \lambda_0 + \delta_1}{2\delta_1}\ln\left(1 + \dfrac{s}{n\delta_1}\right), & |\lambda - \lambda_0| < \delta_1, \\ (n-1)u(\lambda), & otherwise \end{cases}$$

Using the Proposition and Lemma 4 one can estimate the first integral I_1 in the r.h.s. of (3.57) as follows

$$I_1 \leq Cn^{-1/4}\int_{|\lambda-\lambda_0|<\delta_1} ((n-1)|u(\lambda)| + |\phi(\lambda)|)\,d\lambda \leq Csn^{-1/4}\log n \tag{3.58}$$

To estimate the second integral we use inequality (1.8) according to which

$$\left|\int \phi(\lambda)(\rho_n(\lambda) - \rho(\lambda))\,d\lambda\right| \leq Cn^{-1/2}\log^{1/2} n||\phi'||_2^{1/2}||\phi||_2^{1/2} = C\delta_1^{-1}n^{-1/2}\log^{1/2} n \tag{3.59}$$

And the last integral I_3 in the r.h.s. of (3.57) can be calculated by using the result[9], according to which for any λ: $\rho(\lambda) \neq 0$

$$\int \log|\lambda - \lambda'|\rho(\lambda')d\lambda' = \frac{1}{2}V(\lambda) + const \tag{3.60}$$

Thus we have

$$I_3 = \frac{n-1}{2}\left(V(\lambda_0 + s/n) - V(\lambda_0)\right) + O((n\delta_1)^{-1}) \tag{3.61}$$

Relations (3.57)-(3.61) prove (3.54).

To prove (3.55) consider $f(t) \equiv \log\langle \exp\{tU_n(\lambda_0)\}\rangle$. Since $f(t)$ is a convex function

$$f(2) \leq f(0) + 2f'(2) = \log \rho_n(\lambda_0) + 2f'(2) \tag{3.62}$$

In view of (2.16)

$$f'(2) = (n-1)\int u(\lambda) R_{2n}(\lambda,\lambda) d\lambda =$$

$$(n-1)\int u(\lambda) R_{0n}(\lambda,\lambda) d\lambda + (n-1)\int_0^2 dt \int u(\lambda) R'_{tn}(\lambda,\lambda) d\lambda \tag{3.63}$$

where R_{tn} and R'_{tn} are specified by (2.20) and (2.23). According to (2.23) the second integral I_2 in (3.63) can be rewritten as

$$I_2 = (n-1)\int_0^2 dt \left[\int u^2(\lambda) R_{tn}(\lambda,\lambda) d\lambda + (n-1)\int u(\lambda)u(\lambda') R_{tn}^2(\lambda,\lambda') d\lambda d\lambda'\right] \leq$$

$$C\int_{n\delta}^{\infty} \log^2(1+\frac{s}{x})dx + (n-1)^2 \int_0^2 dt \int u(\lambda)u(\lambda') R_{tn}^2(\lambda,\lambda') d\lambda d\lambda' \tag{3.64}$$

where we have used (2.29) for $\lambda = \mu$ and (1.20) according to which $R_{0n}(\lambda,\lambda) \leq K_n(\lambda,\lambda) = \rho_n(\lambda)$, the boundedness of $\rho_n(\lambda)$ and (2.27). Regarding $R_{tn}^2(\lambda,\lambda')$ as a kernel of integral operator $\overline{R}$ in $L_2(R)$ one can estimate its norm as $||\overline{R}|| \leq \max_\lambda\{\int R_{tn}^2(\lambda,\lambda')d\lambda' = n^{-1}$. Thus

$$(n-1)^2\int u(\lambda)u(\lambda') R_{tn}^2(\lambda,\lambda') d\lambda d\lambda' \leq n\int u^2(\lambda) d\lambda \leq C\int_{n\delta}^{\infty} \log^2(1+x^{-1})dx \leq \frac{C_1}{n\delta} \tag{3.65}$$

To estimate the first integral I_1 in the r.h.s. of (3.63) we use again (2.29). Then

$$I_1 = (n-1)\int u(\lambda) R_{0n}(\lambda,\lambda) d\lambda = (n-1)\int u(\lambda) K_n(\lambda,\lambda) d\lambda - \frac{n-1}{\rho_n(\lambda_0)}\int u(\lambda) K_n^2(\lambda_0,\lambda) d\lambda =$$

$$\frac{s}{2}V'(\lambda_0) + O(n^{-1/4}\log n) + O\left(n\frac{\max|u(\lambda)|}{\rho_n(\lambda_0)}\int K_n^2(\lambda_0,\lambda)d\lambda\right) = \frac{s}{2}V'(\lambda_0) + O((n\delta)^{-1}) \tag{3.66}$$

Here we have used (3.54) to calculate $(n-1)\int u(\lambda)K_n(\lambda,\lambda)d\lambda$. Relations (3.62)-(3.66) prove (3.55).

To prove (3.66) let us note that in view of (3.54) and (3.66)

$$E\{U_n(\lambda_0)\} = \left\langle \sum_{j=2}^{n} u(\lambda_j) \right\rangle_0 + O((n\delta)^{-1})$$

where $< ... >_0$ denotes the expectation with respect to density (2.18). This expectation is related to the operation $< ... >= E\{\delta(\lambda_0 - \lambda_1)...\}$ as $< ... >= \rho_n(\lambda_0) < ... >_0$. Thus to estimate the r.h.s. of (3.56) it is enough to estimate

$$\left\langle \left(\sum_{j=2}^{n} u(\lambda_j) - \left\langle \sum_{j=2}^{n} u(\lambda_j) \right\rangle_0 \right)^2 \right\rangle_0 \leq$$

$$(n-1)\int u^2(\lambda)R_{0n}(\lambda,\lambda)d\lambda - (n-1)(n-2)\int u(\lambda)u(\lambda')R^2_{0n}(\lambda,\lambda')d\lambda d\lambda'$$

Combining this inequality with estimate (3.64) we get (3.56). Lemma 10 is proved.

Lemma 11 *Let $X(x)$, $x \in R$ be a smooth enough and fastly decaying function. Then*

$$\int \prod_{j=1}^{l} \frac{dx_j}{x_j} \det ||X(x_j - x_k)||^l_{j,k=0} = \frac{(i\pi)^l X^{l+1}(0)}{l+1} \cdot \frac{1-(-1)^{l+1}}{2} \qquad (3.67)$$

where $x_0 = 0$.

Proof. By using the Fourier integral representation of $X(x)$ we can write

$$\det ||X(x_j - x_k)||^l_{j,k=0} = \int \prod_{j=0}^{l} dp_j \hat{X}(p_j) \det || \exp\{ix_j(p_j - p_k)||^l_{j,k=0}$$

where $\hat{X}(p)$ is the Fourier transform of $X(x)$. This representation and the identity

$$\int \frac{e^{ipx}}{x} dx = i\pi \text{sign} p \equiv i\pi\theta(p)$$

allow us to rewrite the integral in the l.h.s. of (3.64) as follows

$$(i\pi)^l \int \prod_{j=0}^{l} dp_j \hat{X}(p_j) \begin{vmatrix} 1 & 1 & 1 & . & 1 \\ \theta(p_1 - p_0) & 0 & . & . & \theta(p_1 - p_l) \\ \theta(p_2 - p_0) & \theta(p_2 - p_1) & 0 & . & \theta(p_2 - p_l) \\ . & . & . & . & . \\ \theta(p_l - p_0) & . & . & . & 0 \end{vmatrix}$$

Let us compute the determinant in the domain $p_1, ..., p_m < p_0,\ p_{m+1}, ..., p_l \geq p_0$. Without loss of generality we can assume that $p_1 < p_2 ... < p_m < p_0 < p_{m+1} < ... < p_l$. Then the determinant will have the form

$$\begin{vmatrix} +1 & +1 & +1 & +1 & . & . & +1 & +1 \\ -1 & 0 & -1 & -1 & . & . & -1 & -1 \\ -1 & +1 & 0 & -1 & . & . & . & -1 \\ . & . & . & . & . & . & . & . \\ -1 & +1 & +1 & . & . & . & -1 & -1 \\ +1 & +1 & +1 & +1 & . & . & -1 & -1 \\ . & . & . & . & . & . & . & . \\ +1 & +1 & +1 & . & . & +1 & +1 & 0 \end{vmatrix}.$$

Subtracting the first row from l-th, $(l-1)$-th,...$(l-m)$-th ones and then the first column from the second, ..., m-th ones we find that determinant is equal to $(-1)^{l-m}$ Therefore the l.h.s. of (3.67) is equal to

$$(i\pi)^l \sum_{m=0}^{l} C_l^m \int dp_0 \hat{X}(p_0) \left(\int_{-\infty}^{p_0} dp \hat{X}(p) \right)^m \left(\int_{p_0}^{\infty} \hat{X}(p) dp \right)^{l-m} =$$

$$\int dp_0 \hat{X}(p_0) \left(2 \int_{-\infty}^{p_0} dp \hat{X}(p) - X(0) \right)^l = \frac{(i\pi)^l X^{l+1}(0)}{l+1} \cdot \frac{1-(-1)^{l+1}}{2}.$$

Lemma 11 is proved.

4 Discussion.

Let us regard the set $\{\lambda_j^{(n)}\}_{j=1}^n$ of eigenvalues of random matrices as a point process, i.e. as the random counting measure

$$\nu_n(\Delta) \equiv n N_n(\Delta) = \sum_{\lambda_i^{(n)} \in \Delta} 1. \tag{4.1}$$

Keeping in mind that we are studying the asymptotic behavior of the eigenvalue statistics for large n we can define this point process either by system (1.4) of its marginal distributions or by its generating functional

$$\Phi_n[\phi] = E\left\{ \exp\left[\int \phi(\lambda) \nu_n(d\lambda) \right] \right\} \tag{4.2}$$

defined on a suitable space of test functions $\phi(\lambda), \quad \lambda \in R$. We use the simplest case of bounded piece-wise continuous functions with a compact support. Then, by using (1.17) we find that

$$\Phi_n[\phi] = \det(1 - k_n[\phi]) \tag{4.3}$$

where $k_n[\phi]$ is the integral operator defined on the support σ_ϕ of ϕ by the kernel

$$k_n(\lambda,\mu)(1 - e^{\phi(\mu)}). \tag{4.4}$$

According to the Theorem the "scaling" limit (1.9) of all marginal densities (1.4) is given by (1.9) for all unitary invariant ensembles defined by (1.1), (1.7) and (1.11). To find the same limit for the generating functional we have to replace the test function $\phi(\lambda)$ by $\phi_n(x) = \phi(x/n\rho_n(\lambda_0))$. Then

$$\Phi[\phi] \equiv \lim_{n\to\infty} \Phi_n[\phi_n] = \det(1 - Q_\phi) \tag{4.5}$$

where Q_ϕ is the integral operator defined on σ_ϕ by the formula

$$(Q_\phi f)(x) = \int_{\sigma_\phi} S(x-y)(1 - e^{\phi(y)})f(y)dy, \quad x \in \sigma_\phi. \tag{4.6}$$

and $S(x)$ is defined in (1.10). These formulae contain in fact the same information as (1.9) saying that in our case the point process

$$\nu^{(n)}_{\lambda_0}(t) = \nu_n(\lambda_0, \lambda_0 + \frac{t}{n\rho_n(\lambda_0)})) \tag{4.7}$$

converges weakly as $n \to \infty$ to the random process defined by (4.5) and (4.6) or by (1.9).

Consider now the probability

$$R_n\left(\{\Delta_j\}_{j=1}^l\right) = \Pr\{\nu_n(\Delta_j) = 0, \quad j = 1,...,l\} \tag{4.8}$$

that an ordered set of disjoint intervals $\Delta_i = (a_i, b_i)$ does not contain eigenvalues. Then the arguments similar to those proving (4.3) imply that

$$R_n\left(\{\Delta_j\}_{j=1}^l\right) = \det(1 - K_{n\Delta}) \tag{4.9}$$

where $\Delta = \cup_{j=1}^l \Delta_j$ and $K_{n\Delta}$ is the integral operator defined on Δ by the kernel

$$\sum_{j=1}^l \chi_{\Delta_j}(\lambda) k_n(\lambda,\mu)\chi_{\Delta_j}(\mu).$$

Setting

$$a_i = \lambda_0 + \frac{\alpha_i}{n\rho_n(\lambda_0)}, \quad b_i = \lambda_0 + \frac{\beta_i}{n\rho_n(\lambda_0)} \tag{4.10}$$

$$\delta_i = (\alpha_i, \beta_i), \quad \delta = \cup_{j=1}^{l} \delta_j$$

and using the Theorem we obtain that

$$\lim_{n\to\infty} R_n \left(\{\Delta_j\}_{j=1}^{l} \right) = r(\delta) \tag{4.11}$$

where $r(\delta)$ is the Fredholm determinant of the kernel

$$\sum_{j=1}^{l} \chi_{\delta_j}(x) S(x-y) \chi_{\delta_j}(y). \tag{4.12}$$

We can also introduce more general kernel

$$\sum_{j=1}^{l} \tau_j \chi_{\delta_j}(x) S(x-y) \chi_{\delta_j}(y). \tag{4.13}$$

for an arbitrary collection of real τ_j's. Then if for an arbitrary collection $k = (k_1, ...k_l)$ of positive integers we consider the probability

$$R_n \left(\{\Delta_j\}_{j=1}^{l}, \{k_j\}_{j=1}^{l} \right) = \Pr\{\nu_n(\Delta_j) = k_j\}$$

its limit $r(\delta, k)$ is

$$r(\delta, k) = \frac{(-1)^k}{k_1!...k_l!} \cdot \frac{\partial^{k_1+\ldots+k_l}}{\partial\tau_1^{k_1}...\partial\tau_l^{k_l}} r(\delta, \tau) \bigg|_{\tau_j=1} \tag{4.14}$$

where $r(\delta, \tau)$ is the Fredholm determinant of the kernel (4.13).

The case $l = 1$ of (4.8) and (4.11) determines[1] the limiting probability distribution of distances between nearest neighbour eigenvalues (spacings) lying in the $O(n^{-1})$ neighbourhood of λ_0. Thus in the limit (4.10) the spacing probability distribution is the same for all ensembles satisfying the conditions of the Theorem. For the Gaussian case formula (4.14) was obtained in Ref.17 where some other kernels are also considered and various links of determinant (4.13) with integrable systems and related topics are discussed.

We can also consider another asymptotic regime, making "windows" in $O(n^{-1})$ -neighbourhood of different spectral points, i.e. considering joint probability distribution of the counting functions $\nu_{\lambda_1}^{(n)}(t_1), ..., \nu_{\lambda_k}^{(n)}(t_k)$ for distinct n-independent $\lambda_1, ..., \lambda_k$. Take for simplicity $k = 2$. Then we have to consider generating functional (4.2) on functions

$$\phi(\mu) = \phi_1(n\rho_n(\lambda_1)(\mu - \lambda_1)) + \phi_2(n\rho_n(\lambda_2)(\mu - \lambda_2)).$$

Inserting this $\phi(\mu)$ in (4.2) and using the result[9] according to which $p_2^{(n)}(\lambda_1, \lambda_2) \to \rho(\lambda_1)\rho(\lambda_2)$ as $n \to \infty$ for distinct n-independent $\lambda_1,\ \lambda_2$ and the Theorem we obtain

$$\lim_{n\to\infty} \Phi_n[\phi] = \Phi[\phi_1]\Phi[\phi_2]$$

where $\Phi[\phi]$ is defined by (4.5) and (4.6).

We conclude that in the "scaling" limit the local statistics eigenvalues lying in $O(n^{-1})$ -neighbourhoods of distinct spectral points are independent.

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