A HYPERBOLIC INVERSE PROBLEM ARISING IN THE EVOLUTION OF COMBUSTION AEROSOL

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A HYPERBOLIC INVERSE PROBLEM ARISING IN THE EVOLUTION OF COMBUSTION AEROSOL

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§1. **The physical problem.** In this paper we consider an aerosol model describing the evolution of soot in a combustion chamber. If we denote by $n(v,t)$ the density number of the soot particles of "size" v at time t then n satisfies the hyperbolic differential equation [2–4]

$$\frac{\partial n}{\partial t} = S - \frac{\partial}{\partial v}(\psi n) - Rn + \mathcal{C}[n] \tag{1.1}$$

where $S(v,t)$ is a source term, $\psi(v,t)$ is the rate of change of particle size, $R(v,t)$ is a loss rate due to settling, leakage or deposition, and $\mathcal{C}[n]$ is a nonlinear nonlocal functional representing the coagulation rate.

In this paper we make the usual assumption (see [4] [5] [11]) that the particles are spherical, and we shall measure the "size" of a particle by its volume v; the radius of a v size particle, is $(3v/4\pi)^{1/3}$. We shall also assume that

$$S = 0, \quad R = 0, \quad \psi = \frac{Bk(t)}{\rho}\, v^{2/3} \tag{1.2}$$

where $k(t)$ is the surface reaction, ρ is the density of the soot particle, and $Bv^{2/3}$ is the surface area of a ball with volume v, i.e.

$$B = 4\pi\left(\frac{3}{4\pi}\right)^{2/3}.$$

We shall always assume that $k(t) \geq 0$; this means that the surface reaction increases the size of the particle.

The coagulation occurs as size u particles and size v particles collide to form size $u+v$ particles. If we denote the smallest size particles by v_1 and the largest size particles by v_* then the coagulation rate is given by ([4] [5] [11])

$$\begin{aligned}\mathcal{C}[n](v,t) = \frac{1}{2}\int_{v_1}^{v-v_1} \beta(v-u,u)n(v-u,t)n(u,t)\,du \\ - n(v,t)\int_{v_1}^{v_*}\beta(u,v)n(u,t)\,du;\end{aligned} \tag{1.3}$$

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typically

$$v_1 = 10^{-21} cm^3 , \quad v_* = 10^{-16} \ cm^3 .$$

The formula (1.3) presupposes that at each time t the distribution of size v particles is homogeneous in space, for each v. If, further, the mean free path (between collisions) is large with respect to the size of the particles, then one can derive the formula (see [4])

$$\beta(u,v) = \gamma \left(\frac{1}{u} + \frac{1}{v} \right)^{1/2} \left(u^{1/3} + v^{1/3} \right)^2 , \quad \gamma \text{ constant.} \tag{1.4}$$

The final ingredient in the problem is the inception of new particles of smallest size v_1, as a by–product of the ongoing combustion. We shall denote the inception rate by $I(t)$; this translates into the boundary condition

$$n(v_1, t) = \frac{\rho \ I(t)}{B v_1^{2/3} k(t)} . \tag{1.5}$$

For the initial condition we take, for simplicity,

$$n(v, 0) = 0. \tag{1.6}$$

We shall refer to the system (1.1)–(1.6) as the *direct problem.* We shall actually be more interested in the inverse problem: find I and k in terms of the first two moments on $n(v,t)$. More precisely, if we shine a laser beam through the aerosol we can measure both the average absorption $f(t)$ and the average scattering $g(t)$. These quantities are given by (see [2])

$$f(t) = \int_{v_1}^{v_*} v n(v,t) \ dt, \tag{1.7}$$

$$g(t) = \int_{v_1}^{v_*} v^2 n(v,t) \ dt. \tag{1.8}$$

The problem is then:

(1.9) given f and g, determine I and k.

This problem was initiated and studied by Harris, Weiner and Ashcraft [1] who have also carried out the optical measurements mentioned above; the problem was further studied by S.P. Marin and S.J. Harris (private notes) in case of no coagulation (or very small coagulation). Before describing our results let us normalize the problem by introducing the following scaling (as in [11]):

$$v = v_1\hat{v},\ t = t_c\hat{t},\ n(v,t) = n_c\hat{n}(\hat{v},\hat{t}),\ k(t) = k_c\hat{k}(\hat{t}),$$
$$I(t) = I_c\widehat{I}(\hat{t}),\ \mathcal{C}[n](v,t) = C_c\widehat{\mathcal{C}}[\hat{n}](\hat{v},\hat{t})$$

with appropriate constants t_c, n_c, k_c, I_c, C_c. Dropping the superscript "^" we arrive at a similar system

$$\begin{aligned} &\frac{\partial n}{\partial t} + 3k(t)\frac{\partial}{\partial x}(x^{2/3}n) = \mathcal{C}[n],\ 1 < v < \tilde{v}, \quad t > 0, \\ (1.10) \qquad &n(1,t) = \frac{I(t)}{3k(t)}\ ,\ t > 0, \\ &n(v,0) = 0, \quad 1 < v < \infty, \end{aligned}$$

and γ in (1.4) is replaced by another constant γ; k_c and I_c are chosen to have the order of magnitude of k and I, and typically

$$\tilde{v} \equiv \frac{v_*}{v_1} = 10^5\ , \quad \gamma \in [10^{-4}, 10^3].$$

In the sequel we shall not need the special form of the function $\beta(u,v)$ as defined in (1.4) for $1 < v < \tilde{v}$; we shall assume however that for $u \geq 1, v \geq 1$.

$$(1.11) \qquad \beta(u,v) = \beta(v,u),\ 0 \leq \beta \leq M, \qquad M \quad \text{constant}.$$

Since $\tilde{v}$ is a "large" number, it is natural to consider both the direct and inverse problems for $\tilde{v} = \infty$.

In §§2, 3 we consider the direct problem for the case $\tilde{v} = \infty$, making the assumptions in (1.11). We shall prove that this problem has a unique solution n and we shall derive some asymptotic bounds $n(v,t)$ as $v \to \infty$. (Existence and uniqueness for the case $\tilde{v} < \infty$ can similarly be established). In §4 we establish Lipschitz continuity of $n(v,t)$ in v.

In §§5–7 we consider the inverse problem. We assume that $f(t)$ and $g(t)$ are in $C^1[0,\infty)$ and satisfy at $t = 0$ conditions which must necessarily hold if $I(0) > 0, k(0) > 0$. We then reduce the system (1.7), (1.8) to a system of first order nonlinear functional integral equations in the pair (J,k) where $J(t) = (I(t) - I(0))/t$, and prove that the system has a unique Lipschitz solution as long as the resulting I and k satisfy: $I \geq 0, \quad k > 0$.

Finally, §8 we consider the direct problem in case $\beta(u,v)$ is unbounded, and prove existence of a global solution.

There has been quite a bit of work on the transport equation

$$\begin{aligned} (1.12) \qquad &\frac{\partial n}{\partial t} = \mathcal{C}[n]\ , \qquad 1 < v < \infty, t > 0, \\ &n(v,0) = n_0(v)\ ,\ 1 < v < \infty \end{aligned}$$

and its discrete analog

$$\begin{aligned}\frac{\partial n_i}{\partial t} &= \frac{1}{2}\sum_{j=1}^{i-1} \beta_{j,i-j}n_i n_{i-j} - n_i \sum_{j=1}^{\infty}\beta_{ij}n_j,\\ &\qquad n_i(0) = n_{0i}.\end{aligned} \tag{1.13}$$

Melzak [10] proved global existence and uniqueness for (1.12) provided β satisfies the assumptions in (1.11); his proof, which uses power series expansion in t, does not extend to our case. Mcleod [9] established existence and uniqueness for (1.12) in the special case $\beta(u,v) = uv$. Problem (1.13) has been studied under variety of assumptions on the β_{ij}. Mcleod [7] [8] proved local existence and uniqueness provided $n_{0i} = 0$ if $i > 1$ and $\beta_{ij} \le Cij$. White [13] has proved global existence in case $\beta_{ij} \le C(i+j)$, and Leyvraz and Tschudi [6] proved global existence in case $\beta_{ij} \le c_i c_j$ with $c_n = o(n)$. The occurrence of gelation and mass–loss in case $\beta_{ij} = r_i r_j$ with $r_j \ge Cj$ $(C > 0)$ was established by Hendriks, Ernst and Ziff [3]; see also Mcleod [9]. The transport equation with terms accounting for fragmentation was studied by Melzak [10] and Spouge [12]. We finally mention the discrete coagulation–fragmentation equation of Becker–Döring where, in the coagulation coefficients, $\beta_{1j} = \beta_{j1} > 0$ and all other β_{ij} vanish; a detailed study of properties of the solutions is given in Ball, Carr and Penrose [1].

When $\{\beta_{ij}\}$ is unbounded, the only known uniqueness result for (1.13) is that due to Mcleod [7]; it asserts uniqueness in time $\le \dfrac{1}{2e}$ provided $\beta_{ij} \le Cij$ and $n_{0i} = 0$ if $i > 1$. This result can be extended to (1.10) with $\tilde{v} = \infty$ and β unbounded; see §8.

§2. The direct problem; a priori estimates. We shall denote independent space variables by x, y (rather than u, v) and the solution n by u.

Consider the hyperbolic problem:

$$\frac{\partial u}{\partial t} + 3k(t)\frac{\partial}{\partial x}\,(x^{2/3}u) = \mathcal{C}[u](x,t) \qquad \text{if} \qquad 1 < x < \infty, \quad t > 0, \tag{2.1}$$

$$u(1,t) = h(t) \qquad \text{if} \qquad t > 0, \tag{2.2}$$

$$u(x,0) = u_0(x) \qquad \text{if} \qquad 1 < x < \infty \tag{2.3}$$

where

$$\begin{aligned}\mathcal{C}[u](x,t) = \frac{1}{2}&\int_1^{x-1} \beta(x-y,y)u(x-y,t)u(y,t)\,dy\\ &-u(x,t)\int_1^{\infty} \beta(x,y)u(y,t)\,dy\end{aligned} \tag{2.4}$$

and $\beta(x,y)$ is a function satisfying:

$$(2.5)\qquad \begin{array}{rl} \text{(i)} & \beta(x,y) \geq 0 \text{ for } x \geq 1,\ y \geq 1, \\ \text{(ii)} & \beta(x,y) \text{ uniformly continuous for } x \geq 1,\ y \geq 1, \\ \text{(iii)} & \beta(x,y) = \beta(y,x) \text{ for } x \geq 1,\ y \geq 1, \\ \text{(iv)} & \beta(x,y) \leq M \text{ for } x \geq 1,\ y \geq 1;\ M \text{ positive constant.} \end{array}$$

It is to be understood that the first term on the right-hand side of (2.4) is to be dropped out when $1 \leq x \leq 2$.

We assume that

$$(2.6)\qquad \begin{array}{l} h(t),\ k(t) \text{ are continuous for } t \geq 0, \text{ and} \\ u_0(x) \text{ is continuous for } x \geq 1, \end{array}$$

$$(2.7)\qquad h(t) \geq 0,\ k(t) \geq 0 \text{ for } t \geq 0,\ u_0(x) \geq 0 \text{ for } x \geq 1.$$

Note that we do not assume that $h(0) = u_0(0)$.

We introduce the characteristic curves $y = \phi(x,t,s)$ for (1.1); the function $s \to \phi$ satisfies:

$$\begin{aligned} \frac{d\phi}{ds} &= 3k(s)\phi^{2/3} \quad \text{for all } s \geq 0, \\ \phi &= x \quad \text{if } s = t. \end{aligned}$$

We can actually solve ϕ explicitly:

$$(2.8)\qquad \phi(x,t,s) = \left\{ x^{1/3} - \int_s^t k(r)\, dr \right\}^3 .$$

Denote by $\tau(x,t)$ the smallest value of s such that the curve $y = \phi(x,t,s')$ with $s' \geq s$ lies in the domain

$$Q \equiv \{(x,t);\ 1 \leq x < \infty, \quad t \geq 0\}.$$

Thus

$$\text{if } x^{1/3} - \int_0^t k(r)dr \geq 1 \text{ then } \tau(x,t) = 0, \text{ whereas}$$

$$\text{if } x^{1/3} - \int_0^t k(r)dr < 1 \text{ then } \tau \text{ satisfies: } x^{1/3} - \int_{\tau(x,t)}^t k(r)dr = 1.$$

Notice that $\tau(x,t)$ is continuous in (x,t) and is decreasing in x. Also, $\phi(x,t,s) \geq 1$ if $\tau(x,t) \leq s \leq t$ and $\phi(x,t,\tau) = 1$ if $\tau > 0$.

Integrating (2.1)–(2.3) along characteristics we obtain the following equation:

$$u(x,t) = \int_{\tau(x,t)}^{t} x^{-2/3}\phi^{2/3}C[u](\phi,s)ds + u_*(x,t) \tag{2.9}$$

where $\phi = \phi(x,t,s)$ and

$$u_*(x,t) = \begin{cases} h(\tau(x,t))x^{-2/3} & \text{if} \quad \tau(x,t) > 0 \\ u_0(\phi(x,t,0))x^{-2/3}\phi(x,t,0)^{2/3} & \text{if} \quad \tau(x,t) = 0, \end{cases} \tag{2.10}$$

or

$$u_*(x,t) = \begin{cases} h(\tau(x,t))x^{-2/3} \quad \text{if} \quad \tau(x,t) > 0 \\ u_0((x^{1/3} - \int_0^t k(r)dr)^3)x^{-2/3}(x^{1/3} - \int_0^t k(r)dr)^2 \quad \text{if} \quad \tau(x,t) = 0. \end{cases} \tag{2.11}$$

Let

$$Q_1 = \{(x,t); x > 1, t > 0, \quad x < (1 + \int_0^t k(r)dr)^3\},$$

$$Q_2 = \{(x,t); x > 1,\ t > 0, \quad x > (1 + \int_0^t k(r)dr)^3\}.$$

If $h(0) = u_0(0)$ then $u_*(x,t)$ is continuous in Q; if however $h(0) \neq u_0(0)$ then $u_*(x,t)$ is uniformly continuous in bounded subsets of Q_1 and of Q_2, but has a jump discontinuity across $\partial Q_1 \cap \partial Q_2$.

DEFINITION 2.1. By a solution of (2.1)–(2.3) we mean a function u such that u is uniformly continuous in bounded subsets of Q_1 and of Q_2, and (2.9) is satisfied in Q_1 and in Q_2.

It follows that u is differentiable along all characteristic curves in Q_1 and in Q_2; the intersection $\partial Q_1 \cap \partial Q_2$ consists of course of the (single) characteristic curve $x = (1 + \int_0^t k(r)dr)^3$, $t \geq 0$.

LEMMA 2.1. *If u is a solution of (2.1)–(2.3) for $t < T$, satisfying* $\sup_{t\le T}\int_1^\infty |u(x,t)|dx < \infty$, *then*

(a) $u \ge 0$;

(b) $\sup_{t\le T}\int_1^\infty u(x,t)\,dx \le 3\int_0^T k(s)h(s)\,ds + \int_1^\infty u_0(y)dy \equiv \gamma_T$;

(c) *if* $N_l(x,t) \equiv \sup_{\substack{1\le y\le x\\ 0\le r\le t}} [y^l u(y,r)] < \infty$ *for some real number* $l \ge 0$, *then*

$$N_l(x,T) \le e^{(3lK_T+2^l M\gamma_T)T}\left\{\sup_{s\le T} h(s) + \sup_{1\le y\le x} y^l u_0(y)\right\}$$

where $K_T = \sup_{t\le T} k(t)$.

Proof. (a) Set $u(s) = u(\phi(x,t,s),s)\phi(x,t,s)^{2/3}$. Then

$$\frac{du}{ds} = \mathcal{C}[u](\phi(x,t,s),s)\phi(x,t,s)^{2/3}. \tag{2.12}$$

If $x \le 2$ then the first term in $\mathcal{C}[n]$ drops out and we get

$$\frac{du}{ds} = -\rho(s)u(s)$$

where

$$\rho(s) = \int_1^\infty \beta\big(y,\phi(x,t,s)\big)u(y,s)\,dy.$$

Hence

$$u(s) = u(\tau)e^{-\int_\tau^s \rho(r)\,dr}$$

and $u(\tau) \ge 0$ since $h \ge 0$, $u_0 \ge 0$. It follows that $u(x,t) \ge 0$ if $x \le 2$.

Next, if $2 \le x \le 3$ then we can write

$$\frac{du}{ds} = -\rho(s)u(s) + \sigma(s) \tag{2.13}$$

where $\rho(s)$ is defined as before and

$$\sigma(s) = \frac{1}{2}\,\phi^{2/3}\int_1^{\phi-1} \beta(\phi-y,y)u(\phi-y,s)u(y,s)\,dy.$$

But since $u(x,t) \geq 0$ if $x \leq 2$, the function $\sigma(s)$ is ≥ 0 and we can now deduce from (2.13) that $u(s) \geq 0$, i.e., $u(x,t) \geq 0$ if $2 \leq x \leq 3$.

Continuing in the same way step–by–step, we conclude that $u \geq 0$ for $3 \leq x \leq 4$, $4 \leq x \leq 5$, etc.

(b) We begin by showing that

$$\int_1^\infty x^{-2/3} \int_{\tau(x,t)}^t \phi(x,t,s)^{2/3}\mathcal{C}[u](\phi(x,t,s)\; ds\; dx \leq 0. \tag{2.14}$$

Denote the left–hand side of (2.14) by $N(t)$. Then

$$N(t) = \int_0^t \int_{(1+\int_s^t k(r)dr)^3}^\infty x^{-2/3}\phi^{2/3}\; \mathcal{C}[u](\phi,s)\; dx\; ds$$

where $\phi = \phi(x,t,s)$. Changing variables $\phi(x,t,s) = z$ $\;(dz/dx = \phi^{2/3}x^{-2/3})$, we get

$$\begin{aligned} N(t) &= \int_0^t \int_1^\infty \mathcal{C}[u](z;s)\; dz\; ds \\ &= \int_0^t \Big\{ \int_2^\infty \frac{1}{2} \int_1^{z-1} \beta(y,z-y)u(z-y,s)u(y,s)\; dy\; dz \\ &\qquad - \int_1^\infty u(z,s) \int_1^\infty \beta(y,z)u(y,s)\; dy\; dz \Big\}\; ds. \end{aligned}$$

Changing the order of integration in the first integral on the right–hand side we get

$$\begin{aligned} N(t) = \int_0^t \Big\{ &\frac{1}{2} \int_1^\infty \int_{1+y}^\infty \beta(y,z-y)u(z-y,s)u(y,s)\; dz\; dy \\ &- \int_1^\infty \int_1^\infty \beta(y,z)u(y,s)u(z,s)\; dy\; dz \Big\}. \end{aligned}$$

Substituting $z - y = \zeta$ in the first integral we find that

$$N(t) = \int_0^t \left\{ \frac{1}{2} \int_1^\infty \int_1^\infty \beta(y,\zeta)u(\zeta,s)u(y,s)d\zeta\ dy \right.$$

$$\left. - \int_1^\infty \int_1^\infty \beta(y,z)u(y,s)u(z,s)\ dy\ dz \right\} ds \tag{2.15}$$

$$= -\frac{1}{2} \int_0^t \int_1^\infty \int_1^\infty \beta(y,z)u(y,s)u(z,s)\ dy\ dz\ ds \le 0,$$

since $u \ge 0$, and (2.14) is thus proved.

Integrating (2.9) with respect to x and using (2.14), we deduce that

$$\int_1^\infty u(x,t)\ dx \le \int_1^\infty u_*(x,t)\ dx,$$

and easy computation shows that the right–hand side is bounded by γ_T as asserted in (b).

(c) We compute formally

$$\frac{\partial}{\partial t}\ (x^l u) + 3kx^{2/3}\frac{\partial}{\partial x}\ (x^l u) = x^l \left[\frac{\partial u}{\partial t}\ + 3k\ \frac{\partial}{\partial x}\ (x^{2/3}u) \right]$$

$$+ x^{2/3}u \cdot \left(l - \frac{2}{3} \right) 3kx^{l-1}.$$

Consequently

$$\frac{\partial}{\partial t}\ (x^l u) + 3kx^{2/3}\frac{\partial}{\partial x}\ (x^l u) \le x^l \mathcal{C}[u](x,t) + 3k(t)lx^l u. \tag{2.16}$$

One can easily justify (2.16) in the sense of differentiation along characteristics; this is actually the way we shall be using (2.16) in the sequel.

If $x \ge 2$ then

$$x^l \mathcal{C}[u](x,t) \le \frac{1}{2}\ x^l \int_1^{x-1} \beta(y, x-y)u(y,t)u(x-y,t)\ dy$$

$$= \frac{1}{2}\ x^l \int_1^{x/2} \cdots + \frac{1}{2}\ x^l \int_{x/2}^{x-1} \cdots$$

$$\le \frac{1}{2}\ M2^l \left\{ \int_1^{x/2} (x-y)^l u(x-y,t)u(y,t)\ dy \right.$$

$$\left. + \int_{x/2}^{x-1} y^l u(y,t)u(x-y,t)dy \right\}$$

and, using (b),

$$x^l \mathcal{C}[u](x,t) \le M 2^l \gamma_T N_l(x,t). \tag{2.17}$$

Since $x^l\mathcal{C}[u](x,t) \le 0$ for $1 \le x \le 2$ it follows that (2.17) holds for all $x \ge 1$.

Using (2.17) in (2.16) we get

$$\frac{d}{ds}\,(\phi^l u(\phi,s)) \le (M2^l\gamma_T + 3lK_T)N_l(\phi,s)$$

where $\phi = \phi(x,t,s)$; also $N_l(\phi,s) \le N_l(x,s)$ if $s \le t$ since $\phi \le x$. It follows that

$$x^l u(x,t) \le \int_\tau^t (M2^l\gamma_T + 3lK_T)N_l(x,s)\;ds + \tilde{u}_l(x,t)$$

where

$$\tilde{u}_l(x,t) = \begin{cases} h\big(\tau(x,t)\big) & \text{if } \ \tau(x,t) > 0 \\ u_0\big(\phi(x,t,0)\big)\phi(x,t,0)^l & \text{if } \ \tau(x,t) = 0; \end{cases}$$

consequently

$$\begin{aligned} N_l(x,t) \le & \int_0^t (M2^l\gamma_T + 3lK_T)N(x,s)\;ds \\ & + \sup_{s\le t}\; h(s) + \sup_{1\le y\le x}\; y^l u_0(y), \end{aligned}$$

and (c) follows.

§3. Existence and uniqueness.

THEOREM 3.1. *Assume, in addition to (2.5)–(2.7), that* $u_0 \in L^1$. *Then there exists a unique solution* u *of (2.1)–(2.3) such that* $u \ge 0$ *and*

$$\sup_{t\le T}\int_1^\infty u(x,t)\;dx \le \gamma_T \qquad \forall\, T < \infty. \tag{3.1}$$

Recall that γ_T was defined in Lemma 2.1(b).

Proof. For any $\eta > 0$, introduce the Banach space X_η of functions $w(x,t)$ measurable in $\widetilde{Q}_\eta = \{(x,t);\ 1 \le x < \infty,\ 0 \le t \le \eta\}$ and having finite norm

$$\|w\| = \operatorname*{ess\,sup}_{0\le t\le \eta} \int_1^\infty |w(x,t)|\;dx.$$

Denote by X_η^0 the ball $\{w \in X_\eta,\ \|w\| \le 1+\gamma_\eta\}$ where γ_T is defined in Lemma 2.1 (b) for any $T > 0$.

For any $w \in X_\eta^0$ we define

$$(Sw)(x,t) = \int_{\tau(x,t)}^{t} x^{-2/3}\phi^{2/3}\mathcal{C}[w](\phi,s)\,ds + u_*(x,t) \tag{3.2}$$

where $\phi = \phi(x,t,s)$ and u_* is defined by (2.11).

We shall prove that if η is sufficiently small then S maps X_η^0 into itself and is a contraction

We begin by estimating

$$\begin{aligned}\int_1^\infty |Sw(x,t)|\,dx &\le \int_1^\infty\int_\tau^t x^{-2/3}\phi^{2/3}|\mathcal{C}[w](\phi,s)|\,ds\,dx + \int_1^\infty |u_*(x,t)|\,dx\\ &\le \overline{N}(t) + \gamma_\eta\end{aligned}$$

where $\overline{N}(t)$ is defined as $N(t)$ in Lemma 2.1 (b) except that $\mathcal{C}[u]$ is replaced by $|\mathcal{C}[w]|$. Proceeding as in the proof of Lemma 2.1 (b) we obtain (cf. (2.15))

$$\begin{aligned}\overline{N}(t) &\le \int_0^t \Big\{\frac{1}{2}\int_1^\infty\int_1^\infty \beta(y,\zeta)|w(\zeta,s)|\;|w(y,s)|d\zeta\;dy\\ &\quad + \int_1^\infty\int_1^\infty \beta(y,z)\;|w(y,s)|\;|w(z,s)|\;dy\;dz\Big\}ds\\ &\le 2M\int_0^t\Big\{\int_1^\infty\int_1^\infty |w(y,s)|\;|w(z,s)|dy\;dz\Big\}\;ds\\ &\le 2M\eta(1+\gamma_\eta)^2 < 1 \qquad \text{if} \qquad \eta < \frac{1}{4M(1+\gamma_\eta)^2}\;.\end{aligned}$$

For such a choice of η we then have $\|Sw\| < 1+\gamma_\eta$, i.e., S maps X_η^0 into itself.

To prove that S is a contraction, take any $w, \tilde{w}$ in X_η^0 and compute

$$J(t) \equiv \int_1^\infty \big|(Sw - S\tilde{w})(x,t)\big|\;dx = \int_1^\infty\int_\tau^t x^{-2/3}\phi^{2/3}|\mathcal{C}[w](\phi,s) - \mathcal{C}[\tilde{w}](\phi,s)|\;dx\;ds.$$

By a change of variables $\phi(x,t,s) = z$, we get

$$\begin{aligned} J(t) &\le \int_0^t \int_1^\infty |\mathcal{C}[w](z,s) - \mathcal{C}[\tilde{w}](z,s)| \, dz \, ds \\ &\le M \int_0^t \int_1^\infty \Big\{ \int_1^{z-1} |w(z-y,s) - \tilde{w}(z-y,s)| \, |w(y,s)| \, dy \\ &\quad + \int_1^{z-1} |\tilde{w}(z-y,s)| \, |w(y,s) - \tilde{w}(y,s)| \, dy \\ &\quad + |\tilde{w}(z,s) - w(z,s)| \int_1^\infty |\tilde{w}(y,s)| \, dy \\ &\quad + |w(z,s)| \int_1^\infty |\tilde{w}(y,s) - w(y,s)| \, dy \Big\} \, dz \, ds. \end{aligned}$$

It follows that

$$J(t) \le 4M\eta(1+\gamma_\eta) \, \|w - \tilde{w}\| \le \theta \|w - \tilde{w}\| \qquad (\theta < 1)$$

by the choice of η made above. Consequently S is a contraction in X_η^0.

By the Contraction Mapping Theorem we now conclude that S has a unique fixed point in X_η^0. Clearly, if u is a solution of (2.1)–(2.3) for $t \le \eta$ then it is necessarily a fixed point of S.

In order to complete the proof of Theorem 3.1 for $t \le \eta$, it remains to show that the fixed point of S, which we shall designate by $u(x,t)$, is uniformly continuous in bounded subsets of Q_1 and of Q_2.

Set $Q_1^j = Q_1 \cap \{x \le j\} \cap \{t \le \eta\}$, $\quad Q_2^j = Q_2 \cap \{x \le j\} \cap \{t \le \eta\}$. We shall prove that for each $j = 2, 3, \ldots,$

$|u(x,t) - u(x_0,t)| \le \sigma_j(|x - x_0|)$ if both x and x_0
belong to Q_1^j or to Q_2^j, where σ_j is a modulus of continuity.

Consider first the case where $1 \le x, x_0 \le 2$ and let

$$N(s) = u\big(\phi(x,t,s),s\big) - u\big(\phi(x_0,t,s),s\big).$$

Suppose for definiteness that $\tau(x,t) > \tau(x_0,t)$, and set $\phi(s) = \phi(x,t,s)$, $\quad \phi_0(s) = \phi(x_0,t,s)$.

Observe that since $u = Su$, for $1 \le x \le 2$, we have along characteristics, $du/ds = fu$ where f is a bounded function. This implies that there is a version of u such that along any characteristic,

$$|u(\phi(s), s) - u(\phi(s'), s')| \le C|s - s'| \tag{3.5}$$

and such that (2.9) holds everywhere.

Therefore, if $1 \le x, x_0 \le 2$ then we can write

$$\begin{aligned}
N(t) = u(x,t) - u(x_0,t) = &- \int_{\tau(x,t)}^{t} N(s)\phi^{2/3}x^{-2/3} \int_1^{\infty} \beta(\phi, y)u(y,s)\, dy\, ds \\
&+ \int_{\tau(x,t)}^{t} (\phi_0^{2/3}x_0^{-2/3} - \phi^{2/3}x^{-2/3})u(x_0,s) \int_1^{\infty} \beta(\phi,y)u(y,s) dy\, ds \\
&+ \int_{\tau(x,t)}^{t} \phi_0^{2/3}x_0^{-2/3}u(\phi_0,s) \int_1^{\infty} (\beta(\phi_0,y) - \beta(\phi,y))u(y,s)\, dy\, ds \\
&+ \int_{\tau(x_0,t)}^{\tau(x,t)} \phi_0^{2/3}x_0^{-2/3}u(\phi_0,s) \int_1^{\infty} \beta(y,\phi_0)u(y,s)\, dy\, ds \\
&+ \Big[u_*(x,t) - u_*(x_0,t)\Big].
\end{aligned}$$

The second, third and fourth terms on the right–hand side of (3.4) are bounded by $\sigma(|x - x_0|)$, where σ is a modulus of continuity; the last term is also similarly bounded provided (x,t) and (x_0,t) both belong to either Q_1^2 or to Q_2^2. Applying Gronwall's inequality we conclude that

$$|u(x,t) - u(x_0,t)| \le \sigma_2(|x - x_0|). \tag{3.6}$$

Combining (3.5), (3.6) we deduce that u has uniformly continuous versions in Q_1^2 and in Q_2^2, i.e., u is uniformly continuous in Q_1^2 and in Q_2^2.

To prove the continuity in Q_1^3, Q_2^3, we proceed as before. However, if $2 < x, x_0 \le 3$, then there appear additional terms on the right–hand side of (3.4), namely,

$$L(t) \equiv \int_{\tau(x,t)}^{t} \phi_0^{2/3}x_0^{-2/3} \int_1^{x_0 - 1} [\beta(y,\phi_0 - y)u(\phi_0 - y, s) - \beta(y, \phi - y)u(\phi - y, s)]u(y,s)\, dy\, ds$$

plus terms involving

$$\int_{x_0-1}^{x-1} \quad \text{or} \quad \int_{\tau(x_0,t)}^{\tau(x,t)} .$$

The last two terms are clearly bounded by $\sigma(|x-x_0|)$, where σ is a modulus of continuity. As for $L(t)$, as long as the points (ϕ_0-y,s) and $(\phi-y,s)$ both belong to either Q_1^2 or to Q_2^2 we can estimate the integrand by $\sigma(|x-x_0|)$ where σ is a modulus of continuity. On the other hand, for each s, the set of points y such that $\phi_0-y\in Q_l^2$, $\phi-y\notin Q_l^2$ for $l=1$ or for $l=2$ has measure $\le C|\phi-\phi_0|$. It follows that also $L(t)$ is bounded by $\sigma(|x-x_0|)$, with a suitable modulus of continuity σ.

We can therefore proceed as before to establish (3.5) in $Q_1^3\backslash Q_1^2$ and in $Q_2^3\backslash Q_2^2$ with another modulus of continuity σ_3. Since (3.6) remains valid as before, the uniform continuity of $u(x,t)$ in Q_1^3 and in Q_2^3 follows.

Similarly we continue step–by–step, in the same way, to establish the uniform continuity of u in Q_1^j and Q_2^j for $j=4,5,\ldots$.

We have established so far the existence and uniqueness of a solution of (2.1)–(2.3) (in the sense of Definition 2.1) which belongs to X_η. By Lemma 2.1, $u\ge 0$. The choice of η depended only on an upper bound on γ_η. In view of the a priori estimate (b) of Lemma 2.1, we can extend the solution uniquely step-by-step for all $t>0$.

From Theorem 3.1 and Lemma 2.1 (c) we obtain

COROLLARY 3.2. *If* $x^l u_0(x)\le C_0<\infty$ *for some* $l>0$ *then, for any* $T>0$,

$$x^l u(x,t)\le C_{T,l} \quad \text{for all } 1\le x<\infty,\ 0\le t\le T \tag{3.7}$$

where $C_{T,l}$ *is a constant; if in particular,* u_0 *has compact support, then (3.7) holds for all* $l>0$.

The next result is concerned with the total mass

$$m(t)=\int_1^\infty xu(x,t)\,dx. \tag{3.8}$$

THEOREM 3.3. *(a) If* $m(t)<\infty$ *for all* $0\le t\le T$, *then*

$$\begin{aligned} m(t) &\le 3\int_0^t \Big(1+\int_s^t k(r)dr\Big)^3 k(s)h(s)\,ds \\ &+\int_1^\infty \Big(x^{1/3}+\int_0^t k(r)\,dr\Big)^3 u_0(x)\,dx \quad \text{for } 0\le t\le T; \end{aligned} \tag{3.9}$$

(b) if $m(t) < \infty$ for all $0 \le t \le T$, then

$$\begin{aligned} m(t) &= 3\int_0^t k(s)h(s)ds + \int_1^\infty xu_0(x)dx \\ &\quad + \int_0^t 3k(s)\int_1^\infty x^{2/3}u(x,s)dxds \quad \text{for } 0 \le t \le T. \end{aligned} \tag{3.10}$$

Proof. Multiplying (2.9) by x and integrating over x we obtain

$$\int_1^\infty xu(x,t)dx = \int_1^\infty xu_*(x,t)\,dx + J \tag{3.11}$$

where

$$\begin{aligned} J &= \int_1^\infty x\,dx \int_{\tau(x,t)}^t x^{-2/3}\phi^{2/3}\mathcal{C}[u](\phi,s)\,ds \\ &= \int_0^t\int_1^\infty \Big(x^{1/3} + \int_s^t k\Big)^3 \mathcal{C}[u](x,s)dx\,ds \end{aligned} \tag{3.12}$$

by changing variables $\phi(x,t,s) = z$ and then replacing the dummy variable z by x. Also

$$\begin{aligned} \int_1^\infty xu_*(x,t)dx &= \int_1^{(1+\int_0^t k)^3} x^{1/3}h\big(\tau(x,t)\big)dx + \int_{(1+\int_0^t k)^3}^\infty x^{1/3}u_0\big((x^{1/3} - \int_0^t k)^3\big)(x^{1/3} - \int_0^t k)^2dx \\ &= 3\int_0^t (1 + \int_s^t k)^3 k(s)h(s)\,ds + \int_1^\infty (x^{1/3} + \int_0^t k)^3 u_0(x)\,dx \end{aligned}$$

by a change of variables.

Therefore in order to complete the proof (3.9) it is clearly sufficient to show that for

$0 \le \alpha \le 1$

$$J_\alpha(s) \equiv \int_1^\infty x^\alpha \mathcal{C}[u](x,s)\,dx$$

$$= \frac{1}{2}\int_2^\infty dx \int_1^{x-1} x^\alpha \beta(x-y,y)u(x-y,s)u(y,s)\,dy \tag{3.13}$$

$$- \int_1^\infty dx \int_1^\infty x^\alpha \beta(x,y)u(x,s)u(y,s)\,dy \le 0.$$

In the first integral we change the order of integration, obtaining

$$\frac{1}{2}\int_1^\infty dy \int_{1+y}^\infty x^\alpha \beta(x-y,y)u(x-y,s)u(y,s)\,dx.$$

If we now substitute in the inner integral $x = y + z$, we obtain

$$\frac{1}{2}\int_1^\infty dy \int_1^\infty (y+z)^\alpha \beta(z,y)u(z,s)u(y,s)\,dz$$

$$\le \int_1^\infty \int_1^\infty y^\alpha \beta(z,y)u(z,s)u(y,s)\,dz\,dy$$

by symmetry of $\beta(\cdot,\cdot)$. Substituting this into (3.13), we see that $J_\alpha(s) \le 0$.

To prove (3.10), formally, we multiply (2.1) by x and integrate in x to obtain

$$m'(t) - 3k(t)h(t) - 3k(t)\int_1^\infty x^{2/3}u(x,t)dx = 0\ ; \tag{3.14}$$

we have used here the fact, which follows from the proof of (3.13), that

$$\int_1^\infty x\mathcal{C}[u](x,t)dx = 0. \tag{3.15}$$

Integrating (3.14), the assertion (3.10) follows. In order to prove (3.10) rigorously we use the representation (2.9); multiplying both sides by x and integrating over x, we can arrive at (3.10) after some manipulations which involve another application of (2.9) and some integrations by parts. An alternative rigorous proof is as follows: If $\beta(x,y)$ and the initial and boundary data are Lipschitz continuous $\big($and $u_0(0) = h(0)\big)$ then the solution u is Lipschitz continuous (as is proved in §4) and then the above formal proof of (3.10) is valid. By approximation we can then establish the assertion (3.10) for general β and u_0, h.

From Theorem 3.3 (b) we get:

COROLLARY 3.4. *The function $m(t)$ is nondecreasing and, consequently, if $u(x,t_0) \equiv 0$ for some $t_0 > 0$ then $u(x,t) \equiv 0$ for $0 \le t \le t_0$.*

§4. **Lipschitz continuity.** In this section we establish Lipschitz continuity of the solution $u(x,t)$ as a function of x. We shall assume, in addition to (2.5)–(2.7), that

$$|\beta(x,y) - \beta(x',y')| \le M_*(|x-x'| + |y-y'|) \qquad \forall\, x,y,x',y' \text{ in } [1,\infty), \tag{4.1}$$

$$\begin{aligned}
&h(t) \text{ and } k(t) \text{ are Lipschitz continuous for } 0 \le t < \infty,\\
&\text{so that for any } T > 0 \text{ there is a constant } K_T < \infty \text{ such that}\\
&h(t) + k(t) \le K_T \quad \text{for all} \quad 0 \le t \le T,\\
&|h'(t)| + |k'(t)| \le K_T \quad \text{for} \quad a.a. \quad t \in [0,T],\\
&k(t) \ge k_T > 0 \quad \text{for all} \ \ 0 \le t \le T,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&u_0(x) \text{ is Lipschitz continuous in } x, \text{ and there}\\
&\text{exist positive constants } l, U_0 \text{ such that}\\
&x^{l+1} u_0(x) \le U_0 \quad \text{for all} \quad x \ge 1,\\
&x^l |u_0'(x)| \le U_0 \quad \text{for} \quad a.a. \quad x \ge 1.
\end{aligned} \tag{4.3}$$

THEOREM 4.1. *Under the assumptions (4.1)–(4.3), for any $T > 0$ there exists a positive constant C_T depending only on T, K_T, k_T, U_0, M and M_* such that the solution u of (2.1)–(2.3) satisfies:*

$$|u(x+\alpha,t) - u(x,t)| \le \begin{cases} C_T|\alpha|x^{-l} & \text{if} \quad |\alpha| \le \frac{x}{2} \\ C_T|\alpha| & \text{if} \quad |\alpha| \ge \frac{x}{2} \end{cases} \tag{4.4}$$

provided both points $(x+\alpha,t)$ and (x,t) belong to either Q_1 or Q_2 and $t \le T$.

Proof. We shall first establish (4.4) for T sufficiently small. Set $\phi(x,t,s) = \phi_0$, $\phi(x+\alpha,t,s) = \phi$, $\tau(x,t) = \tau_0$, $\tau(x+\alpha,t) = \tau$ and assume for definiteness that $\alpha < 0$; then $\phi_0 \ge \phi$ and $\tau \ge \tau_0$.

Let $N(s) = |u(\phi,s) - u(\phi_0,s)|$. Then, by (2.9),

$$\begin{aligned}
N(t) \le{}& |u_*(x+\alpha,t) - u_*(x,t)|\\
&+ \left| \int_\tau^t \phi^{2/3}(x+\alpha)^{-2/3} \mathcal{C}[u](\phi,s)\, ds - \int_{\tau_0}^t \phi_0^{2/3} x^{2/3} \mathcal{C}[u](\phi_0,s)\, ds \right|,
\end{aligned}$$

and using (2.4) we get

$$
\begin{aligned}
N(t) \le{}& |u_*(x+\alpha,t) - u_*(x,t)| \\
&+ \Bigg| \int\limits_{\tau}^{t} \phi^{2/3}(x+\alpha)^{-2/3} \int\limits_{1}^{\phi-1} \beta(\phi-y,y)u(\phi-y,s)u(y,s)\ dy\ ds \\
&\qquad - \int\limits_{\tau_0}^{t} \phi_0^{2/3} x^{-2/3} \int\limits_{1}^{\phi_0-1} \beta(\phi_0-y,y)u(\phi_0-y,s)u(y,s)\ dy\ ds \Bigg| \\
&+ \Bigg| \int\limits_{\tau}^{t} \phi^{2/3}(x+\alpha)^{-2/3} \int\limits_{1}^{\infty} \beta(\phi,y)u(\phi,s)u(y,s)\ dy\ ds \\
&\qquad - \int\limits_{\tau}^{t} \phi_0^{2/3} x^{-2/3} \int\limits_{1}^{\infty} \beta(\phi_0,y)u(\phi_0,s)u(y,s)\ dy\ ds \Bigg| \\
&\equiv J_1 + J_2 + J_3.
\end{aligned}
\tag{4.5}
$$

We can write (for $|\alpha| \le x/2$)

$$
\begin{aligned}
J_2 \le{}& M \int\limits_{\tau_0}^{\tau} \int\limits_{1}^{\phi_0-1} u(\phi_0-y,s)u(y,s)\ ds \\
&+ M \int\limits_{\tau}^{t} \left| \phi_0^{2/3} x^{-2/3} - \phi^{2/3}(x+\alpha)^{-2/3} \right| \int\limits_{1}^{\phi_0-1} u(\phi_0-y,s)u(y,s)\ dy\ ds \\
&+ M \int\limits_{\tau}^{t} \int\limits_{\phi-1}^{\phi_0-1} u(\phi_0-y,s)u(y,s)\ dy\ ds + M_* \int\limits_{\tau}^{t} \int\limits_{1}^{\phi-1} |\phi_0-\phi| u(\phi_0-y,s)u(y,s) dy\ ds \\
&+ M \int\limits_{\tau}^{t} \int\limits_{1}^{\phi-1} |u(\phi_0-y,s) - u(\phi-y,s)| u(y,s)\ dyds \\
\le{}& \widetilde{C}_0 \left\{ M_l x^{-l} |\alpha| + \int\limits_{\tau}^{t} \int\limits_{1}^{\phi-1} |u(\phi_0-y,s) - u(\phi-y,s)| u(y,s)\ dy\ ds \right\}
\end{aligned}
$$

where $\widetilde{C}_0$ is a constant depending only on M, M_*, K_T, k_T, U_0 and

$$
M_l = \sup_{\substack{1 \le x < \infty \\ 0 \le t \le T}} x^{l+1} u(x,t);
$$

note that $M_l < \infty$ by Corollary 3.2, and M_l depends only on M, K_T, U_0.

A similar bound can be established for J_3. Using these bounds in (4.5), we get

$$N(t) \leq |u_*(x+\alpha,t) - u_*(x,t)| + C_0\left\{x^{-l}|\alpha| + \int_\tau^t \int_1^{\phi-1} |u(\phi_0 - y, s) - u(\phi - y, s)|u(y,s)dy\ ds + \int_\tau^t N(s)\ ds\right\},$$

or

$$N(t) \leq c_0\left\{\int_\tau^t N(s)\ ds + |\alpha|x^{-l}(1 + K_T + U_0) + \int_\tau^t \int_1^{\phi-1} |u(\phi_0 - y, s) - u(\phi - y, s)|u(y,s)\ dy\ ds\right\} \tag{4.6}$$

where c_0 is a constant depending only on K_T, k_T, U_0, M and M_*.

Now let T be such that $(1 + \int_0^T k)^3 \leq 2$ and set

$$R_0 = \{(x,t); t \leq T, x \leq (1 + \int_0^t k)^3\},$$

$$R_1 = \{(x,t); t \leq T, (1 + \int_0^t k)^3 \leq x \leq 2\},$$

$$R_m = \{(x,t); t \leq T, m \leq x \leq m+1\}, m = 2, 3, \ldots\ .$$

Suppose first that both (x,t) and $(x+\alpha,t)$ belong to R_0. Then (4.6) yields

$$N(t) \leq c_0\left\{\int_\tau^t N(s)\ ds + |\alpha|x^{-l}(1 + K_T + U_0)\right\}\ .$$

By Gronwall's inequality,

$$N(t) \leq c_0 e^{c_0 T}|\alpha|x^{-l}(1 + K_T + U_0), \tag{4.7}$$

and (4.4) follows.

In the sequel we shall use the formula

$$
\begin{aligned}
&\int_1^{\phi-1} |u(\phi_0-y,s)-u(\phi-y,s)|u(y,s)\,dy\\
&= \int_1^{\phi-1} |u(z,s)-u(z+\phi_0-\phi,s)|u(\phi-z,s)\,dz \qquad (4.8)\\
&= \int_1^{(1+\int_0^s k)^3-(\phi_0-\phi)} \cdots + \int_{(1+\int_0^s k)^3-(\phi_0-\phi)}^{(1+\int_0^s k)^3} \cdots + \int_{(1+\int_0^s k)^3}^{\phi-1} .
\end{aligned}
$$

Consider next the case where both (x,t) and $(x+\alpha,t)$ belong to $R_1 \cup R_2$. Using (4.7) and Corollary 3.2 we find that the right–hand side of (4.8) is bounded by

$$
\begin{aligned}
&\widetilde{C}_1\left\{ \left(\int_1^{\phi-1} \frac{u(\phi-z,s)}{z^l}\,dz \right) c_0|\phi_0-\phi|e^{c_0T}(1+K_T+U_0) + |\phi_0-\phi|x^{-l} \right\} \qquad (4.9)\\
&\le c_1 x^{-l}|\alpha|e^{c_0T}c_0(1+K_T+U_0) \qquad \text{if} \quad |\alpha| \le \frac{x}{2}\,.
\end{aligned}
$$

Using this estimate in the last integral on the right–hand side of (4.6), we get

$$
\begin{aligned}
N(t) \le c_0\Bigg\{ \int_T^t N(s)\,ds &+ |\alpha|x^{-l}(1+K_T+U_0)\\
&+ |\alpha|x^{-l}c_2Te^{c_0T}(1+K_T+U_0)\Bigg\}.
\end{aligned}
$$

Choosing T such that

$$c_2Te^{c_0T} \le 1 \qquad (4.10)$$

we obtain the inequality

$$N(t) \le 2c_0e^{c_0T}|\alpha|x^{-l}(1+K_T+U_0) \qquad (4.11)$$

provided both (x,t) and $(x+\alpha,t)$ belong to $R_1 \cup R_2$.

Next we consider the case where these two points belong to $R_1 \cup R_2 \cup R_3$. Using (4.11) and Corollary 3.2 we can estimate the last integral on the right-hand side of (4.6), obtaining analogously to (4.9), the bound

$$2c_2|\alpha|x^{-l}e^{c_0T}c_0(1+K_T+U_0).$$

If we choose T (analogously to (4.10)) such that

$$2c_2Te^{c_0T} \leq 1, \tag{4.12}$$

we again get the inequality (4.11). We can now proceed to extend (4.11) to $R_1 \cup R_2 \cup R_3 \cup R_4$ using the fact that this inequality holds in $R_1 \cup R_2 \cup R_3$ (in order to estimate the last integral on the right-hand side of (4.6)) and making the choice (4.12) for T. This results again in the inequality (4.11).

Proceeding in this way step-by-step, the proof of (4.4) follows provided T satisfies (4.12) and $|\alpha| \leq x/2$. The proof for the case $|\alpha| > x/2$ is similar (and slightly simpler).

Finally, to prove (4.4) for any T we proceed step-by-step on time intervals of size η where

$$2c_2\eta e^{c_0T} \leq 1.$$

§5. **Reformulation of the inverse problem.** In studying the inverse problem we assume that, in addition to (2.5), β satisfies a uniform Lipschitz condition:

$$|\beta(x,y)-\beta(x',y')| \leq M_*(|x-x'|+|y-y'|). \tag{5.1}$$

We shall consider (2.1)–(2.3) with

$$h(t) = \frac{I(t)}{3k(t)}\ , \ u_0(x) = 0 \tag{5.2}$$

where $I(t), k(t)$ are continuous functions for $t \geq 0$, and

$$k(t) \geq k_0 > 0\ , \quad I(0) > 0 \tag{5.3}$$

Let

$$f(t) = \int_1^\infty xu(x,t)\,dx\ , \tag{5.4}$$

$$g(t) = \int_1^\infty x^2u(x,t)\,dx \tag{5.5}$$

where u is the solution of (2.1)–(2.3). Our aim is to show that f and g uniquely determine I and k.

LEMMA 5.1. *The functions f, g are in $C^1[0,\infty)$ and*

$$f'(t) = I(t) + \Big\{ \int_0^t 3\Big(1+\int_s^t k(r)dr\Big)^2 I(s)\,ds + \int_0^t\int_1^\infty 3\Big(y^{1/3} + \int_s^t k(r)\,dr\Big)^2 \mathcal{C}[u](y,s)\,dy\,ds \Big\} k(t), \tag{5.6}$$

$$\begin{aligned} g'(t) = I(t) + \Big\{ &\int_0^t 6\Big(1+\int_s^t k(r)dr\Big)^5 I(s)\,ds \\ &+ \int_0^t\int_1^\infty 6\Big(y^{1/3} + \int_s^t k(r)dr\Big)^5 \mathcal{C}[u](y,s)\,dy\,ds \Big\} k(t) \\ &+ \int_1^\infty y^2 \mathcal{C}[u](y,t)\,dy. \end{aligned} \tag{5.7}$$

Proof. We have, by (2.9),

$$\begin{aligned} f(t) &= \int_1^\infty x u(x,t)\,dx = \int_1^\infty x^{1/3}\big(x^{2/3}u(x,t)\big)\,dx \\ &= \int_1^\infty x^{1/3}\left[\int_{\tau(x,t)}^t \phi(x,t,s)^{2/3}\mathcal{C}[u](\phi(x,t,s),s)\,ds + x^{2/3}u_*(x,t)\right] dx \\ &= \int_0^t \int_{(1+\int_s^t k(r)\,dr)^3}^\infty x^{1/3}\phi^{2/3}\mathcal{C}[u](\phi,s)dx ds + \int_1^{(1+\int_0^t k(r)dr)^3} \frac{x^{1/3}\, I\big(\tau(x,t)\big)}{3k\big(\tau(x,t)\big)}\,dx \end{aligned} \tag{5.8}$$

and the last integral is equal to

$$\int_0^t \Big(1+\int_s^t k(r)dr\Big)^3 I(s)\,ds.$$

Substituting $\phi(x,t,s) = y$ into the first integral on the right–hand side of (5.8) we find that

$$f(t) = \int_0^t\int_1^\infty \Big(y^{1/3} + \int_s^t k(r)dr\Big)^3 \mathcal{C}[u](y,s)\,dy\,ds + \int_0^t\Big(1+\int_s^t k(r)dr\Big)^3 I(s)ds. \tag{5.9}$$

It follows that $f \in C^1$ and

$$\begin{aligned} f'(t) = \int\limits_1^\infty y\mathcal{C}[u](y,t)dy + \int\limits_0^t \int\limits_1^\infty 3(y^{1/3} + \int\limits_s^t k(r)dr)^2 k(t)\mathcal{C}[n](y,s)\, dyds \\ +I(t) + \int\limits_0^t 3(1 + \int\limits_s^t k(r)dr)^2 k(t) I(s)\, ds. \end{aligned} \tag{5.10}$$

Recalling (3.15), the assertion (5.6) follows.

The proof of (5.7) is similar.

From (5.6), (5.7) it follows that

$$f'(0) = g'(0) = \alpha \quad \text{where} \quad \alpha = I(0); \tag{5.11}$$

by (5.3), $\alpha > 0$.

Setting

$$f_1(t) = \frac{f'(t) - \alpha}{t}\ , \quad g_1(t) = \frac{g'(t) - \alpha}{t} \tag{5.12}$$

and

$$J(t) = \frac{I(t) - \alpha}{t} \tag{5.13}$$

and introducing the functions

$$A(u)(t) = \frac{1}{t} \int\limits_0^t \int\limits_1^\infty 3k(t) \left(y^{1/3} + \int\limits_s^t k(r)dr \right)^2 \mathcal{C}[u](y,s)dyds, \tag{5.14}$$

$$B(u)(t) = B_1(u)(t) + B_2 u(t) \tag{5.15}$$

where

$$B_1(u)(t) = \frac{1}{t} \int\limits_1^\infty y^2 \mathcal{C}[u](y,t)\, dy, \tag{5.16}$$

$$B_2(u) = \frac{1}{t} \int\limits_0^t \int\limits_1^\infty 6k(t) \left(y^{1/3} + \int\limits_s^t k(r)dr \right)^5 \mathcal{C}[u](y,s)\, dy\, ds, \tag{5.17}$$

we can rewrite (5.6), (5.7) in the form

$$(5.18)\qquad \begin{aligned} &J(t)+\left\{\frac{1}{t}\int_0^t 3\Big(1+\int_s^t k(r)dr\Big)^2 I(s)ds\right\}k(t)=f_1(t)-A(u)(t),\\ &J(t)+\left\{\frac{1}{t}\int_0^t 6\Big(1+\int_s^t k(r)dr\Big)^5 I(s)ds\right\}k(t)=g_1(t)-B(u)(t). \end{aligned}$$

Solving for J, k, we obtain:

(5.19)

$$J(t)=\frac{[f_1(t)-A(u)(t)][\frac{1}{t}\int_0^t 6(1+\int_s^t k)^5(\alpha+J(s)s)ds]-[g_1(t)-B(u)(t)][\frac{1}{t}\int_0^t 3(1+\int_s^t k)^2(\alpha+J(s)s)ds]}{\frac{1}{t}\int_0^t 6(1+\int_s^t k)^5(\alpha+J(s)s)ds-\frac{1}{t}\int_0^t 3(1+\int_s^t k)^2(\alpha+J(s)s)\,ds}$$

(5.20)

$$k(t)=\frac{g_1(t)-f_1(t)+A(u)(t)-B(u)(t)}{\frac{1}{t}\int_0^t 6(1+\int_s^t k)^5(\alpha+J(s)s)ds-\frac{1}{t}\int_0^t 3(1+\int_s^t k)^2(\alpha+J(s)s)ds}$$

LEMMA 5.2. *Given $I(t), k(t)$, let $f(t), g(t)$ be defined by (5.4), (5.5) where $u(x,t)$ is the solution of (2.1)-(2.3), (5.2) (for the given I, k). Then I, k satisfy the system of nonlinear integral equations (5.19), (5.20).*

We remark that the assumption $\alpha = I(0) > 0$ ensures that the denominators in (5.19), (5.20) do not vanish as $t \to 0$.

DEFINITION 5.1. The inverse problem consists of reconstructing the pair I, k in terms of the functions f, g.

In view of Lemma 5.2, the inverse problem has a solution for $0 \le t \le T$ if we can establish that the system (5.19), (5.20) has a solution J, k for $0 \le t \le T$.

In §7 we shall solve the inverse problem globally under the assumptions that $f''(0), g''(0)$ exist and $f''(0) \ne g''(0)$. (Observe that formally $g''(0) - f''(0) = 3\alpha k(0)$.) Some auxiliary results are proved in §6.

§6. Lipschitz dependence of u on I, k.

When (5.2) holds the function u_* defined in (2.11) is given by

$$(6.1)\qquad u_*(x,t)=\begin{cases}\dfrac{I(\tau(x,t))}{3k(\tau(x,t))}\,x^{-2/3} & \text{if}\quad \tau(x,t)>0\\ 0\quad \text{if}\quad \tau(x,t)=0.\end{cases}$$

Lemma 6.1. *The following identity holds:*

(6.2)

$$\int_1^\infty x^2\mathcal{C}[u](x,t)dx = \int_0^t\int_0^t \beta((1+\int_r^t k)^3,(1+\int_s^t k)^3)(1+\int_r^t k)^3(1+\int_s^t k)^3 I(r)I(s)\ dr\ ds$$

$$+2\int_0^t\int_0^t\int_1^\infty \beta((1+\int_r^t k)^3,(y^{1/3}+\int_s^t k)^3)(1+\int_r^t k)^3(y^{1/3}+\int_s^t k)^3 I(r)\mathcal{C}[u](y,s)\ dy\ drds$$

$$+\int_0^t\int_0^t\int_1^\infty\int_1^\infty \beta((z^{1/3}+\int_r^t k)^3,(y^{1/3}+\int_s^t k)^3)(z^{1/3}+\int_r^t k)^3\ (y^{1/3}+\int_s^t k)^3\mathcal{C}[u](y,s)\mathcal{C}[u](z,r)dydzdrds$$

Proof. We have

$$\begin{aligned}\int_1^\infty x^2\mathcal{C}[u](x,t)dx =& \frac{1}{2}\int_2^\infty x^2\int_1^{x-1}\beta(y,x-y)u(x-y,t)u(y,t)\ dy\ dx\\ &-\int_1^\infty x^2u(x,t)\int_1^\infty\beta(x,y)u(y,t)\ dy\ dx,\end{aligned}$$

and the first integral on the right–hand side is equal to

$$\begin{aligned}&\int_1^\infty\int_{1+y}^\infty x^2\beta(y,x-y)u(x-y,t)u(y,t)\ dx\ dy\\ &=\int_1^\infty\int_1^\infty(y+z)^2\beta(y,z)u(z,t)u(y,t)\ dz\ dy.\end{aligned}$$

It follows that

$$\int_1^\infty x^2\mathcal{C}[u](x,t)\ dx = \int_1^\infty\int_1^\infty yx\beta(y,x)u(y,t)u(x,t)\ dy\ dx. \tag{6.3}$$

Substituting u from (2.9) into the right–hand side of (6.3) we get

$$\int_1^\infty x^2\mathcal{C}[u](x,t)\ dx = D_1+D_2+D_3 \tag{6.4}$$

where

$$D_1 = \int_1^\infty \int_1^\infty xy\beta(y,x)\Big\{ \int_{\tau(x,t)}^{t} x^{-2/3}\phi(x,t,s)^{2/3}\mathcal{C}[u]\big(\phi(x,t,r),r\big)\ dr$$

$$\cdot \int_{\tau(y,t)}^{t} y^{-2/3}\phi(y,t,s)^{2/3}\mathcal{C}[u]\big(\phi(y,t,s),s\big)\ ds\Big\} dy\ dx,$$

$$D_2 = 2\int_1^\infty \int_1^\infty yx\beta(y,x)\Big[\int_{\tau(x,t)}^{t} x^{-2/3}\phi(x,t,s)^{2/3}\mathcal{C}[u]\big(\phi(x,t,s),s\big)\ ds\Big] u_*(y,t)dy\ dx,$$

$$D_3 = \int_1^\infty \int_1^\infty xy\beta(y,x)u_*(y,t)u_*(x,t)\ dy\ dx.$$

By the change of variables

$$y \to z = \phi(y,t,r) \quad \text{and} \quad x \to \zeta = \phi(x,t,s)$$

(with r and s fixed) we find that D_1 reduces to the last term on the right–hand side of (6.2) (with y replaced by ζ).

Using the change of variables

$$\tau(y,t) = r \quad \text{and} \quad x \to \zeta = \phi(x,t,s) \quad (s\ \ \text{fixed})$$

we find that D_2 is equal to the second term on the right–hand side of (6.2). Finally, using the substitutions

$$\tau(y,t) = r, \quad \tau(x,t) = s$$

we can show that D_1 is equal to the first term on the right–hand side of (6.2).

LEMMA 6.2. *From any $T > 0$ there exists a constant C_T depending on M, $\|k\|_{L^\infty(0,T)}$, $\|I\|_{L^\infty(0,T)}$ such that*

$$\int_1^\infty x^2|\mathcal{C}[u](x,t)|dx \le C_T t^2 \quad \text{for all} \quad 0 \le t \le T. \tag{6.5}$$

Proof. Let

$$F(t) = \int_1^\infty x^2|\mathcal{C}[u](x,t)|\ dx.$$

Then

$$F(t) \leq \int_2^\infty x^2 \int_1^{x-1} \beta(y, x-y) u(y,t) u(x-y,t)\; dy\; dx$$

$$+ \int_1^\infty x^2 \int_1^\infty u(x,t)\beta(y,x)u(y,t)\; dy\; dx$$

$$\leq M \left\{ \int_1^\infty \int_1^\infty (y+z)^2 u(y,t)u(z,t)\; dy\; dz + \int_1^\infty \int_1^\infty x^2 u(x,t)u(y,t)\; dy\; dx \right\}.$$

Thus

$$F(t) \leq 3M \left\{ \left(\int_1^\infty y^2 u(y,t)\; dy \right) \left(\int_1^\infty u(z,t)\; dz \right) + \left(\int_1^\infty y u(y,t)\; dy \right)^2 \right\}. \tag{6.6}$$

From (2.9) and a change of variables,

$$\int_1^\infty y^2 u(y,t)\; dy = \int_0^t \int_1^\infty (z^{1/3} + \int_s^t k)^6 \mathcal{C}[u](z,s)\; dz\; ds$$

$$+ \int_0^t (1 + \int_s^t k)^6 I(s)\; ds\ ,$$

so that

$$\int_1^\infty y^2 u(y,t)\; dy \leq (1 + T\|k\|)^6 \left\{ \int_0^t F(s)\; ds + \|I\| t \right\}$$

where $\|\cdot\| = \|\cdot\|_{L^\infty(0,T)}$. Also $\int_1^\infty u(z,t)dz \leq t\|I\|$ by Theorem 2.1 (b), and

$$\int_1^\infty y u(y,t)\; dy \leq t(1 + t\|k\|)^3 \frac{\|I\|}{\|k\|}$$

by (3.9).

Using these inequalities we can estimate the right–hand side of (6.6) and thus obtain

$$F(t) \leq 3M \left[(1 + T\|k\|)^6 \int_0^t F(s)\; ds \right] t\|I\| + C_T t^2,$$

and the assertion (6.5) follows.

We now state the main result of this section.

THEOREM 6.3. *Let $\tilde{u}$ be a solution of (2.1)–(2.3) corresponding to $\tilde{k}, \widetilde{I}$ and assume that $\tilde{k}, \widetilde{I}$ satisfy the same assumptions (5.2), (5.3) as k, I, and that $k, I, \tilde{k}, \widetilde{I}$ are Lipschitz continuous for $t \geq 0$. Then for any $T > 0$ there exists a constant C_T depending only on T, M, M_* and the Lipschitz norms of $k, I, \tilde{k}, \widetilde{I}$ in $0 \leq t \leq T$, such that*

$$\int_1^\infty x^2 |\mathcal{C}[u](x,t) - \mathcal{C}[\tilde{u}](x,t)| \, dx \leq C_T t^2 \left(\|k - \tilde{k}\| + \|\frac{I}{k} - \frac{\widetilde{I}}{\tilde{k}}\| \right) \tag{6.7}$$

and

$$\int_1^\infty x^2 |u(x,t) - \tilde{u}(x,t)| \, dx \leq C_T t \left(\|k - \tilde{k}\| + \|\frac{I}{k} - \frac{\widetilde{I}}{\tilde{k}}\| \right) \tag{6.8}$$

where $\|\cdot\|$ refers to the $C^{0,1}$ norm in $0 \leq t \leq T$.

Proof. The left–hand side of (6.7) is bounded by

$$\begin{aligned}
&\frac{1}{2}\int_2^\infty \int_1^{x-1} x^2 \beta(y, x-y) |u(x-y,t) - \tilde{u}(x-y,t)| u(y,t) \, dy \, dx \\
&+ \frac{1}{2}\int_2^\infty \int_1^{x-1} x^2 \beta(y, x-y) \tilde{u}(x-y,t) |u(y,t) - \tilde{u}(y,t)| \, dy \, dx \\
&+ \int_1^\infty \int_1^\infty x^2 \beta(x,y) u(y,t) |u(x,t) - \tilde{u}(x,t)| \, dy \, dx \\
&+ \int_1^\infty \int_1^\infty x^2 \beta(x,y) \tilde{u}(x,t) |u(y,t) - \tilde{u}(y,t)| \, dy \, dx.
\end{aligned}$$

Changing order of integration in the first two integrals and then substituting $x - y = z$ (y fixed), we easily get the crude estimate

$$\begin{aligned}
&\int_1^\infty x^2 |\mathcal{C}[u](x,t) - \mathcal{C}[\tilde{u}](x,t)| \, dx \\
&\leq 3 \int_1^\infty \int_1^\infty x^2 y^2 \beta(x,y) u(y,t)| \; |u(x,t) - \tilde{u}(x,t)| \, dy \, dx \\
&+ 3 \int_1^\infty \int_1^\infty x^2 y^2 \beta(x,y) \tilde{u}(y,t)| \; |u(x,t) - \tilde{u}(x,t)| dy \, dx \\
&\equiv 3E_1 + 3E_2.
\end{aligned} \tag{6.9}$$

In order to estimate E_1, set

$$\phi(x,t,s) = (x^{1/3} - \int_s^t k)^3, \quad \tilde{\phi}(x,t,s) = (x^{1/3} - \int_s^t \tilde{k})^3$$

and define $\widetilde{\tau}(x,t)$ with respect to $\tilde{u}$ in the same way that $\tau(x,t)$ was defined with respect to u. Using the representation (2.9) and (6.1) we obtain, after a change of variables $z = \phi(x,t,s)$,

$$\int_1^\infty x^2 u(x,t)dx = \int_0^t \int_1^\infty (z^{1/3} + \int_s^t k)^6 \mathcal{C}[u](z,s)\ dz\ ds + \int_0^t (1 + \int_s^t k)^6 I(s)\ ds$$
$$\leq (1 + t\|k\|)^6 \left\{ \int_0^t \int_1^\infty z^2 |\mathcal{C}[u](z,s)|\ dz\ ds + t\|I\| \right\}$$

where $\|\cdot\| = \|\cdot\|_{L^\infty(0,t)}$, and applying Lemma 6.2 we get

$$\int_1^\infty x^2 u(x,t)\ dx \leq C_T t. \tag{6.10}$$

To complete the estimate of E_1 we need to estimate

$$\int_1^\infty x^2 |u(x,t) - \tilde{u}(x,t)|\ dx$$
$$\leq \int_1^\infty x^2 \left| \int_\tau^t x^{-2/3} \phi^{2/3} \mathcal{C}[u](\phi,s)\ ds - \int_{\widetilde{\tau}}^t x^{-2/3} \tilde{\phi}^{2/3} \mathcal{C}[\tilde{u}](\tilde{\phi},s)\ ds \right| dx$$
$$+ \int_1^\infty x^2 |u_*(x,t) - \tilde{u}_*(x,t)|\ dx, \quad \text{by (2.9).}$$

We can write

$$
\begin{aligned}
\int_1^\infty x^2|u(x,t)-\tilde{u}(x,t)|dx &\le \int_1^\infty x^2\int_\tau^t x^{-2/3}\phi^{2/3}|\mathcal{C}[u](\phi,s)-\mathcal{C}[\tilde{u}](\phi,s)|\,ds\,dx \\
&+\int_1^\infty x^2\int_\tau^t x^{-2/3}\phi^{2/3}|\mathcal{C}[\tilde{u}](\phi,s)-\mathcal{C}[\tilde{u}](\tilde{\phi},s)|\,ds\,dx \\
&+\int_1^\infty x^2\int_\tau^t x^{-2/3}|\phi^{2/3}-\tilde{\phi}^{2/3}||\mathcal{C}[\tilde{u}](\tilde{\phi},s)|\,ds\,dx \qquad (6.11)\\
&+\int_1^\infty x^2\left|\int_{\tilde{\tau}}^\tau x^{-2/3}\tilde{\phi}^{2/3}|\mathcal{C}[\tilde{u}](\tilde{\phi},s)|\,ds\right|dx \\
&+\int_1^\infty x^2|u_*(x,t)-\tilde{u}_*(x,t)|dx \equiv J_1+J_2+J_3+J_4+J_5.
\end{aligned}
$$

We proceed to estimate the J_i for $i=2,3,4,5$. Since $\phi\le x$.

$$
\begin{aligned}
J_2 \le M\int_1^\infty x^2\int_\tau^t\int_1^{\phi-1}|\tilde{u}(\phi-y,s)-\tilde{u}(\tilde{\phi}-y,s)|\tilde{u}(y,s)\,dy\,ds\,dx \\
+M\int_1^\infty x^2\int_\tau^t\int_1^\infty|\tilde{u}(\phi,s)-\tilde{u}(\tilde{\phi},s)|\tilde{u}(y,s)\,dy\,ds\,dx \\
+M_*\int_1^\infty x^2\int_\tau^t|\tilde{\phi}-\phi|\left|\int_1^{\phi-1}\tilde{u}(\tilde{\phi}-y,s)\tilde{u}(y,s)\,dy+\int_1^\infty\tilde{u}(\phi,s)\tilde{u}(y,s)\,dy\right|ds\,dx \\
+M\int_1^\infty x^2\int_\tau^t\left|\int_{\phi-1}^{\tilde{\phi}-1}\tilde{u}(\tilde{\phi}-y,s)\tilde{u}(y,s)\,dy\right|ds\,dx \equiv L_1+L_2+L_3+L_4.
\end{aligned}
$$

Making a change of variables $\phi(x,t,s) = z$, we find that

(6.13)

$$\begin{aligned} L_1 &\le M \int_0^t \int_1^\infty (z^{1/3} + \int_s^t k)^6 \int_1^{z-1} |\tilde{u}(z-y,s) - \tilde{u}\big((z^{1/3} + \int_s^t (k-\tilde{k}))^3 - y, s\big)|\tilde{u}(y,s)\, dydzds \\ &\le (1+\|k\|)^6 \int_0^t \int_1^\infty \int_1^\infty (x+y)^2 |\tilde{u}(x,s) - \tilde{u}\big((x+y)^{1/3} + \int_s^t (k-\tilde{k})\big)^3 - y, s)|\tilde{u}(y,s)\, dx\, dy\, ds \\ &\le c \int_0^t \int_1^\infty \int_1^\infty x^2 y^2 |\tilde{u}(x,s) - \tilde{u}(x+\alpha)|\tilde{u}(y,s)\, dx\, dy\, ds \end{aligned}$$

where

$$\alpha = 3(x+y)^{2/3} \int_s^t (k-\tilde{k}) + 3(x+y)^{1/3} \left(\int_s^t (k-\tilde{k}) \right)^2 + \left(\int_s^t (k-\tilde{k}) \right)^3 ;$$

clearly

$$|\alpha| \le c\|k-\tilde{k}\| x^{2/3} y^{2/3}. \tag{6.14}$$

We now use Theorem 4.1 and Corollary 3.2 in order to estimate the right–hand side of (6.13); here we should notice that for $|\alpha| \ge x/2$ we have, by (6.14), $x \le cy^2$. We find, after some simple calculations, that

$$L_1 \le ct\|k-\tilde{k}\|.$$

We can estimate L_2 in a similar way. The other terms L_3, L_4 in (6.12) are estimated very easily by $ct\|k-\tilde{k}\|$. Hence

$$J_2 \le ct\|k-\tilde{k}\|. \tag{6.15}$$

Next, it is easy to see that J_3 and J_4 can be estimated by $ct\|k-\tilde{k}\|$, and

$$\begin{aligned} J_5 = \left| \int_1^{(1+\int_0^t k)^3} x^2 \frac{I(\tau(x,t))}{3k(\tau(x,t))}\, x^{-2/3} dx - \int_1^{(1+\int_0^t \tilde{k})^3} x^2 \frac{\tilde{I}(\tilde{\tau}(x,t))}{3\tilde{k}(\tilde{\tau}(x,t))}\, x^{-2/3} dx \right| \\ \le ct \left(\|k-\tilde{k}\| + \|I/k - \tilde{I}/\tilde{k}\| \right). \end{aligned}$$

Combining the estimates on the J_i we deduce from (6.11) that

$$\begin{aligned} \int_1^\infty x^2 |u(x,t) - \tilde{u}(x,t)|\, dx \le c \int_0^t \int_1^\infty z^2 |\mathcal{C}[u](z,s) - \mathcal{C}[\tilde{u}](z,s)|\, dz\, ds \\ + ct\|k-\tilde{k}\| + ct\|I/k - \tilde{I}/\tilde{k}\|. \end{aligned} \tag{6.16}$$

Using (6.16) and (6.10), we can now estimate E_1 in (6.9), and similarly also E_2, and obtain

$$\int_1^\infty x^2 |\mathcal{C}[u](x,t) - \mathcal{C}[\tilde{u}](x,t)|\, dx \le ct^2 \left\{ \|k - \tilde{k}\| + \|I/k - \tilde{I}/\tilde{k}\| \right\}$$
$$+ct \int_0^t \int_1^\infty z^2 |\mathcal{C}[u](z,s) - \mathcal{C}[\tilde{u}](z,s)|\, dz\, ds;$$

the assertion (6.7) then follows by the Gronwall inequality. Finally, (6.8) follows from (6.16) and (6.7).

REMARK 6.1. Later on we shall need the estimate

$$\left| \int_1^\infty y^2 \mathcal{C}[u](y,t)\, dy - \int_1^\infty y^2 \mathcal{C}[u](y,t')\, dy \right| \le C_T t |t - t'| \tag{6.17}$$

for $0 < t' \le t \le T$. This can be established by using Lemmas 6.1 and 6.2; the details are omitted.

§7. Solution of the inverse problem. We now return to the set–up of §5. We shall assume that $f''(0)$ and $g''(0)$ exist. Setting

$$f'' = \gamma_1, \qquad g''(0) = \gamma_2 \tag{7.1}$$

we can formally deduce from (5.19), (5.20) that

$$\begin{aligned} J_0 &\equiv J(0) = 2\gamma_1 - \gamma_2 \equiv \gamma\ , \\ k_0 &\equiv k(0) = \frac{\gamma_2 - \gamma_1}{3\alpha} = \frac{\epsilon_0}{3\alpha} \qquad \text{where } \epsilon_0 = \gamma_2 - \gamma_1 \end{aligned} \tag{7.2}$$

and α is defined in (5.11). We first prove a local result:

THEOREM 7.1. *Assume that f_1, g_1 are Lipschitz continuous and that $f''(0), g''(0)$ exist and $g''(0) - f''(0) = \epsilon_0 > 0$. Then there exists a unique Lipschitz continuous solution (J,k) of (5.19), (5.20) for all $0 \le t \le \eta$, provided η is a sufficiently small positive constant.*

Proof. Introduce the set

$$X_\eta = \Big\{ (J,k) \in C[0,\eta)] \times C[0,\eta], \qquad k(0) = \frac{\epsilon_0}{3\alpha},\ J(0) = 2\gamma_1 - \gamma_2,$$
$$|J(t) - J(t')| \le K_1 |t - t'|,\ \ |k(t) - k(t')| \le K_2 |t - t'| \text{ for all } t, t' \text{ in } [0,\eta] \Big\}$$

as a subset in the Banach space of X of continuous functions (J, k) on $0 \le t \le \eta$ with the $L^\infty(0, \eta)$ norm.

Denote the right–hand sides of (5.19), (5.20), respectively by $S_1(J, k)$, $S_2(J, k)$. We want to show that, for suitable positive constants K_1, K_2 and small $\eta > 0$, $S = (S_1, S_2)$ is a contraction on X_η; this will complete the proof of the theorem

Let

$$\begin{aligned} D_1(t) &= \frac{1}{t} \int_0^t 6\Big(1 + \int_s^t k\Big)^5 \big(\alpha + J(s)s\big)\, ds, \\ D_2(t) &= \frac{1}{t} \int_0^t 3\Big(1 + \int_s^t k\Big)^2 \big(\alpha + J(s)s\big)\, ds, \\ D(t) &= D_1(t) - D_2(t). \end{aligned}$$

Setting $\widetilde{J} = S_1(J, k)$, $\tilde{k} = S_2(J, k)$ we can then write

$$\widetilde{J} = S_1(J, k) = \frac{(f_1 - A)D_1 - (g_1 - B)D_2}{D}\,, \tag{7.3}$$

$$\tilde{k} = S_2(J, k) = \frac{g_1 - f_1 + A - B}{D} \tag{7.4}$$

Lemma 7.2. *(a) D_1 and D_2 are differentiable and*

$$D_1'(0) = 15\alpha k_0 + 3J_0, \quad D_2'(0) = 3\alpha k_0 + \frac{3}{2}\, J_0;$$

(b) for all $t \le \eta \le 1$,

$$\begin{aligned} |D_1'(t) - D_1'(0)| &\le c\|J\|_{C^{0,1}} \left(1 + \|k\|_{C^{0,1}}\right)^5 t, \\ |D_2'(t) - D_2'(0)| &\le c\|J\|_{C^{0,1}} \left(1 + \|k\|_{C^{0,1}}\right)^5 t \end{aligned}$$

where c is a constant independent of η.

Proof. Let $g(t, s) = 6(1 + \int_s^t k)^5$. Then

$$D_1(t) = \frac{1}{t} \int_0^t g(t, s)J(s)s + \frac{\alpha}{t} \int_0^t g(t, s) \equiv E_1(t) + E_2(t).$$

It is easy to see that $E_1'(t), E_2'(t)$ exist for $t > 0$. Further, since $E_1(0) = 0,\ E_2(0) = 6\alpha, g(0,0) = 6$,

$$E_1'(0) = \lim_{t\to 0} \frac{1}{t^2} \int_0^t [g(t,s)J(s)s - g(0,0)J(0)s] + \frac{1}{2}\, g(0,0)J(0) = 3J_0,$$

$$\begin{aligned} E_2'(0) &= \lim_{t\to 0} \frac{1}{t^2} \int_0^t \big(\alpha g(t,s) - 6\alpha t\big) \\ &= \lim_{t\to 0} \frac{1}{t^2} \int_0^t \alpha\big(g(t,s) - g(0,0) - g_t(0,0)t - g_s(0,0)s\big) \\ &+ \alpha g_t(0,0) + \frac{1}{2}\, \alpha g_s(0,0) = 15k_0\alpha. \end{aligned}$$

The proof for D_2 is similar.

To prove (b) for D_1 we write

$$\begin{aligned} &|D_1'(t) - D_1'(0)| \le |E_1'(t) - E_1'(0)| + |E_2'(t) - E_2'(0)| \\ &\le \left| -\frac{1}{t^2} \int_0^t g(t,s)J(s)s + \frac{1}{t}\, g(t,t)J(t)t + \frac{1}{t} \int_0^t g_t(t,s)J(s)s - 3J_0 \right| \\ &+ \left| -\frac{\alpha}{t^2} \int_0^t g(t,s) + \frac{\alpha}{t}\, g(t,t) + \frac{\alpha}{t} \int_0^t g_t(t,s) - 15k_0\alpha \right|. \end{aligned} \tag{7.5}$$

Since $3J_0 = g(0,0)J_0 - \frac{1}{2}\, g(0,0)J_0$ and $g(t,t) = g(0,0)$, the first term on the right-hand side is bounded by

$$\begin{aligned} &\left| -\frac{1}{t^2} \int_0^t [g(t,s)J(s) - g(0,0)J(0)]s \right| + \left| g(t,t)J(t) - g(0,0)J(0) \right| \\ &+ \left| \frac{1}{t} \int_0^t g_t(t,s)J(s)s \right| \le c\|J\|_{C^{0,1}}(1 + \|k\|_{C^{0,1}})^5 t. \end{aligned}$$

Using the relation $g_t(0,0) = -g_s(0,0) = 30k_0$, we can estimate the second term on the right-hand side of (7.5) by

$$\left| -\frac{\alpha}{t^2} \int_0^t [g(t,s) - g(0,0) - g_t(0,0)t - g_s(0,0)s] \right| + \left| \frac{\alpha}{t} \int_0^t (g_t(t,s) - g_t(0,0)) \right|$$
$$\le c\|J\|_{C^{0,1}}(1 + \|k\|_{C^{0,1}})^5,$$

and the asserted bound for D_1 (in (b)) follows.

The proof for D_2 is similar.

Lemma 7.3. *(a) The functions A, B_1, B_2 defined in (5.14)-(5.17) are differentiable, $A(0) = B_1(0) = B_2(0) = 0$, and*

$$A'(0) = 0, \quad B_2'(0) = 0, \quad B_1'(0) = \beta(1,1)\alpha^2;$$

(b) for all $t \leq \eta$,

$$|A_1'(t)| \leq Ct, \quad |B_2'(t)| \leq Ct,$$
$$|B_1'(t) - B_1'(0)| \leq Ct.$$

Proof. Since the proof is similar to that of Lemma 7.2, it will suffice to give it for B_1. We can write (from (6.2))

$$B_1(t) = \frac{1}{t} \int_0^t \int_0^t g(t,s,r)\, ds\, dr$$

where $g \in C^{0,1}$ in (t,s,r) and $g \in C^{1,1}$ in t. This implies that $B_1'(t)$ exists and

$$B_1'(0) = g(0,0,0) = \beta(1,1)I(0)^2 = \beta(1,1)\alpha^2 \quad \text{(from (6.2))}.$$

Setting $g_0 = g(0,0,0)$ we can write

$$\begin{aligned} B_1'(t) &= -\frac{1}{t^2} \int_0^t \int_0^t g\, dr\, ds + \frac{1}{t} \int_0^t g(t,t,r) + \frac{1}{t} \int_0^t g(t,s,t) + \frac{1}{t} \int_0^t \int_0^t g_t\, dr\, ds \\ &= -\frac{1}{t^2} \int_0^t \int_0^t (g - g_0) + \frac{1}{t} \int_0^t \big(g(t,t,r) - g_0\big) + \frac{1}{t} \int_0^t \big(g(t,s,t) - g_0\big) + \frac{1}{t} \int_0^t \int_0^t g_t + g_0 \end{aligned}$$

from which we deduce that $|B_1'(t) - B_1'(0)| \leq Ct$.

Choose

$$\eta < \min\left\{1, \frac{k_0}{2K_2}, \frac{\alpha}{2(K_1 + |\gamma|)}\right\}.$$

If $(J,k) \in X_\eta$ then

$$\frac{k_0}{2} \leq k(t) \leq \frac{3k_0}{2} \tag{7.6}$$

and

$$\frac{\alpha}{2} \leq \alpha + J(t)t \leq \frac{3\alpha}{2}\,. \tag{7.7}$$

Also, by Lemma 7.3,

$$\|A\|_{L^\infty} \le c_1\eta, \|B\|_{L^\infty} \le c_1\eta. \tag{7.8}$$

Using (7.7) we obtain

$$D(t) \ge \frac{1}{t}\int_0^t 3(1 + \int_s^t k)^5(\alpha + J(s)s) \ge \frac{3\alpha}{2}$$

and then, by (7.4),

(7.9)

$$\begin{aligned}
|\tilde{k}(t) - \tilde{k}(t')| &= \left|\frac{1}{D_1(t) - D_2(t)} - \frac{1}{D_1(t') - D_2(t')}\right| \; |g_1(t) - f_1(t) + A(t) - B(t)| \\
&+ \frac{1}{|D_1(t') - D_2(t')|}\; \{|g_1(t) - g_1(t')| + |f_1(t) - f_1(t')| + |A(t) - A(t')| + |B(t) - B(t')|\} \\
&\le \frac{4}{9\alpha^2}\; \{\|g_1\|_{C^{0,1}} + \|f_1\|_{C^{0,1}} + \|A\|_{L^\infty} + \|B\|_{L^\infty}\}\{|D_1(t) - D_1(t')| + |D_2(t) - D_2(t')|\} \\
&+ \frac{2}{3\alpha}\; \{(\|g_1\|_{C^{0,1}} + \|f_1\|_{C^{0,1}})|t - t'| + |A(t) - A(t')| + |B(t) - B(t')|\}.
\end{aligned}$$

By Lemma 7.2,

$$\begin{aligned}
|D_i(t) - D_i(t')| &\le |t - t'|\; |D_i'(\xi)| \le |t - t'|[|D_i'(\xi) - D_i'(0)| + |D_i'(0)|] \\
&\le |t - t'|\{C\eta + |D_i'(0)|\}.
\end{aligned}$$

Similarly, by Lemma 7.3,

$$\begin{aligned}
&|A(t) - A(t')| \le C\eta|t - t'|, \\
&|B(t) - B(t')| \le |t - t'|\{C\eta + |B_1'(0)|\}.
\end{aligned}$$

Substituting these estimates into the right-hand side of (7.9), we get

$$|\tilde{k}(t) - \tilde{k}(t')| \le K_2|t - t'|$$

provided

$$\begin{aligned}
K_2 = &\frac{4}{9\alpha^2}\; (\|g_1\|_{C^{0,1}} + \|f_1\|_{C^{0,1}})(|D_1'(0)| + |D_2'(0)|) \\
&+ \frac{2}{3\alpha}\; (\|g_1\|_{C^{0,1}} + \|f_1\|_{C^{0,1}} + |B_1'(0)| + 1)
\end{aligned}$$

and η is sufficiently small. A similar result can be established for $J, \tilde{J}$. Hence $TX_\eta \subset X_\eta$.

Next, using Theorem 6.3 we can also establish that S is a contraction; more precisely,

$$\|S(J, k) - S(\hat{J}, \hat{k})\|_{L^\infty} \le c\eta\big(\|J - \hat{J}\|_{L^\infty} + \|k - \hat{k}\|_{L^\infty}\big),$$

and $c\eta < 1$ if η is sufficiently small.

In Theorem 7.1 we proved that the inverse problem has a unique Lipschitz solution if $t \le \eta$, η sufficiently small. We shall now extend the solution globally:

THEOREM 7.4. *Assume that the functions* $f(t), g(t)$ *belong to* $C^1[0,\infty)$ *and satisfy the conditions in Theorem 7.1. Then there exists a unique Lipschitz continuous solution* (J,k) *of (5.19), (5.20) as long as* $k(t) > 0, I(t) \geq 0$, *where* $I(t) = I(0) + tJ(t)$.

More precisely: either the inverse problem (5.19), (5.20) has a unique solution for all $0 \leq t < \infty$ and $k(t) \geq 0, I(t) \geq 0$ for $t \geq 0$, or it has a unique solution for all $0 \leq t < T$ for some $0 < T < \infty$, such that $k(t) > 0, I(t) \geq 0$ if $0 \leq t < T$ and either $\liminf_{t \to T} k(t) = 0$ or the solution of (5.19), (5.20) exists for $0 \leq t \leq T + \epsilon$, for some $\epsilon > 0$, and $k(t) > 0$ for $0 \leq t \leq T + \epsilon$ but $I(t) < 0$ for some $t \in (T, T+\epsilon)$. If in particular, the f, g are defined by (5.4), (5.5) where u is a solution corresponding to a pair (I,k), then (5.19), (5.20) has a unique solution for all $t > 0$ which is then, of course, the pair (J,k) where $I(t) = I(0) + tJ(t)$.

Proof. We begin by rewriting (5.6), (5.7) in the form

$$I + E_1 k = F, \; I + E_2 k = G \tag{7.10}$$

where

$$F = F(I,k)(t) = f'(t) - \left(\int_{t_0}^{t} 3(1 + \int_{s}^{t} k)^2 I(s) ds \right) k(t) - \left(\int_{t_0}^{t} \int_{1}^{\infty} 3(y^{1/3} + \int_{s}^{t} k)^2 \mathcal{C}[u](y,s)\, dy\, ds \right) k(t),$$

$$G = G(I,k)(t) = g'(t) - \left(\int_{t_0}^{t} 6(1 + \int_{s}^{t} k)^5 I(s)\, ds \right) k(t)$$

$$- \left(\int_{t_0}^{t} \int_{1}^{\infty} 6(y^{1/3} + \int_{s}^{t} k)^5 \mathcal{C}[u](y,s)\, dy\, ds \right) k(t) - \int_{1}^{\infty} y^2 \mathcal{C}[u](y,t)\, dy,$$

$$E_1 = E_1(I,k)(t) = \int_{0}^{t_0} 3(1 + \int_{s}^{t} k)^2 I(s)\, ds + \int_{0}^{t_0} \int_{1}^{\infty} 3(y^{1/3} + \int_{s}^{t} k)^2 \mathcal{C}[u](y,s)\, dy\, ds,$$

$$E_2 = E_2(I,k)(t) = \int_{0}^{t_0} 6(1 + \int_{s}^{t} k)^5 I(s)\, ds + \int_{0}^{t_0} \int_{1}^{\infty} 6(y^{1/3} + \int_{s}^{t} k)^5 \mathcal{C}[u](y,s)\, dy\, ds.$$

Setting $\Delta = E_2 - E_1$ we solve from (7.10):

$$I(t) = \frac{E_2 F - E_1 G}{\Delta} \;, \tag{7.11}$$

$$k(t) = \frac{G - F}{\Delta} \tag{7.12}$$

provided $\Delta \neq 0$. If we can show that

$$\Delta(t_0) > 0 \quad \text{provided } k(t) > 0, \; I(t) \geq 0 \quad \text{if} \quad t \leq t_0, \tag{7.13}$$

then we can proceed as in Theorem 7.1 to extend the existence and uniqueness of a Lipschitz continuous solution to $[t_0, t_0+\epsilon]$ for some $\epsilon > 0$. Thus the proof of Theorem 7.4 follows once (7.13) is established.

To prove (7.13) we use (2.9) to get

$$\begin{aligned}\int_1^\infty x^{2/3}u(x,t)\,dx &= \int_1^\infty \int_\tau^t \phi^{2/3}\mathcal{C}[u](\phi,s)\,ds\,dx + \int_1^\infty x^{2/3}u_*(x,t)\,dx \\ &= \int_0^t \int_{(1+\int_s^t k)^3}^\infty \phi^{2/3}\mathcal{C}[u](\phi,s)\,dx\,ds + \int_1^{(1+\int_0^t k)^3} \frac{I(\tau)}{3k(\tau)}\,dx \end{aligned} \tag{7.14}$$

where $\phi = \phi(x,t,s) = (x^{1/3} - \int_s^t k)^3$, $1 = x^{1/3} - \int_\tau^t k, \tau = \tau(x,t)$.

Making a change of variables

$$x = (z^{1/3} + \int_s^t k)^3,\ dx = (z^{1/3} + \int_s^t k)^2 z^{-2/3} dz$$

in the first integral on the right–hand side of (7.14) (this means that we take $\phi(x,t,s) = z$), and

$$\tau(x,t) = s,\quad dx = 3(1 + \int_s^t k)^2(-k(s))\,ds$$

in the second integral on the right–hand side of (7.14) and setting $t = t_0$, we get

$$\begin{aligned}\int_1^\infty x^{2/3}u(x,t_0)\,dx &= \int_0^{t_0}\int_1^\infty \mathcal{C}[u](z,s)(z^{1/3} + \int_s^{t_0} k)^2\,dz \\ &\quad + \int_0^{t_0} I(s)(1 + \int_s^{t_0} k)^2 ds = \frac{1}{3}\,E_1(t_0),\end{aligned}$$

by the definition of E_1. Similarly

$$\int_1^\infty x^{5/3}u(x,t_0)\,dx = \frac{1}{6}\,E_2(t_0).$$

Consequently

$$\Delta(t_0) = \int_1^\infty (6x^{5/3} - 3x^{2/3}) u(x, t_0)\, dx > 0 \tag{7.15}$$

since $u(x, t_0) \geq 0$, $u(x, t_0) \not\equiv 0$ by Corollary 3.4 (using the assumption that $I(0) > 0$). This completes the proof of (7.13) and of Theorem 7.4.

REMARK 7.1. Suppose the solution (J, k) exists and $I(t) \geq 0$, $k(t) > 0$ for $0 \leq t \leq T$. Let $(\tilde{f}, \tilde{g})$ be another set of data satisfying the assumptions of Theorem 7.4 and denote by $(\tilde{J}, \tilde{k})$ the corresponding Lipschitz solution of the inverse problem. From the proof of Theorems 7.1, 7.4 it follows that for any $0 < T_0 < T$ there exists a constant $C > 0$ such that, for any sufficiently small $\delta > 0$, if

$$|f'(t) - \tilde{f}'(t)| \leq \delta, \quad |g'(t) - \tilde{g}'(t)| < \delta$$

for $0 \leq t \leq T_0$ and

$$|f''(0) - \tilde{f}''(0)| < \delta, \quad |g''(0) - \tilde{g}''(0)| < \delta,$$

then the solution $(\tilde{J}, \tilde{k})$ of (5.19), (5.20) exists and is unique for $0 \leq t \leq T_0$ and

$$|J(t) - \tilde{J}(t)| \leq C\delta, \qquad |k(t) - \tilde{k}(t)| \leq C\delta.$$

This stability result is very important in applications, since there are always some errors in the measurements which yield the functions $f(t)$ and $g(t)$.

§8. The case where β is unbounded. We consider here the direct problem for the case where $\beta = \beta(x, y)$ is unbounded, and prove the existence of a solution to (2.1)–(2.3). We assume:

$$\begin{aligned} &\text{(i)} && \beta(x,y) \geq 0 \quad \text{for } x \geq 1,\ y \geq 1, \\ &\text{(ii)} && \beta(x,y) \quad \text{is continuous for } x \geq 1,\ y \geq 1, \\ &\text{(iii)} && \beta(x,y) = \beta(y,x) \quad \text{for } x \geq 1,\ y \geq 1, \\ &\text{(iv)} && \limsup_{x^2+y^2 \to \infty} \beta(x,y) = \infty, \quad \text{and} \\ &\text{(v)} && \beta(x,y) \leq \sigma(x+y) \quad \text{if } x \geq y \geq 1 \text{ where } \sigma(r) \text{ is a positive valued} \\ & && \text{function satisfying } \frac{\sigma(r)}{r} \to 0 \text{ if } r \to \infty. \end{aligned} \tag{8.1}$$

Notice that the function β defined in (1.4) satisfies (8.1).

We also assume, in addition to (2.6), (2.7), that

$$c_1 \equiv \int_1^\infty x u_0(x) dx < \infty \quad \text{and} \quad c_2 \equiv \sup_{1 \leq x < \infty} x^\gamma u_0(x) < \infty \quad \text{for some } \gamma \geq 0. \tag{8.2}$$

THEOREM 8.1. *If (2.6), (2.7) and (8.1), (8.2) hold then there exists a solution u of (2.1)–(2.3) satisfying:*

$$\text{(a)} \quad u \geq 0,$$

$$\text{(b)} \quad \sup_{t \leq T} \int_1^\infty x u(x,t)\, dx \leq \nu_T \quad \textit{for any} \quad T > 0,.$$

where $\nu_T = 3 \int_0^T (1 + \int_s^T k)^3 h(s) k(s) ds + \int_1^\infty (x^{1/3} + \int_0^t k)^3 u_0(x) dx$; *if further* $\sigma(r) = c_0 r^{1-\gamma}$ *in (8.1) (v), then*

$$\text{(c)} \quad \sup_{\substack{1 \leq x < \infty \\ 0 \leq t \leq T}} x^\alpha u(x,t) \leq c e^{\nu_T T} (c_2 + \|h\|_{L^\infty(0,T)})$$

where c s a constant depending only on c_0, *and*

$$\alpha = \min \left\{ \frac{2}{3}, \gamma \right\}.$$

To prove the theorem we introduce the solution u_M corresponding to $\beta_M = \min\{\beta, M\}$ $(M = 1, 2, \ldots)$ and proceed to obtain bounds on u_M which are independent on M.

LEMMA 8.2. *For any* $M \geq 1$,

$$\sup_{t \geq T} \int_1^\infty x u_M(x,t)\, dx \leq \nu_T, \tag{8.3}$$

$$\sup_{\substack{1 \leq x < \infty \\ t \leq T}} x^\alpha u_M(x,t) \leq c e^{c \nu_T T} (c_2 + \|h\|_{L^\infty(0,T)}) \quad \textit{if } \sigma(r) = c_0 r^{1-\gamma}, \tag{8.4}$$

where the constants ν_T, c, c_2 *are as in Theorem 8.1.*

Proof. The inequality (8.3) follows from Theorem 3.3. To prove (8.4) notice that since $u_M \geq 0$, $\beta_M \leq \beta$,

$$\frac{\partial u_M}{\partial t} + 3k(t) \frac{\partial}{\partial x} (x^{2/3} u_M) \leq \frac{1}{2} \int_1^{x-1} \beta(x-y, y) u_M(x-y,t) u_M(y,t)\, dy.$$

On the other hand, since $0 < \alpha \leq 2/3$,

$$x^\alpha \left\{ \frac{\partial u_M}{\partial t} + 3k(t) \frac{\partial}{\partial x} (x^{2/3} u_M) \right\} \geq \frac{\partial}{\partial t} (x^\alpha u_M) + 3k(t) x^{2/3} \frac{\partial}{\partial x} (x^\alpha u_M).$$

Combining these inequalities we get

$$(8.5)\quad \frac{\partial}{\partial t}\,(x^\alpha u_M) + 3k(t)x^{2/3}\frac{\partial}{\partial x}(x^\alpha u_M) \le \frac{1}{2}\,x^\alpha \int\limits_1^{x-1} \beta(x-y,y)u_M(x-y,t)u_M(y,t)\;dy.$$

Next, using (8.1), (8.3),

$$\begin{aligned} x^\alpha &\int\limits_1^{x/2} \beta(x-y,y)u_M(x-y,t)u_M(y,t)\;dy \\ &\le c_0 x^\alpha \int\limits_1^{x/2} (x-y)^{1-\gamma}u_M(x-y,t)u_M(y,t)\;dy \\ &\le 2^\alpha c_0 \int\limits_1^{x/2} (x-y)u_M(x-y,t)u_M(y,t)\;dy \le 2^\alpha c_0 \nu_T\;N_M(x,t) \end{aligned}$$

where

$$N_M(x,t) = \sup_{\substack{1\le y\le x\\ t\le T}} y^\alpha u_M(y,t),$$

and similarly

$$\begin{aligned} x^\alpha &\int\limits_{x/2}^{x-1} \beta(x-y,y)u_M(x-y,t)u_M(y,t)\;dy \\ &\qquad \le 2^\alpha c_0 \int\limits_{x/2}^{x-1} y u_M(y,t)u_M(x-y,t)\;dy \le 2^\alpha c_0\nu_T N_M(x,t). \end{aligned}$$

Using these estimates to bound the right–hand side in (8.5), we find that

$$\frac{\partial}{\partial t}\,(x^\alpha u_M) + 3k(t)x^{2/3}\,\frac{\partial}{\partial x}\,(x^\alpha u_M) \le c\nu_T\;N_M(x,t).$$

Integrating this inequality along characteristics we obtain

$$x^\alpha u_M(x,t) \le x^\alpha u_*(x,t) + c\nu_T \int\limits_{\tau(x,t)}^{t} N_M\big(\phi(x,t,s),s\big)\;ds$$

and, since $\phi(x,t,s) \leq x$,

$$x^{\alpha} u_M(x,t) \leq c(c_2 + \|h\|_{L^\infty(0,T)}) + c\nu_T \int_0^t N_M(x,s)\, ds.$$

This easily implies the assertion (8.4).

We return to the proof of Theorem 8.1 and set

$$R_1 = \left\{ (x,t);\ x \geq 1,\ 0 \leq t \leq T,\ x \leq \left(1 + \int_0^t k(r)dr\right)^3 \right\},$$
$$R_2 = \left\{ (x,t);\ x \geq 1,\ 0 \leq t \leq T,\ x \geq \left(1 + \int_0^t k(r)dr\right)^3 \right\},$$
$$R_1^j = R_1 \cap \{x \leq j\},\quad R_2^j = R_2 \cap \{x \leq j\} \quad \text{for } j = 2, 3, \dots\ .$$

We shall first show that the sequence $\{u_M\}_{M=1}^{\infty}$ is uniformly bounded and equicontinuous on R_1^2 and on R_2^2. Let

$$\gamma_M(x,t) = \int_1^\infty \beta_M(x,y) u_M(y,t)\, dy.$$

Using (8.3) of Lemma 8.2 and the assumption (8.1) (v) it is easy to see that for fixed $j \geq 2$:

(8.6) $\gamma_M(\cdot,t)$ is equicontinuous in x for $1 \leq x \leq j$ (with t, M as parameters)

(8.7) $\gamma_M(x,t) \leq 2c_0 \nu_T j$ for $1 \leq x \leq j, M \geq 1,\ t \leq T$.

Since

$$\frac{\partial}{\partial t}\, u_M + 3k\, \frac{\partial}{\partial x}\, (x^{2/3} u_M) = -u_M \gamma_M \quad \text{if } x \leq 2,$$

we have

(8.8) $$u_M(x,t) = u_*(x,t) e^{-\int_0^t \gamma_M(\phi(x,t,s),s)\, ds}.$$

Using (8.6), (8.7) we conclude that $\{u_M\}$ is uniformly bounded and equicontinuous family on R_1^2 and on R_2^2; hence a subsequence $\{u_M^1\}$ is uniformly convergent to some function u^1, which is continuous on R_1^2 and on R_2^2.

Next let $1 \le x \le 3,\ 0 \le t \le T$ and define

$$f_M(x,t) = \begin{cases} 0 \quad \text{if} \quad x \le 2 \\ \dfrac{1}{2} \displaystyle\int_1^{x-1} \beta(x-y,y) u_M^1(x-y,t) u_M^1(y,t)\, dy \text{ if } 2 \le x \le 3. \end{cases}$$

The sequence f_M is uniformly bounded and equicontinuous. Since

$$\begin{aligned} &\frac{d}{ds}\left(\phi(x,t,s)^{2/3} u_M^1(\phi(x,t,s),s))\right) \\ &\quad + \phi(x,t,s)^{2/3} u_M^1(\phi(x,t,s),s)\gamma_M(\phi(x,t,s),s) = \phi(x,t,s)^{2/3} f_M(\phi(x,t,s),s), \end{aligned}$$

we deduce that the sequence $\{u_M^1\}$ has a subsequence which is uniformly convergent on R_1^3 and on R_2^3.

We can proceed in this way step–by–step and obtain a subsequence u_{M^*} of the original sequence u_M, which is uniformly convergent on each R_1^j on each R_2^j. By diagonalization we may, in fact, conclude that the convergence is uniform in every bounded subset of

$$\{x \ge 1,\ t \ge 0, x \le (1 + \int_0^t k)^3\}$$

and in every bounded subset of

$$\{x \ge 1,\ t \ge 0,\ x \ge (1 + \int_0^t k)^3\}.$$

In view of the estimate (8.3), the coagulation operator $\mathcal{C}_M$, corresponding to β_M, satisfies:

$$\mathcal{C}_M[u_M] \to \mathcal{C}[u] \qquad \text{if} \qquad M = M^* \to \infty,$$

uniformly on compact subsets. It follows that u is a solution of (2.1)–(2.3).

Regarding uniqueness we can only establish a very partial result:

THEOREM 8.3. *Under the assumptions of Theorem 8.1, if* $\beta(x,y) \le \lambda xy$ $(\lambda > 0)$ *and* $u_0(x) \equiv 0$ *then the solution is unique for* $0 \le t < c_0$, *where* c_0 *is a universal constant.*

Notice that by scaling u we can always assume that $\lambda = 1$.

Proof. Since the proof is similar to the uniqueness proof of McLeod [7] for (1.13), we describe it only briefly.

Suppose $\tilde{u}$ is another solution. Then

$$
\begin{aligned}
(8.9)\ \frac{\partial}{\partial t}\,(u-\tilde{u})+3k(t)\frac{\partial}{\partial x}\,\big(x^{2/3}(u-\tilde{u})\big) &= -\frac{1}{2}\,(u-\tilde{u})\int\limits_{1}^{\infty}\beta(x,y)\big(u(y,t)+\tilde{u}(y,t)\big)dy \\
&\quad -\frac{1}{2}\,(u+\tilde{u})\int\limits_{1}^{\infty}\beta(x,y)\big((y,t)-\tilde{u}(y,t)\big)dy \\
&\quad +\frac{1}{2}\int\limits_{1}^{x-1}\beta(x-y,y)\big(u(y,t)-\tilde{u}(y,t)\big)\big(u(x-y,t)+\tilde{u}(x-y,t)\big)dy.
\end{aligned}
$$

Multiplying (8.9) by

$$
F(r)=\exp\left\{\int\limits_{\tau(x,t)}^{r}\frac{1}{2}\int\limits_{1}^{\infty}\beta(\phi,y)\big(u(y,s)+\tilde{u}(y,s)\big)dyds\right\}
$$

where $\phi=\phi(x,t,s)$ and integrating along the characteristic $\big(\phi(x,t,s),s\big)$, we obtain (using that F is increasing for $\tau\le r\le t$)

$$
\begin{aligned}
|u(x,t)-\tilde{u}(x,t)| &\le \frac{1}{2}\int\limits_{\tau(x,t)}^{t} x^{-2/3}\phi^{2/3}\big(u(\phi,s) \\
(8.10)\qquad &\quad +\tilde{u}(\phi,s)\big)\int\limits_{1}^{\infty}\beta(\phi,y)|u(y,s)-\tilde{u}(y,s)|dyds \\
&\quad +\frac{1}{2}\int\limits_{\tau}^{t}\int\limits_{1}^{\phi-1} x^{-2/3}\phi^{2/3}\beta(\phi-y,y)|u(y,s)-\tilde{u}(y,s)|\big(u(\phi-y,s)+\tilde{u}(\phi-y,s)\big)dyds.
\end{aligned}
$$

Next, using (2.1)-(2.3) one estimates inductively on i

$$
(8.11)\qquad u(x,t),\tilde{u}(x,t)\le C_i\frac{t^{i-1}}{(i+1)}\qquad\text{for}\qquad i\le x\le i+1
$$

where

$$
(8.12)\qquad \sum_{i=1}^{\infty}C_it^{i-1}\le\gamma_1
$$

if t is sufficiently small, say $t<c_1$; here γ_1, c_1 are universal constants.

Now we can use (8.11) to estimate the first integral on the right-hand side of (8.10) and obtain

$$|u(x,t) - \tilde{u}(x,t)| \leq \frac{K_1 t}{2} \quad \text{for} \quad 1 \leq x \leq 2. \tag{8.13}$$

where $K_1 = \gamma_1 C_1^2$. Using (8.10)-(8.13) we can inductively prove that

$$|u(x,t) - \tilde{u}(x,t)| \leq \frac{K_i t^i}{(i+1)}\ , \ i \leq x \leq i+1 \tag{8.14}$$

where

$$\sum_{j=1}^{\infty} K_j t^j \leq \gamma_2 t \qquad \text{for} \qquad t < c_2. \tag{8.15}$$

Now we can go through the argument again, using (8.14) instead of (8.11); this results in an improvement of (8.14). Continuing in this way step-by-step m times, one arrives at the inequality

$$|u(x,t) - \tilde{u}(x,t)| \leq (c_0^{-1} t)^m \frac{K_i t^i}{(i+1)}\ , \ i \leq x \leq i+1 \tag{8.16}$$

where c_0 is a universal constant and m is any positive integer. Taking $m \to \infty$ we deduce that $u \equiv \tilde{u}$ if $t < c_0$.

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