THE TIME-HARMONIC MAXWELL EQUATIONS IN A DOUBLY PERIODIC STRUCTURE

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THE TIME-HARMONIC MAXWELL EQUATIONS IN A DOUBLY PERIODIC STRUCTURE*

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Abstract. Consider the diffraction of a beam of particles in $\mathbb{R}^3$ when the dielectric coefficient is a constant ε_1 above a surface S and a (different) constant ε_2 below S, and the magnetic permeability is constant μ throughout $\mathbb{R}^3$. S is assumed to be a doubly periodic surface, say $z = f(x, y)$ with $f(x + mL_1, y + nL_2) = f(x, y)$ for all integers m, n. The existence and uniqueness of a solution satisfying a "radiation condition" at infinity is reduced to a system of Fredholm equations. Thus, for all but a discrete set of $\varepsilon's$ there exists a unique solution.

Key words. Maxwell's equations, transmission, reflection, diffraction, Fredholm equations.

AMS(MOS) subject classifications. Primary: 78A10, 78A45. Secondary: 35P25, 47A40.

§0. Introduction. Consider a surface $S : z = f(x, y)$ where f is doubly periodic, i.e., $f(x + mL_1, y + nL_2) = f(x, y)$ for all integers m, n. (Actually we shall consider a more general periodic surface; see §1). Above S the dielectric coefficient is a constant ε_1 and below it the dielectric coefficient is another constant $\varepsilon_2, \varepsilon_2 \neq \varepsilon_1$. The magnetic permeability is assumed to be constant μ throughout $\mathbb{R}^3$. We are interested in solving the scattering problem for a linear electromagnetic wave coming down from $z = +\infty$. The case where f is independent of y, i.e., $f = f(x)$ was recently studied by Chen and Friedman [3]; they reduced the problem to a coupled pair of Fredholm equations.

The problem studied in this paper belongs to the subject which is commonly referred to as diffractive optics in grating material. Interesting engineering and numerical aspects can be found in the collection of articles edited by R. Petit [9]. Related work was done by Nedelec and Starling [8] who studied periodic arrays of antennas, and Bellout and Friedman [1] who considered scattering by stripe grating for the Schrödinger equation.

The approach of this paper, like the one in [3], is based on integral equations: we represent the solution in terms of a "fundamental solution," and use the transmission conditions to derive integral equations for the "surface fields." However, the present paper differs from [3] in a number of important technical details.

Numerical schemes for solving the Maxwell equations by integral equations have been introduced by Benaldi [2].

In §1 we introduce the periodic Maxwell equations. Constructing, in §2, a periodic fundamental solution for the appropriate elliptic equation, we use it in §3 to obtain an integral representation for the periodic components E_α, H_α of the solution. In §4 we study

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integral operators which appear in the representation of E_α, H_α and then, in §5, we derive jump relations. These lead, in §6, to integral equations for the surface field I, J on S. Additional analysis shows that the equations can be reduced to a Fredholm type system of equations. Consequently, when the first alternative holds, there is a unique solution. This solution satisfies also appropriate "radiation conditions" at infinity. Further, when the first alternative holds, there is no other solution of the Maxwell equations which is periodic and satisfies the "radiation condition" at infinity.

We finally mention that the scattering problem studied in this paper is motivated by industrial manufacturing processes which use microelectronic tools to make optical devices [5].

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§1. The Periodic Maxwell Equations. Let S be a surface imbedded in $\mathbb{R}^3$ with C^2 parametrization $F : \mathbb{R}^2 \to \mathbb{R}^3$:

$$F(u) = (f_1(u), f_2(u), f_3(u)),$$

where $u = (u_1, u_2) \in \mathbb{R}^2$ (for piecewise smooth S, see Remark 6.1). Let L_1, L_2 be positive constants. Define the lattice

$$\Lambda = L_1\mathbb{Z} \times L_2\mathbb{Z} \times \{0\} \subset \mathbb{R}^3 , \qquad \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} .$$

We assume that S is periodic with respect to Λ in the sense that

$$(S \cap \{x\}) + n = S \cap \{x + n\}$$

for all $n = (n_1, n_2, 0) \in \Lambda$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We also assume that F is periodic: There exists $l_1, l_2 > 0$ such that

$$F\left(u_1 + \frac{l_1 n_1}{L_1} , u_2 + \frac{l_2 n_2}{L_2}\right) = F(u) + n$$

for all $n \in \Lambda$, $u \in \mathbb{R}^2$. The parametrization F can be viewed as a diffeomorphism from the torus

$$Q = \mathbb{R}^2/(l_1\mathbb{Z} \times l_2\mathbb{Z})$$

onto

$$S_0 = S/\Lambda .$$

Note that there must exist a constant M such that

$$|f_3(u)| \le M \qquad \text{for all } \ u .$$

S divides $\mathbf{R}^3$ into two open connected components. Let Ω_1 denote the component containing some element x such that $x_3 > M$, and let $\Omega_2 = (S \cup \Omega_1)^c$.

Suppose that Ω_1 and Ω_2 are filled with materials in such a way that the magnetic permeability μ is constant throughout $\mathbf{R}^3$, and the dielectric coefficient ε satisfies

$$\varepsilon(x) = \begin{cases} \varepsilon_1 & \text{in} \quad \Omega_1 \\ \varepsilon_2 & \text{in} \quad \Omega_2 \end{cases}$$

where ε_1 and ε_2 are complex constants, $\varepsilon_1 \neq \varepsilon_2$. We assume that $Re\ \varepsilon > 0$, $Im\ \varepsilon \geq 0$. The case $Im\ \varepsilon > 0$ accounts for absorption [4].

Let E, H denote complex vector fields on $\mathbf{R}^3$. Setting

$$E^j = E|_{\Omega_j} \ , \ H^j = H|_{\Omega_j} \qquad (j = 1, 2) \ ,$$

the time-harmonic Maxwell equations in each Ω_j are

$$\nabla \times E^j - i\omega\mu H^j = 0 \quad \text{in} \quad \Omega_j, \tag{1.1}$$

$$\nabla \times H^j + i\omega\varepsilon_j E^j = 0 \quad \text{in} \quad \Omega_j. \tag{1.2}$$

(The customary constant c representing the speed of light has been absorbed into the "frequency" ω.) The weak form of Maxwell's equations in a neighborhood of S give the following jump relations on S:

$$\eta \times (E^1 - E^2) = 0, \tag{1.3}$$

$$\eta \times (H^1 - H^2) = 0, \tag{1.4}$$

$$\eta \cdot (\varepsilon_1 E^1 - \varepsilon_2 E^2) = 0, \tag{1.5}$$

$$\eta \cdot (H^1 - H^2) = 0 \ , \tag{1.6}$$

where η is normal to S. It can be shown [3] that (1.5), (1.6) follow from (1.1)–(1.4). Our goal is to solve (1.1)–(1.4) when a linearly polarized plane wave

$$E_*(x) = se^{iq\cdot x} \ , \ H_*(x) = pe^{iq\cdot x}$$

is incident on S from Ω_1. Here, $p, q \in \mathbf{R}^3$ are constants representing the polarization and incidence, respectively, and $s = \dfrac{1}{\omega\varepsilon_1}\ (p \times q)$. The vectors p, q must satisfy

$$q \cdot q = \omega^2 \varepsilon_1 \mu \ , \quad p \cdot q = 0 \ ,$$

in order for E_*, H_* to satisfy (1.1), (1.2) in Ω_1.

In the particular case $S = \{x \in \mathbb{R}^3 : x_3 = 0\}$, the corresponding solution of Maxwell's equations (1.1)–(1.4) can be found explicitly:

$$
E_0(x) = \begin{cases} s e^{iq\cdot x} + s_r e^{i\tilde{q}\cdot x} & \text{in } \Omega_1 \\ s_t e^{i\hat{q}\cdot x} & \text{in } \Omega_2 \,, \end{cases}
$$

(1.7)

$$
H_0(x) = \begin{cases} p e^{iq\cdot x} + p_r e^{i\tilde{q}\cdot x} & \text{in } \Omega_1 \\ p_t e^{i\hat{q}\cdot x} & \text{in } \Omega_2 \end{cases}
$$

where

$$
\begin{aligned}
&\tilde{q} = (q_1, q_2, -q_3), \\
&\hat{q} = (q_1, q_2, \hat{q}_3) \ , \ \hat{q}_3 = [\nu q \cdot q - (q \cdot \tilde{q})]^{\frac{1}{2}}, \\
&\nu = \frac{\varepsilon_2}{\varepsilon_1} \ .
\end{aligned}
$$

The reflection vectors p_r, s_r and the transmission (or refraction) vectors p_t, s_t are determined as follows. It is convenient to write

$$
p = p' + p''
$$

where $p' \cdot e_3 = 0$ (we denote the standard unit basis vectors for $\mathbb{R}^3$ by e_1, e_2, e_3). In the case $p'' = 0$, the incident wave is called *transverse magnetic* (TM), and we have

$$
\begin{aligned}
p_r &= \frac{q_3\nu - \hat{q}_3}{q_3\nu + \hat{q}_3}\, p \ , \ p_t = \frac{2q_3\nu}{q_3\nu + \hat{q}_3}\, p \ , \\
s_r &= \frac{1}{\omega\varepsilon_1}\,(p_r \times \tilde{q}) \ , \ s_t = \frac{1}{\omega\varepsilon_1}\,(p_t \times \hat{q}) \ .
\end{aligned}
$$

In the case $p' = 0$, the incident wave is called *transverse electric* (TE), and we have

$$
\begin{aligned}
s_r &= \frac{q_3 - \hat{q}_3}{q_3 + \hat{q}_3}\, s \ , \ s_t = \frac{2q_3}{q_3 + \hat{q}_3}\, s \ , \\
p_r &= -\frac{\omega\varepsilon_1}{\tilde{q}\cdot\tilde{q}}\,(s_r \times \tilde{q}) \ , \ p_t = -\frac{\omega\varepsilon_1}{\hat{q}\cdot\hat{q}}\,(s_t \times \hat{q}) \ .
\end{aligned}
$$

In the general case, E_0 and H_0 can be expressed as the sum of TM and TE components.

Given the incidence vector $q = (q_1, q_2, q_3)$, let $\alpha = (q_1, q_2, 0)$ and define

$$E_\alpha(x) = e^{-i\alpha\cdot x}E(x)\ , \quad H_\alpha(x) = e^{-i\alpha\cdot x}H(x)\ .$$

Maxwell's equations (1.1)–(1.4) are then equivalent to

$$\left.\begin{aligned} &(\nabla + i\alpha)\times E_\alpha^j - i\omega\mu H_\alpha^j = 0 && (1.1') \\ &(\nabla + i\alpha)\times H_\alpha^j + i\omega\varepsilon_j E_\alpha^j = 0 && (1.2') \end{aligned}\right\} \quad \text{in} \quad \Omega_j,$$

$$\left.\begin{aligned} &\eta\times(E_\alpha^1 - E_\alpha^2) = 0 && (1.3') \\ &\eta\times(H_\alpha^1 - H_\alpha^2) = 0 && (1.4') \end{aligned}\right\} \quad \text{on} \quad S\ .$$

We are interested in fields E, H for which E_α, H_α are periodic over the lattice Λ. Thus we are led to search for solutions to $(1.1') - (1.4')$ in the quotient space $\mathbb{R}^3/\Lambda$.

§2. Periodic Fundamental Solution. Let $k = \omega(\mu\varepsilon)^{\frac{1}{2}}$ where $Re k > 0$, $Im k \geq 0$. In this section we construct a fundamental solution to the equation

$$\{(\nabla + i\alpha)^2 + k^2\}u = 0 \tag{2.1}$$

on the space $\mathbb{R}^3/\Lambda$; here $(\nabla + i\alpha)^2$ means $(\Delta + 2i\alpha\cdot\nabla - |\alpha|^2)$. It is easily seen that if E_α, H_α satisfy $(1.1'), (1.2')$ then E_α, H_α satisfy (2.1) individually. In the next section we shall use the fundamental solution to obtain an integral representation for E_α, H_α.

For $n = (n_1, n_2, 0) \in \Lambda$, let

$$\alpha_n = \left(\frac{2\pi n_1}{L_1^2}\ ,\ \frac{2\pi n_2}{L_2^2}\ ,\ 0\right)\ . \tag{2.2}$$

We shall henceforth assume that for $j = 1, 2$,

$$k^2|_{\Omega_j} \neq |\alpha_n - \alpha|^2 \qquad \text{for all} \quad n\in\Lambda\ . \tag{2.3}$$

If $Im k > 0$ then (2.3) certainly holds; otherwise, an arbitrarily small perturbation of k satisfies (2.3). We set

$$\beta_n = e^{i\theta/2}|k^2 - |\alpha_n - \alpha|^2|^{\frac{1}{2}}\ , \quad n\in\Lambda\ , \tag{2.4}$$

where

$$\theta = \arg(k^2 - |\alpha_n - \alpha|^2)\ , \quad 0\leq\theta< 2\pi\ ,$$

and define the Λ-periodic sum

$$\psi_k(x) = \frac{i}{2L_1L_2} \sum_{n\in\Lambda} \frac{1}{\beta_n} e^{i\beta_n|x_3|-i\alpha_n\cdot x} \;. \tag{2.5}$$

Notice that if $Imk = 0$ then

$$\beta_n = \begin{cases} (k^2-|\alpha_n-\alpha|^2)^{\frac{1}{2}} & \text{for} \quad k^2 > |\alpha_n-\alpha|^2 \\ i(|\alpha_n-\alpha|^2-k^2|)^{\frac{1}{2}} & \text{for} \quad k^2 < |\alpha_n-\alpha|^2 \;, \end{cases} \tag{2.6}$$

so $Im\beta_n = 0$ for at most a finite set of n, and $Im\beta_n > 0$ for all other n. In the case $Imk > 0$, we have $Im\beta_n > 0$ always. Thus for $x_3 \neq 0$ the terms in the sum (2.5) decay exponentially as $|n| \to \infty$ and the sum converges absolutely.

LEMMA 2.1. *The sum ψ_k satisfies*

$$\{(\nabla + i\alpha)^2 + k^2\}\psi_k = -\sum_{n\in\Lambda} \delta_n \quad in \quad \mathcal{D}'$$

where δ_n is the Dirac measure at n.

Proof. Let $a_n = \dfrac{1}{\beta_n} e^{i\beta_n|x_3|-i\alpha_n\cdot x}$, $\psi^N = \dfrac{i}{2L_1L_2} \displaystyle\sum_{|n|\le N} a_n$. Using the fact that $\beta_n^2 = k^2 - |\alpha_n-\alpha|^2$, we have

$$\begin{aligned} \Delta\psi^N &= \frac{i}{2L_1L_2} \sum_{|n|\le N} (-|\alpha_n|^2 - \beta_n^2)a_n - \frac{1}{L_1L_2} \sum_{|n|\le N} \beta_n a_n \delta(x_3) \\ &= (|\alpha|^2 - k^2)\psi^N - 2i\alpha\cdot\nabla\psi^N - \frac{1}{L_1L_2}\sum \beta_n a_n\delta(x_3) \;. \end{aligned}$$

Hence, for $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} \{(\nabla + i\alpha)^2 + k^2\}\psi^N(\varphi) &= -\frac{1}{L_1L_2} \sum \int_{\mathbb{R}^2} \varphi(x_1,x_2,0)e^{-i\alpha_n\cdot x}dx \\ &= -\frac{1}{L_1L_2} \sum_{|n|\le N} \widehat{\varphi}\left(\frac{2\pi n_1}{L_1^2} , \frac{2\pi n_2}{L_2^2}\right) \end{aligned}$$

(where the Fourier transform is with respect to the first two variables with $x_3 = 0$.) Letting $N \to \infty$ and applying the Poisson summation formula [6; (7.2.1)′] yields

$$\frac{1}{L_2L_2} \sum_{n\in\Lambda} \widehat{\varphi}\left(\frac{2\pi n_1}{L_1^2} , \frac{2\pi n_2}{L_2^2}\right) = \sum_{n\in\Lambda} \varphi(n) \;,$$

which completes the proof. $\square$

REMARK 2.1. The formula (2.5) can be obtained formally by applying the Poisson summation formula directly to the periodic sum

$$\sum_{n\in\Lambda} e^{-i\alpha\cdot(x+n)}\varphi_k(x+n) ,$$

where $\varphi_k(x) = \dfrac{e^{ik|x|}}{4\pi|x|}$ is the fundamental solution of the Helmholtz equation $(\Delta+k^2)u = 0$. This approach can be rigorously justified in the case of a singly-periodic surface S; see [3].

REMARK 2.2. The formula (2.5) is well-known and is related to the so-called Rayleigh expansion for diffraction gratings; see [9].

§3. Periodic Integral Representation. We will show that solutions E_α, H_α to $(1.1'), (1.2')$ can be represented inside each Ω_j in the form

$$\begin{aligned} E_\alpha(x) = \int_{S_0} \{ & i\omega\mu I(y)\psi_k(x-y) - J(y)\times \mathfrak{U}_k(x-y) \\ & + \frac{1}{i\omega\varepsilon}[\nabla_x(I(y)\cdot\mathfrak{U}_k(x-y)) + (I(y)\cdot i\alpha)\mathfrak{U}_k(x-y)]\} dS_y \end{aligned} \tag{3.1}$$

$$\begin{aligned} H_\alpha(x) = \int_{S_0} \{ & i\omega\varepsilon J(y)\psi_k(x-y) + I(y)\times \mathfrak{U}_k(x-y) \\ & + \frac{1}{i\omega\mu}[\nabla_x(J(y)\cdot\mathfrak{U}_k(x-y)) + (J(y)\cdot i\alpha)\mathfrak{U}_k(x-y)]\} dS_y \end{aligned} \tag{3.2}$$

where

$$\mathfrak{U}_k(x-y) = -(\nabla_x + i\alpha)\psi_k(x-y) \tag{3.3}$$

and I, J are "surface fields" on S_0: vector fields on S_0 such that $I\cdot\eta = J\cdot\eta = 0$.

THEOREM 3.1. *For surface fields I, J with integrable components, and for $x \notin S_0$, the fields (3.1), (3.2) satisfy* $(1.1'), (1.2')$.

Proof. We will show $(1.1')$; the proof of $(1.2')$ is similar. Using the relations $\nabla_y\psi_k(x-y) = -\nabla_x\psi_k(x-y)$, $\nabla\times\nabla = 0$, and $\nabla\times(\nabla\times g) = \nabla(\nabla\cdot g) - \Delta g$, one can establish the identities:

$$(\nabla_x + i\alpha)\times\int I\psi_k = \int I\times\mathfrak{U}_k ,$$

$$(\nabla_x + i\alpha)\times\int(-J)\times\mathfrak{U}_k = -\int k^2 J\psi_k + \nabla_x\int J\cdot\mathfrak{U}_k + \int(J\cdot i\alpha)\mathfrak{U}_k ,$$

$$(\nabla_x + i\alpha)\times\nabla_x\int I\cdot\mathfrak{U}_k = 0 , \ (\nabla_x + i\alpha)\times\int(I\cdot i\alpha)\mathfrak{U}_k = 0 .$$

Then from (3.1), (3.2) (noting periodicity)

$$(\nabla_x + i\alpha) \times E_\alpha(x) = \int\limits_{S_0} \{-k^2 J\psi_k + i\omega\mu I \times \mathfrak{U}_k + \nabla_x(J \cdot \mathfrak{U}_k) + (J \cdot i\alpha)\mathfrak{U}_k\} dS_y$$

$$= i\omega\mu H_\alpha(x) \ . \qquad \square$$

The representation (3.1), (3.2) can be simplified a little bit by performing an integration by parts. We need some notation first.

At each point $p = F(u)$ on S, the parametrization F determines the "coordinate frame"

$$F_1 = \left(\frac{\partial f_1}{\partial u_1}, \frac{\partial f_2}{\partial u_1}, \frac{\partial f_3}{\partial u_1}\right)_p \ , \ F_2 = \left(\frac{\partial f_1}{\partial u_2}, \frac{\partial f_2}{\partial u_2}, \frac{\partial f_3}{\partial u_2}\right)_p \ .$$

The induced Riemannian metric tensor on S_0 is given by $\Phi = (g_{ij}) = (F_i \cdot F_j)$, and the volume element is

$$\sigma(u) du = (\det \Phi)^{\frac{1}{2}} du_1 \wedge du_2 \ .$$

Each surface field I on S_0 can be written in the form

$$I = I_1 F_1 + I_2 F_2$$

where I_1, I_2 are scalar functions on the torus Q. We write the covariant divergence of I

$$\nabla_0 \cdot I = \frac{1}{\sigma}\left[\frac{\partial}{\partial u_1}(\sigma I_1) + \frac{\partial}{\partial u_2}(\sigma I_2)\right] \ .$$

LEMMA 3.2. *If $x \notin S_0$ and I is a surface field on S_0 then*

$$\int\limits_{S_0} I(y) \cdot \nabla_y \psi_k(x-y) dS_y = -\int\limits_{S_0} (\nabla_0 \cdot I)(y) \psi_k(x-y) dS_y \ .$$

Proof. Since $x \notin S_0$, $\psi_k(x-y)$ is smooth. Integrating by parts on the torus Q:

$$\int\limits_{S_0} I \cdot \nabla_y \psi_k(x-y) dS_y = \int\limits_Q \{(I_1 F_1 + I_2 F_2) \cdot (\nabla_y \psi_k)(x - F(u))\} \sigma du$$

$$= \int\limits_Q \{\sigma(I_1, I_2) \cdot \nabla_u(\psi_k(x - F(u)))\} du$$

$$= -\int\limits_Q \nabla_u \cdot (\sigma I) \psi_k(x - F(u)) du$$

$$= -\int\limits_{S_0} (\nabla_0 \cdot I) \psi_k(x-y) dS_y \ . \qquad \square$$

Using the identity $\nabla_x \psi_k(x-y) = -\nabla_y \psi_k(x-y)$ and Lemma 3.2, (3.1) and (3.2) can be written in the form

$$E_\alpha(x) = \int_{S_0} \{i\omega\mu I \psi_k - J \times \mathfrak{U}_k + \frac{1}{i\omega\varepsilon} (\nabla_0 \cdot I + i\alpha \cdot I)\mathfrak{U}_k\} dS_y \ , \tag{3.4}$$

$$H_\alpha(x) = \int_{S_0} \{i\omega\varepsilon J \psi_k + I \times \mathfrak{U}_k + \frac{1}{i\omega\mu} (\nabla_0 \cdot J + i\alpha \cdot J)\mathfrak{U}_k\} dS_y \ . \tag{3.5}$$

Finally, note that (3.4), (3.5) are equivalent to

$$E_\alpha(x) = \int_{Q} \{i\omega\mu I \widetilde{\psi}_k - J \times \widetilde{\mathfrak{U}}_k + \frac{1}{i\omega\varepsilon} (\nabla_0 \cdot I + i\alpha \cdot I)\widetilde{\mathfrak{U}}_k\} \sigma du, \tag{3.6}$$

$$H_\alpha(x) = \int_{Q} \{i\omega\varepsilon J \widetilde{\psi}_k + I \times \widetilde{\mathfrak{U}}_k + \frac{1}{i\omega\mu} (\nabla_0 \cdot J + i\alpha \cdot J)\widetilde{\mathfrak{U}}_k\} \sigma du \ , \tag{3.7}$$

where $\widetilde{\psi}_k(x,u) = \psi_k(x - F(u))$, $\widetilde{\mathfrak{U}}_k(x,u) = \mathfrak{U}_k(x - F(u))$.

§4. Some Integral Operators.

We wish to evaluate

$$\lim_{x \to p \in S} E_\alpha(x)$$

using the integral representation of the last section. Singularities occur due to the $\nabla\psi$ term in the kernel $\mathfrak{U}$. In this section we analyze the singular integral operator associated with the kernel $\nabla\psi$. Let

$$\varphi_0(x) = \frac{1}{4\pi|x|} \ ; \text{ so } \ \nabla\varphi_0(x) = \frac{x}{4\pi|x|^3} \ .$$

Denote

$$Q_0 = \left[-\frac{L_1}{2} \ , \ \frac{L_1}{2}\right] \times \left[-\frac{L_2}{2} \ , \ \frac{L_2}{2}\right] \subset \mathbb{R}^2 \ .$$

We define

$$\nabla\psi_0(x) = \sum_{n \in \Lambda} \nabla\varphi_0(x+n) \ ,$$

where convergence is interpreted in the sense

$$\nabla\psi_0^N \equiv \sum_{|n_1|,|n_2| \le N} \nabla\varphi_0(x+n) \to \nabla\psi_0(x) \quad \text{as} \quad N \to \infty \ . \tag{4.1}$$

Convergence is uniform in x over compact sets in the complement of Λ.

LEMMA 4.1. *There exists a constant c such that*

$$|\nabla\psi_k(x) - \nabla\varphi_0(x)| \le \frac{c}{|x|}$$

for all $x \in Q_0 \times \mathbb{R}$.

Proof. Set $\varphi_{k,\alpha} = e^{-i\alpha\cdot x}\varphi_k$, where $\varphi_k = \dfrac{e^{ik|x|}}{4\pi|x|}$. Then

$$\{(\nabla + i\alpha)^2 + k^2\}\varphi_{k,\alpha} = -\delta_0 \quad \text{in} \quad \mathcal{D}' , \tag{4.2}$$

as follows easily from the fact that $(\nabla + k^2)\varphi_k = -\delta_0$. From (4.2) and Lemma 2.1,

$$\{(\nabla + i\alpha)^2 + k^2\}(\psi_k - \varphi_{k,\alpha}) = 0$$

in $Q_0 \times \mathbb{R}$. It follows by elliptic regularity that $(\psi_k - \varphi_{k,\alpha})$ is smooth in $Q_0 \times \mathbb{R}$. Finally, it is easy to estimate

$$|\nabla\varphi_{k,\alpha} - \nabla\varphi_0| \le \frac{c}{|x|} . \qquad \square$$

At each point $p = F(v) \in S_0$, we select the unit normal

$$\eta(v) = \frac{-F_1 \times F_2}{|F_1 \times F_2|}(v) . \tag{4.3}$$

We define the 3×3 matrix A_v by

$$A_v^{-1} = \begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 & \eta_1 \\ \partial_1 f_2 & \partial_2 f_2 & \eta_2 \\ \partial_1 f_3 & \partial_2 f_3 & \eta_3 \end{pmatrix} .$$

Notice that A_v^{-1} maps the coordinate frame (e_1, e_2, e_3) to the frame $(F_1, F_2, -\eta)$ at $p = F(v)$.

Using the notation

$$(v - u, z) = (v_1 - u_1)e_1 + (v_2 - u_2)e_2 + ze_3$$

where $u, v \in Q,\ z \in \mathbb{R}$, and $(v - u)$ means $(v - u)|_{Q_0} \in \mathbb{R}^2$, define

$$\begin{aligned} R_{k,z}(u, v) &= \nabla\psi_0(A_v^{-1}(v - u, z)) \\ &\quad - \nabla\psi_k(F(v) - F(u) + z\eta(v)) . \end{aligned} \tag{4.4}$$

LEMMA 4.2. *For all* $u, v \in Q$, $|z| \le 1$,

$$|R_{k,z}(u,v)| \le \frac{c}{|u-v|} .$$

Proof. From Lemma 4.1, we need only show that

$$|\nabla\varphi_0(F(v) - F(u) + z\eta(v)) - \nabla\varphi_0(A_v^{-1}(v-u,z))| \le \frac{c}{|u-v|} .$$

For each v, we can change coordinate systems and without loss of generality consider the case where $F(v) - F(u) = (u_1, u_2, f(u)), A_v^{-1} = \mathbf{1}$ and $|f(u)| \le c|u|^2$. (Although c may depend on v, there is a uniform bound.) Also, since $\nabla\varphi_0$ is smooth away from the origin, we can assume that $|u| \le 1$. We must estimate

$$\left| \frac{(u_1, u_2, f(u)+z)}{|u_1^2 + u_2^2 + (f(u)+z)^2|^{\frac{3}{2}}} - \frac{(u_1, u_2, z)}{|u_1^2 + u_2^2 + z^2|^{\frac{3}{2}}} \right| .$$

Setting $r^2 = u_1^2 + u_2^2$, it suffices to consider

$$\left| \frac{(r, f+z)}{|r^2 + (f+z)^2|^{\frac{3}{2}}} - \frac{(r,z)}{|r^2 + z^2|^{\frac{3}{2}}} \right| .$$

The first component is

$$K_1 = \frac{r}{N} \left| |r^2 + z^2|^{\frac{3}{2}} - |r^2 + (f+z)^2|^{\frac{3}{2}} \right|$$

where N is the product of denominators. Writing

$$K_1 = \frac{r}{N} |r^2 + z^2|^{\frac{3}{2}} \left| 1 - \left(1 + \frac{2fz + f^2}{r^2 + z^2}\right)^{\frac{3}{2}} \right|$$

and considering separately the cases $|z| < r$ and $|z| > r$, we easily find that $K_1 \le c/r$. The second component

$$K_2 = \frac{|f|}{N} \mid |r^2 + z^2|^{\frac{3}{2}} - |r^2 + (f+z)^2|^{\frac{3}{2}} \mid$$

can be similarly estimated by $K_2 \le c$. $\square$

LEMMA 4.3. *For all $z \in \mathbb{R}$,*

$$\int_{Q_0} \nabla\psi_0((v_1-u_1)F_1(v)+(v_2-u_2)F_2(v)+z\eta(v))du$$

$$= \int_{\mathbb{R}^2} \nabla\varphi_0(A_v^{-1}(v-u,z))du = -\frac{\operatorname{sign} z}{2\sigma(v)}\eta(v) ,$$

where the second integral is in the sense (4.1).

Proof. Let ρ be the rotation in $\mathbb{R}^3$ such that

$$\rho(F_1) = |F_1|e_1,$$

$$\rho(F_2) = |F_2|(\cos\theta e_1 + \sin\theta e_2) ,$$

$$\rho(\eta) = -e_3,$$

where $|F_1|\ |F_2|\cos\theta = F_1\cdot F_2$. Noting that $\nabla\varphi_0$ commutes with $\rho:\ \rho^{-1}\nabla\varphi_0\rho = \nabla\varphi_0$, we find that

$$\begin{aligned}(4.4)\qquad &\int \nabla\varphi_0(z\eta+(v_1-u_1)F_1+(v_2-u_2)F_2)du \\ &= \rho^{-1}\int \nabla\varphi_0((v_1-u_1)|F_1|+(v_2-u_2)\cos\theta|F_2|\ ,\ (v_2-u_2)\sin\theta|F_2|,\ z)du\ .\end{aligned}$$

Changing variables and noting that

$$||F_1|\ |F_2|\sin\theta| = |(F_1\cdot F_1)(F_2\cdot F_2)-(F_1\cdot F_2)^2|^{\frac{1}{2}} = |\det\Phi|^{\frac{1}{2}} = \sigma\ ,$$

we see that (4.4) equals

$$\frac{1}{\sigma(v)}\ \rho^{-1}\int_{\mathbb{R}^2}\nabla\varphi_0(u_1',u_2',z)du'\ .$$

The e_3 component can be computed explicitly:

$$\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{z}{|u_1^2+u_2^2+z^2|^{\frac{3}{2}}}du = \frac{1}{2}\int_0^\infty \frac{zr}{|r^2+z^2|^{\frac{3}{2}}}\ dr = \frac{1}{2}\ \operatorname{sgn} z\ .$$

The e_1, e_2 components

$$\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{u_j}{|u_1^2+u_2^2+z^2|^{\frac{3}{2}}}\ du\ ,$$

taken in the principle value sense (4.1),

$$\lim_{N\to\infty}\int_{-N}^{N} du_j \ ,$$

are zero because the integrands are odd functions. $\square$

We define, for periodic scalar functions $g \in L^2(Q)$, the operator P:

$$(Pg)(v) = \lim_{z\to 0}\int_Q (\sigma(u)g(u) - \sigma(v)g(v))\nabla\psi_0(A_v^{-1}(v-u,z))du \ . \tag{4.5}$$

With this definition, we have by Lemma 4.3,

$$\begin{aligned}\lim_{z\to 0}\int_Q g(u)\nabla\psi_0(A_v^{-1}(v-u,z))\sigma(u)du \\ = (Pg)(v) - \frac{1}{2}\ \operatorname{sign} z\ g(v)\eta(v)\ .\end{aligned} \tag{4.6}$$

From the proof of Lemma 4.3,

$$(Pg)(v) = \lim_{z\to 0}\int_{\mathbb{R}^2} \sigma(u)g(u)\pi_T\nabla\varphi_0(A_v^{-1}(v-u,z))du \ ,$$

where π_T is the projection onto the tangent space of S, and the integral is in the sense of (4.1).

Let χ denote the characteristic function of Q_0 on $\mathbb{R}^2$. We break P into two pieces,

$$P = P\chi\sigma + P(1-\chi)\ .$$

Since $\nabla\varphi_0$ is smooth in $\mathbb{R}^2 \setminus Q_0$,

$$(P(1-\chi)g)(v) = \int_{\mathbb{R}^2\setminus Q_0} \sigma(u)g(u)\pi_T\nabla\varphi_0(A_v^{-1}(v-u))du \ ,$$

(where $v = v|_{Q_0}$) is a compact operator. We denote $K = P(1-\chi)$.

Writing $B = P\chi$, we have

$$\begin{aligned}(B\sigma g)(v) &= \lim_{z\to 0}\int_{Q_0} \sigma(u)g(u)\pi_T\nabla\varphi_0(A_v^{-1}(v-u,z))du \\ &= pv\int_{\mathbb{R}^2} h(v-u)\pi_T\nabla\varphi_0(A_v^{-1}u)du,\end{aligned} \tag{4.7}$$

where $h = \chi\sigma g \in L^2(\mathbf{R}^2)$.

For each $v \in Q$, $u \in \mathbf{R}^2$, $z \in \mathbf{R}$, define

$$\gamma_z(v,u) = \frac{|(u_1,u_2,z)|^3}{|A_v^{-1}(u_1,u_2,z)|^3} \ . \tag{4.8}$$

Notice that γ_0 is homogeneous of degree zero in u: for all real $t \neq 0$,

$$\gamma_0(v,tu) = \gamma_0(v,u) \ .$$

Furthermore, γ_z is continuous, bounded and bounded away from zero in $\{(u,v) : u \neq 0\}$.

From the definition of φ_0 and (4.8) we immediately get:

LEMMA 4.4. *There hold:*

(i) $\nabla\varphi_0(A_v^{-1}(u,z)) = A_v^{-1}\gamma_z(v,u)\nabla\varphi_0(u,z)$,

(ii) $\int\limits_{|u|=1} \gamma_0(v,u)\nabla\varphi_0(u)d\mu(u) = 0$

where μ is the surface measure on $\{|u|=1\}$.

Hold $v_0 \in Q$ fixed; for $j = 1,2$ consider the operator

$$(B^j_{v_0}h)(v) = pv\int\limits_{\mathbf{R}^2} h(v-u)\gamma_0(v_0,u)\pi_j\nabla\varphi_0(u)du \ , \tag{4.9}$$

where π_j is the projection onto the jth coordinate. It follows from Lemma 4.4 (ii) and the general theory of singular integrals (see [10; chap. II §4]) that $B^j_{v_0}$ is bounded on $L^2(\mathbf{R}^2)$, and for $h \in L^2(\mathbf{R}^2)$,

$$(B^j_{v_0}h)\widehat{\ }(\xi) = b_j(v_0,\xi)\hat{h}(\xi),$$

where the symbol $b_j(v_0,\xi)$ is homogeneous of degree 0 in ξ and is given by

$$b_j(v_0,\xi) = \frac{i}{8}\int\limits_{|u|=1} \operatorname{sign}(\xi\cdot u)\gamma_0(v_0,u)u_j d\mu(u) \ , \ |\xi| = 1 \ . \tag{4.10}$$

(here $d\mu$ is the measure on the circle $S^1 = \{|u|=1\}$.)

By the Fourier inversion formula,

$$\begin{aligned}(B^j_{v_0}h)(v_0) &= pv\int h(v_0-u)\gamma_0(v_0,u)\pi_j\nabla\varphi_0(u)du \\ &= \int\limits_{\mathbf{R}^2} b_j(v_0,\xi)\hat{h}(\xi)e^{2\pi i v_0\cdot\xi}d\xi \ . \\ &\equiv (B_jh)(v_0) \qquad (j=1,2) \ .\end{aligned} \tag{4.11}$$

Thus from (4.7) and Lemma 4.4 (i),

$$B\sigma g = (B_1\sigma g)F_1 + (B_2\sigma g)F_2 \ .$$

REMARK 4.1. In the case where $A_v^{-1} = \mathbf{1}$, so that $|F_1| = |F_2| = 1,\ F_1 \cdot F_2 = 0$ (for example, S is flat), we have $\gamma_0(v,u) \equiv 1$ and from (4.10)

$$b_j(v,\xi) = \frac{i\xi_j}{2|\xi|} \ .$$

Then

$$B\sigma g = \left(\frac{1}{2}\ \mathcal{R}_j\sigma g\right) F_1 + \left(\frac{1}{2}\ \mathcal{R}_j\sigma g\right) F_2$$

where the $\mathcal{R}_j$ are Riesz transforms (see, for instance, [10; chap. III §1]).

From (4.10), (4.11) we see that B_j has the form of a standard pseudo-differential operator of order zero. From general theory [11; chapter II, Theorem 6.3]), B_j is bounded on $L^2(Q)$ and

$$\|B_j\|_{L^2(Q),L^2(Q)} \le C \sup_{\substack{v\in Q\\ \xi\in\mathbb{R}^2}} |b_j(v,\xi)| \ . \tag{4.12}$$

From (4.10), $\|B_j\| \ \le C \sup\limits_{v\in Q} \|A_v\|^3$. Summarizing:

LEMMA 4.5. *The operator P defined in (4.5) can be written*

$$Pg = B_1\sigma g F_1 + B_2\sigma g F_2 + Kg$$

where K is compact and B_1, B_2 are bounded on $L^2(Q)$. The operators B_j are "order zero", admit the representation (4.11), and satisfy the bound (4.12).

§5. The Jump Relations. We note that the parametrization F is oriented so that $\eta(v)$ defined in (4.3) points into Ω_2.

Let $v \in Q$, so $F(v) = p \in S$. Define

$$E_\alpha^1(v) = \lim_{z\downarrow 0} E_\alpha(F(v) + z\eta(v)) \ ,$$

$$E_\alpha^2(v) = \lim_{z\uparrow 0} E_\alpha(F(v) + z\eta(v)) \ ,$$

and define H_α^1 , H_α^2 similarly.

Let $k_1 = \omega(\varepsilon_1\mu)^{\frac{1}{2}}$, $k_2 = \omega(\varepsilon_2\mu)^{\frac{1}{2}}$. We denote, for $j = 1,2$, $\psi_j(v,u) = \psi_{k_j}(F(v) - F(u))$, $R_j(v,u) = R_{k_{j,0}}(v,u)$, and define

$$G_j = R_j - i\alpha\psi_j \ .$$

The kernel G_j is the "regular part" of the kernel $\mathfrak{U}_{k_j}$ defined in (3.3) in the sense that $\mathfrak{U}_{k_j} = G_j + \nabla\psi_0 A_v^{-1}$, and by Lemma 4.2 we have

$$|G_j(v,u)| \ \le \frac{c}{|v-u|} \ .$$

From the representation (3.6) we get

$$E_\alpha^1(v) = \int_Q \{i\omega\mu I\psi_1 - J \times G_1 + \frac{1}{i\omega\varepsilon_1}(\nabla_0 \cdot I + i\alpha \cdot I)G_1\}\sigma du$$

$$+ \lim_{z\downarrow 0} \int_Q (-J) \times \nabla\varphi_0(A_v^{-1}(v-u,z))\sigma du$$

$$+ \lim_{z\downarrow 0} \frac{1}{i\omega\varepsilon_1} \int_Q (\nabla_0 \cdot I + i\alpha \cdot I)\nabla\varphi_0(A_v^{-1}(v-u,z))\sigma du \ .$$

Thus from (4.6),

$$E_\alpha^1(v) = \int_Q \{i\omega\mu I\psi_1 - J \times G_1 + \frac{1}{i\omega\varepsilon_1}(\nabla_0 \cdot I + i\alpha \cdot I)G_1\}\sigma du \tag{5.1}$$

$$- P_1(J) \times F_1 - P_2(J) \times F_2 + \frac{1}{2}\ J \times \eta$$

$$+ \frac{1}{i\omega\varepsilon_1}\ [P(\nabla_0 \cdot I + i\alpha \cdot I) - \frac{1}{2}\ (\nabla_0 \cdot I + i\alpha \cdot I)\eta] \ .$$

Similarly,

$$E_\alpha^2(v) = \int_a \{i\omega\mu I\psi_2 - J \times G_2 + \frac{1}{i\omega\varepsilon_2}(\nabla_0 \cdot I + i\alpha \cdot I)G_2\}\sigma du \tag{5.2}$$

$$- P_1(J) \times F_1 - P_2(J) \times F_2 - \frac{1}{2}\ J \times \eta$$

$$+ \frac{1}{i\omega\varepsilon_2}\left[P(\nabla_0 \cdot I + i\alpha \cdot I) + \frac{1}{2}\ (\nabla_0 \cdot I + i\alpha \cdot I)\eta\right] \ .$$

In the same way, we have for $j = 1, 2$,

$$
\begin{aligned}
(5.3) \qquad H^j_\alpha(v) = \int\limits_Q \{ i\omega\varepsilon_j J\psi_j + I \times G_j + \frac{1}{i\omega\mu}(\nabla_0 \cdot J + i\alpha \cdot J) G_j \} \sigma du \\
+ P_1(I) \times F_1 + P_2(I) \times F_2 \mp \frac{1}{2}\, I \times \eta \\
+ \frac{1}{i\omega\mu}\, [P(\nabla_0 \cdot J + i\alpha \cdot J) \mp \frac{1}{2}(\nabla_0 \cdot J + i\alpha \cdot J)\eta]
\end{aligned}
$$

where " $-$ " is for $j = 1$, " $+$ " is for $j = 2$.

From (5.1), (5.2), (5.3) we see that

$$
\begin{aligned}
[\![E_\alpha]\!] &\equiv (E^1_\alpha - E^2_\alpha) \\
&= \int\limits_Q \left\{ i\omega\mu I(\psi_1 - \psi_2) - J \times (G_1 - G_2) + \frac{1}{i\omega}(\nabla_0 + i\alpha) \cdot I \left(\frac{G_1}{\varepsilon_1} - \frac{G_2}{\varepsilon_2} \right) \right\} \sigma du \\
(5.4) \qquad & + J \times \eta + \frac{1}{i\omega} \left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 \varepsilon_2} \right) P(\nabla_0 \cdot I + i\alpha \cdot I) \\
& - \frac{1}{i\omega} \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) (\nabla_0 \cdot I + i\alpha \cdot I)\eta
\end{aligned}
$$

and

$$
\begin{aligned}
[\![H_\alpha]\!] &\equiv (H^1_\alpha - H^2_\alpha) \\
(5.5) \qquad &= \int\limits_Q \left\{ i\omega J(\varepsilon_1\psi_1 - \varepsilon_2\psi_2) + I \times (G_1 - G_2) + \frac{1}{i\omega\mu}(\nabla_0 + i\alpha) \cdot J(G_1 - G_2) \right\} \sigma du \\
& - I \times \eta - \frac{1}{i\omega\mu}(\nabla_0 \cdot J + i\alpha \cdot J)\eta \ .
\end{aligned}
$$

Denoting the integral expression in (5.4) by $T_1(I, J)$, the integral expression in (5.5) by $T_2(I, J)$ and using the identities $\eta \times (J \times \eta) = J$ and $\eta \times \eta = 0$, we see that

$$
\begin{aligned}
(5.6) \qquad \eta \times [\![E_\alpha]\!] &= J + \frac{1}{i\omega} \left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 \varepsilon_2} \right) \eta \times P(\nabla_0 \cdot I + i\alpha \cdot I) + \eta \times T_1(I, J) \ , \\
\eta \times [\![H_\alpha]\!] &= -I + \eta \times T_2(I, J) \ .
\end{aligned}
$$

§6. **Reduction to Integral Equations.** Recall the fields E_0, H_0 defined in (1.7). We shall try to solve (1.1)–(1.4) in the form

$$(E,H) = (E_0, H_0) + (E_\alpha, H_\alpha) e^{i\alpha\cdot x} \tag{6.1}$$

with E_α, H_α periodic over the lattice Λ.

Denote

$$I_0 = \eta \times (H_0^1 - H_0^2) e^{-i\alpha\cdot x} \ ,$$

$$J_0 = \eta \times (E_0^1 - E_0^2) e^{-i\alpha\cdot x} \ ,$$

where $E_0^j = E_0|_{\Omega_j}$, $H_0^j = H_0|_{\Omega_j}$. We want to solve

$$\begin{cases} \eta \times [\![H_\alpha]\!] + I_0 = 0 \\ \eta \times [\![E_\alpha]\!] + J_0 = 0 \end{cases} \tag{6.2}$$

where $\eta \times [\![H_\alpha]\!]$, $\eta \times [\![E_\alpha]\!]$ are given by (5.6). In particular, we shall find conditions under which the system (6.2) is Fredholm over $L^2(Q)$ (or more accurately, over $X = [L^2(Q)]^4$.) We can integrate by parts and write

$$T_1 = \frac{1}{i\omega} \frac{(\varepsilon_2 - \varepsilon_1)}{\varepsilon_1 \varepsilon_2} \int (\nabla_0 \cdot I) G_1 + \widehat{T}_1 \ ,$$

where

$$\widehat{T}_1 = \int_Q \left\{ i\omega\mu I(\psi_1 - \psi_2) - J \times (G_1 - G_2) + \left(\frac{\alpha}{\omega} \cdot I\right) \left(\frac{G_1}{\varepsilon_1} - \frac{G_2}{\varepsilon_2}\right) \right.$$

$$\left. + \frac{1}{i\omega\varepsilon_2} \int I \cdot \nabla_u (G_1 - G_2) \right\} \sigma du \ ,$$

also

$$T_2 = i\omega(\varepsilon_1 - \varepsilon_2) \int J\psi_1 \sigma du + \widehat{T}_2$$

where

$$\widehat{T}_2 = \int_Q \left\{ i\omega\varepsilon_2 J(\psi_1 - \psi_2) + I \times (G_1 - G_2) + \frac{1}{i\omega\mu} J \cdot \nabla_u (G_1 - G_2) \right.$$

$$\left. + \frac{\alpha}{\omega\mu} \cdot J (G_1 - G_2) \right\} \sigma du \ .$$

Then from (5.6) the system is

$$(6.3)\qquad I - i\omega(\varepsilon_1-\varepsilon_2)\int(\eta\times J)\psi_1\sigma du - \eta\times\widehat{T}_2(I,J) = I_0\ ,$$

$$(6.4)\qquad J + \frac{1}{i\omega}\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_1\varepsilon_2}\right)\eta\times[P(\nabla_0\cdot I + i\alpha\cdot I) + \int(\nabla_0\cdot I)G_1] + \eta\times\widehat{T}_1(I,J) = -J_0\ .$$

LEMMA 6.1. *There holds,*

$$\nabla_0\cdot\int_Q(\eta\times J)\psi_1\sigma du = -P\cdot(\eta\times J) - \int_Q(\eta\times J)\cdot R_1\sigma du\ .$$

Proof. Integrating by parts as in Lemma 3.2,

$$\nabla_0\cdot\int_Q(\eta\times J)\psi_1\sigma du = -\int_{S_0}\{(\eta\times J)\cdot\nabla_y\psi_1\}dS_y$$

$$= -\int_Q(\eta\times J)\cdot\nabla\varphi_0 A_v^{-1}\sigma du - \int_Q(n\times J)\cdot R_1\sigma du\ . \qquad \square$$

From (6.3) and Lemma 6.1,

$$\nabla_0\cdot I = i\omega(\varepsilon_2-\varepsilon_1)[P\cdot(\eta\times J) + \int_Q(\eta\times J)\cdot R_1\sigma du] + K_1 + \nabla_0\cdot I_0\ ,$$

where K_1 is compact. Substituting into (6.4), we get

$$(6.5)\qquad J + \frac{(\varepsilon_1-\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\ \eta\times P(P\cdot(\eta\times J)) + K_2 = -\widetilde{J}_0$$

where K_2 is the sum of all the compact operators and $\widetilde{J}_0$, the inhomogeneous term, depends only on I_0, J_0.

By Lemma 4.5, $P(P\cdot(\eta\times J))$ is a bounded operator in $L^2(Q)$.

Consider the resolvent operator

$$(6.6)\qquad R(\lambda) = [\mathbf{1} + \lambda\eta\times P(P\cdot(\eta\times\mathbf{1}))]^{-1}$$

This is holomorphic function in the complex λ-plane. Denote its poles by $\lambda_1, \lambda_2, \ldots$; then $|\lambda_j|\to\infty$ if $j\to\infty$. We shall henceforth assume that

$$(6.7)\qquad \frac{(\varepsilon_1-\varepsilon_2)^2}{\varepsilon_1\varepsilon_2} \neq \lambda_j \qquad \text{for all}\quad j\ .$$

Set $\overline{\lambda} = (\varepsilon_1-\varepsilon_2)^2/(\varepsilon_1\varepsilon_2)$ and apply $R(\overline{\lambda})$ to (6.5):

$$(6.8)\qquad J + R(\overline{\lambda})K_2 = -R(\overline{\lambda})\widetilde{J}_0$$

Clearly (6.3), (6.8) form a Fredholm system of integral equations equivalent to (6.3), (6.4). Summarizing:

THEOREM 6.2. *Assume that (6.7) holds. Then the Maxwell equations (1.1)–(1.4) have a solution of the form (6.1) with E_α, H_α periodic if for the Fredholm system (6.3), (6.8) the first alternative holds; further, the field (E_α, H_α) has the form*

$$(6.9) \qquad \begin{aligned} (E_\alpha, H_\alpha) &= \sum_{n\in\Lambda} A_n^{\pm} e^{-i\alpha_n\cdot x \pm i\beta_n(k)x_3} \quad \text{if} \quad |x_3| > |f_3|_{L^\infty} , \\ |A_n^{\pm}| &\le \frac{C}{\sqrt{1+|n|}} \end{aligned}$$

where $\pm\beta_n(k)$ is $+\beta_n(k_1)$, $A_n^{\pm}$ is A_n^{+} above S and β_n is $-\beta_n(k_2)$, $A_n^{\pm}$ is A_n^{-} below S, and $\beta_n = \beta_n(k)$ is defined in (2.4).

Observe that the series in (6.9) represents back-scattered (or reflected) waves above S and forward scattered (or transmitted) waves below S. Thus it is natural to refer to the condition (6.9) as the "radiation condition" at infinity.

We supplement Theorem 6.2 with a uniqueness result:

THEOREM 6.3. *Assume that (6.7) holds and that the first alternative holds for the Fredholm system (6.3), (6.8). Then the solution of the Maxwell equations having the form (6.1) with E_α, H_α periodic over Λ and satisfying the radiation condition (6.9) is unique.*

Proof. Following [3] we use the integral representation [7; p. 130] for E_α, H_α in $Q_0 \times \{-M < x_3 < M\}$. The boundary integrals on $\partial Q_0 \times \{-M < x_3 < M\}$ cancel out by the periodicity. If we let $M \to \infty$ and use (6.9), we find that the integrals over $\{x_3 = \pm M\}$ tend to zero. In the limit we obtain precisely the same integral representation as in §3 (with $I = -\eta \times H_\alpha$, $J = \eta \times E_\alpha$). This then leads to the Fredholm integral equations (6.3), (6.8) whose solution is unique by the first alternative assumption.

$\square$

REMARK 6.1. We have assumed so far that S is a C^2 surface. However all our results extend to the case where S is piecewise C^2, i.e., the surface S consists of portions, each being uniformly in C^2. Indeed the main point to check is that the operator Pg defined in (4.5) remains bounded in L^2. But this follows by decomposing g into a sum of functions g_j each supported on a C^2 portion S_j of S. Since g_j may be viewed as extended by 0 into a C^2 surface $\widetilde{S}_j$ containing the closure of S_j in its interior, $\|Pg_j\|_{L^2} \le C\|g_j\|_{L^2}$ and the same then holds for Pg.

REFERENCES

[1] H. BELLOUT and A. FRIEDMAN, *Scattering by stripe grating*, J. Math. Anal. Appl., 147 (1990), 228–248.

[2] A. BENALDI, *Numerical analysis of the exterior boundary value problem for the time-harmonic Maxwell equations by a boundary finite element method. Part I: The continuous problem. Part II: The discrete problem*, Math. of Computation, 167 (1984), 29–46 and 47–68.

[3] X. Chen and A. Friedman, *Maxwell's equations in a periodic structure*, Trans. Amer. Math. Soc., to appear.

[4] C.A. Coulson and A. Jeffrey, Waves, Second edition, Longman, London, 1977.

[5] A. Friedman, *Mathematics in Industrial Problems, Part 3*, Springer–Verlag, Heidelberg, 1990.

[6] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer–Verlag, Heidelberg, 1983.

[7] C. Muller, *Foundation of Mathematical Theory of Electromagnetic Waves*, Springer–Verlag, Berlin, 1969.

[8] J.C. Nedelec and F. Starling, *Integral equation methods in quasi periodic diffraction problems for the time harmonic Maxwell's equations*, Ecole Polytechnique, Center de Mathematiques Appliquees, Tech. Report #756, April 1988.

[9] *Electromagnetic Theory of Gratings*, Topics in Current Physics, Vol. 22, edited by R. Petit, Springer–Verlag, Heidelberg, 1980.

[10] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, N.J., 1970.

[11] M. Taylor, *Pseudodifferential Operators*, Princeton University Press, Princeton, N.J., 1981.

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