THE LINEARIZING PROJECTION, GLOBAL THEORIES

BY

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THE LINEARIZING PROJECTION, GLOBAL THEORIES

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THE LINEARIZING PROJECTION, GLOBAL THEORIES

1. Introduction

A linearizing operator or projection is a device which converts nonlinear information into linear form. A well-known example of a linearizing projection is the Shapley value, both in the discrete case (Shapley, 1953) and the continuous case (Aumann and Shapley, 1974). A linearizing projection usually satisfies certain axioms of rationality which insure that it is the "unique, fair" allocation or distribution. Thus it is an admiriable bookkeeping device because bookkeeping must be linear.

In Ruckle (1982) a first attempt was made to treat the Aumann-Shapley theory of values in the setting of functionals defined in an arbitrary Banach space E. Unfortunately, except for the space $\mathcal{P}(E')$ of polynomials of continuous linear functionals on E, the spaces of functionals considered there are somewhat contrived. Hence it is not easy to determine whether a functional encountered in an attempted application is in one of the spaces or not. Also note that the space $\mathcal{U}(E', x_0)$ in Ruckle (1982) consists entirely of bounded functionals and cannot, therefore, contain E'. This means that the operator P from $\mathcal{U}(E', x_0)$ onto E' is not a projection.

This paper continues the effort begun in Ruckle (1982) by constructing three global theories of linearizing projections (or x_0-values). These theories are called "global" because they refer to spaces of functionals which are defined on the entire Banach space E. In Section 6 we shall describe what we mean by a "local" theory and explain why such theories are needed.

The polynomial theory described in Section 2 is more complete than that given in Ruckle (1982) for $\mathcal{P}(E')$, not only because it avoids the technical hypothesis that E have dimension greater than three, but more significantly because it applies to polynomials of the form

$$G(x) = \sum_{k=0}^{n} F_k(x) \tag{1-1}$$

where each F_k is a symmetric multilinear form defined on E. The basic theory of such polynomials is treated in Chapter 26 of Hille and Phillips (1957). When E has infinite dimension not all polynomials of the form (1-1) are polynomials of linear functionals. For example, the functional T on $C[0,1]$ defined by

$$T(x) = \int_0^1 \left(x(t)\right)^3 g(t)dt$$

where g is a fixed integrable function is a polynomial of type (1-1) but not a polynomial in a linear functional.

It is a short step from a polynomial theory to a theory for entire real-analytic functions on E. Such a theory is briefly described in Section 3.

The main contribution of this paper, found in Section 4, is a theory of linearizing projections defined on the space $C_0^1(E)$ of (locally uniformly) continuously differentiable functionals on E. This is a natural space of functionals on E, and it is usually possible to verify a functional's membership in the space.

In Section 5 we describe two sample applications of the mathematical theories constructed. In the final section we mention some lines of future investigation.

Given a arbitrary real Banach space E, let X be a linear space of real-valued functions (functionals) on E which includes the space E^* of continuous linear functionals on E and has the property that whenever F is in X and T is a continuous linear mapping from E into E the functional $F \circ T$, defined by $F \circ T(x) = F(Tx)$ for x in E, is also in X.

1.1. <u>Definition</u>. Let x_0 be a distinguished nonzero vector in E. An <u>x_0-value</u> is a function P from X onto E^* which satisfies the following conditions:

(L_1) P is linear; i.e.,

$$P(au+bv) = aPu + bPv;\ a,b \text{ in } R;\ u,v \text{ in } X\ ;$$

(L_2) P is x_0-symmetric; i.e., if T is a bicontinuous linear isomorphism from E onto E such that $Tx_0 = x_0$, then

$$P(u \circ T) = (Pu) \circ T \qquad u \in X\ ;$$

(L_3) P is x_0-efficient; i.e.,

$$(Pu)(x_0) = u(x_0) \qquad u \in X\ ;$$

(L_4) P is idempotent; i.e.,

$$Pu = u \qquad u \in E^*\ .$$

Briefly, an x_0-value is a projection from X onto E^* which is x_0-symmetric and x_0-efficient. If some topology is placed upon X so that E^* becomes a closed subspace it may also be required that P be continuous. In this case we shall call P a continuous x_0-value.

2. The Polynomial Theory

Let E be a real linear space. A symmetric multilinear form of degree n is a real-valued function $\hat{F}$ on $X \times E \times \dots \times E$ (n factors) such that (a) $\hat{F}$ is separately linear on each compoment, i.e.,

$$\hat{F}(x_1, x_2, \dots, ax_j + by_j, x_{j+1}, \dots, x_n)$$
$$= a\hat{F}(x_1, x_2, \dots, x_j, x_{j+1}, \dots, x_n) + b\hat{F}(x_1, x_2, \dots, y_j, x_{j+1}, \dots, x_n)$$

for all $j = 1, 2, \dots, n$; all $x_1, \dots, x_n, y_j$ in E; all reals a and b and all $j = 1, 2, \dots, n$; (b) the value of $\hat{F}$ does not depend upon the order of it components, i.e.,

$$\hat{F}(x_1, x_2, \dots, x_n) = \hat{F}(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$$

for all $x_1, x_2, \dots, x_n$ in E and all permutations π on $\{1, 2, \dots, n\}$. For x and y in E and k a positive integer less than n, $\hat{F}(x^k, y^{n-k})$ denotes the value of

$\hat{F}(x_1, \ldots, x_n)$ where for k components we have $x_j = x$ and for n-k components we have $x_j = y$; $\hat{F}(x^n)$ denotes $\hat{F}(x, x, \ldots, x)$. The nth degree homogeneous polynomial F derived from the multilinear form $\hat{F}$ is defined by the equation

$$F(x) = \hat{F}(x^n) .$$

The multilinear form $\hat{F}$ is called the polar form of F. A polynomial functional H on E is a linear combination of homogeneous polynomials. The degree of a polynomial function is the highest degree of the homogeneous polynomials in its expansion. Since each linear combination of homogeneous polynomials of degree n is again a homogeneous polynomial of degree n (or the zero polynomial) each polynomial H on E can be expressed in the form

$$H(x) = \sum_{k=1}^{n} F_k(x) \tag{2-1}$$

where each F_k is a homogeneous polynomial of degree k (or 0).

Suppose the linear space E has a topology derived from a norm $\| \ \|$. Then a homogeneous polynomial F of degree n is continuous if and only if there is $M > 0$ such that $|F(x)| \leqslant M\|x\|^n$ for all x in E. A general polynomial H is continuous if each summand in its expansion is continuous. Let $\mathcal{P}(E)$ denote the linear space of all continuous real-valued polynomials H on E. By our definition polynomials do not include constant terms so $H(0) = 0$ for every H in $\mathcal{P}(E)$.

2.1 Theorem. Let x_0 be a distinguished nonzero point in a Banach space E. There exists a unique x_0-value P on $\mathcal{P}(E)$ given by the formulas

$$(PH)(x) = \int_0^1 \left[\frac{\partial}{\partial t} H(sx_0 + tx)\right]_{t=0} ds \quad , \qquad x \in E \ , \quad H \in \mathcal{P}(E) \tag{2-2}$$

and

$$(PH)(x) = \sum_{k=1}^{n} \hat{F}_k(x_0^{k-1}, x) \quad , \qquad x \in E \tag{2-3}$$

when H has the form (2-1).

Proof. Existence. It suffices to verify that the linear operator P given by formula (2-2) satisfies conditions (L_1) - (L_4) of Definition 1.1. That P is linear on $\mathcal{P}(E)$ results from the linearity of the operations of partial differentiation and integration. To show P is x_0-symmetric assume T is a bicontinuous linear isomorphism from E onto E such that $Tx_0 = x_0$. Then we have

$$\big(P(H \circ T)\big)(x) = \int_0^1 \left[\frac{\partial}{\partial t} H\big(T(sx_0 + tx)\big)\right]_{t=0} ds$$

$$= \int_0^1 \left[\frac{\partial}{\partial t} H\big(sx_0 + tT(x)\big)\right]_{t=0} ds$$

$$= (PH)(Tx) \ .$$

To show P is x_0-efficient we calculate

$$(PH)(x_0) = \int_0^1 \left[\frac{\partial}{\partial t} H(sx_0 + tx_0)\right]_{t=0} ds$$

$$= \int_0^1 \frac{\partial}{\partial s} H(sx_0)\, ds$$

$$= \Big[H(sx_0)\Big]_0^1 = H(x_0) \ .$$

Finally, to see P is idempotent, suppose $H \in E^*$, then $H(sx_0 + tx) = sH(x_0) + tH(x)$ so we have

$$PH(x) = \int_0^1 \frac{\partial}{\partial t}\Big[sH(x_0) + tH(x)\Big]_{t=0} ds$$

$$= \int_0^1 H(x)\, dx = H(x) \ .$$

Uniqueness. Suppose Q is any operator from $\mathcal{P}(E)$ onto E^* which satisfies $(L_1) - (L_4)$. It suffices to prove that $Q(F) = \hat{F}(x_0^{n-1}, \cdot)$ for every homogeneous polynomial F. We first treat the special case when F has the form $F(x) = (u'(x))^n$ where u' is in E^*. Let v' be any member of E^* such that $v'(x_0) = 1$. Define S_t from E into E by

$$S_t(x) = (1-t)v'(x)x_0 + tx \qquad x \in E \,. \tag{2-4}$$

For $t \neq 0$, S_t is a bicontinuous isomorphism which preserves x_0. For x in E

$$(F \circ S_t)(x) = \{u'((1-t)v'(x)x_0 + tx)\}^n$$

$$= \sum_{k=0}^{n} \binom{n}{k}(1-t)^k u'(x_0)^k t^{n-k} v'(x)^k u'(x)^{n-k} \,.$$

In other words, we have

$$F \circ S_t = \sum_{k=0}^{n} \binom{n}{k}(1-t)^k u'(x_0)^k t^{n-k} v'(\cdot)^k u'(\cdot)^{n-k} \tag{2-5}$$

where $v'(\cdot)^k u'(\cdot)^{n-k}$ is a homogeneous polynomial of degree n. The value of $P(F \circ S_t)$ at x is

$$(PF)(S_t x) = (PF)((1-t)\, v'(x)x_0 + tx)$$

$$= (1-t)F(x_0)v'(x) + t(PF)(x) \,.$$

Applying P to both sides of (2-5) we obtain

$$(1-t)F(x_0)v' + t(PF) = \sum_{k=0}^{n} \binom{n}{k}(1-t)^k u'(x_0)^k t^{n-k} P(v'(\cdot)^k u'(\cdot)^{n-k}) \,. \tag{2-6}$$

By letting $t \rightarrow 0$ in (2-6) we find

$$F(x_0)v' = u(x_0)^n P(v'(\cdot)^n) \,. \tag{2-7}$$

If $u'(x_0) \neq 0$ we may take v' to be $u'/u'(x_0)$. Then by substituting this value in (2-7) and using the linearity of P we conclude

$$P\left(u'(\cdot)^n\right) = u'(x_0)^{n-1} u'$$

which is required. If $u'(x_0) = 0$, $F(x_0)$ is also 0 so (2-6) becomes

$$t(PF) = t^n P(F)$$

for all t so $P(F) = 0 = u'(x_0)^{n-1} u'$.

We have shown that if H has the form $H(x) = (u'(x))^n$ then (2-2) is valid. Since the expression on the right hand side of (2-2) is linear in F it follows that $(PH)(x)$ has the form (2-2) for every linear combination of powers of linear functionals. It is known that functionals of the form $u'(\cdot)v'(\cdot)^n$ where u' and v' are in E^* can be written as such a linear combination. See Aumann and Shapley (1974). Therefore,

$$\text{(2-8)} \qquad P\left(u'(\cdot)v'(\cdot)^n\right) = \int_0^1 \left[\frac{\partial}{\partial t}\left(u'(sx_0 + tx)v'(sx_0 + tx)^n\right)\right]_{t=0} ds$$

$$= (n+1)^{-1} v'(x_0)^n u'(x) + n(n+1)^{-1} u'(x_0)v'(x_0)^{n-1} v'(x) \ .$$

Now let F be an arbitrary homogeneous polynomial from E into the scalar field such that the degree of F is n. Let v' in E^* be such that $v'(x_0) = 1$ and let S_t be defined by (2-4). For each x in E we have

$$\text{(2-9)} \qquad F \circ S_t(x) = F\left((1-t)v'(x)x_0 + tx\right) = \sum_{k=0}^{n} \binom{n}{k}(1-t)^k t^{n-k} F(x_0^k, x^{n-k}) v'(x)^k$$

and

$$\text{(2-10)} \qquad (PF)(S_t x) = (PF)\left((1-t)v'(x)x_0 + tx\right) = (1-t)F(x_0)v'(x) + t(PF)(x) \ .$$

Applying P to both sides of (2-9) we obtain

$$(1-t)F(x_0)v' + t(PF) = (1-t)^n F(x_0) P\big(v'(\cdot)^n\big) + \sum_{k=0}^{n-1} \binom{n}{k} (1-t)^k t^{n-k} P\big(\hat{F}(x_0^k, \cdot) v'(\cdot)^k\big) \quad . \tag{2-11}$$

Since $P\big(v'(\cdot)^n\big) = v'(x_0)^{n-1} v' = v'$ we can subtract the first two terms from both sides of (2-11) to get

$$(t-t^n)P(F) + \big((1-t) - (1-t)^n\big)F(x_0)v' = \sum_{k=1}^{n-1} \binom{n}{k} (1-t)^k t^{n-k} P\big(\hat{F}(x_0^k, \cdot) v'(\cdot)^k\big) \quad . \tag{2-12}$$

If we divide (2-12) by $t(1-t)$ we find

$$(1+t+\ldots+t^{n-2})P(F) + \big(1+(1-t)+\ldots+(1-t)^{n-2}\big)F(x_0)v' = \sum_{k=1}^{n-1} \binom{n}{k} (1-t)^{k-1} t^{n-k-1} P\big(\hat{F}(x_0^k, \cdot) v'(\cdot)^k\big) \quad . \tag{2-13}$$

Taking the limit of both sides of (2-13) as $t \to 0$ we conclude

$$P(F) + (n-1)F(x_0)v' = nP\big(\hat{F}(x_0^{n-1}, \cdot) v'(\cdot)^{n-1}\big) \quad . \tag{2-14}$$

But $\hat{F}(x_0^{n-1}, \cdot)$ is in E^* so by (2-8) we have

$$\begin{aligned} P\big(\hat{F}(x_0^{n-1}, \cdot) v(\cdot)^{n-1}\big) &= n^{-1} h(x_0)^n \hat{F}(x_0^{n-1}, \cdot) + n^{-1}(n-1)F(x_0)v'(x_0)^{n-1} v' \\ &= n^{-1} \hat{F}(x_0^{n-1}, \cdot) + n^{-1}(n-1)F(x_0)v' \quad . \end{aligned}$$

By substituting this in (2-14) we finally conclude that

$$P(F) = \hat{F}(x_0^{n-1}, \cdot) \ . \tag{2-15}$$

Since $P(F)$ satisfies (2-2) and (2-3) for each homogeneous polynomial F it follows that (2-2) and (2-3) are also true for all H in $\mathcal{P}(E)$. ///

3. The Analytic Theory

An analytic theory of linearizing projections can be quickly obtained as a bonus to the polynomial theory. However, it is necessary to introduce a topology and require that the x_0-value be continuous with respect to this topology.

A real-valued function H on a real Banach space E is called <u>real analytic</u> if there is a sequence (F_n) of homogeneous polynomials with F_n having degree n (or being 0) and

$$(3\text{-}1) \qquad M_n = \sup\left\{ \left|\hat{F}_n(x_1, x_2, \dots, x_n)\right| \Big/ (\|x_1\| \, \|x_2\| \cdots \|x_n\|) : x_j \neq 0 \text{ for each } j \right\} < \infty$$

such that

$$(3\text{-}2) \qquad H(x) = \sum_n F_n(x)$$

for all x in E. Because of (3-1) the series in (3-2) converges absolutely and uniformly on bounded subsets of E. Moreover, the sequence (F_n) is uniquely determined by H. Let $\mathcal{A}(E)$ denote the space of all real analytic functions on E. A topology on $\mathcal{A}(E)$ is determined by the sequence of seminorms

$$(3\text{-}3) \qquad p_k(H) = \sum_{n=1}^{\infty} M_n k^n \qquad k = 1, 2, \dots ,$$

where each M_n is given by (3-1). With this topology $\mathcal{A}(E)$ is a complete metric linear space, i.e., an (F)-space which is, however, not a Banach space.

Let $\mathfrak{F}_n$ denote the linear space of all homogeneous polynomials of degree n along with the zero polynomial considered as a degenerate homogeneous polynomial of degree n. Each $\mathfrak{F}_n$ is then a closed subspace of $\mathcal{A}(E)$. The sequence $(\mathfrak{F}_n)$ forms a Schauder decomposition of $\mathcal{A}(E)$ in view of the uniqueness of the expansion (3-1). See Ruckle (1964).

3.1 Theorem. Let x_0 be a distinguished nonzero vector in a Banach space E. There exists a unique x_0-value P on $\mathfrak{A}(E)$ which is continuous with respect to the topology determined by the seminorms (3-3). The x_0-value can be given by the formula

$$(PH)(x) = \int_0^1 \left[\frac{\partial}{\partial t} H(sx_0 + tx)\right]_{t=0} ds \tag{3-4}$$

or the formula

$$(PH)(x) = \sum_n \hat{F}_n(x_0^{n-1}, x) \tag{3-5}$$

for H given by (3-2).

Proof. Existence. It will be shown that PH as determined by formula (3-4) is defined for all H in $\mathfrak{A}(E)$, satisfies Definition 1.1 and is continuous.

If H is determined by (3-2) then for all s and t

$$H(sx_0 + tx) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k} \hat{F}_n(x_0^k; x^{n-k}) \ . \tag{3-6}$$

The series (3-6) converges absolutely and uniformly on bounded subsets of E because the sequence (F_n) satisfies (3-1). Thus

$$\frac{\partial}{\partial t} H(sx_0 + tx) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k}(n-k) s^k t^{n-k-1} \hat{F}_n(x_0^k; x^{n-k})$$

$$\left[\frac{\partial}{\partial t} H(sx_0 + tx)\right]_{k=0} = \sum_{n-1}^{\infty} n s^{n-1} \hat{F}(x_0^{n-1}; x)$$

$$\int_0^1 \left[\frac{\partial}{\partial t} H(sx_0 + tx)\right]_{t=0} ds = \sum_{n=1}^{\infty} \hat{F}(x_0^{n-1}; x) \quad . \tag{3-7}$$

All of the series encountered converge absolutely because the sequence (F_n) satisfies (3-1). Therefore, PH exists for all H in $\mathcal{A}(E)$.

It is clear that P is linear. Because of (3-7), PH satisfies (3-5) so that P must be x_0-efficient. If T is a bicontinuous linear isomorphism from E into E such that $Tx_0 = x_0$

$$
\begin{aligned}
P(H \circ T)(x) &= \int_0^1 \left[\frac{\partial}{\partial t} H \circ T(sx_0 + tx)\right]_{t=0} ds \\
&= \int_0^1 \left[\frac{\partial}{\partial t} H\Big(T(sx_0) + T(tx)\Big)\right]_{t=0} ds \\
&= \int_0^1 \left[\frac{\partial}{\partial t} H(sx_0 + tTx)\right]_{t=0} ds \\
&= P(H)(Tx) \quad .
\end{aligned}
$$

Therefore P is x_0-symmetric.

Suppose $\|x_0\| \leqslant k$; then for each x in E we have

$$
|P(H)(x)| \leqslant \sum_n |\hat{F}_n(x_0^{n-1}, x)| \leqslant \sum_n M_n \|x_0\|^{n-1} \|x\|
$$

$$
\leqslant \left(\sum_n M_n \|x_0\|^n\right) \|x\| / \|x_0\| \leqslant \|x_0\|^{-1} p_k(H) \|x\| \quad .
$$

Therefore we conclude that for all $H \in \mathcal{A}(E)$

$$
\|P(H)\| \leqslant \|x_0\|^{-1} p_k(H)
$$

which implies that P is continuous from $\mathcal{A}(E)$ onto E^*.

Uniqueness. By Theorem 2.1 an x_0-value is uniquely defined by (3-4) for H in $\mathcal{P}(E)$. But $\mathcal{P}(E)$ is dense in $\mathcal{Q}(E)$ so that if Q is any continuous x_0-value it must coincide with P on all of $\mathcal{Q}(E)$. ///

4. C^1 Theory

Let E denote a real Banach space and E^* its dual space. A real-valued function f on E is sait to be Fréchet differentiable at x if there is a continuous linear functional x'_x on E such that for each u in E

$$x'_x(u) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t} .$$

For a general discussion of Fréchet differentiability, see Dieudonné (1971). The functional x'_x will be written $df(x;\cdot)$ and $x'_x(u)$ its value at u, $df(x;u)$.

Let $C_0^1(E)$ be the space of all real-valued functions f on E such that $f(0) = 0$ and the correspondence $x \to df(x;\cdot)$ is locally bounded and locally uniformly continuous from E into E^*. That the correspondence is locally bounded means that for each $n = 1, \eta, \dots,$ $\sup\{\|df(x;\cdot)\| : \|x\| \leqslant n\} < \infty$. That the correspondence is locally uniformly continuous means that for each n and each $\epsilon > 0$ there is $\delta = \delta(n, \epsilon)$ such that

$$\|d(x;\cdot) - d(y;\cdot)\| < \epsilon$$

whenever $\|x\|$ and $\|y\|$ are $\leqslant n$ and $\|x - y\| < \epsilon$. For example, $C_0^1(E)$ contains all real-valued continuous polynomials on E. See Chapter 26 of Hille and Phillips (1957). If E has finite dimension then local uniform continuity is equivalent to continuity since E is then locally compact.

The space $C_0^1(E)$ can be given the topology of uniform convergence on bounded sets. This topology is determined by the sequence of seminorms

$$p_n(f) = \sup\{\|df(x;\cdot)\| : \|x\| \leqslant n\} , \qquad n = 1, 2, \dots .$$

This topology makes $C_0^1(E)$ a complete metric linear space which is, however, not a Banach space. The Hausdorff property of this topology follows from the fact that if $p_n(f) = 0$ for each n then $df(x;\cdot) = 0$ for all x. Thus f is constant; but since $f(0) = 0$ it follows that $f(x) = 0$ for all x.

The space of all continuous linear operators from E into E is denoted by L(E). The space L(E) is a Banach space when given the uniform operator topology determined by the norm

$$\|T\| = \sup\left\{\|Tx\| : x \in E,\ \|x\| \leqslant 1\right\} \quad .$$

4.1 Lemma. If T is a continuous linear operator from E into E, then for f in $C_0^1(E)$, $f \circ T$ is in $C_0^1(E)$ and

$$(1) \qquad df \circ T(x; u) = df(Tx; Tu) \qquad x, u \in E \ .$$

Proof. By definition

$$\begin{aligned} df \circ T(x;u) &= \lim_{t \to 0} \frac{f \circ T(x+tu) - f \circ T(x)}{t} \\ &= \lim_{t \to 0} \frac{f\big(T(x+tu)\big) - f(Tx)}{t} \\ &= df(Tx; Tu) \quad . \end{aligned}$$

Therefore, formula (1) is valid.

The correspondence $x \to df \circ T(x;\cdot)$ is locally bounded because if $\|u\| \leqslant n$,

$$|df \circ T(x;u)| = df(Tx; Tu) \leqslant \|df(Tx;\cdot)\| \, \|Tu\| \leqslant np_m(f) \|T\|$$

where m is such that $n\|T\| \leqslant m$.

To show the correspondence $x \to df \circ T(x;\cdot)$ is locally uniformly continuous when $T \neq 0$, let $\epsilon > 0$ and n be given. Let δ be such that $\|df(x,\cdot) - df(y,\cdot)\| < \epsilon/\|T\|$ when $\|x\|$ and $\|y\|$ are $\leqslant n\|T\|$ and $\|x-y\| < \delta$. Then if $\|u\|$ and $\|v\|$ are $\leqslant n$ and $\|u-v\| < \delta/\|T\|$, $\|Tu - Tv\| < \delta$ and $\|Tu\|$ and $\|Tv\| \leqslant n\|T\|$. Thus we have

$$\|df(Tu, \cdot) - df(Tv, \cdot)\| < \epsilon/\|T\|$$

so that if $\|w\| \leqslant 1$,

$$\|df(Tu, Tw) - df(Tv, Tw)\| \leqslant \|df(Tu; \cdot) - df(Tv; \cdot)\| \, \|T\| < \epsilon \quad .$$

Consequently we see

$$\|df \circ T(u; \cdot) - df \circ T(v; \cdot)\| < \epsilon \quad . \qquad ///$$

4.2 Lemma. Suppose f is in $C_0^1(E)$ and (T_n) is a sequence in $L(E)$ which converges in norm to T. Then $f \circ T_n$ converges in $C_0^1(E)$ to $f \circ T$.

Proof. Since (T_n) converges in norm to T there is $M > 0$ such that $\|T\|$ and each $\|T_n\|$ is no greater than M. Given k and $\epsilon > 0$ let δ be such that $\|df(x; \cdot) - df(y; \cdot)\| < \epsilon/(2M)$ when $\|x - y\| < \delta$ and $\|x\|$ and $\|y\| < Mk$. Let N be such that $\|T - T_n\| < \epsilon/(2p_{Mk}(f))$ and $\|T - T_n\| < \delta/(Mk)$ when $n > N$. When $n > N$, $\|x\| \leqslant k$ and $\|u\| \leqslant 1$ we have

$$\begin{aligned} |df(Tx; Tu) - df(T_n x, T_n u)| &\leqslant |df(Tx; Tu) - df(Tx, T_n u)| + |df(Tx, T_n u) - df(T_n x, T_n u)| \\ &\leqslant \|df(Tx; \cdot)\| \, \|T_u - T_n u\| + \|df(Tx; \cdot) - df(T_n x, \cdot)\| \, \|T_n u\| \\ &< p_{Mk}(f) \|T - T_n\| + \epsilon M/(2M) \leqslant \epsilon/2 + \epsilon/2 = \epsilon \quad . \end{aligned}$$

Consequently for $n > N$ we have

$$p_k(f \circ T - f \circ T_n) < \epsilon \quad . \qquad ///$$

4.3 Theorem. Let x_0 be a distinguished nonzero point in a real Banach space E. There is a unique continuous x_0-value P on $C_0^1(E)$ given by the formula

$$(4\text{-}1) \qquad (Pf)(u) = \int_0^1 df(sx_0; u)\,ds \qquad u \in E \quad .$$

Proof. Existence. We show that the mapping P defined by (4-1) is a continuous projection from $C_0^1(E)$ onto E^* which satisfies (L_2) and (L_3). Since

$$d(f+g)(x;\cdot) = df(x;\cdot) + dg(x;\cdot)$$

and

$$d(af)(x;\cdot) = a\,df(x;\cdot)$$

for all f, g in $C_0^1(E)$, a in R and $x \in E$, it follows that P is linear.

The inequality

$$\begin{aligned}\|Pf(u)\| &\leqslant \sup_{0 \leqslant s \leqslant 1} \|df(sx_0;\cdot)\|\,\|u\| \\ &\leqslant p_M(f)\|u\|\end{aligned}$$

where $M \geqslant \|x_0\|$ shows that P is continuous. If x' is in E^*

$$(Pf)(x')(u) = \int_0^1 dx'(sx_0; u)\,ds = \int_0^1 x'(u)\,ds = x'(u)$$

for all u in E. Therefore, P is idempotent (L_4).

To show P satisfies (L_3) we calculate

$$(4\text{-}2) \qquad \begin{aligned}(Pf)(x_0) &= \int_0^1 df(sx_0, x_0)\,ds \\ &= \int_0^1 \lim_{t \to 0} (1/t)\Big(f\big((s+t)x_0\big) - f(sx_0)\Big)ds \\ &= f(x_0) - f(0) = f(x_0)\ .\end{aligned}$$

To check condition (L_2) let T be a continuous linear mapping from E into E such that $Tx_0 = x_0$. By Lemma 4.1 we have

$$P(f \circ T)(u) = \int_0^1 d(f \circ T)(sx_0; u)ds$$

$$= \int_0^1 df(sx_0; Tu)ds$$

$$= (Pf)(Tu) .$$

Uniqueness. If E is a space of two dimensions the theorem is true since the mapping P is continuous on the set $\mathcal{P}(E)$ of polynomials and this set is dense in $C_0^1(E)$. See Kingsley (1951) or Butzer (1953) where it is shown that if f has a continuous kth derivative then the Bernstein polynomial for f along with its first k derivatives converges uniformly to f on compact sets. Now assume **E** is an arbitrary real Banach space. Let u be any vector in E. If u is a multiple of x_0, say $u = ax_0$, then

$$(Pf)(u) = a(Pf)(x_0) = af(x_0) = a \int_0^1 df(sx_0, s_0)ds$$

because of Lemma 4.1), (L_1) and (L_3).

If u is not a multiple of x_0 let E_1 denote the two dimensional space spanned by x_0 and u. Let H be any continuous linear projection from E onto E_1. Such a projection can always be found for a finite dimensional subspace. For f in $C_0^1(E_1)$ we extend f to a function $\hat{f}$ on all of E by the formula

$$\hat{f}(x) = f(Hx) .$$

The extension $\hat{f}$ is in $C_0^1(E)$ because for x and v in E

$$\lim_{t \to 0} \frac{\hat{f}(x+tv) - \hat{f}(x)}{t} = \lim_{t \to 0} \frac{f(Hx+tHv) - f(Hx)}{t} = df(Hx; Hu) .$$

Suppose P is any continuous mapping from $C_0^1(E)$ onto E^* which satisfies $(L_1) - (L_4)$. Define Q from $C_0^1(E_1)$ onto E_1^* by

$$Qf = P\hat{f} = P(f \circ H) \ .$$

Then Q is a continuous operator from $C_0^1(E_1)$ onto E_1^* which satisfies Definition 1.1. Therefore, for f in $C_0^1(E_1)$

$$\big(P(f \circ H)\big)(u) = (Qf)(u) = \int_0^1 df(sx_0; u)ds = \int_0^1 df \circ H(sx_0; u)ds \ .$$

If f is in $C_0^1(E)$, f_1 the restriction of f to E_1 is in $C_0^1(E_1)$ and $f_1(v) = (f \circ H)(v)$ for v in E_1. Since u is in E_1 and $(f \circ H)(x) = (f_1 \circ H)(x)$ for x in E

$$\begin{aligned} P(f \circ H)(u) &= \big(P(f_1 \circ H)\big)(u) \\ &= (Qf_1)(u) \\ &= \int_0^1 df(sx_0; u)ds \ . \end{aligned}$$

For each $t > 0$, $H + t(I - H)$ is a continuous isomorphism from E onto E which preserves x_0. Therefore, for f in $C_0^1(E)$ we have

$$\begin{aligned} P\Big(f \circ \big(H + t(I - H)\big)\Big) &= (Pf) \circ \big(H + t(I - H)\big) \\ &= (Pf) \circ H + t(Pf) \circ (t - H) \ . \end{aligned}$$

The last equality holds because Pf is linear. As t converges to 0, $H + t(I - H)$ converges in norm to H so that $f \circ \big(H + t(I - H)\big)$ converges in $C_0^1(E)$ to $f \circ H$ by Lemma 4.2. Therefore, we see

$$\begin{aligned} (Pf)(u) &= \big((Pf) \circ H\big)(u) \\ &= \lim_{t \to 0} \big(Pf \circ H + t(Pf) \circ (I - H)\big)(u) \\ &= P(f \circ H)(u) \\ &= \int_0^1 df(sx_0; u)ds \ . \qquad /// \end{aligned}$$

5. Examples

In this section we present two examples which illustrate potential application of the theory described above. The functions which appear are given specific forms not because they realistically apply to the situations encountered, but so we can display the method of calculation.

Example 1: Risk. In a certain population there are four components. If a member of this population is exposed to x units of Hazard A and y units of Hazard B, the expected number of days of life lost is

$f_1(x,y) = 3x^2y + 2xy^2$ for one from component I

$f_2(x,y) = 4x^2y^2$ for one from component II

$f_3(x,y) = 2x^2y + 3xy^2$ for one from component III

$f_4(x,y) = 2xy + x^2y$ for one from component IV .

Each member of the population is exposed to 1 unit of Hazard A and 2 units of Hazard B and is to be compensated for expected life loss due to source A by Agency α and life loss due to source B by Agency β. What is an equitable division of the compensation for the four segments of the population between Agencies α and β?

In this case E is R^2, the space X is the space of polynomials in two variables and the distinguished vector x_0 is the point $(1,2)$. By using the formula of Theorem 2.1 to compute Pf_1 we obtain

$$Pf_1(x,y) = \int_0^1 \left[\frac{\partial}{\partial t}\, 3(s+tx)^2(2s+ty) + 2(s+tx)(2s+ty)^2\right]_{t=0} ds$$

$$= \int_0^1 20s^2x + 11s^2y$$

$$= (20/3)x + (11/3)y \ .$$

Similarly

$$Pf_2 = 8x + 4y$$

$$Pf_3 = (26/3)x + (11/3)y$$

$$Pf_4 = (16/3)x + (4/3)y \quad .$$

When $x = 1$ and $y = 2$ these formulae lead to the following results. Members of component

I receive compensation for 20/3 days from Agency α and for 22/3 days from Agency β.

II receive compensation for 8 days from Agency α and 8 days from Agency β.

III receive compensation for 26/3 days from Agency α and 22/3 days from Agency β.

IV receive compensation for 16/3 days from Agency α and 8/3 days from Agency β.

Example 2: Cost Allocation. Suppose the cost of generating electricity at the rate of $x(t)$ KW/day for one day is given by the functional

$$C(x) = \int_0^1 \big(x(t)\big)^2 g(t)dt$$

where g is a fixed continuous functional on $[0, 1]$. If there are n users whose rates of use are $x_1, x_2, \ldots, x_n$ respectively, what is a reasonable (continuous) rate of charge for each user by a supplier who must recover his costs but not obtain a profit?

Here E is the space $C[0, 1]$ the space of continuous functions on $[0, 1]$; the space X is $\mathcal{P}$; the distinguished vector x_0 is given by

$$x_0 = \sum_{j=1}^{n} x_j \quad .$$

We compute the PC by the same procedure as in the previous example:

$$PC(x_j) = \int_0^1 \left[\frac{\partial}{\partial k} P(sx_0 + tx_j)\right]_{t=0} ds$$

$$= \int_0^1 \left[\frac{\partial}{\partial k} \int_0^1 \left(sx_0(u) + tx_j(u)\right)^2 g(u)du\right]_{t=0} ds$$

$$= \int_0^1 x_0(t)g(t)x_j(t)dt \quad .$$

Thus the rate of charge is given by the function $x_0(t)g(t)$ which is intuitively reasonable.

6. Comments, Directions for Further Investigation

In contrast to the global theory of linearizing operators, a local theory applies to a space X of functionals defined on a subset S of a Banach space. The x_0-value would now be a projection from X onto the space E_s^* consisting of the restrictions of continuous linear functionals to S. The computation of x_0-values according to formula (4-1) involves only the value of $df(x;\cdot)$ on the line segment $\{sx_0 : 0 \leqslant s \leqslant 1\}$. Thus we anticipate that S will be a set containing this line segment. The construction of a viable local theory is essential for applications because the global requirement of continuous differentiability is not usually met in functionals arising in applied problems.

A second direction for future inquiry is the interpretation of conditions (L_2) and (L_3) of Definition 1.1 in particular spaces. The abstract condition of x_0-symmetry (L_2) is especially hard to justify as an economic axiom. This condition is a generalization of the "change of name" axiom of Shapley (1953) and the invariance under measurable permutation axiom of Aumann and Shapley (1974). The x_0-value constructed in Theorems 2.1, 3.1 and 4.3 satisfies the stronger condition

(L_2') If T is a continuous linear mapping from E into E such that $Tx_0 = x_0$, then

$$P(u \circ T) = (Pu) \circ T \qquad u \in X .$$

In specific spaces it should be possible to replace invariance under all bicontinuous homomorphisms which preserve x_0 with invariance under a smaller group of continuous linear operators. Invariance under this group could then be interpreted in a realistic way.

The existence argument in Theorem 2.1 can be modified to show the existence of a mapping P satisfying (L_1) - (L_4) from the space $\mathcal{P}(E, F)$ of continuous polynomials from the Banach space E into the Banach space F onto L(E, F) the space of continuous linear operators from E into F. It is not known, however, whether this P is unique.

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