# SMOOTHING AND DECAY PROPERTIES OF SOLUTIONS OF THE KORTEWEG-DE VRIES EQUATION ON A PERIODIC DOMAIN WITH POINT DISSIPATION

By

**David L. Russell**

and

**Bing-Yu Zhang**

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# Smoothing and Decay Properties of Solutions of the Korteweg-de Vries Equation on a Periodic Domain with Point Dissipation

David L. Russell
Department of Mathematics
Virginia Tech
Blacksburg, VA 24061

Bing-Yu Zhang *
Institute of Mathematics and Its Applications
University of Minnesota
Minneapolis, Minnesota 55455

*On leave of absence from: Department for Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221

**Abstract**

We consider the Korteweg-de Vries (KdV) equation

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0$$

for $0 \leq x \leq 1$ and $t \geq 0$, with $x = 0$ identified with $x = 1$ via periodic boundary conditions

$$u(1,t) = u(0,t), \qquad \partial_x^2 u(1,t) = \partial_x^2 u(0,t)$$

while an $L^2$–stabilizing control input, implemented by a feedback mechanism, provideds a third boundary condition

$$\partial_x u(1,t) = \alpha \partial_x u(0,t), \quad |\alpha| < 1.$$

It is readily verified that smooth solutions of this system conserve the "volume"

$$[u] \equiv \int_0^1 u(x,t)dx.$$

For initial states $u_0$ in appropriate spaces we are concerned with smoothing properties of solutions of this system with asymptotic decay properties corresponding to approach, in the space $L^2[0,1]$, to the (constant) state $u \equiv [u_0]$ as $t \to \infty$, a property made plausible by the fact that the norm $||u(.,t)||_{L^2[0,1]}$ is monotonically decreasing in these circumstances.

These results are obtained using an integral equation based on the the "variation of parameters" formula and explicit representation of the operator semigroup associated with the linearized equation

$$\partial_t u + \partial_x^3 u = 0$$

with the same boundary conditions and by use of Lyapounov techniques based on properties of this linear third order dispersion system.

# 1 Introduction

In the present work we consider a class of equations which may be described as being of *Korteweg-de Vries* (KdV) type. These have the general form

$$\partial_t u + \gamma u \partial_x u + \partial_x^3 u = 0, \tag{1.1}$$

with $\gamma$ a non-negative real number. For $\gamma = 0$ we have the *third order linear dispersion equation*, which the authors have studied in [27]. References in this paper to the *linear case* refer primarily to that work. All cases $\gamma > 0$ are essentially equivalent, involving only a change of scale in the dependent variable, and may thus be covered by setting $\gamma = 1$.

The literature pertaining to this equation both on a periodic domain and a domain $-\infty < x < \infty$, $-\infty < t < \infty$, is absolutely enormous; we refer the reader to [1], [14], [17] and [24] for a beginning collection of references. Our particular interests in this paper on the periodic case is related to smoothing properties of the KdV equation for the infinite interval case due initially by Cohen [2], Kato [14] and developed subsequently in papers by Constantin and Saut [4], Sjölin [30] as well as many others ( The reader is referred to [16], [5], [9] and the references therein for various smoothing properties of dispersive wave equations and their applications). It is well known in this now "standard" theory of the KdV equation that evolution of solutions smoothes compactly supported initial data. This occurs through the dispersive effects associated with the third order operator, resulting in solution components of increasingly high "spatial" frequency being more rapidly dispersed to infinity as the solution evolves. There is no counterpart of this smoothing action on strictly periodic domains where boundary conditions

$$\partial_x^k u(1,t) = \partial_x^k u(0,t), \tag{1.2}$$

minimally applying for $k = 0, 1, 2$, are imposed on the process. Intuitively one could say that in this compact domain situation there is nowhere for the high frequency solution components to "escape to". More rigorously one would observe that since the process is invariant under simultaneous time ($t$) and space ($x$) reversal, each instance of a non-smooth initial state evolving into a later smooth state is matched by a corresponding smooth initial state evolving into a non-smooth state (cf. [28]). The only way to restore the smoothing property is to build into the system a selective dissipative mechanism which increasingly attenuates higher frequency solution components. The behavior of the system (1.1), (1.2) modified with such a dissipative mechanism having single point support is the subject of the present work.

To explain the *point dissipation* situation studied here, some background is required. For appropriately smooth solutions of (1.1) on the indicated domain with the periodic boundary conditions (1.2) it is well known [23], [25] that the following integral quantities (and infinitely many others) are conserved:

$$\int_0^1 u(x,t)dx, \tag{1.3}$$

$$\int_0^1 w(x,t)^2 dx, \tag{1.4}$$

$$\int_0^1 \left( (\partial_x u(x,t))^2 - \frac{\gamma}{3} u^3(x,t) \right) dx. \tag{1.5}$$

From the historical origins [22], [29] of the KdV equation, involving the behavior of solitary water waves in a shallow channel, it is natural to think of (1.3) as corresponding to fluid volume or mass. That context suggests that the system should continue to be studied in a volume conserving context. Let us observe that, for a smooth solution of (1.1)

$$\begin{aligned} \partial_t \int_0^1 u(x,t) &= \int_0^1 \left( -\gamma u(x,t) \partial_x u(x,t) - \partial_x^3 u(x,t) \right) dx \\ &= \int_0^1 \partial_x \left( -\frac{\gamma}{2} u(x,t)^2 - \partial_x^2 u(x,t) \right) dx \\ &= \left[ -\frac{\gamma}{2} u(x,t)^2 - \partial_x^2 u(x,t) \right]_{x=0}^{x=1}. \end{aligned} \tag{1.6}$$

This computation allows us to see that if we enforce the periodic boundary conditions (1.2) for $k = 0$ and $k = 2$ and introduce an input into the system via

$$\partial_x u(1,t) - \partial_x u(0,t) = v(t)) \tag{1.7}$$

then volume is still conserved because the last quantity in (1.6) involves only the function and second derivative values at $x = 0, 1$. The *open loop control system* of interest thus consists of (1.1), (1.2) for $k = 0$ and 2, and (1.7). For the linear case,

$\gamma = 0$, this control system has been studied in [27]. However, the open loop system is not the specific object of study in this paper.

In control theory, *controllability problems* are associated with corresponding *stabilization problems* in which the input is required, for each $t$, to be a functional of fixed form (i.e., independent of $t$) of the system state $u(.,t)$ and the objective of the control action is to cause the system to approach, as $t \to \infty$, a particular set of equilibrium states; if (1.3) is conserved it is natural to require the controlled solution to approach the constant state $\tilde{u}$ for which $\tilde{u} = [u_0]$, i.e., the equilibrium state of the same volume.

To make plausible the *closed loop point dissipation* process to be studied in the present article we first remind the reader of the readily verified fact that among all functions $u \in L^2(0,1)$ for which $[u] = c$, $c$ given, $\|u\|_{L^2(0,1)}$ is uniquely minimized by the constant function $\bar{u} \equiv c$; indeed, for all complex valued $u \in L^2(0,1)$, if we set $\bar{u} = [u]$, $x \in (0,1)$, we have the identity

$$\int_0^1 |u(x)^2| dx = \int_0^1 |u(x) - \tilde{u}|^2 \, dx + \int_0^1 |\tilde{u}(x)|^2 dx. \tag{1.8}$$

We may therefore anticipate, for a solution $u(x,t)$ of (1.1) satisfying (1.2) for $k = 0, 2$, and thus such that $\tilde{u}(x,t) \equiv \tilde{w}_0(x) \equiv [u_0]$, that $u(.,t)$ may be caused to approach this constant state as $t \to \infty$ through use of a control process designed so that $\|u(.,t)\|^2_{L^2(0,1)}$ is non-increasing. Indeed, using the fact that

$$u(.,t) - \tilde{u}_0(x) \equiv u(.,t) - \tilde{u}(.,t)$$

is orthogonal in $L^2(0,1)$ to the constant function $\tilde{w}_0(x)$, we observe that for smooth, real solutions $u$ of the indicated system

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\int_0^1 (u(x,t) - \tilde{u}_0(x))^2 dx\right) &= \int_0^1 u(x,t)\partial_t u(x,t) dx \\ &= \int_0^1 u(x,t)\left(-\gamma u(x,t)\partial_x u(x,t) - \partial_x^3 u(x,t)\right) dx \end{aligned}$$

(integrating the last term by parts, using (1.2), $k = 0, 1$, and the fact that $u(x,t) - [u](t)$ is orthogonal to constant functions on $(0,1)$)

$$= \int_0^1 \left(-\frac{\gamma}{3}\partial_x (u(x,t)^3) + \frac{1}{2}\partial(\partial_x u(x,t))^2\right) dx$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\partial_x u(x,t)\right)^2 \Big|_{x=0}^{x=1} \\
&= \frac{1}{2}\left(\partial_x u(1,t) + \partial_x u(0,t)\right)\left(\partial_x u(1,t) - \partial_x u(0,t)\right) \\
&= \frac{1}{2}\left(\partial_x u(1,t) + \partial_x u(0,t)\right) v(t).
\end{aligned} \tag{1.9}
$$

Thus, if for $\Re > 0$ we set

$$
v(t) = -\Re\left(\partial_x u(1,t) + \partial_x u(0,t)\right), \tag{1.10}
$$

which is clearly the same as imposing the *closed loop boundary condition*

$$
\partial_x u(1,t) = \alpha \partial_x u(0,t), \quad \alpha = \frac{1-\Re}{1+\Re}, \tag{1.11}
$$

(clearly $0 < |\alpha| < 1$) we obtain

$$
\begin{aligned}
\frac{d}{dt}\left(\int_0^1 u(x,t)^2 dx\right) &= -\Re\left(\partial_x u(1,t) + \partial_x u(0,t)\right)^2 \\
&= \frac{1}{2}(\alpha^2 - 1)(\partial_x u(0,t))^2 \leq 0.
\end{aligned} \tag{1.12}
$$

Equality obtains in (1.12) if and only if

$$
\partial_x u(1,t) - \partial_x u(0,t) \equiv v(t) \equiv -\Re(\partial_x u(1,t) + \partial_x u(0,t)) = 0,
$$

which is the same as

$$
\partial_x u(1,t) = -\partial_x u(0,t) = 0.
$$

For complex valued solutions of the corresponding linear ($\gamma = 0$) system the counterpart to (1.12) is readily seen to be

$$
\frac{d}{dt}\left(\int_0^1 |u(x,t)^2| dx\right) = \frac{1}{2}(\alpha^2 - 1)\left|\partial_x u(0,t)\right|^2 \leq 0. \tag{1.13}
$$

The closed loop point dissipation process thus consists of the equation (1.1), (1.2) for $k = 0$ and 2 and the further condition (1.11).

We shall begin by considering the associated linear problem

$$
\begin{cases}
\partial_t w + \partial_x^3 w = 0, \quad 0 < x < 1, \ t \geq 0 \\
w(x,0) = \phi(x) \\
w(1,t) = w(0,t), \quad \partial_x w(1,t) = \alpha \partial_x w(0,t), \quad \partial_x^2 w(1,t) = \partial_x^2 w(0,t)
\end{cases} \tag{1.14}
$$

whose solution is given by the semigroup $S_\alpha(t)$ in $L^2(0,1)$, where $S_\alpha(t)$ is generated by the operator $A_\alpha$ defined by

$$(A_\alpha u)(x) = -u'''(x), \tag{1.15}$$

with

$$\begin{aligned}\mathcal{D}(A_\alpha) &\equiv \{w \in H^3(0,1) \mid w(1) = w(0), \quad w'(1) = \alpha w'(0), \quad w''(1) = w''(0)\} \\ &\equiv H^3_\alpha \end{aligned} \tag{1.16}$$

In [27] we have already seen that $S_\alpha(t)$ is strongly differentiable semigroup. Thus any solution $w(x,t) = S_\alpha(t)\phi$ of (1.14) with $\phi \in L^2(0,1)$ is infinitely smooth in both the time and the space variables for $t > 0$, which compares with the smoothing property we mentioned earlier for the KdV equation posed on the whole line $R$. In this paper we shall establish a smoothing property of Kato type for (1.14), i.e. for any $T > 0$,

$$\int_0^T \|S_\alpha(t)\phi\|^2_{H^1(0,1)} dt \le C\|\phi\|^2_{L^2(0,1)} \tag{1.17}$$

and

$$\sup_{0\le t\le T} \| \int_0^t S_\alpha(t-\tau) f(.,\tau) d\tau \|_{H^1(0,1)} \le C \left( \int_0^T \|f(.,\tau)\|^2_{H^1(0,1)} d\tau \right)^{\frac{1}{2}}. \tag{1.18}$$

Some other type of smoothing properties are also given in the paper such as the global smoothing effect of Stricharz type. These smoothing properties will be used to prove the well posedness of the initial boundary value problem

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x^3 u = 0, \qquad 0 \le x \le 1, \ t \ge 0 \\ u(x,0) = \phi(x) \\ u(1,t) = u(0,t), \quad \partial_x u(1,t) = \alpha u(0,t), \quad \partial_x^2 u(1,t) = \partial_x^2 u(0,t) \end{cases} \tag{1.19}$$

Then we shall show that the small amplitude solutions of (1.19) decay exponentially to the mean value of its initial state $\phi$ as $t \to +\infty$. This of course is due to the dissipation mechanism we have introduced into the boundary. conditions

The paper is organized as follows. In section 2, the spectral properties of the operator $A_\alpha$ defined by (1.15) and (1.16) are studied in detail. In particular, if

$\alpha \neq -1/2$, $A_\alpha$ is shown to be a discrete spectral operator. The asymptotic form of the eigenvalues is given and the corresponding eigenfunctions are shown to form a Riesz basis for the space $L^2(0,1)$. These results are important for us in establishing the smoothing properties (1.17) and (1.18). In section 3, the various smoothing properties are given. In section 4, we prove that the initial boundary value problem (1.19) is well posed in the space $H^{3n+1}(0,1)$ for any $n \geq 0$. In section 5, the small amplitude solutions of (1.19) are shown to decay exponentially to the mean value of the initial state in the space $H^1(0,1)$ as $t \to \infty$.

# 2 Spectral properties of the third order linear system

In [27] we have studied the third order linear dispersion equation

$$\partial_t w + \partial_x^3 w = 0, \qquad 0 \leq x \leq 1,\ t \geq 0, \tag{2.1}$$

the "$x$" domain is treated as being periodic with the boundary conditions

$$w(1,t) \equiv w(0,t), \qquad \partial_x^2 w(1,t) \equiv \partial_x^2 w(0,t) \tag{2.2}$$

applying. As we have seen in section 1, these guarantee that smooth solutions conserve the volume integral

$$[w](t) \equiv \int_0^1 w(x,t)dx.$$

With the further boundary condition on the first order derivative

$$\partial_x w(1,t) \equiv \alpha \partial_x w(0,t), \qquad |\alpha| < 1 \tag{2.3}$$

we have also seen in section 1 that

$$\|w(.,t)\|_{L^2} \equiv \int_0^1 |w(x,t)|^2 dx$$

is non-increasing and is, in fact, strictly decreasing on any t-interval during which either of the derivatives in (2.3) fails to vanish almost everywhere. In [27] we showed that the operator $A_\alpha$ generates a strongly continuous semigroup $S_\alpha(t)$ on $L^2 \equiv L^2[0,1]$ for $t \geq 0$. It was further shown there that $S_\alpha(t)$ is, in fact, a $C^\infty$ semigroup for $t > 0$.

For the work to be done in this paper we require very detailed information about the eigenvalues of $A_\alpha$ and its adjoint, $A_\alpha^*$, which agrees with (2.4) except for removal of the "-" sign there and reversal of the roles played by 0 and 1 in (2.3). For use in the propositions to follow, which detail some of that information, and for subsequent use in this article we note that a vector $\phi \in L^2$ can be thought of as a linear map from from $E^1$ into $L^2$, associating with each complex number $f$ the vector $f\phi$ in $L^2$. We denote the corresponding Hilbert space adjoint by $\phi^*$; it is clear for two vectors $\phi$ and $\psi$ in $L^2$ that

$$(\phi, \psi)_{L^2} = \psi^* \phi \; ; \qquad \text{strictly speaking, } (\psi^*\phi)(1)$$

**Proposition 2.1** *Assume that $\alpha \neq -\frac{1}{2}$ or $-2$. Then the operator $A_\alpha$ defined by (2.4) and (2.5) is a discrete spectral operator, all but a finite number of whose eigenvalues $\lambda$ corresponds to one-dimensional projections $E(\lambda; T)$.*

**Proof:** The results follow directly from ([6] , pp. 2341, Theorem 13) if the hypotheses there are verified. In fact, in the case here, using the notations in [6], we have $p = 3$, $n = 3$, $\nu = 1$, $\omega_0 = 0$,

$$\omega_1 = \exp(\frac{2\pi i}{3}), \quad \omega_2 = \omega_1^2, \quad \sigma_0(x,\mu) = e^{i\mu x}$$

and

$$\sigma_1(x,\mu) = e^{i\mu\omega_1 x}, \quad \sigma_2(x,\mu) = e^{i\mu\omega_2(x-1)}.$$

Direct computations shows that

$$\begin{aligned} N(\mu) &= \det \begin{bmatrix} e^{i\mu} - 1 & -1 & 1 \\ i\mu e^{i\mu} - \alpha i\mu & -i\alpha\mu\omega_1 & i\mu\omega_2 \\ -\mu^2 e^{i\mu} + \mu^2 & \mu^2\omega_1^2 & -\mu^2\omega_2^2 \end{bmatrix} \\ &= i\mu^3(2+\alpha)\omega_1(\omega_1 - 1)e^{i\mu} + i\mu^3\omega_1(\omega_1 - 1)(2\alpha + 1) \end{aligned}$$

and

$$\begin{aligned} \hat{N}(\mu) &= \det \begin{bmatrix} e^{i\mu} - 1 & 1 & -1 \\ i\mu e^{i\mu} - \alpha i\mu & i\mu\omega_1 & -\alpha i\mu\omega_2 \\ -\mu^2 e^{i\mu} + \mu^2 & -\mu^2\omega_1^2 & \mu^2\omega_2^2 \end{bmatrix} \\ &= i\mu^3(2+\alpha)\omega_1(\omega_1 - 1)e^{i\mu} + i\mu^3\omega_1(1 - \omega_1)(2\alpha + 1). \end{aligned}$$

Hence

$$\pi_1(\mu) = i\mu^3(2+\alpha)\omega_1(\omega_1 - 1), \quad \pi_2(\mu) = \mu^3\omega_1(\omega_1 - 1)(2\alpha + 1)$$

and

$$\hat{\pi}_1(\mu) = i\mu^3(2+\alpha)\omega_1(\omega_1 - 1), \quad \hat{\pi}_2(\mu) = i\mu^3(2\alpha + 1)\omega_1(1 - \omega_1).$$

Thus, if $\alpha \neq -1/2$, or $-2$, then the hypothesis 9 and 10 in ([6], pp. 2336-2337 ) are satisfied and the proof is completed by using ([6] pp..2341, Theorem 13). □

**Proposition 2.2** *The operators $A_\alpha$, $A_\alpha^*$ with the indicated domains, have compact resolvents and, for $\alpha \neq -1/2$ and $|\alpha| < 1$, complete sets of eigenvectors, respectively,*

$$\{\phi_k \mid -\infty < k < +\infty\}, \qquad \{\psi_k \mid -\infty < k < +\infty\}$$

*which, normalized so that (with $\delta_{k,j}$, the Kronecker delta )*

$$\psi_j^* \phi_k = \delta_{k,j} \tag{2.4}$$

*form dual Riesz basis for $L^2[0,1]$. The corresponding eigenvalues of $A_\alpha$ have the asymptotic form*

$$\lambda_k = \left(8\pi^3 k^3 + O(k^2)\right) i - 12\pi^2 k^2 r + O(k), \quad k \to \infty \tag{2.5}$$

*where*

$$r = -\log\left|\frac{1+2\alpha}{2+\alpha}\right| > 0.$$

**Proof:** It is trivial that $\phi_0(x)$, $\phi_0(x) \equiv 1$ are eigenfunctions of $A_\alpha$, $A_\alpha^*$, respectively, corresponding to the eigenvalue $\lambda_0 = 0$ and that 0 is a simple eigenvalue in each case. For $Im\,\lambda > 0$ we denote the three cube roots of $-\lambda$ by $\mu_0$, $\mu_1$, $\mu_2$. These must have distinct real parts; we let $\mu_0$ be the unique root such that $\pi/3 < arg(\mu_0) < (2\pi)/3$ and see readily that among the three $|Re\,\mu_0|$ is minimal with $Re\,\mu_1 < 0$ and $Re\,\mu_2 > 0$ if we set

$$\mu_1 = e^{\frac{2\pi i}{3}}\mu_0 \equiv e_1\mu_0, \qquad \mu_2 = e_1^2\mu_0 \equiv e_2\mu_0. \tag{2.6}$$

The general solution of the characteristic equation

$$\phi'''(x) + \lambda\phi(x) = 0 \tag{2.7}$$

is then

$$\phi(x) = c_0 e^{\mu_0 x} + c_1 e^{\mu_1 x} + c_2 e^{\mu_2 (x-1)} \tag{2.8}$$

where $c_0$, $c_1$ and $c_2$ are arbitrary coefficients. Substituting (2.10) into the boundary conditions (cf. (2.2) and (2.3)

$$\phi(1) = \phi(0), \quad \phi'(1) = \alpha\phi'(0), \quad \phi''(x) = \phi''(0) \tag{2.9}$$

and neglecting the terms $e^{\mu_1}$ and $e^{-\mu_2}$ , which are very small for large $|\lambda|$, we arrive at the system of equations

$$\begin{bmatrix} 1 - e^{\mu_0} & 1 & -1 \\ (\alpha - e^{\mu_0})\mu_0 & \alpha\mu_1 & -\mu_2 \\ (1 - e^{\mu_0})\mu_0^2 & \mu_1^2 & -\mu_2^2] \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = 0. \tag{2.10}$$

Setting the determinant of the matrix equal to zero we have

$$\begin{aligned} &[1 - e^{\mu_0}] \left[-\alpha\mu_1\mu_2^2 + \alpha\mu_1\mu_0^2 + \mu_0\mu_2^2 + \mu_2\mu_1^2 - \mu_2\mu_0^2 - \mu_0\mu_1^2\right] \\ &\quad = \mu_0(1-\alpha)\left[\mu_2^2 - \mu_1^2\right] \end{aligned}$$

Making the substitutions indicated by (2.8) and using the identities $e_1^3 = e_2^3 = e_2 e_1 = 1$ we arrive at

$$[1 - e^{\mu_0}]\,\mu_0^3\,[-\alpha e_2 + \alpha e_1 + e_1 + e_1 - e_2 - e_2] = \mu_0^3(1-\alpha)[e_2 - e_1].$$

Dividing by $\mu_0^3(e_1 - e_2)$ and simplifying we have

$$e^{\mu_0} = \frac{1 + 2\alpha}{2 + \alpha}. \tag{2.11}$$

Setting

$$\hat{\mu}_{0,k} = 2\pi i[k + \epsilon_k], \qquad k = 1, 2, ...,$$

we see that $\hat{\mu}_{0,k}$ is a solution of the approximation equation (2.10) if

$$i\epsilon_k = \begin{cases} \frac{1}{2\pi}\log\left|\frac{1+2\alpha}{2+\alpha}\right|, & \text{if } \frac{1+2\alpha}{2+\alpha} > 0 \\ \frac{1}{2\pi}\log\left|\frac{1+2\alpha}{2+\alpha}\right| + \frac{1}{2}, & \text{if } \frac{1+2\alpha}{2+\alpha} < 0. \end{cases}$$

Thus

$$\hat{\mu}_{0,k} = 2\pi\hat{k}i - r \tag{2.12}$$

with

$$-r = \log \left| \frac{1+2\alpha}{2+\alpha} \right|$$

and $\hat{k} = k$ or $k + 1/2$ as required.

The exceptional value for which $\epsilon_k$ is not defined corresponds to $\alpha \neq -\frac{1}{2}$. For values of $\alpha$ with $|\alpha| < 1$ other than $-1/2$ we have

$$0 < \left| \frac{1+2\alpha}{2+\alpha} \right| < 1$$

so that the indicated logarithm is negative. Then we see, using the implicit function theorem to correct for the neglected terms referred to following (2.11), that $A_\alpha$ has eigenvalues, with $r$ given by (2.14),

$$\begin{aligned} \lambda_k &= -\mu_{0,k}^3 = 8\pi^3 \hat{k} i - 12\pi \hat{k}^2 r + O(k) \\ &= \left[ 8\pi^3 k^3 + O(k^2) \right] i - 12\pi^2 k^2 r + O(k) \end{aligned} \tag{2.13}$$

as $k \to \infty$. This gives the formula (2.7); the one to one relationship between the eigenvalues and the indices $k$ can be established using Rouché's theorem. An entirely similar argument gives the eigenvalues $\lambda_{-k}$ as the conjugates of the $\lambda_k$ shown.

We see now that the corresponding eigenfunctions of $A_\alpha$ take the form

$$\phi_0(x) = c_0 \neq 0$$

and

$$\phi_k(x) = c_k^0 e^{\mu_k^0 x} + c_k^1 e^{\mu_k^2 x} + c_k^2 e^{\mu_k^2 (x-1)}, \quad k \neq 0 \tag{2.14}$$

Dividing the third equation of the system (2.12) by $\mu_0^2$ we have

$$c_1 - c_0 = \left[ e^{\mu_0} - 1 \right] c_0,$$

$$e_2 c_2 - e_1 c_2 = \left[ e^{\mu_0} - 1 \right] c_0,$$

which gives the relationships

$$c_1 = \left[ e^{\mu_0} - 1 \right] \left[ \frac{e_1 - 1}{e_1 - e_2} \right] c_0, \tag{2.15}$$

$$c_2 = \left[ e^{\mu_0} - 1 \right] \left[ \frac{e_2 - 1}{e_1 - e_2} \right]. \tag{2.16}$$

We must, of course, remember that these are only approximate relationships obtained by neglecting exponentially small terms. Again correcting with use of the implicit function theorem we see that these relationships are, in fact, asymptotically valid as $|k| \to \infty$. From this we see that $c_k^1$ and $c_k^2$ are uniformly bounded relative to $c_k^0$, $|k| \to \infty$.

The eigenvalues of $A_\alpha^*$ corresponding to $\bar{\lambda}_k$, which we designate by $\psi_k(x)$, are easily seen to have the form

$$\psi_k(x) = \overline{\phi_k(1-x)}, \qquad -\infty < k < \infty \tag{2.17}$$

From the representation (2.10) and the boundedness of $c_k^1$ and $c_k^2$ relative to $c_k^0$, a simple computation of the integral involved shows that we may normalize the $\phi_k(x)$ by choice of the coefficients $c_k^0$, and correspondingly the $\psi_k(x)$, so that

$$\psi_j^* \phi_k \equiv (\psi_k, \phi_j)_{L^2} = \delta_{kj} \tag{2.18}$$

for $-\infty < k, j < \infty$, and in so doing, the coefficients $c_k^m$ $(m = 0, 1, 2)$ will be determined so as to be the uniformly bounded and uniformly bounded away from 0. That being the case, application of the Carleson theory [10], or the more recent extension by Komornik [19] of Ingham's classical work [11], shows that both the $\phi_k$ and the $\psi_k$ sequences have the uniform $l^2$ convergence property

$$\| \sum_{k=-\infty}^{\infty} f_k \phi_k \|^2 \le D^2 \sum_{k=-\infty}^{\infty} |f_k|^2, \tag{2.19}$$

$$\| \sum_{j=-\infty}^{\infty} g_j \psi_j \|^2 \le D^2 \sum_{j=-\infty}^{\infty} |g_j|^2, \tag{2.20}$$

relative to the indicated square summable coefficient sequences, for some $D > 0$ independent of the particular coefficient sequences $f_k$, $g_k$ in question. But then since (2.20) with the Schwarz inequality gives

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |f_k|^2 &= \left( \sum_{k=-\infty}^{\infty} f_k \phi_k, \sum_{k=-\infty}^{\infty} f_j \psi_j \right)_{L^2} \\ &\le \| \sum_{k=-\infty}^{\infty} f_k \phi_k \|_{L^2} \| \sum_{j=-\infty}^{\infty} f_j \psi_j \|_{L^2}, \end{aligned}$$

we quickly have, using (2.22) with the $g_j$ replaced by the $f_j$,

$$\| \sum_{k=-\infty}^{\infty} f_k \phi_k \|^2 \geq D^{-2} \sum_{k=-\infty}^{\infty} |f_k|^2, \tag{2.21}$$

and an entirely similar argument gives

$$\| \sum_{j=-\infty}^{\infty} g_j \psi_k \|^2 \geq D^{-2} \sum_{j=-\infty}^{\infty} |g_j|^2, \tag{2.22}$$

both valid for arbitrary coefficient sequences in $l^2$. According to Proposition 2.1, the $\phi_k$, and also the $\psi_j$, are complete in $L^2$ since both $A_\alpha$ and $A_\alpha^*$ are discrete spectral operators. We conclude that $\{\phi_k\}$, $\{\psi_j\}$ are Riesz bases for that space, dual to each other as indicated by the biorthogonality relation (2.20) and the proof of Proposition 2.2 is complete. □

Let us denote by $\phi_k'$, $\phi_k''$, $\psi_j'$, $\psi_j''$ etc., the derivatives of the indicated function with respect to $x$. Because the $\mu_k^j$ appearing in (2.16) are proportional in magnitude to $|k|$ as $k \to \infty$, the Carleson, Ingham-Komornik results cited earlier can be used to obtain inequalities, valid for any positive $n$,

$$\| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)} \|_{L^2}^2 \leq D_n^2 \sum_{k=-\infty}^{\infty} |k^n f_k|^2 \tag{2.23}$$

$$\| \sum_{j=-\infty}^{\infty} g_j \psi_j^{(n)} \|_{L^2}^2 \leq D_n^2 \sum_{j=-\infty}^{\infty} |j^n g_j|^2 \tag{2.24}$$

Thus the function $\phi_k'/k$, $\psi_j'/j$ possess the $l_n^2-$ uniform convergence property in $L^2$, where

$$l_n^2 = \left\{ a_k : \sum_{k=-\infty}^{\infty} |k^n a_k|^2 < \infty \right\}.$$

In general there is no completeness result but, integrating by parts and using the boundary condition (2.2), we can see that

$$\begin{aligned} -\left(\phi_k'', \psi_j'\right)_{L^2} &= \int_0^1 \phi_k''(x)\overline{\psi_j'(x)}dx = \int_0^1 \phi_k'''(x)\overline{\psi_j(x)}dx \\ &= (A_\alpha \phi_k, \psi_j) = \lambda_k (\phi_k, \psi_j) \\ &= \lambda_k \delta_{kj}, \qquad \text{for } -\infty \leq k,\ j < \infty \end{aligned}$$

and the fact that $|\lambda_k|$ are bounded above and below by fixed positive multiples of $k^3$, we see that we can find normalizing coefficients $d_k$, bounded and bounded below such that

$$|d_k|^2 \left(\phi_k''/k^2, \psi_j'/j\right)_{L^2} = \delta_{kj}, \quad k,\, j \neq 0.$$

Then an argument quite similar to that used to obtain (2.23), (2.24) allows us to see, for $n = 1,\, 2$, that

$$\begin{aligned} \| \sum_{k=-\infty}^{\infty} f_k \phi_k^{(n)} \|_{L^2}^2 &= \| \sum_{k=-\infty,\, k\neq 0}^{\infty} k^n f_k \phi_k^{(n)}/k^n \|_{L^2}^2 \\ &\geq \hat{D}_n^2 \sum_{k=-\infty,\, k\neq 0}^{\infty} \left|k^n f_k\right|^2, \end{aligned} \tag{2.25}$$

$$\begin{aligned} \| \sum_{j=-\infty}^{\infty} g_j \psi j k^{(n)} \|_{L^2}^2 &= \| \sum_{j=-\infty,\, j\neq 0}^{\infty} j^n g_j \psi_j^{(n)}/j^n \|_{L^2}^2 \\ &\geq \hat{D}_n^2 \sum_{j=-\infty,\, j\neq 0}^{\infty} \left|j^n g_j\right|^2, \end{aligned} \tag{2.26}$$

for positive coefficients $\hat{D}_n$, independent of the sequences $f_k$, $g_j$. Thus the functions $\phi_k'/k$, $\phi_k''/k^2$, $\psi_j'/j$, $\psi_j''/j^2$ for $k,\, j \neq 0$, also possess the $l_n^2$-uniform independence property, as expressed by these inequalities, failing to be Riesz bases only because they are not complete in the space $L^2$. While these results, as developed here, are valid only for $n = 1,\, 2$, by using the relationship implied by the characteristic equation (2.9), it is not hard to extend the work done here to prove

**Proposition 2.3** *With $\phi_k^{(n)}$, $\psi_j^{(n)}$ denoting the n-th derivatives of the eigenfunctions $\phi_k$ and dual eigenfunctions $\psi_j$, of $A_\alpha$ and $A_\alpha^*$, respectively, for $\alpha \neq -1/2$, there exist positive $D_n$, $\hat{D}_n$ such that the inequalities ( 2.25), (2.26) and (2.27), (2.28) are valid for all positive n.*

**Remark 2.1** *It is easy to see from the structure of $\phi_k(x)$ (cf. (2.10)) that there exist $D_n > 0$ depending only on $\alpha$ such that*

$$\sup_{0\leq x\leq 1} |k^n \phi_k^{(n)}(x)| < D_n, \quad \textit{for any } -\infty < k < \infty.$$

**Definition 2.1** *We denote by $H^n_\alpha$ the Hilbert space consisting of functions $w$ in the Sobolev space $H^n[0,1]$ which obey boundary conditions of the norm (cf. (2.2), (2.3))*

$$w^{(3j)}(1) = w^{(3j)}(0), \quad w^{(3j+2)}(1) = w^{(3j+2)}(0), \quad w^{(3j+1)}(1) = \alpha w^{(3j+1)}(0), \tag{2.27}$$

*as long as the indicated derivatives are of order $\leq n-1$. The norm and inner product in $H^n_\alpha$ are $\| \ \|_n$, $(\,.\,)_n$, inherited from $H^n[0,1]$.*

**Proposition 2.4** *For $\alpha \neq -1/2$, a function $w \in L^2$ also lies in $H^n_\alpha$ if and only if, represented in the form*

$$w = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

*we have*

$$\sum_{k=-\infty}^{\infty} |k^n c_k|^2 < \infty. \tag{2.28}$$

*Moreover, $\|w\|^2_{H^n_\alpha}$ is a equivalent to the sum*

$$\sum_{k=-\infty}^{\infty} \left[ |c_k|^2 + |k^n c_k|^2 \right],$$

*i.e., there are positive constants $e_n$, $E_n$ (these will, in general, depend on $\alpha$) such that, uniformly for $w \in H^n_\alpha$,*

$$e_n^2 \sum_{k=-\infty}^{\infty} \left[ |c_k|^2 + |k^n c_k|^2 \right] \leq \|w\|^2_{H^2_\alpha} \leq E_n^2 \sum_{k=-\infty}^{\infty} \left( |c_k|^2 + |k c_k|^2 \right). \tag{2.29}$$

**Proof:** The sufficiency of the condition (2.30) is an immediate consequence of Proposition 2.3. The necessity follows from

$$c_k = \int_0^1 w(x)\overline{\psi_k(x)}dx,$$

the form (2.8), (2.17) of $\psi_k(x)$ and repeated integration by parts, the original boundary conditions (2.2) and (2.3) ) satisfied by $w$ and the corresponding dual boundary conditions satisfied by $\psi_k(x)$ (cf. paragraph following (2.5) combining to annihilate the boundary terms arising in the course of these integration by parts operations. The proof is then complete. □

These results will be used extensively in the work to follow in later sections of this paper.

To end this section we define a class of Banach spaces which will be used in the next section to measure smoothing effects of the KdV system we are studying.

Let $\{\phi_k(x)\}$ be the Riesz basis of $L^2(0,1)$ defined in Proposition 2.2. For any $s \geq 0$ and $p \geq 1$, define

$$H_\alpha^{s,p} = \left\{ w = \sum_{k=-\infty}^{\infty} c_k \phi_k; \qquad \sum_{k=-\infty}^{\infty} (1 + |k|^{ps}) \, |c_k|^p < \infty \right\}$$

with the norm

$$\|w\|_{H_\alpha^{s,p}}^p \equiv \sum_{k=-\infty}^{\infty} |c_k|^p (1 + |k|^{ps}).$$

$H_\alpha^{s,p}$ is continuously imbedded into the space $H_\alpha^{s',p}$ if $s' < s$. Obviously, if $p \geq 2$, then

$$H_\alpha^s \subset H_\alpha^{s,p}, \qquad \text{for any } s \geq 0. \tag{2.30}$$

In the case that $p = 2$, we denote $H_\alpha^{s,p}$ by $H_\alpha^s$. Clearly, if $s = n$ is an integer, then the space $H_\alpha^s$ agrees with the space $H_\alpha^n$ defined by Definition 2.1 with the equivalent norm.

# 3 Smoothing properties of the third order linear (Airy) system

The resolution of the identity associated with the operator $A_\alpha$ defined in the preceding section is the sum, strongly convergent in $\mathcal{L}(L^2, L^2)$,

$$I = \sum_{k=-\infty}^{\infty} P_k$$

where $P_k$, $-\infty < k < \infty$, is the (generally non-orthogonal) projection

$$P_k = \phi_k \psi_k^* : \quad L^2 \to L^2.$$

The corresponding strongly convergent representation of the semigroup generated by $A_\alpha$ is

$$S_\alpha(t) = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} \phi_k \psi_k^* = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} P_k. \tag{3.1}$$

The solution of the initial value problem (IVP) associated with

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \quad x \in (0,1), \quad t \geq 0 \\ v(x,0) = \phi(x) \\ v(0,t) = v(1,t), \quad v_x(1,t) = \alpha(0,t), \quad v_{xx}(1,t) = v_{xx}(0,t) \end{cases} \tag{3.2}$$

is represented by

$$v(t) = S_\alpha(t)\phi \qquad \text{for any } t \geq 0$$

and the solution of the inhomogeneous problem

$$\begin{cases} \partial_t v + \partial_x^3 v = f, \quad x \in (0,1), \quad t \geq 0 \\ v(x,0) = 0 \\ v(1,t) = v(0,t), \quad v_x(1,t) = \alpha(0,t), \quad v_{xx}(1,t) = v_{xx}(0,t) \end{cases} \tag{3.3}$$

is given by

$$v(t) = \int_0^t S_\alpha(t-\tau) f(.,\tau) d\tau$$

In the definitions to follow $\| \; \|_s$ refers to the norm in $H_\alpha^s$ and $\| \; \|_{s,p}$ refers to the norm in $H_\alpha^{s,p}$ as introduced in the previous section.

**Proposition 3.1** *For any $s \geq 0$,*

$$\|S_\alpha(t) w_0\|_s \leq \|w_0\|_s \tag{3.4}$$

**Proof:** As we have seen in the previous section, if

$$w_0 = \sum_{k=-\infty}^{\infty} c_k \phi_k,$$

then

$$S_\alpha(t) w_0 = \sum_{k=-\infty}^{\infty} e^{\lambda_n t} c_k \phi_k.$$

Thus, according to the definition of the space $H^s_\alpha$, using the fact that $Re\,\lambda_k \le 0$ for all $k$,

$$\begin{aligned} \|S_\alpha(t)w_0\|_s^2 &= \sum_{k=-\infty}^{\infty} (1+|k|^{2s}) e^{2Re\,\lambda_k t} |c_k|^2 \\ &\le \sum_{k=-\infty}^{\infty} (1+|k|^{2s})|c_k|^2 \\ &= \|w_0\|_s^2 \end{aligned}$$

for any $t \ge 0$. The proof is completed. □

**Proposition 3.2** *Let $s \ge 0$ and $T > 0$ be given. Then for any $w_0 \in H^s_\alpha$,*

$$\int_0^T \|S_\alpha(t)w_0\|_{s+1}^2 dt \le B_n^2 \|w_0\|_s^2 \tag{3.5}$$

*for some $B_n >$ depending only on $n$.*

**Remark 3.1** *Except for the $\phi_0$ component of $S_\alpha(t)$, which remains constant, it would be possible to replace $T$ by $\infty$.*

**Remark 3.2** *The inequality (3.5) is analogous to the Kato type smoothing effect (but is global). We refer to [14] for the local Kato type smoothing property for solutions of the KdV equation posed on the whole real line $R$. The result (3.5) together with the results of [27] to the effect that $S_\alpha$ is a $C^\infty$ semigroup, (easily re-proved with the framework used here except for the singular case $\alpha = -1/2$) expresses the fundamental smoothing property of the linear equation (2.1) with the boundary condition (2.2) and (2.3).*

**Remark 3.3** *The results of [27] also apply in the singular case ($\alpha = -1/2$); in fact, most of the results of the present paper should be extendable to that case once the appropriate framework for proving them has been found since the singular case should have even stronger regularity and smoothing properties than have been obtained in the "regular" case studied here.*

**Proof of Proposition 3.2:** From (2.7) we see that there is a positive $\beta > 0$ such that

$$Re\,\lambda_k \le -\beta k^2, \qquad \text{for any integer } k \ne 0. \tag{3.6}$$

Since

$$S_\alpha(t)w_0 = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} c_k \phi_k$$

where

$$w_0 = \sum_{k=-\infty}^{\infty} c_k \phi_k,$$

$$\begin{aligned}
\int_0^T \|S_\alpha(t)w_0\|_{s+1}^2 dt &\leq \sum_{k=-\infty}^{\infty} \int_0^T e^{2Re\,\lambda_k t} dt (1+|k|^{2s+2})|c_k|^2 \\
&\leq \sum_{k=-\infty}^{\infty} |c_k|^2 (1+|k|^{2s+2}) \int_0^T e^{-\beta k^2 t} dt \\
&\leq T|c_0|^2 + \sum_{k=-\infty,\,k\neq 0}^{\infty} |c_k|^2 \left[\frac{1+|k|^{2s+2}}{2\beta k^2}\right] \\
&\leq B_n^2 \sum_{k=-\infty}^{\infty} (1+|k|^{2s})|c_k|^2 \\
&= B_n^2 \|w_0\|_s^2.
\end{aligned}$$

This completes the proof of the proposition. □

**Proposition 3.3** *Let $s \geq 0$ and $T > 0$ be given. Then, uniformly for $f \in L^2(0,T;H_\alpha^s)$,*

$$\sup_{0\leq t\leq T} \| \int_0^t S_\alpha(t-\tau) f(.,\tau) d\tau \|_{s+1} \leq B_s \left[ \int_0^T \|f(.,\tau)\|_s^2 d\tau \right]^{\frac{1}{2}} \tag{3.7}$$

*for some $B_s > 0$ (depending on $\alpha$ and $T$ in general).*

**Remark 3.4** *If the $\phi_o$ component $f_0(t)$ is zero, we can replace $T$ in (3.7) by $+\infty$.*

**Proof of Proposition 3.3:** Note that

$$\int_0^t S_\alpha(t-\tau) f(.,\tau) d\tau = \sum_{k=-\infty}^{\infty} \int_0^t e^{\lambda_k (t-\tau)} f_k(\tau) d\tau \phi_k$$

where

$$f(x,t) = \sum_{k=-\infty}^{\infty} f_k(t) \phi_k(x).$$

Using (3.6) we see that

$$\sup_{0\le t\le T} \| \int_0^t S_\alpha(t-\tau) f(.,\tau) d\tau \|_{s+1}^2$$

$$\le \sum_{k=-\infty}^{\infty} \sup_{0\le t\le T} \left( \int_0^t e^{Re\,\lambda_k(t-\tau)} |f_k(\tau)| d\tau \right)^2 \left(1+|k|^{2s+2}\right)$$

$$\le \sum_{k=-\infty}^{\infty} \sup_{0\le t\le T} \int_0^t e^{2Re\,\lambda_k(t-\tau)} d\tau \int_0^T |f_k|^2 d\tau \left(1+|k|^{2s+2}\right)$$

$$\le T\int_0^T |f_0(\tau)|^2 d\tau + \sum_{k=-\infty,\,k\neq 0}^{\infty} \sup_{0\le t\le T} \int_0^t e^{-2\beta k^2(t-\tau)} d\tau \int_0^T |f_k(\tau)|^2 d\tau (1+|k|^{2s+2})$$

$$\le T\int_0^T |f_0|^2 d\tau + \sum_{k=-\infty,\,k\neq 0}^{\infty} \frac{1+|k|^{2s+2}}{2\beta k^2} \int_0^T |f_k(\tau)|^2 d\tau$$

$$\le B_s^2 \sum_{k=-\infty}^{\infty} (1+|k|^{2s}) \int_0^T |f_k(\tau)|^2 d\tau$$

$$= B_s^2 \int_0^T \|f(.,\tau)\|_s^2 d\tau.$$

The proof is completed. □

**Proposition 3.4** *Let $s\ge 0$ be given. Then there exists a $B_s>0$ such that for any $f\in L^\infty(0,\infty;H_\alpha^s)$ with*

$$\int_0^1 f(x,t)dx \equiv 0,$$

$$\sup_{0\le t<\infty} \| \int_0^t S_\alpha(t-\tau) f(.,\tau) d\tau \|_{s+1} \le B_s \sup_{0\le t<+\infty} \|f(.,t)\|_s. \tag{3.8}$$

**Proof:** Defining

$$t,1 = \max\{t-1,0\},$$

we are concerned with

$$\begin{aligned} \int_0^t S_\alpha(t-\tau)f(\tau)d\tau &= \int_{t,1}^t S_\alpha(t-\tau)f(\tau)d\tau + \int_0^{t,1} S_\alpha(t-\tau)f(\tau)d\tau \\ &\equiv \hat{h}(.,t) + h(.,t). \end{aligned}$$

In view of (3.7),

$$\begin{aligned} \|\hat{h}(.,t)\|_{s+1}^2 &\le B^2 \int_{t,1}^t \|f(.,\tau)\|_s^2 d\tau \\ &\le B^2 \sup_{0\le\tau<\infty} \|f(.,\tau)\|_s^2. \end{aligned} \tag{3.9}$$

For $t \leq 1$ this is all that is required, otherwise we also need to obtain a comparable estimate for

$$h(.,t) \equiv \int_0^{t,1} S_\alpha(t-\tau)f(.,\tau)d\tau = \int_0^{t-1} S_\alpha(t-\tau)f(.,\tau)d\tau.$$

Since $\int_0^1 f(x,t)dx \equiv 0$,

$$f(.,\tau) = \sum_{j=-\infty,\, j\neq 0}^{\infty} f_j(\tau)\phi_j$$

and

$$h(.,t) = \sum_{j=-\infty,\, j\neq 0}^{\infty} \int_0^{t-1} e^{\lambda_j(t-\tau)} f_j(\tau)d\tau\phi_j.$$

Now for $j \neq 0$, we can obtain, for some $\beta > 0$ and for $\tau \in [0, t-1]$, a uniform estimate

$$|(1+|k|^2)|e^{\lambda_k(t-\tau)}|^2 \leq G^2|e^{\lambda_k(t-1-\tau)}|^2$$

where

$$G^2 = \sup_{j\neq 0}\left\{j^2 e^{2Re\,\lambda_j}\right\} < \infty.$$

Then

$$\begin{aligned}
&\sum_{j=-\infty,\, j\neq 0}^{\infty} (1+|j|^{2s+2})|e^{\lambda_j(t-\tau)}f_j(\tau)|^2 \\
&\qquad \leq \sum_{j=-\infty,\, j\neq 0}^{\infty} (1+|j|^2)|e^{\lambda_j(t-\tau)}|^2(1+|j|^{2s})|f_j(\tau)|^2 \\
&\qquad \leq G^2 e^{-2\beta(t-1-\tau)} \sum_{k=-\infty}^{\infty} (1+|k|^{2s})|f_k(\tau)|^2 \\
&\qquad = G^2 e^{-2\beta(t-1-\tau)} \|f(.,\tau)\|_s^2
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
\| \int_0^{t-1} \sum_{k=-\infty}^{\infty} e^{\lambda_k(t-\tau)} f_k(\tau)d\tau\phi_k\|_{s+1}^2 &\leq \sum_{k=-\infty}^{\infty} \left|\int_0^{t-1} e^{\lambda_k(t-\tau)} f_k(\tau)d\tau\right|^2 (1+|k|^{2s+2}) \\
&\leq G^2 \left(\int_0^{t-1} e^{-\beta(t-1-\tau)}d\tau \sum_{k=-\infty}^{\infty} \sup_{0\leq\tau\leq t-1} (1+|k|^{2s})|f_k(\tau)|^2\right) \\
&\leq \frac{G^2}{\beta^2} \sup_{0\leq t<\infty} \|f(.,t)\|_s. \qquad (3.10)
\end{aligned}$$

Combining (3.9) and (3.10) yields

$$\sup_{0\le t<\infty} \| \int_0^t S_\alpha(t-\tau)f(.,\tau)d\tau\|_{s+1} \le B_s \sup_{0\le t<\infty} \|f(.,t)\|_s$$

with some $B_s > 0$. The proof is completed. □

The next proposition will be used to establish a smoothing property for solutions of the nonlinear system in the next section.

**Proposition 3.5** *Let $s \ge 0$ and $T > 0$ be given. Then for any $s' > s + 3/4$ there is a constant $B_{s'} > 0$ such that for any $w_0 \in H_\alpha^{s',4}$,*

$$\int_0^T \|S_\alpha(t)w_0\|_{s+1}^4 dt \le B_{s'}^4 \|w_0\|_{s',4}^4 \tag{3.11}$$

*where $B_{s'} \to +\infty$ as $s' \to s + 3/4$.*

**Proof:** As we have already seen,

$$S_\alpha(t)w_0 = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} c_k \phi_k$$

if

$$w_0 = \sum_{k=-\infty}^{\infty} c_k \phi_k.$$

Accordingly,

$$\begin{aligned}
\|S_\alpha(t)w_0\|_{s+1}^4 &\le \left( \sum_{k=-\infty}^{\infty} (1+|k|^{2s+2})|c_k|^2 e^{2Re\,\lambda_k t} \right)^2 \\
&\le C_{s'} \sum_{k=-\infty}^{\infty} e^{4Re\,\lambda_k t}|c_k|^4 (1+|k|^{2s+2})^2 (1+|k|^\epsilon)^2
\end{aligned}$$

where

$$C_{s'} = \sum_{k=-\infty}^{\infty} \frac{1}{(1+|k|^\epsilon)^2}$$

with $\epsilon = 2(s'-s)-1$. Thus

$$\begin{aligned}
\int_0^T \|S_\alpha(t)w_0\|_{s+1}^4 dt &\le C_{s'} \left( |c_0|^4 + \sum_{k=-\infty,\, k\ne 0}^{\infty} |c_k|^4 \frac{(1+|k|^{2s+2})^2}{4|Re\,\lambda_k|} (1+|k|^\epsilon)^2 \right) \\
&\le B_{s'}^4 \sum_{k=-\infty}^{\infty} (1+|k|^{4(s+\frac{1+\epsilon}{2})})|c_k|^4 \\
&= B_{s'}^4 \|w_0\|_{s',4}^4
\end{aligned}$$

for some $B_{s'} > 0$ which only depends on $s'$ and tends to $+\infty$ as $s' \to s + 3/4$. The proof is completed. □

To end this section we give two more smoothing properties of solutions of the linear system which are analogous to smoothing properties established in [17].

**Proposition 3.6** *Let $n \geq 0$, an integer, $T > 0$ and $s > n + 1/2$ be given. Then there exists a constant $B > 0$ depending only on $s$ such that for any $w_0 \in H^s_\alpha$,*

$$\sup_{0 \leq x \leq 1} \int_0^T \left|\partial_x^{n+1} S_\alpha(t) w_0\right|^2 dt \leq B^2 \|w_0\|_s^2 \tag{3.12}$$

**Proof:** Since

$$S_\alpha w_0 = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} c_k \phi_k(x)$$

where

$$w_0 = \sum_{k=-\infty}^{\infty} c_k \phi_k,$$

$$\partial_x^{n+1} S_\alpha(t) w_0 = \sum_{k=-\infty}^{\infty} e^{\lambda_k} c_k \phi^{(n+1)}(x).$$

Thus

$$\begin{aligned}
&\sup_{0 \leq x \leq 1} \int_0^T \left|\partial_x^{(n+1)} S_\alpha(t) w_0\right|^2 dt \leq \sup_{0 \leq x \leq 1} \int_0^T \left(\sum_{k=-\infty}^{\infty} e^{Re\,\lambda_k t} |c_k| \left|\phi^{(n+1)}(x)\right|\right)^2 dt \\
&\leq \sup_{0 \leq x \leq 1} \int_0^T \left(\sum_{k=-\infty}^{\infty} e^{2Re\,\lambda_k t} |c_k|^2 \left|\phi^{(n+1)}(x)\right| (1 + |K|^{2(s-n)})\right) \sum_{k=-\infty}^{\infty} (1 + |k|^{2(s-n)})^{-1} \\
&\leq \left(\sum_{k=-\infty}^{\infty} (1 + |k|^{2(s+1)}) |c_k|^2 \int_0^T e^{2Re\,\lambda_k t} dt \sup_{0 \leq x \leq 1} \frac{|\phi^{(n+1)}(x)|^2}{1 + |k|^{2(n+1)}}\right) \left(\sum_{k=-\infty}^{\infty} (1 + |k|^{2(s-n)})^{-1}\right) \\
&\leq B^2 \sum_{k=-\infty}^{\infty} |c_k|^2 (1 + |k|^{2s}) \\
&= B^2 \|w_0\|_s^2
\end{aligned}$$

for some $B > 0$ depending on $s$, $n$ and $T$. The proof is completed. □

**Proposition 3.7** *Let $n \geq 0$ and $T > 0$ be given and $s > n + 1/2$. Then for any $w_0 \in H^s_\alpha$,*

$$\int_0^1 \sup_{0 \leq t \leq T} |\partial_x^n S_\alpha(t) w_0|^2 \, dx \leq B^2 \|w_0\|_s^2 \tag{3.13}$$

*where $B > 0$ depends only on $s$, $T$ and $n$.*

**Proof:** Let

$$w_0 = \sum_{k=-\infty}^{\infty} c_k \phi_k.$$

Then

$$\begin{aligned}
\int_0^1 \left( \sup_{0 \leq t \leq T} |\partial_x^n S_\alpha(t) w_0| \right)^2 dx &\leq \int_0^1 \left( \sum_{k=-\infty}^{\infty} |c_k| \left| \phi_k^{(n)}(x) \right| \right)^2 dx \\
&\leq B' \left( \sum_{k=-\infty}^{\infty} (1 + |k|^n)|c_k| \right)^2 \\
&\leq B'' \sum_{k=-\infty}^{\infty} (1 + |k|^{2(s-n)})^{-1} \sum_{k=-\infty}^{\infty} (1 + |k|^{2s})|c_k|^2 \\
&= B^2 \|w_0\|_s^2.
\end{aligned}$$

The proof is completed.

# 4 Well-posedness of the nonlinear system

We propose now to study the initial value problem for the KdV equation

$$\begin{cases} \partial_t w + w \partial_x w + \partial_x^3 w = 0, & 0 < x < 1, \ t \geq 0 \\ w(x,0) = w_0(x) \end{cases} \tag{4.1}$$

with the "$\alpha$" boundary condition

$$w(1,t) = w(0,t), \quad \partial_x(1,t) = \alpha \partial_x(0,t), \quad \partial_x^3 w(1,t) = \partial_x^3(0,t), \tag{4.2}$$

where $\alpha \neq -1/2$ and $|\alpha| < 1$.

In the equation (4.1), if we move the nonlinear term $w\partial_x w$ to the right side of the equation and treat it as an inhomogeneous term, then the *variation of parameters* formula yields for $w$ the equivalent integral equation

$$w(.,t) = S_\alpha(t)w_0 - \int_0^t S_\alpha(t-\tau)(w\partial_x w)(.,\tau)d\tau. \tag{4.3}$$

We study this equation by the familiar method of looking for a fixed point of the corresponding map

$$\begin{aligned} w(.,t) &= (Fu)(.,t) \\ &:= S_\alpha(t)w_0 - \int_0^t S_\alpha(t-\tau)\left((u\partial_x u)(.,\tau)d\tau\right. \end{aligned} \tag{4.4}$$

in which $w_0$ appears as a parameter and is considered fixed (cf. [17] and [18]).

First we prove that the IVP of (4.1) - (4.2) is well-posed in the space $H^1_\alpha$.

**Theorem 4.1** *For any $w_0 \in H^1_\alpha$ there exists a $T = T(\|w_0\|_1) > 0$ such that the IVP (4.1) - (4.2) has a unique solution*

$$u \in X_T := C(0,T;H^1_\alpha) \cap L^\infty(0,T;H^1_\alpha)$$

*where $T \to \infty$ as $\|w_0\|_1 \to 0$; (ii). For any $T' < T$, there exists a neighborhood $U$ of $\phi$ in $H^1_\alpha$ such that the map*

$$K: \quad w_0 \to u(.,t)$$

*from $U$ to $X_{T'}$ is Lipschitz continuous.*

**Remark 4.1** *In fact we are able to show that the map $K$ is analytic from $U$ to $X_{T'}$ (cf. [36]).*

**Proof of Theorem 4.1:** Let

$$S_{T,b} = \left\{ v \in X_T | \quad \sup_{0 \le t \le T} \|v(.,t)\|_1 \le b \right\} \tag{4.5}$$

for some $b > 0$, $T > 0$ to be determined. We shall show that for appropriately chosen $b$ and $T$, the map defined by (4.4) is a contraction from $S_{T,b}$ to $S_{T,b}$ and as a consequence, its fixed point is the desired solution of the IVP (4.1) - (4.2).

Applying (3.06) and (3.11) to (4.04) yields

$$\begin{aligned}\sup_{0\le t\le T}\|Fv\|_1 &\le c\|w_0\|_1 + c\left(\int_0^T \|v\partial_x v\|^2 dt\right)^{1/2} \\ &\le c\|w_0\|_1 + cT^{/12}\left(\sup_{0\le t\le T}\|v(t)\|_1\right)^2 \end{aligned} \tag{4.6}$$

for some $c > 0$ independent of $T$ and $v$.

If one chooses

$$b = 2c\|w_0\|_1 \tag{4.7}$$

and $T > 0$ such that

$$cT^{1/2}b < 1/2 \tag{4.8}$$

then it follows from (4.6) that

$$\sup_{0\le t\le T}\|Fv\|_1 \le b$$

and therefore $F$ is a map from $S_{T,b}$ into $S_{T,b}$. For $v_1$, $v_2 \in S_{T,b}$, let

$$w = v_1 - v_2.$$

Then

$$Fv_1 - Fv_2 = -\int_0^t S_\alpha(t-\tau)\left(v_1\partial_x w + w\partial_x v_2\right)(\tau)d\tau.$$

Using (3.7) we obtain

$$\begin{aligned}\sup_{0\le t\le T}\|Fv_1 - Fv_2\|_1 &\le c\left(\int_0^T \|v_1\partial_x w + w\partial_x v_2\|^2 dt\right)^{1/2} \\ &\le cT^{1/2}\left(\sup_{0\le t\le T}\|v_1(t)\|_1 + \sup_{0\le t\le T}\|v_2(t)\|_1\right)\sup_{0\le t\le T}\|w\|_1 \\ &\le 2cbT^{1/2}\sup_{0\le t\le T}\|w\|_1 \\ &< \sup_{0\le t\le T}\|v_1 - v_2\|_1 \qquad \text{by (4.8).}\end{aligned}$$

Hence $F$ is a contraction map from $S_{T,b}$ to $S_{T,b}$. As a consequence, it has a unique fixed point which is then the unique solution of (4.1) - (4.2).

According to (4.7) and (4.8), for a given $w_0 \in H^1_\alpha$, one may choose

$$T = \rho\,(4c\|w_0\|_1)^{-2}$$

with $\rho$ being a given number between 0 and 1. Using this choice of $T$ we $T \to \infty$ as $\|w\|_1 \to 0$.

Finally, it is obvious that for any $T' < T$ there is a neighborhood $U$ of $w_0$ in $H^1_\alpha$ such that the map $K$ is well-defined from $U$ to $X_{T'}$. For any $w_1$, $w_2 \in U$, let

$$u_1 = Kw_1, \qquad u_2 = Kw_2$$

and $w = u_1 - u_2$. Then

$$w = S_\alpha(t)(w_1 - w_2) - \int_0^t S_\alpha(t-\tau)(u_1\partial_x w + w\partial_x u_2)(\tau)d\tau$$

Using (3.4) and (3.7) we have

$$\begin{aligned}\sup_{0\le t\le T}\|w(t)\|_1 &\le c\|w_1 - w_2\|_1 + cT^{1/2}\left(\sup_{0\le t\le T'}\|u_1(t)+\right.\\ &+ \left.\sup_{0\le t\le T'}\|u_2(t)\|_1\right)\sup_{0\le t\le T}\|w(t)\|_1\end{aligned}$$

which, together with (4.07) and (4.08) with $T$ replaced by $T' < T$, shows $K$ is a Lipschitz continuous from $U$ to $X_{T'}$. The proof is completed.

The next theorem shows that the solution of the nonlinear system (4.1) - (4.2) also possesses a smoothing property.

**Theorem 4.2** *Let $s > 1/2$ be given. (i) . For any $w_0 \in H^{s,4}_\alpha$ there exists a $T = T(\|w_0\|_{s,4}) > 0$ such that (4.1) - (4.2) has a unique solution*

$$u \in Y_T \equiv C(0,T; L^2 \cap L^4(0,T; H^1_\alpha)$$

*and $T \to \infty$ as $\|w_0\|_{s,4} \to 0$; (ii) For any $T' < T$, there exists a neighborhood $U$ of $w_0$ in $H^{s,4}_\alpha$ such that the map*

$$K:\ w_0 \to u$$

*from $U$ to $Y_T$ is Lipschitz continuous.*

**Proof:** It follows from (4.04) , using (3.11), that

$$\begin{aligned}\left(\int_0^T \|Fu\|_1^4 dt\right)^{1/4} &\le c\|w_0\|_{s,4} + \left(\int_0^T \|\int_0^t S_\alpha(t-\tau)(u\partial_x u)d\tau\|_1^4 dt\right)^{1/4}\\ &\le c\|w_0\|_{s,4} + T^{1/4}\sup_{0\le t\le T}\|\int_0^t S_\alpha(t-\tau)(u\partial_x u)(\tau)d\tau\|_1\\ &\le c\|w_0\|_{s,4} + cT^{1/4}\left(\int_0^T \|u\partial u\|^2 dt\right)^{1/2}\\ &\le c\|w_0\|_{s,4} + cT^{1/4}\left(\int_0^T \|u\|_1^4 dt\right)^{1/2}.\end{aligned}$$

Thus if one chooses

$$b = 2c\|w_0\|_{s,4}$$

and $T > 0$ such that

$$2cT^{1/4} < 1$$

and if one defines

$$S_{T,b} = \left\{ v \in Y_t;\quad \left(\sup_{0\le t\le T}\|v(.,t)\|^2 + \left(\int_0^T \|v\|_1^4\right)^{1/2}\right)^{1/2} < b \right\},$$

then $F$ is a map from $S_{T,b}$ into $S_{T,b}$. The rest of the proof is similar to that of Theorem 4.1 and is therefore omitted here. □

In the remaining part of this section, we discuss the regularity of the IVP (4.1) - (4.2). More precisely, does the solution $u(.,t) \in H_\alpha^n$ for any $n > 0$ if its initial state $\phi \in H_\alpha^n$? The answer is affirmative for the corresponding linear case (see Proposition 3.1 in section 3). However, in the nonlinear case, because of the special structure of the space $H_\alpha^n$, $u \in H_\alpha^n$ does not imply that $u_x \in H_\alpha^{n-1}$ when $n \ge 2$, we cannot apply the estimates (3.7)-(3.8) directly to (4.4) and use the same argument in the proof of Theorem 4.1 to obtain the existence of the solution in the space $H_\alpha^n$. Instead, from inspection of the equation, it is easy to see that $\partial_t u$ and $\partial_x^3 u$ should be in the same space $H^n$, and therefore, alternatively, we may establish the regularity of the solution by obtaining the regularity of $\partial_t u$.

The following lemma is needed for our purpose.

**Lemma 4.1** *Suppose that $f \in C[0,T;H^1_\alpha]$ and $\partial_t f \in L^2[0,T;L^2]$. Then*

$$\partial_t \left(\int_0^t S_\alpha(t-\tau)f(.,\tau)d\tau\right) = S_\alpha(t)f(.,0) + \int_0^t S_\alpha(t-\tau)\dot{f}(.,\tau)d\tau, \tag{4.9}$$

*where*

$$\dot{f}(.,t) = \partial_t f(.,t),$$

*and, as a consequence,*

$$\sup_{0\le t\le T} \|\partial_t \int_0^t S_\alpha(t-\tau)f(.,\tau)\|^1_\alpha \le c\|f(.,0)\|_1 - c\left(\int_0^T \|\dot{f}(.,\tau)\|^2_{L^2}dt\right)^{1/2} \tag{4.10}$$

*where $c > 0$ is independent of $T$. Moreover if $\int_0^1 \dot{f}(x,t)dx \equiv 0$, then*

$$\sup_{0\le t<\infty} \|\partial_t \int_0^t S_\alpha(t-\tau)f(.,\tau)d\tau\|_1 \le c\|f(.,0)\|_1 + c \sup_{0\le t<\infty} \|\dot{f}(.,\tau)\|_{L^2}. \tag{4.11}$$

**Proof:** Denote by

$$u = \int_0^t S_\alpha(t-\tau)f(.,\tau)d\tau,$$

then

$$\begin{cases} \partial_t u + \partial_x^3 u = f \\ u(x,0) = 0 \\ u(1,t) = u(0,t), \quad \partial_x u(1,t) = \alpha\partial_x(0,t), \quad \partial_x^2 u(1,t) = \partial_x^2 u(0,t) \end{cases} \tag{4.12}$$

and $v = \partial_t u$ solves

$$\begin{cases} \partial_t v + \partial_x^3 v = \dot{f} \\ v(x,0) = f(x,0) \\ v(1,t) = v(0,t), \quad \partial_x v(1,t) = \alpha\partial_x(0,t), \quad \partial_x^2 v(1,t) = \partial_x^2 v(0,t) \end{cases} \tag{4.13}$$

from which (4.9) follows. As for (4.10) and (4.11), they follow from applying (3.7) and (3.8) to (4.12) and (4.13) respectively. The proof is completed. □

For a given $T > 0$ we introduce a Banach space $Y_T$ as follows:

$$Y_T = \left\{ u \in C^1(0,T;H^1_\alpha); \quad \sup_{0\le t\le T} \|u(t)\|_1 < \infty, \quad \sup_{0\le t\le T} \|\dot{u}(t)\|_1 < \infty \right\}$$

with the norm

$$\|u\|_{Y_T} := \left( \sup_{0\le t\le T} \|u(t)\|_1^2 + \sup_{0\le t\le T} \|\dot{u}(t)\|_1^2 \right)^{1/2}.$$

In addition, let

$$X = \{\phi \in H^4 \cap H^3_\alpha, \quad \phi''' + \phi'\phi \in H^1_\alpha\}$$

with inherited topology from $H^4$. Note that $X$ is not necessarily a Banach space. But it is a closed subset of $H^4$ and is a complete metric space with ithe nherited metric from the norm of $H^4$.

**Theorem 4.3** *For any $\phi \in X$ there exists a $T = T(\|\phi\|_X) > 0$ such that (4.1)-(4.2) has a unique solution $u \in Y_T$ and for any $0 < T' < T$, there is a neighborhood $U$ of $\phi$ in $X$ such that the map $K: \phi \to u$ is Lipschitz continuous from $U$ to $Y_T$.*

**Proof:** Let

$$S_{T,B} = \{u \in Y_T| \quad \|u\|_{Y_T} \le b, \quad u(x,0) = \phi(x)\}$$

for some $T > 0$ and $b > 0$ to be determined. Obviously $S_{T,b}$ is a closed covex subset of $Y_T$. We define a nonlinear map $F$ on $S_{T,b}$:

$$Fv = S_\alpha(t)\phi - \int_0^t S_\alpha(t-\tau)(v\partial_x v)(\tau)d\tau, \qquad v \in S_{T,b}.$$

As in the proof of Theorem 4.1,

$$\sup_{0\le t\le T} \|Fv\|_1 \le c\|\phi\|_1 + cT^{1/2}\left(\sup_{0\le t\le T} \|v(t)\|_1\right)^2 \tag{4.14}$$

According to Lemma 4.1,

$$\partial_t(Fv) = -S_\alpha(t)(\phi''' + \phi\phi') - \int_0^t S_\alpha(t-\tau)\left(\partial_t(v\partial_x v)\right)(\tau)d\tau$$

and

$$\begin{aligned} \sup_{0\le t\le T} \|\partial_t F(v)\|_1 &\le c\|\phi''' + \phi\phi'\|_1 + c\left(\int_0^T \|\partial_t(v\partial_x v)\|_{L^2}^2 dt\right)^{1/2} \\ &\le c\left(\|\phi\|_4 + \|\phi\|_2^2\right) + cT^{1/2} \sup_{0\le t\le T}\sup_{0\le t\le T} \|\dot{v}\|_1\|v\|_1 \end{aligned} \tag{4.15}$$

Combining (4.9) and (4.10) yields

$$\|Fv\|_{Y_T} \leq c\left(\|\phi\|_4 + \|\phi\|_2^2\right) + cT^{1/2}\|v\|_{Y_T}^2$$

for some $c > 0$ independent of $T$ and $v$. Choosing

$$b = 2c(\|\phi\|_4 + \|\phi\|_2^2) \tag{4.16}$$

and $T > 0$ such that

$$cT^{1/2}b < 1/2, \tag{4.17}$$

$F$ is seen to be a map from $S_{T,b}$ into $S_{T,b}$. The contraction property of the map $F$ follows similarly. As a consequence, the map $F$ has a fixed point $u \in Y_T$ which is the unique solution of (4.1)-(4.2). It follows from the equation

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0$$

that $u \in C(0,T;H^4)$ since $\partial_t u \in C(0,T;H_\alpha^1)$. Besides , $u \in C(0,T;H_\alpha^3)$ since

$$u(t) = S_\alpha(t)\phi - \int_0^t S_\alpha(t-\tau)(u\partial_x u)(\tau)d\tau.$$

The Lipschitz continuity of the map $K$ follows similarly as in the proof of Theorem 4.1 and is therefore omitted here. The proof is completed. □

The method used in the above proof can be used to obtain smooth solutions of (4.1)-(4.3) while assuming the initial value $\phi$ is smooth and satisfies the appropriate compatibility conditions. We will only state these results here.

Let us define a series of differential operators $\{P_k\}_0^n$ as follows:

$$\begin{cases} P_0(\phi) = \phi \\ P_1(\phi) = \frac{1}{2}\partial_x(\phi^2) + \partial_x^3\phi \\ \cdots\cdots\cdots \\ P_n(\phi) = \frac{1}{2}\partial_x\left(\sum_{k=0}^{n-1}\begin{pmatrix} k \\ n-1 \end{pmatrix} P_k(\phi)P_{n-k}(\phi)\right) + \partial_x^3 P_{n-1}(\phi) \end{cases} \tag{4.18}$$

for any $\phi \in H^{3n}$.

**Theorem 4.4** *Let $n \geq 1$ be given. For any $\phi \in H^3_\alpha \cap H^{3n+1}$ satisfying*

$$P_k(\phi) \in H^3_\alpha \cap H^{3(n-k)+1}, \quad k = 0, 1, 2, ..., n-1$$

*and*

$$P_n(\phi) \in H^1_\alpha,$$

*there exists a $T = T(\|\phi\|_{3n+1}) > 0$ such that (4.1)-(4.2) has a unique solution*

$$u \in C(0, T; H^3_\alpha \cap H^{3n+1}).$$

*Moreover,*

$$\partial^k_t u \in C(0, T; H^3_\alpha \cap H^{3(n-k)+1}), \quad k = 1, 2, ..., n-1$$

*and*

$$\partial^n_t u \in C(0, T; H^1_\alpha).$$

# 5 Global existence and exponential decay of small amplitude solutions

The theorems proved in the previous section are local results since the time interval $(0, T)$ on which the solution exists depends on the size of its initial value. The further question is whether we have global existence results valid on an interval $(0, T)$ independent of the initial data. Usually, the global existence results follow from the local existence results and some global a priori estimate of solutions. Unfortunately, those needed global a priori estimates are not available so far. Hence, in general, we do not know whether the solutions of (4.1)-(4.3) exist for all time $t > 0$ or if they blow up at some finite time. This is not uncommon in nonlinear dispersive wave systems where a global priori estimate is not available (see [18], for instance, for the generalized Korteweg-de Vries equation). In this section, we shall see that if a solution blows up in finite time, then there exists a $T^* > 0$ such that the solution exists on the interval $(0, T^*)$ and the norm of the solution tends to $+\infty$ as $t \to T^*$. This $T^*$ is usually called the life span of the solution. On the other hand, if the size of the initial value is small enough, then the solution exists for all time $t > 0$ and , in addtion, decays exponentially to a constant as $t \to \infty$.

**Theorem 5.1** *For any $\phi \in H^1_\alpha$, there exists a $T^* \in (0, +\infty]$ such that (4.1)-(4.2) has a unique solution*

$$u \in C\left([0, T^*); H^1_\alpha\right)$$

*and*

$$\lim_{t \to T^{*+}} \|u(., t)\|_1 = +\infty$$

*if $T^* < +\infty$.*

**Proof:** According to Theorem 4.1, for any given $\phi \in H^1_\alpha$, the corresponding solution exists on an interval $(0, T)$ where $T$ depends continously only on $\|\phi\|_1$. Thus a standard argument can be used to extend the time existence interval of the solution as long as the $H^1_\alpha$ norm of the solution remains bounded. So either the solution exists for all time $t > 0$ or its $H^1_\alpha$ norm blows up at some $T^* > 0$.

**Theorem 5.2** *There exists a $\beta > 0$ such that for any $\phi \in H^1_\alpha$ with $\|\phi\|_1 \leq \beta$, (4.1)-(4.2) has a unique solution*

$$u \in C(R^+; H^1_\alpha) \cap L^\infty(R^+; H^1_\alpha)$$

*where $R^+ = [0, \infty)$.*

**Proof:** Denote by

$$S_{b,\infty} = \left\{ v \in C(R^+; H^1_\alpha) \mid \sup_{0 \leq t \leq \infty} \|v(., t)\|_1 < b \right\}$$

with $b > 0$ to be determined. Consider the map $F$ defined on $S_{b,\infty}$ as follows:

$$Fv = S_\alpha(t)\phi - \int_0^t S_\alpha(t - \tau)(v\partial_x v)(\tau) d\tau$$

Applying (3.4) and (3.8) yields

$$\begin{aligned} \sup_{0 \leq t < +\infty} \|Fv\|_1 &\leq c\|\phi\|_1 + c \sup_{0 \leq t < \infty} \|v\partial_x v\|_{L^2} \\ &\leq c\|\phi\|_1 + c\left(\sup_{0 \leq t < \infty} \|v\|_1\right)^2. \end{aligned}$$

Choose $b > 0$ and $\beta > 0$ such that

$$cb^2 \leq \frac{1}{2}b$$

and

$$c\beta \leq \frac{1}{2}b.$$

Then we have

$$\sup_{0 \leq t < \infty} \|Fv\|_1 < b$$

if $\|\phi\|_1 \leq \beta$. Thus $F$ is a map from $S_{b,\infty}$ into $S_{b,\infty}$. The contraction property of the map $F$ is obtained similarly. Its fixed point is the solution we are looking for. The proof is completed. □

A proof similar to that of Theorem 4.2 gives us the following more general result.

**Theorem 5.3** *For any $n \geq 1$, there exists a $\beta = \beta(n) > 0$ such that if $\phi \in H^3_\alpha \cap H^{3n+1}$ with $\|\phi\|_{3n+1} \leq \beta(n)$ and satisfies the compatibility condition (4.18) of Theorem 4.4 , then (4.1)-(4.2) has a unique solution*

$$u \in C(0, \infty; H^3_\alpha \cap H^{3n+1}) \cap L^\infty(0, \infty; H^{3n+1}).$$

*In addition,*

$$\partial_t^k u \in C(0, \infty; H^3_\alpha \cap H^{3(n-k)+1}), \quad k = 1, 2, ..., n-1$$

*and*

$$\partial_t^n u \in C(0, \infty; H^1_\alpha) \cap C(0, \infty; H^1_\alpha).$$

In the remaining part of this section we prove that a small amplitude solution decays exponentially to the mean value of its initial state.

**Theorem 5.4** *There exists a $\delta > 0$ such that for any $\phi \in H^1_\alpha$ with $\|\phi\|_1 < \delta$, the corresponding unique solution of (4.1)-(4.3) satisfies*

$$\|u(.,t) - [\phi]\|_{L^2} \leq ce^{-\rho t}\|\phi - [\phi]\|_{L^2}, \quad t \geq 0, \tag{5.1}$$

*where $c > 0$ and $\rho > 0$ are independent of $\phi$.*

**Proof:** We will use an infinite dimensional version of the second method of Lyapounov [12], [31] to establish this result. First of all, it will be convenient to indicate by $L_0^2$ the closed subspace of $L^2[0,1]$ consisting of functions w such that

$$\int_0^1 w(x)dx = 1$$

which can be identified in a natural way with $L^2$ modulo constant functions. Without loss of generality, we may think of the solution $u(.,t)$ as lying in the corresponding subspace $H^1_{\alpha,0}$ of $H^1_\alpha$ and we may change the inequality to be proved to

$$\|u(.,t)\|_{L_0^2} \leq ce^{-\rho t}\|\phi\|_{L_0^2}, \quad t \geq 0. \tag{5.2}$$

We first note that the operator

$$Y: \quad L_0^2 \to L_0^2$$

defined by the strongly convergent series

$$Y = \sum_{k\neq 0} Y_k, \qquad Y_k = \psi_k \psi_k^*,$$

the $\psi_k$ being, as in the section 1, the normalized eigenvector of $A_\alpha^*$, is bounded and positive define on $L_0^2$. Correspondingly we represent $w \in L_0^2$ as

$$w = \sum_{j\neq 0} c_j \phi_j$$

and see immediately that

$$w^* Y w = \sum_{j\neq 0} |c_j|^2$$

so that the boundness and strict positivity follow from (2.19) and (2.10) . If we further define

$$X: \; L_0^2 \to L_0^2$$

by

$$X = \sum_{k\neq 0} \xi_k Y_k, \qquad \xi_k = -\frac{1}{2Re\lambda_k}, \tag{5.3}$$

so that $X$ is bounded, symmetric and non-negative, then

$$\begin{aligned} A_\alpha^* X + X A_\alpha + Y &= \sum_{k\neq 0} \left[\xi_k \left((A_\alpha^* \psi_k)\psi_k^* + \psi_k(\psi_k^* A_\alpha)\right) + \psi_k \psi_k^*\right] \\ &= \sum_{k\neq 0} \left[\xi_k (2Re\lambda_k) + 1\right] \psi_k \psi_k^* \\ &= 0. \end{aligned} \tag{5.4}$$

With $u(.,t)$ the subject solution of (4.1)-(4.2), we compute

$$\begin{aligned} \frac{d}{dt}\left[u(.,t)^* X u(.,t)\right] &= \left[A_\alpha u(.,t) - u(.,t)u'(.,t)\right]^* X u(.,t) + \\ &+ u(.,t)^* X \left[A_\alpha u(.,t) - u(.,t)u'(.,t)\right] = -u(.,t)^* Y u(.,t) - \\ &- \left[u(.,t)u'(.,t)^* X u(.,t) + u(.,t)^* X (u(.,t)u'(.,t)\right]. \end{aligned} \tag{5.5}$$

For any $d > 0$ we have an estimate

$$\begin{aligned} &\left|(u(.,t)u'(.,t))^* X u(.,t) + u(.,t)^* X \left(u(.,t)u'(.,t)\right)\right| \\ &\leq d^2 u(.,t)^* X u(.,t) + d^{-2} \left(u(.,t)u'(.,t)\right)^* X \left(u(.,t)u'(.,t)\right). \end{aligned} \tag{5.6}$$

We fix $d$ so that, in the sense of quadratic forms on $L_0^2$,

$$d^2 X \leq \frac{1}{2} Y.$$

Turning our attention to the last term in (5.5), we note first of all that, since $u(1,t) \equiv u(0,t)$,

$$u(.,t)u'(.,t) = \frac{1}{2}\left(u(.,t)^2\right)' \in L_0^2.$$

We also know that $u(.,t) \in H_\alpha^1$ and that each $\psi_k$ satisfies the adjoint boundary conditions, including $\psi_k(1,t) = \psi_k(0,t)$. Thus

$$\begin{aligned} \left(u(.,t)u'(.,t)\right)^* X \left(u(.,t)u'(.,t)\right) &= \frac{1}{4}\left(u(.,t)^2\right)'^* X \left(u(.,t)^2\right)' \\ &= \frac{1}{4}\sum_{k\neq 0} \xi_k \left(u(.,t)^2\right)'^* \psi_k \psi_k^* \left(u(.,t)^2\right)' \\ &= \frac{1}{4}\sum_{k\neq 0} \xi_k \left|\int_0^1 \overline{\psi_k(x)} \partial x \left(u(x,t)^2\right) dx\right|^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{k\neq 0}\xi_k\left|-\int_0^1 \overline{\psi_k'(x)}u(x,t)^2dx + \overline{\psi_k(x)}u(x,t)^2 \,\Big|_{x=0}^{x=1}\right|^2 \\
&= \frac{1}{4}\sum_{k\neq 0}\xi_k(u(.,t)^2)^*\psi_k'\psi_k'^*(u(.,t)^2) \\
&\equiv (u(.,t)^2)^*\hat{z}(u(.,t)^2) \qquad (5.7)
\end{aligned}
$$

where

$$
\hat{z} = \sum_{k\neq 0}\xi_k\psi_k'\psi_k'^* \leq \sum_{k\neq y0}\frac{1}{2rk^2}\psi_k'\psi_k'^* \equiv z. \qquad (5.8)
$$

Now for any $z \in L_0^2$ the uniform $l^2$ independence property (2.23) of the $\psi_k'$ shows that

$$
\sum_{k\neq 0}\frac{1}{k^2}\left|\psi_k'^* z\right|^2 \leq \hat{D}_1^{\,2}\|z\|_{L_0^2}^2.
$$

Since Theorem 5.2 shows that

$$
\sup_{0\leq t<\infty}\|u(.,t)\|_1 \leq b
$$

if $\|\psi\|_1 \leq \beta$, where $\beta$ is as in Theorem 5.2 . Thus

$$
\|u(.,t)^2\|_{L_0^2} \leq b\|u(.,t)\|_{L_0^2}
$$

and we see from (5.6), (5.7) and (5.8) that

$$
\begin{aligned}
&\left|(u(.,t)u'(.,t))^*\,Xu(.,t) + u(.,t)^*X\,(u(.,t)u'(.,t))\right| \\
&\leq d^2u(.,t)^*Xu(.,t) + \hat{D}_1^{\,2}\frac{b^2}{2rd^2}\|u(.,t)\|_{L_0^2}^2 \\
&\leq d^2u(.,t)^*Xu(.,t) + \frac{1}{4}u(.,t)^*Yu(.,t) \\
&\leq \frac{3}{4}u(.,t)^*Yu(.,t)
\end{aligned}
$$

if we choose $b$ sufficiently small. Then (5.5) gives

$$
\frac{d}{dt}\left[u(.,t)^*Xu(.,t)\right] \leq -\frac{1}{4}u(.,t)^*Yu(.,t).
$$

Since we also know, taking $I_0$ to be the identity on $L_0^2$, that

$$
\frac{d}{dt}\left[u(.,t)^*I_0u(.,t)\right] \leq 0,
$$

we in fact have

$$\frac{d}{dt}\left[u(.,t)^*(I_0+X)u(.,t)\right] \leq -\frac{1}{4}u(.,t)^*Yu(.,t).$$

Since both $I_0+X$ and $Y$ are positive definite in $L_0^2$ the inequality (5.1) (equivalently (5.2) ) now follows from a familiar argument. The proof is completed. □

# References


[1] J. L. Bona and R. Smith, The initial value problem for the Korteweg-de Vries equation, *Roy. Soc. London*, Ser. A, **278**(1978), 555 – 601

[2] A. Cohen, Solutions of the Korteweg-de Vries equation from irregular data, *Duke Math. J.,* **45**(1978), 149 – 181

[3] A. Cohen and T. Kappeler, Solution to the Korteweg-de Vries equation with initial profile in $L_1^1(R)\cap L_N^1(R^+)$, *SIAM J. Math. Anal.* , **18**(1987), 991 – 1025

[4] P. Constantin and J. C. Saut, Local smoothing properties of dispersive equations, *J. Amer. Math. Soc.* **1**(1988), 413 – 446

[5] W. Craig, T. Kappeler and W. A. Strauss, Gain of regularity for equations of KdV type, *Analyse Non Linéaire, Ann. Inst. Henri Poincaré*, **9**(1992), 147 – 186

[6] N. Dunford and J. T. Schwartz, *Linear Operators,* part III, Wiley-interscience, 1971

[7] C. S. Gardner, J. M. Green, M. D. Kruskal & R. M. Miura, A method for solving the Korteweg-de Vries equation, *Phys. Rev. Letters*, **19**(1967), 1095 – 1097

[8] ————-, The Korteweg-de Vries equation and generalizations VI. Method for exact solutions, *Comm. Pure and Math.*, **27**(1974), 97 – 133

[9] J. Ginibre and G. Velo, Smoothing properties and retarded estimates for some dispersive evolution equations, preprint

[10] L. F. Ho and D. L. Russell, Admissible input elements for systems in Hilbert space and Carlson measure criterion, *SIMA J. Contr. & Opt.* , **21**(1981), 614–640

[11] A. E. Ingham, Some trigonometrical inequalities wiht application to the theory of series, *Math. Zeitschr.*, *41*(1936), 367 – 379

[12] J. P. LaSalle and S. Lefschetz, *Stability by Liapunov's direct method with applicatrions*, Academic Press, New York, 1961

[13] T. Kato, On the Korteweg-de Vries equation, *Manuscripta Math.*, **29**(1979), 89 – 99

[14] ———, On the Cauchy problem for the (generalized) Korteweg-de Vries equations, *Advances in Mathematics Supplementary Studies, Studies in Applied Math.*, **8**(1983), 93 – 128

[15] T. Kappeler, Solutions to the Korteweg-de Vries equation with irregular initial data, *Comm. P. D. E.*, **11**(1986), 927 – 945

[16] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integral and regularity of dispersive equations, *Indiana University Math. J.,* **40**(1991), 37 – 69

[17] ———, Well-posedness of the initial value problem for the Korteweg-de Vries, *J. Amer. Math. Soc.*, **4**(1991), 323 – 347

[18] ————-, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, to appear in *Comm. Pure Appl. Math.*

[19] V. Komornik, A generalization of Ingham's inequality, *Colloq. Math. Soc. Jànos Bolgai, Diff. Equations and Applications*, Budapest, Hungary, 1991, North-Holland, to appear

[20] V. Komornik, D. L. Russell and B.-Y. Zhang, Stabilisation de l'equation de Korteweg-de Vries, *C. R. Acad. Sci. Paris*, **312**(1991), 841 – 843

[21] V. Komornik, D. L. Russell and B.-Y. Zhang, Control and stabilization of the Korteweg-de Vries equation on a periodic domain, to appear in *J. Diff. Eqns.*

[22] D. J. Korteweg & G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Philos, Mag.*, 5, **39**(1895), 422 – 423

[23] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure and Appl. Math.*, **21**(1968), 467 – 490

[24] R. M. Miura, The Korteweg-de Vries equation: A survey of results, *SIAM Review*, **18**(1976), 412 – 459

[25] R. M. Miura, S. C. Gardner & M. D. Kruskal, Korteweg-de Vries equation and generalizations, II. Existence of conservation laws and constant of motion, *J. Math. Phys.*, **9**(1968), 1204 – 1209

[26] D. L. Russell, *Mathematics of Finite Dimensional Control Systems; Theory and Design.* Marcel Dekker, Inc., New York, 1979

[27] D. L. Russell and B.-Y. Zhang, Controllability and stabilizability of the third order linear dispersion equation on a periodic domain, to appear in *SIMA Contrl. & Opt.* , 1992

[28] J. C. Saut and R. Temam, Remarks on the Korteweg-de Vries equation, *Israel J. Math.*, **24**(1976), 78 – 87

[29] A. C. Scott, F. Y. Chu and D. W. McLaughin, The soliton: A new concept in applied sciences, *Proc. IEEE*, **61**(1973), 1443 – 1483

[30] P. Sjölin, Regularity of solutions to the Schrödinger equations, *Duke Math. J.* , **55**(1987), 699 – 715

[31] M. Slemrod, A note on complete controllability and stabilizability for linear control systems in HIlbert space, *SIAM J. Contr. & Opt.* , **12**(1974), 500–508

[32] R. S. Strichartz, Multipliers in fractional Sobolev spaces, *J. Math. Mech.*, **16**(1967), 1031 – 1060

[33] ———, Restriction of Fourier transform to quadratic surface and decay of solutions of wave equations, *Duke Math. J.*, **44**(1977), 705 – 714

[34] Y. Tsutsumi, The Cauchy problem for the Korteweg-de Vries equation with measure as initial data, *SIAM J. Math. Anal.*, **20**(1989), 582 – 588

[35] L. Vega , The Schrödinger equation: pointwise convergence to the initial data, *Proc. Amer. Math. Soc.*, **102**(1988), 874 – 878

[36] B.-Y. Zhang, Taylor series expansion for solutions of the Koteweg-de Vries equation with respect to their initial values, *IMA preprint*, series # 1015, August 1992

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