The Limit of the Small I-function for Calabi-Yau GIT Targets and an Application to the Integrality of the Mirror Map

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Dedication

To my family.

Abstract

The aim of this thesis is to prove a limit relation between the K-theoretic and cohomological small I-function for aptly defined Calabi-Yau GIT targets. The proof derives from description of inertia stack of the moduli of $(0, 2, d)$-quasimaps with 0+-stability. As a corollary, we obtain an application to the integrality of the mirror map for a family of Calabi-Yau GIT targets.

Contents

Chapter 1

Introduction

1.1 Overview

Given a nonsingular quasi-projective variety X, the J-function is a well-studied generating function of Gromov-Witten invariants with one gravitational insertion $1/(1-qL)$ and a variable amount of primary insertions of a parameter t. First defined in quantum cohomology and then adapted to quantum K-theory, J is in general hard to compute in either theory. In order to overcome this difficulty, Givental introduced the "small" I-function for toric targets, which is easier to compute and related to the J-function by a mirror theorem [Giv98]. Subsequently Ciocan-Fontanine and Kim defined the "big" I-function for nice GIT targets as a generalization [CFK10, CFKM14], and proved that it is computable for various families of targets in cohomology [CFK16]. Furthermore we now understand that I and J are related by a wall-crossing formula in stability parameter $\varepsilon \in \mathbb{Q}_{\geq 0}$ in both cohomology and K-theory [CFK20, CFK17, CJR17, Zho22, TY16, ZZ20]. In fact, the coefficients of I-function initially appeared in the physics literature, as the period solutions to the Picard-Fuchs equation of the mirror Calabi-Yau manifold. The following transformation, called the mirror map, therefore appears in mirror symmetry:

$$Q_i \to Q_i e^{I_{1,i}/I_0}$$

where Q_i are the Novikov variables, and $I_{1,i}$ and I_0 are specific coefficients in the I-function. This transformation has been heavily studied for quintic 3-folds for example, and plays an important role in the study of mirror symmetry.

Finding a relationship between the K-theoretic and cohomological J-functions has been of interest in order to compare each theory's respective Gromov-Witten invariants. Using the confluence of the q-difference equations which describe the J-function, it was shown that a value of the cohomological J-function can be obtained by a limit K-theoretical J for projective targets [Roq19, Wen22]. Consequently Milanov and Roquefeuil [MR21] proved a confluence result for smooth projecive varieties with nef anticanonical bundle using the reconstruction of K-theoretic J from cohomological correlators due to Givental and Tonita [GT14].

In this work, we prove an analogous confluence result for the I-function with Calabi-Yau GIT targets, adapting the methods of Milanov, Roquefeuil, Givental, and Tonita, as well as the description for the I-function as a generating function of invariants in $\varepsilon = 0+$ stability due to Ciocan-Fontanine and Kim in cohomology, and Tseng and You in K-theory. [CFK14, TY16]. The precise set up can be described as follows.

Given a triple (W, G, θ) defining a GIT quotient $X := W /\!\!/_\theta G$ with certain nice conditions, there exists a compact moduli space $Q^\varepsilon_{g,k,d}(X)$ of ε-stable (g, k, d)-quasimaps to X, where ε is the stability parameter in $\mathbb{Q}_{\geq 0}$. The theory of moduli of ε-stable curves be traced back to the work of [MM07, CFK10, MOP11, Tod11], but we employ the fully developed version of the theory in [CFKM14]. For any ε, including the asymptotics $\varepsilon = 0+, \infty$, $Q^\varepsilon_{g,k,d}(X)$ is proper Deligne-Mumford stack with perfect obstruction theory, and therefore carries a virtual fundamental class which allows for the definition of the permutation equivariant Gromov-Witten invariants in K-theory and cohomology (Definitions 2.8.1 and 2.8.2).

$$\langle \Gamma_1 L_1^{a_1}, \ldots, \Gamma_k L_k^{a_k}, t(L), \ldots, t(L)\rangle^{\varepsilon, S_n}_{g,k+n,d} := \chi\left(\prod_{i=1}^{k} \mathrm{ev}_i^*(\Gamma_i) L_i^{a_i} \prod_{j=k+1}^{k+n} \sum_a \mathrm{ev}_j^*(t_a) L_j^a; \mathscr{O}^{vir}_{Q^\varepsilon_{g,k+n,d}(X)/S_n}\right)$$

$$\langle \gamma_1 \psi_1^{a_1}, \ldots, \gamma_k \psi_k^{a_k}, t(\psi), \ldots, t(\psi)\rangle^{\varepsilon, S_n}_{g,k+n,d} := \int_{[Q^\varepsilon_{g,k,d}(X)/S_n]^{vir}} \prod_{i=1}^{k} \mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i} \prod_{j=k+1}^{k+n} \sum_a \mathrm{ev}_j^*(t_a)\psi_j^a$$

for free parameters $t(L)$, $t(\psi)$ in a suitably well-defined subalgebra of $K^0(X) \otimes$

$\mathbb{Q}[[q, Q, L_i]]$ (resp. $H^*(X) \otimes \mathbb{Q}[[z, Q, \psi_i]]$) and evalution maps $\mathrm{ev}_i : Q^\varepsilon_{g,k,d}(X) \to X$ for $i = 1, \ldots, k+n$.

To define the J^ε-function in general, introduce a new moduli called the graph space $QG^\varepsilon_{g,k,d}(X)$, which parametrizes certain quasimaps to $X \times \mathbb{P}^1$. This space carries a natural torus action and a distinguished fixed subspace F_0 with evaluation map $\mathrm{ev}_\bullet : F_0 \to X$ which does not extend to the graph sapce. The J^ε-function for each stability paramater ε is defined as follows. (Equations (2.10.3) and (2.10.4))

$$\begin{aligned}
J^\varepsilon(t, q, Q) &:= \sum_{k,d \geq 0} Q^d (\mathrm{ev}_\bullet)_* \left(\frac{\prod_{i=1}^k \mathrm{ev}_i^*(t)}{k!} \otimes \frac{\mathscr{O}^{\mathrm{vir}}_{F_0}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_0})^*} \right) \\
&= \mathbb{1} + \frac{t}{1-q} + \sum_a \sum_{d \neq 0, d(L_\theta) \leq 1/\varepsilon} Q^d \Phi^a \chi \left(F_0; \mathscr{O}^{\mathrm{vir}}_{F_0} \otimes \frac{\mathrm{ev}_\bullet^*(\Phi_a)}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_0})^\vee} \right) \\
&\quad + \sum_a \sum_{\substack{k>0,d \\ d(L_\theta)>1/\varepsilon}} Q^d \Phi^a \langle \frac{\Phi_a}{(1-q)(1-qL)}, t, \ldots, t \rangle^\varepsilon_{0,1+k,d} \\
J^{coh,\varepsilon}(t, z, Q) &:= \sum_{k,d \geq 0} Q^d (\mathrm{ev}_\bullet)_* \left(\frac{\prod_{i=1}^k \mathrm{ev}_i^*(t)}{k!} \cap \mathrm{Res}_{F_0} [QG^\varepsilon_{0,k,d}(X)]^{\mathrm{vir}} \right) \\
&= \mathbb{1} + \frac{t}{z} + \sum_a \sum_{d \neq 0, d(L_\theta) \leq 1/\varepsilon} Q^d \phi^a \int_{F_0^{\mathrm{vir}}} \frac{\mathrm{ev}_\bullet^*(\phi_a)}{e^{\mathbb{C}^*}(N_{F_0})} \\
&\quad + \sum_a \sum_{\substack{k>0,d \\ d(L_\theta)>1/\varepsilon}} Q^d \phi^a \langle \frac{\phi_a}{z(z-\psi_\bullet)}, t, \ldots, t \rangle^\varepsilon_{0,1+k,d}.
\end{aligned}$$

The formulations in terms of a bases are calculated by the known formulae for the Euler classes of $N^{\mathrm{vir}}_{F_0}$ for various k, d in genus 0. The I and J functions are the asymptotic cases, $J = J^\infty$ and $I = J^{0+}$. In the $\varepsilon = \infty$ case, all $d(L_\theta) > 1/\varepsilon$, so that every term is a correlator. Therefore Milanov and Roquefeuil were able to apply the Givental-Tonita methods for writing the K-theoretic correlators as cohomological ones to prove their confluence of the J-function theorem. But that in the $\varepsilon = 0+$ case, none of the $k = 0$ terms are defined by a correlator, since $d(L_\theta) \geq 1/\varepsilon$ for all d, making applying the Givental-Tonita theory not obvious. The key to proving the confluence of the I-function is the Birkhoff factorization

$$J^{\varepsilon} = S^{\varepsilon}(P^{\varepsilon})$$

where S^{ε} is an operator with coefficients in $1/(1-q)$ (resp. $1/z$), while P is a generating function in terms of $(1-q)$ (resp. z) [CFK14, TY16]. This is proven by $\mathbb{C}^*$-equivariant localization on the graph space $QG^{\varepsilon}_{0,k,d}(X)$. Both the S-operator and P-series are defined in terms of correlators on $Q^{0+}_{0,k,d}(X)$ and $QG^{0+}_{0,k,d}(X)$, making a similar argument to Milanov and Roquefeuil possible.

The integrality result that appears in this thesis is due to an essential observation by Jockers and Mayr [JM20]. They first note that the connection between the confluence of I and the integrality of the mirror map is Givental's Explicit Reconstruction theorem in K-theory [Giv16]. Write $\mathcal{L}$ as the image (perhaps up to a shift by $1-q$) of the J-function in its codomain $\mathcal{K}$ (see Definition 2.7.2), under different values of the parameter $t \in \mathcal{K}^+$. The reconstruction theorems describe how to generate new series in the image given a known element in $\mathcal{L}$, typically when $t = 0$. In particular, given a value of J written out as $\sum_d I_d Q^d \in \mathcal{L}$, then the family of series

$$\sum_d I_d Q^d \exp\left(\sum_{k>0} \frac{\Psi^k(\varepsilon_a) P^{ka} q^{k(a\cdot d)}}{k(1-q^k)}\right)$$

also lies on $\mathcal{L}$, where $P_1, \ldots, P_r$ are suitably defined line bundles and ε_a series in Q with Laurent polynomial in q coefficients. Given that the P_i generate $K^0(X)$, the entirety of $\mathcal{L}$ can be generated in this way. In particular, this is known for toric varieties, so that any value of J can be generated from this reconstruction operator. In cohomology, this was proven in various forms by Iritani [IMT15] and then subsequently by Ciocan-Fontanine and Kim [CFK16]. The current formulation in both cohomology and K-theory by Givental.

Assuming the confluence of I and applying the Explicit Reconstruction theorem, Jockers and Mayr then mention that for the quintic threefold, the exponent of the mirror map can be written in terms of a natural logarithm.

$$I_1/I_0 = \lim_{q\to 1}(1-q)\sum_{k>0} \frac{\Psi^k(\varepsilon_1)}{1-q^k} = -\sum_d n_d^i \ln(1-Q^d)$$

where $I_{1,i}/I_0$ is the ratio of I-function coefficients appearing in the mirror map, Ψ^k is the kth Adams operation, ε_i is a Laurent series in q and Q with integer coefficients n_d^i in the $q \to 1$ limit. Then of course the mirror map becomes

$$Q_i \mapsto Q_i \prod_d (1 - Q^d)^{-n_d^i}$$

which has integer coefficients when expanded. The purpose of this work is to prove the confluence of the I-function in the Calabi-Yau GIT target case, and then apply it to the integrality for the mirror map for a subclass of said Calabi-Yau GIT targets. The proof outlined by Jockers and Mayr holds when $K^0(X)$ is generated by line bundles, which is true for some Calabi-Yau GIT targets, but not all. We expect that it readily extends to all Calabi-Yau GIT targets.

The integrality of mirror maps has been of course been of great interest since the origin of mirror symmetry. The p-adic methods of finding solutions to differential equations due to Dwork [Dwo69, Dwo73] have yielded rich results since the advent of mirror symmetry. Candelas, de las Ossa, Green, and Parkes proved integrality of the mirror map for the quintic 3-fold in 1990 [CdlOGP91]. Within mathematics, the results in this field are generalization of a result due to Lian and Yau [LY96], who prove integrality of the mirror map for prime degree p Calabi-Yau hypersurfaces in $\mathbb{P}^{p-1}$ and proved a template for further work in the toric case. Mirror map integrality for prime power degrees was subsequently prove by Zudilin [Zud02]. Building off of further work by Lian and Yau on nth roots of mirror maps [LY03], Krattenthaler and Rivoal proved strong integrality results in the toric case [KR10, KR09, KR11] and there continues to be developments on this problem within the p-adic and toric context [Del13, Zho12, AS17, AS20, KSV06, BV21].

1.2 Main Results

The results of this work are proved for the "nice" GIT quotients X in the sense of [CFK10], see Section 2.1. Let $\deg : H^*(X)_{\mathbb{Q}} \to \mathbb{Z}$ denote the operator which sends a homogeneous element $\phi_a \in H^*(X)_{\mathbb{Q}}$ of cohomology to its complex codimension $|\phi_a| = \frac{1}{2}\deg(\phi_a)$. In this way, let $(1-q)^{\deg}\operatorname{ch}$ be the operator $K^0(X)_{\mathbb{Q}} \to H^*(X)_{\mathbb{Q}} \otimes \mathbb{Q}[q]$ which sends $\Phi_a \mapsto (1-q)^{|\phi_a|}\phi_a$ where $\operatorname{ch}(\Phi_a) = \phi_a$. Note that typically one considers

$\mathrm{ch}(q) = e^{-z}$ when comparing quantum K-theory and cohomology in the $\mathbb{C}^*$-equivariant context, but we do not use that fact here. In the confluence theorem, ch only acts on the $K^0(X)$.

Theorem 1.2.1 (Theorem 5.0.1)**.** *For a Calabi-Yau GIT target $X = W /\!\!/ G$,*

$$\lim_{q \to 1} (1-q)^{\deg} \mathrm{ch}(I(0,q,Q)) = I^{coh}(0,1,Q).$$

In particular, the coefficient of the $\Phi_a Q^d$ in the I-function are rational functions in q, with pole at $q = 1$ of order $|\phi_a|$. The subsequent integrality result can be stated thus.

Theorem 1.2.2 (Theorem 6.0.1)**.** *For X a Calabi-Yau GIT target such that $K^0(X)$ is generated by line bundles, the mirror map $Q_i \to Q_i e^{I_{1,i}(1,Q)/I_0(1,Q)}$ has integer coefficients in Q.*

1.3 Outline

Chapter 2 covers all the necessary preliminaries. Chapter 3 describes the Birkhoff factorization for J^ε in K-theory and cohomology, then proves some necessary formulae as corollaries. Then in Chapter 4, we prove a recursion relation on correlators using Givental's description of inertia for $\varepsilon = 0+$. The confluence of the I-function is proven in Chapter 5. Finally in Chapter 6, Givental's Explicit Reconstruction theorem is described and a proof of integrality of the mirror map is given.

Chapter 2

Preliminaries

2.1 Class of GIT Quotients

Fix the base field to be $\mathbb{C}$. Let (W, G, θ) be a triple with $W = \operatorname{Spec} A$ an affine variety, G a reductive algebraic group with action on W, and $\theta \in \chi(G)$ a character of G. Let $[W/G]$ be the associated quotient stack and $W/_{\mathrm{aff}}G = \operatorname{Spec} A^G$ the affine quotient. Finally let $X := W \mathbin{/\!\!/}_\theta G$ be the GIT quotient defined by

$$X := W \mathbin{/\!\!/}_\theta G := \operatorname{Proj}\left(\bigoplus_{n\geq 0} \Gamma(W, L_\theta^{\otimes n})\right)^G$$

where L_θ the trivial line bundle with linearization defined by θ.

Let $W^s(\theta)$ and $W^{ss}(\theta)$ denote the open subsets of stable and semistable points of W determined by L_θ. Assume for the rest of the thesis that (i) $\emptyset \neq W^s(\theta) = W^{ss}(\theta)$, (ii) the stable locus is nonsingular, (iii) G acts freely on the stable locus, and (iv) W has at worst l.c.i. singularities. In this situation the quotient map $W^s \to X$ is a principle G-bundle in the etale topology by Luna's slice theorem. Therefore this quotient induces an isomorphism $[W^s/G] \to X$, so that X can be considered an open substack of $[W/G]$ and proper over the affine quotient. The relationship under assumptions (i) - (iv) between these quotients can be summarized by the following diagram.

$$\begin{array}{ccc} X := [W^s/G] & \hookrightarrow & [W/G] \\ \downarrow \simeq & & \downarrow \\ X := W /\!\!/_\theta G & \longrightarrow & W/_{\text{aff}}G \end{array}$$

For the remainder of the paper, we say X is a *GIT target* to mean $X = W /\!\!/ G$ is a GIT quotient (W, G, θ) with properties (i) - (iv).

2.2 K-Theory and Cohomology

Let $K^0(X)$ be the K-theory of the GIT quotient X, defined as the Grothendieck group of vector bundles on X. Let $H^*(X)$ denote the even degree cohomology theory of X. Let $K^0(X)_{\mathbb{Q}}$ and $H^*(X)_{\mathbb{Q}}$ be the corresponding rings with rational coefficients. The Chern character is an isomorphism $\text{ch} : K_0(X)_{\mathbb{Q}} \to H^*(X)_{\mathbb{Q}}$ defined on line bundles by $\text{ch}(L) = e^{c_1(L)}$. We are concerned with both integer and rational coefficients, so we specify what coefficients we are using whenever possible.

Fix once and for all the following bases of $K^0(X)_{\mathbb{Q}}$ and $H^*(X)_{\mathbb{Q}}$. Let $\{\phi_a\}$ be a homogeneous basis of $H^*(X)_{\mathbb{Q}}$ such that $\phi_0 = [X]$. Let $\{\Phi_a\}$ be the basis of $K^0(X)_{\mathbb{Q}}$ such that $\text{ch}(\Phi_a) = \phi_a$, and hence $\Phi_0 = \mathscr{O}_X$. On $K^0(X)_{\mathbb{Q}}$ define the Euler pairing with respect to this basis by

$$g_{ab} := \langle \Phi_a, \Phi_b \rangle = \chi(\Phi_a \otimes \Phi_b; X) = \int_X \text{ch}(\Phi_a \otimes \Phi_b)\,\text{td}(T_X). \tag{2.2.1}$$

For X nonsingular this is a perfect pairing, so define a Poincare dual basis $\{\Phi^a\}$ by

$$\Phi^a = \sum_b g^{ab} \Phi_b \tag{2.2.2}$$

where g^{ab} denotes the ab-entry of the inverse matrix to g_{ab}. On cohomology, define

$$\gamma_{ab} := \langle \phi_a, \phi_b \rangle = \int_X \phi_a \phi_b. \tag{2.2.3}$$

Define a dual basis $\{\phi^a\}$ by $\phi^a = \text{ch}(\Phi^a)\,\text{td}(T_X)$. This satisfies the property of being a Poincare dual since

$$\langle \phi_a, \phi^b \rangle = \int_X \mathrm{ch}(\Phi_a)\,\mathrm{ch}(\Phi^b)\,\mathrm{td}(T_X) = \delta_{ab} \tag{2.2.4}$$

by definition of $\{\Phi^a\}$. Note in particular that $\mathrm{ch}(\Phi^a) = \phi^a / \mathrm{td}(T_X)$.

Remark 2.2.1. Poincare dual bases live strictly over the rationals in general. For example, take X to be the quintic 3-fold. Then the free part of $K^0(X)$, and hence $K^0(X)_{\mathbb{Q}}$, is generated as a free $\mathbb{Z}$-module by $\Phi_k = (1-P)^k$ where $k \in \{0,1,2,3\}$, $P = \mathscr{O}(-H)$, and H is the hyperplane class in $H^*(\mathbb{P}^4)$ restricted to X. The perfect pairing on $K^0(X)_{\mathbb{Q}}$ is determined by the matrix

$$g_{ab} = \begin{pmatrix} 0 & 5 & -5 & 5 \\ 5 & -5 & 5 & 0 \\ -5 & 5 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is not invertible over the integers, and so Φ^k lives strictly in $K^0(X)_{\mathbb{Q}}$. One could change the basis to be $\Phi_0, \Phi_1, \frac{1}{5}\Phi_2, \frac{1}{5}\Phi_3$, for example, to make g_{ab} invertible over the integers, but then the original basis will live only over the rationals.

To simplify notation, we denote the complex degree of a homogeneous element $\phi_a \in H^*(X)_{\mathbb{Q}}$ to be $|\phi_a| = \frac{1}{2}\deg \phi_a$. To extend the our notion of degree to K-theory using the Chern character, we will often write $|\Phi_a|$ for the degree $\frac{1}{2}\deg \mathrm{ch}(\Phi_a)$ in the case when $\mathrm{ch}(\Phi_a)$ is homogeneous. Let $|\phi^a| := \dim X - |\phi_a|$ denote the degree of the ϕ^k, where $\dim X$ is always over $\mathbb{C}$. Similarly define $|\Phi^a| = \dim X - |\Phi_a|$.

2.3 Moduli of Quasimaps

Let C be a reduced, projective, at worst nodal curve. A map $[u] : C \to [W/G]$ is equivalent to a pair (P, u) where $P \to C$ is a principle G-bundle and $u : C \to P \times_G W$ is a section of the mixed stack quotient

$$P \times_G W := [(P \times W)/G]$$

where G acts on $P \times W$ diagonally.

To such a map we can associate a numerical class (or degree) as follows. An equivariant line bundle $L \to W$, or equivalently a bundle on $[W/G]$, induces a bundle on the mixed quotient $P \times_G L \to P \times_G W$. Pulling back along the section u, we get a line bundle $u^*(P \times_G L) \to C$. Since C is a curve, then the bundle carries a natural notion of a degree in the integers.

Definition 2.3.1. *The numerical class of a pair* $(P, u) \in Map(C, [W/G])$ *is the homomorphism* $d : \mathrm{Pic}([W/G]) \to \mathbb{Z}$ *defined by*

$$d(L) = \deg_C u^*(P \times_G L) = \deg_C [u]^*(L).$$

The value $d(L_\theta)$ *is the usual degree of the map.*

Now we define the notion of prestable maps, prestable quasimaps, and stable quasimaps to $W /\!\!/_\theta G$.

Definition 2.3.2. *Let* $(C, x_1, \ldots, x_k)$ *be a prestable genus* g *curve with marked points. A map* $[u] : C \to [W/G]$ *or equivalently a pair* (P, u) *with* $u : C \to P \times_G W$ *is called a* k*-pointed* prestable map *of genus* g *to* $[W/G]$.

Definition 2.3.3. *Let* $(C, x_1, \ldots, x_k, [u])$ *be a prestable map to* $[W/G]$.

(i) The base locus *of* $[u]$ *with respect to* θ *is the reduced scheme* $[u]^{-1}([W^{us}(\theta)/G])$.

(ii) The data $(C, x_1, \ldots, x_k, [u])$ *is called a* θ-quasimap *to* $X := W /\!\!/_\theta G$ *if the base locus with respect to the* θ *is purely 0-dimensional.*

(iii) A θ*-quasimap* $(C, x_1, \ldots, x_k, f)$ *is called* θ-prestable *if the base locus contains no nodes or marked points of* C.

Definition 2.3.4. *A homomorphism* $d \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}([W/G]), \mathbb{Z})$ *is called* θ-effective *if it can be realized as a numerical classes of a* θ*-quasimap, possibly with disconnected domain curve. Denote the semigroup of* θ*-effective classes by* $\mathrm{Eff}(W, G, \theta)$.

If the character θ is understood, we drop the θ from the nomenclature.

Definition 2.3.5. *Let $x \in C$. The* length $\ell(x)$ *at x of a prestable quasimap (C, x_i, P, u) to X is*

$$\ell(x) = \mathit{length}_x(\mathit{coker}(u^*\mathscr{J} \to \mathscr{O}_C))$$

where $\mathscr{J}$ is the ideal sheaf of $P \times_G W^{us}$ in $P \times_G W$. In other words, $\ell(x)$ is the order of contact of $u(C)$ with the unstable locus $P \times_G W^{us}$ at $u(x)$.

Definition 2.3.6. *Let $\varepsilon \in \mathbb{Q}_{>0}$ be the stability parameter. We say a quasimap (C, x_i, P, u) is* ε-stable *if it is prestable and*

(i) $\omega_C(\sum_{i=1}^k x_i) \otimes (\mathcal{L}_\theta)^\varepsilon$ is ample

(ii) $\varepsilon\ell(x) \leq 1$ for all points $x \in C$

where $\mathcal{L}_\theta := P \times_G \mathbb{C}_\theta = u^*(P \times_G L_\theta)$.

Definition 2.3.7. *Denote by $Q^\varepsilon_{g,k,d}(X)$ the moduli space of ε-stable quasimaps of class d to $X := W /\!\!/_\theta G$.*

It was shown by Ciocan-Fontanine, Kim, and Maulik that this is a moduli with good properties [CFKM14].

Theorem 2.3.8. *Let $(g, k) \neq (0, 0)$. Under the assumptions of from Section 2.1, the space $Q^\varepsilon_{g,k,d}(X)$ is a separated, finite type, Deligne-Mumford stack, which is proper over the affine quotient $W/_{\mathit{aff}}G$, and carries a canonical perfect obstruction theory. The moduli space is empty for $2g - 2 + k + \varepsilon d(L_\theta) \leq 0$ and has virtual dimension*

$$\operatorname{vdim} Q^\varepsilon_{g,k,d}(X) = d(\det T_W) + (1 - g)(\dim X - 3) + k. \tag{2.3.1}$$

Proof: See [CFKM14] for details. □

The following wall-and-chamber structure on the parameter space $\varepsilon \in \mathbb{Q}_{>0}$ is well-known.

Proposition 2.3.9. *For a fixed $(g, k, d) \neq (0, 0, d), (0, 1, d)$, the parameter space $\mathbb{Q}_{>0}$ carries a chamber structure, where two $\varepsilon, \varepsilon'$ in the same chamber induce the same stability condition. The chambers are formed by intervals of the form $\left(\frac{1}{e+1}, \frac{1}{e}\right]$ where e is an integer $1 \leq e \leq d(L_\theta) - 1$, and curves with stability in this interval are allowed only* *rational tails of total degree at least* $e+1$ *with respect to* L_θ. *The two extreme chambers* $(1,\infty)$ *and* $\left(0, \frac{1}{d(L_\theta)}\right]$ *correspond to stable maps and stable quasimaps respectively.*

In light of this proposition, if $\varepsilon > 1$, we say that such curves are ∞-stable, which is equivalent to stable maps in the sense of Kontsevich. For all ε, there exist evaluation maps $\mathrm{ev}_i : Q^{\varepsilon}_{g,k,d}(X) \to X$. In general, the evaluation map lands in the codomain of the curve $C \to [W/G]$, but note by definition a quasimap has image disjoint from the base locus, so evaluation lands in the GIT quotient X.

2.4 0+-stability

The above definitions work well for stability conditions $\varepsilon > 0$, but of importance in this work is the case when $\varepsilon = 0+$. Conceptually, this is the stability condition formed by taking the limit as $\varepsilon \to 0$ from the right. To define this stability condition, we extend the above notions from integral characters in $\chi(G)$ to rational ones.

First, fix the notation that $\theta' \in \chi(G)$ is an integral character. Given a parameter $\lambda \in \mathbb{Q}_{>0}$, let $\theta = \lambda\theta'$ becomes a rational character in $\chi(G)_{\mathbb{Q}}$.

Definition 2.4.1. *Given a rational character* $\theta \in \chi(G)_{\mathbb{Q}}$, *define the corresponding line bundle* L_θ *by*

$$L_\theta := (W \times C_{m\theta})^{\otimes 1/m}$$

where m *is any positive integer making* $m\theta \in \chi(G)$.

This definition is independent of m since the rays of θ' in the character group produce identical GIT quotients. Similarly for a rational character θ, define $W^{us}(\theta) = W^{us}(m\theta')$ and $W^{ss}(\theta) = W^{s}(\theta) = W \backslash W^{us}(\theta)$ as usual.

From these definitions, the concepts a θ-prestable k-pointed genus g curve to $[W/G]$, the base locus, a θ-quasimap, and a θ-effective numerical class follow exactly as before. For the length of a θ-prestable quasimap, we have the following extension.

Definition 2.4.2. *The* θ*-length* $\ell_\theta(x)$ *for a quasimap* (C, x_i, P, u) *at a point* $x \in C$ *is defined as*

$$\ell_\theta(x) := \lambda \ell_{\theta'}(x)$$

where $\theta' = \theta/\lambda$ *is integral.*

Definition 2.4.3. *For a rational character θ, we say a θ-prestable quasimap (C, x_i, P, u) is* θ-stable *if*

(i) $\omega_C(\sum_i x_i) \otimes [u]^*(L_\theta)$ *is ample*

(ii) for every smooth point $x \in C$,

$$\ell_\theta(x) + \sum_i \delta_{x_i,x} \leq 1$$

where δ denote the Kronecker delta.

In particular, if $x = x_i$, then we must have that $\ell_\theta(x_i) = 0$, so that the base locus of a θ-stable quasimap does not intersect the markings of C. Finally we come to 0+-stability.

Definition 2.4.4. *A prestable map (C, x_i, P, u) to $[W/G]$ is called a $(0+) \cdot \theta -$ stable quasimap to X if it is $\lambda\theta$-stable for all $0 < \lambda << 1$.*

Remark 2.4.5. More conceptually, we can characterize 0+-stable quasimaps as follows. Given an effective numerical class d, we describe what happens to the stability conditions if $\varepsilon \to 0+$. First, $\varepsilon \leq \frac{1}{d(L_\theta)}$ so that the length condition is always satisfied. Second, the ampleness condition implies that C has no rational tails and $\mathcal{L}_\theta$ has positive degree on rational bridges. We can summarize these conditions by the following.

We say a curve (C, x_i, P, u) is a 0+-stable quasimap with respect to θ if $\omega_C(\sum_i x_i) \otimes \mathcal{L}_\theta^\varepsilon$ is ample for all $\varepsilon \in \mathbb{Q}_{>0}$. Such maps are also called simply *stable quasimaps.* With this nomenclature, we have on the one hand classical stable maps when $\varepsilon = \infty$, and on the other hand stable quasimaps when $\varepsilon = 0+$.

Definition 2.4.6. *Denote the space of genus g, k-pointed, 0+-stable quasimaps to X of numerical class d by $Q^{0+}_{g,k,d}(X)$.*

As above, Ciocan-Fontanine, Kim, and Maulik proved it is also a separated, finite type, DM stack with perfect obstruction theory and the same virtual dimension.

2.5 Graph Spaces of Quasimaps

There are also moduli of quasimaps where the domain curve C contains a parametrized $\mathbb{P}^1$. Starting with a prestable map (C, x_i, P, u), we can add a datum φ, which is a map

$\varphi : C \to \mathbb{P}^1$ of degree 1. Putting φ together with $[u]$, we obtain a map $([u], \varphi) : C \to [W/G] \times \mathbb{P}^1$ of numerical class $(d, 1)$. We say C contains a parametrized $\mathbb{P}^1$ since under these assumptions, C has a single component C_0 such that $\varphi|_{C_0} : C_0 \simeq \mathbb{P}^1$, while φ contracts the rest of the curve to a point. We call the data (C, x_i, P, u, φ) are graph quasimap.

Definition 2.5.1. *Let $\varepsilon \geq 0+$. Let $QG^{\varepsilon}_{g,k,d}(X)$ denote the moduli of ε-stable graph quasimaps (C, x_i, P, u, φ), called the ε-stable quasimap graph space.*

Specifically, the stability conditions imposed on graph quasimaps are:

(i) $\varepsilon\ell(x) \geq 1$

(ii) $\omega_{C'}(\sum x_i + \sum y_j) \otimes \mathcal{L}^{\varepsilon}_{\theta}$ *is ample*

where $C' = \overline{C \backslash C_0}$, x_i *are the markings on* C' *and* y_j *are the nodes on* $C' \cap C_0$.

It was proven in [CFKM14] that this is "good" moduli space as well. The graph spaces also carry evaluation maps $\mathrm{ev}_i : QG^{\varepsilon}_{g,k,d}(X) \to X \times \mathbb{P}^1$, since by definition graph quasimaps land in $[W/G] \times \mathbb{P}^1$ and the marked points are disjoint from the base locus. Since a graph quasimap is a map $[u] : C \to X \times \mathbb{P}^1$ of degree $(d, 1)$, the virtual dimension of the graph space is:

$$\begin{aligned} \mathrm{vdim}\, QG^{\varepsilon}_{g,k,d}(X) &= \int_{(d,1)} c_1(T_{X \times \mathbb{P}^1}) + (1-g)(\dim (X \times \mathbb{P}^1) - 3) + k \\ &= d(\det T_W) + 2 + (1-g)(\dim X - 2) + k. \end{aligned}$$

2.6 $\mathbb{C}^*$-localization of Graph Spaces

The ε-stable graph space carries additional structure of a torus action, which means we can apply torus equivariant localization in K-theory and intersection theory when working with this space. We describe the torus action and its fixed loci.

Consider the one dimensional torus $\mathbb{C}^*$ and its standard action on projective space $\mathbb{P}^1$, written in coordinates as

$$t \cdot [x : y] = [tx : y].$$

This induces an action on the ε-stable graph quasimaps $C \to [W/G] \times \mathbb{P}^1$ where $\mathbb{C}^*$ acts trivially on $[W/G]$, which in turn becomes an action on $QG^{\varepsilon}_{g,k,d}(X)$.

Now suppose a graph quasimap (C, x_i, P, u, φ) is $\mathbb{C}^*$-fixed. Then the points of $C' \cap C_0$ can only be 0 and ∞, else the torus action will shift C' along $C_0 \simeq \mathbb{P}^1$. In fact, all the markings, nodes, and base points are supported over 0 and ∞ of C_0, so g, k, and d split over the curves supported over 0 and ∞. The parametrized C_0 does not contribute to the data of the fixed loci, so we have the following decomposition.

$$(QG^{\varepsilon}_{g,k,d}(X))^{\mathbb{C}^*} = \coprod F^{g_1,k_1,d_1}_{g_2,k_2,d_2} \tag{2.6.1}$$

where the disjoint union is over all possible splittings $g_1 + g_2 = g$, $k_1 + k_2 = k$, and $d_1 + d_2 = d$, and $F^{g_1,k_1,d_1}_{g_2,k_2,d_2}$ consists of all fixed graph quasimaps with a (g_1, k_1, d_1)-quasimap over 0 and a (g_2, k_2, d_2)-quasimap over ∞. The components $F^{g_1,k_1,d_1}_{g_2,k_2,d_2}$ have a nice description. By applying the splitting principle to the ε-quasimaps supported over 0 and ∞ from C_0, we can write:

$$F^{g_1,k_1,d_1}_{g_2,k_2,d_2} = Q^{\varepsilon}_{g_1,k_1+\bullet,d_1}(X) \times_X Q^{\varepsilon}_{g_2,k_2+\bullet,d_2}(X) \tag{2.6.2}$$

where the fiber product is over the evaluation maps to X along the extra marked points $\bullet$ and $\star$ lying above $C' \cap C_0$. Note that if $(g_1, k_1, d_1) = (0, 0, 0)$ or $(0, 1, 0)$, and likewise for the second curve, the moduli space $Q^{\varepsilon}_{g,k,d}(X)$ is unstable, even though this splitting is possible in the graph space. We include these cases by the following conventions:

$$Q^{\varepsilon}_{0,0+\bullet,0}(X) := X \quad Q^{\varepsilon}_{0,1+\bullet,0}(X) := X \quad \mathrm{ev}_\bullet = \mathrm{id}_X \tag{2.6.3}$$

informed by the fact that $QG_{0,1,0}(X) = X \times \mathbb{P}^1$ with diagonal action and $QG_{0,0,0}(X) = X$ with trivial action.

For $\varepsilon \leq 1$, if $(g_1, k_1, d_1) = (0, 0, d_1)$ with $d_1 \neq 0$ and $\varepsilon \leq 1/d_1(L_\theta)$, then we have an unstable case for the fixed locus since the stability condition would require $k_1 > 0$ in this case. This is true symmetrically for $(g_0, k_0, d_0) = (0, 0, d_1)$. In this situation, we write

$$\begin{aligned} F^{g_0,k_0,d_0}_{0,0,d_1} &= Q^{\varepsilon}_{g_0,k_0+\bullet,d_0}(X) \times_{ev} Q^{\varepsilon}_{0,\bullet,d_1}(X)_\infty \\ F^{0,0,d_0}_{g_1,k_1,d_1} &= Q^{\varepsilon}_{0,\bullet,d_0}(X)_0 \times_{ev} Q^{\varepsilon}_{g_1,k_1+\bullet,d_1}(X) \\ F^{0,0,d_0}_{0,0,d_1} &= Q^{\varepsilon}_{0,\bullet,d_0}(X)_0 \times_{ev} Q^{\varepsilon}_{0,\bullet,d_1}(X)_\infty \end{aligned}$$

where $Q^{\varepsilon}_{0,\bullet,d_0}(X)_0$ is the moduli of graph quasimaps with a basis point of length $d_0(L_\theta)$ at $0 \in \mathbb{P}^1$ and no other base points on the parametrized component. Similarly $Q^{\varepsilon}_{0,\bullet,d_1}(X)_\infty$ consists of graph quasimaps with base point at $\infty \in \mathbb{P}^1$ and no other base points.

For further use, we give special attention the distinguished locus where all the data is concentrated over $0 \in \mathbb{P}^1$.

Definition 2.6.1. *Let $F_0 := F^{g,k,d}_{0,0,0}$ be the fixed locus of $QG^{\varepsilon}_{g,k,d}(X)$ where the entire curve is concentrated over $0 \in \mathbb{P}^1$.*

By the above construction of F_0 as a quasimap moduli, there is an evaluation map $\text{ev}_\bullet : F_0 \to X$, which does not extend to a map on all of $QG^{\varepsilon}_{g,k,d}(X)$. This map will be important in the upcoming definition of the J^{ε}-function. As a final note, any evaluation map on the entire graph space, by definition of the torus action, are torus equivariant. Therefore any appropriately defined invariants on the graph space can be endowed with a torus equivariant structure.

2.7 The Loop Space

Now we develop the necessary formalism for quantum K-theory and cohomology. By definition, an effective numerical class d is an element of $\text{Eff}(W, G, \theta) \subset \text{Hom}(\text{Pic}([W/G]), \mathbb{Z})$. Under the natural isomorphism between $\text{Eff}(W, G, \theta)$ and a subsemigroup of $H_2(X; \mathbb{Z})$, we can view the numerical classes as elements of the homology group instead. Thus, given a numerical class d, define the Novikov variables to be

$$Q^d = Q_1^{\langle p_1, d\rangle} \dots Q_r^{\langle p_r, d\rangle}$$

where $p_i = c_1(P_i)$ which form a nef integral basis of the free part of $H^2(X;\mathbb{Z})$, and $\langle -, - \rangle$ denotes the pairing $H^2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$. The grading within the Novikov variables is defined as $\deg Q_i = m_i$ where $c_1(T_X) = \sum_i m_i p_i$.

Let q be the equivariant parameter, i.e. a generator of the equivariant K-ring $K^0_{\mathbb{C}^*}(\text{pt}) = \mathbb{C}[q, q^{-1}]$. We often use q as a formal variable, except for situations where otherwise noted. The cohomological analog will be denoted $z = c_1^{\mathbb{C}^*}(T_0\mathbb{P}^1)$ which generates $H^*_{\mathbb{C}^*}(\text{pt})$. Generally, the generating functions in this work live in S_n-equivariant quantum K-theory and cohomology. There exists a S_N-equivariant framework for quantum K-theory and cohomology which we present here. However for the majority of the work, will suppress this notation since we will work with the "small" theory, i.e. when $N = 0$. We begin with the K-theoretic (symplectic) loop space.

Definition 2.7.1. *Define Λ to be the λ-algebra over $\mathbb{Q}$ containing Q, and the symmetric polynomials in N variables with Adams operations $\Psi^k : \Lambda \to \Lambda$ such that and $\Psi^k(Q) = Q^k$. Write Λ_+ for the ideal of positive degree symmetric functions and monomials in the Novikov variables, and equip Λ with the Λ_+-adic topology to ensure well-behaved convergence.*

Definition 2.7.2. *Define the loop space $\mathcal{K}$ to be the sympletic vector space of Λ-valued functions rational in q with coefficients from $K^0(X) \otimes \Lambda$. The equipped symplectic form is*

$$\Omega(f, g) := -[Res_{q=0} + Res_{q=\infty}](f(q^{-1}), g(q))\frac{dq}{q}.$$

Since modulo any power the ideal generated by Q the coefficients will be rational functions in q, the loop space $\mathcal{K}$ can be decomposed via partial fractions into a Lagrangian polarization

$$\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^- \tag{2.7.1}$$

where $\mathcal{K}^+$ consists of the series with Laurent polynomials in q and $\mathcal{K}^-$ consists of rational functions regular at $q = 0$ and vanishing at $q = \infty$.

Definition 2.7.3. *In cohomology, we can define $\mathcal{K}^{coh}$ analagously with the paramater z replacing q, however modulo powers of Q, only Laurent polynomials in z appear as coefficients, instead of rational functions. Therefore the symplecctic form Ω on $\mathcal{K}^{coh}$ need only take residue at $z = 0$, and the Lagrangian polarization is defined by $\mathcal{K}^{coh}_+ = \Lambda[[z]]$ and $\mathcal{K}^{coh}_- = z^{-1}\Lambda[z^{-1}]$.*

2.8 K-theoretic and Cohomological Correlators

Now we define correlator invariants used throughout this work, which are known for $\varepsilon = \infty$ as Gromov-Witten invariants. However they exist for every $\varepsilon \geq 0+$ and in S_n-equivariant quantum theory, and we will present them in this generality here.

The moduli space $Q^{\varepsilon}_{g,k,d}(X)$ carries canonical line bundles $L_i \in K^0(Q^{\varepsilon}_{g,k,d}(X))$ for every marking x_i, whose fiber over a curve $(C, x_1, \ldots, x_k, P, u)$ is the cotangent line to the marking $x_i \in C$. Let $\psi_i = c_1(L_i) \in H^*(Q^{\varepsilon}_{g,k,d}(X))$. Now let $t \in \mathcal{K}^+$ be a free parameter. Since $\mathcal{K}^+$ consists of Laurent polynomials in q, we can write $t = \sum_m t_a q^a$ where t_a is free in $\Lambda/q \otimes K^0(X)_{\mathbb{Q}}$. Hence rewrite t as $t(q)$. Denote $t(L_i)$ to the parameter t with L_i substituted for q. When the marked point x_i associated to L_i is clear from context (as it is in the following definition), we simply write $t(L)$. We can similarly define $t(\psi)$ in cohomology.

Definition 2.8.1. *Let $\Gamma_1, \ldots, \Gamma_n$ be classes in $K_0(X)$. Define the S_n-equivariant correlator in K-theory to be*

$$\langle \Gamma_1 L_1^{a_1}, \ldots, \Gamma_k L_k^{a_k}, t(L), \ldots, t(L)\rangle^{\varepsilon, S_n}_{g,k+n,d} := \chi(Q_{g,k+n,d}(X)/S_n; \prod_{i=1}^{k} \mathrm{ev}_i^*(\Gamma_i) L_i^{a_i} \otimes \prod_{j=k+1}^{k+n} \sum_a \mathrm{ev}_j^*(t_a) L_i^a \otimes \mathscr{O}_Q^{vir})$$

where $\mathrm{ev}_i : Q^{\varepsilon}_{g,k+n,d}(X) \to X$ is the ith evaluation map.

Definition 2.8.2. *Let $\gamma_1, \ldots, \gamma_k \in H^*(X)$ and $t \in \mathcal{K}^{coh}_+$. Define the cohomological correlators by*

$$\langle \gamma_1 \psi_1^{a_1}, \ldots, \gamma_k \psi_k^{a_k}, t(\psi), \ldots, t(\psi)\rangle^{\varepsilon, S_n}_{g,k+n,d} = \int_{[Q^{\varepsilon}_{g,k+n,d}(X)/S_n]^{vir}} \prod_{i=1}^{k} \mathrm{ev}_i^*(\gamma_i) \psi_i^{a_i} \cup \prod_{j=k+1}^{k+n} \sum_a \mathrm{ev}_j^*(t_a) \psi_j^a.$$

Since the last n insertions are all equal as polynomials in L or ψ, the K-theoretic permutation equivariant correlator takes values in S_n-modules, determined by the alternating sum in cohomology used to defined the Euler characteristic. In cohomology, the S_n-equivariant structure results in adjusting by a factor of $1/n!$. The purpose of this definition is that the correlators can be expanded multilinearly to encode all correlators of a certain form. To simplify notation for the resulting generating functions, we define the following double bracket notation.

Definition 2.8.3. *Define the double bracket correlator as:*

$$\langle\!\langle \Gamma_1 L_1^{a_1}, \ldots, \Gamma_k L_k^{a_k} \rangle\!\rangle_{g,k}^{\varepsilon} := \sum_{n,d\geq 0} Q^d \langle \Gamma_1 L_1^{a_1}, \ldots, \Gamma_k L_k^{a_k}, t(L), \ldots, t(L) \rangle_{g,k+n,d}^{\varepsilon, S_n} \tag{2.8.1}$$

$$\langle\!\langle \gamma_1 \psi_1^{a_1}, \ldots, \gamma_k \psi_k^{a_k} \rangle\!\rangle_{g,k}^{\varepsilon,} := \sum_{n,d\geq 0} \frac{Q^d}{n!} \langle \gamma_1 \psi_1^{a_1}, \ldots, \gamma_k \psi_k^{a_k}, t(\psi), \ldots, t(\psi) \rangle_{g,k+n,d}^{\varepsilon} \tag{2.8.2}$$

Definition 2.8.4. *For each of the above definitions, there exist variants for the graph spaces $QG^{\varepsilon}_{g,k,d}(X)$, where the quasimap space $Q^{\varepsilon}_{g,k,d}(X)$ is replaced by its graph space counterpart. Denote those correlators by $\langle - \rangle^{QG,\varepsilon}_{g,k,d}$ and $\langle\!\langle - \rangle\!\rangle^{\varepsilon,QG}_{g,k}$. Recall that the evaluation maps on $QG^{\varepsilon}_{g,k,d}(X)$ land in $X \times \mathbb{P}^1$, so graph space correlators take insertions from K-theory/cohomology on this target.*

2.9 ABC-correlators in Cohomology

Now we describe the ABC-twisted cohomological correlators of Givental and Tonita [GT14]. These correlators are associated to orbifold targets, and so require the Gromov-Witten theory for orbifolds developed by Abramovich, Graber, and Vistoli [AGV08] for DM stacks and by Chen and Ruan for orbifolds [CR04]. In this thesis, we will consider the algebraic case and will consider the orbifolds to be quotient stacks. Below, we follow Milanov and Roquefeuil's presentation of ABC-correlators in the stable map case. The theory can be defined in general, however the DM stacks we consider will be simply $Y = [X/\mu_m] \simeq X \times B\mu_m$, where X is smooth projective, μ_m is the multiplicative group of order m, and it acts on X trivially. The components of the rigidified inertia are indexed by $\nu \in \{1, \ldots, m\}$ in this case.

Definition 2.9.1. *Let $Y = [X/\mu_m]$. Let ℓ be a formal variable, which will take values in $K^0(\overline{M}_{g,k,d}(Y))$ when appropriate. Define 3 types of data as follows.*

(A) A finite number of orbifold vector bundles $E_\alpha \to Y$, a series of constants $s^A_{\alpha,i} \in \mathbb{C}$,

and corresponding multiplicative classes in cohomology

$$A_\alpha(\ell) = \exp\left(\sum_i s^A_{\alpha,i}\,\mathrm{ch}_i(\ell)\right)$$

(B) *A finite number of polynomials $f_\beta \in K^0(Y)[\ell]$, constants $s^B_{\beta,i} \in \mathbb{C}$, and multiplicative classes*

$$B_\beta(\ell) := \exp\left(\sum_i s^B_{\beta,i}\,\mathrm{ch}_i(\ell)\right)$$

(C) *A finite number of orbifold vector bundles $E_{\nu,\gamma} \to Y$ where $1 \leq \nu \leq m$, $1 \leq \gamma \leq k_\nu$, constants $s^C_{\nu,\gamma,i} \in \mathbb{C}$, and multiplicative classes*

$$C_{\nu,\gamma}(\ell) := \exp\left(\sum_i s^C_{\nu,\gamma,i}\,\mathrm{ch}_i(\ell)\right)$$

Definition 2.9.2. *Let $\pi : U \to \overline{M}_{g,k,d}(Y)$ be projection from the unviersal curve to the moduli of stable maps. Present U as the moduli of stable maps with an extra marking, say $\bullet$, so that the universal stable map coincides with the evaluation map $\mathrm{ev}_\bullet : U \to Y$. Let Z_ν be the closed substack of U consisting of stable maps such that the component C' carrying $\bullet$ has the following properties: C' carries two nodes and no other marked points, $f : C' \to Y$ is degree 0, and evaluation at one (and hence both) of the nodes lands in the rigified inertia component indexed by ν. Given the characteristic classes above, define three cohomology classes in $\overline{M}_{g,k,d}(Y)$ as*

$$\Theta^A_{g,k,d} := \prod_\alpha A_\alpha(\pi_*\,\mathrm{ev}^*_\bullet(E_\alpha))$$
$$\Theta^B_{g,k,d} := \prod_\beta B_\beta(\pi_*(\mathrm{ev}^*_\bullet(f_\beta)(L^{-1}_\bullet) - \mathrm{ev}^*_\bullet(f_\beta)(1)))$$
$$\Theta^C_{g,k,d} := \prod_{\nu,\gamma} C_{\nu,\gamma}(\pi_*(\mathrm{ev}^*_\bullet(E_{\nu,\gamma}) \otimes i_*\mathscr{O}_{Z_\nu})).$$

Put $\Theta^{ABC}_{g,k,d} = \Theta^A_{g,k,d}\Theta^B_{g,k,d}\Theta^C_{g,k,d}$.

Definition 2.9.3. *Define the ABC-twisted cohomological correlators as*

$$\langle \gamma_1\psi_1^{a_1}, \ldots, \gamma_k\psi_k^{a_k}\rangle_{g,k,d}^{ABC} = \int_{[\overline{M}_{g,k,d}(Y)]^{vir}} \Theta_{g,k,d}^{ABC} \prod_i \mathrm{ev}_i^*(\gamma_i)\psi_i^{a_i}. \tag{2.9.1}$$

For this work, we also need to extend the definition to stability conditions with $\varepsilon \leq 1$. The definition will differ in a few key points.

First, we need to address the extension of the quasimap theory we described so far to orbifold targets. The theory of stable quasimaps to orbifolds has in fact been developed by Cheong, Ciocan-Fontanine, and Kim [CCFK15], so we can rely on their framework to define ε-stable ABC-correlators. Next, even considering simply the GIT quotient X, since $Q^{\varepsilon}_{g,k,d}(X)$ is a fine moduli space, it does carry a universal curve which we denote UQ^{ε} with univeral quasimap, and projection $\pi : UQ^{\varepsilon} \to Q^{\varepsilon}_{g,k,d}(X)$. Indeed, UQ^{ε} can be presented as the quasimap moduli with an extra marked point, but it may occur at a base point, which is not true for standard marked points [CFKM14]. This extra marking $\bullet$, referred to as an infinitesimal marking, does carry an evaluation map $\mathrm{ev}_{\bullet} : QU^{\varepsilon} \to [W/G]$ which coincides with the universal quasimap and now lands in the quotient stack in general. In this way, the pullbacks of $E_\alpha, E_{\nu,\gamma} \to X$ and $f_\beta \in K^0(X)[\ell]$ are not well-defined for ε-stable quasimaps since they don't live over the quotient stack. Therefore we also need to put some stipulations on the data in the definition to make sure the constructions live over $[W/G]$ as opposed to simply $X = W /\!\!/ G$.

All in all, let μ_m act trivially on the affine variety W. Since the actinos of G and μ_m commute, we can equivalently consider $[[W/G]/\mu_m] \simeq [W/(G \times \mu_m)] \simeq [W/G] \times B\mu_m$. We will consider $G \times \mu_m$-equivariance on W. Let $E \to W$ be a $G \times \mu_m$-equivariant bundle. Pull back this bundle along the universal quasimap $QU^{\varepsilon} \to [W/(G \times \mu_m)]$ by the mixed quotient construction $\mathcal{P} \times_{G\times\mu_m} E$, where $\mathcal{P}$ is the universal principal $G \times \mu_m$-bundle.

Now we define the data $A^{\varepsilon}_{\alpha}(\ell)$, $B^{\varepsilon}_{\beta}(\ell)$, and $C^{\varepsilon}_{\nu,\gamma}(\ell)$ in the same way as Definition 2.9.1 except E_α and $E_{\nu,\gamma}$ are $G \times \mu_m$-equivariant bundles on W, f_β are polynomials in $K^0([W/(G \times \mu_m)])[\ell]$, and ℓ will take values in $Q^{\varepsilon}_{g,k,d}(Y)$. Define the series $\Theta^{A,\varepsilon}_{g,k,d}$, $\Theta^{B,\varepsilon}_{g,k,d}(X)$, and $\Theta^{C,\varepsilon}_{g,k,d}$ in the same way except the pullback to the universal curve $\mathrm{ev}_{\bullet}^*$ is replaced by the mixed quotient construction above. Similarly replace $\mathrm{ev}_{\bullet}^*(f_\beta)$ with $\mathcal{P} \times_{G\times\mu_m} f_\beta \in K^0(QU^{\varepsilon})[\ell]$. Finally, $\Theta^{ABC,\varepsilon}_{g,k,d}$ is the product of the three ABC series as before. Then

$$\langle \gamma_1 \psi_1^{a_1}, \ldots, \gamma_k \psi_k^{a_k} \rangle_{g,k,d}^{ABC,\varepsilon} = \int_{[Q^{\varepsilon}_{g,k,d}(X/H)]^{vir}} \Theta^{ABC,\varepsilon}_{g,k,d} \prod_i \mathrm{ev}_i^*(\gamma_i)\psi^{a_i} \tag{2.9.2}$$

is a well-defined extension of ABC-correlators to the ε-stable case. For insertions $\gamma_i t(\psi_i)$, extend the definition multilinearly.

While we will not use the specifics of this definition to "untwist" ABC-correlators, the aforementioned fake correlator for stable maps and the stem correlator for 0+-stable maps do appear later in this work. The main property of those ABC-correlators in this work is that the cohomology twist has the form $\Theta^{ABC,\varepsilon}_{g,k,d} = 1 +$ (higher degree terms). Therefore the twisted correlators behave the same as classic GW ones if we are counting dimensions to check for vanishing, i.e. the lowest degree terms will agree with or without the twist.

2.10 The J^ε function

From here on, we will only be interested in the genus $g = 0$ case, which contains the much studied J^ε generating function of invariants in both K-theory and cohomology. This section presents their definitions for both theories. We begin with more theory of graph space localization.

Since $g = 0$ has no nontrivial splitting, we drop it from the super and subscript notation, so the fixed loci are denoted $F^{k_1,d_1}_{k_2,d_2}$. One can show that when $k \geq 1$ or $k = 0$ and $d(L_\theta) > 1/\varepsilon$, then $F_0 := F^{k,d}_{0,0} \simeq Q^{\varepsilon}_{0,k+\bullet,d}(X)$ and has equivariant Euler class of the virtual normal bundle given by:

$$e^{\mathbb{C}^*}(N_{F_0}) = \mathrm{cont}_{k,d}(q) := \begin{cases} 1 & (k,d) = (0,0) \\ 1-q & (k,d) = (1,0) \\ (1-q)(1-qL) & \text{otherwise} \end{cases} . \tag{2.10.1}$$

The corresponding formula in cohomology is

$$e^{\mathbb{C}^*}(N_{F_0}) = \mathrm{cont}_{k,d}(z) := \begin{cases} 1 & (k,d) = (0,0) \\ z & (k,d) = (1,0) \\ z(z-\psi_\bullet) & \text{otherwise} \end{cases}. \tag{2.10.2}$$

Note that the formula is the same for all targets. However, in the case when $k = 0$ and $d(L_\theta) \leq 1/\varepsilon$, the moduli space $Q^\varepsilon_{0,\bullet,d}(X)$ does not exist any more and F_0 has a different description. In this situation, both F_0 and the Euler class $e^{\mathbb{C}^*}(N_{F_0}) = \mathrm{cont}(0,d)(q)$ depends on the target X, and similarly in cohomology. If $\varepsilon = \infty$ then all d fall under the first case when $k = 0$, while if $\varepsilon = 0+$, we have the second. In general, F_0 is a union of flag bundles over subvarieties in X with evaluation $\mathrm{ev}_\bullet$ is the projection map on each component. See [Web18] for details.

Now we come to the definition of the J-function for ε-stable quasimaps. This definition appears in [CFKM14] for cohomology and was extended to K-theory in [TY16]. In this thesis we use the normalized convention for the J^ε-function, which is equivalent the definition of J in Givental's work scaled by $1/(1-q)$ and $1/z$ respectively.

Definition 2.10.1. *Let $t(L)$ be a parameter from $\mathcal{K}^+[L_i]$ where the L_i are the cotangent bundle for each marking on $Q^\varepsilon_{0,k,d}(X)$. Define the K-theoretic permutation equivariant function $J^\varepsilon_X(-,q,Q) : \mathcal{K}^+[L_i] \to \mathcal{K}$ by*

$$J^\varepsilon(t,q,Q) = \sum_{n,d\geq 0} Q^d(\mathrm{ev}_\bullet)_* \left(\prod_{i=1}^n \sum_a \mathrm{ev}_i^*(t_a) L_i^a \otimes \frac{\mathscr{O}^{vir}_{F_0}}{e^{\mathbb{C}^*}(N^{vir}_{F_0})^\vee} \right) \tag{2.10.3}$$

where $(\mathrm{ev}_\bullet)_*$ is the S_n-equivariant pushforward to X, $(N^{vir}_{F_0})^*$ is the virtual conormal bundle of F_0 in $QG^\varepsilon_{g,k,d}(X)$, and $e^{\mathbb{C}^*}(-) = \mathrm{tr}_{\mathbb{C}^*} \bigwedge^\bullet(-)$ denotes the torus-equivariant Euler class.

Similarly in cohomology,

$$J^\varepsilon_{coh}(t,z,Q) := \sum_{n,d\geq 0} Q^d(\mathrm{ev}_\bullet)_* \left(\frac{1}{n!} \prod_{i=1}^n \sum_a \mathrm{ev}_i^*(t_a)\psi_i^a \cap \frac{[QG^\varepsilon_{0,k,d}(X)]^{vir}}{e^{\mathbb{C}^*}(N^{vir}_{F_0})^\vee} \right) \tag{2.10.4}$$

Of importance to this work is the stable quasimap $\varepsilon = 0+$ case. Let $J^{0+}(t,q,Q) =$

$I(t,q,Q)$ (resp. $J^{0+}_{coh}(t,z,Q) = I^{coh}(t,z,Q)$), which we call the I-function. If $t=0$, then $J^\varepsilon(0,q,Q)$ (resp. $J^\varepsilon_{coh}(0,z,Q)$) is no longer S_n-equivariant and we refer to it as the *small* J^ε-function (and resp. *small* I-function). The well-known J-function of Givetnal is the $\varepsilon=\infty$ case, so define $J(t,q,Q) := J^\infty(t,q,Q)$ and similarly in cohomology.

Proposition 2.10.2. *Let $\{\Phi_a\}$ be a basis of $K^0(X)_{\mathbb{Q}}$. Then*

$$\begin{aligned} J^\varepsilon(t,q,Q) = \Phi_0 + \frac{t(q)}{1-q} &+ \sum_a \sum_{d\neq 0, d(L_\theta)\leq 1/\varepsilon} \Phi_a Q^d \chi\left(F_0; \mathscr{O}^{vir}_{F_0} \otimes \frac{\mathrm{ev}^*_\bullet(\Phi^a)}{e^{\mathbb{C}^*}(N^{vir}_{F_0})^\vee}\right) \\ &+ \sum_a \sum_{n>0 \ or \ d(L_\theta)>1/\varepsilon} Q^d \Phi_a \langle \frac{\Phi^a}{(1-q)(1-qL)}, t(L), \ldots, t(L)\rangle^{\varepsilon, S_n}_{0,1+n,d} \\ = \sum_a \Phi_i \langle\!\langle \frac{\Phi^i}{1-qL}\rangle\!\rangle^\varepsilon_{0,1}. \end{aligned}$$

Likewise for a basis $\{\phi_a\}$ of cohomology:

$$\begin{aligned} J^\varepsilon_{coh}(t,z,Q) = \phi_0 + \frac{t(z)}{z} &+ \sum_a \sum_{d\neq 0, d(L_\theta)\leq 1/\varepsilon} \phi_a Q^d \int_{F_0^{vir}} \frac{\mathrm{ev}^*_\bullet(\phi^a)}{e^{\mathbb{C}^*}(N_{F_0})} \\ &+ \sum_a \sum_{\substack{n>0 \ or \ d \\ d(L_\theta)>1/\varepsilon}} \phi_a \frac{Q^d}{n!} \langle \frac{\phi^a}{z(z-\psi_\bullet)}, t(\psi), \ldots, t(\psi)\rangle^\varepsilon_{0,1+n,d} \\ = \sum_a \phi_a \langle\!\langle \frac{\phi^a}{z-\psi_\bullet}\rangle\!\rangle^\varepsilon_{0,\bullet} \end{aligned}$$

where the unstable terms are implicitly understood in the $\langle\!\langle - \rangle\!\rangle$ notation.

For the dilaton shifts Φ_0 and ϕ_0, we will from here on drop the basis notation a simply write "1" for these initial terms.

Proof: First partition the sum over (n,d) by the two cases outlined in equation (2.10.1) and equation (2.10.2). Writing the residues as linear combinations of the dual basis using the Poincare pairing and applying the projection formula, one obtains the result. □

When $\varepsilon = 0+$, then $d(L_\theta) \leq 1/\varepsilon$ for all d, so the terms of the small I-function consist of irregular residues of F_0 only. Therefore we do not use the double bracket notation for the small I-function as none of terms are written as correlators. The small I-function takes the form

$$I(0,q,Q) = 1 + \sum_{a,d} \Phi_a Q^d \chi\left(\frac{\mathrm{ev}_\bullet^* \Phi^a}{\mathrm{tr}_{\mathbb{C}^*} \bigwedge^*(N_{F_0}^{vir})^\vee}\right). \tag{2.10.5}$$

Similarly, $d(L_\theta) > 1/\varepsilon$ when $\varepsilon = \infty$ for $d \neq 0$ and only terms of the second type appear, and so that J is entirely given by the double bracket, which expands to the classical formula for the small J-function.

$$J(0,q,Q) = 1 + \sum_{a,d} \Phi_a Q^d \langle \frac{\Phi^a}{(1-q)(1-qL)} \rangle_{0,1,d}^{\varepsilon} \tag{2.10.6}$$

Example 2.10.3. As stated in the introduction, the I-function was introduced by Givental to be a more easily computable generating function related to J which can be studied either geometrically or using q-finite difference methods. When $t = 0$, the toric case is well-understood [Giv, Part V]. If $X = W /\!\!/ G$ is toric, then

$$I(0,q,Q) = \sum_{d \in \mathbb{Z}^K} Q^d \prod_{j=1}^{N} \frac{\prod_{r=-\infty}^{0} (1 - U_j(P) q^r)}{\prod_{r=-\infty}^{D_j(d)} (1 - U_j(P) q^r)}$$

where $W = \mathbb{C}^N$, $G = (\mathbb{C}^*)^K$, and P_i and U_j, subject to Kirwan's relations $U_j(P) = 0$, generate $K^0_{(\mathbb{C}^*)^N}(X)$, which determine the integers $D_j(d)$ as well. Taking X to be the quintic 3-fold, we obtain

$$I(0,q,Q) = \sum_{d \geq 0} Q^d \frac{\prod_{k=1}^{5d} (1 - P^5 q^k)}{\prod_{k=1}^{d} (1 - P q^k)^5}.$$

where $P = \mathscr{O}(-H)$, see Remark 2.2.1. For the corresponding formulae in cohomology, see [CFK10]. In fact, given simple enough cases Theorem 5.0.1 is computable using elementary series expansion. For example in the quintic 3-fold case, one can expand the numerator in terms of powers of $(1 - P)$, and expand the denominator similarly by $1/(1-x)^5 = \sum_n \binom{n+4}{n} x^n$. Both will terminate since $(1 - P)$ is nilpotent in $K^0(X)$. Such expansion are not easy to see in general, so we employ geometric methods to prove confluence for Calabi-Yau GIT targets.

2.11 Examples of GIT Targets

We list classes of examples of GIT targets to which this theory applies. Additionally, since we give attention to Calabi-Yau GIT targets, we make note on examples of that type as well.

Example 2.11.1 (**Toric Varieties**). The quasimap theory for case was first developed in [CFK10]. For toric varieties, W is isomorphic to a complex vector space $\mathbb{C}^N$ and G an abelian torus $(\mathbb{C}^*)^K$ with action given by weights on coordinates in the torus. This is a well-understood class of varieties, containing projective space $\mathbb{P}^n = \mathbb{C}^{n+1} /\!\!/ \mathbb{C}^*$, Hirzebruch surfaces, total spaces of bundles $\mathscr{O}_{\mathbb{P}^n}(-m)$, and many more. The Calabi-Yau condition translates nicely to toric varieties: a smooth toric variety X will be Calabi-Yau iff the sum of the weights of the action vanishes.

Example 2.11.2 (**Flag Varieties of type** A). Here W is a vector space of the form $\prod_i \operatorname{Hom}(V_{i-1}, V_i)$, and the group G is a product of general linear groups $GL_{k_i}(\mathbb{C})$ and the character θ is the determinant character on each factor. Of course Grassmannians and projective space fall into this class of GIT quotients.

Example 2.11.3 (**Projective Varieties**). Suppose X is a smooth projective variety say $X \subseteq \mathbb{P}^n$. We can view X as a GIT target by considering $W = C(X)$, where $C(X)$ is the affine cone over X in $\mathbb{C}^{n+1}$. Then $X = C(X) /\!\!/ \mathbb{C}^*$ where the $\mathbb{C}^*$-action is induced from the action on $\mathbb{C}^{n+1}$. Indeed the unstable locus is the singular point of $C(X)$ at the origin, while $W^s = W^{ss} = C(X)\backslash\{0\}$. When $\varepsilon > 1$, the moduli spaces of stable maps to X are proper and with perfect obstruction theory. However in the projective case when $\varepsilon \leq 1$, $Q^{\varepsilon}_{g,k,d}(X)$ is proper and does carry an obstruction theory, but it is only perfect for X a complete intersection [CFKM14].

Example 2.11.4 (**Zero Locus of Sections of Homogeneous Bundles**). Perhaps the most interesting class of examples are GIT quotients which are zero loci of a section of a homogeneous bundle and in fact this generalizes the previous classes. Specifically, suppose $Y = (W, G, \theta)$ is a general GIT quotient. Given a finite dimensional representation V of G, and let $W \times V$ be the corresponding equivariant vector bundle on W. This induces a vector bundle $\overline{E} := W \times_G E$ on Y by the mixed quotient construction. Take a section $s \in H^0(W, W \times E)^G$, or equivalently, a section $\overline{s} \in H^0(Y, \overline{E})$. If *overlines* is regular and its zero locus $X := Z(\overline{s})$ is smooth, then $X := Z(s) /\!\!/_{\theta} G$. Since Z contains only l.c.i singularities, and they are in the unstable locus, X satisfies condition (i) - (iv) from Section 2.1.

Locally, s will be rank(E) number of functions which form a regular sequence. Therefore in the case G is abelian, then E will split into one-dimensional subrepresentations and thus s will be a regular section, and this construction generates complete intersections $X \subseteq Y$ in this case. Of course this is true for toric varieties.

There exist many interesting cases when G is nonabelian as well. For example, flag varieties of type B, C, and D can be realized as zero sections of homogeneous bundles on the type A flag variety. To mention an example noted in where X is not a complete intersection [CFK14, Example 2.8.4], take $Y = G(2,6)$, the Grassmannian of 2-dimensional subspaces in a $\mathbb{C}^6$. Let $\overline{E} = Sym^3(S^\vee)$ where $S^\vee$ is the dual of the tautological rank 2 subbundle. Take a regular section of $\overline{E}$, and the zero locus defines a Calabi-Yau fourfold which is not a complete intersection.

This leads into how to generate Calabi-Yau zero loci of regular sections. By the adjunction formula, if $c_1(\overline{E}) = c_1(\det N_{Y/X})$ is inverse to $K_Y = c_1(\omega_Y)$, then X will be Calabi-Yau. This relation is not an unusual occurrence. In the projective case, take $X \subseteq Y = \mathbb{P}^n$ to be a vanishing locus of k homogeneous equations of degree $d_1, \ldots, d_k$. Then $\omega_X = \mathscr{O}(d_1 + \cdots + d_k - n - 1)$, and X will be Calabi-Yau when $\sum_i d_i = n + 1$. A famous example is the vanishing of a section of $\mathscr{O}(5)$ on $\mathbb{P}^4$, which is the quintic 3-fold. However this process generates many interesting Calabi-Yau varieties, as we have seen with the $Y = G(2,6)$ example. These are the Calabi-Yau GIT targets to which this thesis is concerned.

More specifically for the integrality of the mirror map, there is currently the additional assumption on the result that $K^0(X)$ is generated by line bundles. This is true for some of the classes of examples above but not all. In particular, if Y is toric or a flag manifold of type A, and X is a zero locus of a vector bundle $E \to Y$ coming from a representation G, then $K^0(X)$ is generated by line bundles.

Chapter 3

Birkhoff Factorization for $J^{\mathcal{E}}$

In order to prove the main theorem of the thesis, we use the Birkhoff factorization formula for the J^ε-function,

$$J^\varepsilon(t,q,Q) = S^\varepsilon(t,q,Q)(P^\varepsilon(t,q,Q))$$

where S^ε is an operator series in $1/(1-q)$ (resp. $1/z$) while P^ε is a series in $1-q$ (resp. z). Our goal for this section is the prove this result for both K-theory and cohomology, and state some corollaries which will be useful. The techniques in this section were originally developed for cohomology by Ciocan-Fontanine and Kim [CFK14] and adapted for K-theory by Tseng and You [TY16]. Since we work with S and P in K-theory primarily, we present the proofs in those cases, while noting that the cohomology proofs are analogous.

For the rest of the thesis, we will denote $t(L)$ by t and $\mathrm{ev}_i^*(t) := \sum_a \mathrm{ev}_i(t_a)L^a$ to shorten notation, and similarly for $t(\psi)$. But note that the free parameter insertions will depend on the L_i or ψ_i in general. We will make note when the canonical bundle or class is relevant, such as in Lemma 4.4.7.

3.1 A Non-constant Metric in K-theory

First we define a non-constant metric G^ε_{ab} on $\mathcal{K}$ which appears in the definitions of $(S^\varepsilon)^*$ and P^ε for K-theory. We use the adjective "non-constant" since it is a perturbation of g_{ij} in the variables t, q, and Q. As described in [Lee04], G^ε appears only in K-theory as a consequence of the modified contraction axiom. As a result, it appears in the K-theoretic WDVV equation, which in equivalent to the associativity of the quantum tensor product [Giv00]. In particular G_{ij} appears as the derivatives $d/dt_i dt_j$ in the coordinates $t = \sum_a t_a \Phi_a$ of the quantum K-potential in genus 0

$$G^{\varepsilon}(t,Q) = \frac{1}{2}\langle t, t\rangle + \sum_n \sum_d Q^d \langle t, \ldots, t\rangle_{0,n,d}^{\varepsilon, S_n}$$

and the inverse tensor arises from appyling the alternating series $1 - f + f^2 - \cdots = 1/(1+f)$ to G^{ε}_{ij}.

Definition 3.1.1. *For $t \in K^+$, the non-constant metric G^{ε} is defined by*

$$G^{\varepsilon}_{ab} := g_{ab} + \langle\!\langle \Phi_a, \Phi_b \rangle\!\rangle^{\varepsilon}_{0,2} \tag{3.1.1}$$

with inverse tensor

$$G^{ab}_{\varepsilon} = g^{ab} + \sum_{i \geq 0} \sum_{m_1, \ldots, m_i} (-1)^{i+1} \langle\!\langle \Phi^a, \Phi^{m_1} \rangle\!\rangle^{\varepsilon}_{0,2} \langle\!\langle \Phi_{m_1}, \Phi^{m_2} \rangle\!\rangle^{\varepsilon}_{0,2} \cdots \langle\!\langle \Phi_{m_i}, \Phi^b \rangle\!\rangle^{\varepsilon}_{0,2}. \tag{3.1.2}$$

Using Lee's and Givental's description, we present G^{ij} in a more compact form in the following lemma.

Lemma 3.1.2. *There exists a space $\mathcal{Z}^{\varepsilon}$ with evaluation maps $\hat{\mathrm{ev}}_i : \mathcal{Z}^{\varepsilon} \to X$ and a sheaf $\mathscr{O}^G \in \mathcal{K}_{\mathcal{Z}^{\varepsilon}}$ such that*

$$G^{ij} = \langle\!\langle \Phi^a, \Phi^b \rangle\!\rangle^{\mathscr{O}^G}_{0,2} := \sum_{n,d} Q^d \chi(\hat{\mathrm{ev}}_1^* \Phi^a \hat{\mathrm{ev}}_2^* \Phi^b \prod_{k=3}^{n+2} \hat{\mathrm{ev}}_k^*(t) \otimes \mathscr{O}^G ; \mathcal{Z}^{\varepsilon}/S_n) \tag{3.1.3}$$

Proof: Consider the rth term of the series of G^{ij} without the alternating sign:

$$\begin{aligned}(-1)^{r+1} G^{ab}_r &= \sum_{m_1, \ldots, m_r} \langle\!\langle \Phi^a, \Phi^{m_1} \rangle\!\rangle_{0,2} \langle\!\langle \Phi^{m_1}, \Phi_{m_2} \rangle\!\rangle_{0,2} \cdots \langle\!\langle \Phi_{m_r}, \Phi^b \rangle\!\rangle_{0,2} \\ &= \sum_{n,d} \sum_{\substack{\sum_j n_j = n \\ \sum_j d_j = d}} Q^d \sum_{m_1, \ldots, m_r} \prod_{j=1}^{r+1} \chi(\mathrm{ev}_1^*(\Phi_{m_{j-1}}) \, \mathrm{ev}_2^*(\Phi^{m_j}) \prod_{k=3}^{2+n_i} \mathrm{ev}_k^*(t); \mathscr{O}^{vir}_{Q^{\varepsilon}_{0,2+n_j,d_j}(X)}/S_{n_j})\end{aligned}$$

where by an abuse of notation $\Phi_{m_0} := \Phi^a$ and $\Phi^{m_{r+1}} := \Phi^b$. Let p_n^j denote the nth marked point on the jth quasimap space above. Now since the fundamental class of the diagonal $\Delta \subset X \times X$ is $\sum_k \Phi_k \boxtimes \Phi^k$, glue one by one the marked points p_2^j and p_1^{j+1} since they carry a sum of a product Poincare dual insertions. Then the sum over the $m_1, \ldots, m_r$ of products of Euler characteristics becomes one Euler characteristics over the space

$$Q^r_{0,\vec{n},\vec{d}} := Q^\varepsilon_{0,2+n_1,d_1}(X) \times_{\text{ev}} Q^\varepsilon_{0,2+n_2,d_2} \times_{\text{ev}} \cdots \times_{\text{ev}} Q^\varepsilon_{0,2+n_r,d_r} \subset Q^\varepsilon_{0,2+\sum_j n_j, \sum_j d_j}$$

where the product over ev denotes the gluing diagram:

$$\begin{array}{ccc} Q^\varepsilon_{0,2+n_j,d_j} \times_{\text{ev}} Q^\varepsilon_{0,2+n_{j+1},d_{j+1}} & \xrightarrow{\text{ev}} & X \\ \downarrow i & & \downarrow \Delta \\ Q^\varepsilon_{0,2+n_j,d_j}(X) \times Q^\varepsilon_{0,2+n_{j+1},d_{j+1}} & \xrightarrow{\text{ev}_{p_2^j} \times \text{ev}_{p_1^{j+1}}} & X \times X \end{array} \tag{3.1.4}$$

where the top arrow denotes evaluation at the preimage of p_2^j and p_1^{j+1}. The marked points p_1^1 and p_2^r corresponding to Φ^a and Φ^b do not get glued and so induce evaluation maps on $Q^r_{0,\vec{n},\vec{d}}$ which are restrictions from evaluation maps on $Q^\varepsilon_{0,2+n,d}$.

$$\text{ev}_{p_1^1}, \text{ev}_{p_2^r} : Q^r_{0,\vec{n},\vec{d}} \to X$$

Similarly none of the marked points with insertion t are glued, and so they carry evaluation maps $\hat{\text{ev}}_k$ on $Q^r_{0,\vec{n},\vec{d}}$ as well. All in all,

$$\begin{aligned} (-1)^{r+1} G_r^{ab} &= \sum_{n,d} \sum_{\substack{\sum_j n_j = n \\ \sum_j d_j = d}} Q^d \chi(\text{ev}^*_{p_1^1}(\Phi^a)\, \text{ev}^*_{p_2^r}(\Phi^b) \prod_k^n \text{ev}^*_k(t); (Q^r_{0,\vec{n},\vec{d}})^{vir}/S_{n_1} \times \cdots \times S_{n_r}) \\ &= \sum_{n,d} \sum_{\substack{\sum d_j = d \\ r \text{ subsets of } \{1,\ldots,n\}}} Q^d \chi(\text{ev}^*_{p_1^1}(\Phi^a)\, \text{ev}^*_{p_2^r}(\Phi^b) \prod_k^n \text{ev}^*_k(t); (Q^r_{0,\vec{n},\vec{d}})^{vir}/S_n) \\ &:= \sum_{\vec{n},\vec{d}} \langle\langle \Phi^a, \Phi^b \rangle\rangle_{0,2}^{Q^r_{0,\vec{n},\vec{d}}}. \end{aligned}$$

Now finally we can see how to define $\mathscr{O}^G$ and $\mathcal{Z}^\varepsilon$. Taking the alternating sum over all of the $Q^r_{0,\vec{n},\vec{d}}$, define $\mathscr{O}^G$ to be the structure sheaf of the alternating sum of spaces:

$$\sum_r (-1)^{r+1} \left(\sum_{\vec{n},\vec{d}} Q^r_{0,\vec{n},\vec{d}} \right) \tag{3.1.5}$$

By rearranging the first sum to over n, d, we can embed this sum into $K^0(\mathcal{Z}^\varepsilon) \otimes \mathbb{Q}[[Q,t]]$ where $\mathcal{Z}^\varepsilon$ is the disjoint union over all $Q^\varepsilon_{0,2+n,d}(X)$, $n, d \geq 0$. We can define evaluation maps on $\mathcal{Z}^\varepsilon$ corresponding to the first two marked points for each component, and we refer to those by $\hat{\mathrm{ev}}_1, \hat{\mathrm{ev}}_2 : \mathcal{Z}^\varepsilon \to X$. The claim follows by linearity. □

We have stated this lemma in a way which mimics the proof of the WDVV identity in K-theory (see [Giv00]), but for simpliciy one might want a sheaf in a K-ring instead of a K-ring module such as $\mathcal{K}_{\mathcal{Z}^\varepsilon}$. To this end, one can simply consider G^{ij} to be a double bracket correlator $\langle\langle \Phi^a, \Phi^b \rangle\rangle^{\mathscr{O}^G}_{0,2}$ by fixing n, d in equation (3.1.5) and letting the double brackets denote a sum over n and d. Of course the first two terms $G^{ij} = g^{ij} + \langle \Phi^a, \Phi^b \rangle_{0,2} + \ldots$ are the unstable case and when $r = 0$, which are included implicitly in the double bracket notation. In any case by definition, evaluation maps $\mathcal{Z}^\varepsilon \to X$ induce morphisms on loop spaces

$$(\hat{ev}_1)_*, (\hat{\mathrm{ev}}_2)_* : \mathcal{K}_{\mathcal{Z}^\varepsilon} \to \mathcal{K}_X.$$

This analysis will help give basis free definitions of operators and series in K-theory in the next section.

3.2 Definition of S and P

Now we present the definition of $S^\varepsilon(-, q, Q)$ and its inverse operator $(S^\varepsilon)^*(-, 1/q, Q)$

Definition 3.2.1. *The operators* $S^\varepsilon(t, q, Q)(-) : \mathcal{K} \to \mathcal{K}$ *and* $(S^\varepsilon)^*(t, q, Q)(-) : \mathcal{K} \to \mathcal{K}$

are defined:

$$S^{\varepsilon}(t,q,Q)(\Gamma) := \Gamma + \sum_{(n,d)\neq(0,0)} Q^d (\mathrm{ev}_1)_* \left(\frac{\mathscr{O}^{vir}_{Q^{\varepsilon}_{0,2+k,d}(X)}}{1-qL} \mathrm{ev}_2^*(\Gamma) \prod_{j=3}^{2+k} \mathrm{ev}_j^*(t) \right)$$

$$= \sum_{\alpha,\beta} \Phi_\beta g^{\alpha\beta} \left(\langle \Phi_\alpha, \Gamma \rangle + \sum_{(k,d)\neq(0,0)} Q^d \left\langle \frac{\Phi_\alpha}{1-qL}, \Gamma, t, \ldots, t \right\rangle^{\varepsilon, S_n}_{0,2+k,d} \right)$$

$$(S^{\varepsilon})^*(t,q,Q)(\Gamma) := \sum_{\alpha,\beta} \Phi_\beta G^{\alpha\beta}_{\varepsilon} \left(\langle \Phi_\alpha, \Gamma \rangle + \sum_{(n,d)\neq(0,0)} Q^d \left\langle \frac{\Gamma}{1-qL}, \Phi_\alpha, t, \ldots, t \right\rangle^{\varepsilon, S_n}_{0,n+2,d} \right)$$

When $t = 0$, we call these the small *S-operators.*

The P-series is a generating function of correlators on graph spaces $QG_{0,1+n,d}(X)$ with one distinguished marking, which we define by the equivariant K-theory on $\mathbb{P}^1$. Note that we have a $\mathbb{C}^*$-equivariant map $QG_{0,1+n,d}(X) \to \mathbb{P}^1$, which induces a map $K^0(\mathbb{C}^*, \mathbb{P}^1) \to K^0(\mathbb{C}^*, QG_{0,1+n,d}(X))$, making the $\mathbb{C}^*$-equivariant K-theory of the graph spaces a $K^0(\mathbb{C}^*, \mathbb{P}^1)$-module. By $\mathbb{C}^*$-equivariant localization there is an injective map $K^0(\mathbb{C}^*, [0]) \oplus K^0(\mathbb{C}^*, [\infty]) \to K^0(\mathbb{C}^*, \mathbb{P}^1)$ since the fixed points of the action are $0, \infty \in \mathbb{P}^1$. Recall that the equivariant K-theory of a point is simply the representation ring of $\mathbb{C}^*$, so $K^0(\mathbb{C}^*, pt) = \mathbb{C}[q, q^{-1}]$. Finally, let $p_0, p_\infty \in K^0(C^*, \mathbb{P}^1)$ be defined by

$$p_0|_0 = q \quad p_0|_\infty = 1 \quad p_\infty|_0 = 1 \quad p_\infty|_\infty = 1/q.$$

Definition 3.2.2. *The P-series in K-theory is defined as:*

$$P^{\varepsilon}(t,q,Q) := \sum_{\alpha\beta} \Phi_\beta G^{\alpha,\beta}_{\varepsilon} \sum_{(n,d)\neq(0,0)} Q^d \langle \Phi_\alpha (1 - p_\infty), t, \ldots, t \rangle^{QG, \varepsilon, S_n}_{0,1+n,d}.$$

In the case when $t = 0$, this will be called the small P-series.

Now we present the cohomological definitions. Similarly, the equivariant cohomology $H^*(\mathbb{C}^*, QG_{0,2,d}(X))$ is a $H^*(\mathbb{C}^*, \mathbb{P}^1)$-module. By a slight abuse of notation, let $p_0, p_\infty \in H^*(\mathbb{C}^*, \mathbb{P}^1)$ be defined by

$$p_0|_0 = z \quad p_0|_\infty = 0 \quad p_\infty|_0 = 0 \quad p_\infty|_\infty = -z.$$

Definition 3.2.3. *Let ϕ_i be a basis of $H^*(X;\mathbb{Q})$. The S^ε, $(S^\varepsilon)^*$ operators and P-series in cohomology are defined:*

$$S^\varepsilon_{coh}(t,z)(\gamma) := \sum_a \phi^a \langle\!\langle \frac{\phi_a}{z-\psi}, \gamma \rangle\!\rangle^\varepsilon_{0,2}$$
$$(S^\varepsilon_{coh})^*(t,z)(\gamma) := \sum_a \phi^a \langle\!\langle \phi_a, \frac{\gamma}{z-\psi} \rangle\!\rangle^\varepsilon_{0,2}$$
$$P^\varepsilon_{coh}(t,z) := \sum_a \phi^a \langle\!\langle \phi_a p_\infty \rangle\!\rangle^{QG,\varepsilon}_{0,1}$$

Now we present basis free formulations of the small P-series in K-theory and cohomology, the former of which will ultimately be useful in proving integrality.

Proposition 3.2.4.

$$P^\varepsilon(0,q,Q) = \sum_d Q^d \hat{\mathrm{ev}}_{2,*}\left(\mathscr{O}^G \otimes \hat{\mathrm{ev}}_1^*\left((p_X \circ \mathrm{ev})_*(\mathscr{O}^{vir}_{QG})\chi((p_{\mathbb{P}^1}\circ \mathrm{ev})^*(1-p_\infty)\otimes \mathscr{O}^{vir}_{QG})\right)\right)$$
$$P^\varepsilon_{coh}(0,z,Q) = \sum_d Q^d (p_X\circ \mathrm{ev})_*[QG]^{vir} \int_{QG^{vir}} (p_{\mathbb{P}^1}\circ \mathrm{ev})^*(1-p_\infty)$$

where $\mathscr{O}^G$, $\hat{\mathrm{ev}}_1$ and $\hat{\mathrm{ev}}_2$ were defined in equation (3.1.3) *and then restricted to $t = 0$.*

Proof: Let $\pi_Y : Y \to \mathrm{pt}$ be the morphism to the point from any space Y. Note that $\pi_{QG} = \pi_X \circ \mathrm{ev}$. Both cases follow from the definition of the perfect pairing on their respective theories, and the projection formula on ev. Recall that $\mathrm{ev} : QG^\varepsilon_{g,k,d}(X) \to X \times \mathbb{P}^1$ and is equivariant with respect to the $\mathbb{C}^*$ action on $\mathbb{P}^1$. Let p_X and $p_{\mathbb{P}^1}$ denote the projections on $X \times \mathbb{P}^1$, the latter of which is $\mathbb{C}^*$ equivariant. In cohomology,

$$\begin{aligned}
P^\varepsilon_{coh}(0,q,Q) &= \sum_{a,d} Q^d \phi^a (\pi_{X\times\mathbb{P}^1})_*(\mathrm{ev}_*(\mathrm{ev}^*(\phi^a p_\infty) \cap [QG]^{vir})) \\
&= \sum_{a,d} Q^d \phi^a (\pi_{X\times\mathbb{P}^1})_*(\phi^a(1-p_\infty) \cap \mathrm{ev}_*[QG]^{vir}) \\
&= \sum_{a,d} Q^d \phi_a (\pi_X)_*(\phi^a (p_X)_*\, \mathrm{ev}_*[QG]^{vir})(\pi_{\mathbb{P}^1})_*((1-p_\infty)(p_{\mathbb{P}^1})_*[QG]^{vir}) \\
&= \sum_d Q^d (p_X\circ \mathrm{ev})_*[QG]^{vir} \int_{QG^{vir}} (p_{\mathbb{P}^1}\circ \mathrm{ev})^*(1-p_\infty)
\end{aligned}$$

In K-theory, apply the same calculation but incorporate $\mathscr{O}^G$.

$$\begin{aligned}
P^\varepsilon(0,q,Q) &= \sum_d \sum_{\alpha,\beta} \Phi_\beta G^{\alpha\beta} Q^d \langle \Phi_\alpha(1-p_\infty)\rangle^{QG}_{0,1,d} \\
&= \sum_{\alpha,\beta,d} Q^d \Phi_\beta \chi(\hat{\mathrm{ev}}_1^*(\Phi^\alpha)\hat{\mathrm{ev}}_2^*(\Phi^\beta)\otimes \mathscr{O}^G; \mathcal{Z})\chi(\mathrm{ev}^*\Phi_\alpha(1-p_\infty)\otimes \mathscr{O}^{vir}_{QG}; QG_{0,1+n,d}(X)) \\
&= \sum_d \sum_{\alpha,\beta} Q^d \Phi_\beta (\pi_X)_*(\Phi^\beta \hat{\mathrm{ev}}_{2,*}(\hat{\mathrm{ev}}_1^*\Phi^\alpha \otimes \mathscr{O}^G))\chi(\mathrm{ev}^*\Phi_\alpha(1-p_\infty)\otimes \mathscr{O}^{vir}_{QG}; QG_{0,n,d}(X)) \\
&= \sum_d \sum_\alpha Q^d \hat{\mathrm{ev}}_{2,*}(\hat{\mathrm{ev}}_1^*\Phi^\alpha \otimes \mathscr{O}^G)\chi(\mathrm{ev}^*\Phi_\alpha(1-p_\infty)\otimes \mathscr{O}^{vir}_{QG}; QG_{0,n,d}(X)) \\
&= \sum_d Q^d \hat{\mathrm{ev}}_{2,*}\left(\mathscr{O}^G \otimes \hat{\mathrm{ev}}_1^* \sum_\alpha \Phi_\alpha \chi(\mathrm{ev}^*\Phi_\alpha(1-p_\infty)\otimes \mathscr{O}^{vir}_{QG}; QG_{0,n,d}(X))\right) \\
&= \sum_d Q^d \hat{\mathrm{ev}}_{2,*}\left(\mathscr{O}^G \otimes \hat{\mathrm{ev}}_1^*\left((p_X\circ \mathrm{ev})_*(\mathscr{O}^{vir}_{QG})\chi((p_{\mathbb{P}^1}\circ \mathrm{ev})^*(1-p_\infty)\otimes \mathscr{O}^{vir}_{QG})\right)\right).
\end{aligned}$$

□

3.3 Birkhoff Factorization

We have the following relations and factorizations, proven by Ciocan-Fontanine and Kim for cohomology [CFK14], and then extended to K-theory by Tseng and You [TY16].

Proposition 3.3.1. *We have the following equalities.*

(a) $(S^\varepsilon)^*(t,q,Q)(S^\varepsilon(t,1/q)(-)) = Id$

(b) $(S^\varepsilon_{coh})^*(t,z,Q)(S^\varepsilon_{coh}(t,-z)(-)) = Id$

(c) $J^\varepsilon(t,q,Q) = S^\varepsilon(t,q,Q)(P^\varepsilon(t,q,Q))$

(d) $J^\varepsilon_{coh}(t,z,Q) = S^\varepsilon_{coh}(t,z,Q)(P^\varepsilon_{coh}(t,z,Q))$

Proof of (a): For any elements $\Gamma,\Delta \in K^0(X)_{\mathbb{Q}}$, consider

$$\sum_{(n,d)\geq 0} Q^d \langle \Gamma(1-p_0), t, \ldots, t, \Delta(1-p_\infty)\rangle^{QG,S_n}_{0,1+n+1,d}.$$

This is a generating series in variables q, Q but with no poles at $q = 1$, since p_0 and p_∞ are defined in terms of q and $1/q$ respectively. Applying $\mathbb{C}^*$-localization to graph spaces,

$$\begin{aligned}\langle\Gamma(1-p_0),&t,\dots,t,\Delta(1-p_\infty)\rangle^{QG,S_n}_{0,1+n+1,d}\\ &= \chi\left(\mathrm{ev}_1^*(\Gamma(1-p_0))\,\mathrm{ev}_{n+2}^*(\Delta(1-p_\infty))\prod_i \mathrm{ev}_i^*(t),(QG^{\varepsilon}_{0,1+n+1,d}(X))^{vir}/S_n\right)\\ &= \sum_{\substack{k_1+k_2=n+2\\ d_1+d_2=d}}\int_{(F^{k_1,d_1}_{k_2,d_2})^{vir}} i^*\frac{\mathrm{ch}\,\mathrm{tr}(\mathrm{ev}_1^*(\Gamma(1-p_0))\,\mathrm{ev}_{n+2}^*(\Delta(1-p_\infty))\prod_i \mathrm{ev}_i^*(t))}{e(N^{vir}_{F^{k_1,d_1}_{k_2,d_2}})}\end{aligned}$$

Note that when the first insertion appears in k_2, the term vanishes by definition of p_0, similarly for the last insertion and k_1 by definition of p_∞. Therefore we can separate out the Γ and Δ terms into a product of generating functions of $Q^{\varepsilon}_{0,2+n_1,d_1}(X)$ and $Q^{\varepsilon}_{0,2+n_2,d_2}(X)$ correlators. Writing out the Euler class of the virtual normal bundle $(N^{k_1,d_1}_{k_2,d_2})^{vir}$ yields rational functions with terms of the form $1/(1-qL)$ and $1/(1-L/q)$. Expanding these as Laurent series about $q = 1$:

$$\tfrac{1}{1-qL} = \textstyle\sum_{n\geq 0}\tfrac{q^n(L-1)^n}{(1-q)^{n+1}} = \tfrac{1}{1-q} + O\left(\tfrac{q}{1-q}\right) \quad \tfrac{1}{1-L/q} = \textstyle\sum_{n\geq 0}\tfrac{q^n(L-1)^n}{(q-1)^{n+1}} = 1 + \tfrac{1}{1-q} + O\left(\tfrac{q}{1-q}\right)$$

Now putting all the algebra together:

$$\begin{aligned}\sum_{(n,d)\geq 0} &Q^d\langle\Gamma(1-p_0),t,\dots,t,\Delta(1-p_\infty)\rangle^{QG,S_n}_{0,1+n+1,d}\\ &= \sum_{\alpha,\beta} g^{\alpha\beta}\left(\langle\Phi_a,\Gamma\rangle + \langle\!\langle\frac{\Phi_\alpha}{1-qL},\Gamma\rangle\!\rangle^{\varepsilon}_{0,2}\right)\left(\langle\Phi_\beta,\Delta\rangle + \langle\!\langle\Delta,\frac{\Phi_\beta}{1-L/q}\rangle\!\rangle^{\varepsilon}_{0,2}\right)\\ &= \sum_{\alpha,\beta} g^{\alpha\beta}\langle\Phi_\alpha,\Gamma\rangle\left(\langle\Delta,\Phi_\beta\rangle + \langle\!\langle\Delta,\Phi_\beta\rangle\!\rangle^{\varepsilon}_{0,2}\right) + O\left(\frac{1}{1-q}\right)\\ &= \langle\Delta,\Gamma\rangle + \langle\!\langle\Delta,\Gamma\rangle\!\rangle^{\varepsilon}_{0,2} + O\left(\frac{1}{1-q}\right)\end{aligned} \tag{3.3.1}$$

But recall the the generating function had no poles at $q = 1$, so it simplifies to $\langle\Delta,\Gamma\rangle + \langle\!\langle\Delta,\Gamma\rangle\!\rangle^{\varepsilon}_{0,2}$. In fact this is a general case for simplifying $(S^{\varepsilon})^*(t,q,Q)(S^{\varepsilon}(t,1/q,Q)(\Delta))$.

Indeed:

$$(S^\varepsilon)^*(t,q,Q)(S^\varepsilon(t,1/q,Q)(\Delta)) = \sum_{\alpha,\beta} \Phi_\beta G_\varepsilon^{\alpha\beta} \left(\langle S^\varepsilon(t,1/q)(\Delta), \Phi_\alpha\rangle + \langle\langle \frac{S^\varepsilon(t,1/q)(\Delta)}{1-qL}, \Phi_\alpha \rangle\rangle_{0,2}^\varepsilon \right)$$
$$= \sum_{\alpha,\beta} G_\varepsilon^{\alpha\beta}\Phi_\beta \sum_{\alpha',\beta'} \left(\langle \Phi_{\beta'}, \Phi_\alpha\rangle + \langle\langle \frac{\Phi_{\beta'}}{1-qL}, \Phi_\alpha\rangle\rangle_{0,2}^\varepsilon \right) \left(\langle \Phi_{\alpha'}, \Delta\rangle + \langle\langle \frac{\Phi_\alpha}{1-L/q}, \Delta\rangle\rangle_{0,2}^\varepsilon \right).$$

This is exactly equation (3.3.1) when $\Phi_\alpha = \Gamma$, $\Phi_{\beta'} = \Phi_\alpha$, and $\Phi_{\alpha'} = \Phi_\beta$. Therefore the poles at $q = 1$ disappear and:

$$\begin{aligned} (S^\varepsilon)^*(t,q,Q)(S^\varepsilon(t,1/q,Q)(\Delta)) &= \sum_{\alpha,\beta} \Phi_\beta G_\varepsilon^{\alpha,\beta} \left(\langle \Delta, \Phi_\alpha\rangle + \langle\langle \Delta, \Phi_\alpha\rangle\rangle_{0,2}^\varepsilon \right) \\ &= \sum_{\alpha,\beta} \Phi_\beta G_\varepsilon^{\alpha\beta} \sum_i G_{\alpha i}^\varepsilon \langle \Phi^i, \Delta\rangle \\ &= \sum_\beta \Phi_\beta \langle \Phi^\beta, \Delta\rangle \\ &= \Delta. \end{aligned}$$

Therefore the composition of the operators is the identity.

Proof of (b): See the [CFK14, Proposition 5.3.1] for details.

Proof of (c): It suffices to show that $P^\varepsilon(t,q,Q) = (S^\varepsilon)^*(t,1/q,Q)(J^\varepsilon)$. For then part (a) proves the result. To prove the equation for $P^\varepsilon(t,q,Q)$, again apply equivariant localization to the correlators in P^ε. As in part (a), the first insertion must appear in k_2, otherwise the term vanishes. Furthermore, the unstable cases appear as a basis element and intersection pairing with t.

$$
\begin{aligned}
\langle\!\langle \Phi_\alpha(1-p_\infty)\rangle\!\rangle_{0,1}^{QG^\varepsilon} &= \sum_d Q^d \chi\left(\mathrm{ev}_1^*(\Phi_\alpha)(1-p_\infty)\prod_i \mathrm{ev}_i^*(t); QG_{0,1+n,d}(X)^{vir}/S_n\right)\\
&= \sum_d Q^d \sum_{\substack{k_1+k_2=1+n, 1\in k_2\\ d_1+d_2=d}} \int_{(F^{k_1,d_1}_{k_2,d_2})^{vir}} i^* \frac{\mathrm{ch}\,\mathrm{tr}\,\mathrm{ev}_1^*(\Phi_\alpha(1-1/q)\prod_i \mathrm{ev}_i^*(t)}{e(N_F^{vir})}\\
&= \sum_{\alpha',\beta'} g^{\alpha'\beta'}\left(\Phi_{\alpha'} + \frac{1}{1-q}\langle\Phi_{\alpha'}, t\rangle + \langle\!\langle \frac{\Phi_{\alpha'}}{(1-q)(1-qL)}\rangle\!\rangle_{0,1}^{\varepsilon}\right)\left(g_{\alpha\beta'} + \langle\!\langle \Phi_\alpha, \frac{\Phi_{\beta'}}{1-L/q}\rangle\!\rangle_{0,2}^{\varepsilon}\right)\\
&= \langle \Phi_\alpha, J^\varepsilon\rangle + \langle\!\langle \Phi_\alpha, \frac{J^\varepsilon}{1-L/q}\rangle\!\rangle_{0,2}^{\varepsilon}
\end{aligned}
$$

Then

$$
\begin{aligned}
P^\varepsilon(t,q,Q) &= \sum_{\alpha,\beta} \Phi_\beta G_\varepsilon^{\alpha\beta}\left(\langle\Phi_\alpha, J^\varepsilon\rangle + \langle\!\langle \Phi_\alpha, \frac{J^\varepsilon}{1-L/q}\rangle\!\rangle\right)\\
&= (S^\varepsilon)^*(t,1/q,Q)(J^\varepsilon)
\end{aligned}
$$

Proof of (d): Similar to (c), see [CFK14, Equation 5.4.1] for details. □

Proposition 3.3.2. *We have the following expressions for J^ε and P^ε.*

(a) For all $\varepsilon \geq 0+$, $P^\varepsilon = 1 + O(Q)$ where the coefficients of Q^d live in $K^0(X)[q,q^{-1}]$.

(b) If $\varepsilon > 1$, then $P^\varepsilon(t,q,Q) = \Phi_0$.

Proof of (a): By Definition 2.10.1, the J^ε-function has the expansion

$$
J^\varepsilon(t,q,Q) = 1 + O(Q) + O\left(\frac{1}{1-q}\right)
$$

where $O(Q)$ denotes the unstable terms and $O(1/(1-q))$ denotes the correlator terms. By definition the inverse S-operator has the form

$$
(S^\varepsilon)^*(t,q,Q)(\Gamma) = \Gamma + O\left(\frac{1}{1-q}\right)
$$

since it is defined by a constant term then correlators in $1/(1-q)$. Plugging in J^ε, then

$$\begin{aligned} P^\varepsilon(t,q,Q) &= (S^\varepsilon)^*(t,q,Q)(J^\varepsilon) \\ &= 1 + O(Q) + O\left(\frac{1}{1-q}\right) \end{aligned}$$

But by definition, P^ε has no poles at $q=1$ so $P^\varepsilon(t,q,Q) = 1 + O(Q)$, where the coefficients of the asymptotic Q lie in $K^0(X)[q,q^{-1}]$ and contribute to the unstable terms of J^ε.

Proof of (b): Specifically, if $\varepsilon > 1$ there are no unstable terms so $P^\varepsilon = \Phi_0$.
□

3.4 Formulae for Semipositive Targets

A GIT quotient $X = (W, G, \theta)$ is said to be *semi-positive* if $d(\det T_W) \geq 0$ for all θ-effective classes d. Of course all Calabi-Yau X are semi-positive since for all d,

$$d(\det T_W) = \int_d c_1(T_X) = 0.$$

Ciocan-Fontanine and Kim prove further results for S and P in this case in [CFK14][CFK17].

Proposition 3.4.1. *If X is semi-positive, then we have the following two formulae:*

$$\frac{J^\varepsilon_{coh}(0,z,Q)}{\langle \phi^0, P^\varepsilon_{coh}(0,q,Q)\rangle} = \phi_0 + \sum_{a,d\neq 0} \phi_a Q^d \langle \frac{\phi^a}{z-\psi}, \phi_0\rangle^\varepsilon_{0,2,d} \tag{3.4.1}$$

and

$$P^\varepsilon_{coh}(0,z,Q) = \langle \phi^0, J^\varepsilon_{coh}(0,z,Q)\rangle. \tag{3.4.2}$$

Proof: The semi-positive assumption allows us to put a lower bound on the virtual dimension of the graph space. In particular,

$$\operatorname{vdim} QG^\varepsilon_{0,1,d}(X) = d(\det T_W) + (1-0)(\dim X - 2) + 3 \geq \dim X + 1 \tag{3.4.3}$$

But the insertion $\phi_i p_\infty$ has degree $\deg \phi_i p_\infty = \deg \phi_i + \deg p_\infty \leq \dim X + 1$. Therefore the coefficient of ϕ^i in $P^\varepsilon_{coh}(0,q)$ is nonvanishing only when $d(\det T_W) = 0$ and $\deg \phi_i p_\infty = \dim X + 1$, i.e. if $i = \dim X$. Writing out the coefficients of P using the intersection pairing:

$$\langle \phi^0, P^\varepsilon_{coh}(0,z)\rangle = \sum_{d(\det T_W)=0} Q^d \langle \phi^0 p_\infty \rangle^{QG}_{0,1,d} = 1 + (\text{higher degree terms})$$

Since the constant term is nonzero, this series is invertible. Using the Birkhoff factorization:

$$J^\varepsilon(0,z,Q) = S^{coh}(0,z,Q)(P^{coh}(0,z,Q)) = S^{coh}(0,z,Q)(\phi_0) \cdot \langle \phi^0, P^\varepsilon_{coh}(0,z,Q)\rangle \quad (3.4.4)$$

Therefore:

$$\begin{aligned}
\frac{J^\varepsilon(0,z,Q)}{\langle \phi^0, P^\varepsilon_{coh}(0,z)\rangle} &= S^\varepsilon_{coh}(0,z,Q)(\phi_0) = \sum_a \phi_a \langle\!\langle \frac{\phi^a}{z-\psi}, \phi_0 \rangle\!\rangle^\varepsilon_{0,2} \\
&= 1 + \sum_{a,d>0} \phi_a Q^d \langle \frac{\phi^a}{z-\psi}, \phi_0 \rangle^\varepsilon_{0,2,d} \\
&= 1 + \sum_{a,d>0,k} \frac{\phi_a Q^d}{z^{k+1}} \langle \psi^k \phi^a, \phi_0 \rangle^\varepsilon_{0,2} \\
&= 1 + \frac{1}{z}\left(\sum_{a,d>0} \phi_a Q^d \langle \phi^a, \phi_0 \rangle^\varepsilon_{0,2} \right) + O\left(\frac{1}{z^2}\right)
\end{aligned}$$

The second line is the first formula we wanted to prove. For the second claim, write out J in terms of powers of $1/z$ by:

$$J^\varepsilon(0,z,Q) = J^\varepsilon_0 \cdot \phi_0 + \frac{1}{z} J^\varepsilon_1 + O\left(\frac{1}{z^2}\right)$$

so that $\langle \phi^0, P^\varepsilon_{coh}\rangle = J^\varepsilon_0$ which is crucially independent of z. Further, the operator $(S^\varepsilon_{coh})^*$ has the general form $\phi_0 + O(1/z)$ as well, so that:

$$P^\varepsilon_{coh}(0,z,Q) = (S^\varepsilon_{coh})^*(0,-z,Q)(J^\varepsilon(0,z,Q)) = J^\varepsilon_0 \cdot \phi_0 + O\left(\frac{1}{z}\right)$$

But P has no poles at $z = 0$, so therefore $P^{\varepsilon}_{coh}(0, z, Q) = J^{\varepsilon}_0 \cdot \phi_0 = \langle \phi^0, J^{\varepsilon}_{coh}(0, q, Q) \rangle$ as desired. $\square$

Chapter 4

Givental's Formalism for Inertia and Splitting Curves

In order to prove the confluence result for the small I-function, Theorem 5.0.1, ultimately we need to simplify the Euler characteristic of a virtual bundle above an orbifold (or Deligne-Mumford stack) $Q^{\varepsilon}_{0,k,d}(X)$ which appears in Birkhoff factorization $I = S^{0+}(P^{0+})$. As with any equivariant GRR type theorem for a DM stack $\mathcal{X}$, the Euler characteristic becomes an integral over the inertia $I\mathcal{X}$, the stack of pairs (x, g) with $x \in \mathcal{X}$ and $g \in \mathrm{Aut}(x)$. With the addition of the virtual structure sheaf and/or fundamental class, computing the Euler characteristic can, for example, be accomplished using a generalized Kawasaki-Riemann-Roch theorem for virtual orbibundles, due to Tonita [Ton14].

Theorem 4.0.1 (Kawasaki-Riemann-Roch for Virtual Sheaves)**.** *Let $\mathcal{X}$ be a compact, complex orbifold with perfect obstruction theory and inertia $I\mathcal{X} = \coprod_\mu \mathcal{X}_\mu$ and multiplicity m_μ of the suborbifold $i_\mu : \mathcal{X}_\mu \to \mathcal{X}$. Then*

$$\chi(\mathcal{X}, j^*(V) \otimes \mathscr{O}^{vir}_{\mathcal{X}}) = \sum_\mu \frac{1}{m_\mu} \int_{\mathcal{X}_\mu} \mathrm{ch}\,\mathrm{tr} \left(\frac{V_\mu \otimes \mathscr{O}^{vir}_{\mathcal{X}_\mu}}{\bigwedge^\bullet (N^{vir}_\mu)^\vee} \right) \mathrm{td}_{T_{\mathcal{X}}} . \qquad (4.0.1)$$

Another useful version of virtual equivariant GRR formulated in terms of DM stacks is due to Ravi and Sreedhar.

Theorem 4.0.2 (Corollary 4.10 in [RS21])**.** *Let $[Y/G]$ be a global quotient DM stack* *with perfect obstruction theory and inertia stack* $I[Y/G] = \coprod_\mu I_\mu[Y/G]$*. Then there is a map on inertia* $I\tau : K^0([Y/G])_{\mathbb{Q}} \to H^*(I[Y/G])_{\mathbb{Q}}$ *such that we have the following equality in* $H^*(I[Y/G])_{\mathbb{Q}}$*.*

$$I\tau(\mathscr{O}^{vir}_{[Y/G]}) = \sum_{\mu} \frac{\operatorname{td}(T^{vir}_{I\mu[Y/G]})}{e((N^{vir}_{I\mu[Y/G]})^\vee)} \cap [I_\mu[Y/G]]^{vir}$$

Composing with the pushforward to the point yields the desired VEHRR formula. Often the "virtual" part of the integrals will be understood, so we drop the notation in long computations.

In order to apply KRR or VEHRR to $[Y/G] = Q^{0+}_{0,k,d}(X)$, we describe the inertia space $IQ^{0+}_{0,k,d}(X)$, and in particular a classification of curves with automorphism. For simplicity, we refer to an ε-stable quasimap and automorphism pair $((C, x_i, P, u), g)$ as an *inertia quasimap*. Before getting into the details, we begin with some remarks about inertia quasimaps in the permutation equivariant theory. While we for the most part work in the small case with no S_n action, this does affect the terminology we present in this section.

In the small case, the integrals we need to simplify will be of the following form:

$$\left\langle \frac{\Phi_a}{1-qL}, \Phi_b \right\rangle^{0+}_{0,2,d} = \int_{IQ^{0+}_{0,2,d}(X)^{\mathrm{vir}}} \frac{\operatorname{td} T_{IQ}}{\operatorname{ch}\operatorname{tr}(\bigwedge^\bullet N^\vee_{0,2,d})} \frac{\operatorname{ev}^*_{s_1}(\operatorname{ch}(\Phi_a))\operatorname{ev}^*_{s_2}(\operatorname{ch}(\Phi_b))}{1-q\operatorname{ch}\operatorname{tr}(L_{s_1})}. \quad (4.0.2)$$

But more generally, the big J^ε function or S and P are built of correlators with 1 or 2 special insertions from $K^0(X)_{\mathbb{Q}}[L_i]$ and n insertions of the parameter t, on which S_n acts. Or more more accurately small P is made up of correlators in $QG_{0,1,d}(X)$. In any case, principles of this decomposition work most generally for $Q^\varepsilon_{0,k+n,d}(X)$ with S_n-action, but we focus on the small case $IQ_{0,2,d}(X)$, and extend the results to the graph space moduli $IQG_{0,1,d}(X)$ subsequently.

4.1 Givental's Classification of Inertia Quasimaps

Givental presents these terms in the $\varepsilon = \infty$ stable maps case [Giv, Part III] [GT14]. However, the terminology can be adapted to any ε-stability condition just as well, and ultimately we will use it in the 0+-stability case in the next section.

Suppose we have an inertia quasimap $(C, x_i, P, u, g) \in IQ^{\varepsilon}_{0,2+n,d}(X)$. Consider the first marked points x_1, which by convention is preserved under the automorphism g. The moduli is DM, so g induces an finite order action on the cotangent space L_1 to x_1, and so acts by a root of unity ζ on L_1. Since families of curves must share the same eigenvalue at the first marked point, then there is a disjoint union decomposition

$$IQ_{0,k,d} = \coprod_{\zeta} I_{\zeta} Q_{0,k,d}(X) \tag{4.1.1}$$

where $I_{\zeta} Q_{0,k,d}(X)$ consists of the inertia quasimaps $((C, x_i, P, u), g)$ with g acting on L_1 by root of unity eigenvalue ζ.

Now, the connected components are separated into two cases naturally: when $\zeta = 1$ and $\zeta \neq 1$. If $\zeta = 1$, then g must at induce the trivial action on the irreducible curve that contains x_1. The maximally connected part of C containing the horn on which g acts trivially is called the *head*. The rest of the curve decomposes into subcurves, which are called *arms*, coming off the head. By definition, the node which connects an arm to the head has a different eigenvalue than $\zeta = 1$.

When $\zeta \neq 1$, the component containing x_1 will not have a trivial action by g. But we can identify a useful component containing x_1 as follows. Call a node in C *balanced* if the two eigenvalues on each intersecting component are equal to $1/\zeta$ and ζ. Now consider the maximally connected curve containing x_1 and only balanced nodes, which we call C_0. Now ζ is an m-th primitive root of unity for some $m > 1$, and g^m acts trivially on C_0. Therefore C_0 carries an action by the finite multiplicative cycling group μ_m defined by $1, g, \ldots, g^{m-1}$. Define the *stem* of the C to be $\tilde{C}_0 := [C_0/\mu_m]$. Morevoer, there is an induced ε-stable orbifold quasimap from $\tilde{C}_0 \to B\mu_m$ in the sense of Jarvis and Kimura [JK02]. More on this in the next subsection.

Coming off C_0 are components with unbalanced nodes. If C_0 is a chain curves with the horn at one end, we call the final point where C_0 is unbalanced the *butt*. The butt may be an unspecial fixed point, the other permutation invariant marked point, or a node. In the case of a node, the curve coming off of the butt is called the *tail*. But in any case, both the horn and butt are ramified points on the cover $C_0 \to \tilde{C}_0$. Marked points on the stem $\tilde{C}_0$ may occur at unramified points if g cyclically permutes a set of marked points carrying insertions t upstairs on C_0. In the quotient curve $\tilde{C}_0$, the curves coming off of the unramified marked points are called *legs*.

4.2 Decomposition of $0+$-stable 2-pointed quasimap Inertia

For the remainder of this work, we consider primarliy $\varepsilon = 0+$ stability. Hence we drop the $0+$ from the notation, and if $\varepsilon \neq 0+$, we will be specific. We apply Givental's description of curves with inertia to further decompose $I_\zeta Q_{0,2,d}(X)$ into substrata. The method takes its structure from Milanov and Roquefeuil [MR21], who write out the results of Givental explicitly in the stable map case [Giv, Part III][GT14]. The moduli of $(0,2,d)$-stable quasimaps work much the same way.

Let (C, s_1, s_2, P, u, g) be a $(0,2,d)$-stable quasimap. The domain curve is a tree of genus 0 curves. By $0+$-stability, each irreducible component must have either 2 special points and nonzero degree, or 3 special points if it is contracted to a point under $[u]$. With $k = 2$, the possibilities are limited to C being a chain of genus 0 curves, each with nonzero degree map to X. On each end of the chain is a marked point, with simple nodes along each "link" of the chain.

Definition 4.2.1. *Consider the trivial action of μ_m on a GIT target X. Let $Q^{\zeta,\zeta^{-1}}_{0,2,d}([X/\mu_m])$ be the moduli of $(0,2,d)$-stable quasimaps to $[X/\mu_m]$ with automorphism g such that g acts by eigenvalues ζ and ζ^{-1} on the cotangent lines to the two marked points. Of course, ζ is required to be an m-th root of unity by definition.*

For $\zeta \neq 1$, the moduli space parametrizes the stems $\tilde{C}_0$ with no legs or tail in $I_\zeta Q_{0,2,d}$, where the first marked point is the horn while the second is a butt. Of course if $\zeta = 1$, then $Q^{1,1}_{0,2,d}(X) = Q_{0,2,d}(X)$.

Note that $[X/\mu_m]$ is endowed with an DM stack (or equivalently orbifold) structure, and so the moduli $Q^{\zeta,\zeta^{-1}}_{0,2,d}([X/\mu_m])$ parametrizes orbifold stable quasimaps, which is described in [CCFK15]. These moduli have connected components with index set depending on the components of the rigidified cyclotomic inertia. In particular if Q^ν is a connected component of $Q^{\zeta,\zeta^{-1}}_{0,2,d}([X/\mu_m])$, then the Riemann-Roch theorem for twisted curves implies

$$\operatorname{vdim} Q^{\nu} = \dim X - 1 + \int_d c_1(TX) - \sum \iota(\nu_i) \leq \operatorname{vdim} Q_{0,2,d}(X)$$

where the error term is given by the sum of the nonnegative rational *ages*. For our purposes, we need only that the virtual dimension is bounded above by vdim $Q_{0,2,d}(X)$.

Proposition 4.2.2. *The inertia stratum $I_\zeta Q_{0,2,d}(X)$ has the following decomposition.*

$$I_\zeta Q_{0,2,d}(X) = Q^{\zeta,\zeta^{-1}}_{0,2,d}([X/\mu_m]) \sqcup \left(\coprod_{\eta} \coprod_{\substack{d_0,d_1>0 \\ d_0+d_1=d}} Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m]) \times_{\mathrm{ev}} I_\eta Q_{0,2,d_1}(X) \right) \tag{4.2.1}$$

where the first disjoint union is indexed by roots of unity η.

Proof: If $(C, s_1, s_2, f, g) \in I_\zeta Q_{0,2,d}(X)$, denote the components of the chain C by $C_1, \dots, C_r$, beginning with $s_1 \in C_1$ and ending with $s_2 \in C_r$. By 0+-stability, $\deg f|_{C_i} > 0$. Let the n_i be the node formed by the intersection of C_i and C_{i+1} for $i = 1, \dots, r-1$. The automorphism g acts on each C_i individually, and preserves marked points and the nodes.

By definition of the stratum, g_1 acts on the contangent line to C_1 at s_1 by ζ, and so acts by ζ^{-1} at n_1. We let C_0 be the maximally connected curve containing C_1 and with balanced nodes (or on which g acts trivially). Let $C_\ell = \overline{C \backslash C_0}$. Note that C_0 is always nonempty since it contains C_1. But C_ℓ is potentially empty if all nodes are balanced. If $\zeta \neq 1$, $\tilde{C}_0 := [C_0/\mu_m]$ is the *stem* of C, while we have one singular *tail* C_ℓ, since the first unbalanced note is ramified, and therefore is classified as a butt. If $\zeta = 1$, then C_0 is classified as a *head* and C_ℓ an *arm*.

If $C_0 \subsetneq C$, let $\eta \neq \zeta$ be the eigenvalue of the action of g at the butt cotangent to C_ℓ. We can unglue the curves C_0 and C_ℓ there, making two quasimaps with automorphism $(C_0 s_1, p_0, f|_{C_0}, g_0)$ and $(C_\ell, p_1, s_2, f|_{C_\ell}, g_\ell)$, where $p_0 \in C_0$ and $p_1 \in C_\ell$ glue to form the node $n_k \in C_0 \cap C_\ell$. The following commutative diagram summarizes the situation.

$$\begin{array}{ccc} C = C_0 \times_{\mathrm{ev}} C_\ell & \xrightarrow{\mathrm{ev}_{\mathrm{butt}}} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \Delta} \\ C_0 \times C_\ell & \xrightarrow{\mathrm{ev}_{p_0} \times \mathrm{ev}_1} & X \times X \end{array}$$

Given the data $(C, s_1, s_2, f, g) \in I_\zeta Q_{0,2,d}(X)$, we form the map

$$(C, s_1, s_2, f, g) \mapsto ((C_0, s_1, p_0, f|_{C_0}, g_0), (C_\ell, p_1, s_2, f|_{C_\ell}, g_l))$$

where the second tuple isn't written if $C_0 = C$ (i.e. the case where all the nodes are balanced). The first quasimap f_0 will have some degree d_0 so that the first term is parametrized by

$$(f|_{C_0}, C_0, s_1, p_0, g_0) \in \{I_\zeta Q_{0,2,d_0}(X) \mid g \text{ acts by eigenvalues } (\zeta, \zeta^{-1})\} = Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m]).$$

To explain the appearance of the μ_m-quotient, the added data of the g-action with restricted eigenvalues is equivalent to a $(0, 2, d)$-stable quasimap which carries a m-fold cover $C_0 \to [C_0/\mu_m]$, ramified at the special points. As written in Jarvis-Kimura [JK02], this is equivalent to 0+-stable quasimaps to $X \times B\mu_m = [X/\mu_m]$, where μ_m acts on X trivially. Then the quotient $\tilde{C}_0$ has stable quasimap $\tilde{f} : \tilde{C}_0 \to [X/\mu_m]$, and thus

$$(f_0, C_0, s_1, p_0, g_0) = (\tilde{f}, \tilde{C}_0, s_1, p_0) \in Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m]).$$

Next, suppose f_ℓ has degree d_ℓ and g_ℓ acts on L_{p_1} with eigenvalue $\eta \neq \zeta$. Then $(f_\ell, C_\ell, p_1, s_2, g_\ell) \in I_\eta Q_{0,2,d_\ell}(X)$ by definition. Recall that p_0 and p_1 glue back to a node in C, so that these curves decompose in products over evaluation $\mathrm{ev}_{\text{node}}$. Renaming d_ℓ to d_1 for ease of notation and considering all tuples η, d_0, d_1, and the case when every node is balanced, the ungluing procedure becomes a map

$$I_\zeta Q_{0,2,d}(X) \to Q^{\zeta,\zeta^{-1}}_{0,2,d}([X/\mu_m]) \sqcup \left(\coprod_{\eta} \coprod_{\substack{d_0, d_1 > 0 \\ d_0 + d_1 = d}} Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m]) \times_{\mathrm{ev}} I_\eta Q_{0,2,d_1}(X) \right).$$

The reverse map is clear since g_0, g_ℓ can be glued back to an automorphism g. This works for families of curves in each moduli, so we obtain a well defined isomorphism of DM stacks. $\square$

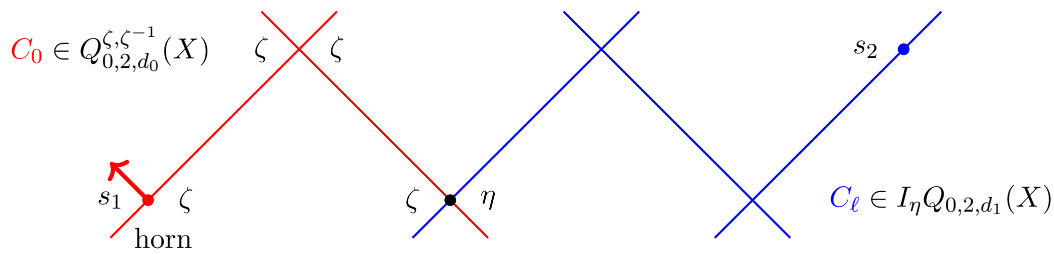

Figure 4.1: The decomposition of an example curve $C \in I_\zeta Q_{0,2,d}(X)$. The red subcurve denotes the cover over the spine (or head), while the blue curve is the single tail. To decompose the curve, unglue at the first unbalanced node.

4.3 Inertia for Graph spaces

We state and outline the proof for a similar result for 1-pointed graph spaces.

Proposition 4.3.1. *For all d and smooth varieties X, we have the following disjoint union decomposition of the Kawasaki strata of $IQG_{0,1,d}(X)$. When $\zeta = 1$*

$$I_1QG_{0,1,d}(X) = QG_{0,1,d}(X) \sqcup \coprod_{\eta \neq 1} \coprod_{\substack{d_0,d_1>0 \\ d_0+d_1=d}} (Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m]) \times \mathbb{P}^1) \times_{\mathrm{ev}} I_\eta QG_{0,1,d_1}(X)$$

and when $\zeta \neq 1$

$$I_\zeta QG_{0,1,d}(X) = \coprod_{\eta} \coprod_{\substack{d_0,d_1>0 \\ d_0+d_1=d}} (Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m]) \times \mathbb{P}^1) \times_{\mathrm{ev}} I_\eta QG_{0,1,d_1}(X).$$

Proof: This follows from the Givental formalism as well. A graph space curve $((f,\phi), C, s_1, g)$ with a single marked point contains a parametrized curve $\phi : P \to \mathbb{P}^1$ and a chain of curves $\tilde{C}$ with the marked point at the other end. The extra curve intersects the parametrized $\mathbb{P}^1$ at a point $x \in \mathbb{P}^1$. As long as ζ and η are not both unity, the tail of C is always nonempty when $\zeta \neq 1$, since the eigenvalue along P will be equal to 1, while the head contains an eigenvalue ζ, so a node is unbalanced somewhere in between. The only other case is when all eigenvalues on the special points are 1, and so such a curve lies on the trivial stratum. The head carries an extra piece of data: a point $x \in \mathbb{P}^1$ where it lied over the parametrized line. Hence, the stem moduli becomes $Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m]) \times \mathbb{P}^1$, and the tail still parametrizes $I_\eta Q_{0,1,d_1}(X)$. □

4.4 Recursion relation for 0+-stability 2-pointed correlators

In this section we prove a recursion relation for contributions of $I_\zeta Q^{0+}_{0,2,d}(X)$ to the two pointed correlator, which becomes a translation from a K-theoretic correlator to a sum of cohomological ones. This result is an adaptation of the calculation in Appendix A in [MR21], which in turn is due to Givental, to the 0+-stable $(0,2,d)$-quasimaps of the last section.

Definition 4.4.1. *Consider the K-theoretic correlator $\left\langle \frac{\Phi_a}{1-L/q}, \Phi_b \right\rangle_{0,2,d}$. Define $\tau^{\zeta}_{a,b,d}(q)$ to be the contribution of the stratum $I_\zeta Q_{0,2,d}(X)$:*

$$\tau^{\zeta}_{a,b,d}(q) = \int_{I_\zeta Q_{0,2,d}(X)^{vir}} \frac{\operatorname{td} T_{IQ}}{\operatorname{ch}\operatorname{tr}(\bigwedge^\bullet N^\vee_{IQ})} \frac{\operatorname{ev}^*_{s_1}(\operatorname{ch}(\Phi_a))\operatorname{ev}^*_{s_2}(\operatorname{ch}(\Phi_b))}{1-q\operatorname{ch}\operatorname{tr}(L_{s_1})}.$$

By KRR, we know that $\left\langle \frac{\Phi_a}{1-qL}, \Phi_b \right\rangle_{0,2,d} = \sum_\zeta \tau^{\zeta}_{a,b,d}(q)$.

We also wish to define contributions τ for when the above two pointed correlator has Poincare dual insertions.

Definition 4.4.2. *Let $\tau^{\zeta,a}_{-,b,d}(q)$ and $\tau^{\zeta,b}_{a,-,d}(q)$ be the contributions from $I_\zeta Q_{0,2,d}(X)$ to the above two pointed correlators for insertions $\Phi^a/(1-L/q)$ and Φ^b respectively. In other words by linearity,*

$$\tau^{\zeta,a}_{-,b,d}(q) = \sum_i g^{ai}\tau^{\zeta}_{i,b,d}$$
$$\tau^{\zeta,b}_{a,-,d}(q) = \sum_j g^{bj}\tau^{\zeta}_{a,j,d}.$$

To simplify the contrubitions τ, first we describe the Chern roots of T^{vir}_{IQ} and $(\bigwedge^\bullet N^\vee_{0,2,d})^{\mathrm{vir}}$ which appear in the integral.

Definition 4.4.3. *Let $T(C)$ be the virtual tangent space to $C \in Q_{0,2,d}(X)$. Given an automorphism g of C, g will act on $T(C)$. Let $T(C,g)_\lambda$ be the eigenspace of the action of g on $T(C)$ with eigenvalue λ. Let $I_\lambda TQ_{0,2,d} \to IQ_{0,2,d}(X)$ be the bundle on the inertia stack with fiber $T(C,g)_\lambda$ over a point (C, s_1, s_2, f, g).*

Since the virtual tangent bundle is the fixed part of the action on inertia, then $T_{IQ}^{vir} = I_1 TQ_{0,2,d}$. Similarly, the virtual normal bundle is the moving part, so that $N_{IQ}^{\mathrm{vir}} = \bigoplus_{\lambda\neq 1} I_\lambda TQ_{0,2,d}$.

Definition 4.4.4. *In light of the above description of the virtual normal and tangent bundles, name the Chern roots of $I_\lambda TQ_{0,2,d}$ by L_λ^i for index set $i \in R_\lambda$, which is possibly empty. In particular, $T_{IQ}^{vir} = \bigoplus_{i\in R_1} L_1^i$ and $N_{IQ}^{vir} = \bigoplus_{\lambda\neq 1}\bigoplus_{i\in R_\lambda} L_\lambda^i$.*

Definition 4.4.5. *We define the* moving Todd class $\mathrm{td}_\lambda(E)$ *by the formula on Chern roots*

$$\mathrm{td}_\lambda(L) = \begin{cases} \dfrac{c_1(L)}{1-e^{-c_1(L)}} & \text{if } \lambda = 1 \\ \dfrac{1}{1-\lambda e^{-c_1(L)}} & \text{if } \lambda \neq 1 \end{cases}$$

and expand multiplicatively. In particular $\mathrm{td}_1 = \mathrm{td}$.

Proposition 4.4.6. *The first fraction in the above definition of $\tau_{a,b,d}^\zeta$ simplifies to:*

$$\tilde{\mathrm{td}}(T_{0,2,d}) := \frac{\mathrm{td}\, T_{IQ}}{\mathrm{ch}\,\mathrm{tr}(\bigwedge^\bullet N_{0,2,d}^\vee)} = \prod_\lambda \mathrm{td}_{\lambda^{-1}}(I_\lambda TQ_{0,2,d}).$$

Proof: This equality is the result of expanding out the Chern roots of denominator in terms of L_λ^i and then noting that the whole expansion is equal to the right hand side. First, since $\mathrm{td}_1 = \mathrm{td}$, then the $\mathrm{td}(T_{IQ})$ on the left exactly matches the term $\mathrm{td}_1(I_1TQ_{0,2,d})$ on the right. Now consider the $\lambda \neq 1$ case. Since the exterior product is multiplicative on summands, and then tr and ch preserve products, it suffices to consider $I_\lambda TQ_{0,2,d}$ for a fixed λ. Let $c_1(L_\lambda^i) = x_i$. Then:

$$\begin{aligned}\mathrm{ch}\,\mathrm{tr}\left(\bigwedge^\bullet N_{I_\lambda TQ_{0,2,d}}^\vee\right) &= \sum_{k=0}^{n_\lambda}(-1)^k\lambda^{-k}\sum_{i_1<\cdots<i_k} e^{-x_{i_1}-\cdots-x_{i_k}} \\ &= \sum_{k=0}^{n_\lambda}(-1)^k\sum_{i_1<\cdots<i_{k_\lambda}}\prod_{j=1}^{k}\lambda^{-1}e^{-x_{i_j}} \\ &= \prod_{i=1}^{n_\lambda}(1-\lambda^{-1}e^{-x_i})\end{aligned}$$

Finally we can multiply over all $\lambda \neq 1$ and take the reciprocal to get the desired formula. □

Now we are in a position to prove a recursion relation in the $\tau^{\zeta}_{a,b,d}$. This recursion involves an extension to 0+-stability of the so-called *stem correlator*, which is an example of an ABC-correlator introduced by Givental and Tonita [GT14] in the stablel map case. Milanov and Roquefeuil denote this with square brackets $[-]^{ABC}$, to which we follow suit. The nomenclature comes from the fact that it is an integral over the moduli parametrizing the stems of inertia curves when $\zeta \neq 1$.

Lemma 4.4.7. *Let X be a smooth variety and $[-]^{ABC}$ the stem correlator. Then*

$$\tau^{\zeta}_{a,b,d}(q) = \left[\frac{\phi_a}{1-\zeta q e^{\psi_{s_1}}}, \phi_b\right]^{ABC}_{0,2,d} + \sum_{\substack{c,\eta \\ d_0,d_1>0 \\ d_0+d_1=d}} \left[\frac{\phi_a}{1-\zeta q e^{\psi_{s_1}}}, \tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0}.$$

Note that $\tau^{\zeta}_{a,b,d}(\zeta^{-1}e^{\psi_{p_0}})$ is an insertion of the form $t(\psi)$. This is defined for ABC-correlators by expanding multilinearly. See the note after equation (2.9.2).

Proof: By Theorem 4.0.1 and the definitions of τ

$$\begin{aligned}\left\langle \frac{\Phi_a}{1-qL}, \Phi_b\right\rangle_{0,2,d} &= \sum_{\zeta} \tau^{\zeta}_{a,b,d} = \sum_{\zeta} \int_{I_\zeta Q_{0,2,d}(X)^{\mathrm{vir}}} \tilde{\mathrm{td}}(T_{0,2,d}) \frac{\mathrm{ev}^*_{s_1}(\phi_a)\,\mathrm{ev}^*_{s_2}(\phi_b)}{1-q\,\mathrm{ch}\,\mathrm{tr}(L_{s_1})} \\ &= \sum_{\substack{\zeta,\eta \\ d_0+d_1=d \\ d_0>0}} \int_{Q^{\zeta,\zeta^{-1}}_{0,2,d_0}(X/\mu_m)^{\mathrm{vir}} \times_{\mathrm{ev}} I_\eta Q^{\mathrm{vir}}_{0,2,d_1}} \tilde{\mathrm{td}}(T_{0,2,d}) \frac{\mathrm{ev}^*_{s_1}(\phi_a)\,\mathrm{ev}^*_{s_2}(\phi_b)}{1-q\,\mathrm{ch}\,\mathrm{tr}(L_{s_1})}\end{aligned}$$

The strategy to prove the recursion is to split the integral into a product of integrals on $Q^{\zeta,\zeta^{-1}}_{0,2,d_0}(X)$ and $I_\eta Q_{0,2,d_1}(X)$, and simplify the latter term on in terms of τ_{d_1}.

First, recall that we have one distinguished substratum $Q^{\zeta,\zeta^{-1}}_{0,2,d}(X) \subseteq I_\zeta Q_{0,2,d}(X)$ corresponding to when the curve C has all balanced nodes. This part of the sum is in fact independent of the second eigenvalue η. Note that in this case $\tilde{\mathrm{td}}(T_{0,2,d}) = \Theta^{ABC} =$

$1 + \dots$, where ABC denotes the stem term. Therefore the integral on this substrata is

$$\int_{Q^{\zeta,\zeta^{-1}}_{0,2,d}(X/\mu_m)} \tilde{\mathrm{td}}(T_{0,2,d}) \frac{\mathrm{ev}^*_{s_1}(\phi_a)\,\mathrm{ev}^*_{s_2}(\phi_b)}{1 - q^{-1}\,\mathrm{ch}\,\mathrm{tr}(L_{s_1})}$$

$$= \int_{Q^{\zeta,\zeta^{-1}}_{0,2,d}(X/\mu_m)} \Theta^{ABC} \left(\frac{\mathrm{ev}^*_{s_1}(\phi_a)}{1 - \zeta q e^{\psi_{s_1}}} \right) \mathrm{ev}^*_{s_2}(\phi_b)$$

$$= \left[\frac{\phi_a}{1 - \zeta q e^{\psi_{s_1}}}, \phi_b \right]^{ABC}_{0,2,d}$$

For the remainder of the strata where $d_0, d_1 > 0$, decompose the $\tilde{\mathrm{td}}$ term as a product of terms in $Q^{\zeta,\zeta^{-1}}_{0,2,d_0}(X/\mu_m)$ and $I_\eta Q_{0,2,d_1}$. Recall that the virtual tangent space to a curve $C \in I_\zeta Q_{0,2,d}(X)$ has decomposition in K-theory

$$T(C) = T(C_0) + T(C_1) + T_{p_0}C_0 \otimes T_{p_1}C_1 - T_{f(p)}X.$$

Therefore, an eigenspace $T(C,g)_\lambda \subset T(C)$ has an corresponding decomposition. Denote by g_0 and g_1 the restrictions of the automorphism g of C restricts to the head and the tail respectively. Then

$$T(C,g)_\lambda = T(C_0,g_0)_\lambda + T(C_1,g_1)_\lambda + \left((T_{p_0}C_0, g) \otimes (T_{p_1}C_1, g)\right)_\lambda - T_{f(p)}(X)_\lambda.$$

Since $T(C,g)_\lambda$ is a fiber of $I_\lambda T_{0,2,d}$, we obtain multiplicative decomposition of $\tilde{\mathrm{td}}(T_{0,2,d})$,

$$\tilde{\mathrm{td}}(T_{0,2,d}) = \prod_\lambda \left(\mathrm{td}_{\lambda^{-1}} T(C_0,g_0)_\lambda\right) \left(\mathrm{td}_{\lambda^{-1}} T(C_1,g_1)_\lambda\right)$$
$$\mathrm{td}_{\lambda^{-1}} \left((T_{p_0}C_0, g)_\lambda \otimes (T_{p_1}C_1, g)_\lambda\right) \left(\mathrm{td}_{\lambda^{-1}} T_{f(p)}(X)_\lambda\right)^{-1}$$

where we abuse notation by conflating a vector bundle with its fiber.

(1) One can consider this term

$$\prod_\lambda \mathrm{td}_{\lambda^{-1}} T(C_0,g_0)_\lambda = \Theta^{\mathrm{ABC}} = 1 + \text{higher order terms}$$

to be the stem correlator of Givental and Tonita extended to 0+-stability, since it is associated with the target $Q^{\zeta,\zeta^{-1}}_{0,2,d}([X/\mu_m])$. But for this proof, we only need that it is of the form $1 + \dots$, which is clear by the series expansion of the moving Todd class.

(2) The second term $\prod_\lambda T(C_1, g_1)_\lambda$ is equal to $\tilde{\operatorname{td}}(T_{0,2,d_1})$. For $T(C_1, g_1)_\lambda$ is just the λ-eigenspace for $T(C_1)$ on $I_\eta Q_{0,2,d_1}$, and the claim follows by definition.

(3) For the third term, note that by definition of the strata, g acts by eigenvalues ζ on $T_{p_0}(C_0)$ and η^{-1} on $T_{p_1}(C_1)$. Thus

$$(T_{p_0}C_0 \otimes T_{p_1}C_1, g)_\lambda = \begin{cases} 0 & \text{if } \lambda \neq \zeta\eta^{-1} \\ T_{p_0}C_0 \otimes T_{p_1}C_1 & \text{if } \lambda = \zeta\eta^{-1} \end{cases}.$$

Since ζ and η opposite nodes on an unbalanced node by definition, then $\zeta^{-1} \neq \eta$, so that $\lambda = \zeta^{-1}\eta \neq 1$. Therefore,

$$\prod_\lambda \operatorname{td}_{\lambda^{-1}}(T_{p_0}C_0, g)_\lambda \otimes (T_{p_1}C_1, g)_\lambda = \frac{1}{1 - \zeta^{-1}\eta \operatorname{ch}(L_0 \otimes L_1)}.$$

As it is written, this is a term on $Q^{\zeta,\zeta^{-1}}_{0,2,d_0}(X) \times I_\eta Q_{0,2,d_1}(X)$. Expanding it as a infinite series, to splits it over the product.

(4) Finally $T_{f(p)}X$ inherits the trivial action of g, since g is an automorphism on the domain curve C, and not on the target. More accurately this tangent bundle represents the tangent space to $f(p)$ pulled back to $I_\eta Q_{0,2,d_1}$. Therefore this term contributes $\operatorname{ev}^*_{p_1}(1/\operatorname{td} T_X)$ since the twisted todd class for $\lambda = 1$ is just the regular todd class.

Now consider the gluing commutative diagram.

$$\begin{array}{ccc} Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m]) \times_{\operatorname{ev}} I_\eta Q_{0,2,d_1} & \xrightarrow{\operatorname{ev}} & X \\ \downarrow i & & \downarrow \Delta \\ Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m]) \times I_\eta Q_{0,2,d_1} & \xrightarrow{\operatorname{ev}_0 \times \operatorname{ev}_1} & X \times X \end{array} \tag{4.4.1}$$

Putting together the last 4 calculations, the integrand splits (up to the third term) over the fiber product as

$$i^* \left(\Theta^{ABC} \boxtimes 1\right) \left(1 \boxtimes \tilde{\mathrm{td}}(T_{0,2,d_1})\right) \left(\frac{1}{1-\zeta^{-1}\eta \operatorname{ch}(L_0 \otimes L_1)}\right) \left(\frac{1}{1 \boxtimes \mathrm{ev}_{p_1}(T_X)}\right)$$
$$\left(\frac{\mathrm{ev}_{s_1}^*(\phi_a)}{1-q \operatorname{ch}\operatorname{tr}(L_{s_1})} \boxtimes \mathrm{ev}_{s_2}^*(\phi_b)\right) := i^*(\alpha_0 \boxtimes \alpha_1).$$

Applying the Thom isomorphism for the commutative diagram Equation (4.4.1):

$$\int_{Q^{\zeta,\zeta^{-1},\mathrm{vir}}_{0,2,d_0}([X/\mu_m]) \times_{\mathrm{ev}} I^{\mathrm{vir}}_{\eta} Q_{0,2,d_1}} i^*(\alpha_0 \boxtimes \alpha_1)$$
$$= \int_{Q^{\zeta,\zeta^{-1},\mathrm{vir}}_{0,2,d_0}([X/\mu_m]) \times I^{\mathrm{vir}}_{\eta} Q_{0,2,d_1}} (\alpha_0 \boxtimes \alpha_1) \cup \theta(N_i)$$

where $\theta(N_i)$ is the Thom class of the normal bundle of i. But we know from the diagram it is the pullback of the Thom class of the normal bundle to the diagonal embedding, which is just the diagonal class's Poincare dual. Since $N_\Delta = \sum_c \phi_c \otimes \phi^c$, then we obtain

$$\theta(N_i) = \sum_c \mathrm{ev}_{p_0}^*(\phi_c) \boxtimes \mathrm{ev}_{p_1}^*(\phi^c).$$

Then all in all, the integral over the substratum for (ζ, η, d_0, d_1) is

$$\int_{Q^{\zeta,\zeta^{-1},\mathrm{vir}}_{0,2,d_0}([X/\mu_m]) \times_{\mathrm{ev}} I^{\mathrm{vir}}_{\eta} Q_{0,2,d_1}} i^*(\alpha_0 \boxtimes \alpha_1)$$
$$= \sum_c \left(\int_{Q^{\zeta,\zeta^{-1},\mathrm{vir}}_{0,2,d_0}([X/\mu_m])} \alpha_0 \, \mathrm{ev}_{p_0}^*(\phi_c)\right) \left(\int_{I^{\mathrm{vir}}_{\eta} Q_{0,2,d_1}} \alpha_1 \, \mathrm{ev}_{p_1}^*(\phi^c)\right)$$

To see how the mixed term contributes, let

$$\frac{1}{1-\zeta^{-1}\eta \operatorname{ch}(L_0 \otimes L_1)} = \sum_j \gamma_j^0 \boxtimes \gamma_j^1$$

where again the γ_j^- can be found by the $1/(1-x)$ series expansion. Writing out the product of the integrals and passing the j sum and I integrals in the Q one, we obtain

$$\sum_{c,j}\left(\int_{Q^{\zeta,\zeta^{-1}}_{0,2,d_0}(X)} \Theta^{ABC}\frac{\mathrm{ev}^*_{s_1}(\phi_a)}{1-q\,\mathrm{ch}\,\mathrm{tr}(L_{s_1})}\,\mathrm{ev}^*_{p_0}(\phi_c)\gamma^0_j\right)$$
$$\times\left(\int_{I_\eta Q_{0,2,d_1}} \tilde{\mathrm{td}}(T_{0,2,d_1})\gamma^1_j\,\mathrm{ev}^*_{p_1}(\phi^c/\,\mathrm{td}(T_X))\,\mathrm{ev}^*_{s_2}(\phi_b)\right)$$
$$=\sum_c\int_{Q^{\zeta,\zeta^{-1}}_{0,2,d_0}(X)} \Theta^{ABC}\frac{\mathrm{ev}^*_{s_1}(\phi_a)}{1-q\,\mathrm{ch}\,\mathrm{tr}(L_{s_1})}\,\mathrm{ev}^*_{p_0}(\phi_c)\sum_j\gamma^0_j\left(\int_{I_\eta Q_{0,2,d_1}(X)}\gamma^1_j\otimes\dots\right)$$

Now we can justify passing the γ^0_j term into the $I_\eta Q_{0,2,d_1}(X)$ integral as follows. Since integrating is notation for pushing-forward to a point, we also have a map $\pi\times\mathrm{id}: Q\times I\to Q\times\mathrm{pt}=Q$. So bringing in the γ^0_j term changes the integral from π_* to $(\mathrm{id}\times\pi)_*$, i.e. now we are starting with an object in $H^*(Q)\otimes H^*(I)$, taking the degree of the I-term, and outputing something in $H^*(Q)_{\mathbb{Q}}$. In particular,

$$\sum_j\gamma^0_j\int_{I_\eta Q_{0,2,d_1}(X)}(\gamma^1_i\otimes\dots)=\int_I\left(\sum_j\gamma^0_j\boxtimes\gamma^1_j\right)\otimes\dots$$
$$=\int_{I_\eta Q_{0,2,d_1}(X)}\frac{1}{1-\zeta^{-1}\eta\,\mathrm{ch}(L_0\otimes L_1)}\tilde{\mathrm{td}}(T_{0,2,d_1})\,\mathrm{ev}^*_{p_1}(\phi^c/\,\mathrm{td}(T_X))\,\mathrm{ev}^*_{s_2}(\phi_b)$$

Now we can split the $\zeta^{-1}\eta$ term by remembering that ζ^{-1} and η are the eigenvalues at the cotangent lines L_0 and L_1. Then

$$\zeta^{-1}\eta\,\mathrm{ch}(L_0\otimes L_1)=(\zeta^{-1}\,\mathrm{ch}(L_0))(\eta\,\mathrm{ch}(L_1))$$

so that the above integral becomes by definition:

$$\int_{I_\eta Q_{0,2,d_1}(X)}\frac{1}{1-(\zeta^{-1}\,\mathrm{ch}(L_0))(\mathrm{ch}\,\mathrm{tr}(L_1))}\tilde{\mathrm{td}}(T_{0,2,d_1})\,\mathrm{ev}^*_{p_1}(\phi^c/\,\mathrm{td}(T_X))\,\mathrm{ev}^*_{s_2}(\phi_b)$$
$$=\tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}\,\mathrm{ch}(L_0))\in H^*(Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m])).$$

Plugging everything back into the $Q^{\zeta,\zeta^{-1}}_{0,2,d_0}([X/\mu_m])$ integral, we obtain

$$\begin{aligned}
&\sum_c \int_{Q^{\zeta,\zeta^{-1},\mathrm{vir}}_{0,2,d_0}([X/\mu_m])} \Theta^{ABC} \tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}\operatorname{ch}(L_0)) \frac{\mathrm{ev}^*_{s_1}(\phi_a)}{1-q\operatorname{ch}\operatorname{tr}(L_{s_1})}\,\mathrm{ev}^*_{p_0}(\phi_c) \\
&\quad = \sum_c \int_{Q^{\zeta,\zeta^{-1},\mathrm{vir}}_{0,2,d_0}([X/\mu_m])} \Theta^{ABC} \left(\frac{\mathrm{ev}^*_{s_1}(\phi_a)}{1-\zeta q e^{\psi_{s_1}}}\right)\left(\tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\,\mathrm{ev}^*_{p_0}(\phi_c)\right) \\
&\quad = \sum_c \left[\frac{\phi_a}{1-\zeta q e^{\psi_{s_1}}}, \tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0}.
\end{aligned}$$

Adding everything back together,

$$\tau^{\zeta}_{a,b,d}(q) = \left[\frac{\phi_a}{1-\zeta q e^{\psi_{s_1}}}, \phi_b\right]^{ABC}_{0,2,d} + \sum_{\substack{\eta,c\\ d_0,d_1>0\\ d_0+d_1=d}} \left[\frac{\phi_a}{1-\zeta q e^{\psi_{s_1}}}, \tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0}$$

as desired. □

4.5 Corollaries

This formula yields a host of useful corollaries. Recall that when we say $X = W /\!\!/ G$, we mean X is a GIT quotient (W, G, θ) satisfying the properties listed in Section 2.1.

Corollary 4.5.1. *For $X = W /\!\!/ G$, then*

$$\begin{aligned}
\left\langle \frac{\Phi_a}{1-qL}, \Phi_b \right\rangle_{0,2,d} &= \sum_{\zeta} \left[\frac{\phi_a}{1-q\zeta e^{\psi_{s_1}}}, \phi_b\right]^{ABC}_{0,2,d} \\
&\quad + \sum_{\substack{\zeta,\eta,c\\ d_0+d_1=d\\ d_0>0}} \left[\frac{\phi_a}{1-\zeta q e^{\psi_{s_1}}}, \tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0}.
\end{aligned}$$

Proof: This formula immediately follows from summing over all contributions of the strata $I_\zeta Q_{0,2,d}(X)$ to the 2-pointed K-theoretic correlator and applying Lemma 4.4.7. □

Proposition 4.5.2. *For $X = W /\!\!/ G$ and $F(q, L_{s_1})$ any series in q and L_{s_1}, a similar formula holds.*

$$\langle F(q, L_{s_1})\Phi_a, \Phi_b\rangle_{0,2,d} = \sum_{\zeta} \left[F(q, \zeta e^{\psi_{s_1}})\phi_a, \phi_b\right]^{ABC}_{0,2,d}$$
$$+ \sum_{\substack{\zeta,\eta,c \\ d_0+d_1=d \\ d_0>0}} \left[F(q, \zeta e^{\psi_{s_1}}))\phi_a, \tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0}$$

Proof: One can repeat the same argument as before with $1/(1 - qL)$ replaced by F. The series in the first insertion was independent of simplifying the second insertion to a τ function. $\square$

Now we are in a position to prove important vanishing lemmas. Recall that $|\phi_a| = \frac{1}{2} \deg \phi_a$ denotes the complex degree of a homogeneous element $\phi_a \in H^{2a}(X)_{\mathbb{Q}}$ while $|\Phi_a|$ refers to $|\operatorname{ch}(\Phi_a)|$.

Lemma 4.5.3. *Let $X = W /\!\!/ G$ be a Calabi-Yau GIT target. Then given K-theory basis elements Φ_a, Φ_b with $\operatorname{ch}(\Phi_a)$, $\operatorname{ch}(\Phi_b)$ homogeneous, the following statements hold.*

(a) If $|\Phi_a| + |\Phi_b| > \dim X - 1$, then $\tau^{\zeta}_{a,b,d}(q) = 0$

(b) If $|\Phi_a| + |\Phi_b| > \dim X - 1$, then $\langle \Phi_a, \Phi_b\rangle_{0,2,d} = 0$.

(c) If $|\Phi_a| + |\Phi_b| < \dim X + 1$, then $G^{ab}|_{t=0} = g^{ab}$, where G^{ab} was the inverse of the non-constant metric, see equation (3.1.2).

Proof (a): We prove this by contrapositive, so assume $\tau^{\zeta}_{a,b,d}(q) \neq 0$. Applying the recursion, one of the terms must be nonzero. First, the distinguished term where $d_1 = 0$, which has the form

$$\left[\frac{\phi_a}{1 - \zeta q e^{\psi}}, \phi_b\right]^{ABC}_{0,2,d}.$$

We can expand the first insertion about $q - 1$ as $1 + O(q, \psi)$, and so we see that this correlator is nonvanishing only when $|\Phi_a| + |\Phi_b| \leq \dim X - 1$ by a simple dimension count.

For the other terms where $d_1 \neq 0$, we do an induction on d using the lexicographical order on the semigroup $\operatorname{Eff}(W, G, \theta)$. We say $d > d'$ if $d - d'$ is effective and nonzero.

Of course if d_0, d_1 are effective, nonzero, and $d_0 + d_1 = d$ then $d_1 < d$, which is how we apply this lexicographical ordering. Specically induction hypothesis to the term $\tau^{\eta,c}_{-,b,d_1}$ in the recursion. Then the correlator vanishes either when $|\Phi_a| + |\phi_c| > \dim X - 1$ (by dimension considerations) or $|\phi^c| + |\Phi_b| > \dim X - 1$ (when τ vanishes by hypothesis). If this term is nonvanishing when for some c we have $|\Phi_a| + |\phi_c| \leq \dim X - 1$ and $\dim X - |\phi_c| + |\Phi_b| \leq \dim X - 1$. Rearranging, we obtain $|\phi_c| \leq \dim X - 1 - |\Phi_a|$ and $|\phi_c| \geq |\Phi_b| + 1$. All together this implies $|\Phi_b| + 1 \leq \dim X - 1 - |\Phi_a|$, or in other words $|\Phi_a| + |\Phi_b| \leq \dim X - 2$. Therefore these terms have a slightly stronger nonvanishing condition than the distinguished term. All in all, in order for one of these terms to be nonvanishing, then $|\Phi_a| + |\Phi_b| \leq \dim X - 1$ as desired.

Proof (b): By Proposition 4.5.2, we can apply the exact same argument as (a). More generally $\langle F(q, L_{s_1})\Phi_a, \Phi_b\rangle_{0,2,d}$ vanishes under these conditions.

Proof (c): This follows from expanding the double bracket correlators by definition when $t = 0$ and then simplifying by a dimension count and part (a). In particular the first few terms of the nonconstant metric can be written out:

$$\begin{aligned}
G^{ab} = g^{ab} &- \sum_d Q^d \langle \Phi^a, \Phi^b\rangle_{0,2,d} \\
&+ \sum_m \left(\sum_{d_1} Q^{d_1}\langle \Phi^a, \Phi^m\rangle_{0,2,d_1}\right)\left(\sum_{d_1} Q^{d_2}\langle \Phi_m, \Phi^b\rangle_{0,2,d_2}\right) + / - \dots \\
= g^{ab} &- \sum_d Q^d \langle \Phi^a, \Phi^b\rangle_{0,2,d} \\
&+ \left(\sum_d Q^d \sum_{d_1+d_2=d}\left(\sum_m \langle \Phi^a, \Phi^m\rangle_{0,2,d_1}\langle \Phi_m, \Phi^b\rangle_{0,2,d_2}\right)\right) + / - \dots .
\end{aligned}$$

Let r denote the index of the terms in G^{ab}, i.e. the r-th term is a product of r double bracket correlators. The degree d coefficient of the r-th term is

$$\sum_{d_1+\dots+d_r=d}\ \sum_{m_1,\dots,m_{r-1}} \langle \Phi^a, \Phi^{m_1}\rangle_{0,2,d_1}\langle \Phi_{m_1}, \Phi^{m_2}\rangle_{0,2,d_2}\dots\langle \Phi_{m_{r-1}}, \Phi^b\rangle_{0,2,d_r}.$$

Of course for 2 pointed correlators in 0+-stability, the degree must be strictly positive, so we must have that $d_i \geq 1$. And so therefore the sum is empty if $d < r$. In total, summing over d for each r and combining, we obtain:

$$G^{ab} = g^{ab} + \sum_{d \geq r \geq 1} \sum_{\substack{d_1+\cdots+d_r=d \\ m_1,\ldots,m_{r-1}}} Q^d \langle \Phi^a, \Phi^{m_1} \rangle_{0,2,d_1} \langle \Phi_{m_1}, \Phi^{m_2} \rangle_{0,2,d_2} \ldots \langle \Phi_{m_{r-1}}, \Phi^b \rangle_{0,2,d_r}.$$

First, part (a) covers the $r = 1$ term $\langle \Phi^a, \Phi^b \rangle_{0,2,d}$. Indeed it vanishes for $|\Phi^a| + |\Phi^b| > \dim X - 1$, which becomes $|\Phi_a| + |\Phi_b| < \dim X + 1$.

Next when $r > 1$, we have a product of terms of the form $\langle \Phi_{m_i}, \Phi^{m_{i+1}} \rangle_{0,2,d_{i+1}}$. To find the vanishing condition, again use a contrapositive argument. In order for these terms to be nonzero, by part (a)

$$|\Phi_{m_i}| + |\Phi^{m_{i+1}}| \leq \dim X - 1$$

so that $|\Phi_{m_{i+1}}| \geq |\Phi_{m_i}| + 1$ for all $i = 1, \ldots, r-1$. For the initial term to be nonvanishing, $|\Phi^a| + |\Phi^{m_1}| \leq \dim X - 1$ which implies that

$$|\Phi_{m_1}| \geq \dim X + 1 - |\Phi_a|.$$

Correspondingly, $|\Phi^b| + |\Phi_{m_{r-1}}| \leq \dim X - 1$ implies

$$|\Phi_b| \geq |\Phi_{m_{r-1}}| + 1.$$

Therefore the product of correlators is nonvanishing when the following system of inequalities

$$\dim X - |\Phi_a| < |\Phi_{m_1}| < |\Phi_{m_2}| < \cdots < |\Phi_{m_{r-1}}| < |\Phi_b|$$

has solutions from the basis of K-theory formed by $\{\Phi_i\}$. All in all this implies $|\Phi_a| + |\Phi_b| \geq \dim X + 1$. Then of course by contrapositive if $|\Phi_a| + |\Phi_b| < \dim X + 1$, there are no basis elements $\Phi_{m_1}, \ldots, \Phi_{m_{r-1}}$ which fit the criteria so the whole term vanishes. $\square$

To end this section, we prove a K-theoretic version of Proposition 3.4.1(b) using Lemma 4.5.3(c).

Proposition 4.5.4. *Let $X = W /\!\!/ G$ be a Calabi-Yau GIT target and denote $P_0(0,q,Q)$ (resp. $I_0(0,q,Q)$) to be the coefficient of Φ_0 in P (resp. in I). Then $P_0(0,q,Q) = I_0(0,q,Q)$.*

Proof: This can be computed by the Birkhoff factorization and applying Lemma 4.5.3. Beginning with $P(0,q,Q) = S^*(0,1/q)(I(0,q,Q))$, we consider only the terms of S^* with $d \geq 1$ first. (Of course P will have other powers of Q from the I-function insertion.) Denote the $d \geq 1$ terms of the S^*-operator by $S^*_{d\geq 1}$. Let $P_{d\geq 1}$ be the subseries of P defined by $S^*(0,q,Q)_{d\geq 1}$ applied to $I(0,q,Q)$. Then

$$\begin{aligned} P_{d\geq 1}(0,q,Q) &= S^*(0,1/q,Q)_{d\geq 1}(I(0,q,Q)) \\ &= \sum_{a,b} \Phi_a \sum_{d\geq 1} Q^d G^{ab} \left\langle \frac{I}{1-L/q}, \Phi_b \right\rangle_{0,2,d}. \end{aligned}$$

Taking the coefficient of Φ_0 only, by part Lemma 4.5.3(c)

$$\begin{aligned} \Phi_0 \cdot P_{0,d\geq 1} &= \sum_b \Phi_0 \sum_{d\geq 1} g^{0b} \left\langle \frac{I}{1-L/q}, \Phi_b \right\rangle_{0,2,d} \\ &= \Phi_0 \cdot \sum_{d\geq 1} \left\langle \frac{I}{1-L/q}, \Phi^0 \right\rangle_{0,2,d} \end{aligned}$$

Writing out I in terms of $\{\Phi_a\}$ and the Novikov variables and applying the estimate in Lemma 4.5.3(b):

$$\begin{aligned} P_{0,d\geq 1} &= \sum_{d\geq 1} \left\langle \frac{1}{1-L/q} \sum_{a,\delta} Q^\delta \Phi_a I_a^\delta, \Phi^0 \right\rangle_{0,2,d} \\ &= \sum_{a,d\geq 1,\delta} Q^{d+\delta} I_a^\delta \left\langle \frac{\Phi_b}{1-L/q}, \Phi^0 \right\rangle_{0,2,d} = 0 \end{aligned}$$

since $|\Phi^0| + |\Phi_b| > \dim X - 1$. Then P_0 is determined by the degree $d = 0$ terms of S^*

applied to I. Let $\Gamma = \sum_a \Gamma_a \Phi_a$. Again by Lemma 4.5.3(c),

$$\begin{aligned} S^*(0,1/q,Q)(\Gamma)_{d=0} &= \sum_{\alpha,\beta} \Phi_\beta G^{\alpha\beta} \langle \Phi_\alpha, \Gamma \rangle \\ &= \sum_{\alpha} \Phi_0 g^{\alpha 0} \langle \Phi_\alpha, \Gamma \rangle + (\beta > 0 \text{ terms}) \\ &= \Phi_0 \cdot \Gamma_0 + (\beta > 0 \text{ terms}). \end{aligned}$$

Letting $\Gamma = I(0,q,Q)$ proves result. $\square$

Chapter 5

The $q \to 1$ limit of the small K-theoretic I-function

In this section we prove the main result of the thesis.

Theorem 5.0.1. *Let $X = W /\!\!/ G$ be a Calabi-Yau GIT target. Then*

$$\lim_{q \to 1} (1-q)^{\deg} \operatorname{ch}(I(0,q,Q)) = I^{coh}(0,1,Q).$$

5.1 Lemmas

We prove preliminary lemmas which do significant computational work towards the main result.

Lemma 5.1.1. *Let $X = W /\!\!/ G$ be a Calabi-Yau GIT target. Then*

$$\lim_{q \to 1} (1-q)^{|\Phi_k|} \left\langle \frac{\Phi^k}{1-qL}, \Phi_b \right\rangle_{0,2,d} = \begin{cases} 0 & \text{if } b \neq 0 \text{ or } k = 0 \\ \langle \frac{\phi^k}{1-\psi_{s_1}}, 1 \rangle_{0,2,d} & \text{else} \end{cases}$$

Proof: We compute the limit along all the Kawasaki strata listed in Proposition 4.2.2 and see that the only nonvanishing contributions occur along the stratum $Q^{1,1}_{0,2,d}(X) = Q_{0,2,d}(X)$. First consider all the strata for $\zeta \neq 1$ and $d_1 > 0$. Using the recursion from

Lemma 4.4.7, the $I_\zeta Q_{0,2,d}(X)$ contribute to the limit

$$(1-q)^{|\Phi_k|}\sum_a g^{a,k}\sum_{\substack{\zeta,\eta,c\\ d_0>0\\ d_0+d_1=d}}\left[\frac{\phi_a}{1-\zeta q e^{\psi_{s_1}}},\tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0}.$$

If $k>0$, the $(1-q)^{|\Phi_k|}$ makes the limit is zero since those correlators have no pole at $q=1$. If $k=0$, then

$$\begin{aligned}\lim_{\substack{q\to 1\\ \zeta\neq 1}}\left\langle\frac{\Phi^0}{1-qL},\Phi_b\right\rangle_{0,2,d} &= \sum_{\substack{\zeta\neq 1\\ \eta,c\\ d_0>0\\ d_0+d_1=d}}\left[\frac{[\mathrm{pt}]}{1-\zeta e^{\psi_{s_1}}},\tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0}\\ &=0\end{aligned}$$

The last step is justified by a dimension count: the degree of the point class is $\dim X$ and $\operatorname{vdim} Q^{\zeta,\zeta^{-1}}_{0,2,d_0}(X)=\dim X-1$. For the exact same reasons in the stratum with only balanced nodes,

$$\lim_{q\to 1}(1-q)^{|\Phi_k|}\sum_a g^{ak}\left[\frac{\phi_a}{1-q\zeta e^{\psi_{s_1}}},\phi_b\right]=0.$$

Therefore the $I_\zeta Q_{0,2,d}(X)$ contributions become zero in the limit when $\zeta\neq 1$. For $\zeta=1$, we expand the rational part of the correlator about $q=1$ for the terms where $I_\eta Q_{0,2,d_1}(X)$ is nonempty by

$$\begin{aligned}\frac{1}{1-qe^{\psi}} &= \frac{1}{1-q}\frac{1}{1-\frac{q}{1-q}(e^{\psi}-1)}\\ &=\frac{1}{1-q}\sum_\ell\left(\frac{q}{1-q}\right)^\ell(e^{\psi}-1)^\ell\\ &=\sum_\ell(q^\ell\psi^\ell+\dots)(1-q)^{-\ell-1}.\end{aligned}$$

Therefore, contributions from the substrata of $I_1Q_{0,2,d}(X)$ with nontrivial automorphisms are:

$$(1-q)^{|\Phi_k|}\sum_a g^{ak} \sum_{\substack{\eta,c\\ d_0,d_1>0\\ d_0+d_1=d}} \left[\frac{\phi_a}{1-qe^{\psi_{s_1}}}, \tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0}$$
$$= \sum_{\substack{\ell,a,\eta,c\\ d_0,d_1>0\\ d_0+d_1=d}} (1-q)^{|\Phi_k|-\ell-1} g^{ak}\left[\phi_a(q^\ell\psi^\ell+\dots), \tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0}.$$

To show existence of the limit of this term as $q \to 1$, it suffices to show the negative powers of $(1-q)$ have vanishing coefficient. So assume $|\Phi_k|-\ell-1<0$, i.e. $\ell > |\Phi_k|-1$. The degrees of the insertions are at least $|\Phi^k|+\ell$ in the first position and at least $|\phi_c| \geq 0$ in the second insertion. But then by assumption

$$\begin{aligned}\dim X - |\Phi_k| + \ell + |\phi_c| &> \dim X - |\Phi_k| + (|\Phi_k|-1) + |\phi_c| \\ &= \dim X - 1 + |\phi_c| \geq \dim X - 1.\end{aligned}$$

All in all the degrees of the insertions exceed $\operatorname{vdim} Q_{0,2,d_0}(X) = \dim X - 1$ for all Φ_b, and so the correlator vanishes for negative powers of $(1-q)$. Therefore the limit exists.

To actually compute the limit, note that if $k=0$, then $|\Phi_k|-\ell-1<0$ for all ℓ so this case vanishes. Otherwise if $|\Phi_k|>0$, then the only nonvanishing term is if $\ell = |\Phi_k|-1$, so that the contribution to the limit becomes

$$\sum_{\substack{\eta,c\\ d_0,d_1>0\\ d_0+d_1=d}} \left[\phi^k\psi^{k-1}, \tau^{\eta,c}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})\phi_c\right]^{ABC}_{0,2,d_0} = 0.$$

Indeed this contribution vanishes, for the degrees of the insertions are at least $\dim X - |\Phi_k| + |\Phi_k| - 1 + |\phi_c|$. By dimension count, the correlator vanishes for $|\phi_c|>0$. But when $|\phi_c|=0$, $\tau^{\eta,0}_{-,b,d_1}(\zeta^{-1}e^{\psi_{p_0}})=0$ by Lemma 4.5.3 and $\dim X - 0 + |\phi_j| > \dim X - 1$. All in all, the whole term vanishes for the substrata of $I_1Q_{0,2,d}(X)$ with nontrivial automorphisms, i.e. when the inertia quasimap has a tail with $\eta \neq 1$.

Finally, the only remaining stratum to consider is the trivial stratum $Q^{1.1}_{0,2,d}(X) =$

$Q_{0,2,d}(X)$, when $g = \mathrm{id}$. This is the only stratum that contributes to the limit.

$$\begin{aligned}\lim_{q\to 1}(1-q)^{|\Phi_k|}\int_{Q_{0,2,d}(X)} &\Theta^{ABC}\frac{\mathrm{ev}_{s_1}^*(\mathrm{ch}(\Phi^k))\,\mathrm{ev}_{s_2}^*(\mathrm{ch}(\Phi_b))}{1-qe^{\psi}} \\ &= \lim_{q\to 1}(1-q)^{|\Phi_k|}\left\langle \frac{\phi^k/\,\mathrm{td}(X)}{1-qe^{\psi_{s_1}}}, \phi_b\right\rangle_{0,2,d}^{fake} \\ &= \sum_{\ell}\lim_{q\to 1}(1-q)^{|\Phi_k|-\ell-1}\langle(\phi^k+\dots)(q^\ell\psi^\ell+\dots),\phi_b\rangle_{0,2,d}^{fake}\end{aligned}$$

Just as above, if $|\Phi_k|-\ell-1<0$, then the correlator vanishes so the limit exists, and if $|\Phi_k|-\ell-1>0$, those terms vanish in the limit. So again the only remaining term corresponds to when $\ell = |\Phi_k|-1$. In this case the first insertion already has degree at least $\dim X - 1$, so if $|\Phi_b| = |\phi_b| > 0$, the whole term vanishes. Finally if $|\Phi_b| = 0$, the limit when $\ell = |\Phi_k|-1$ is $\sum_\ell \langle \phi^k\psi^\ell, 1\rangle_{0,2,d}$. On the other hand $\langle \phi^k/(1-\psi), 1\rangle_{0,2,d} = \sum_\ell\langle\phi^k\psi^\ell,1\rangle_{0,2,d}$ only has nonvanishing term when $\ell = |\Phi_k|-1$ as well. This finishes the proof. □

Now we turn to the P-series.

Lemma 5.1.2. *For $X = W /\!\!/ G$ a Calabi-Yau GIT target,* $\lim_{q\to 1}\mathrm{ch}(P_0(0,q,Q)) = P_0^{coh}(0,1,Q)$*, or equivalently* $\lim_{q\to 1}\mathrm{ch}(I_0(0,q,Q)) = I_0^{coh}(0,1,Q)$.

Proof: First of all, by definition P has no pole at $q=1$, so this limit exists without the $(1-q)^{\deg}$ adjustment. The equivalence of the limits follows from Proposition 4.5.4 and Proposition 3.4.1. Hence we would like to compute the limit of the graph space correlator

$$P_0(0,q,Q) = \langle \Phi^0(1-p_\infty)\rangle_{0,1,d}^{QG}$$

Consider the diagram below.

$$\begin{array}{ccccc} K^0(QG_{0,1,d}(X)) & \xrightarrow{\mathrm{ev}_*} & K^0_{\mathbb{C}^*}(X\times\mathbb{P}^1) & \xrightarrow{p_*} & K^0_{\mathbb{C}^*}(pt) \\ \downarrow{\scriptstyle I\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(IQG_{0,1,d}(X)) & \xrightarrow{\mathrm{ev}_*} & H^*_{\mathbb{C}^*}(X\times\mathbb{P}^1) & \xrightarrow{p_*} & H^*_{\mathbb{C}^*}(pt)\end{array} \qquad (5.1.1)$$

where $\mathbb{C}^*$ acts diagonally on $X\times\mathbb{P}^1$, with the trivial action on X. This is not torus equivariant virtual GRR, but the torus actions are important information we want to keep track of. Factoring the pushforward to the point and applying the projection formula and GRR

$$\begin{aligned}\langle (1-p_\infty)\Phi^0\rangle^{QG}_{0,1,d} &= p_* \operatorname{ev}_*(\operatorname{ev}^*(\Phi^0(1-q^{-1}))\otimes \mathscr{O}^{vir}_{QG}) \\ &= \int_{IQG^{vir}} \operatorname{ch}\operatorname{tr} \frac{ev^*(\Phi^0)(1-p_\infty)}{\bigwedge^*(N^{vir}_{IQG})^\vee} \operatorname{td}(T^{vir}_{IQG})\end{aligned}$$

where $\tilde{\operatorname{td}}$ is the moving Todd class from the previous section. Now we show that all the contributions from the twisted sectors of inertia vanish, leaving only the term from the trivial stratum. We apply Proposition 4.3.1 in much the same way as the proof of Lemma 4.4.7.

$$\begin{aligned}&\int_{IQG^{vir}} \operatorname{ch}\operatorname{tr}\frac{\operatorname{ev}^*(\Phi^0)(1-p_\infty)}{\bigwedge^*(N^{vir}_{IQG})^\vee} \operatorname{td}(T^{vir}_{IQG}) \\ &\qquad = \sum_\zeta \int_{I_\zeta QG_{0,1,d}(X)} \operatorname{ch}\operatorname{tr}\operatorname{ev}^*((1-p_\infty)\phi^0/\operatorname{td}(T_X))\tilde{\operatorname{td}}(T^{QG}_{0,1,d})\end{aligned}$$

If $\zeta = 1$ and $\eta = 1$, we have the trivial stratum, which we show becomes P_0^{coh} at the end of the proof. Otherwise, we can simplify other terms by decomposing the moving Todd class and applying the Thom isomorphism. The terms γ_c runs over a basis of $H^*(X\times\mathbb{P}^1)_\mathbb{Q}$ since that is target of evaluation.

$$\begin{aligned}&\int_{I_\zeta QG_{0,1,d}(X)} \operatorname{ch}\operatorname{tr}\operatorname{ev}^*((1-p_\infty)\phi^0/\operatorname{td}(T_X))\tilde{\operatorname{td}}(T^{QG}_{0,1,d}) \\ &\quad = \sum_{\substack{\eta,c\\ d_0,d_1>0\\ d_0+d_1=d}} \int_{Q^{\zeta,\zeta^{-1}}_{0,1,d_0}(X)\times\mathbb{P}^1} \Theta^{ABC}\operatorname{ev}^*((1-p_\infty)\phi^0/\operatorname{td}(T_X))\operatorname{ev}^*_{p_0}(\gamma_c) \\ &\qquad\quad \times \int_{I_\eta QG_{0,1,d_1}(X)} \operatorname{ev}^*_{p_1}(\gamma^c/\operatorname{td}(X\times\mathbb{P}^1))(1-\zeta^{-1}\eta\operatorname{ch}\operatorname{tr}(L_{p_0}\otimes L_{p_1}))^{-1} \\ &\quad = \sum \int_{Q^{\zeta,\zeta^{-1}}_{0,2,d_0}(X)\times\mathbb{P}^1} \tau^{QG}_{\eta,c}(\zeta^{-1}\psi_{p_0})\operatorname{ev}^*((1-p_\infty)\phi^0/\operatorname{td}(T_X))\operatorname{ev}^*(\gamma_c)\Theta^{ABC} = 0\end{aligned}$$

We denote the integral over I_η, which is a series in ψ_{p_0}, by τ^{QG} in much the same way as Lemma 4.4.7. This integral vanishes by a dimension count. Indeed, we split the pushforward into a product of integrals over $Q^{\zeta,\zeta^{-1}}_{0,1,d_0}(X)$ and $\mathbb{P}^1$. The virtual dimension of the former is $\dim X - 1$, while we have a term ϕ^0 which has codimension $\dim X$. Now for the contribution from the trivial stratum. The normal bundle is trivial since the embedding from this stratum into $QG_{0,1,d}(X)$ is an open embedding. Applying the projection formula and pushing forward to $X \times \mathbb{P}^1$

$$\begin{aligned}\int_{QG^{vir}} \mathrm{ev}^*((1-p_\infty)\phi^0/\operatorname{td}(T_X))\operatorname{td}(T^{vir}) &= \int_{QG^{vir}} \mathrm{ev}^*(\phi^0)(1-p^\infty)\\ &= \int_{X\times\mathbb{P}^1} \phi^0(1-p_\infty)\cap \mathrm{ev}_*([QG]^{vir}).\end{aligned}$$

The higher terms of $\operatorname{td}(T^{vir})$ and $1/\operatorname{td}(T_X)$ contribute only unity since the virtual dimension of $QG_{0,1,d}(X)$ is $\dim X + 1$, and $\phi^0(1-p_\infty)$ is already full codimension. Under the identification $\operatorname{ch}(q) = e^{-z}$, taking the limit as $q \to 1$ and then taking Chern is equivalent to taking Chern and then taking the limit as $z \to 0$. Then the limit of a Laurent polynomial in $1 - e^{-z}$ is the same as that of $-z$.

$$\begin{aligned}\lim_{q\to 1}\operatorname{ch}(\Phi_0\cdot P_0(0,q,Q)) &= \phi_0\cdot\lim_{q\to 1}\langle\Phi^0(1-e^{-z})\rangle^{QG}_{0,1,d}\\ &= \phi_0\cdot\lim_{z\to 0}\int_{X\times\mathbb{P}^1}\phi^0(1-e^{-z})\cap\mathrm{ev}_*([QG]^{vir})\\ &= \phi_0\cdot\lim_{z\to 0}\int_{X\times\mathbb{P}^1}\phi^0(-z)\cap\mathrm{ev}_*([QG]^{vir})\\ &= \lim_{z\to 0}\phi_0\int_{QG^{vir}}\mathrm{ev}^*(\phi^0 p_\infty)\\ &= \lim_{z\to 0}\phi_0\cdot\langle\phi^0 p_\infty\rangle^{QG}_{0,1,d} = \phi_0\cdot P_0^{coh}(0,1,Q)\end{aligned}$$

where the last step follows from the fact that P_0^{coh} is independent of z. □

5.2 Proof of Main Theorem

By Lemma 5.1.1, for some basis element Φ_b:

$$\lim_{q\to 1}(1-q)^{\deg}\operatorname{ch}(S(0,q,Q)(\Phi_b)) = \lim_{q\to 1}\sum_{\alpha,\beta} g^{\alpha\beta}(1-q)^{|\phi_\beta|}\phi_\beta\left(g_{\alpha b}+\sum_{d\geq 1}Q^d\left\langle\frac{\Phi_\alpha}{1-qL},\Phi_b\right\rangle_{0,2,d}\right)$$

$$= \sum_{\beta}\phi_\beta\left(\left(\sum_{\alpha} g^{\alpha\beta}g_{\alpha b}\lim_{q\to 1}(1-q)^{\beta}\right)+\sum_{d}Q^d\lim_{q\to 1}(1-q)^{\beta}\left\langle\frac{\Phi^\beta}{1-qL},\Phi_b\right\rangle_{0,2,d}\right)$$

$$= \sum_{\beta}\phi_\beta\left(\begin{cases}1 & \text{if } b=\beta=0\\ 0 & \text{else}\end{cases}+\sum_{d}Q^d\begin{cases}\langle\phi^\beta/(1-\psi),1\rangle_{0,2,d} & \text{if } b=0 \text{ and } \beta>0\\ 0 & \text{else}\end{cases}\right).$$

If $\phi_b \neq \phi_0$ then the limit vanishes. Otherwise, the term simplifies to

$$\lim_{q\to 1}(1-q)^{\deg}\operatorname{ch}(S(0,q,Q)(\Phi_0)) = \phi_0+\sum_{\beta>0}\phi_\beta\sum_{d}Q^d\left\langle\frac{\phi^\beta}{1-\psi},1\right\rangle_{0,2,d}$$
$$= \frac{I^{\mathrm{coh}}(0,1,Q)}{\langle\phi^0,P^{\mathrm{coh}}(0,1,Q)\rangle} = \frac{I^{\mathrm{coh}}(0,1,Q)}{I_0^{\mathrm{coh}}(0,1,Q)}$$

by equation (3.4.1). Finally by equation (3.4.2) and Lemma 5.1.2,

$$\lim_{q\to 1}(1-q)^{\deg}\operatorname{ch}(I_K(0,q,Q)) = \lim_{q\to 1}(1-q)^{\deg}\operatorname{ch}(S(0,q)(\Phi_0)I_0(0,q,Q))$$
$$= \frac{I^{coh}(0,1,Q)}{I_0^{coh}(0,1,Q)}I_0^{\mathrm{coh}}(0,1,Q) = I^{coh}(0,1,Q)$$

as desired. □

Corollary 5.2.1. *The limit preserves coefficients of Φ_i in I, i.e.*

$$\lim_{q\to 1}(1-q)^{|\Phi_a|}\operatorname{ch}(\Phi_a\cdot I_a(0,q,Q)) = \phi_a\cdot I_a^{coh}(0,1,Q).$$

Proof:

$$\begin{aligned}
\lim_{q\to 1}(1-q)^{|\phi_a|}\,\mathrm{ch}(\Phi_a\cdot I_a) &= \lim_{q\to 1}(1-q)^{|\Phi_a|}\phi_a\cdot I_a\\
&= \langle \phi^a, \lim_{q\to 1}(1-q)^{\deg}\,\mathrm{ch}(I(0,q,Q))\rangle\\
&= \langle \phi^a, I^{coh}(0,1,Q)\rangle\\
&= \phi_a\cdot I^{coh}_a(0,1,Q) \qquad \square
\end{aligned}$$

Chapter 6

Integrality of Mirror Map

In this section we give a pr whoof of the integrality of the mirror map for a class of Calabi-Yau GIT targets.

Theorem 6.0.1. *Let $X = W /\!\!/ G$ be a Calabi-Yau GIT target for which $K^(X)$ is generated by line bundles. Then the mirror map transformation*

$$Q_i \to Q_i e^{I_{1,i}(1,Q)/I_0(1,Q)}$$

has integer coefficients of Q.

The proof comes from a key observation made by Jockers and Mayr in [JM20, Section 8.3], with the details completely filled in. In this section we describe the terms of this mirror map, prove a formula for the exponent using Givental's reconstruction theorem, and then finish the proof using Theorem 5.0.1. It is expected that this readily generalizes to all Calabi-Yau GIT targets.

6.1 A Nonhomogeneous Basis

In order to define and expand the exponent of the mirror map $I_{1,i}(1, q, Q)/I_0(1, q, Q)$, we require another basis of K-theory. Define $P_1, \ldots, P_k$, where $P_i = \mathscr{O}(-H_i)$ and $H_1, \ldots, H_k$ form a numerically effective integer basis of the free part of $H^2(X)$. Let the primitive part of K-theory, $K^0(X)_{pr}$, be the subring of $K^0(X)$ generated by the P_i. In other words, the assumption that $K^0(X)$ is generated by line bundles is equivalent to

$K^0(X) = K^0(X)_{pr}$. Denote the monomials in P_i by $P^a = P_1^{a_1} \dots P_k^{a_k}$. Recall that the effective classes in K-theory can be interpreted as degree d curves in $H_2(X)$. Therefore we can use the pairing $H^2(X) \otimes H_2(X) \to \mathbb{Z}$ to form a product

$$a \cdot d = \sum_i a_i d_i = -\int_d c_1(P^a)$$

which will appear in Theorem 6.3.2. Finally, another set of generators of $K^0(X)_{pr}$ we will use is $1 - P_1, \dots, 1 - P_k$. We write $(1 - P^\ell)^a = \prod_i (1 - P_i^\ell)^{a_i}$.

Since H_i generate a free subgroup of $H^2(X)$, we can consider them as elements in rational cohomology as well. By the usual identification of the divisor class group with the Picard group, the bundles P_i also generate a free subgroup of $\mathrm{Pic}(X)$, and so we can consider them in $K^0(X)_\mathbb{Q}$ as well. Let $\Phi_1, \dots, \Phi_k$ with $k \leq n$ be the basis elements of $K^0(X)_\mathbb{Q}$ such that $\mathrm{ch}(\Phi_i) = H_i$ for $1 \leq i \leq k$. Now,

$$\mathrm{ch}(1 - P_i) = 1 - e^{-H_i} = H_i + O(H_i).$$

Therefore $\mathrm{ch}(1 - P_i - \Phi_i)$ lives in cohomology of complex degree 2 and higher so that

$$1 - P_i - \Phi_i = \sum_{j>k} a_{ij} \Phi_j.$$

This has interesting implications for the coefficients of the small I-function. The terms $I_{1,i}$ in the mirror map are defined to be the coefficients of $(1 - P_i)$ for $i = 1, \dots, k$ in small I. The notation $1, i$ reflects that the coefficients in cohomology are associated to complex degree 1 classes, the H_i. In fact these coefficients $I_{1,i}$ are equal to those of $\mathrm{ch}^{-1}(H_i) = \Phi_i$. Writing out small I in $K^0(X)_\mathbb{Q}$ homogeneous coordinates and changing bases from $\Phi_1, \ldots, \Phi_k$ to $1-P_1, \ldots, 1-P_k$:

$$\begin{aligned} I(0,q,Q) &= \sum_a I_a \Phi_a = I_0 \cdot \Phi_0 + \sum_{i=1}^k I_i \cdot \Phi_i + \sum_{j>k} I_j \cdot \Phi_j \\ &= I_0 \cdot \Phi_0 + \sum_{i=1}^k I_i \cdot \left((1-P_i) - \sum_{j>k} a_{ij} \Phi_j \right) + \sum_{j>k} J_j^{\varepsilon} \cdot \Phi_j \\ &= I_0 \cdot \Phi_0 + \sum_{i=1}^k I_i \cdot (1-P_i) + \sum_{j>k} \left(I_j - \sum_i a_{ij} I_i \right) \cdot \Phi_j . \end{aligned}$$

Hence $I_{1,i} = \langle \Phi^i, I \rangle$. However, the bundles $1-P_i$ live in the Picard group with integer coefficients, so similar observations can be made about the coefficients $I_{1,i}$ if we make clear in what sense small I has integer coefficients as a whole. We address this issue in the next section.

6.2 Integer Coefficients of the J^{ε}-function

In this section, we clarify in what sense $J^{\varepsilon}(0,q,Q)$ has integer coefficients. Recall the basis free presentations of the small S-operator and the P-series from Definition 3.2.1 and Proposition 3.2.4 in ε-stability.

$$S^{\varepsilon}(0,q,Q)(\Gamma) = \Gamma + \sum_{d>0} Q^d (\mathrm{ev}_1)_* \left(\frac{\mathscr{O}^{vir}_{Q_{0,2,d}(X)}}{1-qL} \mathrm{ev}_2^*(\Gamma) \right)$$

$$P^{\varepsilon}(0,q,Q) = \sum_d Q^d \hat{\mathrm{ev}}_{2,*} \left(\mathscr{O}^G \otimes \hat{\mathrm{ev}}_1^* \left((p_X \circ \mathrm{ev})_* (\mathscr{O}^{vir}_{QG}) \chi ((p_{\mathbb{P}^1} \circ \mathrm{ev})^* (1-p_\infty) \otimes \mathscr{O}^{vir}_{QG}) \right) \right)$$

First inspect the P^{ε}-series. Since $(p_{\mathbb{P}^1} \circ \mathrm{ev})_*(1-p_\infty) \in K^0(QG^{\varepsilon}_{0,1,d}(X))[q,q^{-1}]$ then it's Euler characteristic lives in $\mathbb{Z}[q,q^{-1}]$. Recall the from Section 3.1 that $\mathcal{Z}^{\varepsilon}$ is a stack with evaluation maps $\hat{\mathrm{ev}}_1$ and $\hat{\mathrm{ev}}_2$, and sheaf $\mathscr{O}^G \in \mathcal{K}^0(\mathcal{Z}^{\varepsilon})$ when $t=0$. Hence tracing through the diagram

$$\begin{array}{ccc} K^0(\mathcal{Z})[q,q^{-1}] & \xrightarrow{\mathscr{O}^G\otimes -} & K^0(\mathcal{Z})[q,q^{-1}] \\ {\scriptstyle (\hat{\mathrm{ev}}_1)^*}\nwarrow & & \swarrow{\scriptstyle (\hat{\mathrm{ev}}_2)_*} \\ & K^0(X)[q,q^{-1}] & \end{array} \tag{6.2.1}$$

the summands of P^ε lie in $\mathcal{K}^0(X)[q,q^{-1}]$, and thus $P^\varepsilon(0,q,Q) \in K^0(X)[q,q^{-1}][[Q]]$.

Moving to the S^ε-operator, we describe the insertion $1/(1-qL)$ and its affect on the poles of coefficients of I as a rational functions in q, which is explained in [GT14, Part III]. In general, suppose L satisfies a reduced polynomial $p(L^{-1}) = 0$ in $K^0(Q_{0,2,d}(X))$ with $p(0) \neq 0$. Write $p \in K^0(Q_{0,2,d}(X))[V]$ for a formal variable V. Then $p(q) - p(V)$ has root $V = q$, so there exists a formal polynomial $F(q,V)$ in q and V such that $p(q) - p(V) = F(q,V)(V-q)$ with $\deg F < \deg p$. Putting $V = L^{-1}$, then

$$\frac{1}{1-qL} = \frac{1}{L^{-1}}\frac{1}{L^{-1}-q} = \frac{LF(q,L^{-1})}{p(q)}. \tag{6.2.2}$$

Hence, the poles will occur at roots of $p(q)$, which in general are roots of unity [Giv, Part VIII]. The expansion is regular at $q = 0$ and the function vanishes at $q = \infty$. However crucially, the numerators will live in $K^0(Q_{0,2,d}(X))$ and so we can expect the numerators to carry integer coefficients when written out in basis of the K-ring. Moreover, since $\deg F < \deg p$, the Laurent contribution to $J^\varepsilon(0,q,Q)$ comes from the $d = 0$ term in $S^\varepsilon(P)$, which is simply the P^ε-series. Plugging this expansion of $1/(1-qL)$ into $J^\varepsilon = S^\varepsilon(P^\varepsilon)$ we obtain:

$$\begin{aligned} J^\varepsilon &= P^\varepsilon + \sum_{d>0} Q^d (\mathrm{ev}_1)_* \left(\frac{1}{1-qL}\, \mathrm{ev}_2^*(P^\varepsilon) \otimes \mathscr{O}_Q^{vir} \right) \\ &= P^\varepsilon + \frac{1}{p(q)} \sum_{d>0} Q^d (\mathrm{ev}_1)_* \left(LF(q,L^{-1})\, \mathrm{ev}_2^*(P^\varepsilon) \otimes \mathscr{O}_Q^{vir} \right) \end{aligned}$$

By inspection, both P^ε and the numerators of $p(q)$ are in $K^0(X)[q,q^{-1}][[Q]]$, which converge in the Q-adic topology. For the proof of the integrality of the mirror map we will also need the following lemma.

Lemma 6.2.1. *For a GIT target $X = W /\!/ G$, the Laurent polynomial part and the numerators of the rational part of the small J^ε-function lies in $K^0(X)[q, q^{-1}][[Q]]$ endowed with the Q-adic topology.*

Proof: Take the expansion of J^ε above using $LF(q, L^{-1})/p(q)$. Write the $d > 0$ terms as $\sum_{d>0} Q^d f_d/p$ where $f_d \in K^0(X)[q, q^{-1}]$ and $\deg f_d$ may be larger than $\deg p$. Clearing denominators in the Laurent terms of f^d and applying the Euclidean algorithm to each free part of $K^0(X)$, then $f_d = E_d + g_d p$ where $\deg E_d < \deg p$ and $g_d \in K^0(X)[q, q^{-1}]$. The coefficients of Q^d reduce to $g_d + \frac{E_d}{p}$ and the claim is proven. □

Remark 6.2.2. It may be tempting to expand $S(0, q, Q)(P(0, q, Q))$ in this way to prove intergality of the mirror map, but not much more can be said given the tools presented so far, even for X Calabi-Yau. The canonical line bundle L associated to the first marking is not unipotent since $Q_{0,2,d}(X)$ is a DM stack, so we cannot expect the Laurent expansion to be finite. Equivalently, the polynomial $p(q)$ will vanish at roots of unity, not just $q = 1$. Moreover, expanding $LF(q, L^{-1})/p(q)$ using partial fractions (or equivalently VEGRR) will in general yield rational numerators.

6.3 The Reconstruction Theorem and the Mirror Exponential

The next step in the proof of integrality is the formula for $I_{1,i}/I_0$, originally written down by Jockers and Mayr [JM20].

Proposition 6.3.1. *For $X = W /\!/ G$ a Calabi-Yau GIT target with $K^0(X) = K^0(X)_{pr}$,*

$$\frac{I_{1,i}(0, q, Q)}{I_0(0, q, Q)} = \sum_{k>0} \frac{\Psi^k(\varepsilon_i)}{1 - q^k} + c_i \tag{6.3.1}$$

where $\varepsilon_i \in \mathbb{Z}[Q]$, $c_i \in \Lambda[q, q^{-1}]$, and Ψ^k is the kth Adams operation.

Before we can prove Proposition 6.3.1, we require a few preliminary results. The first is Givental's Explicit Reconstruction theorem [Giv16][Giv, Part VIII].

Theorem 6.3.2. *Denote by $\mathcal{L} \subset \mathcal{K}$ the range of the J-function in classical ∞-stability. For each line bundle P_a, let $\bar{\varepsilon}_a$ be a free parameter in the maximal ideal Λ_+, $c_a \in$*

$\Lambda[q, q^{-1}]$. *If a series* $I = \sum_d I_d Q^d$ *lies in* $\mathcal{L}$*, then so does the following family*

$$\sum_d I_d Q^d e^{\sum_{k>0,a} \Psi^k(\overline{\varepsilon}_a) P^{ka} q^{k(a\cdot d)}/k(1-q^k)} \sum_a c_a P^a q^{ad}. \tag{6.3.2}$$

If $K^0(X) = K^0(X)_{pr}$*, then all of* $\mathcal{L}$ *is generated in this way.*

Since we assumed that $K^0(X)_{pr}$ was generated by monomials in the P_i, the exponent can be rewritten in terms of

$$(1 - q^{kd} P^k)^a := \prod_i (1 - q^{k(a\cdot d)} P_i^{ka_i})$$

with a change of basis in the ε_a. In particular if $\overline{\varepsilon}_a$ are the parameters associated to P^a, and ε_a with $(1-P)^a$ then

$$\varepsilon_0 = \textstyle\sum_a \overline{\varepsilon}_a \quad \varepsilon_{e_i} = \sum_a a_i \overline{\varepsilon}_a.$$

Similarly rewrite c_a terms as $\sum_a c_a(1 - Pq^d)^a$. The choice of variable I in the above is not a coincidence. Zhang and Zhou proved for general Calabi-Yau GIT targets that the I-function lies in the image of J, stated in a different form in [ZZ20, Theorem 5.15].

Theorem 6.3.3. *For any GIT target* X*, the* I*-function lies in* $\mathcal{L}$*. In particular, there exists a* $\tau^{0,\infty}(q) \in \mathcal{K}^+$ *such that* $I(0, q, Q) = J(\tau^{0,\infty}(q), q, Q)$*.*

Therefore in the case when $K^0(X) = K^0(X)_{pr}$, we can reconstruct $I(0, q, Q)$ from $J(0, q, Q)$ using the reconstruction operator. In particular there exist $\varepsilon_a \in \Lambda_+$ and $c_a \in \Lambda[q, q^{-1}]$, determined by $\tau^{0,\infty}(q)$, such that

$$I(0, q, Q) = \sum_d J_d(0, q, Q) Q^d e^{\sum_{k>0,a} \Psi^k(\varepsilon_a)(1-P^{ka} q^{k(a\cdot d)})/k(1-q^k)} \sum_a c_a (1 - Pq^d)^a \tag{6.3.3}$$

The final piece we need to prove Proposition 6.3.1 is a lemma from Milanov and Roquefeuil, whose result shows that the degree 0 and 1 coefficients of J vanish for X Calabi-Yau.

Proposition 6.3.4. *Let* X *be a Calabi-Yau manifold. Write the un-normalized small* J*-function as* $J(0, q, Q) = 1 - q + \sum_a \Phi_a J_a(0, q, Q)$*. Then* $J_a(0, q, Q)$ *vanishes when* $|\Phi_a| = 0, 1$*.*

Proof: This follows exactly from the proof of Theorem 3.4.1 in [MR21] applied to the Calabi-Yau case. Recall that by definition

$$J(0,q,Q) = 1 - q + \sum_d \sum_a Q^d \Phi_a \left\langle \frac{\Phi^a}{1-qL_1} \right\rangle_{0,1,d}.$$

Therefore $J_a(0,q,Q) = \sum_d Q^d \left\langle \frac{\Phi^a}{1-qL_1} \right\rangle_{0,1,d}$. By Proposition 1 in [GT14], rewrite J using fake correlators (defined using the fake Euler characteristic):

$$J(0,q,Q) = 1 - q + \tau^f + \sum_{d,a} \sum_\ell \frac{Q^d \Phi_a}{\ell!} \left\langle \frac{\Phi^a}{1-qL_1}, \tau^f, \ldots, \tau^f \right\rangle^{\text{fake}}_{0,1+\ell,d}$$

where

$$\tau^f = \sum_{m>1} \sum_\zeta \sum_{a,d} Q^d \Phi_a \int_{I_\zeta X_{0,1,d}} \operatorname{td}(T_{IX_{0,1,d}}) \operatorname{ch} \operatorname{tr} \frac{\operatorname{ev}_1^* \Phi^a/(1-qL_1)}{\bigwedge^\bullet N^\vee_{IX_{0,1,d}}}.$$

Decompse τ^f by

$$\tau^f = \sum_{a,d} Q^d \Phi_a \tau^f_{a,d}$$

so that we can equate coefficients of the two forms of J. Then

$$\begin{aligned} J(0,q,Q) &= 1 - q + \sum_{a,d} Q^d \Phi_a \tau^f_{a,d}(q) + \sum_{a,d} Q^d \Phi_a \sum_\ell \frac{1}{\ell!} \left\langle \frac{\Phi^a}{1-qL_1}, \tau^f, \ldots, \tau^f \right\rangle^{\text{fake}}_{0,1+\ell,d} \\ &= 1 - q + \sum_{a,d} Q^d \Phi_a \left(\tau^f_{a,d}(q) + \sum_\ell \frac{1}{\ell!} \left\langle \frac{\Phi^a}{1-qL_1}, \tau^f, \ldots, \tau^f \right\rangle^{\text{fake}}_{0,1+\ell,d} \right) \end{aligned}$$

Therefore

$$\left\langle \frac{\Phi^a}{1-qL_1} \right\rangle_{0,1,d} = \tau^f_{a,d}(q) + \sum_\ell \frac{1}{\ell!} \left\langle \frac{\Phi^a}{1-qL_1}, \tau^f, \ldots, \tau^f \right\rangle^{\text{fake}}_{0,1+\ell,d}.$$

To prove vanishing, we need to show that $\tau^f_{a,d}(q)$ vanishes and the fake correlator sum vanishes. First by Lemma 3.4.2 in [MR21], $\tau^f_{a,d}(q) = 0$ for $|\Phi_a| - 1 + \int_d c_1(T_X) \leq 0$. But X is CY, so we conclude that $\tau^f_{a,d} = 0$ for $|\Phi_a| \leq 1$ as desired.

Second, write out the insertions τ in the fake correlator and write it as an ABC-twisted cohomological stem correlator.

$$\sum_\ell \frac{1}{\ell!}\left\langle \frac{\Phi^a}{1-qL_1}, \tau^f, \dots, \tau^f \right\rangle_{0,1+\ell,d}^{\text{fake}}$$
$$= \sum_\ell \sum_{d_1,\dots,d_\ell} \sum_{a_1,\dots,a_\ell} \frac{Q^{d_1+\dots+d_\ell}}{\ell!} \left\langle \frac{\Phi^a}{1-qL_1}, \tau^f_{a_1,d_1}\Phi_{a_1}, \dots, \tau^f_{a_\ell,d_\ell}\Phi_{a_\ell} \right\rangle_{0,1+\ell,d}^{\text{fake}}$$
$$= \sum_\ell \sum_{\substack{d_1,\dots,d_\ell\\ a_1,\dots,a_\ell}} \frac{Q^{d_1+\dots+d_\ell}}{\ell!} \left\langle \frac{\phi^a/\operatorname{td} X}{1-qe^{\psi_1}}, \tau^f_{a_1,d_1}\phi_{a_1}, \dots, \tau^f_{a_\ell,d_\ell}\phi_{a_\ell} \right\rangle_{0,1+\ell,d}^{ABC}$$
$$= \sum_\ell \sum_{\substack{d_1,\dots,d_\ell\\ a_1,\dots,a_\ell}} \sum_k \frac{Q^{d_1+\dots+d_\ell}}{\ell!} \left\langle \frac{(\phi^a+\dots)(q^k\psi^k+\dots)}{(1-q)^{k+1}}, \tau^f_{a_1,d_1}\phi_{a_1}, \dots, \tau^f_{a_\ell,d_\ell}\phi_{a_\ell} \right\rangle_{0,1+\ell,d}^{ABC}$$

Isolate one correlator in the sum and assume it is nonvanishing. By the first case, $|\Phi_{a_i}| > 1$. The lowest degrees of the insertions sum to $\dim X - |\Phi_a| + k + \sum_\ell |\Phi_{a_\ell}|$. Therefore nonvanishing implies

$$\dim X - |\Phi_a| + k + 2\ell \le \dim X - |\Phi_a| + k + \sum_\ell |\Phi_{a_i}| \le \operatorname{vdim} \overline{M}_{0,1+\ell,d}(X) = \ell - 2 + \dim X.$$

Rearranging, we obtain (correlator) $\neq 0$ implies $|\Phi_a| \ge k+\ell+2 \ge 2$. By contrapositive, $a \le 1$ implies the correlator vanishes. This works for all of them, so if $|\Phi_a| \le 1$ then

$$\left\langle \frac{\Phi^a}{1-qL_1} \right\rangle_{0,1,d} = 0$$

for stable maps. $\square$

Proof of Proposition 6.3.1: By Theorem 6.3.3 amd the assumption that $K^0(X) = K^0(X)_{pr}$,

$$I(0,q,Q) = \sum_d J_d(0,q,Q) Q^d e^{\sum_{k,\vec{a}} \frac{\Psi^k(\varepsilon_a)(1-q^{kd}P^k)^a}{k(1-q^k)}} \sum_a c_a(1-P^a q^{a\cdot d}).$$

Since $J_0(0,q,Q) = 1$ and $J_{1,i}(0,q,Q) = 0$, then the contributions to I_0 and I_1 come only from the reconstruction term in degree 0. Expanding the terms $(1-P^k)^a$ in the exponent for each a in terms of $1-P_i$,

$$\begin{aligned}(1-P^k)^a &:= \prod_i (1-P_i^k)^{a_i} \\ &= \prod_i (1-(1-(1-P_i))^k)^{a_i} \\ &= \prod_i (k(1-P_i)+O((1-P_i)^2)^{a_i}.\end{aligned}$$

The only a which contribute to I_0 and I_1 are $a=0$ and $a=e_i$ applied to the dilaton shift 1 which appears at the front of J. Write $\varepsilon_{e_i}=\varepsilon_i$.

$$\begin{aligned}e^{\sum_{k>0,a}\frac{\Psi^k(\varepsilon_a)(1-P_i^k)}{k(1-q^k)}} &= \left(\prod_k e^{\frac{\Psi^k(\varepsilon_0)}{k(1-q^k)}}\right)\prod_i\left(\prod_k e^{\frac{\Psi^k(\varepsilon_i)(k(1-P_i)+\dots)}{k(1-q^k)}}\right)(1+\dots) \\ &= \left(\prod_k e^{\frac{\Psi^k(\varepsilon_0)}{k(1-q^k)}}\right)\prod_i\left(1+\sum_k\frac{\Psi^k(\varepsilon_i)(1-P_i)}{1-q^k}+\dots\right)(1+\dots) \\ &= e^{\sum_{k>0}\frac{\Psi^k(\varepsilon_0)}{k(1-q^k)}}\left(1+\sum_i(1-P_i)\sum_k\frac{\Psi^k(\varepsilon_i)}{1-q^k}+\dots\right)\end{aligned}$$

Taking into account the c terms,

$$\sum_a c_a(1-P)^a = c_0+\sum_i c_i(1-P_i)+\dots.$$

Multiplying together, the relevant terms to the reconstruction operator are

$$\begin{aligned}&e^{\sum_{k>0}\frac{\Psi^k(\varepsilon_0)}{k(1-q^k)}}\left(1+\sum_i(1-P_i)\sum_k\frac{\Psi^k(\varepsilon_i)}{1-q^k}+\dots\right)\left(c_0+\sum_i c_i(1-P_i)+\dots\right) \\ &\quad= e^{\sum_{k>0}\frac{\Psi^k(\varepsilon_0)}{k(1-q^k)}}\left(c_0+\sum_i(1-P_i)\left(c_0\sum_{k>0}\frac{\Psi^k(\varepsilon_i)}{1-q^k}+c_i\right)\right)+\dots.\end{aligned}$$

Now we can read off I_0 and $I_{1,i}$ as they are coefficients of $\Phi_0=\mathscr{O}_X=1$ and $1-P_i$. First, $I_0(0,q,Q)=c_0e^{\sum_{k>0}\frac{\Psi^k(\varepsilon_0)}{k(1-q^k)}}$ and the coefficient of $(1-P_i)$ is therefore $I_{1,i}/I_0$. In total,

$$I_1/I_0=\sum_i(1-P_i)(I_{1,i}/I_0)=\sum_i(1-P_i)\left(\sum_{k>0}\frac{\Psi^k(\varepsilon_i)}{1-q^k}+\frac{c_i}{c_0}\right).$$

Since $I_0 = 1 + O(Q)$ and $\varepsilon_0 \in \Lambda_+$, then we can consider $c_0 = 1 + O(Q)$ and hence $1/c_0$ is well defined. Renaming c_i/c_0 to be c_i we obtain the formula in the proposition.

To see that the reconstruction parameters ε_i live over the integers, we can apply the analysis from Section 6.2. Let $\varepsilon_i = \sum_{d>0} n_d^i Q^d$. Expand I_1 and inspect the coefficients.

$$\begin{aligned} I_1 &= I_0 \sum_i (1 - P_i) \left(\sum_k \frac{\Psi^k(\varepsilon_i)}{1 - q^k} + c_i \right) \\ &= I_0 \sum_i (1 - P_i) \left(\sum_k \frac{\sum_d n_d^i Q^{kd}}{1 - q^k} + c_i \right) \\ &= \sum_i (1 - P_i) \left(\sum_k \frac{\sum_d n_d^i Q^{kd}}{1 - q^k} + c_i \right) + \dots \end{aligned}$$

Since c_i is a Laurent polynomial, it only contributes to the Laurent polynomial part of I_1. Therefore the n_d^i appear in the numerators of the rational parts which have roots of unity in the denominator. By Lemma 6.2.1, we can conclude that $n_d^i \in \mathbb{Z}$ as desired. □

6.4 Proof of Mirror Map Integrality

Per Proposition 6.3.1, as above let $\varepsilon_i = \sum_{d,r} n_d^i Q^d$ where $d \in \mathrm{Eff}(W, G, \theta)$ and $n_{d,}^i \in \mathbb{Z}$. Since $I_{1,i}$ is also a coefficient of Φ_i where $\mathrm{ch}(\Phi_i) = H_i$ then we can apply Corollary 5.2.1 to compute $I_{1,i}^{coh}/I_0^{coh}$. Since c_i is a Laurent polynomial in q, $(1-q)c_i$ vanishes in the limit.

$$\begin{aligned} \frac{I_{1,i}^{coh}(1,Q)}{I_0^{coh}(1,Q)} &= \lim_{q\to 1}(1-q)\frac{I_{1,i}(q,Q)}{I_0(q,Q)} = \lim_{q\to 1}(1-q)\sum_{k>0}\frac{\Psi^k(\varepsilon_i)}{1-q^k} \\ &= \sum_{k>0}\frac{\sum_d n_d^i \left(Q^{kd}\right)}{k} = \sum_d (-n_d^i) \sum_{k>0} -\frac{(Q^d)^k}{k} \\ &= -\sum_d n_d^i \ln(1 - Q^d) \end{aligned}$$

Then the mirror map becomes

$$Q_i \mapsto Q_i e^{\frac{I_i^{coh}(1,Q)}{I_0^{coh}(1,Q)}} = Q_i e^{-\sum_d n_d^i \ln(1-Q^d)} = Q_i \prod_d (1-Q^d)^{-n_d^i}$$

which has integer coefficients since n_d^i are integers. □

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