BIPARTITE GRAPHS AND INVERSE SIGN PATTERNS OF STRONG SIGN-NONSINGULAR MATRICES

By

Richard A. Brualdi
Keith L. Chavey
and
Bryan L. Shader

IMA Preprint Series # 1109
March 1993

Bipartite Graphs and Inverse Sign patterns of Strong Sign–nonsingular Matrices

Richard A. Brualdi*
Department of Mathematics
University of Wisconsin
Madison, WI 53706

Keith L. Chavey
Department of Mathematics
and Computer Systems
University of Wisconsin-River Falls
River Falls, WI 54022

Bryan L. Shader†
Department of Mathematics
University of Wyoming
Laramie, WY 82071

February 8, 1993

Abstract

A sign-nonsingular matrix is a matrix B such that each matrix X with the same sign pattern as B is nonsingular. If, in addition, the sign pattern of the inverse of X is the same for all X, then B is a strong sign–nonsingular matrix. A fully indecomposable matrix is a matrix whose associated bipartite graph is connected and (perfect) matching covered. The bipartite graphs of fully indecomposable, strong sign–nonsingular matrices are characterized and a

*Research partially supported by NSF Grant DMS-9123318. This paper was written partly while the author was a member of the Institute for Mathematics and its Applications (IMA), University of Minnesota. I thank the IMA for its support.

†This paper was written while the author was a Postdoctoral Fellow at the IMA.

recursive construction is given. This characterization is used to determine the sign patterns of the inverses of fully indecomposable, strong sign–nonsingular matrices, and to develop a recognition algorithm for such sign patterns. Those maximal strong sign–nonsingular matrices B whose sign patterns are uniquely determined by the sign patterns of their inverses are also characterized in terms of bipartite graphs.

1 Introduction

The *sign* of a real number a is defined by

$$\text{sign } a = \begin{cases} +1 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -1 & \text{if } a < 0. \end{cases}$$

Let $B = [b_{ij}]$ be a real matrix. The *sign pattern* of B is the $(0, 1, -1)$–matrix obtained from B by replacing each entry by its sign. The *zero pattern* of B is the $(0, 1)$–matrix obtained from B by replacing each nonzero entry by a 1. If $A = [a_{ij}]$ is also a real matrix with the same dimensions as B, then we write $A \leq B$ provided $a_{ij} \leq b_{ij}$ for all i and j. The set of real matrices with the same sign pattern as B is called the *qualitative class of* B and is denoted by $\mathcal{Q}(B)$. Since each qualitative class contains exactly one $(0, 1, -1)$-matrix, each qualitative class equals $\mathcal{Q}(B)$ for some $(0, 1, -1)$-matrix B. Qualitative matrix theory involves the study of properties which hold for all matrices in a qualitative class of matrices, that is, properties which depend only on the signs of the entries and not on their magnitudes. This paper is concerned with qualitative properties related to nonsingularity.

A square matrix B for which each matrix in $\mathcal{Q}(B)$ is nonsingular is called a *sign–nonsingular* matrix, abbreviated SNS–matrix. The class of SNS–matrices has been extensively studied (see [BS] for a list of references). Clearly, the property of being an SNS–matrix is preserved under row and column permutations, and by multiplication of rows and columns by -1. Using such operations any $(0, 1, -1)$ SNS–matrix may be brought to the form where each of its main diagonal entries equals -1.

Let B be a $(0, 1, -1)$ SNS–matrix. Each matrix in $\mathcal{Q}(B)$ has an inverse, but the inverses of the matrices in $\mathcal{Q}(B)$ may have many sign patterns. For example, the matrices

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & -1 & -1 \end{bmatrix} \qquad (1)$$

belong to the same qualitative class. However, the classical determinant formula for the inverse of a matrix shows that the signs of the entries in the $(3, 1)$ position of their inverses are different. An SNS–matrix B with the property that the inverses of the matrices in $\mathcal{Q}(B)$ all have the same sign pattern is called a *strong* SNS–matrix, abbreviated S^2NS–matrix. As for SNS–matrices, the property of being an S^2NS–matrix is preserved under row and column permutations, and by multiplication of rows and columns by -1. It can be verified that

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad (2)$$

is an SNS–matrix and the sign pattern of the inverse of each matrix in $\mathcal{Q}(A)$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Hence, A is an S^2NS–matrix. An S^2NS–matrix is a *maximal* S^2NS–matrix provided that no matrix obtained by replacing at least one zero entry by a nonzero is an S^2NS–matrix. The matrix in (2) is an example of a maximal S^2NS–matrix.

Let $A = [a_{ij}]$ be a matrix of order $n \geq 2$. Then A is *fully indecomposable* provided there do not exist permutation matrices P and Q such that PAQ has the form

$$\left[\begin{array}{c|c} A_1 & 0 \\ \hline * & A_2 \end{array}\right]$$

where A_1 and A_2 are square (nonvacuous) matrices. In [BS] it is shown that if B_1 and B_2 are fully indecomposable $(0,1,-1)$ SNS–matrices with the same zero pattern, then there exist diagonal matrices D and E each of whose diagonal entries equal ± 1 such that $DB_1E = B_2$. It follows that up to multiplication of certain rows and columns by -1, there is at most one SNS–matrix and at most one S^2NS–matrix with any given fully indecomposable zero pattern. This motivates the study of the zero patterns of fully indecomposable SNS–matrices and of S^2NS–matrices

Although the $(0,1)$–matrices that are zero patterns of fully indecomposable SNS–matrices have been shown to satisfy several restrictive properties (see [L,ST,TH92, BS]), the complexity of the problem of recognizing whether a given $(0,1)$–matrix is the zero pattern of an SNS–matrix has not yet been determined. However, in [TH89] the $(0,1)$–matrices which are the zero patterns of fully indecomposable S^2NS–matrices are characterized in terms of digraphs, and a polynomial–time algorithm for recognizing such matrices is given. Let $A = [a_{ij}]$ be a fully indecomposable $(0,1)$–matrix of order n. The *digraph of the matrix* A is the digraph with vertices $1, 2, \ldots, n$ and an arc from i to j if and only if $a_{ij} \neq 0$ $(i \neq j)$. By permuting the rows of A we may assume that each of the main diagonal entries of equals 1. Under this assumption, it is shown that A is the zero pattern of a S^{2}NS–matrix if and only if A does not contain a digraph obtained by inserting new vertices on the arcs of the following digraph.

In section 2, we characterize zero patterns of S^{2}NS–matrices in terms of bipartite graphs. The *bipartite graph of the matrix* $A = [a_{ij}]$ is the bipartite graph with vertices $1, 2, \ldots, n$ and $1', 2', \ldots, n'$, such that there is an edge joining i to j' if and only if $a_{ij} \neq 0$. The matrix A is fully indecomposable if and only if its bipartite graph is connected and each edge is in some perfect matching. We show that A is the zero pattern of an S^{2}NS–matrix if and only if the bipartite graph of B does not contain an even subdivision of $K_{2,3}$, that is, a subdigraph that can be obtained from the complete bipartite graph $K_{2,3}$ by inserting an even number of new vertices on each edge. One advantage of this bipartite characterization is that it provides a simple recursive description of the zero patterns of fully indecomposable S^{2}NS–matrices.

In the final section, the results of section 2 are used to show how to recursively construct the sign patterns of the inverses of fully indecomposable S^{2}NS–matrices. A polynomial–time algorithm for determining whether or not a given $(0, 1, -1)$–matrix is the sign pattern of the inverse of a fully indecomposable S^{2}NS–matrix is given. Two S^{2}NS–matrices with different sign patterns may have inverses with the same sign pattern. Indeed, if B is a fully indecomposable S^{2}NS–matrix, then any fully indecomposable matrix obtained from B by replacing a nonzero entry by zero is an S^{2}NS–matrix whose inverse has the same sign pattern as B^{-1}. We conclude the paper by studying properties shared by two S^{2}NS–matrices whose inverses have the same sign pattern. In particular, we identify the fully indecomposable, $(0, 1, -1)$ maximal S^{2}NS–matrices the sign pattern of whose inverse is not the sign pattern of the inverse of a different $(0, 1, -1)$ maximal S^{2}NS–matrix.

2 Fully indecomposable S^{2}NS–matrices

Let $A = [a_{ij}]$ be a matrix of order n. If α and β are subsets of $\{1, 2, \ldots, n\}$, then $A[\alpha, \beta]$ denotes the submatrix of A with rows whose index is in α and with columns whose index is in β, and $A(\alpha, \beta)$ denotes the complementary submatrix with rows whose index is not in α and with columns whose index is not in β. If $\alpha = \{i\}$, and $\beta = \{j\}$, then we write $A(i, j)$ instead of $A(\{i\}, \{j\})$.

We first recall some known facts about SNS–matrices and S^{2}NS–matrices. The determinant of A is given by

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \tag{3}$$

where the summation is taken over all permutations σ of $1, 2, \ldots, n$. If each of the $n!$ terms in the determinant expansion (3) of A equals 0, then we say that A has an *identically zero determinant.* By the Frobenius–Kõnig theorem, A has an identically zero determinant if and only if there exist positive integers p and q with $p+q = n+1$ such that A contains a zero submatrix of size p by q. It is well known and easy to verify that A is an SNS–matrix if and only if A does not have an identically zero determinant and each nonzero term in its determinant expansion (3) has the same sign. The matrix A is an S^2NS–matrix if and only if A is an SNS–matrix and each submatrix of A of order $n-1$ either is an SNS–matrix or has an identically zero determinant. The Frobenius–Kõnig theorem implies that each submatrix of order $n-1$ of a fully indecomposable S^2NS–matrix is an SNS–matrix, and consequently the inverse of a fully indecomposable S^2NS–matrix has no zero entries. Also, if A is an S^2NS–matrix and B is a fully indecomposable matrix obtained from A by replacing some of its nonzero entries by zeros, then B is an S^2NS–matrix and the sign patterns of A^{-1} and B^{-1} are the same.

We now describe a method for constructing S^2NS–matrices of order $n+1$ from S^2NS–matrices of order n. Let $A = [a_{ij}]$ be a matrix of order n such that $a_{pq} \neq 0$ and let B be the matrix of order $n+1$ obtained from A by bordering as shown below:

$$\left[\begin{array}{c|c|c|c} & & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline & a_{pq} & & a_{pq} \\ \hline & & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline 0 \;\cdots\; 0 & a_{pq} & 0 \;\cdots\; 0 & -a_{pq} \end{array}\right].$$

Then B is the matrix obtained from A by *conformally copying the entry* a_{pq}. More generally, we say that a matrix B can be obtained from A by *conformally copying entries* provided that there is a sequence of matrices

$$A = A_0, A_1, \ldots, A_k = B \quad (k \geq 0)$$

such that A_i can be obtained by conformally copying a (nonzero) entry of A_{i-1} ($i = 1, \ldots, k$).

Lemma 2.1 *Let B be a matrix which is obtained from a matrix A by conformally copying entries.*

(a) *If A is fully indecomposable, then B is fully indecomposable.*

(b) *If A is an SNS–matrix, then B is an SNS–matrix*

(c) *If A is a fully indecomposable S^2NS–matrix then B is a fully indecomposable S^2NS–matrix.*

Proof. It suffices to assume that B is obtained from A by conformally copying an entry a_{pq}. Let n be the order of A. Statement (a) is an easy consequence of the definitions. Now assume that A is an SNS–matrix. Since each of the the nonzero terms in the determinant expansion of A has the same sign, it is easily verified that each of the nonzero terms in the determinant expansion of B has the same sign. Hence B is an SNS–matrix and (b) holds. Finally, assume that A is a fully indecomposable S^2NS–matrix. We verify that B is also an S^2NS–matrix by showing that each submatrix $B(i,j)$ of B of order n is an SNS–matrix. If $i = j = n+1$ then $B(i,j) = A$, and hence $B(i,j)$ is an SNS–matrix. If $i = p$ and $j \neq n+1$ or if $i = n+1$ and $j \neq n+1$, then the conclusion holds since $A(p,j)$ is an SNS–matrix. If $j = q$ and $i \neq n+1$ or if $j = n+1$ and $i \neq n+1$, then $B(i,j)$ is an SNS–matrix because $A(i,q)$ is an SNS–matrix. Otherwise, $B(i,j)$ can be obtained from the SNS–matrix $A(i,j)$ by conformally copying the entry a_{pq}, and hence is an SNS–matrix. Therefore, B is an S^2NS–matrix, and (c) holds. □

It follows from (c) of Lemma 2.1 that every matrix obtained from identity matrix I_1 of order 1 by conformally copying entries is a fully indecomposable S^2NS–matrix. The main result of this section is that up to permutation of rows and columns, and multiplication of rows and columns by -1, the fully indecomposable, maximal $(0,1,-1)$ S^2NS–matrices are exactly those matrices obtained from I_1 by conformally copying entries.

Lemma 2.2 *Let $B = [b_{ij}]$ be a matrix of order $n \geq 2$ such that*

$$B = \left[\begin{array}{cccc|c} & & & & 0 \\ & & A & & \vdots \\ & & & & 0 \\ & & & & x \\ \hline 0 & \cdots & 0 & y & z \end{array}\right]$$

where $A = B(n,n)$ is fully indecomposable, $b_{n-1,n-1} = 0$, and x, y, and z are nonzero numbers. Let B' be the matrix obtained from B by replacing $b_{n-1,n-1}$ by $-xyz$. If B is an SNS–matrix then B' is an SNS–matrix, and if B is a fully indecomposable S^2NS–matrix then B' is fully indecomposable S^2NS–matrix.

Proof. First assume that B is an SNS–matrix. Then B, and hence B', does not have an identically zero determinant. Also, each nonzero term in the determinant expansion of B has the same sign. Each nonzero term in the determinant expansion of B' which contains the $(n-1,n-1)$-entry also contains the (n,n)-entry. It now follows that each nonzero term in the determinant expansion of B' has the same sign as a nonzero term in the determinant expansion of B, implying that B' is an SNS–matrix.

Now assume that B is a fully indecomposable S²NS–matrix. Then by what we have already shown, B' is an SNS–matrix. We verify that B' is an S²NS–matrix by showing that each submatrix $C = B'(i,j)$ of B' of order $n-1$ is an SNS–matrix. If $i = n$ and $j = n$, then $C = B'(n,n)$, and the conclusion follows from the facts that B' is an SNS–matrix and $b_{n,n} = z \neq 0$. If $i \neq n$ and $j = n-1$ or if $i = n-1$ and $j \neq n$, then since $C = B(i,j)$, the conclusion holds. If $i \neq n$ and $j = n$, then the conclusion follows because $B(i,n-1)$ is an SNS–matrix. A similar argument handles the case that $i = n$, and $j \neq n$. Otherwise, C can be obtained from $B(i,j)$ by replacing the entry in position $(n-2,n-2)$ by $-xyz$. Since B is a fully indecomposable S²NS–matrix, $B(i,j)$ is an SNS–matrix. The fact that C is an SNS–matrix now follows from the first assertion of the theorem. □

Let B be a fully indecomposable S²NS–matrix of order n, and let A be an $n-1$ by n submatrix of B. Then each submatrix of A of order $n-1$ is an SNS–matrix.[1]

We will use the following lemma from ([BCS,M]).

Lemma 2.3 *Let A be an $n-1$ by n such that each submatrix of order $n-1$ is an SNS–matrix and the matrix of order $n-1$ obtained from A by deleting column n is fully indecomposable. Then column n of A contains exactly one nonzero entry.*

We also make use of the following structure result for fully indecomposable matrices.

Lemma 2.4 *Let A be a fully indecomposable $(0,1)$–matrix of order $n \geq 2$. Then there exists a positive integer $m < n$ and permutation matrices P and Q such that*

[1] An $n-1$ by n matrix is an S^**-matrix* provided each of its submatrices of order $n-1$ is an SNS–matrix (see [KLM]).

PAQ has the form

$$\left[\begin{array}{c|cccccc} A' & & & G & & & \\ \hline & 1 & 1 & 0 & \cdots & 0 & 0 \\ & 0 & 1 & 1 & \cdots & 0 & 0 \\ & 0 & 0 & 1 & \cdots & 0 & 0 \\ F & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & 0 & \cdots & 1 & 1 \\ & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \tag{4}$$

where A' is a fully indecomposable matrix of order m, the only nonzero entries of F are in its last row and the only nonzero entries of G are in its first column.

Proof. Let $A' = A[\alpha, \beta]$ be a fully indecomposable, proper submatrix of A of maximal order $m < n$ such that $A(\alpha, \beta)$ has a nonzero term in its determinant expansion. Let $\overline{\alpha} = \{1, 2, \ldots, n\} \setminus \alpha$. By permuting the rows and columns of A we may assume that $\alpha = \beta = \{1, 2, \ldots, m\}$ and that $I_{n-m} \leq A(\alpha, \alpha)$. By the full indecomposability of A, there exists a nonzero entry in $G = A[\alpha, \overline{\alpha}]$ and this nonzero entry is contained in a nonzero term of the determinant expansion of A. This nonzero term and the maximal property of A' imply that after simultaneous permutations of the last $n - m$ rows and columns of A we may assume that

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \leq A(\alpha),$$

G has a nonzero entry in its first column and $F = A[\overline{\alpha}, \alpha]$ has a nonzero entry in its last row. The maximal property of A' immediately implies that the only nonzero entries of F and G are in their last row and first column respectively, and that the entries of $A(\alpha)$ in the positions (i, j) with $j \geq i + 2$ are all equal to zero. Suppose that $A(\alpha)$ has a nonzero entry in position (p, q) where $q < p$. Let

$$\gamma = \{1, 2, \ldots, q - 1, p, p + 1, \ldots, n\} \text{ and } \delta = \{1, 2, \ldots, q, p + 1, p + 2, \ldots, n\}.$$

Then $A[\gamma, \delta]$ is a fully indecomposable proper submatrix of A such that $A(\gamma, \delta)$ has a nonzero term in its determinant expansion, contrary to the choice of A'. Hence all of the entries of $A(\alpha)$ in positions (p, q) with $q < p$ are equal to zero. □

The following theorem characterizes fully indecomposable, maximal S^2NS–matrices.

Theorem 2.5 *Up to row and column permutations and multiplication of rows and columns by* -1*, the set of fully indecomposable, maximal* $(0,1,-1)$ *S^2NS–matrices is precisely the set of matrices that can be obtained from* I_1 *by conformally copying entries.*

Proof. Let $A = [a_{ij}]$ be a matrix of order n which is obtained from I_1 by conformally copying entries. By Lemma 2.1, A is a fully indecomposable S^2NS–matrix. We show that A is maximal by induction on n. If $n = 1$, this is clear. Assume $n \geq 2$. Without loss of generality we assume that A is obtained from $A(n,n)$ by conformally copying the entry $a_{n-1,n-1}$. By the induction hypothesis, $A(n,n)$ is a fully indecomposable, maximal S^2NS–matrix. Thus, no matrix obtained from A by replacing a zero in $A(n,n)$ by a 1 or a -1 is an S^2NS–matrix. Applying Lemma 2.3 to the matrix obtained from A by deleting its last row, we conclude that no matrix obtained from A by replacing a zero in column n by a 1 or a -1 is an S^2NS–matrix. By a similar argument applied to the transpose of A, we conclude that no matrix obtained from A by replacing a zero in row n by a 1 or a -1 is an S^2NS–matrix. Therefore, A is a maximal S^2NS–matrix.

Now let $A = [a_{ij}]$ be a fully indecomposable, $(0,1,-1)$ maximal S^2NS–matrix of order n. We prove by induction on n that up to row and column permutations and multiplication by rows and columns by -1, A can be obtained from I_1 by conformally copying entries. This is clear, if $n = 1$. Assume that $n \geq 2$. By Lemma 2.4 (applied to the zero pattern of A) and by multiplying certain rows and columns of A by -1, we may assume that A has the form

$$\left[\begin{array}{c|cccccc} A' & & & G & & & \\ \hline & -1 & 1 & 0 & \cdots & 0 & 0 \\ & 0 & -1 & 1 & \cdots & 0 & 0 \\ & 0 & 0 & -1 & \cdots & 0 & 0 \\ F & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 0 & 0 & 0 & \cdots & -1 & 1 \\ & 0 & 0 & 0 & \cdots & 0 & -1 \end{array}\right] \qquad (5)$$

where A' is a fully indecomposable matrix of order $m < n$, and the only nonzero entries of F and G are in their last row and first column, respectively. The maximal property of A, and Lemma 2.2 now imply that $n - m = 1$. Since A' is fully indecomposable, Lemma 2.3 implies that column n of A has exactly two nonzero entries. Similarly, row n of A has exactly two nonzero entries. Without loss of generality we assume that these nonzero entries occur in row $n-1$ and column $n-1$, respectively. The maximal property of A and Lemma 2.2 now imply that $a_{n-1,n-1} \neq 0$. Since A is a fully indecomposable, $(0,1,-1)$ S^2NS–matrix, it follows that $a_{n-1,n-1} = -a_{n,n}a_{n-1,n}a_{n,n-1}$. Thus A can be obtained from $A(n,n)$ by conformally copying $a_{n-1,n-1}$. The maximal property of A now implies that $A(n,n)$ is a fully indecomposable, maximal S^2NS–matrix, and the conclusion follows by induction. □

Theorem 2.5 immediately implies the following corollaries, the first of which is contained in [TH89].

Corollary 2.6 *Every fully indecomposable, maximal S^2NS–matrix of order n has exactly $3n-2$ nonzero entries.*

Corollary 2.7 *Up to row and column permutations and multiplication of rows and columns by -1, the fully indecomposable $(0,1,-1)$ S^2NS–matrices are precisely the fully indecomposable matrices that can be obtained from I_1 by conformally copying entries and then replacing some of the nonzero entries by zeros.*

We conclude this section by classifying the zero patterns of fully indecomposable S^2NS–matrices in terms of their bipartite graphs. We recall the following concepts from graph theory. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the edges of G. A *subdivision* of G is a graph H obtained from G by the insertion of $k_i \geq 0$ new vertices on the edge α_i of G. If each of the k_i is even, then H is an *even subdivision* of G. Note that if G is a bipartite graph, then every even subdivision of G is a bipartite graph. A *perfect matching* of a graph is a set of edges such that each vertex is contained in exactly one of the edges.

Let G_1 and G_2 be graphs with disjoint vertex sets. An *edge–join* of G_1 and G_2 is a graph obtained by identifying an edge of G_1 with an edge of G_2. A graph which is either an edge or can be obtained from an edge by a sequence of edge–joins with cycles of length 4 is called a *4–cockade.* Note that if the matrix A can be obtained from the matrix B by conformally copying an entry, then the bipartite graph of A can be obtained by an edge–join of the bipartite graph of B and a cycle of length 4. It follows that the bipartite graphs of the set of matrices which can be obtained from I_1 by conformally copying entries, are precisely the 4–cockades. Moreover, Theorem 2.5 implies that the $(0,1)$–matrix A is the zero pattern of a fully indecomposable, maximal S^2NS–matrix if and only if the bipartite graph of A is a 4–cockade. The following facts about 4–cockades are contained in [TH89,HLM]. Any edge of a 4–cockade can be used as the first edge in a construction of the 4–cockade. A connected graph G is a subgraph of a 4–cockade if and only if G is bipartite and does not contain an even subdivision of $K_{2,3}$, or equivalently if and only if G is bipartite and there do not exist two vertices u and v not joined by a path of even length and three paths from u and v no two of which have a common internal vertex. The Frobenius–König theorem implies that a square matrix is fully indecomposable if and only if its bipartite graph is connected and each edge is contained in a perfect matching. The next theorem characterizes the zero patterns of fully indecomposable S^2NS–matrices in terms of bipartite graphs and follows from the preceding discussion and Corollary 2.7.

Theorem 2.8 *Let A be a $(0,1)$-matrix of order n. Then the following are equivalent:*

(i) *A is the zero pattern of a fully indecomposable S^2NS–matrix;*

(ii) *The bipartite graph of A is a connected spanning subgraph of a 4-cockade and each edge is contained in a perfect matching;*

(iii) *The bipartite graph of A is connected, does not contain an even subdivision of $K_{2,3}$ and each edge is contained in a perfect matching.*

3 Inverse sign patterns

Let $u = (u_1, u_2, \ldots, u_n)$ be a real vector. Then u is *balanced* provided that either $u = 0$, or u contains both a positive entry and a negative entry. The vector u is *unsigned* provided it is not balanced. Thus u is unsigned if and only if $u \neq 0$ and the nonzero entries of u have the same sign. A *signing of order n* is a $(0, 1, -1)$ diagonal matrix D such that $D \neq O$. A signing is a *strict signing* provided it is nonsingular. Let A be a matrix of order n. We recall the following basic facts concerning SNS–matrices and S^2NS–matrices [BCS].

(*) A is an SNS–matrix if and only if for each signing D of order n, AD contains at least one unsigned row.

(**) If A is an SNS–matrix, then A is an S^2NS–matrix if and only if for each i $(1 \le i \le n)$ there is a unique signing D_i such that row i is the only unsigned row of AD_i and the nonzero entries in row i are positive. Moreover, if A is fully indecomposable S^2NS–matrix, then each of the D_i is a strict signing.

(***) If A is an S^2NS–matrix and $D_1, D_2, \ldots, D_n$ are the signings described in (**), then the sign pattern of A^{-1} is the matrix whose entry in the (i,j)-position equals the (i,i) entry of D_j $(1 \le i,j \le n)$.

We begin this section by describing the relationship between the inverse sign pattern of an S^2NS–matrix A and that of an S^2NS–matrix B obtained from A by conformally copying an entry. First we note that (**) and (***) imply the following lemma.

Lemma 3.1 *If $A = [a_{ij}]$ is a fully indecomposable S^2NS–matrix and $a_{pq} \neq 0$, then the sign of the entry in the (q,p)-position of A^{-1} equals the sign of a_{pq}.*

Theorem 3.2 *Let $A = [a_{ij}]$ be a fully indecomposable S^2NS–matrix of order n and assume that the (p, q)–entry of A is nonzero. Let $B = [b_{ij}]$ be the fully indecomposable S^2NS–matrix obtained by conformally copying the (p, q)–entry of A. Then the sign pattern of B^{-1} equals the sign pattern of the matrix*

$$\left[\begin{array}{c|c} A^{-1} & v \\ \hline u & -a_{pq} \end{array}\right], \tag{6}$$

where $u = (u_1, \ldots, u_n)$ is the qth row of A^{-1} and $v = (v_1, \ldots, v_n)^T$ is the pth column of A^{-1}.

Proof. Let D_j be the strict signing of order n whose ith diagonal entry equals the sign of the entry in position (i, j) of A^{-1} for each $i = 1, 2, \ldots, n$, and let $E_j = D_j \oplus (u_j)$ $(j = 1, \ldots, n)$. Since A is an S^{2}NS–matrix, row j is the only unsigned row of AD_j and its nonzero entries are positive. It follows from the definitions of B and E_j that row j is the only unsigned row of BE_j and that its nonzero entries are positive. Thus, the sign patterns of the first n columns of B^{-1} are equal to the sign patterns of the corresponding columns of (6). A similar argument applied to A^T shows that the sign patterns of the first n rows of B^{-1} are equal to the sign patterns of the corresponding rows of (6). Since $b_{n+1,n+1} = -a_{pq}$, the entry in $(n+1, n+1)$–position of B^{-1} equals $-a_{pq}$ and the theorem follows. □

Let $B = [b_{ij}]$ be the sign pattern of the inverse of a fully indecomposable S^{2}NS–matrix. It follows from (**) and (***) that no two columns of B are equal. However, Corollary 2.7 and Theorem 3.2 imply that B has two columns that are almost equal or are almost opposite. More precisely, there exists an integer r such that the matrix B_r obtained from B by deleting row r contains two columns which either are equal or are negatives of one another. The following lemma shows that if columns p and q of B_r satisfy this property, then the signs of the (p, r) and (q, r) entries of every S^{2}NS–matrix whose inverse has sign pattern equal to B are equal to b_{rp} and b_{rq}, respectively.

Lemma 3.3 *Let $A = [a_{ij}]$ be a fully indecomposable S^2NS–matrix of order n and let $B = [b_{ij}]$ be the sign pattern of A^{-1}. Suppose that p, q and r are integers with $1 \le p < q < n$ and $1 \le r \le n$ such that the vectors u and v obtained from columns p and q of B by deleting their rth entries satisfy $u = \pm v$. Then*

$$\text{sign } a_{pr} = b_{rp} \text{ and sign } a_{qr} = b_{rq}.$$

In addition, the matrix $\widehat{A}$ obtained from A by replacing each entry of the form a_{sr} $(s \neq p, q)$ by zero is a fully indecomposable S^2NS–matrix.

Proof. Let D and E be the (strict) signings of order n whose ith diagonal entry equals b_{ip} and b_{iq} ($i = 1, \ldots, n$), respectively. The assumptions on columns p and q of B imply that $D(r,r) = \pm E(r,r)$. Since row p of AD is unisigned and row q of AD is balanced it now follows that $a_{pr} \neq 0$. Since the nonzero entries of row p of AD are positive, the sign of a_{pr} equals b_{rp}. A similar argument shows that $a_{qr} \neq 0$, and that the sign of a_{qr} equals b_{rq}.

Since A is a fully indecomposable S^{2}NS–matrix, $\widehat{A}$ is an S^{2}NS–matrix. Since columns p and q of B are neither equal nor opposite, and since u and v are equal or opposite, we have that

$$v = -\frac{b_{rp}}{b_{rq}} u. \tag{7}$$

For $i = 1, 2, \ldots, n$ let F_i be the strict signing whose jth diagonal entry equals b_{ij} ($1 \leq j \leq n$). Since A is an S^{2}NS–matrix, column i is the only unisigned column of $F_i A$, and its nonzero entries are positive. Using (7) it is now easy to verify that column i is the only unisigned column of $F_i \widehat{A}$, and its nonzero entries are positive ($1 \leq i \leq n$). It now follows from $(***)$ that the sign pattern of $\widehat{A}^{-1}$ equals B, and hence that $\widehat{A}$ is fully indecomposable. □

If $X = [x_{ij}]$ is a $(0, 1, -1)$–matrix of order n and $A = [a_{ij}]$ is a matrix of order n, then we say that A *conforms to* X provided sign $a_{ij} = x_{ij}$ for all i and j with $x_{ij} \neq 0$. The next theorem provides necessary and sufficient conditions for the existence of a fully indecomposable S^{2}NS–matrix which conforms to X and whose inverse sign pattern is prescribed. We remark that the assumption in the theorem that B has no zero entries implies by the Frobenius-König theorem that any S^{2}NS–matrix whose inverse has sign pattern equal to B is fully indecomposable and by Lemma 3.1 that the inverse conforms to the transpose X^T of X.

Theorem 3.4 *Let $B = [b_{ij}]$ be a $(1, -1)$–matrix of order $n \geq 2$ and let $X = [x_{ij}]$ be a $(0, 1, -1)$–matrix of order n such that $b_{\ell k} = x_{k\ell}$ for each k and ℓ with $x_{k\ell} \neq 0$. Then*

(a) *If B is the sign pattern of the inverse of an S^{2}NS–matrix A which conforms to X, then there exist integers p, q, r, s such that*

 (i) *the vectors $u^{(p)}$ and $u^{(q)}$ obtained from rows p and q of B by deleting their rth entry satisfy $u^{(p)} = \pm u^{(q)}$,*

 (ii) *the vectors $v^{(r)}$ and $v^{(s)}$ obtained from columns r and s of B by deleting their pth entry satisfy $v^{(r)} = \pm v^{(s)}$,*

 (iii) *$x_{rj} \neq 0$ only if $j = p$ or $j = q$, and*

 (iv) *$x_{ip} \neq 0$ only if $i = r$ or $i = s$.*

(b) *B is the sign pattern of the inverse of an S^2NS–matrix which conforms to X if and only if there exist integers p, q, r, s satisfying (i)–(iv) such that $B(p, r)$ is the sign pattern of the inverse of an S^2NS–matrix whose sign pattern conforms to $\widehat{X}(r, p)$ where $\widehat{X}$ is obtained from X by replacing the entry in position (s, q) with $-a_{rp}a_{sp}a_{rq}$.*

Proof. First suppose that there exists an S^2NS-matrix A whose sign pattern conforms to X and whose inverse has sign pattern B. As remarked above, A is fully indecomposable. Without loss of generality we may assume that A is a maximal S^2NS–matrix. Theorem 2.5 implies that there exist integers p, q, r, s such that the nonzero entries in row p of A occur in columns r and s, and the nonzero entries in column r of A occur in rows p and q. Thus the integers p, q, r, s satisfy (i) and (ii), and by Theorem 3.2 also satisfy (iii) and (iv).

Now suppose that p, q, r, s are integers satisfying (i)–(iv). Lemma 3.3, implies that B is the sign pattern of an S^2NS–matrix if and only if B is a sign pattern of an S^2NS–matrix such that its nonzero entries in row r are in columns p and q, its nonzero entries in column p are in rows r and s, and its (s, q) entry in nonzero. Statement (b) now follows from Theorem 3.2 □

We now give a polynomial–time algorithm for determining whether a $(1, -1)$–matrix is the sign pattern of the inverse of a fully indecomposable S^2NS–matrix. Its validity follows from Theorem 3.4.

Algorithm

Let B be a $(1, -1)$–matrix of order n.

1. Let $E = \emptyset$, $\alpha = \{1, 2, \ldots, n\}$ and $\beta = \{1, 2, \ldots, n\}$.
2. If α contains only one element, then B is the sign pattern of an inverse of an S^2NS–matrix. Otherwise
3. If there exist integers p, q, r, and s with $p, q \in \alpha$ and $r, s \in \beta$ such that
   (i) rows p and q of $B[\alpha, \beta \setminus \{r\}]$ are equal or negatives of one another,
   (ii) columns r and s of $B[\alpha \setminus \{p\}, \beta]$ are equal or negatives of one another,
   (iii) if $(x, r) \in E$, and $x \in \alpha$, then $x = p$ or $x = q$, and if
   (iv) if $(p, y) \in E$, and $y \in \beta$, then $y = r$ or $y = s$,

   then replace E by $E \cup \{(q, s)\}$, α by $\alpha \setminus \{p\}$, and β by $\beta \setminus \{r\}$, and then go to 2. Otherwise,
4. B is not the sign pattern of the inverse of an S^2NS–matrix.

We conclude this section by characterizing the fully indecomposable, maximal S^2NS–matrices that can be uniquely determined by the sign pattern of their inverse.

Theorem 3.5 *Let $A = [a_{ij}]$ be a fully indecomposable, $(0,1,-1)$ maximal S^2NS–matrix of order n. Then A is the unique fully indecomposable, $(0,1,-1)$ maximal S^2NS–matrix whose inverse has the same sign pattern as A^{-1} if and only if no edge of the bipartite graph of A is contained in exactly two cycles of length 4.*

Proof. Let G be the bipartite graph of A. By Theorem 2.8, G is a 4–cockade, and hence G does not contain an even subdivision of $K_{2,3}$. First assume that G contains an edge, say $\{1,1'\}$, which is contained in exactly two cycles of length 4. Since G does not contain an even subdivision of $K_{2,3}$, the two cycles of length 4 which contain $\{1,1'\}$ have exactly one edge in common. Without loss of generality we assume that the cycles are $(1,2',2,1')$ and $(1,3',3,1')$. Let G_1 be the graph obtained from G by removing the edge $\{1,1'\}$, and let G_2 be the graph obtained from G_1 by inserting the edge $(2,3)$. The graph G can be constructed from the edge $\{1,1'\}$ by a sequence of edge-joins with cycles $C_1 = (1,2',2,1')$, $C_2 = (1,3',3,1')$, C_3, $\ldots$, C_{n-1} of length 4. Since G_2 can be constructed from the edge $\{2,3'\}$ by a sequence a edge–joins with the cycles $(2,2',1,3')$, $(2,3',3,1')$, $C_3, \ldots, C_{n-1}$, G_2 is a 4–cockade. It is easy to verify that G_1 is a connected spanning subgraph of both G and G_2, and that each edge of G_1 is contained in a perfect matching. Theorem 2.8 implies that there exists a $(0,1,-1)$ S^2NS–matrix A' whose bipartite graph equals G_2. Since each edge of G_1 is contained in a perfect matching, the matrix obtained from A' by replacing the entry in position $(2,3)$ is a fully indecomposable matrix whose graph equals G_1. Since, up to multiplication of rows and columns by -1, there is a unique (0,1,-1) S^2NS–matrix whose graph equals G_1, we may assume without loss of generality that the corresponding entries of A' and A are equal except for those in their $(1,1,)$– and $(2,3)$–positions. It now follows that A'^{-1} and A^{-1} have the same sign pattern.

Now assume that no edge of G is contained in exactly two cycles of length 4. We use induction on n to show that A is the only fully indecomposable, $(0,1,-1)$ maximal S^2NS–matrix whose inverse has the same sign pattern as A^{-1}. Let $X = [x_{ij}]$ be a $(0,1,-1)$ maximal S^2NS–matrix such that X^{-1} and A^{-1} have the same sign pattern. The assumption on G allows us to assume without loss of generality that for some integer $k \geq 2$,

G is obtained by successively appending the cycles $((n-k),(n-k+i)',(n-k+i),(n-k)')$ for $i=1,\ldots,k$, to a 4-cockade G' with vertices $1,2,\ldots,(n-k),\ 1',2',\ldots(n-k)'$ where $\{(n-k),(n-k)'\}$ is an edge of $\widetilde{G}$ which is contained in exactly one cycle of length 4 of $\widetilde{G}$. Note that $k<n-1$. It follows from Lemma 3.3 that $x_{ii}=a_{ii}$, $x_{n-k,i}=a_{n-k,i}$, and $x_{i,n-k}=a_{i,n-k}$ for $i=n-k+1,n-k+2,\ldots,n$. Let $Y=[y_{ij}]$ be the matrix obtained from X by replacing each entry in row i other than x_{ii} and $x_{i,n-k}$, and each entry in column i other than x_{ii} and $x_{n-k,i}$ by 0 $(i=n-k+1,n-k+2,\ldots,n)$. By repeated applications of Lemma 3.3, Y is a fully indecomposable S^{2}NS–matrix. The matrix Y has the form

$$\left[\begin{array}{cccc|cccc}
 & & & & 0 & 0 & \cdots & 0\\
 & & & & 0 & 0 & \cdots & 0\\
 \multicolumn{4}{c|}{Y[\{1,\ldots,n-k\},\{1,\ldots,n-k\}]} & \vdots & \vdots & \vdots & \vdots\\
 & & & & 0 & 0 & \cdots & 0\\
 & & & & y_{n-k,n-k+1} & y_{n-k,n-k+2} & \cdots & y_{n-k,n}\\
\hline
0 & 0 & \cdots & 0 \quad y_{n-k+1,n-k} & y_{n-k+1,n-k+1} & 0 & \cdots & 0\\
 & & & & 0 & y_{n-k+2,n-k+2} & \cdots & 0\\
\vdots & \vdots & \cdots & \vdots \quad \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & 0 \quad y_{n,n-k} & 0 & 0 & \cdots & y_{n,n}
\end{array}\right].$$

Let H be the bipartite graph of X. Since X is a fully indecomposable, maximal S^{2}NS–matrix, H is a 4–cockade. The full indecomposability of Y implies that bipartite subgraph $\widetilde{H}$ of H induced by the vertices $1,2,\ldots,n-k,\ 1',2',\ldots,(n-k)'$ is connected. Since H does not contain an even subdivision of $K_{2,3}$ and since H_1 is connected, H does not contain any edge of the form $\{p,q'\}$ where $n-k+1\leq p,q\leq n$ and $p\neq q$. The edge $\{n,n'\}$ of H belongs to a cycle (n,n',p,q') of length 4, and by the preceding remark, $p\leq n-k$ and $q\leq n-k$. Suppose that either $p\neq n-k$ or $q\neq n-k$. Then there exists a a path from p to $(n-k)$ in $\widetilde{H}$ which does not contain $(n-k)'$ or there exists a path from p to $(n-k)'$ which does not contain $(n-k)$. In the former case there are three disjoint paths from $(n-k)$ to n in H and in the latter case there are three disjoint paths from $(n-k)'$ to n' in H. In either case, we contradict the fact that H is a 4–cockade. We conclude that $p=q=n-k$. In particular $\{(n-k),(n-k)'\}$ is an edge of H. It can now be verified that since Y is fully indecomposable, the graphs obtained from $\widetilde{H}$ by deleting vertex $(n-k)$ or vertex $(n-k)'$ are both connected. This fact implies that if H contains an edge e of the form $\{p,q'\}$ where $p<n-k$ and $q'\geq n-k+1$, or $p\geq n-k+1$ and $q<n-k$, then H contains an even

subdivision of $K_{2,3}$. Therefore, H does not contain edges of that form, and $X = Y$. It follows that both $\widehat{X} = X[\{1, 2, \ldots, n-k\}, \{1, 2, \ldots, n-k\}]$ and $\widehat{A} = A[\{1, 2\ldots, n-k\}, \{1, 2, \ldots, n-k\}]$ are fully indecomposable maximal, S^2NS–matrices of order $n - k < n$, whose inverses have the same sign pattern. Since $\{1, 1'\}$ is an edge of $\widetilde{G}$ which is contained in a unique cycle of length 4, $\widehat{A}$ satisfies the inductive hypothesis. Therefore, $\widehat{X} = \widehat{A}$, and $X = A$. □

References

[BS] R.A. Brualdi and B.L. Shader, On converting the permanent into the determinant and sign–nonsingular matrices, in *Applied Geometry and Discrete Mathematics* (P. Gritzmann and B. Sturmfels, Eds.), Amer. Math. Soc., Providence, 1991, pp. 117–134.

[BCS] R.A. Brualdi, K.L. Chavey and B.L. Shader, Rectangular *L*–matrices, *Linear Algebra Appls.*, to appear.

[HLM] F. Harary, J.R. Lundgren and J.S. Maybee, On signed digraphs with all cycles negative, *Discrete Appl. Math.*, 12:155–164 (1985).

[KLM] V. Klee, R. Ladner and R. Manber, Signsolvability revisited, *Linear Algebra Appl.*, 159:131–157 (1983)

[L] C.H.C. Little, A characterization of convertible (0, 1)–matrices, *J. Combin. Theory Ser. B.,* 18:187–208 (1975).

[M] R. Manber, Graph–theoretical approach to qualitative solvability of linear systems, *Linear Algebra and its Appls.*, 48:457-470 (1982).

[ST] P. Seymour and C. Thomassen, Characterization of even directed graphs, *J. Combin. Theory Ser. B.*, 42:36-45 (1987).

[Th86] C. Thomassen, Sign–nonsingular matrices and even cycles in directed graphs, *Linear Algebra Appls.*, 75:27–41, (1986).

[TH89] C. Thomassen, When the sign pattern of a square matrix determines uniquely the sign pattern of its inverse, *Linear Algebra and its Appls.*, 119:27-34, (1989).

[TH92] C. Thomassen, The even cycle problem for directed graphs, *J. American Math. Soc.*, to appear.