DETERMINING CRACKS BY BOUNDARY MEASUREMENTS

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DETERMINING CRACKS BY BOUNDARY MEASUREMENTS*

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§0 Introduction. A practically and theoretically very interesting inverse problem concerns the determination of an unknown internal conductivity profile from boundary measurements of voltage potentials and corresponding current fluxes. Most of the theoretical work connected with this problem has been based on the assumption that one has available information about all possible voltage potentials and their corresponding current fluxes. With such complete information it is known that an isotropic conductor is uniquely determined under appropriate smoothness conditions, cf. [6-11]. When some structural information is already known about the conductivity profile then it is conceivable that one can determine characteristic features by making only a finite number of boundary measurements. One such situation where a single measurement suffices has recently been studied in [4]. The physical situation there corresponds to finitely many small inclusions of zero or infinite conductivity imbedded in a known reference conductor. The results assert that it is possible to gain information about the location of the inhomogeneities, their relative sizes and their orientations. In this paper we consider the practically very important case of a curve, σ, of zero or infinite conductivity located inside a 2-dimensional medium with a known reference conductivity. We prove that the presence of σ and its actual shape and location may be determined from exactly two boundary measurements. We construct two specific boundary voltage fluxes ψ_1, and ψ_2 with the property that these fluxes together with the corresponding boundary voltage potentials uniquely determine σ. This result is best possible; it is simple to see that any one measurement will not suffice. We also construct two specific boundary voltage potentials ϕ_1 and ϕ_2 such that these potentials together with the corresponding boundary fluxes uniquely determine σ. By duality the problem involving a perfectly insulating curve is equivalent to a problem involving the same curve, as a perfect conductor. In our analysis we make extensive use of this duality, since it is somewhat easier to examine the case of a perfectly conducting curve by direct methods. A perfectly insulating curve is normally called a crack; by a slight extension of notation we also refer to a perfectly conducting curve as a crack. The uniqueness results in combination with a simple compactness argument leads to a very weak type of stability results. It is a much harder problem to obtain actual estimates of the distance between two cracks σ and σ' in terms of the deviation in boundary measurements. In the last section of this paper we prove a Lipschitz estimate for (partial information about) the location of cracks which are a priori known to be line segments.

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§1. **Statement of the uniqueness results.** Let Ω be a simply connected, bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$ and let γ: $\overline{\Omega} \to \mathbf{R}$ be a positive smooth function (the known reference conductivity). By a C^2 curve, σ, we understand a twice continuously differentiable map : $[a, b] \to \Omega$, $a \neq b$. We do allow the degenerate case that $a > b$, in which case the curve σ is empty. In the following we shall only be interested in non self-intersecting curves; this in particular means that the initial and terminal points do not coincide. Let $\psi \not\equiv 0$ be a current density along $\partial\Omega$ satisfying

$$\int_{\partial\Omega} \psi \, ds = 0 , \tag{1.1}$$

and let J denote the functional

$$J(v) \equiv \frac{1}{2} \int_{\Omega\setminus\sigma} \gamma|\nabla v|^2 - \int_{\partial\Omega} \psi v \tag{1.2}$$

We consider the following two minimization problems:

(I) Find a minimizer of J in the class
$K_1 = \{v \in H^1(\Omega \setminus \sigma),\ v = \text{ constant on } \sigma\}$,

and

(II) Find a minimizer of J in the
class $K_2 = \{v \in H^1(\Omega \setminus \sigma)\}$.

The problem (I) corresponds to a perfectly conducting curve, the problem (II), with the natural boundary condition $\gamma\frac{\partial u}{\partial \nu} = 0$ on σ, corresponds to a perfectly insulating curve. Notice that for both problems the solution, u, is determined uniquely up to an additive constant. One may, e.g., normalize by

$$\int_{\partial\Omega} u \, ds = 0,$$

whenever we make this or any other normalization we shall explicitly say so.

We remark that the set K_1 appearing in problem (I) may alternately be characterized as

$$K_1 = \{v \in H^1(\Omega),\ v = \text{ constant on } \sigma\}. \tag{1.3}$$

Indeed, if $v \in H^1(\Omega \setminus \Omega_0)$ and $v =$ constant in Ω_0, where Ω_0 is a compact subdomain of Ω with Lipschitz boundary, then $v \in H^1(\Omega)$ (cf. [5; p. 34]). By application of an approximation argument we obtain the same result when Ω_0 degenerates to a curve σ. Let $\phi = \int \psi \, ds$, where the indefinite integral is taken counterclockwise on $\partial\Omega$. Convex duality asserts that the problems (I) and (II) are equivalent to

$$(\tilde{I}) \qquad \text{Find a minimizer of } \tilde{J}(v) \equiv \frac{1}{2} \int\limits_{\Omega\setminus\sigma} \gamma^{-1} |\nabla v|^2 \text{ in the class } \tilde{K}_1 = \{v \in H^1(\Omega \setminus \sigma),\ v = \phi \text{ on } \partial\Omega\}.$$

and

$$(\widetilde{II}) \qquad \text{Find a minimizer of } \tilde{J} \text{ in the class } \tilde{K}_2 = \{v \in H^1(\Omega),\ v = \text{ constant on } \sigma \text{ and } v = \phi \text{ on } \partial\Omega\}$$

respectively (for more details see §4). If u denotes the solution to (I) or (II) and w denotes the solution to $(\tilde{I})$ or $(\widetilde{II})$ then u and w are in both cases related by

$$\left(\frac{\partial}{\partial x_2} w, -\frac{\partial}{\partial x_1} w\right) = -(\nabla w)^{\perp} = \gamma \nabla u,$$

and therefore

$$\frac{\partial w}{\partial s} = -\nu \cdot (\nabla w)^{\perp} = \gamma \frac{\partial u}{\partial \nu} \quad \text{and} \quad \frac{\partial u}{\partial s} = -\nu \cdot (\nabla u)^{\perp} = -\gamma^{-1} \frac{\partial w}{\partial \nu} \quad \text{on } \partial\Omega,$$

where $\frac{\partial}{\partial s}$ is the counterclockwise tangential derivative along $\partial\Omega$.

We shall assume that

$$\gamma \text{ is real analytic on } \overline{\Omega} \quad \text{and} \quad \partial\Omega \text{ is real analytic.} \tag{1.4}$$

In the following, whenever we use the term analytic, we always mean real analytic. For any point $R \in \partial\Omega$ and any small positive number ϵ, we denote by $\lambda_{R,\epsilon}$ the curve on $\partial\Omega$ having arclength 2ϵ with R at its midpoint. Let P and Q be two distinct points on $\partial\Omega$ and denote by $\widehat{PQ}$ ($\widehat{QP}$) the curve which one traverses in going from P to Q (from Q to P) counterclockwise along $\partial\Omega$. We define two boundary current densities ψ_i as follows:

$$\psi_1(x) = \begin{cases} 1 & \text{on } \widehat{PQ} \\ -|\widehat{PQ}|/|\widehat{QP}| & \text{on } \widehat{QP} \end{cases} \tag{1.5}$$

and

$$\psi_2(x) = \epsilon^{-1} \cdot 1_{\lambda_{P,\epsilon}} - \epsilon^{-1} \cdot 1_{\lambda_{Q,\epsilon}} \tag{1.6$_\epsilon$}$$

where $|\widehat{PQ}|$ denotes the length of the curve $\widehat{PQ}$, and $1_{\lambda_{P,\epsilon}}$ is the characteristic function of the set $\lambda_{P,\epsilon}$. The constant ϵ is chosen sufficiently small so that

$$\mathrm{dist}(\lambda_{P,\epsilon}, \lambda_{Q,\epsilon}) > \frac{1}{2}|P-Q|.$$

In the following Γ denotes a nonempty open subset of $\partial\Omega$. A main result of this paper is

THEOREM 1.1. *Let K compact subdomain of Ω and let ψ_i $(i = 1, 2)$ be the fluxes defined by (1.5) and (1.6$_\epsilon$). There exists a positive constant ϵ_0 such that the following holds for any $\epsilon \le \epsilon_0$:*

If σ, σ' are (possibly empty) non self-intersecting C^2 curves in K and $u_\sigma^{(i)}, u_{\sigma'}^{(i)}$ denote the solutions of problem (I) corresponding to the curves σ, σ' and fluxes ψ_i, and if

$$\frac{\partial}{\partial s} u_\sigma^{(i)} = \frac{\partial}{\partial s} u_{\sigma'}^{(i)} \quad \text{on } \Gamma \subset \Omega \quad \text{for } i = 1 \text{ and } i = 2$$

then $\sigma = \sigma'$.

REMARK 1.1. Theorem 1.1 remains true (with the same proof) if ψ_1 is any piecewise smooth function such that $\psi_1(x) \neq 0$ for $x \neq P$ and $x \neq Q$ and $\int_{\partial\Omega} \psi_1 ds = 0$.

REMARK 1.2. It is easy to see that any one measurement is not enough to determine σ. Indeed, if u is the solution to

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega,$$
$$\gamma \frac{\partial u}{\partial \nu} = \psi \quad \text{on } \partial\Omega$$

for a given ψ, then u is also the solution to problem (I) for any curve σ which lies on a level curve for u. Thus there is a continuum of curves which yield the same values $\gamma \partial u / \partial \nu$ and u on $\partial\Omega$.

We can formulate another version of Theorem 1.1 in which ψ_2 is defined by

$$\psi_2 = \delta_P - \delta_Q \tag{1.7}$$

where δ_P is the Dirac measure at P. The function $u_\sigma^{(2)}$ grows like $|\log r|$ near P and Q and it solves

$$\nabla \cdot (\gamma \nabla u_\sigma^{(2)}) = 0 \text{ in } \Omega, \quad u_\sigma^{(2)} = \text{constant on } \sigma, \quad \gamma \frac{\partial u_\sigma^{(2)}}{\partial \nu} = \delta_P - \delta_Q \text{ on } \partial\Omega. \tag{1.8}$$

Note that $u_\sigma^{(2)}$ is not in H^1 in any neighborhood of P or Q, so this is not a variational solution in the sense of (I). The function $u_\sigma^{(1)}$ is a solution to (I) as before. By taking first variations we get that $u_\sigma^{(1)}$ solves

$$
\nabla \cdot (\gamma \nabla u_\sigma^{(1)}) = 0 \text{ in } \Omega, \quad u_\sigma^{(1)} = \text{constant on } \sigma, \quad \gamma \frac{\partial u_\sigma^{(1)}}{\partial \nu} = \psi_1 \text{ on } \partial\Omega. \tag{1.9}
$$

THEOREM 1.1'. *Let σ and σ' be (possibly empty) C^2 curves in Ω and denote by $u_\sigma^{(i)}, u_{\sigma'}^{(i)}$ the solutions to problems (1.8) – (1.9) corresponding to σ and σ' respectively. If*

$$
\frac{\partial}{\partial s} u_\sigma^{(i)} = \frac{\partial}{\partial s} u_{\sigma'}^{(i)} \quad \text{on } \Gamma \subset \partial\Omega \quad \text{for } i = 1 \text{ and } i = 2
$$

then $\sigma = \sigma'$.

Due to the the duality mentioned earlier it follows that Theorem 1.1 and Theorem 1.1' (with reference conductivity γ^{-1}) lead to similar results for perfectly insulating curves with fixed Dirichlet conditions ϕ_1 , ϕ_2 on $\partial\Omega$ and measurements of the form $\gamma \frac{\partial w_\sigma^{(i)}}{\partial \nu}|_\Gamma$. The boundary voltages ϕ_i are simply $\int \psi_i ds$, where the indefinite integral is computed counterclockwise on $\partial\Omega$.

We next proceed to formulate an analog of Theorem 1.1 (that is a result with fixed Neumann conditions) for the case of perfectly insulating curves. Let P, Q, R and S be four distinct points on $\partial\Omega$ listed in order of counterclockwise appearance. We define

$$
\tilde{\psi}_1(x) = \begin{cases} 1 & \text{on } \widehat{PQ} \\ -|\widehat{PQ}|/|\widehat{RS}| & \text{on } \widehat{RS} \\ 0 & \text{otherwise,} \end{cases} \tag{1.10}
$$

and

$$
\tilde{\psi}_2(x) = \begin{cases} 1 & \text{on } \widehat{QR} \\ -|\widehat{QR}|/|\widehat{SP}| & \text{on } \widehat{SP} \\ 0 & \text{otherwise.} \end{cases} \tag{1.11}
$$

Let $\tilde{\phi}_i$ denote the counterclockwise integral $\tilde{\phi}_i = \int \tilde{\psi}_i ds$. We now notice that:

(i) the function $\tilde{\phi}_1$ is constantly equal to its maximal value on $\widehat{QR}$ and is constantly equal to its minimal value on $\widehat{SP}$, and

(ii) the function $\tilde{\phi}_2$ is constantly equal to its maximal value on $\widehat{RS}$ and is constantly equal to its minimal value on $\widehat{PQ}$.

Based on (i) and (ii) we conclude that

(1.12) if z_0 is an arbitrary point on $\partial\Omega$, then one of the functions $\tilde{\phi}_1$ or $\tilde{\phi}_2$ attains an extremal value there.

The property (1.12) is essential for the proof of the following uniqueness theorem.

THEOREM 1.2. *Let σ, σ' be (possibly empty) non self-intersecting C^2 curves in Ω and let $u_\sigma^{(i)}, u_{\sigma'}^{(i)}$ denote the solutions to problem (II) corresponding to the curves σ, σ' and the fluxes $\tilde{\psi}_i$ defined by (1.10) and (1.11). If*

$$\frac{\partial}{\partial s} u_\sigma^{(i)} = \frac{\partial}{\partial s} u_{\sigma'}^{(i)} \quad \text{on } \Gamma \subset \partial\Omega \quad \text{for } i = 1 \text{ and } i = 2$$

then $\sigma = \sigma'$.

We note that the solutions $u_\sigma^{(i)}, u_{\sigma'}^{(i)}$ are variational but nontheless there is no need in this case to consider curves inside a fixed compact subdomain K.

Due to the duality, Theorem 1.2 leads to a similar result for perfectly conducting curves with fixed Dirichlet boundary conditions $\tilde{\phi}_i = \int \tilde{\psi}_i ds$ on $\partial\Omega$ and measurements of the form $\gamma \frac{\partial w_\sigma^{(i)}}{\partial \nu}|_\Gamma$.

The outline of this paper is as follows. In §2 we establish four auxiliary results needed in the proofs of our main theorems. Theorem 1.1 and Theorem 1.1' are proved in §3 and Theorem 1.2 is proved in §4. The uniqueness theorems immediately imply some very weak type of stability results, but they do not provide any concrete estimates. In §5 we establish a Lipschitz continuous dependence estimate for the case where σ and σ' are a priori known to be line segments. More specifically we prove that

$$\text{dist}_\Omega(l_\sigma, l_{\sigma'}) \le C \sum_{i=1}^{2} \left\| \frac{\partial}{\partial s}(u_\sigma^{(i)} - u_{\sigma'}^{(i)}) \right\|_{L^\infty(\Gamma)}$$

where l_σ and $l_{\sigma'}$ are the entire lines passing through σ and σ', respectively (provided $\overline{\Gamma}$ does not contain any of the points of discontinuity of ψ_i).

§2. **Auxiliary results.** We begin by studying the regularity of the solution u_σ of problem (I). Clearly u_σ is smooth (in fact, analytic) in $\Omega \setminus \sigma$. Furthermore u is $C^{1+\alpha}$ from each side of σ, uniformly up to σ, except for any small neighborhood of the two endpoints of σ.

LEMMA 2.1. *Let x^* denote any of the two endpoints of σ. The function u_σ, which solves (I), is continuous in Ω,*

(2.1a) $$|u_\sigma(x) - u_\sigma(x^*)| \le C|x - x^*|^{1/2} \text{ for } x \text{ near } x^*,$$

and

(2.1b) $$|\nabla u_\sigma(x)| \le C|x - x^*|^{-1/2} \text{ for } x \in \Omega \setminus \sigma,\ x \text{ near } x^*.$$

Proof. Assume for simplicity that $x^* = 0$ and that, near $x = 0$, the curve σ is given by

$$x_2 = h(x_1)\ ,\ -\delta < x_1 < 0\ ,\ h'(0) = 0.$$

Let $\zeta = \xi_1 + i\xi_2$ denote the conformal transformation $\zeta(x) = \sqrt{z}$, $(z = x_1 + ix_2)$ and set $w(\xi) = u_\sigma(x)$. We compute

$$\begin{aligned}\gamma(x)\Delta_\xi w &= \gamma(x)\Delta_x u_\sigma |\frac{dz}{d\zeta}|^2 \\ &= -\nabla_x\gamma \cdot \nabla_x u_\sigma |\frac{dz}{d\zeta}|^2,\end{aligned}$$

and since $\nabla_x u_\sigma = (\frac{D\xi}{Dx})^T \nabla_\xi w$ it follows that

$$\Delta_\xi w + \sum \beta_j \frac{\partial}{\partial \xi_j} w = 0, \tag{2.2}$$

where β_j are continuous near $\xi = 0$. The original curve σ, which in polar coordinates is represented as

$$\phi = \pm\pi + f(r)\ ,\ f(0) = 0\ ,\ f'(0) = 0,$$

$(z = re^{i\phi})$ is after the transformation $z \to \zeta$ represented as

$$\theta = \pm\frac{\pi}{2} + \frac{1}{2}f(\rho^2),$$

(remember $\theta = \frac{\phi}{2}$ and $\rho = \sqrt{r}$). The function $g(\rho) = \frac{1}{2}f(\rho^2)$ satisfies $g(0) = 0$, $g'(0) = 0$ and $g''(0) = 0$; it therefore follows that the image of σ, σ_ζ, is a C^2 curve. The function w is constant on σ_ζ and it now follows from (2.2) and elliptic regularity theory that

$$w \in W^{2,p}(B) \text{ for any } p < \infty,$$

where B is a neighborhood of $0 \in \sigma_\zeta$ relative to the transformed domain. Using Sobolev's imbedding theorem we get that $\nabla w \in L^\infty(B)$ and therefore, going back to the original $x-$ coordinates

$$\begin{aligned}|\nabla_x u_\sigma(x)| &= |(\frac{D\xi}{Dx})^T \nabla_\xi w| \\ &\le C|x|^{-1/2}\ ,\ x \in \Omega\setminus\sigma\ ,\ x \text{ near } 0.\end{aligned}$$

This verifies (2.1b); the estimate (2.1a) and the fact that u_σ is continuous at x^* are immediate corrollaries of this. □

The most difficult case among the uniqueness results we have formulated in §1 is the case when the curves are perfectly conducting and Neumann data is fixed on the boundary (or by duality the case when the curves are perfectly insulating and Dirichlet data is fixed on the boundary). In this case we must ensure that no level curve for u_σ reaches the boundary at P or Q. This is exactly the purpose of the following two lemmas; the first lemma is related to the ϵ-dependent flux, defined in (1.6_ϵ), and it will be used in the proof of Theorem 1.1.

LEMMA 2.2. *Let ω be a smooth subdomain of Ω for which $\partial\omega \cap \partial\Omega$ is a nonempty curve containing the point $P \in \partial\Omega$ in its interior. Select δ sufficiently small so that*

$$\lambda_\delta \equiv \{x \in \partial\Omega; |x - P| < \delta\} \quad \textit{is contained in } \partial\omega \cap \partial\Omega.$$

Suppose $h_\epsilon \in L^\infty(\partial\omega)$ with $\|h_\epsilon\|_{L^\infty} \le C_0$ and $\int\limits_{\partial\omega\setminus\lambda_\delta} h_\epsilon = -2$. Let g_ϵ denote the function

$$g_\epsilon = \begin{cases} \epsilon^{-1} & \text{on } \lambda_{P,\epsilon} \\ 0 & \text{on } \lambda_\delta \setminus \lambda_{P,\epsilon} \\ h_\epsilon & \text{on } \partial\omega \setminus \lambda_\delta, \end{cases}$$

defined for any $\epsilon < \delta$, and suppose $u_\epsilon \in H^1$ is a solution of

$$\begin{aligned} \nabla\cdot(\gamma\nabla u_\epsilon) &= 0 \quad \textit{in } \omega, \\ \gamma\frac{\partial u_\epsilon}{\partial\nu} &= g_\epsilon \quad \textit{on } \partial\omega, \end{aligned} \tag{2.3}$$

with $\int\limits_{\partial\omega} u_\epsilon = 0$. Then, for any $c > 0$, there exist positive constants c_0, ϵ_0 depending only on c, C_0, δ, ω and γ such that

$$\forall\, \epsilon < \epsilon_0,\ x \in \omega, \qquad \mathrm{dist}(x, P) < c\epsilon \Rightarrow u_\epsilon(x) > c_0\ |log\epsilon| \tag{2.4}$$

Proof. Let $N_\omega(x;y)$ be the Neumann function for the domain ω (see e.g. [2; Chap.5]). Then u_ϵ may be represented as

$$\begin{aligned} u_\epsilon(x) &= \int\limits_{\lambda_{P,\epsilon}} \epsilon^{-1} N_\omega(x;y)\, ds_y + \int\limits_{\partial\omega\setminus\lambda_\delta} h_\epsilon N_\omega(x;y)\, ds_y \\ &= u_{1,\epsilon}(x) + u_{2,\epsilon}(x). \end{aligned} \tag{2.5}$$

Noting that $|N_\omega(x;y)| \le D$ if $|x - y| \ge d > 0$, we see that

$$|u_{2,\epsilon}(x)| \le C \tag{2.6}$$

for x in a neighborhood of P, uniformly in ϵ. To study the behavior of $u_{1,\epsilon}$ near P we introduce, for some small δ' $(0 < \delta' < \delta)$, a diffeomorphism $x' = \Phi(x)$ mapping $W_{\delta'} \equiv \{x \in \omega : |x - P| < \delta'\}$ onto a domain in $\{x_2' > 0\}$ which contains a semicircle

$$S^+ = \{x_1'^2 + x_2'^2 < \rho^2,\ x_2' > 0\}$$

such that $\partial S^+ \cap \{x_2' = 0\}$ is in the image of $\overline{W}_{\delta'} \cap \partial\omega$. The function $v_\epsilon(x') = u_{1,\epsilon}(\Phi^{-1}(x'))$ solves

$$\nabla\cdot(\gamma'(x')\nabla v_\epsilon(x')) = 0, \tag{2.7}$$

where $\gamma'(x')$ is the matrix $(\gamma'_{ij}(x'))$ given by

$$\gamma' \circ \Phi = \frac{D\Phi}{Dx}\,\gamma\left(\frac{D\Phi}{Dx}\right)^T (|\det \frac{D\Phi}{Dx}|)^{-1}.$$

We can choose Φ so that $\frac{D\Phi}{Dx}|_{x=P}$ is orthogonal and

$$\gamma'_{12}(x'_1, 0) = 0. \tag{2.8}$$

Indeed, picking a coordinate system with origin at P and x_1−axis tangential to $\partial\omega$ we can achieve (2.8) by first choosing

$$y_1 = x_1,\ y_2 = x_2 - k(x_1) \tag{2.9}$$

where $x_2 > k(x_1)$ defines ω in a neighborhood of P, and then second setting

$$x'_1 = y_1 + y_2 m(y_1),\ x'_2 = y_2$$

where

$$m(y_1) = -\frac{\tilde{\gamma}_{12}(y_1, y_2)}{\tilde{\gamma}_{22}(y_1, y_2)},$$

and $(\tilde{\gamma}_{ij}(y))$ is obtained from transformation of $\gamma(x)$ according to the change of variables (2.9), cf. [2; p. 196] (note that (2.9) guarantees that m is a function of y_1 alone). We now extend the coefficients γ'_{ij} into all of $S = \{x_1'^2 + x_2'^2 < \rho^2\}$ by defining

$$\gamma'_{12}(x'_1, x'_2) = -\gamma'_{12}(x'_1, -x'_2),$$
$$\gamma'_{ii}(x'_1, x'_2) = \gamma'_{ii}(x'_1, -x'_2)$$

for $x'_2 < 0$. In view of (2.8) the extended coefficients, which we again denote by γ'_{ij}, are Lipschitz continuous across $\{x'_2 = 0\}$. We can therefore construct in S a fundamental solution $K(x';\xi')$ by the parametrix method, i.e., we can construct a solution to

$$\nabla_{x'} \cdot (\gamma'(x')\nabla_{x'} K(x';\xi')) = -\delta(x' - \xi') \qquad \forall\, x', \xi' \in S \tag{2.10}$$

of the form

$$K(x';\xi') = -\mu(\xi')\log[(x' - \xi')^T \gamma'(\xi')^{-1}(x' - \xi')]^{1/2} + k(x';\xi') \tag{2.11}$$

where

$$\begin{aligned} &\mu(\xi') = \frac{1}{2\pi}[\det(\gamma'_{ij}(\xi'))]^{-1/2}, \\ &k \quad \text{is bounded, with } |\nabla_{x'} k(x';\xi')| \le \frac{C_t}{|x' - \xi'|^t} \quad \forall\, t > 0; \end{aligned} \tag{2.12}$$

for details see, for instance, [2; Chap 5]. The function

$$N(x';\xi') = K(x_1', x_2'; \xi') + K(x_1', -x_2'; \xi') \tag{2.13}$$

satisfies (2.10) for x', ξ' in S^+ and

$$\frac{\partial N(x';\xi')}{\partial x_2'} = 0 \qquad \text{on} \quad \{x_2' = 0\}. \tag{2.14}$$

We recall that $v_\epsilon(x') = u_{1,\epsilon}(\Phi^{-1}(x'))$ satisfies (2.7) and furthermore

$$\gamma'(x')\nabla v_\epsilon(x') \cdot \nu = \begin{cases} \epsilon^{-1} n(x_1') & \text{on } \lambda_\epsilon' \equiv \{(x_1', 0), -\theta_1(\epsilon) < x_1' < \theta_2(\epsilon)\} \\ 0 & \text{if } x' = (x_1', 0) \in \partial S^+ \setminus \lambda_\epsilon' \end{cases} \tag{2.15}$$

for some smooth function $n(\cdot)$ and some $\theta_i(\epsilon), \frac{\epsilon}{2} < \theta_i(\epsilon) < 2\epsilon$ provided δ' and ϵ are sufficiently small. The fact that $\frac{D\Phi}{Dx}|_{x=P}$ is orthogonal implies that n(0)=1. Since $\gamma_{12}'(x_1', 0) = 0$ and $\nu(x_1', 0) = (0, -1)^T$, (2.15) may be rewritten

$$\gamma_{22}'(x_1', 0) \frac{\partial v_\epsilon(x_1', 0)}{\partial x_2'} = \begin{cases} -\epsilon^{-1} n(x_1') & \text{if} \quad (x_1', 0) \in \lambda_\epsilon' \\ 0 & \text{if} \quad (x_1', 0) \in \partial S^+ \setminus \lambda_\epsilon'. \end{cases} \tag{2.16}$$

Consider the function

$$w_\epsilon(x') = \int_{-\theta_1(\epsilon)}^{\theta_2(\epsilon)} \epsilon^{-1} n(\xi_1') N(x_1', x_2'; \xi_1', 0)\, d\xi_1' \, . \tag{2.17}$$

Using well known properties of single layer potentials in combination with (2.11), (2.12) and (2.13) we get

$$\begin{aligned} &\gamma_{22}(x_1', 0) \lim_{x_2' \downarrow 0} \frac{\partial w_\epsilon(x_1', x_2')}{\partial x_2'} \\ &\qquad = -\epsilon^{-1} n(x_1') + \epsilon^{-1} \gamma_{22}(x_1', 0) \int_{-\theta_1(\epsilon)}^{\theta_2(\epsilon)} \frac{\partial N}{\partial x_2}(x_1', 0; \xi_1', 0) n(\xi_1')\, d\xi_1' \quad (x_1', 0) \in \lambda_\epsilon', \end{aligned}$$

where the last integral is an improper integral with integrand equal to zero, by (2.14). It follows that w_ϵ satisfies the first relation in (2.16); the second relation is verified in a similar manner. In combination with (2.15) this shows that the function $U_\epsilon = v_\epsilon - w_\epsilon$ satisfies

$$(\gamma' \nabla U_\epsilon) \cdot \nu = 0 \quad \text{on} \quad \partial S^+ \cap \{x_2' = 0\}. \tag{2.18}$$

At the same time

$$\nabla \cdot (\gamma' \nabla U_\epsilon) = 0 \quad \text{in} \quad S^+. \tag{2.19}$$

Because of the formulas (2.5) and (2.17) and the fact that $|N_\omega(x;\xi)| \le D$ for $|x-\xi| \ge d > 0$, $|N(x';\xi')| \le D$ for $|x'-\xi'| \ge d > 0$ it follows that if ϵ is sufficiently small

$$|DU_\epsilon(x')| + |U_\epsilon(x')| \le C \qquad \forall x' \in \partial S^+ \cap \{x_1'^2 + x_2'^2 = \rho^2\}. \tag{2.20}$$

with C independent of ϵ. From elliptic regularity theory it now follows that $\|U_\epsilon\|_{H^{3/2}(S^+)} \le C$ independently of ϵ and therefore, by Sobolev's imbedding theorem

$$|U_\epsilon(x')| \le C \quad \text{in} \quad S^+ \tag{2.21}$$

Consider now the function $w_\epsilon(x')$. From (2.11)–(2.13) we see that if (x_1', x_2') belongs to a $(c\epsilon)$–neighborhood of 0 then, for ϵ sufficiently small

$$\begin{aligned} w_\epsilon(x') &\ge \tilde{c} \int_{-\theta_1(\epsilon)}^{\theta_2(\epsilon)} \epsilon^{-1} |\log\ C\epsilon|\, d\xi_1' - \tilde{C} \\ &\ge \tilde{c}|\log\ \epsilon| - \tilde{\tilde{C}} \end{aligned}$$

where $\tilde{c}$, $\tilde{C}$ and $\tilde{\tilde{C}}$ are positive constants (independent of ϵ). A Combination of this estimate and (2.21), gives that $v_\epsilon(x') \ge c_*\ |log\ \epsilon|$ provided $(x_1'^2 + x_2'^2)^{1/2} < c\epsilon$. The assertion of the lemma immediately follows by switching back to the original coordinates and using (2.6). $\square$

REMARK 2.1. If $v_\epsilon \in H^1$ is any solution of

$$\begin{aligned} \nabla \cdot (\gamma \nabla v_\epsilon) &= 0 \quad \text{in } \omega, \\ \gamma \frac{\partial v_\epsilon}{\partial \nu} &= g_\epsilon \quad \text{on } \partial\omega, \end{aligned}$$

and u_ϵ denotes the solution to (2.3) (with $\int_{\partial\omega} u_\epsilon = 0$), then $v_\epsilon = u_\epsilon + \int_{\partial\omega} v_\epsilon$ and therefore it follows from Lemma 2.2 that

$$\forall\ \epsilon < \epsilon_0,\ x \in \omega, \qquad \text{dist}(x,P) < c\epsilon \Rightarrow v_\epsilon(x) > c_0\ |log\epsilon| + \int_{\partial\omega} v_\epsilon.$$

We shall next establish the limiting case of Lemma 2.2 corresponding to $\epsilon = 0$. This result will be used in the proof of Theorem 1.1'.

LEMMA 2.2'. *Let ω be a smooth subdomain of Ω for which $\partial\omega \cap \partial\Omega$ is a nonempty curve containing the point $P \in \partial\Omega$ in its interior. Let g be a function in $L^2(\partial\omega)$, with $\int_{\partial\omega} g = -1$. Let u denote a solution to*

$$\begin{aligned} \nabla\cdot(\gamma\nabla u) &= 0 \quad \text{in } \omega\,, \\ \gamma\frac{\partial u}{\partial\nu} &= \delta_P + g \quad \text{on } \partial\omega\,. \end{aligned}$$

Then

$$\lim_{\substack{x\to P\\ x\in\omega}} u(x) = \infty$$

Proof. By a smooth change of variables $x' = \Phi(x)$, ω may be mapped to a domain ω' in such a way that P is mapped to the origin and a neighborhood of P in ω is mapped to $\{x' : x_2' > 0, x_1'^2 + x_2'^2 < \rho^2\}$. The equations for $v(x') = u(\Phi^{-1}(x'))$ become

$$\begin{aligned} \nabla\cdot(\tilde\gamma\nabla v) &= 0 \quad \text{in} \quad \omega'\,, \\ (\tilde\gamma\nabla v)\cdot\nu &= \delta_0 + \tilde g \quad \text{on} \quad \partial\omega' \end{aligned}$$

where

$$\begin{aligned} \tilde\gamma\circ\Phi &= \left(\frac{D\Phi}{Dx}\right)\gamma\left(\frac{D\Phi}{Dx}\right)^T |\det\frac{D\Phi}{Dx}|^{-1}, \\ \tilde g\circ\Phi &= |\left(\frac{D\Phi}{Dx}\right)\tau|^{-1} g, \end{aligned} \tag{2.22}$$

τ is the unit tangent to $\partial\omega$. It is always possible to construct Φ so that

$$\left.\frac{D\Phi}{Dx}\right|_{x=P} \quad \text{is orthogonal ;} \tag{2.23}$$

for instance if we pick the coordinate system with origin at P and x_1-axis tangent to $\partial\omega$, then $\partial\omega$ is locally given by $x_2 = h(x_1)$, with $h'(0) = 0$. Near the origin we may now "flatten" the boundary by mapping $(x_1, x_2) \to (x_1, x_2 - h(x_1))$.

From (2.22) and (2.23) it follows that $\tilde\gamma(0)$ is a scalar. Let v_0 denote the function

$$v_0(x') = \frac{1}{\tilde\gamma(0)\pi}\log|x'|. \tag{2.24}$$

Then

$$\Delta v_0 = 0 \text{ in } \omega' \text{ and } \frac{\partial v_0}{\partial\nu} = -\tilde\gamma(0)^{-1}(\delta_0 + k) \text{ on } \partial\omega'$$

where $k \in C^\infty(\partial\omega')$ and $k \equiv 0$ in neighborhood of 0. We therefore get

$$
\begin{aligned}
\int_{\omega'} \tilde{\gamma}\nabla(v+v_0)\cdot\nabla\phi\; dx' &= \phi(0) + \int_{\partial\omega'} \tilde{g}\phi\; ds' \\
&\quad + \int_{\omega'} (\tilde{\gamma}-\tilde{\gamma}(0))\nabla v_0\cdot\nabla\phi\; dx' \\
&\quad - \phi(0) - \int_{\partial\omega'} k\phi\; ds' \\
&= \int_{\partial\omega'} (\tilde{g}-k)\phi\; ds' + \int_{\omega'} (\tilde{\gamma}-\tilde{\gamma}(0))\nabla v_0\cdot\nabla\phi dx',
\end{aligned}
\tag{2.25}
$$

for any $\phi \in C^\infty(\omega')$. Concerning the last term on the right- hand side,

$$
\begin{aligned}
\int_{\omega'} (\tilde{\gamma}-\tilde{\gamma}(0))\nabla v_0\cdot\nabla\phi\; dx' &= \lim_{\epsilon\to 0} \int_{\omega'\cap\{|x'|>\epsilon\}} (\tilde{\gamma}-\tilde{\gamma}(0))\nabla v_0\cdot\nabla\phi dx' \\
&= \lim_{\epsilon\to 0} \Big(- \int_{\omega'\cap\{|x'|>\epsilon\}} \nabla\cdot[(\tilde{\gamma}-\tilde{\gamma}(0))\cdot\nabla v_0]\phi\; dx' \\
&\quad + \int_{\partial(\omega'\cap\{|x'|>\epsilon\})} [(\tilde{\gamma}-\tilde{\gamma}(0))\nabla v_0]\cdot\nu\phi\; ds' \Big)
\end{aligned}
$$

and since $\nabla\cdot[(\tilde{\gamma}-\tilde{\gamma}(0))\cdot\nabla v_0]$ is bounded by $C|x'|^{-1}$ in ω' and $(\tilde{\gamma}-\tilde{\gamma}(0))\nabla v_0$ is uniformly bounded on $\omega'\cap\{|x'|>\epsilon\}$, it follows that

$$
\Big|\int_{\omega'} (\tilde{\gamma}-\tilde{\gamma}(0))\nabla v_0\cdot\nabla\phi\; dx'\Big| \le C_q(\|\phi\|_{L^q(\omega')} + \|\phi\|_{L^2(\partial\omega')}),
\tag{2.26}
$$

for any $\phi \in C^\infty(\omega')$ and any $q>2$. From Sobolev's imbedding theorem we have

$$
H^s \hookrightarrow L^q \quad \text{continuously,}
$$

provided $s > 1 - \dfrac{2}{q}$, and the well known trace theorem asserts that

$$
\|\phi\|_{L^2(\partial\omega')} \le C_s\|\phi\|_{H^s(\omega')}
$$

provided $s > 1/2$. Based on (2.26) it now follows that

$$
\Big|\int_{\omega'} (\tilde{\gamma}-\tilde{\gamma}(0))\cdot\nabla v_0\nabla\phi\; dx'\Big| \le C_s\|\phi\|_{H^s(\omega')}
\tag{2.27}
$$

for any $\phi \in C^\infty(\omega')$ and any $s > 1/2$. By insertion of (2.27) into (2.25), using that $\tilde{g} - k \in L^2(\partial\omega')$, we get

$$\Big| \int_{\omega'} \tilde{\gamma}\nabla(v + v_0) \cdot \nabla\phi dx' \Big| \leq C_s \|\phi\|_{H^s(\omega')}$$

for any $\phi \in C^\infty(\omega')$ and any $s > 1/2$. Elliptic regularity theory now immediately gives that

$$v + v_0 \in H^t(\omega') \quad \text{for any} \quad t < 3/2,$$

and thus according to Sobolev's imbedding theorem

$$v + v_0 \in L^\infty(\omega').$$

It follows that $\lim\limits_{x' \to z'} v = \infty$ whenever $\lim\limits_{x' \to z'} v_0 = -\infty$, and as a consequence

$$\lim_{\substack{x \to P \\ x \in \omega}} u(x) = \lim_{\substack{x' \to 0 \\ x \in \omega'}} v(x') = \infty \ .$$

□

LEMMA 2.3. *Let $v_\sigma \in H^1(\Omega)$ be a solution to $\nabla \cdot (\gamma \nabla v_\sigma) = 0$ in $\Omega \setminus \sigma$, $v_\sigma =$ constant on σ and $\gamma\nabla v_\sigma \cdot \nu = \psi$ on $\partial\Omega$ for some piecewise constant function ψ, which does not vanish everywhere. Let ρ be a non self-intersecting analytic curve in $\Omega \setminus \sigma$ on which v_σ is constant. Then there exists an open analytic curve $\rho' \subset \Omega \setminus \sigma$ parametrized, say, by $x = \lambda(t), -\infty < t < \infty$, such that*

(2.28) $\qquad \rho \subset \rho'$ *and v_σ is constant on ρ',*

(2.29) $\qquad$ *ρ' does not self-intersect and it has endpoints*
$\qquad z_0 = \lim\limits_{t \to -\infty} \lambda(t),\ z_1 = \lim\limits_{t \to \infty} \lambda(t)$;

furthermore

(2.30) $\qquad$ *one of the endpoints of ρ' lies on $\partial\Omega$*
$\qquad$ *and the other lies on $\partial\Omega$ or on σ.*

Proof. Let D be any compact subdomain of $\Omega \setminus \sigma$ which contains ρ. We shall extend ρ step-by-step at both endpoints until the extended curve exits D (at both ends). To perform such an extension about an endpoint x^* of ρ, notice that if $\nabla v_\sigma(x^*) \neq 0$ then the extension is uniquely established by the implicit function theorem applied to the solution of $v_\sigma(x) = v_\sigma(x^*)$.

Suppose now that $\nabla v_\sigma(x^*) = 0$. Let $(r,\theta) \in [0,\epsilon] \times [0,2\pi]$ denote polar coordinates near x^*. Since $\nabla v_\sigma(x^*) = 0$ we know that $\frac{\partial}{\partial r} v_\sigma(0,\theta) \equiv 0$, and by expanding in a Taylor series in r,

$$v_\sigma(x) = v_\sigma(x^*) + r^n(a \sin n\theta + b \cos n\theta + rA(r,\theta))$$

for some a, b (not both zero) and some $n \geq 2$. Here we have used that $v_\sigma(x)$ satisfies $\nabla \cdot (\gamma \nabla v_\sigma) = 0$ near x^*, and is therefore analytic near x^*. We may without loss of generality assume that the (half) tangent to ρ at x^* is $\{(r,0),\ r > 0\}$. It follows that $b = 0$, i.e.,

$$v_\sigma(x) = v_\sigma(x^*) + ar^n(\sin n\theta + rA(r,\theta)) \ , \quad a \neq 0.$$

Since v_σ is analytic near x^* it is well known that v_σ as a function of (r,θ) is analytic on $[0,\epsilon] \times [0,2\pi]$, the main point being that it is analytic at $r = 0$ (indeed the analytic extension for negative r is given by $\tilde{u}(r,\theta) = u(-r,\theta+\pi)$). It follows that $A(r,\theta)$ also has an analytic extension near $r = 0$; we denote this extension by $\tilde{A}(r,\theta)$, $(r,\theta) \in [-\epsilon,\epsilon] \times [0,2\pi]$. The function $F(r,\theta) = \sin n\theta + r\tilde{A}(r,\theta)$ satisfies

$$F(0,0) = 0 \quad \text{and} \quad \frac{\partial}{\partial\theta} F(0,0) = n \ ,$$

and therefore, by the implicit function theorem, it is possible to find a unique analytic function $\theta(r)$ such that $\theta(0) = 0$ and $\{(r,\theta) : F(r,\theta) = 0\}$ equals $\{(r,\theta(r))\}$ in a neighborhood of $(0,0)$. The curve

$$(r \cos\theta(r), r \sin\theta(r)) + x^*$$

is an analytic curve through x^*. For r positive this curve (due to the implicit function theorem) coincides with ρ, and it therefore represents an analytic extension beyond x^*.

The above argument shows that the extension can be carried out step–by–step, and each time, if the old endpoint x^* is in D, then the extension can be achieved in a δ–neighborhood of x^* with δ independent of x^*; δ will depend on D, of course. It is easy to see that a level curve of v_σ inside D will have finite length; it follows that after a finite number of steps we get an extension ρ'_D of ρ with both endpoints on ∂D. The level curve thus constructed cannot have self–intersections. Indeed, if there were self-intersections then there would be a subdomain of Ω bounded by a portion of the level curve inside which, by the maximum principle, $v_\sigma \equiv$ constant. By unique continuation v_σ would therefore be constant in all of Ω and this is a contradiction.

We now repeat the process with a sequence of increasing domains D_m $(D_1 = D)$ which exhaust $\Omega \setminus \sigma$. We denote the curve thus obtained by ρ' and we parametrize it by $x = \lambda(t), -\infty < t < \infty$, such that at the points $x = \lambda(\pm t_m)$, (with $t_m \to \infty$) it exits D_m for the first time. We claim that there cannot exist a sequence $\tau_j \to +\infty$ (or $-\infty$) such that

$$\lambda(\tau_j) \to \overline{x},\ \overline{x} \in \Omega \setminus \sigma. \tag{2.31}$$

Indeed, otherwise there is, in any δ-neighborhood of $\overline{x}$, an infinite number of distinct subarcs of ρ' which enter and exit this neighborhood. This however is a contradiction to the construction of ρ', if δ is small enough.

We wish to prove that $\lim_{t\to\infty} \lambda(t)$ exists. If this is not so, then by the previous paragraph it follows that $\lambda(t)$ oscillates near the boundary of $\Omega \setminus \sigma$ and, in fact, either

$$\lambda(t) \quad \text{oscillates near a curve } \mu \text{ on } \partial\Omega, \text{ as } t \to \infty, \tag{2.32}$$

or

$$\lambda(t) \quad \text{oscillates near a curve } \mu \text{ on } \sigma, \text{ as } t \to \infty. \tag{2.33}$$

We may without loss of generality assume that v_σ is smooth up to the curve μ, approaching from one side. Taking a line normal to μ which intersects μ at a point y^*, we then see that $v_\sigma(x) =$ constant at a sequence of points on this line converging to y^*. Consequently v_σ and $\partial v_\sigma/\partial\nu$ vanish at y^*, and since y^* is arbitrary,

$$v_\sigma = \frac{\partial v_\sigma}{\partial \nu} = 0 \quad \text{on} \quad \mu\,. \tag{2.34}$$

We can now apply the unique continuation theorem to deduce that $v_\sigma = 0$ in Ω, which is a contradiction.

This shows that $z_1 \equiv \lim_{t\to\infty} \lambda(t)$ exists. The point $z_0 = \lim_{t\to-\infty} \lambda(t)$ exists for similar reasons. Furthermore, z_0 and z_1 belong to $\partial(\Omega \setminus \sigma)$ for otherwise we shall be in a situation where (2.31) holds, and this was already excluded. Finally, at least one of the points z_i must belong to $\partial\Omega$. Otherwise $v_\sigma =$ constant in the domain bounded by ρ' and a portion of σ (between z_0 and z_1), and this would be a contradiction. □

§3. Proof of Theorem 1.1. Since the boundary data is entirely in terms of derivatives we may without loss of generality assume that it concerns the solutions for which $\int_{\partial\Omega} u_\sigma^{(i)} = \int_{\partial\Omega} u_{\sigma'}^{(i)} = 0$. Let O be the open set enclosed by σ and σ'. Since $\Omega \setminus (O \cup \sigma \cup \sigma')$ has only one connected component, it follows from the assumptions about the boundary data (via the unique continuation theorem) that $u_\sigma^{(i)} - u_{\sigma'}^{(i)}$, $i = 1,2$ are constant in $\Omega \setminus (O \cup \sigma \cup \sigma')$. Due to the normalization of $u_\sigma^{(i)}$ and $u_{\sigma'}^{(i)}$ it follows furthermore that these constants are zero, i.e.,

$$u_\sigma^{(i)} - u_{\sigma'}^{(i)} = 0 \quad \text{in} \quad \Omega \setminus (O \cup \sigma \cup \sigma') \quad i = 1,2. \tag{3.1}$$

If O is nonempty then ∂O consists of pieces of σ and σ' on which $u_\sigma^{(i)}$, respectively $u_{\sigma'}^{(i)}$, is a constant. Due to (3.1) it now follows that each function $u_\sigma^{(i)}$ takes at most two values on ∂O. Recalling that $u_\sigma^{(i)}$ is continuous in Ω (Lemma 2.1), we get that $u_\sigma^{(i)}$ is constant on each connected component of ∂O. It follows, by the maximum principle, that $u_\sigma^{(i)}$ is constant on each connected component of O, and this implies that $u_\sigma^{(i)}$ is constant in all of Ω – a contradiction. We thus conclude O is empty so $u_\sigma^{(i)} - u_{\sigma'}^{(i)} = 0 \quad \text{in} \quad \Omega \setminus (\sigma \cup \sigma')$; by continuity it now follows that

$$u_\sigma^{(i)} - u_{\sigma'}^{(i)} = 0 \quad \text{in} \ \ \Omega \quad i = 1,2. \tag{3.2}$$

From Lemma 2.1,

$$|u_\sigma^{(i)}| \le \tilde{C} \quad \text{in} \quad K. \tag{3.3}$$

The constant $\tilde{C} = \tilde{C}(K)$ is easily seen to be independent of σ, it is also independent of ϵ due to interior elliptic regularity results and the fact that $\int_{\partial\Omega} |\psi_2| = 4$.

Let ω be a smooth subdomain of Ω for which $\partial\omega \cap \partial\Omega$ is a nonempty curve containing P in its interior and $\overline{\omega} \cap K = \emptyset$. By elliptic regularity theory it follows that $u_\sigma^{(2)}$ satisfies equations of the type (2.3), *but not* with the boundary integral on $\partial\omega$ vanishing. According to Lemma 2.2 and the remark following it there exist ϵ_0 and c_0 such that

$$u_\sigma^{(2)} > c_0 |\log \epsilon| + \int_{\partial\omega} u_\sigma^{(2)} \tag{3.4}$$

in a (2ϵ)–neighborhood of P, for any $\epsilon < \epsilon_0$. By elliptic estimates we get for any $1/2 < t < 1$

$$\begin{aligned} \Big|\int_{\partial\omega} u_\sigma^{(2)}\Big| &\le C_t \|u_\sigma^{(2)}\|_{H^t(\omega)} \\ &\le C_t' \|u_\sigma^{(2)}\|_{H^t(\Omega)} \\ &\le C_t'' \|\psi_2\|_{H^{t-3/2}(\partial\Omega)}. \end{aligned} \tag{3.5}$$

From Sobolev's imbedding theorem and the fact that $\|\psi_2\|_{L^1(\partial\Omega)} = 4$ we get that

$$\Big|\int_{\partial\Omega} \psi_2 v\Big| \le \|\psi_2\|_{L^1(\partial\Omega)} \|v\|_{L^\infty(\partial\Omega)} \le C_s \|v\|_{H^s(\partial\Omega)},$$

for any $v \in H^s(\Omega)$ and any $1/2 < s$. By duality we thus conclude that $\|\psi_2\|_{H^{-s}(\partial\Omega)}$ is bounded independently of ϵ for any $-s < -1/2$. If $t < 1$ then $t - 3/2 < -1/2$ and it now follows from the estimate (3.5) that

$$\Big|\int_{\partial\omega} u_\sigma^{(2)}\Big| \text{ is bounded independently of } \epsilon.$$

Using (3.4) we now conclude that there exists ϵ_0 such that

$$(3.6) \qquad |u_\sigma^{(2)}| > \tilde{C}$$

in a (2ϵ)–neighborhood of P for $\epsilon < \epsilon_0$ ($\tilde{C}$ is the same constant as in (3.3)). A similar statement holds with P replaced by Q. It follows by a combination of (3.3) and (3.6) that

(3.7) any level curve of $u_\sigma^{(2)}$ $(= u_{\sigma'}^{(2)})$ which intersects K cannot intersect $\overline{\lambda}_{P,\epsilon}$ or $\overline{\lambda}_{Q,\epsilon}$.

We shall now assume that $\sigma \neq \sigma'$ and proceed to derive a contradiction. Without loss of generality we may assume that there exists a curve $\rho \subset \sigma'$, which lies in $\Omega \setminus \sigma$. Based on (3.2) we now conclude that $u_\sigma^{(i)}$ are constant on ρ, $i = 1, 2$. There must exist a point on ρ where $\nabla u_\sigma^{(1)} \neq 0$, since otherwise $u_\sigma^{(1)}$ is constant in Ω (by unique continuation); near this point ρ must be analytic because of the implict function theorem. By shortening if necessary we may assume that the entire curve ρ is analytic. We are now in a situation where we can apply Lemma 2.3. Corresponding to each $u_\sigma^{(i)}$ we get an open analytic curve such that (2.28)–(2.30) are satisfied, but due to the analyticity the two curves must coincide. In summary, there exists an open analytic curve $\rho' \subseteq \Omega \setminus \sigma$ such that

$$(3.8) \qquad \rho \subseteq \rho' \quad \text{and} \quad u_\sigma^{(i)} \quad \text{is constant on } \rho', i = 1, 2,$$

(3.9) ρ' has one endpoint, z_0, on $\partial\Omega$ and the other endpoint lies on σ or on $\partial\Omega$

From (3.7) it follows that z_0 is not contained in $\overline{\lambda}_{P,\epsilon} \cup \overline{\lambda}_{Q,\epsilon}$. Therefore, by (1.5), $\dfrac{\partial}{\partial\nu} u_\sigma^{(1)} \neq 0$ in a neighborhood of z_0 and we conclude that ρ' (a level curve of $u_\sigma^{(1)}$) meets $\partial\Omega$ at an angle different from $\dfrac{\pi}{2}$. At z_0 we now have that $\dfrac{\partial}{\partial\nu} u_\sigma^{(2)}$ as well as the tangential derivative of $u_\sigma^{(2)}$ along ρ' vanish. Since these derivatives are in linearly independent directions, we see that

$$(3.10) \qquad z_0 \text{ is a critical point for } u_\sigma^{(2)}.$$

Let (r, θ) denote the polar coordinates around z_0 chosen so that the (half) tangent to ρ' is given by $\{(r, 0),\ r > 0\}$. Since $\partial u_\sigma^{(2)} / \partial\nu = 0$ in a neighborhood of z_0 on $\partial\Omega$ and since $\partial\Omega$ is analytic, $u_\sigma^{(2)}$ has an analytic extension into a full neighborhood of z_0 in $\mathbf{R}^2$ (here is the only place in the proof of Theorem 1.1 where the analyticity of $\partial\Omega$ is required; elsewhere $\partial\Omega \in C^{2+\alpha}$ suffices). We now get, as in the proof of Lemma 2.3,

$$u_\sigma^{(2)}(x) = u_\sigma^{(2)}(z_0) + a r^n (\sin n\theta + r A(r, \theta)) \ , \ a \neq 0, n \geq 2,$$

near z_0. Let $\tilde{A}$ denote the analytic extension of A to $[-\epsilon, \epsilon] \times [0, 2\pi]$. The function $F(r, \theta) = \sin n\theta + r\tilde{A}(r, \theta)$ satisfies

$$F(0, \pm\frac{\pi}{n}) = 0 \quad \text{and} \quad \frac{\partial}{\partial\theta} F(0, \pm\frac{\pi}{n}) \neq 0 \,;$$

therefore, by the implicit function theorem, it is possible to find unique analytic functions $\theta_{\pm}(r)$ such that $\theta_{\pm}(0) = \pm\pi/n$ and $\{(r, \theta) : F(r, \theta) = 0\}$ equals $\{(r, \theta_{\pm}(r))\}$ in a neighborhood of $(0, \pm\frac{\pi}{n})$. The curves

$$\rho_{\pm} : (r\cos\theta_{\pm}(r), r\sin\theta_{\pm}(r)) + z_0$$

are analytic curves through z_0 and since ρ' is not normal to $\partial\Omega$ at z_0 and $\frac{\pi}{n} \leq \frac{\pi}{2}$, it is clear that at least one of these curves, say ρ_-, is contained in Ω for small positive r. The function $u_\sigma^{(2)}$ takes a constant value, the same as on ρ', on the curve ρ_-. Assume that ρ_- is taken small enough that it does not intersect σ. We now use Lemma 2.3 to find a curve $\rho'' \subset \Omega \setminus \sigma$, which contains ρ_-, such that

(3.11) $\qquad u_\sigma^{(2)}$ is constant along ρ'' (with the same constant value as on ρ'),

(3.12) $\qquad \rho''$ has one endpoint at z_0, the other endpoint, z_1, lies on σ or $\partial\Omega$.

It is not possible for both ρ' and ρ'' to have an endpoint on σ. Indeed, if this were the case then $\overline{\rho'} \cup \overline{\rho''} \cup \sigma$ would enclose an open set on which by necessity $u_\sigma^{(2)} =$ constant; this would in turn lead to $u_\sigma^{(2)} \equiv$ constant in Ω, a contradiction. We shall assume for definiteness that ρ'' has its other endpoint, z_1, on $\partial\Omega$. The curve ρ'' does not touch ρ', except at z_0, and neither ρ'' nor ρ' have any self intersections, for the same reasons that ρ' and ρ'' did not both touch σ. The curve ρ'' divides Ω into two subdomains $\tilde{\Omega}$ and $\tilde{\tilde{\Omega}}$. $\tilde{\Omega}$ is bounded by $\widehat{z_1 z_0}$ and ρ'', and $\tilde{\tilde{\Omega}}$ is bounded by $\widehat{z_0 z_1}$ and ρ'' . We may assume that ρ' lies in $\tilde{\Omega}$. By (3.7) both points z_0 and z_1 lie outside $\lambda_{P,\epsilon}$ and $\lambda_{Q,\epsilon}$.

If $\lambda_{P,\epsilon}$ and $\lambda_{Q,\epsilon}$ both lie in $\widehat{z_0 z_1}$ or $\widehat{z_1 z_0}$, then it follows immediately that $u_\sigma^{(2)}$ is constant in one of the domains $\tilde{\Omega}$ or $\tilde{\tilde{\Omega}}$, which is a contradiction. Without loss of generality we may therefore assume that $\lambda_{P,\epsilon}$ is contained in $\partial\tilde{\Omega}$ and does not intersect $\partial\tilde{\tilde{\Omega}}$ (the case where $\lambda_{Q,\epsilon} \subset \partial\tilde{\Omega}$, $\lambda_{P,\epsilon} \subset \partial\tilde{\tilde{\Omega}}$ can be treated in the same way). The situation is as described in Figure 1.a or 1.b.

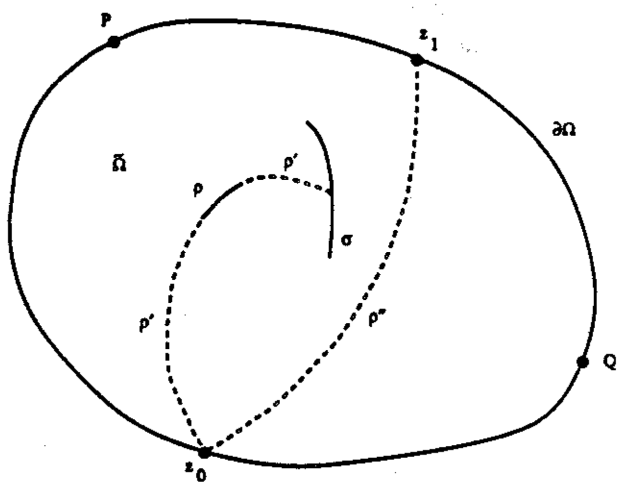

Figure 1.a

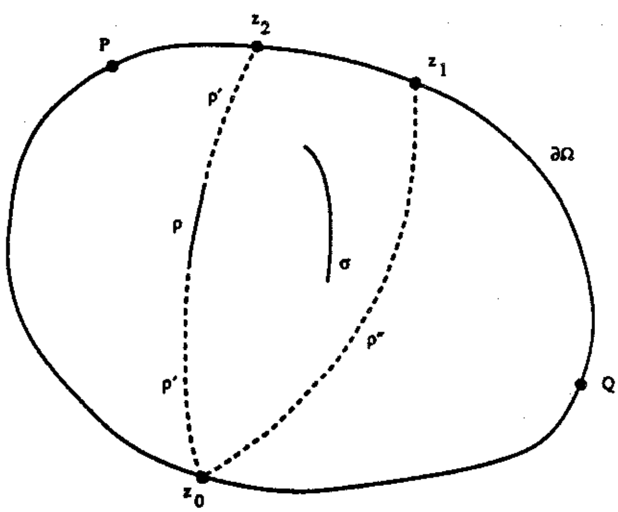

Figure 1.b

Consider first the case where both endpoints of ρ', z_0 and z_2, lie on $\partial\Omega$ (Figure 1.b). By (3.7) $\lambda_{P,\epsilon}$ lies either in $\widehat{z_1 z_2}$ or $\widehat{z_2 z_0}$. It follows that $u_\sigma^{(2)} \equiv$ constant in one of the two subdomains that ρ' divides $\tilde{\Omega}$ into, and we get a contradiction. We now know that ρ' must end on σ as in Figure 1.a. Let z be a point on ρ' (in $\Omega \setminus \sigma$), then the function $u_\sigma^{(2)}$ is a solution to $\nabla \cdot (\gamma \nabla v) = 0$ in a neighborhood of z. From the maximum principle it follows immediately that $u_\sigma^{(2)}$ attains values strictly above and strictly below $u_\sigma^{(2)}(z) = u_\sigma^{(2)}(z_0)$ in any neighborhood of z. As a consequence

$$\inf_{x\in\tilde\Omega} u_\sigma^{(2)}(x) < u_\sigma^{(2)}(z_0) < \sup_{x\in\tilde\Omega} u_\sigma^{(2)}(x) \tag{3.13}$$

The strong maximum principle applied to the function $u_\sigma^{(2)}$ in the domain $\Omega' = \tilde\Omega\setminus(\rho'\cup\sigma)$ asserts that minimal value $\inf_{x\in\Omega'} u_\sigma^{(2)}(x) = \inf_{x\in\tilde\Omega} u_\sigma^{(2)}(x)$ cannot be attained on $\widehat{z_1 z_0}$, since $\partial u_\sigma^{(2)}/\partial\nu$ is non-negative there. It follows that the minimal value must be attained on ρ', σ or ρ'', and therefore

$$u_\sigma^{(2)}(z_0) = \inf_{x\in\tilde\Omega} u_\sigma^{(2)}(x).$$

This contradicts (3.13), and so we conclude that the initial assumption $\sigma \neq \sigma'$ is invalid. This completes the proof of the theorem. □

Proof of Theorem 1.1'. The proof is nearly identical to the proof of Theorem 1.1 provided we use Lemma 2.2' instead of Lemma 2.2. □

§4. **Proof of Theorem 1.2.** Consider a solution $u_\sigma^{(i)}$ of problem (II) and set

$$\tau_\sigma^{(i)} \equiv \gamma\nabla u_\sigma^{(i)}.$$

Then $\nabla\cdot\tau_\sigma^{(i)} = 0$ in $\Omega\setminus\sigma$. Since $\tau_\sigma^{(i)}$ and $\nabla\cdot\tau_\sigma^{(i)}$ belong to $L^2(\Omega\setminus\sigma)$, $\tau_\sigma^{(i)}\cdot\nu$ is well defined on both sides of σ by means of Green's formula. From the variational formulation of the solution to problem (II) we find that $\tau_\sigma^{(i)}\cdot\nu = 0$ on σ. This allows us to conclude that $\nabla\cdot\tau_\sigma^{(i)} = 0$ in Ω, in the sense of distributions. Consequently

$$\tau_\sigma^{(i)} = -\left(\nabla w_\sigma^{(i)}\right)^\perp \equiv \left(\frac{\partial w_\sigma^{(i)}}{\partial x_2}, -\frac{\partial w_\sigma^{(i)}}{\partial x_1}\right),$$

for some $w_\sigma^{(i)} \in H^1(\Omega)$. The function $w_\sigma^{(i)}$ is uniquely determined by the normalization $\int_{\partial\Omega} w_\sigma^{(i)} = 0$, and it satisfies $\partial w_\sigma^{(i)}/\partial s = -(\nabla w_\sigma^{(i)})^\perp\cdot\nu = \psi_i$ on $\partial\Omega$, $w_\sigma^{(i)}$ = constant on σ; here $\partial/\partial s$ denotes the counterclockwise tangential derivative on $\partial\Omega$. We easily find that $\nabla\cdot(\gamma^{-1}\nabla w_\sigma^{(i)}) = 0$ in $\Omega\setminus\sigma$, and in summary $w_\sigma^{(i)}$ therefore satisfies

$$\nabla\cdot\left(\gamma^{-1}\nabla w_\sigma^{(i)}\right) = 0 \quad \text{in} \quad \Omega\setminus\sigma,$$
$$w_\sigma^{(i)} = \text{constant on } \sigma, \quad w_\sigma^{(i)} = \phi_i \text{ on } \partial\Omega.$$

with $\phi_i \equiv \int \psi_i ds$, a counterclockwise indefinite integral along $\partial\Omega$.

We initially proceed as in the proof of Theorem 1.1 working with $w_\sigma^{(i)}$ in place of $u_\sigma^{(i)}$. The level curve ρ' meets the boundary $\partial\Omega$ at z_0. According to (1.12) z_0 is either an extremal point for $\tilde\phi_1$ or $\tilde\phi_2$. Without loss of generality we shall assume z_0 is a point where

$\tilde{\phi}_1$ attains its maximal value c_m (in other words z_0 lies on the curve $\widehat{QR}$). The function $w_\sigma^{(1)}$ is constantly equal to c_m along ρ'; if ρ' had both endpoints on $\partial\Omega$ then these would have to lie on the curve $\widehat{QR}$, and there would be a subdomain of Ω enclosed by $\widehat{QR}$ and ρ' in which $w_\sigma^{(1)}$ was constant – a contradiction. We therefore conclude that ρ' has one endpoint (z_0) on $\partial\Omega$ the other on σ, and in particular $w_\sigma^{(1)} = c_m$ on σ. By application of the maximum principle we see that

$$\sup_{x\in\Omega\setminus\sigma} w_\sigma^{(1)}(x) = c_m. \tag{4.1}$$

Let z be a point on ρ' near z_0, such that $w_\sigma^{(1)}$ satisfies $\nabla\cdot(\gamma^{-1}\nabla w_\sigma^{(1)}) = 0$ in a neighborhood of z. From the maximum principle it immediately follows that $w_\sigma^{(1)}$ takes values above $w_\sigma^{(1)}(z) = c_m$ in any neighborhood of z, and this contradicts (4.1). The initial assumption that $\sigma \neq \sigma'$ is therefore invalid, and this concludes the proof of Theorem 1.2.

□

§5. **Stability.** In the remainder of this paper we shall for definiteness only consider solutions to the problem (I), although completely parallel results can be obtained for solutions to problem (II). By our conventional abuse of notation we refer to the curves σ and σ' as cracks even though they are peferctly conducting. Let M, L_0, L_1 and α, $0 < \alpha < 1$, be positive constants and K a compact subdomain of Ω. Consider the class Σ of all $C^{2+\alpha}$ curves in K, of length $L_0 \le l(\sigma) \le L_1$, which when parametrized by arclength s satisfy

$$\sigma = \{x(s),\ 0 \le s \le l\} \quad \text{with} \quad \|x(\cdot)\|_{C^{2+\alpha}} \le M, \tag{5.1}$$

and

$$\int_{s_1}^{s_2} |x'(s)|\, ds \le M|x(s_1) - x(s_2)| \quad \text{for any} \quad s_1, s_2 \in [0, l]. \tag{5.2}$$

Let Γ be a nonempty open portion of $\partial\Omega$, so that $\overline{\Gamma}$ does not contain P or Q. If $\sigma_n = \{x_n(s),\ 0 \le s \le l_n\}$ is a sequence of curves in Σ converging to $\sigma = \{x(s),\ 0 \le s \le l\}$ in the sense that

$$l_n \to l \quad \text{and} \quad \|x_n(\cdot) - x(\cdot)\|_{C^2[\delta, l-\delta]} \to 0 \quad \forall\, \delta > 0, \tag{5.3}$$

then σ will again be in Σ and the corresponding solutions $u_{\sigma_n}^{(i)}$, $u_\sigma^{(i)}$ satisfy

$$\|\frac{\partial}{\partial s}(u_{\sigma_n}^{(i)} - u_\sigma^{(i)})\|_{L^\infty(\Gamma)} \to 0 \quad \text{as} \quad n \to \infty \tag{5.4}$$

for $i = 1, 2$, provided ϵ is sufficiently small that $\lambda_{P,\epsilon} \cup \lambda_{Q,\epsilon}$ does not intersect Γ.

Theorem 1.1 allows us to establish the following stability result:

THEOREM 5.1. *Suppose $\epsilon < \epsilon_0$ as in Theorem 1.1. If σ_n, $\sigma \in \Sigma$ and*

$$\sum_{i=1}^{2} \|\frac{\partial}{\partial s}(u^{(i)}_{\sigma_n} - u^{(i)}_{\sigma})\|_{L^\infty(\Gamma)} \to 0 \text{ as } n \to \infty, \tag{5.5}$$

then

$$\mathrm{dist}(\sigma_n, \sigma) \to 0$$

where $\mathrm{dist}(\sigma_n, \sigma) = \max_{x \in \sigma_n} \min_{y \in \sigma} |x - y|$.

Proof. By compactness, the sequence σ_n converges to a curve $\tilde{\sigma} = \{\tilde{x}(s), 0 \le s \le \tilde{l}\}$ in the sense of (5.3). From (5.4) (with $\sigma = \tilde{\sigma}$) and the assumption (5.5) we get that

$$\frac{\partial}{\partial s}(u^{(i)}_{\sigma} - u^{(i)}_{\tilde{\sigma}}) = 0 \quad \text{on} \quad \Gamma, \quad i = 1, 2.$$

Both curves σ and $\tilde{\sigma}$ are C^2 and they are non self-intersecting, due to (5.2). Therefore Theorem 1.1 asserts that $\sigma = \tilde{\sigma}$ and it immediately follows that $\mathrm{dist}(\sigma_n, \sigma) = \mathrm{dist}(\sigma_n, \tilde{\sigma}) \to 0$. $\square$

Theorem 5.1 does not provide any estimates. We now proceed to establish a partial stability estimate in case the curves are line segments.

Suppose a curve σ_t is given by

$$\sigma_t = (I + t\delta A)\sigma + t\delta b\ ,$$

where δA is a matrix of the form

$$\delta A = \begin{pmatrix} 0 & -\delta\theta \\ \delta\theta & 0 \end{pmatrix},$$

$\delta\theta$ is a scalar and δb is a vector. The curve $\sigma \subset \Omega$ is fixed. Let $N_t(\cdot\,; y)$, $y \in \Omega \setminus \sigma_t$ be the solution to

$$\begin{aligned} -\nabla \cdot (\gamma \nabla N_t(\cdot\,; y) &= \delta_y\ , \quad \text{in } \Omega \setminus \sigma_t\ , \\ N_t(\cdot\,; y) &= 0 \quad \text{on } \sigma_t\ , \\ \gamma \frac{\partial N_t}{\partial \nu}(\cdot\,; y) &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{5.6}$$

(t is sufficiently small that $\sigma_t \subset \Omega$). We define

$$\begin{aligned} \phi_t(x) &= (I + t\delta A)x + t\delta b\ , \text{ and} \\ M_t(x; y) &= N_t(\phi_t(x); y). \end{aligned}$$

A simple change of variables gives

$$\int\limits_{\phi_t^{-1}(\Omega)} \gamma(\phi_t(x))\nabla M_t(x;y)\cdot\nabla v(x)\,dx = v(\phi_t^{-1}(y)) \tag{5.7}$$

for any $v \in H^1(\phi_t^{-1}(\Omega)), v = 0$ on σ. It is not difficult to see that the derivative

$$DM_{t_0}(\cdot\,;y) \equiv \frac{\partial}{\partial t}M_t\big|_{t=t_0}(\cdot\,;y)$$

exists as a function in $H^1_{loc}(\phi_{t_0}^{-1}(\Omega)\setminus\phi_{t_0}^{-1}(y))$. By the chain rule

$$DM_{t_0}(x;y) = \frac{\partial}{\partial t}N_t\Big|_{t=t_0}(\phi_{t_0}(x);y) + \nabla_x N_{t_0}(\phi_{t_0}(x);y)\frac{\partial}{\partial t}\phi_t\Big|_{t=t_0}(x),$$

i.e.,

$$\frac{\partial}{\partial t}N_t\Big|_{t=t_0}(x;y) = DM_{t_0}(\phi_{t_0}^{-1}(x);y) - \nabla_x N_{t_0}(x;y)\cdot(\delta A\phi_{t_0}^{-1}(x)+\delta b)\ . \tag{5.8}$$

Using the relation (5.7) it is fairly easy to obtain an explicit equation for $DM_{t_0}(\cdot\,;y)$. We shall here only need the following two obvious properties in case $t_0 = 0$:

$$\begin{aligned}&DM_0(\cdot\,;y) \text{ is constant on } \sigma, \text{ and}\\ &\nabla\cdot(\gamma\nabla DM_0(\cdot\,;y)) = 0 \text{ in } \Omega\setminus(y\cup\sigma).\end{aligned}$$

For a given linear crack σ inside Ω let l_σ denote the entire line in Ω that passes through σ. If σ' is some other linear crack inside Ω then we define

$$\text{dist}_\Omega(l_\sigma, l_{\sigma'}) = \max_{x\in l_\sigma\cap\overline{\Omega}}\ \min_{y\in l_{\sigma'}\cap\overline{\Omega}} |x-y|\ .$$

THEOREM 5.2. *Let K be a compact subdomain of Ω. Let $u_\sigma^{(i)}$ and $u_{\sigma'}^{(i)}$ be solutions to problem (I) corresponding to the two linear cracks σ and $\sigma' \subset K$, and the fluxes ψ_i $(i = 1,2)$ defined by (1.5) and (1.6$_\epsilon$). Suppose that ϵ is fixed and $\epsilon < \epsilon_0$, where ϵ_0 is as in Theorem 1.1. Suppose that the linear cracks have lengths at least $L_0 > 0$. Then there exists a constant C independent of σ,σ' such that*

$$\text{dist}_\Omega(l_\sigma, l_{\sigma'}) \le C\sum_{i=1}^{2}\|\frac{\partial}{\partial s}(u_\sigma^{(i)} - u_{\sigma'}^{(i)})\|_{L^\infty(\Gamma)}. \tag{5.9}$$

Proof. Since the right hand side in the estimate (5.9) involves a derivative we may normalize $u_\sigma^{(i)}$ and $u_{\sigma'}^{(i)}$ any way we choose. It is convenient to work with $u_\sigma^{(i)}$ and $u_{\sigma'}^{(i)}$ that satisfy

$$u_\sigma^{(i)} = 0 \text{ on } \sigma \ , \ u_{\sigma'}^{(i)} = 0 \text{ on } \sigma'$$

Suppose the assertion (5.9) is not true. Then there exist sequences σ_k, σ'_k such that the corresponding solutions $u_{\sigma_k}^{(i)}$, $u_{\sigma'_k}^{(i)}$ satisfy

$$\text{(5.10)} \qquad \frac{\|\frac{\partial}{\partial s}(u_{\sigma_k}^{(i)} - u_{\sigma'_k}^{(i)})\|_{L^\infty(\Gamma)}}{\operatorname{dist}_\Omega(l_{\sigma_k}, l_{\sigma'_k})} \to 0 \quad \text{as } k \to \infty \qquad (i = 1,2).$$

Since $\operatorname{dist}_\Omega(l_{\sigma_k}, l_{\sigma'_k})$ is bounded it follows that

$$\|\frac{\partial}{\partial s}(u_{\sigma_k}^{(i)} - u_{\sigma'_k}^{(i)})\|_{L^\infty(\Gamma)} \to 0 \quad \text{as } k \to \infty.$$

By extraction of a subsequence (for simplicity also indexed by k) we obtain that

$$\text{(5.11)} \qquad \sigma_k \to \sigma,\ \sigma'_k \to \sigma',\ u_{\sigma_k}^{(i)} \to u_\sigma^{(i)},\ u_{\sigma'_k}^{(i)} \to u_{\sigma'}^{(i)} \quad \text{as } k \to \infty.$$

From (5.11) we immediately get

$$\|\frac{\partial}{\partial s}(u_\sigma^{(i)} - u_{\sigma'}^{(i)})\|_{L^\infty(\Gamma)} = \lim_{k\to\infty} \|\frac{\partial}{\partial s}(u_{\sigma_k}^{(i)} - u_{\sigma'_k}^{(i)})\|_{L^\infty(\Gamma)} = 0 \qquad (i = 1,2),$$

and Theorem 1.2 now yields $\sigma = \sigma'$, and in particular $\sigma_k \to \sigma$ and $\sigma'_k \to \sigma$ as $k \to \infty$.

For a moment we consider the special case where one of the sequences is a fixed crack, say $\sigma'_k = \sigma$. The statement (5.10) reads

$$\frac{\|\frac{\partial}{\partial s}(u_{\sigma_k}^{(i)} - u_{\sigma}^{(i)})\|_{L^\infty(\Gamma)}}{\operatorname{dist}_\Omega(l_{\sigma_k}, l_{\sigma})} \to 0 \quad \text{as } k \to \infty \qquad (i = 1,2)\ .$$

For simplicity we pick the coordinate system so that σ lies on the x_1-axis (from the origin to $(|\sigma|, 0)$). The crack σ may be mapped onto σ_k by a rotation followed by a dilatation and a translation:

$$\sigma_k = \lambda_k \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix} \sigma + b_k, \quad -\pi/2 \le \theta_k < \pi/2.$$

It is not difficult to see that $\lim_{k\to\infty} \lambda_k = 1$, $\lim_{k\to\infty} \theta_k = 0$, $\lim_{k\to\infty} b_k = 0$ and

$$\begin{aligned} c(|\theta_k| + |b_{k,2}|) &\le \operatorname{dist}_\Omega(l_{\sigma_k}, l_\sigma) \\ &\le C(|\theta_k| + |b_{k,2}|) \quad (C > c > 0), \end{aligned}$$

uniformly in k. Let

$$\delta\theta = \lim_{k\to\infty} \frac{\theta_k}{|\theta_k| + |b_{k,2}|} ,$$

$$\delta b_2 = \lim_{k\to\infty} \frac{b_{k,2}}{|\theta_k| + |b_{k,2}|}$$

(we may assume these limits exist, if not we just extract a subsequence). Note that $|\delta\theta| + |\delta b_2| = 1$. It now follows that

$$(5.12) \qquad 0 = \lim_{k\to\infty} \frac{\frac{\partial}{\partial s}(u^{(i)}_{\sigma_k} - u^{(i)}_{\sigma})}{|\theta_k| + |b_{k,2}|} = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial t} u^{(i)}_t\big|_{t=0}\right) \quad \text{on } \Gamma$$

with $u^{(i)}_t$ given by $u^{(i)}_t = u^{(i)}_{\sigma_t}$ ($u^{(i)}_{\sigma_t} = 0$ on σ_t), and σ_t denoting the crack

$$\sigma_t = \left[I + t\begin{pmatrix} 0 & -\delta\theta \\ \delta\theta & 0 \end{pmatrix}\right]\sigma + t\begin{pmatrix} 0 \\ \delta b_2 \end{pmatrix}.$$

The function $\frac{\partial}{\partial t} u^{(i)}_t\big|_{t=0}$ satisfies

$$\nabla \cdot (\gamma \nabla (\frac{\partial}{\partial t} u^{(i)}_t\big|_{t=0})) = 0 \text{ in } \Omega \setminus \sigma \quad \text{with} \quad \frac{\partial}{\partial \nu}(\frac{\partial}{\partial t} u^{(i)}_t\big|_{t=0}) = 0 \text{ on } \partial\Omega.$$

From (5.12) and the unique continuation theorem it now follows that

$$\frac{\partial}{\partial t} u^{(i)}_t\big|_{t=0} = C_0 \text{ (a constant)} \quad \text{in } \Omega \setminus \sigma.$$

The function $u^{(i)}_t$ may be represented as

$$(5.14) \qquad \begin{aligned} u^{(i)}_t(y) &= \int_{\Omega} u^{(i)}_t \delta_y = \int_{\partial\Omega} \gamma \frac{\partial u^{(i)}_t(x)}{\partial \nu} N_t(x;y)\, ds_x \\ &= \int_{\partial\Omega} \psi_i(x) N_t(x;y)\, ds_x \end{aligned}$$

where N_t is the Green–Neumann function from (5.6) with $\gamma \partial N_t(x;y)/\partial \nu_x = 0, x \in \partial\Omega$ and $N_t(x;y) = 0, x \in \sigma_t$. To derive (5.14) we use that $u^{(i)}_t$ vanishes on σ_t. Differentiating (5.14) with respect to t and combining with (5.13) we get

$$(5.15) \qquad C_0 = \frac{\partial}{\partial t} u^{(i)}_t\big|_{t=0} = \int_{\partial\Omega} \psi_i \frac{\partial}{\partial t} N_t\big|_{t=0}(x;y)\, ds_x \quad \forall y \in \Omega \setminus \sigma .$$

If the sequence $\{\sigma'_k\}$ does not consist of one fixed line segment, then we may still proceed along the lines of the argument above using the fact that $\partial u_t^{(i)}/\partial t$ is uniformly continuous as a function of σ, y and t. We again arrive at the assertion (5.15). Based on (5.15) and integration by parts we deduce that

$$C_0 = \int_{\partial\Omega} \psi_i \frac{\partial}{\partial t} N_t\big|_{t=0}(x;y)\, ds_x = -\int_{\tau_\delta} \gamma \frac{\partial u_\sigma^{(i)}}{\partial \nu} \frac{\partial}{\partial t} N_t\big|_{t=0}(x;y)\, ds_x$$
$$+ \int_{\tau_\delta} u_\sigma^{(i)} \gamma \frac{\partial}{\partial \nu}\left(\frac{\partial}{\partial t} N_t\big|_{t=0}\right)(x;y)\, ds_x + u_\sigma^{(i)}(y)\ ,$$

where τ_δ is the curve consisting of the two circles $\partial B_1^\delta, \partial B_2^\delta$ of radius δ centered at the endpoints of σ plus the line segment $J_\delta = [\delta, |\sigma| - \delta] \subset \sigma$ traversed twice (cf. Figure 2).

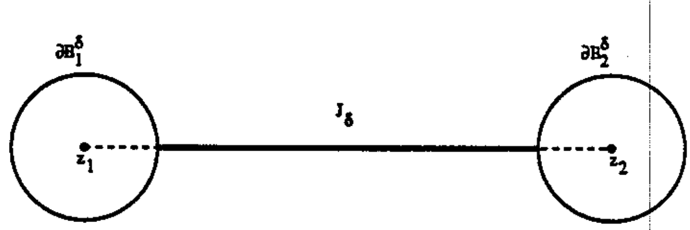

Figure 2

Using (5.8) and the fact that $u_\sigma^{(i)}$, $DM_0(\cdot\,;y)$ vanish on σ we get

$$C_0 - \int_{J_\delta} \gamma \frac{\partial u_\sigma^{(i)}}{\partial \nu} \nabla_x\, N_0(x;y) \cdot \left(\begin{pmatrix} 0 & -\delta\theta \\ \delta\theta & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \delta b_2 \end{pmatrix} \right) ds_x - u_\sigma^{(i)}(y)$$

(5.16)

$$= \sum_{j=1}^{2} \int_{\partial B_j^\delta} \left(u_\sigma^{(i)} \gamma \frac{\partial}{\partial \nu} \frac{\partial}{\partial t} N_t\big|_{t=0} - \gamma \frac{\partial u_\sigma^{(i)}}{\partial \nu} \frac{\partial}{\partial t} N_t\big|_{t=0} \right)(x;y)\, ds_x \qquad ;$$

the integral over J_δ is actually two integrals, one over the top (say J_δ^+) and one over the bottom (say J_δ^-), with the normal ν pointing in opposite directions.

Any point $y_0 \in J_\delta$ corresponds to the two points $y_0^+ \in J_\delta^+$ and $y_0^- \in J_\delta^-$. Let $y_0 \in J_\delta$ and let $y \to y_0$ from above σ. Then for any continuous function ϕ on J_δ^+

(5.17)
$$-\int_{J_\delta^+} \gamma \frac{\partial N_0(x;y)}{\partial \nu_x} \phi(x)\, ds_x \to \phi(y_0^+)\ ,$$

in fact, the proof of this is the same as for the usual Green function. At the same time we know that for y above σ and x below σ the function $N_0(x;y)$ is a harmonic function in x

vanishing as $y \to y_0 \in J_\delta$ (by symmetry of N_0). It follows that $\nabla_x N_0(x;y)$ converges to zero uniformly in $x \in J_\delta$ as $y \to y_0$ from above σ and thus

$$\int_{J_\delta^-} \gamma \frac{\partial N_0(x;y)}{\partial \nu_x} \psi(x)\, ds_x \to 0 , \tag{5.18}$$

for any continuous function ψ on J_δ^- . Using (5.17), (5.18) and noting that the tangential derivative $\dfrac{\partial N_0}{\partial s_x} = 0$ on J_δ^+ and J_δ^-, we see that

$$\begin{cases} \text{the left–hand side of (5.16) converges to} \\ C_0 + \dfrac{\partial u_\sigma^{(i)}(y_0^+)}{\partial \nu} \left(\begin{pmatrix} 0 & -\delta\theta \\ \delta\theta & 0 \end{pmatrix} y_0 + \begin{pmatrix} 0 \\ \delta b_2 \end{pmatrix} \right) \cdot \nu_{y_0^+} , \\ \text{as } y \to y_0 \in J_\delta \text{ from above } \sigma . \end{cases} \tag{5.19a}$$

The corresponding statement if y approaches y_0 from below σ reads

$$\begin{cases} \text{the left–hand side of (5.16) converges to} \\ C_0 + \dfrac{\partial u_\sigma^{(i)}(y_0^-)}{\partial \nu} \left(\begin{pmatrix} 0 & -\delta\theta \\ \delta\theta & 0 \end{pmatrix} y_0 + \begin{pmatrix} 0 \\ \delta b_2 \end{pmatrix} \right) \cdot \nu_{y_0^-} , \\ \text{as } y \to y_0 \in J_\delta \text{ from below } \sigma . \end{cases} \tag{5.19b}$$

The terms $u_\sigma^{(i)}(y_0^\pm)$ disappear since $u_\sigma^{(i)}$ vanishes on σ.

We next analyze the behavior of the right–hand side of (5.16) as $y \to y_0$. By symmetry of N_t, $N_t(x;y) = 0$ if $y \in \sigma, x \in \Omega \setminus \sigma$, and thus also

$$\nabla_x N_t(x;y) = 0, \quad D_x^2 N_t(x;y) = 0 \quad \text{if } y \in J_\delta, x \in \Omega \setminus \sigma .$$

Due to (5.8) it follows that if $y \to y_0$ then the right–hand side of (5.16) converges to

$$\sum_{j=1}^{2} \int_{\partial B_j^\epsilon} \left(u_\sigma^{(i)} \gamma \frac{\partial}{\partial \nu} DM_0 - \gamma \frac{\partial u_\sigma^{(i)}}{\partial \nu} DM_0 \right) . \tag{5.20}$$

Let z_1 $(= (0,0))$ and z_2 $(= (|\sigma|, 0))$ denote the endpoints of σ. Since the function $DM_0(x; y_0)$ is both harmonic and in H^1 in $\Omega \setminus \sigma$-neighborhoods of z_j, it has the following asymptotic behavior near z_j :

$$\begin{aligned} |DM_0(x;y_0)| &\le C\, r^{1/2} \\ |\nabla_x DM_0(x;y_0)| &\le C\, r^{-1/2} \qquad r = |x - z_j|, \end{aligned}$$

(we obtain these estimates by application of Lemma 2.1). Similarly

$$(5.21) \qquad \begin{aligned} |u_\sigma^{(i)}(x)| &\le C\, r^{1/2} \\ |\nabla_x u_\sigma^{(i)}(x)| &\le C\, r^{-1/2}\ , \end{aligned}$$

near z_j. The expression (5.20) is therefore of order $O(\delta)$ and so it converges to zero as $\delta \to 0$. We conclude that the right–hand side of (5.16) converges to zero as $y \to y_0$ followed by $\delta \to 0$. This limit is the same whether $y \to y_0$ from above σ or from below σ. Recalling (5.19a), (5.19b) and the fact that $|\delta\theta| + |\delta b_2| = 1$ we now get that

$$(5.22) \qquad \partial u_\sigma^{(i)}(y_0^+)/\partial\nu = -\partial u_\sigma^{(i)}(y_0^-)/\partial\nu \quad \text{for } y_0 \in \operatorname{int} \sigma.$$

Since we also know that $u_\sigma^{(i)} = 0$ on σ it follows immediately by integration by parts and application of (5.21) and (5.22) that $u_\sigma^{(i)}$, $i = 1, 2$ satisfy the equation $\nabla \cdot (\gamma \nabla u_\sigma^{(i)}) = 0$ in all of Ω (not just in $\Omega \backslash \sigma$). The crack σ thus produces the same boundary measurements as the empty crack; from Theorem 1.1 it now follows that σ is the empty crack, and this contradicts the fact that σ has length at least $L_0 > 0$. Consequently we get that the estimate (5.9) holds and this concludes the proof of Theorem 5.2. □

REFERENCES

[1] H. BELLOUT AND A. FRIEDMAN, *Identification problem in potential theory*, Archive Rat. Mech. Anal. 101 (1988), 143–160.

[2] A FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice–Hall, Englewood Cliffs, 1964. (Reprinted by Krieger).

[3] A. FRIEDMAN AND B. GUSTAFSSON, *Identification of the conductivity coefficient in an elliptic equations*, SIAM J. Math. Anal., 18 (1987), 777–784.

[4] A. FRIEDMAN AND M. VOGELIUS, *Identification of small inhomogeneities of extreme conductivity by boundary measurements: a continuous dependence result*, Archive Rat. Mech. Anal., to appear.

[5] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.

[6] R. KOHN AND M. VOGELIUS, *Determining conductivity by boundary measurements*, Comm. Pure Appl. Math, 37(1984), 289–298.

[7] R. KOHN AND M. VOGELIUS, *Determining conductivity by boundary measurements II, Interior Results*, Comm. Pure Appl. Math., 38 (1985), 643–667.

[8] A. NACHMAN, *Reconstruction from boundary measurements*, Preprint, 1987.

[9] J. SYLVESTER AND G. UHLMANN, *A uniqueness theorem for an inverse boundary value problem in electrical prospection*, Comm. Pure Appl. Math., 39 (1986), 91–112.

[10] J. SYLVESTER AND G. UHLMANN, *A global uniqueness theorem for an inverse boundary value problem, Annals of Math., 125 (1987), 153–169.*

[11] J. SYLVESTER AND G. UHLMANN, *Inverse boundary value problems at the boundary–continuous dependence*, Comm. Pure Appl. Math., to appear.