A NOTE ON THE SOLUTIONS OF TWO RECURSIVE RELATIONS *

by

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In this note we state some results concerning two recursive relations (1) and (2) below which has come up in different connection: in group testing [4] and [5], in ranking n numbers by binary comparisons [1], and in other related problems [3], [6]. We omit the proofs here since they are given elsewhere. Some of the proofs are already published in [3], [4] and [5]; other proofs are given in [6] and [1], where (1) and (2) appear frequently. This note was motivated by [2] and it is of interest to point out that the results in [2] are less specific than the result of Theorem 1 in this note.

Let $A(1) = M(1) = 0$ and for $n = 2,3,\ldots$ let $A(n)$ and $M(n)$ satisfy, respectively, the relations:

$$(1) \qquad A(n) = 1 + \min_{1\le x\le n-1}\left[\frac{x}{n}A(x) + \frac{n-x}{n}A(n-x)\right] \text{ and}$$

$$(2) \qquad M(n) = 1 + \min_{1\le x\le n-1}\left[\max(M(x),\, M(n-x))\right].$$

Let $\Delta_n^{(A)}$ be the set of all integers x such that there is no power of 2 strictly between x and $n - x$. Let $\{y\}$ denote the smallest integer not smaller than y. All logs in this note are to the base 2.

Lemma 1. An integer x belongs to $\Delta_n^{(A)}$ if and only if both x and $n-x$ are in the closed interval

$$[2^{\{\log \frac{n}{4}\}},\, 2^{\{\log \frac{n}{2}\}}].$$

Let $\Delta_n^{(M)}$ denote the set of all integers in the closed interval

$$[n - 2^{\{\log \frac{n}{2}\}}, 2^{\{\log \frac{n}{2}\}}] .$$

Lemma 2. $\{\frac{n}{2}\} \in \Delta_n^{(A)} \subset \Delta_n^{(M)}$.

Theorem 1. The set of integers that minimizes the RHS of (1) is $\Delta_n^{(A)}$ and

$$A(n) = \{\log n\} - \frac{2^{\{\log n\}} - n}{n} . \tag{3}$$

Theorem 2. The set of integers that minimize the RHS of (2) is $\Delta_n^{(M)}$ and

$$M(n) = \{A(n)\} = \{\log n\}. \tag{4}$$

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References

[1] Hadian, A. (1969). Optimality properties of various procedures for ranking n different numbers using only binary comparisons. Technical Report No. 117, Department of Statistics, Univ. of Minn. (See pp. 12-17 and 26-29.)

[2] Morris, Robert (1969). Some theorems on sorting. SIAM J. APPL. MATH. 17 No. 1, pp. 1-6.

[3] Sandelius, M. (1961). On an optimal search problem, Amer. Math. Monthly 68, 134-138.

[4] Sobel, Milton (1967). Optimal group testing. Proceedings of the Colloquium on Information Theory at Debrecen, Hungary. pp. 411-488 (see Section 4).

[5] Sobel, Milton (1968). Binomial and hypergeometric group-testing. Studia Scientiarum Mathematicarum Hungarica III, pp. 19-42 (see p. 24).

[6] Sobel, Milton (1968). On the ordering of the t best of n stems using binary comparisons. Technical Report No. 113, Department of Statistics, University of Minnesota (see pp. 32-34).