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Abstract. A nonstationary 2-dimensional filtration in a porous medium is considered, whereby part of the medium is saturated, another part is unsaturated but wet, and the remaining part is dry. The saturated/unsaturated and unsaturated/dry interfaces are free boundaries. It is shown that there exists a unique solution and that the saturation function is continuous in the wet portion of the medium; this implies that the two interfaces are separated. Under some (monotonicity type) conditions on the initial and boundary data it is shown that the free boundaries are continuous.

§1. The model. We consider a non-stationary flow in porous medium. Denote by u the hydrostatic potential due to capillary suction, i.e., the pressure, and denote by $c(u)$ the relative volumetric moisture content, i.e., the saturation. If u exceeds a critical value, which we take to be 0, then the porous medium becomes saturated and $c(u)$ is equal to a constant, which is taken to be 1. In the dry portion of the porous medium the pressure is also a constant, which is taken to be -1. The function $c(u)$ is monotone increasing for $-1 < u < 0$ and, for simplicity, we take it to be $1 + u$. Thus,

$$c(u) = \begin{cases} 1+u & \text{if} \quad -1 < u < 0 \\ 1 & \text{if} \quad u \geq 1 \,. \end{cases}$$

The pressure satisfies the equation [3; p. 488]

$$\frac{\partial}{\partial t}\, c(u) = \Delta u \tag{1.1}$$

in the flow region. To account for the free boundary condition at the wet/dry interface (where $u = -1$) we extend the definition of $c(u)$ to $u = -1$ by taking

$$c(u) = \min\{u, 0\} + H(u+1) \quad \text{for} \quad u \geq -1 \tag{1.2}$$

where H is the Heaviside function. As shown in [7], equation (1.1) across the wet/dry interface represents the (physical) equation of continuity, provided gravity is ignored. To

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include gravity in the downward direction $-y$, the term $\frac{\partial}{\partial y} H(u+1)$ needs to be added to the right hand side of (1.1); see [7].

We note that in the unsaturated region equation (1.1) is the same equation satisfied by the temperature in the one-phase Stefan problem.

In the unsaturated/saturated region equation (1.1) becomes

$$\frac{\partial \alpha(u)}{\partial t} = \Delta u \qquad \alpha(u) = 1 + \min\{u, 0\} \tag{1.3}$$

and the condition one imposes at the interface $u = 0$ is that $\alpha(u)$ be continuous.

The one-phase Stefan problem has been studied extensively in the literature; see [9] and the references therein; under some initial and boundary conditions the free boundary is C^∞. Equation (1.3) has been studied in recent years by several authors: [6] [11] [14] [18] [19] for one dimension and [1] [2] [8] [10] [12] [13] [15] for higher dimensions; in the case of 1-dimension the free boundary is known to be in C^∞.

In this paper we consider the filtration problem in the 2-dimensional half-space

$$\mathbf{R}_2^+ = \{(x, y) \; ; \; x \in \mathbf{R}^1, \; y > 0\}$$

and assume that

$$\frac{\partial u(x, 0, t)}{\partial y} = -g(x, t) \tag{1.4}$$

where

$$\begin{aligned} &g(x,t) = g(-x,t) \geq 0 \; , \\ &g(x,t) = 0 \quad \text{if} \quad |x| > x_0 \qquad (0 < x_0 < \infty) \; . \end{aligned} \tag{1.5}$$

This means that the fluid is being injected across the boundary $y = 0$. We also prescribe the initial moisture content

$$c(u(x, y, 0)) = \gamma_0(x, y) \; , \; \gamma_0(x, y) = \gamma_0(-x, y) \; . \tag{1.6}$$

Finally, we assume that the medium is dry at ∞:

$$\lim_{x^2+y^2 \to \infty} u(x, y, t) = -1 \; , \; t > 0 \; . \tag{1.7}$$

Since the data are symmetric with respect to x, we also seek a solution which is symmetric with respect to x (Actually by the uniqueness proof given in §2, no other solutions exist), that is, $u(x,y,t) = u(-x,y,t)$. We can then restrict the problem to $x > 0$, by imposing the boundary condition

$$\frac{\partial u(0,y,t)}{\partial x} = 0 \ . \tag{1.8}$$

We refer to (1.1), (1.2), (1.4)–(1.8) as *problem* (P). The model may represent, for example, a filtration of water into soil and a spill over a table cloth, or over other porous media.

We shall need the following notation:

$$\Omega_R = \{(x,y); x > 0, y > 0, x^2 + y^2 < R^2\} \ ,$$

$$\Gamma_{1,R} = \{(x,0), \ 0 < x < R\}, \ \Gamma_{2,R} = \{(0,y); \ 0 < y < R\} \ ,$$

$$\Gamma_R = \{(x,y); \ x > 0, y > 0, x^2 + y^2 = R^2\} \ ,$$

$$\Omega_{R,T} = \Omega_R \times \{0 < t < T\}, \ \Gamma_{i,R,T} = \Gamma_{i,R} \times \{0 < t < T\} \ ,$$

$$\Gamma_{R,T} = \Gamma_R \times \{0 < t < T\} \ .$$

Definition 1.1. A pair (u, γ) is called a *weak solution* of problem (P) if

$$u \in L^2(0,T; H^1(\Omega_R)) \qquad \forall \, R > 0, \ T > 0 \ ,$$

$$\gamma \in c(u) \ ,$$

$$u = -1, \ \gamma = -1 \quad \text{if} \quad x^2 + y^2 > \sigma(t)$$

where $\sigma(t)$ is a continuous monotone increasing function of t and, for any $R > \sigma(T)$,

$$\iint\limits_{\Omega_{R,T}} (\nabla u \cdot \nabla \zeta - \gamma \zeta_t) = \iint\limits_{\Omega_R} \gamma_0 \zeta(\cdot, 0) + \iint\limits_{\Gamma_{1,R,T}} g\zeta \qquad \forall \, \zeta \in V_{R,T} \tag{1.9}$$

where

$$V_{R,T} = \{\zeta \in H^1(\Omega_{R,T}), \ \zeta(\cdot, T) = 0 \ , \ \zeta|_{\Gamma_{R,T}} = 0\} \ .$$

Note that $\gamma \in c(u)$ means that $\gamma(x,y,t) = c(u(x,y,t))$ if $u(x,y,t) > -1$,

$$-1 \le \gamma(x,y,t) \le 0 \quad \text{if} \quad u(x,y,t) = -1 \ ,$$

and $u \ge -1$ a.e. (i.e., the graph $c(u)$ is empty if $u < -1$).

In §2 we define a truncated problem (P_R) and prove that it has a unique solution. In §3 we let $R \to \infty$ and establish the existence and uniqueness of a weak solution to problem

(P). Next, in §4 we prove that $\max\{c(u), 0\} = 1 + \min\{u, 0\}$ is a continuous function. This implies that the two interfaces $S_1 = \partial\{u = 0\} \cap \partial\{u < 0\}$ and $S_2 = \partial\{u > -1\} \cap \partial\{u = -1\}$ are separated by positive distance. In §5 we make the assumptions

$$\frac{\partial \gamma_0}{\partial x} \leq 0 \ , \ \frac{\partial \gamma_0}{\partial y} \leq 0 \ , \ \frac{\partial g(x,t)}{\partial x} \leq 0 \ , \ g(0,t) > 0$$

and prove that S_1 and S_2 have continuous representations $r = \zeta_1(\theta, t)$ and $r = \zeta_2(\theta, t)$ (in polar coordinates (r, θ)); ζ_2 is actually C^∞, whereas for ζ_1 we only get a logarithmic-type modulus of continuity.

§2. **The truncated problem.** In this section we assume that

$$g \in L^2(\Gamma_{1,R,T}), \ g \geq 0 \ , \tag{2.1}$$

$$-1 \leq \gamma_0 \leq 1 \ . \tag{2.2}$$

Definition 2.1. A pair (u, γ) is called a *weak solution* to problem (P_R) if for any $T > 0$

$$u \in L^2(0, T; H^1(\Omega_R)), \quad u = -1 \quad \text{on} \quad \Gamma_{R,T} \ , \tag{2.3}$$

$$\gamma \in c(u) \tag{2.4}$$

and (1.9) is satisfied for any $\zeta \in V_{R,T}$.

THEOREM 2.1. *There exists at most one weak solution to problem (P_R).*

Proof. Suppose there exist two solutions (u_1, γ_1) and (u_2, γ_2). For any $t_0 \in (0, T)$ set

$$\zeta_{t_0} = \begin{cases} \displaystyle\int_t^{t_0} [u_1(\cdot, \tau) - u_2(\cdot, \tau)] d\tau & \text{if} \quad t < t_0 \\ 0 \quad \text{if} \quad t > t_0 \ . \end{cases}$$

Then $\zeta_{t_0} \in V_{R,T}$. Taking $\zeta = \zeta_{t_0}$ in (1.9) for both $u = u_1$ and $u = u_2$ and subtracting one equation from the other, we get

$$0 = \int_0^{t_0} \int_{\Omega_R} \left[\nabla(u_1 - u_2) \cdot \nabla \int_t^{t_0} (u_1(\cdot, \tau) - u_2(\cdot, \tau)) + (\gamma_1 - \gamma_2)(u_1 - u_2) \right]$$

$$= \frac{1}{2} \int_{\Omega_R} \left| \nabla \int_0^{t_0} (u_1(\cdot, \tau) - u_2(\cdot, \tau)) \right|^2 + \iint_{\Omega_{R,t_0}} (\gamma_1 - \gamma_2)(u_1 - u_2) \ .$$

Since $\gamma_i \in c(u_i)$, $(\gamma_1 - \gamma_2)(u_1 - u_2) \geq 0$. Hence

$$\nabla \int_0^{t_0} [u_1(\cdot, \tau) - u_2(\cdot, \tau)] = 0 \qquad \text{a.e.}$$

which implies that $u_1 - u_2 = 0$ a.e. and then, by (1.9), $\gamma_1 - \gamma_2 = 0$ a.e. .

THEOREM 2.2. *There exists a unique weak solution to problem* (P_R).

Proof. Let c_ε $(\varepsilon > 0)$ be smooth approximations to c, as $\varepsilon \to 0$, so that $c_\varepsilon(0) = 1$, and

$$c'_\varepsilon(u) = \begin{cases} \dfrac{\varepsilon}{(1+u)^3} & \text{if} \quad \varepsilon < u < \infty \\ 1 & \text{if} \quad -1 \leq u \leq 0 \\ \dfrac{1}{\varepsilon} & \text{if} \quad -\infty < u < -1-\varepsilon\ , \end{cases} \tag{2.5}$$

$$c''_\varepsilon(u) \leq 0\ . \tag{2.6}$$

Let δ_ε be such that

$$c_\varepsilon(-1-\delta_\varepsilon) = -1\ .$$

Note that $\delta_\varepsilon \leq 2\varepsilon$, and $c_\varepsilon \leq 1 + 2\varepsilon$.

Let $\gamma_{0\varepsilon}$ be C^∞ functions that

$$-1 \leq \gamma_{0\varepsilon} \leq 1\ , \quad \gamma_{0\varepsilon} = -1 \quad \text{in a neighborhood of } \Gamma_R\ ,$$

$$\frac{\partial \gamma_{0\varepsilon}}{\partial x} = 0 \quad \text{on} \quad \Gamma_{2,R}\ ,$$

$$\gamma_{0\varepsilon} \to \gamma_0 \quad \text{in} \quad L^2(\Omega_R) \quad \text{and a.e.}$$

as $\varepsilon \to 0$. Set $u_{0\varepsilon} = c_\varepsilon^{-1}(\gamma_{0\varepsilon})$. Then

$$-1-\delta_\varepsilon \leq u_{0\varepsilon} \leq 0\ ,$$

$$u_{0\varepsilon} = -1-\delta_\varepsilon \quad \text{in a neighborhood of } \Gamma_R\ ,$$

$$\frac{\partial u_{0\varepsilon}}{\partial x} = 0 \quad \text{on} \quad \Gamma_{2,R}\ .$$

Let g_ε be smooth functions such that

$$g_\varepsilon \geq 0 \ , \ -g_\varepsilon = \frac{\partial u_{0\varepsilon}}{\partial y} \quad \text{on} \quad \Gamma_{1,R} \cap \{t=0\} \ ,$$

$$g_\varepsilon \to g \quad \text{in} \quad L^2(\Gamma_{1,R,T}) \ .$$

Consider the parabolic problem

$$\frac{\partial}{\partial t} c_\varepsilon(u) = \Delta u \quad \text{in} \quad \Omega_{R,T} \ , \tag{2.7}$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{on} \quad \Gamma_{2,R,T} \ , \tag{2.8}$$

$$-\frac{\partial u}{\partial y} = g_\varepsilon \quad \text{on} \quad \Gamma_{1,R,T} \ , \tag{2.9}$$

$$u = -1-\delta_\varepsilon \quad \text{on} \quad \Gamma_{R,T} \ , \tag{2.10}$$

$$u = u_{0\varepsilon} \quad \text{at} \quad t=0 \ . \tag{2.11}$$

Since the initial and boundary conditions are smooth and satisfy the compatibility conditions for a classical solution, there exists a unique solution u_ε such that

$$u_\varepsilon \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}_{R,T}) \cap C^\infty(\Omega_{R,T}) \ .$$

By the maximum principle

$$u_\varepsilon \geq -1-\delta_\varepsilon \ . \tag{2.12}$$

Multiplying (2.7) by $u_\varepsilon + 1 + \delta_\varepsilon$ and integrating over $\Omega_{R,T}$ we get

$$\int\limits_{\Omega_R} W_\varepsilon(u_\varepsilon(\cdot,T)) + \iint\limits_{\Omega_{R,T}} |\nabla u_\varepsilon|^2 = \int\limits_{\Omega_R} W_\varepsilon(u_\varepsilon(\cdot,0)) + \iint\limits_{\Gamma_{1,R,T}} g_\varepsilon(u_\varepsilon + 1 + \delta_\varepsilon) \tag{2.13}$$

where

$$W_\varepsilon(u) = \int\limits_{-1-\delta_\varepsilon}^{u} c'_\varepsilon(s)(s+1+\delta_\varepsilon) \ .$$

Notice that $W_\varepsilon(u) \geq 0$ and

$$W_\varepsilon(u_\varepsilon(\cdot,0)) \leq 1 + C\varepsilon \quad (C \text{ constant})$$

Since by Sobolev's inequalities

$$\|f\|_{2,\Omega_R}^2 + \|f\|_{2,\Gamma_{1,R}}^2 \leq C_R(\|\nabla f\|_{2,\Omega_R}^2 + \|f\|_{2,\Gamma_R}^2) ,$$

we deduce from (2.13) that

$$\int_0^T \|u_\varepsilon\|_{H^1(\Omega_R)}^2 \leq C_R \left[1 + \int_0^T \|g_\varepsilon\|_{2,\Gamma_{1,R}}^2 \right] . \tag{2.14}$$

The function $\gamma_\varepsilon = c_\varepsilon(u_\varepsilon)$ satisfies

$$-1 - \delta_\varepsilon \leq \gamma_\varepsilon \leq 1 + 2\varepsilon . \tag{2.15}$$

We can now choose a sequence $\varepsilon \to 0$ such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u && \text{weakly in} \quad L^2(\Omega_{R,T}) , \\ \nabla u_\varepsilon &\rightharpoonup \nabla u && \text{weakly in} \quad L^2(\Omega_{R,T}) , \\ \gamma_\varepsilon &\rightharpoonup \gamma && \text{weakly in} \quad L^2(\Omega_{R,T}) , \end{aligned} \tag{2.16}$$

for all $T > 0$ where u, γ are functions satisfying

$$u \in L^2[0,T; H^1(\Omega_R)] , \quad u \geq -1 ,$$

$$\gamma \in L^\infty(\Omega_{R,T}) , \quad -1 \leq \gamma \leq 1$$

for all $T > 0$. Multiplying (2.7) by $\zeta, \zeta \in V_{R,T}$, integrating over $\Omega_{R,T}$, and passing to the limit with $\varepsilon \to 0$, we obtain the relation (1.9). In order to complete the proof that (u, γ) is a weak solution it remains to show that (2.4) holds.

Set

$$w_\varepsilon = \max\{c_\varepsilon(u_\varepsilon), 0\} = 1 + \min\{u, 0\} . \tag{2.17}$$

Then $|\nabla w_\varepsilon| \leq |\nabla u_\varepsilon|$ and therefore, by (2.14), for any $0 \leq t_1 < t_2 \leq T$,

$$\int_{t_1}^{t_2} \|w_\varepsilon(\cdot, \tau)\|_{H^1(\Omega_R)} \leq C_{R,T} < \infty . \tag{2.18}$$

Next, for any small $h > 0$,

$$\int_0^{T-h}\int_{\Omega_R} |w_\varepsilon(\cdot,t+h) - w_\varepsilon(\cdot,t)|^2$$

$$\leq \int_0^{T-h}\int_{\Omega_R} [c_\varepsilon(u_\varepsilon(\cdot,t+h)) - c_\varepsilon(u_\varepsilon(\cdot,t))][u_\varepsilon(\cdot,t+h) - u_\varepsilon(\cdot,t)]$$

$$= \int_0^{T-h}\int_{\Omega_R}\int_t^{t+h} \frac{\partial}{\partial\tau} c_\varepsilon(u_\varepsilon(\cdot,\tau)) \cdot [u_\varepsilon(\cdot,t+h) - u_\varepsilon(\cdot,t)]$$

$$= \int_0^{T-h}\int_{\Omega_R}\int_t^{t+h} \Delta u_\varepsilon(\cdot,\tau)[u_\varepsilon(\cdot,t+h) - u_\varepsilon(\cdot,t)]$$

$$= \int_0^{T-h}\int_t^{t+h} \left\{ \int_{\Gamma_{1,R}} g(\cdot,\tau)[u_\varepsilon(\cdot,t+h) - u_\varepsilon(\cdot,t)] - \int_{\Omega_R} \nabla u_\varepsilon(\cdot,\tau)\cdot[\nabla u_\varepsilon(\cdot,t+h) - \nabla u_\varepsilon(\cdot,t)] \right\}$$

and after applying the Schwarz inequality on the right-hand side,

$$\int_0^{T-h}\int_{\Omega_R} |w_\varepsilon(\cdot,t+h) - w_\varepsilon(\cdot,t)|^2 \leq Ch \tag{2.19}$$

where C is a constant independent of ε. The estimate (2.18) implies that

$$\left\{ \int_{t_1}^{t_2} w_\varepsilon(\cdot,\tau)d\tau \right\} \quad \text{is precompact in } L^2(\Omega_R)$$

for any $0 \leq t_1 < t_2 \leq T$. This together with (2.19) can be used to easily show (see [17]) that $\{w_\varepsilon\}$ is precompact in $L^2(\Omega_{R,T})$. It follows that, for a subsequence,

$$w_\varepsilon \to w \quad \text{in} \quad L^2(\Omega_{R,T}) \quad \text{and a.e.}$$

for some function $w \in L^2(\Omega_{R,T})$, for any $T > 0$. Consider the set in $\Omega_{R,T}$,

$$C_\mu \equiv \{\mu < w < 1-\mu\}\ , \quad \mu > 0\ .$$

For any small $\lambda > 0$ there is a set A_λ of measure $< \lambda$ such that (for a subsequence) $w_\varepsilon \to w$ uniformly in $C_\mu \backslash A_\lambda$. Therefore

$$\frac{\mu}{2} < w_\varepsilon < 1 - \frac{\mu}{2} \quad \text{in} \quad C_\mu \backslash A_\lambda$$

if ε is small and, consequently,

$$\frac{\mu}{2} < c_\varepsilon(u_\varepsilon) < 1 - \frac{\mu}{2} \ .$$

From the definition of c_ε it then follows that u_ε is uniformly convergent on $C_\mu \backslash A_\lambda$. The limiting function is of course u (since $u_\varepsilon \rightharpoonup u$). Since λ and μ are arbitrary,

$$\gamma_\varepsilon = c_\varepsilon(u_\varepsilon) \to c(u) \quad \text{a.e. in} \quad 0 < w < 1 \ . \tag{2.20}$$

The same argument shows that, on $\{w = 1\} \backslash \widetilde{A}_\lambda$, $c_\varepsilon(u_\varepsilon) > 1 - \mu/2$ and $u_\varepsilon > -\frac{\mu}{2}$ if ε is small enough where $\text{meas}(\widetilde{A}_\lambda) < \lambda$. Thus $\liminf_{\varepsilon \to 0} c_\varepsilon(u_\varepsilon) \geq 1$ and $\liminf_{\varepsilon \to 0} u_\varepsilon \geq 0$ on $w = 1$. Since $c_\varepsilon(u) \leq 1 + 2\varepsilon$ for all u, we conclude that

$$\gamma = 1 \quad \text{on} \quad \{w = 1\} = \{c(u) = 1\} \ . \tag{2.21}$$

Next, for any $\delta > 0$,

$$\begin{aligned} 0 = \lim_{\varepsilon \to 0} |\{w_\varepsilon > \delta, \ w = 0\}| &= \lim_{\varepsilon \to 0} |\{c_\varepsilon(u_\varepsilon) > \delta, \quad w = 0\}| \\ &= \lim_{\varepsilon \to 0} |\{u_\varepsilon \geq -1 + \delta, \ w = 0\}| \ . \end{aligned} \tag{2.22}$$

Since $u_\varepsilon \rightharpoonup u$,

$$\int_{\{w=0\}} (u + 1 - \delta)^+ = \int_{\{w=0\}} (u + 1 - \delta) \operatorname{sgn}(u + 1 - \delta)^+ = \lim_{\varepsilon \to 0} \int_{\{w=0\}} (u_\varepsilon + 1 - \delta) \operatorname{sgn}(u + 1 - \delta)$$

$$\leq \lim_{\varepsilon \to 0} |\{u_\varepsilon + 1 - \delta > 0, w = 0\}|^{1/2} \left\{ \int_{\{w=0\}} (u_\varepsilon + 1 - \delta)^2 \right\}^{1/2} \to 0$$

by (2.22) and the fact that $\|u_\varepsilon\|_{L^2} \leq C$. It follows that $u \leq -1 + \delta$ on $\{w = 0\}$ for any $\delta > 0$, so that

$$u = -1 \quad \text{on} \quad w = 0 \ ;$$

in other words, $c(u) = c(-1) = [-1, 0]$ on $\{w = 0\}$. On the other hand on $\{w = 0\}$ we have $w_\varepsilon \to 0$ a.e. so that $\overline{\lim}\ c_\varepsilon(u_\varepsilon) \leq 0$, and thus $\gamma \leq 0$. Since also $\gamma \geq -1$ we conclude that $\gamma \in c(u)$ on $\{w = 0\}$. Recalling (2.20), (2.22), the assertion (2.4) follows.

§3. The complete problem. In this section we extend Theorem 2.2 to problem (P). We shall need the following comparison result.

LEMMA 3.1. *Let* $(u^1, \gamma^1), (u^2, \gamma^2)$ *satisfy*

$$u^i \in L^2(0,T; H^1(\Omega_R)), \quad u_i = -1 \quad \text{on} \quad \Gamma_{R,T} ,$$

$$\gamma^i \in c(u^i) ,$$

and

$$\iint\limits_{\Omega_{R,T}} (\nabla u^1 \cdot \nabla\zeta - \gamma^1\zeta_t) \geq \int\limits_{\Omega_R} \gamma_0^1 \zeta(\cdot, 0) + \iint\limits_{\Gamma_{1,R,T}} g^1\zeta , \tag{3.1}$$

$$\iint\limits_{\Omega_{R,T}} (\nabla u^2 \cdot \nabla\zeta - \gamma^2\zeta_t) \leq \int\limits_{\Omega_R} \gamma_0^2 \zeta(\cdot, 0) + \iint\limits_{\Gamma_{1,R,T}} g^2\zeta \tag{3.2}$$

for any $\zeta \in V_{R,T}$, $\zeta \geq 0$, *where* $\gamma_0^2 \leq \gamma_0^1$, $g^2 \leq g^1$. *Then either* $u^2 \leq u^1$ *or* $\gamma^2 \leq \gamma^1$.

Proof. The proof is an extension of the proof of stability for the Stefan problem [9; Chap. 5]. Set

$$u = u^2 - u^1, \ \gamma = \gamma^2 - \gamma^1, \ \gamma_0 = \gamma_0^2 - \gamma_0^1, \ g = g^2 - g^1 .$$

Taking ζ in $V_{R,T}$, $\zeta \geq 0$, $[\partial\zeta/\partial n]_{\Gamma_{i,R,T}} = 0 \quad (i = 1, 2)$ and substituting it in (3.1), (3.2), we get

$$\iint\limits_{\Omega_{R,T}} (u\Delta\zeta + \gamma\zeta_t) + \int\limits_{\Omega_R} \gamma_0\zeta(\cdot, 0) + \iint\limits_{\Gamma_{1,R,T}} g\zeta \geq 0 . \tag{3.3}$$

We shall apply (3.3) to functions $\zeta = \zeta_m$ constructed as follows: Let a_m, b_m be smooth strictly positive functions and let f_m be smooth nonnegative functions. Then the ζ_m are the solutions of

$$a_m \frac{\partial\zeta_m}{\partial t} + b_m\Delta\zeta_m = -f_m \quad \text{in} \quad \Omega_{R,T} , \tag{3.4}$$

$$\zeta_m(\cdot, T) = 0 , \tag{3.5}$$

$$\zeta_m\big|_{\Gamma_{R,T}} = 0 , \ \frac{\partial\zeta_m}{\partial n}\big|_{\Gamma_{i,R,T}} = 0 \qquad (i = 1, 2) . \tag{3.6}$$

Clearly $\zeta_m \geq 0$. Multiplying (3.4) by $\dfrac{1}{a_m} \Delta\zeta_m$ and integrating, we easily obtain after using (3.5), (3.6) and the Schwarz inequality,

$$\sup_{0 \leq t \leq T} \int\limits_{\Omega_R} |\nabla\zeta_m(\cdot, t)|^2 + \iint\limits_{\Omega_{R,T}} \frac{b_m}{a_m} |\Delta\zeta_m|^2 \leq C \iint\limits_{\Omega_{R,T}} \frac{f_m^2}{a_m b_m} . \tag{3.7}$$

Similarly, if we multiply (3.4) by $\dfrac{1}{b_m}\ \dfrac{\partial \zeta_m}{\partial t}$ and integrate, we get

$$(3.8)\qquad \sup_{0\le t\le T}\int\limits_{\Omega_R} |\nabla\zeta_m(\cdot,t)|^2 + \iint\limits_{\Omega_{R,T}} \frac{a_m}{b_m}\ \left|\frac{\partial\zeta_m}{\partial t}\right|^2 \le C\iint\limits_{\Omega_{R,T}} \frac{f_m^2}{a_m b_m}\ .$$

Substituting $\zeta=\zeta_m$ into (3.3) we obtain, for any function $\alpha\in L^\infty(\Omega_{R,T})$, $|\alpha|\le 1$,

$$0\le \int\limits_{\Omega_R}\gamma_0\zeta_m(\cdot,0)+\iint\limits_{\Gamma_{1,R,T}} g\zeta_m+\iint\limits_{\Omega_{R,T}}(-\alpha)f_m+\iint\limits_{\Omega_{R,T}}\left[(u-\alpha b_m)\Delta\zeta_m+(\gamma-\alpha a_m)\frac{\partial\zeta_m}{\partial t}\right]\ .$$

Hence

$$(3.9)\quad \iint\limits_{\Omega_{R,T}}\alpha f_m\le\int\limits_{\Omega_R}\gamma_0^+\zeta_m(\cdot,0)+\iint\limits_{\Gamma_{1,R,T}} g^+\zeta_m+\iint\limits_{\Omega_{R,T}}\left[|u-\alpha b_m||\Delta\zeta_m|+|\gamma-\alpha a_m|\left|\frac{\partial\zeta_m}{\partial t}\right|\right]\ .$$

Using (3.7), (3.8) we can estimate the last integral by

$$\left\{\iint\limits_{\Omega_{R,T}}\frac{f_m^2}{a_m b_m}\right\}^{1/2}\left\{\left(\iint\limits_{\Omega_{R,T}}\frac{a_m}{b_m}\ |u-\alpha b_m|^2\right)^{1/2}+\left(\iint\limits_{\Omega_{R,T}}\frac{b_m}{a_m}|\gamma-\alpha a_m|^2\right)^{1/2}\right\}\ .$$

Similarly,

$$\int\limits_{\Omega_R}\gamma_0^+\zeta_m(\cdot,0)\le\|\gamma_0^+\|_{p,\Omega_R}\|\zeta_m\|_{p',\Omega_R}\le C\|\gamma_0^+\|_{p,\Omega_R}\|\nabla\zeta_m(\cdot,0)\|_{2,\Omega_R}$$

$$\le C\|\gamma_0^+\|_{p,\Omega_R}\left\{\iint\limits_{\Omega_{R,T}}\frac{f_m^2}{a_m b_m}\right\}^{1/2}\ ,$$

$$\int\limits_{\Gamma_{1,R,T}} g^+\zeta_m\le\int_0^T\|g^+(\cdot,t)\|_{p,\Gamma_{1,R}}\sup_{0\le t\le T}\left\{\int\limits_{\Gamma_{1,R}}\zeta_n^{p'}(\cdot,t)\right\}^{1/p'}$$

$$\le C\int_0^T\|g^+(\cdot,t)\|_{p,\Gamma_{1,R}}\left\{\iint\limits_{\Omega_{R,T}}\frac{f_m^2}{a_m b_m}\right\}^{1/2}\ .$$

Take

$$\alpha = \begin{cases} 1 & \text{if} \quad u > 0 \quad \text{or if} \quad u = 0,\ \gamma > 0 \\ -1 & \text{if} \quad u < 0 \quad \text{or if} \quad u = 0,\ \gamma < 0 \\ 0 & \text{if} \quad u = 0,\ \gamma = 0 \end{cases}$$

and $\widehat{f}_m = a_m b_m \chi_{\{\alpha>0\}}$. Then

$$\iint_{\Omega_{R,T}} \alpha \widehat{f}_m = \iint_{\Omega_{R,T}} a_m b_m \chi_{\{\alpha>0\}} = \iint_{\Omega_{R,T}} \frac{\widehat{f}_m^2}{a_m b_m} .$$

We can approximate $\widehat{f}_m$ by smooth functions f_m to which (3.9) and the subsequent estimates of the right-hand side of (3.9) hold and then, by taking the limit, obtain

$$\iint_{\Omega_{R,T}} a_m b_m \chi_{\{\alpha>0\}} \le C \left\{ \|\gamma_0^+\|_{p,\Omega_R} + \int_0^T \|g^+(\cdot,t)\|_{p,\Gamma_{1,R}} + J_m \right\} \tag{3.10}$$

where

$$J_m = \left(\iint_{\Omega_{R,T}} \frac{a_m}{b_m}(u - \alpha b_m)^2 \right)^{1/2} + \left(\iint_{\Omega_{R,T}} \frac{b_m}{a_m}(\gamma - \alpha a_m)^2 \right)^{1/2}$$

$$\le \|a_m\|_\infty^{1/2} \left(\iint_{\Omega_{R,T}} \frac{(|u| - b_m)^2}{b_m} \right)^{1/2} + \|b_m\|_2^{1/2} \left(\iint_{\Omega_{R,T}} \frac{(|\gamma| - a_m)^4}{a_m^2} \right)^{1/2}$$

by the definition of α.

To further estimate J_m we need the following lemma.

LEMMA 3.2. *If $a \in L^\infty(\Omega_{R,T})$, $a \ge 0$ and $b \in L^2(\Omega_{R,T})$, $b \ge 0$, then there exist sequences of C^∞ functions a_m, b_m in $\overline{\Omega}_{R,T}$ satisfying:*

$$a_m > \frac{1}{m} , \ b_m > \frac{1}{m} , \tag{i}$$

$$\|a_m\|_\infty \le \|a\|_\infty + 1 , \ \|b_m\|_2 \le \|b\|_2 + 1 , \tag{ii}$$

$$a_m \to a , \ b_m \to b \quad in \quad L^2(\Omega_{R,T}) , \tag{iii}$$

and

(iv)
$$\iint\limits_{\Omega_{R,T}} \frac{(a-a_m)^4}{a_m^2} \to 0 \ , \ \iint\limits_{\Omega_{R,T}} \frac{(b-b_m)^2}{b_m} \to 0 \ ,$$

as $m \to \infty$.

Proof. Let
$$\widehat{a}_m = \max\left\{a, \frac{1}{m}\right\} \ , \ \widehat{b}_m = \max\left\{b, \ \frac{1}{m}\right\}$$

and let
$$a_{m\varepsilon} = \rho_\varepsilon * \widehat{a}_m \ , \ b_{m\varepsilon} = \rho_\varepsilon * \widehat{b}_m \qquad (\varepsilon > 0)$$

be mollifiers of $\widehat{a}_m, \widehat{b}_m$. Then
$$\frac{1}{m} \le a_{m\varepsilon} \le \|a\|_\infty + 1 \ , \qquad b_{m\varepsilon} \ge \frac{1}{m}$$

and

$$a_{m\varepsilon} \to \widehat{a}_m \quad \text{in} \quad L^4 \qquad \text{(in fact in any } L^p\text{)},$$
$$b_{m\varepsilon} \to \widehat{b}_m \quad \text{in} \quad L^2 \ .$$

We take $\varepsilon = \varepsilon_m$ so that
$$\|a_{m\varepsilon} - \widehat{a}_m\|_4 < \frac{1}{m} \ , \ \|b_{m\varepsilon} - \widehat{b}_m\|_2 < \frac{1}{m} \ ,$$

and set $a_m = a_{m\varepsilon_m}$, $b_m = b_{m\varepsilon_m}$. Then (i)–(iii) clearly hold, whereas (iv) is proved as follows:

$$\iint \frac{(b-b_m)^2}{b_m} = 2\iint \frac{(\widehat{b}_m - b_m)^2}{b_m} + 2\iint \frac{(b-\widehat{b}_m)^2}{b_m}$$
$$\le 2m\|b_{m\varepsilon_m} - \widehat{b}_m\|_2^2 + 2m \iint\limits_{\{b<\frac{1}{m}\}} \left(b - \frac{1}{m}\right)^2 \le \frac{C}{m} \to 0 \ ,$$
$$\iint \frac{(a-a_m)^4}{a_m^2} \le 16 \iint \frac{(\widehat{a}_m - a_m)^4}{a_m^2} + 16 \iint \frac{(a-\widehat{a}_m)^4}{a_m^2}$$
$$\le 16m^2\|\widehat{a}_m - a_{m\varepsilon_m}\|_4^4 + 16m^2 \iint\limits_{\{a<\frac{1}{m}\}} \left(a - \frac{1}{m}\right)^4 \le \frac{C}{m^2} \to 0 \ .$$

We apply Lemma 3.2 to $a = |\gamma|$, $b = |u|$ and conclude that $J_m \to 0$ if $m \to \infty$. Then (3.10) yields

$$\iint\limits_{\Omega_{R,T}} u^+\gamma^+ \leq C(\|\gamma_0^+\|_{p,\Omega_R} + \int_0^T \|g^+(\cdot,t)\|_{p,\Gamma_{1,R}})^2 .$$

Since, by assumption, the right-hand side is equal to zero, it follows that

$$u^+\gamma^+ = 0$$

This implies that either $u^2 \leq u^1$ or $\gamma^2 \leq \gamma_1$.

In the sequel we assume that

$$\gamma_0(x,y) = -1 \quad \text{if} \quad (x^2+y^2)^{1/2} \geq x_0, \quad -1 \leq \gamma_0 \leq 1 , \tag{3.11}$$

$$0 \leq g(x,t) \leq M , \; g(x,t) = 0 \quad \text{if} \quad x \geq x_0 , \tag{3.12}$$

where M and x_0 are positive constants.

THEOREM 3.3. *There exists a continuous monotone increasing function $t \to \sigma(t)$ such that the solutions (u_R, γ_R) of (P_R) satisfy*

$$u_R = -1 , \; \gamma_R = -1 \quad \text{if} \quad x^2+y^2 > \sigma(t) .$$

Proof. For any $r > x_0 + 1$ denote by w_r the solution to

$$\begin{aligned} &\Delta w = 0 \quad \text{in} \quad \Omega_r , \\ &\frac{\partial w}{\partial x} = 0 \quad \text{on} \quad \Gamma_{2,r} , \\ &-\frac{\partial w}{\partial y} = \begin{cases} N & \text{if} \quad 0 \leq x \leq x_0 \\ 0 & \text{if} \quad x_0 < x < r , \end{cases} \\ &w = -1 \quad \text{if} \quad x^2+y^2 = r^2 . \end{aligned}$$

If $r_1 > r_2$ then $w_{r_1} > w_{r_2} > 0$ in Ω_{r_2}, so that $\partial w_r/\partial r \geq 0$ if $x^2+y^2 \leq r^2$. Note that, by comparison,

$$w_r(x,y) \leq N \int_{-x_0}^{x_0} \Gamma(x,y;\xi,0)d\xi$$

where Γ is the Neumann function for Δ in $\{y > 0\}$. Hence

$$\lim_{r\to\infty} w_r(x,y) \equiv W(x,y) \quad \text{exists and is finite.}$$

We choose $N > M$ and so large that

$$W(x,y) > \gamma_0(x,y) \quad \text{in} \quad \Omega_\infty ,$$

and then also

$$c(w_{\tilde r}) \geq \gamma_0 \quad \text{for some} \quad \tilde r \in (x_0, \infty) . \tag{3.13}$$

The function $w_r(r\cos\theta, r\sin\theta)$ (for $r > \tilde r$) is smooth in (r,θ) (also at $\theta = 0$) and therefore

$$K(r) = \sup_{0 \leq \theta \leq \frac{\pi}{2}} |\nabla w_r(r\cos\theta, r\sin\theta)|$$

is Lipschitz continuous in r. Let $s(t)$ be the solution to

$$\frac{ds}{dt} = K(s) , \qquad s(0) = \tilde r$$

and define

$$\overline{u}(x,y,t) = w_{s(t)}(x,y),$$

$$\overline{\gamma}(x,y,t) = c(\overline{u})$$

if $x^2 + y^2 \leq s^2(t)$. Extend $\overline{u}$ by -1 and $\overline{\gamma}$ by -1 to $x^2 + y^2 > s^2(t)$. Then, by (3.13), $\overline{\gamma} \geq \gamma_0$ everywhere.

If $R > s(T)$ then for any $\zeta \in V_{R,T}$, $\zeta \geq 0$,

$$\iint\limits_{\Omega_{R,T}} (\nabla \overline{u} \cdot \nabla \zeta - \overline{\gamma}\zeta_t) - \iint\limits_{\Omega_R} \gamma_0 \zeta(\cdot, 0) - \iint\limits_{\Gamma_{1,R,T}} g\zeta$$

$$= \int_0^T \int\limits_{\Omega_{s(t)}} (c(\overline{u})_t - \Delta \overline{u}) + \int_0^T \int\limits_{\Gamma_{s(t)}} \left(\frac{\partial \overline{u}}{\partial n} + \dot s \right) \zeta$$

$$+ \int\limits_{\Omega_{s(0)}} [c(\overline{u}(\cdot,0)) - \gamma_0]\zeta(\cdot,0) + \int\limits_{\Gamma_{1,x_0,T}} (N - g)\zeta \geq 0$$

since

$$c(\overline{u})_t - \Delta \overline{u} = c'(\overline{u}) \frac{\partial w}{\partial r} \frac{ds(t)}{dt} \geq 0 \quad \left(w = w_r \Big|_{r = s(t)} \right) ,$$

$$- \frac{\partial \overline{u}}{\partial n} \leq K(s(t)) = \dot s \quad \text{on} \quad \Gamma_{s(t)}$$

and $\overline{\gamma} \geq \gamma_0$. Applying Lemma 3.1 we deduce that $u_R \leq \overline{u} = -1$ if $x^2 + y^2 \geq s^2(t)$. Using (1.9) with ζ supported in $x^2 + y^2 > s^2(t)$ we find that also $\gamma_R = -1$ if $x^2 + y^2 > s^2(t)$.

Suppose (u_R, γ_R) and $(u_{R'}, \gamma_{R'})$ are weak solutions to (P_R) with $R' > R > s^2(T)$. Then by the proof of Theorem 2.1 we find that $u_R = u_{R'}, \gamma_R = \gamma_R$, if $t < T$. Hence

$$u = \lim_{R \to \infty} u_R \ , \ \gamma = \lim_{R \to \infty} \gamma_R \tag{3.14}$$

form a weak solution to problem (P). The uniqueness of the solution follows by the proof of Theorem 2.1. We summarize:

THEOREM 3.3. *If (3.11), (3.12) hold then there exists a unique weak solution (u, γ) to problem (P); it is given by (3.14). Furthermore, u is uniformly bounded.*

The last statement needs to be proved. Let w be the solution to

$$\Delta w = 0 \quad \text{in} \quad \Omega_R \ ,$$

with boundary conditions

$$w_x(0, y) = 0 \quad \text{if} \quad 0 < y < R \ ,$$

$$-\ w_y(x, 0) = \begin{cases} M_0 & \text{if} \quad 0 < x < x_0 \\ 0 & \text{if} \quad x_0 < x < R \ , \end{cases}$$

$$w = 0 \quad \text{if} \quad x^2 + y^2 = R^2 \ .$$

If M_0 is large enough then $c(w) > \gamma_0$ and then, by comparison $u_\varepsilon \leq w$. It follows that

$$u_\varepsilon \leq C \quad \text{in} \quad \Omega_{R,T} \tag{3.15}$$

where C a constant is independent of ε, R, T, and then also $u \leq C$.

§4. **Continuity of max{c(u), 0}.** In this section we assume, in addition to (3.11), (3.12), that

$$\gamma_0 \in c(u_0) \quad \text{where} \quad \int |\nabla u_0|^2 < \infty \ , \tag{4.1}$$

$$g \in BV(0, T; L^1(\Omega_R)) \qquad \forall\, R > 0 \ , \ T > 0 \ . \tag{4.2}$$

Multiplying (2.7) (for $u = u_\varepsilon$) by $\partial u_\varepsilon / \partial t$ and integrating yields

$$\begin{aligned} 0 &= \iint\limits_{\Omega_{R,t}} c_\varepsilon'(u_\varepsilon) u_{\varepsilon,t}^2 + \frac{1}{2} \int\limits_{\Omega_R} |\nabla u_\varepsilon(\cdot, \tau)|^2 \Big|_{\tau=0}^{\tau=t} - \iint\limits_{\Gamma_{1,R,t}} u_{\varepsilon,t} g_\varepsilon \\ &\geq \iint\limits_{\Omega_{R,t}} c_\varepsilon'(u_\varepsilon) u_{\varepsilon,t}^2 + \frac{1}{2} \int\limits_{\Omega_R} |\nabla u_\varepsilon(\cdot, t)|^2 - \frac{1}{2} \int\limits_{\Omega_R} |\nabla u_\varepsilon(\cdot, 0)|^2 \\ &\quad - \|u_\varepsilon\|_{L^\infty} (2\|g_\varepsilon\|_{L^\infty} + \|g_\varepsilon\|_{BV[0,T;L^1(\Omega_R)]}) \ . \end{aligned} \tag{4.3}$$

By (4.1), (4.2) we can define $\gamma_{0\varepsilon}$ and g_ε such that the last two terms are bounded independently of ε (here we use also (3.15)). Defining

$$w_\varepsilon = \max\{c_\varepsilon(u_\varepsilon), 0\} \tag{4.4}$$

and noting that

$$|\nabla w_\varepsilon| \leq |\nabla u_\varepsilon| \ , \ \ |w_{\varepsilon,t}| \leq |u_{\varepsilon,t}|$$

we then obtain from (4.3),

$$\iint\limits_{\Omega_{R,T}} |w_{\varepsilon,t}|^2 + \sup_{0\leq t\leq T} \int\limits_{\Omega_R} |\nabla w_\varepsilon(\cdot,t)|^2 \leq C \ . \tag{4.5}$$

This estimate will be used to prove the following theorem.

THEOREM 4.1. *The function* $w = \max\{c(u), 0\}$ *is continuous.*

Proof. Introduce the notation: $X = (x, y)$,

$$\begin{aligned}
&B_\rho(X) = \ \text{ball of radius } \rho \text{ and center } X \ , \\
&Q_\rho(X,t) = B_\rho(X) \times \left\{t - \frac{7}{8}\rho^2 \ , \ t + \frac{1}{8}\rho^2\right\} \ , \\
&\fint\limits_{B_\rho} = \frac{1}{|B_\rho|} \int\limits_{B_\rho} \ , \ \ \fint\!\!\fint\limits_{Q_\rho} = \frac{1}{|Q_\rho|} \iint\limits_{Q_\rho} \ , \\
&w_{B_\rho} = \fint\limits_{B_\rho} w \ , \ \ w_{Q_\rho} = \fint\!\!\fint\limits_{Q_\rho} w
\end{aligned}$$

where $B_\rho = B_\rho(X), Q_\rho = Q_\rho(X,t)$. From (4.5) we have

$$\iint\limits_{\Omega_{R,T}} |w_t|^2 + \sup_{0\leq t\leq T} \int\limits_{\Omega_R} |\nabla w|^2 \leq C \ . \tag{4.6}$$

LEMMA 4.2. *If a function* w *satisfies (4.6) then for any* $(X,t) \in \Omega_{R,T}$

$$\lim_{\rho\to 0} \fint\!\!\fint\limits_{Q_\rho(X,t)} |w - w_{Q_\rho(X,t)}|^2 = 0 \ . \tag{4.7}$$

Proof. Since $H^1(0,T;L^2(\Omega_R))$ is continuously imbedded in $C^{1/2}(0,T;L^2(\Omega_R))$, there is a version $t \to w^*(\cdot,t)$ such that for any $0<t<T$

$$\frac{1}{t}\int_t^{t+h} w(\cdot,\tau) - w^*(\cdot,t) \to 0 \quad \text{in} \quad L^2(\Omega_R) \tag{4.8}$$

and therefore also a.e. in (X,t). Since $w \in L^\infty(0,T;H^1(\Omega_R))$, (4.8) implies that $w^*(\cdot,t) \in H^1(\Omega_R)$ for all $t \in (0,T)$ and

$$\|w^*(\cdot,t)\|_{H^1(\Omega_R)} \le C\ . \tag{4.9}$$

By Sobolev's imbedding

$$\int_{B_\rho(X)} |w^*(\cdot,t) - w^*_{B_\rho(X)}(t)|^2 \le C\rho^2 \int_{B_\rho(X)} |\nabla w^*(\cdot,t)|^2\ ,$$

$$\sup_{t-\frac{7}{8}\rho^2<\tau<t+\frac{1}{8}\rho^2} |w(y,\tau) - w(y,t)|^2 \le C\rho^2 \int_{t-\frac{7}{8}\rho^2}^{t+\frac{1}{8}\rho^2} w_t^2(y,s)ds\ .$$

Therefore, with $Q_\rho = Q_\rho(X,t), B_\rho = B_\rho(X)$,

$$\iint_{Q_\rho} |w - w_{Q_\rho}|^2 = \inf_{c\in\mathbf{R}^1} \iint_{Q_\rho} |w^* - c|^2 \le \iint_{Q_\rho} |w^* - w^*_{B_\rho}(t)|^2$$

$$\le \iint_{Q_\rho} \left\{2|w^*(y,\tau) - w^*(y,t)|^2 + 2|w^*(y,t) - w^*_{B_\rho}(t)|^2\right\}$$

$$\le C\rho^4 \left\{\iint_{Q_\rho} |w_t^*|^2 + \int_{B_\rho} |\nabla w^*(\cdot,t)|^2\right\}\ .$$

Hence

$$\lim_{\rho\to 0} ⨌_{Q_\rho} |w - w_{Q_\rho}|^2 \le C\overline{\lim_{\rho\to 0}} \left\{\iint_{Q_\rho} |w_t^*|^2 + \int_{B_\rho(X)} |\nabla w^*(\cdot,t)|^2\right\}\ . \tag{4.10}$$

From the estimate (4.9) and the estimate on w_t in (4.6), which must hold also for w_t^*, it follows that the right-hand side of (4.10) is equal to zero.

LEMMA 4.3. *There exists a small positive constant μ (independent of ρ) such that*
(i) if

$$\iint\limits_{Q_\rho(X,t)} \left|\left(w-\frac{1}{2}\right)^+\right|^2 < \mu \tag{4.11}$$

then $w \le \frac{3}{4}$ *in* $Q^*_{\rho/2}(X,t) \equiv B_{\rho/2}(X) \times \left\{t - \frac{\rho^2}{8}\ ,\ t + \frac{\rho^2}{8}\right\}$, *and*
(ii) if

$$\iint\limits_{Q_\rho(X,t)} \left|\left(w-\frac{1}{2}\right)^-\right|^2 < \mu \tag{4.12}$$

then $w \ge \frac{1}{4}$ *in* $Q^*_{\rho/2}$.

Proof. For the purpose of this proof only we modify the definition of the $c_\varepsilon(u)$ (defined in (2.5)) for $u \ge 0$, taking

$$\begin{aligned} &c_\varepsilon(u) = 1 + u && \text{if} \quad -1 \le u \le 0 \quad \text{(as before)}, \\ &c'_\varepsilon(u) = \frac{\varepsilon}{(1+u)^3} && \text{if} \quad u \ge 0 \qquad \text{(instead of } u \ge \varepsilon). \end{aligned} \tag{4.13}$$

The new functions c_ε are not differentiable at $u = 0$, so that also u_ε are not as smooth as before. This however does not affect any of the previous results.

Let ζ be a smooth function satisfying:

$$\zeta = 1 \quad \text{in} \quad B_s \times \left[t + \frac{\rho^2}{8} - s^2\ ,\ t + \frac{\rho^2}{8}\right] \equiv \widetilde{Q}_s\ ,$$

$$\zeta = 0 \quad \text{on the parabolic boundary of } Q_\rho,$$

$$\zeta \ge 0\ ;$$

we can choose ζ of the form $\zeta_0(t)\zeta_1(r)$, $r = \sqrt{x^2+y^2}$, with $\zeta_1' \le 0, \zeta_1'' \le 0$, so that $\Delta\zeta \le 0$. Here $s \in (0,\rho)$ is still to be determined. Let k be a real number, $k \in \left[\frac{1}{4}\ ,\ \frac{3}{4}\right]$. Multiplying (2.7) (with $u = u_\varepsilon$) by $(w_\varepsilon - k)^+\zeta^2$ and integrating over $Q_{\rho,\tau} \equiv \{(X',t')\ ;\ (X',t') \in Q_\rho\ ,\ t' < \tau\}$ we get

$$\begin{aligned} &\iint\limits_{Q_{\rho,\tau}} \frac{\partial}{\partial t}\, c_\varepsilon(u_\varepsilon)\cdot(w_\varepsilon - k)^+\zeta^2 + \iint\limits_{Q_{\rho,\tau}} [\nabla u_\varepsilon \cdot \nabla(w_\varepsilon - k)^+]\zeta^2 \\ &\qquad = -\iint\limits_{Q_{\rho,\tau}} \nabla u_\varepsilon \cdot (w_\varepsilon - k)^+ \nabla\zeta^2\ . \end{aligned} \tag{4.14}$$

We claim that

$$\left|\iint\limits_{Q_{\rho,\tau}} [\nabla u_\varepsilon \cdot \nabla (w_\varepsilon - k)^+]\zeta^2 - \iint\limits_{Q_{\rho,\tau}} |\nabla (w_\varepsilon - k)^+|^2\zeta^2\right| \le C\varepsilon \iint\limits_{Q_{\rho,\tau}} |\nabla u_\varepsilon|^2 . \tag{4.15}$$

Indeed, if $w_\varepsilon > k, u_\varepsilon < 0$ then $w_\varepsilon = c(u_\varepsilon) = u_\varepsilon + 1$ (by (4.13)) so that

$$\nabla u_\varepsilon \cdot \nabla (w_\varepsilon - k)^+ = \nabla (w_\varepsilon - k) \cdot \nabla (w_\varepsilon - k)^+ = |\nabla (w_\varepsilon - k)^+|^2 .$$

These equalities hold trivially also if $w_\varepsilon \le k$. Finally, if $w_\varepsilon > k$ and $u_\varepsilon > 0$ then $|\nabla w_\varepsilon| \le \varepsilon |\nabla u_\varepsilon|$, by (4.13), and this gives the error term on the right-hand side of (4.15).

Next we have, since $c_\varepsilon(u_\varepsilon) = w_\varepsilon$ when $w_\varepsilon > k$,

$$\begin{aligned}\iint\limits_{Q_{\rho,\tau}} \frac{\partial}{\partial t}\, c_\varepsilon(u_\varepsilon) \cdot (w_\varepsilon - k)^+\zeta^2 &= \iint\limits_{Q_{\rho,\tau}} \frac{\partial}{\partial t}\, (w_\varepsilon - k) \cdot (w_\varepsilon - k)^+\zeta^2 \\ &= \frac{1}{2} \int\limits_{B\rho} [(w_\varepsilon - k)^+]^2\zeta^2(\cdot,\tau) - \iint\limits_{Q_{\rho,\tau}} |(w_\varepsilon - k)^+]^2\zeta\zeta_t .\end{aligned} \tag{4.16}$$

Consider finally the right-hand side of (4.14). We may assume that $w_\varepsilon > k$ in the integrand; then $u_\varepsilon + 1 > 0$, $w_\varepsilon = c_\varepsilon((u_\varepsilon + 1)^+ - 1)$ and

$$\nabla u_\varepsilon (w_\varepsilon - k)^+ = \nabla (u_\varepsilon + 1)^+ (w_\varepsilon - k)^+ = \nabla G_\varepsilon(u_\varepsilon) \quad \text{where}$$

$$G_\varepsilon(u_\varepsilon) = \int\limits_{\overline{u}}^{u_\varepsilon + 1} (c_\varepsilon(s-1) - k)^+ ds \le C[(c_\varepsilon(u_\varepsilon) - k)^+]^2 \le C[(w_\varepsilon - k)^+]^2 ;$$

here $\overline{u}$ is determined by $c_\varepsilon(\overline{u} - 1) = k$. We then have (cf. [8; p. 7])

$$\begin{aligned}-\iint\limits_{Q_{\rho,\tau}} \nabla u_\varepsilon \cdot (w_\varepsilon - k)^+ \nabla\zeta^2 &= -\iint\limits_{Q_{\rho,\tau}} \nabla G_\varepsilon(u_\varepsilon) \cdot \nabla \zeta^2 \\ &= -2 \iint\limits_{Q_{\rho,\tau}} \nabla[\zeta G_\varepsilon(u_\varepsilon)] \cdot \nabla\zeta + 2 \iint\limits_{Q_{\rho,\tau}} G_\varepsilon(u_\varepsilon)|\nabla\zeta|^2 \le 2 \iint\limits_{Q_{\rho,\tau}} G_\varepsilon(u_\varepsilon)|\nabla\zeta|^2 ,\end{aligned}$$

since $\Delta\zeta \le 0$. Combining this with (4.16), (4.15), we obtain from (4.14), after taking $\varepsilon \to 0$,

$$\begin{aligned}\iint\limits_{Q_\rho} |\nabla((w-k)^+\zeta)|^2 &+ \sup_{t - \frac{7}{8}\rho^2 \le \tau \le t + \frac{1}{8}\rho^2} \int\limits_{B_\rho} |(w-k)^+|^2\zeta^2(\tau) \\ &\le C \iint\limits_{Q_\rho} |(w-k)^+|^2[|\zeta_t| + |\nabla\zeta|^2] .\end{aligned} \tag{4.17}$$

Notice that ζ can be chosen so that

$$|\zeta_t| + |\nabla\zeta|^2 \le \frac{C}{(\rho - s)^2} \ .$$

We can now proceed as in [16; Chap. 2, §6]. Indeed we apply (4.17) recursively with ρ replaced by $\rho_n, s = \rho_{n+1}, k = k_n$ where

$$\rho_n = \left(\frac{1}{2} + \frac{1}{2^n}\right)\rho \ , \quad k_n = \frac{3}{4} - \frac{1}{2^n} \ , \quad n \ge 2$$

and obtain (cf. [16; Lemma 6.1, p. 106])

$$\iint\limits_{Q_{\rho_{n+1}}} |(w - k_{n+1})^+|^2 \le C_0 C^n \left\{ \iint\limits_{Q_{\rho_n}} |(w - k_n)^+|^2 \right\}^{3/2} \qquad (n \ge 2)$$

where C_0, C are constants independent of μ. Therefore, by [16; Lemma 5.7, p. 96], if (4.11) holds with μ small enough then

$$\iint\limits_{Q_{\rho_\infty}} |(w - k_\infty)^+|^2 = 0 \ , \quad \text{i.e.,} \quad w \le \frac{3}{4} \quad \text{in} \quad Q_{\rho/2} \ .$$

The proof of (ii) is similar.

We return to the proof of Theorem 4.1. For any $(X,t) \in \Omega_{R,T}$ and for any small $\mu > 0$ we have, by Lemma 4.2,

$$\iint\limits_{Q_\rho(X,t)} |w - w_{Q_\rho(X,t)}|^2 < \mu$$

if ρ is small enough. Consider separately the cases:

$$\text{(i)} \qquad w_{Q_\rho}(X,t) < \frac{1}{2} \ ,$$

$$\text{(ii)} \qquad w_{Q_\rho}(X,t) \ge \frac{1}{2} \ .$$

In case (i),

$$\iint\limits_{Q_\rho(X,t)} \left| \left(w - \frac{1}{2}\right)^+ \right|^2 \le \iint\limits_{Q_\rho(X,t)} \left| (w - w_{Q_\rho(X,t)})^+ \right|^2 < \mu$$

and Lemma 4.3 gives

$$w \le \frac{3}{4} \quad \text{in} \quad Q^*_{\rho/2} = Q^*_{\rho/2}(X,t) \ ,$$

so that $u \le -\dfrac{1}{4}$ in $Q^*_{\rho/2}$. Similarly, in case (ii), $w \ge \dfrac{1}{4}$ and $u \ge -\dfrac{3}{4}$ in $Q^*_{\rho/2}$.

In case (i) u satisfies

$$\frac{\partial}{\partial t}\,(u + H(u+1)) = \Delta u \quad \text{in} \quad Q^*_{\rho/2}\ .$$

By a result of Caffarelli and Friedman [5] u is a continuous function with modulus of continuity

$$\omega(r) = C2^{-|\log r|^\lambda} \quad \text{for any} \quad 0 < \lambda < \frac{1}{2} \qquad (C > 0)\ .$$

This implies the continuity of $\max\{c(u), 0\}$ with the same modulus of continuity. In case (ii) u satisfies (1.1) in $Q^*_{\rho/2}$ where $c(u)$ is continuous, $c(u) = 1$ if $u \ge 1$, and $c'(u) > 0$ if $u < 1$. By a result of DiBenedetto and Gariepy [8], $c(u)$ is continuous, and this concludes the proof of Theorem 4.1.

§**5. Properties of the free boundary.** In this section we assume that

$$\frac{\partial \gamma_0}{\partial x} \le 0\ , \ \frac{\partial \gamma_0}{\partial y} \le 0\ , \ \frac{\partial g}{\partial x} \le 0\ . \tag{5.1}$$

THEOREM 5.1. *Under the assumptions (5.1),*

$$\frac{\partial u}{\partial y} \le 0\ , \ \frac{\partial \gamma}{\partial y} \le 0 \tag{5.2}$$

and

$$\frac{\partial u}{\partial x} \le 0\ , \ \frac{\partial \gamma}{\partial x} \le 0 \tag{5.3}$$

where the derivatives are taken in the distribution sense.

Proof. Differentiating (2.7) with respect to y we get for $v = \partial u_\varepsilon / \partial y$ the equation

$$c'_\varepsilon(u_\varepsilon) v_t - \Delta v + c''_\varepsilon(u_\varepsilon) u_{\varepsilon,t} v = 0\ .$$

By the maximum principle we then conclude that $v \le 0$. Similarly one shows that $\partial u_\varepsilon / \partial x \le 0$. Taking $\varepsilon \to 0$, the assertions of the theorem follow.

Remark 5.1. If $\dfrac{\partial}{\partial \theta}\,\gamma_0(r\cos\theta, r\sin\theta) \le 0$ then also $\dfrac{\partial}{\partial \theta}\,u \le 0$.

By Theorem 5.1 there is a version of γ which is everywhere monotone decreasing in x and in y; hence also in $r = \sqrt{x^2 + y^2}$. Set

$$\zeta_1(\theta, t) = \sup\{r;\ \gamma(r\cos\theta, r\sin\theta, t) = 1\}\ , \tag{5.4}$$

$$\zeta_2(\theta, t) = \inf\{r;\ \gamma(r\cos\theta, r\sin\theta, t) \le 0\}\ , \tag{5.5}$$

and introduce the free boundaries

$$
\begin{aligned}
S_1 &\ : \ r = \zeta_1(\theta, t) \ , \\
S_2 &\ : \ r = \zeta_2(\theta, t) \ .
\end{aligned}
$$

By Theorem 4.1, the sets S_1, S_2 are separated by a positive distance, in any compact set in $\Omega_\infty \cap \{0 < t < \infty\}$.

By Theorem 5.1, for each $t > 0$, $\zeta_1(\theta, t)$ and $\zeta_2(\theta, t)$ are Lipschitz continuous in θ. From a result of Caffarelli [4] it also follows that

$$
\zeta_2(\theta, t) \in C^\infty\left(0, \frac{\pi}{2}\right] \times (0, \infty) \ . \tag{5.6}
$$

It is also well known (see, for example, [9]) that

$$
\frac{\partial \zeta_2}{\partial t} \geq 0 \ . \tag{5.7}
$$

We proceed to study the free boundary S_1.

THEOREM 5.2. *If $g(x,t) \equiv 0$ for $t_0 \leq t \leq t_0 + h$ then $\zeta_1(\theta, t) = 0$ for $0 < \theta < \dfrac{\pi}{2}$, $t_0 < t < t_0 + h$.*

Proof. Let $\overline{u}$ be the solution to the Stefan problem

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\overline{u} + H(\overline{u}+1)) = \Delta \overline{u} \quad \text{in} \quad \Omega_R \times (t_0, t_0 + h) \ , \quad R \gg 1 \ , \\
&c(\overline{u}(\cdot, t_0)) = c(\min[u(\cdot, t_0), 0]) \ , \\
&\overline{u}_x(0, y) = 0 \ , \ \overline{u}_y(x, 0) = 0 \ , \\
&\overline{u} = -1 \quad \text{on} \quad \Gamma_R \times (t_0, t_0 + h) \ .
\end{aligned}
$$

By the maximum principle $-1 \leq \overline{u} < 0$ in $\Omega_{R,T}$, so that $\overline{u}$ is actually a solution of

$$
\frac{\partial}{\partial t} \ c(\overline{u}) = \Delta \overline{u} \ .
$$

Since $c(u(X, t_0)) = c(\overline{u}(X, t_0))$ if $u(X, t_0) \leq 0$ and $c(u(X, t_0)) = 1 = c(0) = c(\overline{u}(X, t_0))$ if $u(X, t_0) > 0$, we have $c(u) = c(\overline{u})$ at $t = t_0$ and by uniqueness (Theorem 2.1) $u \equiv \overline{u}$. It follows that $u < 0$ in $\Omega_R \times (t_0, t_0 + h)$ and therefore $\zeta_1(\theta, t) = 0$ if $0 < \theta < \dfrac{\pi}{2}$, $t_0 < t < t_0 + h$.

THEOREM 5.3. *If $g(x,t) \geq b > 0$ for $0 \leq x \leq x_1,\ t_1 < t < t_2$ $(x_1 > 0,\ t_2 > t_1 \geq 0)$ then $\zeta_1(\theta,t)$ is continuous in (θ,t) for $t_1 < t < t_1,\ 0 \leq \theta \leq \frac{\pi}{2}$.*

The proof depends on two lemmas.

LEMMA 5.4. *Let $r_0 = \zeta_1(\theta_0,t_0) > 0$ where $0 < \theta_0 < \dfrac{\pi}{2},\quad t_1 < t_0 < t_2$. Then there is a constant $C > 0$ such that for all τ positive and sufficiently small*

$$\zeta_1(\theta_0,t_0+\tau) \geq \zeta_1(\theta_0,t_0) - C\tau^{1/6}\ . \tag{5.8}$$

Proof. Set $x^0 = r_0\cos\theta_0,\ y^0 = r_0\sin\theta_0$. For any small $\rho \geq 0$ define

$$D_\rho = \{(x,y);\ 0 < x < x^0-\rho,\ 0 < y < y^0-\rho\}\ ,$$

$$\ell_\rho = \partial D_\rho \cap \{x > 0,\ y > 0\}\ .$$

Take any small $\delta > 0$ and let w_1 be the harmonic function in D_0 such that $w_1 = -\delta$ on ℓ_0, $\partial w_1/\partial x = 0$ on $\partial D_0 \cap \{x = 0\}$, and $\partial w_1/\partial y = -g_0$ on $\partial D_0 \cap \{y = 0\}$ where

$$0 \leq g_0 \leq b,\ g_0 = b \quad \text{in} \quad \left[\frac{1}{4}\overline{x},\ \frac{3}{4}\overline{x}\right]\ ,$$

$$g = 0 \quad \text{if} \quad x < \frac{1}{8}\overline{x} \quad \text{or} \quad x > \frac{7}{8}\overline{x} \qquad (\overline{x} = \min\{x^0,x_1\})\ ,$$

and $g \in C^3[0,\infty)$. Then $w_1 \in C^2(\overline{D})$ and, by comparison,

$$-\delta + \frac{1}{C_0}(x-x_0)(y-y_0) \leq w_1(x,y) \leq -\delta + C_0(x-x_0)(y-y_0) \quad \text{in} \quad D_0 \tag{5.9}$$

if C_0 is a sufficiently large positive constant. It follows that

$$w_1 < 0 \quad \text{in} \quad D_0 \backslash D_{A\delta}\ ,\ A = [2C_0\max\{x^0,y^0\}]^{-1}\ . \tag{5.10}$$

Let $\zeta \in C^2(\overline{\Omega}_R)$ $\quad (R \gg 1)$ be a function satisfying:

$$\begin{aligned} &\zeta = 0 \quad \text{in} \quad D_{A\delta}\ ,\ \zeta = 1 \quad \text{in} \quad \Omega_R\backslash D_0\ , \qquad 0 \leq \zeta \leq 1\ ,\\ &\zeta_y(x,0) = 0,\ \zeta_x(0,y) = 0\ ,\\ &|D\zeta| \leq \frac{C_1}{\delta}\ ,\ |D^2\zeta| \leq \frac{C_1}{\delta^2} \quad \text{in} \quad \Omega_R \end{aligned} \tag{5.11}$$

where C_1 is a positive constant independent of δ. Extend w_1 into $\Omega_R\backslash D_0$ by $w_1 = -\delta$ and define

$$w = (1-\zeta)w_1 - \zeta \quad \text{in} \quad \Omega_R\ . \tag{5.12}$$

Since w_1 is in $C^2(\overline{D}_0)$,

$$\|w\|_{C^2(\Omega_R)} \leq \frac{C_2}{\delta^2} \; . \tag{5.13}$$

We now define

$$\overline{u} = \max\left\{-1, w - \frac{C_2}{\delta^2}(t - t_0)\right\} \quad \text{in} \quad \Omega_R \times [t_0, t_2] \; ,$$

$$\underline{\gamma} = \begin{cases} c(\overline{u}) & \text{if} \quad \overline{u} > -1 \\ -1 & \text{if} \quad \overline{u} = -1 \; . \end{cases}$$

We wish to apply Lemma 3.1 to show that $u \geq \overline{u}$ in $\{\overline{u} < 0\}$.

If $\overline{u}(x,y,t) \geq 0$ then $w(x,y) \geq 0$ and, by (5.10)–(5.12), $(x,y) \in D_{A\delta}$, so that

$$\underline{\gamma}_t - \Delta\overline{u} \leq -\Delta\overline{u} = -\Delta w = 0 \; .$$

If $-1 < \overline{u}(x,y,t) < 0$ then $\overline{u} = w - \dfrac{C_2}{\delta^2}(t - t_0)$ and hence

$$\underline{\gamma}_t - \Delta\overline{u} = u_t - \Delta\overline{u} = -\frac{C_2}{\delta^2} - \Delta w \leq 0$$

by (5.13). Observe next that $\partial\overline{u}/\partial n \leq 0$ on $\partial\{\overline{u} > -1\} \cap \{\overline{u} = -1\}$ where n is the normal pointing into $\{\overline{u} = -1\}$ and, therefore $\Delta\overline{u}$ is a positive measure on $\partial\{\overline{u} \geq -1\} \cap \{\overline{u} = -1\}$. It then easily follows that

$$\underline{\gamma}_t - \Delta\overline{u} \leq 0$$

in the distribution sense, in a neighborhood of $\partial\{\overline{u} \geq -1\} \cap \{\overline{u} = -1\}$. In conclusion, $\underline{\gamma}_t - \Delta\overline{u} \leq 0$ in $\Omega_R \times [t_0, t_2)$ in the distribution sense.

Since $u(\cdot, t_0) \geq 0$ in D_0, $\gamma(\cdot, t_0) = 1$ and therefore $\gamma(\cdot, t_0) \geq \underline{\gamma}(\cdot, t_0)$ in D_0. Also $\gamma(\cdot, t_0) \geq -1 = \underline{\gamma}(\cdot, t_0)$ in $\Omega_R \backslash D_0$ (since $\overline{u}(\cdot, t_0) = -1$ in $\Omega_R \backslash D_0$). Finally $\overline{u}_x = 0$ on $\{x = 0\}$ and $-u_y \leq g$ on $\{y = 0\}$. We can therefore apply Lemma 3.1 with $u^2 = \overline{u}$ and conclude that either $u \geq \overline{u}$ or $\gamma \geq \underline{\gamma}$; in particular, $u \geq 0$ if $\overline{u} \geq 0$.

From (5.9) we know that for any $\lambda > 0$ and $0 < \tau < t_2 - t_0$,

$$\begin{aligned} \overline{u}((r_0 - \lambda)\cos\theta_0, (r_0 - \lambda)\sin\theta_0, \; t_0 + \tau) &\geq w(x_0 - \lambda\cos\theta_0, y_0 - \lambda\sin\theta_0) - \frac{C_2}{\delta^2}\tau \\ &\geq -\delta + \frac{1}{C_0}\lambda^2 \sin\theta_0\cos\theta_0 - \frac{C_2}{\delta^2}\tau > 0 \end{aligned}$$

if we take $\delta = \tau^{1/3}$, $\lambda = C_3\tau^{1/6} = C_3\delta^{1/2}$ where

$$C_3 = \left[\frac{1+C_2}{\sin\theta_0\cos\theta_0}\;C_0\right]^{1/2}$$

(notice that $(x_0 - \lambda\cos\theta_0, y_0 - \lambda\sin\theta_0) \in D_{A\delta}$ so that $\zeta = 0$ at this point). We conclude that

$$u((r_0 - C_3\tau^{1/6})\cos\theta_0, (r_0 - C_3\tau^{1/6})\sin\theta_0,\ \tau) \geq 0$$

and the assertion (5.8) thus follows.

Remark 5.2. The constant C in (5.8) is uniform in (x^0, y^0, t_0) provided (x^0, y^0, t_0) remains in a compact subset of $\{x > 0, y > 0,\ t_1 < t < t_2\}$.

LEMMA 5.5. *Let $r_0 = \zeta_1(\theta_0, t_0) > 0$, $0 < \theta_0 < \frac{\pi}{2}$, $t_1 < t_0 < t_2$. Then there is a positive constant C such that*

$$\zeta_1(\theta_0, t_0 + \tau) \leq \zeta_1(\theta_0, t_0) + C\left(\log\frac{1}{\tau}\right)^{-1/6} . \tag{5.14}$$

Proof. By Lemma 5.4, if $t_0 - h \leq t \leq t_0$ (h small) then

$$u(x^0 + \delta, y^0 + \delta, t) < 0 \qquad (\delta = Ch^{1/6}) .$$

Hence

$$u(x, y^0 + \delta, t) \leq 0\ ,\ u(x^0 + \delta, y, t) \leq 0$$

for $x^0 + \delta \leq x < \infty$, $y^0 + \delta \leq y < \infty$, $t_0 - h \leq t \leq t_0$.

Denote by $\widetilde{R}$ the interior of the subset in $t_0 - h \leq t \leq t_0$ where $x \geq x^0 + \delta$ or $y \geq y^0 + \delta$, and $u > -1$. Denote by $\partial_0\widetilde{R}$ the lateral boundary of $\widetilde{R}$ on which $u = -1$ and by $\partial_1\widetilde{R}$ the lateral boundary of $\widetilde{R}$ where either $x = x_0 + \delta$ or $y = y_0 + \delta$. Let U be the solution of

$$\begin{aligned} &U_t = \Delta U \quad \text{in} \quad \widetilde{R}\ , \\ &U = -1 \quad \text{on} \quad \partial_0\widetilde{R}\ ,\ U = 0 \quad \text{on} \quad \partial_1\widetilde{R}\ , \\ &U = 0 \quad \text{at} \quad t = t_0 - h\ . \end{aligned}$$

Then $u \leq U$. Hence, by parabolic estimates on U,

$$u(x^0 + 2\delta, y^0 + 2\delta, t_0) \leq U(x^0 + 2\delta, y^0 + 2\delta, t_0) \leq -Ce^{-\frac{c}{h}} \equiv -\widetilde{\delta} \tag{5.15}$$

where C, c are positive constants.

Let R be a large constant such that $u=-1$ if $x^2+y^2>R^2$, $t\in[t_1,t_2]$. For each small $\rho\geq 0$ define

$$D_\rho=(0,R+\rho)\times(0,R+\rho)\backslash\{x\geq x^0+\rho,\ y\geq y^0+\rho\}\ .$$

Let w_1 be harmonic function in $D_{3\delta}$ satisfying:

$$w_{1,x}=0 \quad \text{on} \quad \partial D_{3\delta}\cap\{x=0\},\ -w_{1,y}=G\geq \sup_{t_1\leq t\leq t_2} g(\cdot,t) \quad \text{on} \quad \partial D_{3\delta}\cap\{y=0\}\ ,$$

and $w_1=-\widetilde{\delta}$ on the remaining part of $\partial D_{3\delta}$.

Using the conformal mapping $z\to z^{2/3}$ near the corner of $D_{3\delta}$ at $(x_0+3\delta,y_0+3\delta)$ we can deduce that

$$-\widetilde{\delta}+\frac{d^{2/3}}{C_0}\leq w_1(x,y)\leq-\widetilde{\delta}+C_0d^{2/3} \quad \text{in} \quad D_{3\rho}\cap B_\eta(x_0+3\delta,y_0+3\delta) \tag{5.16}$$

where $B_\eta(x_1,y_1)$ is a disc with center (x_1,y_1) and radius η,d is the distance from (x,y) to $\partial D_{3\delta}$, and C_0,η are some positive constants.

We can use reflection to estimate w_1 near the other corners of $D_{3\delta}$ and thus deduce that there is a constant $A>0$ such that

$$w_1<-\frac{\widetilde{\delta}}{2} \quad \text{in} \quad D_{3\delta}\backslash D_{3\delta-A\widetilde{\delta}^{3/2}}\ . \tag{5.17}$$

and (since $\widetilde{\delta}<\delta^3$ if $\delta\ll 1$) that

$$w_1>0 \quad \text{in} \quad D_{2\delta}\ . \tag{5.18}$$

From interior estimates for harmonic functions,

$$\|w_1\|_{C^k(D_{3\delta-\frac{1}{2}A\widetilde{\delta}^{3/2}})}\leq\frac{C_1}{\widetilde{\delta}^{\frac{3k}{2}}} \qquad (k=1,2)\ . \tag{5.19}$$

Let $\zeta\in C^3(\Omega_{R+1})$ be a function satisfying:

$$\begin{aligned}
&\zeta=0 \quad \text{in} \quad D_{3\delta-A\widetilde{\delta}^{3/2}}\ ,\ \zeta=1 \quad \text{in} \quad \Omega_{R+1}\backslash D_{3\delta-\frac{1}{2}A\widetilde{\delta}^{3/2}}\ ,\\
&\zeta_x=0 \quad \text{on} \quad x=0 \quad \text{and} \quad \zeta_y=0 \quad \text{on} \quad y=0\ ,\\
&0\leq\zeta\leq 1,\ |D\zeta|\leq\frac{C_2}{\widetilde{\delta}^{3/2}}\ ,\ |D^2\zeta|\leq\frac{C}{\widetilde{\delta}^3}\ .
\end{aligned} \tag{5.20}$$

Extend w_1 into $\Omega_{R+1}\backslash D_{3\delta}$ by $w_1=-\widetilde{\delta}$ and define

$$w=(1-\zeta)w_1-\widetilde{\delta}\zeta\ . \tag{5.21}$$

Since $w_1 \geq -\widetilde{\delta}$, also $w \geq -\widetilde{\delta}$. Further, by (5.19), (5.20)

$$|D^2 w|_{C^0(\Omega_{R+1})} \leq \frac{C_3}{\widetilde{\delta}^3} \ . \tag{5.22}$$

We now choose

$$\overline{u} = w + \frac{C_3}{\widetilde{\delta}^3}(t - t_0) \quad \text{in} \quad \Omega_R \times \left[t_0, t_0 + \frac{\widetilde{\delta}^4}{2C_3}\right) \ ,$$

$$\overline{\gamma} = c(\overline{u}) \ .$$

Since $w > -\widetilde{\delta}$ in Ω_R and $w \geq 0$ in $D_{2\delta}$ we have, by (5.15), that $\gamma(\cdot, t_0) \leq \overline{\gamma}(\cdot, t_0)$ in Ω_R (indeed $\overline{\gamma}(\cdot, t_0) = 1$ in $D_{2\delta}$ and $\overline{\gamma}(\cdot, t_0) > -1 = \gamma(\cdot, t_0)$ in $\Omega_R \backslash D_{2\delta}$). Next $\overline{u}_x = 0$ on $\{x = 0\}$ and $-\overline{u}_y = G > -u_y$ on $y = 0$. We finally show that $\overline{u}$ is a supersolution.

If $\overline{u}(x,y,t) > 0$, $t \in [t_0, \ t_0 + \widetilde{\delta}^4/(2C_3))$ then $w \geq -\dfrac{\widetilde{\delta}}{2}$ and, by (5.17), $(x,y) \in D_{3\delta - A\widetilde{\delta}^{3/2}}$. Consequently

$$\overline{\gamma}_t - \Delta \overline{u} \geq -\Delta \overline{u} = 0 \ .$$

If $\overline{u}(x,y,t) \leq 0$ then, since $\overline{u} \geq w > -\widetilde{\delta}$,

$$\overline{\gamma}_t - \Delta \overline{u} = \overline{u}_t - \Delta \overline{u} = \frac{C_3}{\widetilde{\delta}^3} - \Delta w > 0$$

by (5.22). This completes the proof that $\overline{u}$ is a supersolution.

We can then apply Lemma 3.1 to conclude that $u \leq \overline{u}$ in $\{\overline{u} \leq 0\}$. Since for $x > x^0 + 3\delta, \ y > y^0 + 3\delta$ and $\tau = \widetilde{\delta}^4/(2C_3)$,

$$\overline{u}(x,y,t_0 + \tau) = w(x,y) + \frac{\widetilde{\delta}}{2} = -\widetilde{\delta} + \frac{\widetilde{\delta}}{2} < 0 \ ,$$

it follows that $u(x,y,t_0 + \tau) < 0$. This implies that

$$\zeta_1(\theta_0, t_0 + \tau) \leq \zeta_1(\theta_0) + \frac{3\delta}{C_4}$$

where $C_4 = \min\{\cos\theta_0, \ \sin\theta_0\}$. Choosing $\widetilde{\delta}$ as in (5.15) and recalling that $\delta = Ch^{1/6}$, we easily deduce the relation

$$\delta = C_5 \left(\log \frac{C_6}{\tau}\right)^{-1/6} \ ,$$

and (5.14) follows.

From Lemmas 5.4, 5.5 we get

$$|\zeta_1(\theta_0, t_0 + \tau) - \zeta_1(\theta_0, t_0)| \leq C \left(\log \frac{1}{\tau}\right)^{-1/6} ; \tag{5.23}$$

the constant C is independent of θ_0, t_0 so long as θ_0, t_0 vary in a compact subset of $\left\{0 < \theta_0 < \frac{\pi}{2} \, , \, t_1 < t_0 < t_2\right\}$. The constant C degenerates as $\theta_0 \to 0$ or as $\theta_0 \to \frac{\pi}{2}$. However the proofs of the lemmas can be modified to establish the continuity of $\zeta_1(\theta, t)$ in t also at $\theta = 0, \pi$.

Since the function $\theta \to \zeta_1(\theta, t)$ is Lipschitz continuous in θ, uniformly in t, the proof of Theorem 5.3 is complete.

Remark 5.3. If $g(x,t) \geq b > 0$ for $0 \leq t < t_2$ then the continuity of $\zeta_1(\theta, t)$ at $t = 0$ can be established by slight modifications in the previous proofs.

Remark 5.4. The 1-d analog of the problem studied in this paper is:

$$\begin{aligned}
&u_{xx} = 0, \ u > 0 \quad \text{if} \quad 0 < x < \zeta_1(t), \ t > 0 \, , \\
&u_t = u_{xx}, \ -1 < u < 0 \quad \text{if} \quad \zeta_1(t) < x < \zeta_2(t), \ t > 0 \, , \\
&u = -1 \quad \text{if} \quad x > \zeta_2(t), \ t > 0 \, , \\
&u(\zeta_1(t), t) = 0, \ u(\zeta_2(t), t) = -1 \quad \text{if} \quad t > 0 \, , \\
&-u_x(0,t) = g(t) \quad \text{if} \quad t > 0 \quad (g \geq 0) \, , \\
&u(x,0) = u_0(x) \quad \text{if} \quad 0 < x < \infty \quad (\partial u_0 / \partial x \leq 0) \, .
\end{aligned} \tag{5.24}$$

All the results established in this paper are clearly valid for the problem (5.24). We shall now show that if $g(t) > 0$ for $0 \leq t < T$, $g(T) = 0$ then, $\zeta_1(t)$ may become discontinuous at $t = T$ even when $g(t)$ is continuous at $t = T$. (Thus the assumption $g \geq b > 0$ made in Theorem 5.3 cannot be dropped, in general.) Let

$$W(x,t,\delta) = \frac{1}{2\delta\sqrt{t}} \, e^{-x^2/(4t)} - \frac{1}{\delta} - 1 \quad (\delta > 0, \ \delta \ll 1) \, .$$

Then

$$W = 0 \quad \text{on} \quad x = s_1(t) = 2t \left[\log \frac{1}{4t(1+\delta)^2}\right]^{1/2} , \ 0 < t < \frac{1}{4(1+\delta)^2} \, ,$$

$$W = -1 \quad \text{on} \quad x = s_2(t) = \left[2t \log \frac{1}{4t}\right]^{1/2} , \quad 0 < t < \frac{1}{4} \, .$$

Set

$$\widehat{G}(\delta,t) = -\frac{\partial W}{\partial x}\bigg|_{x=s_1(t)} = \frac{1+\delta}{\delta\sqrt{2t}}\left[\log\frac{1}{4t(1+\delta)^2}\right]^{1/2} \quad , \ 0 < t < \frac{1}{4(1+\delta)^2} \ .$$

Define, for $0 < t < 1/[4(1+\delta)^2]$,

$$\widehat{u}(x,t) = \begin{cases} -\widehat{G}(\delta,t)x & \text{if} \quad x < s_1(t) \\ W(x,t,\delta) & \text{if} \quad s_1(t) < x < s_2(t) \\ -1 & \text{if} \quad x > s_2(t)\ , \end{cases}$$

and set

$$\underline{u}(x,\tau) = \widehat{u}\left(x + x_0, \frac{1}{4(1+\delta)^2} + \tau\right) \ ,$$

$$G(\delta,\tau) = \widehat{G}\left(\delta, \frac{1}{4(1+\delta)^2} + \tau\right)$$

where

$$x_0 = -\zeta_1(T-\delta) + s_2\left(\frac{1}{4(1+\delta)^2} - \delta\right) \ .$$

Notice that $G(\delta,\tau) > 0$ if $-\delta < \tau < 0$, $G(\delta,0) = 0$. The function $\underline{u}$ is a subsolution for the differential equations in (5.24) (we use here the fact that $\dot{s}_2(t) < 0$). Hence if

$$g(T+\tau) \geq G(\delta,\tau)\ , \ -\delta < \tau < 0 \tag{5.25}$$

and

$$u(x, T-\delta) > \underline{u}(x,-\delta) \tag{5.26}$$

then, by comparison,

$$u(x, T+\tau) \geq \underline{u}(x,\tau) \quad (-\delta < \tau < 0)\ . \tag{5.27}$$

To prove (5.26) we note that $\underline{u}(x,-\delta) = -1$ if $x = \zeta_1(T-\delta)$ so that (5.26) is valid for $x \geq \zeta_1(T-\delta)$. Next, if

$$\zeta_1(T-\delta) > 2C\delta\ , \tag{5.28}$$

where C is large enough constant, then $\underline{u}(0,-\delta) < 0$ and therefore (5.26) holds also for $0 \leq x < \zeta_1(T-\delta)$. We conclude that if (5.25) and (5.28) hold then (5.27) is valid and, in particular,

$$\underline{u}(\zeta_1(T), 0) < 0 \quad \text{if} \quad \zeta_1(T) > 0\ ,$$

that is,

$$\zeta_1(T) + x_0 > s_1\left(\frac{1}{4(1+\delta)^2}\right) = 0 \quad \text{if} \quad \zeta_1(T) > 0\ .$$

Recalling the definition of x_0 we conclude that

$$\zeta_1(T) \geq \zeta_1(T-\delta) - C_1\delta \quad \text{if} \quad \zeta_1(T) > 0\ .$$

where C_1 is a positive constant independent of δ and C. Choosing C so large that $C > C_1$ and recalling (5.28), it follows that

$$\zeta_1(T) > C\delta \quad \text{if} \quad \zeta_1(T) > 0\ . \tag{5.29}$$

Consider now a solution of (5.24) with $g(t) > 0$ for $0 < t < T - \delta$. We can choose δ so small that (5.28) is satisfied and then extend $g(t)$ to $[T-\delta, T]$ so that it satisfies (5.25) and $g(T) = 0$. We can then apply (5.29) for each T' smaller than T; since $\zeta_1(T') > 0$ we then have $\zeta_1(T') \geq C\delta$. Since however $\zeta_1(T) = 0$, $\zeta_1(t)$ is discontinuous at $t = T$. Note that $g(t)$ decreases linearly to zero as $t \uparrow T$. Perhaps a faster decrease will not produce a discontinuity for $\zeta_1(t)$ at $t = T$.

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