STABILITY WITH REGIME SWITCHING

by

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ABSTRACT

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The purpose of this paper is to analyze stability of a system of piecewise continuous differential equations, and its application to disequilibrium economic models. A unique solution in the sense of Filippov for such a system is defined and claimed to exist. This problem frequently appears in disequilibrium models, since the so-called "short-side" rule assigns either demand or supply to the transaction amount which is a state variable of an economic system. The concept of Filippov solution makes it possible to analyze a dynamic evolution of such a model. This paper demonstrates that (i) stability conditions for each piecewise system of differential equations are neither necessary nor sufficient for the overall stability with regime switching, except special cases such as a system of linear differential equations in $\mathbb{R}^2$, with two regimes separated by a linear boundary; (ii) several sufficient conditions for an overall stability with many regimes are available, making use of a Lyapunov function common to all regimes; (iii) stability theorems with regime switching are useful for disequilibrium economic models with several regimes.

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I. Introduction

The purpose of this paper is to analyze mathematically a system of piecewise continuous differential equations, and its applications to several economic models. Mathematically the problem is viewed as a special case of differential equations with discontinuous right-hand side or discontinuous vector fields. In mathematical economics, this problem frequently appears in disequilibrium analysis, since the so-called "short-side" rule assigns either demand or supply to the transaction amount which is a state variable of an economic system. The problem of stability with regime switching has been explicitly investigated by Veendorp (1975), Ito (1979), Löfgren (1979), Picard (1979), Aoki (1976: pp. 202 233) and Eckalbar (1979, 1980) in the context of disequilibrium analysis. Their problems turn out to be a special case of a problem studied in dynamic control theory.[1] When a control variable is assumed to have an effect on differential equations, a discontinuous change in the feedback control gives a similar problem to the one in disequilibrium economics. Henry (1972) (1973) and Champsaur et. al. (1977) investigated a system of differential equations with multivalued right-hand side. The Filippov solution we propose in the following is different from their solution concept in that the Filippov solution ignores vectors of direction on an arbitrary set of measure zero near a point of discontinuity. This makes it possible for us to establish, with mild conditions, the uniqueness of a trajectory which Henry and others do not have. We explicitly consider a solution path which travels across the boundary between different regimes. The convergence of such a solution to the unique equilibrium is a main concern of this paper. Henry and others consider a model with a boundary which a solution path cannot cross. Therefore although we share similar solution concepts for a system of differential equations with the discontinuous (or multivalued) right-hand side, purposes of study are different. Eckalbar (1980) considers a model where two regimes in $\mathbb{R}^2$ have different systems of differential equations but they are continuous across the boundary. He showed the stability of each system is sufficient for the overall stability. However, the continuity across the boundary is a very restrictive assumption and often violated in disequilibrium models. Laroque (1979) shows a particular discontinuous model by Veendorp to be geometrically equivalent to a continuous model and then combined it with Eckalbar's result, i.e., the stability for each regime is sufficient for the overall stability. However, the class of models which Laroque's method applies is very limited. A stability investigation for discontinuous systems is acutely needed, since the behavior of such a system is radically different from the continuous models.

Introductory examples in the rest of this section illustrate two results, provided that we have a rule to "switch" regimes at a boundary between them when both vectors near a point on the boundary are directed toward the same one of the two regimes. Two results are: (i) Stable regimes (if they each were defined for an entire domain) may be "patched" into one unstable system, and (ii) it is possible to create an example that two unstable regimes are "patched" into one stable regime. Therefore stability for each regime is neither necessary nor sufficient for the overall stability with regime switching. Section II is devoted to defining a reasonable solution concept for a system of differential equations with a discontinuous right - hand side. We employ the Filippov solution, that is, a class of classical solutions which is robust against small perturbations to the system. The third section includes main results of this paper on the overall stability of discontinuous system: (i) In a case of two regimes in $\mathbb{R}^2$ with a smooth boundary, the stability of each regime is sufficient for the local stability of the overall system with regime switching; and (ii) if a Jacobian matrix of each regime near the equilibrium is quasi-negative definite with a common positive definite matrix, then the equilibrium is locally stable. The latter result holds for a case of many regimes in $\mathbb{R}^n$. The last section of this paper shows that the results in the preceding sections are useful for a couple of economic models. Especially, a result in the third section is shown essential to an idea that Veendorp tried but failed to show in his paper.

Introductory Examples

We will give an example to motivate further details of problems associated with regime switching. Consider the following systems of differential equations.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \tag{*}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \tag{**}$$

where the motion of trajectories (solution paths, integral curves) are illustrated in Figures I.1 and I.2, respectively. Now impose the following "patchwork" on (*) and (**): cut off the fourth quadrant from (*) and "glue" the fourth quadrant of (**) to the rest of (*). In other words, for the first, second and third quadrants, the system of (*) regulates the motion and for the fourth quadrant, (**) regulates the motion. At the boundary, differential equations of the "overall"

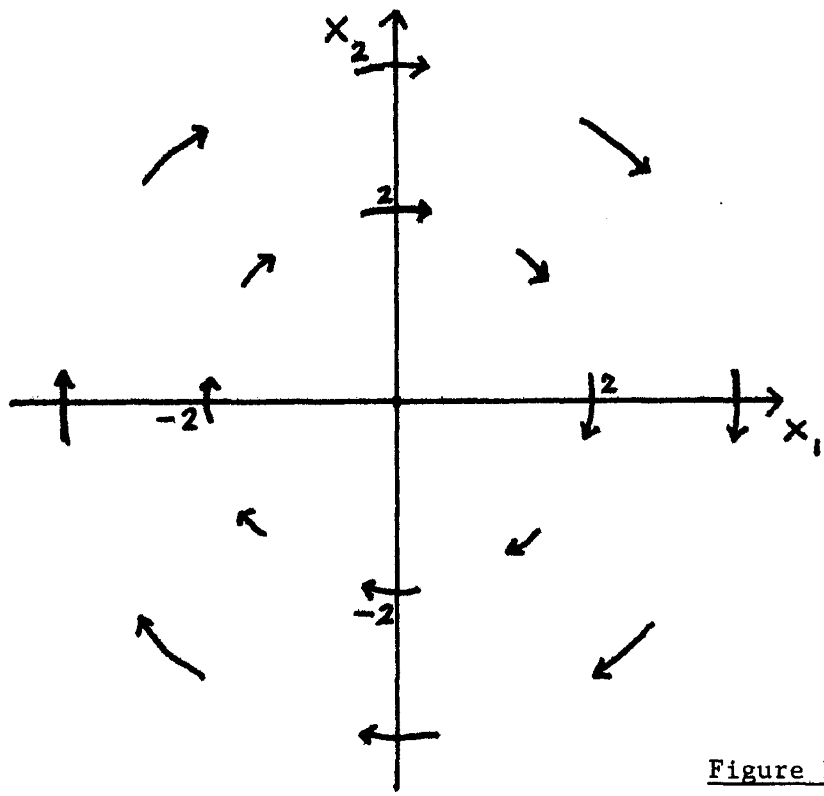


Figure I.1

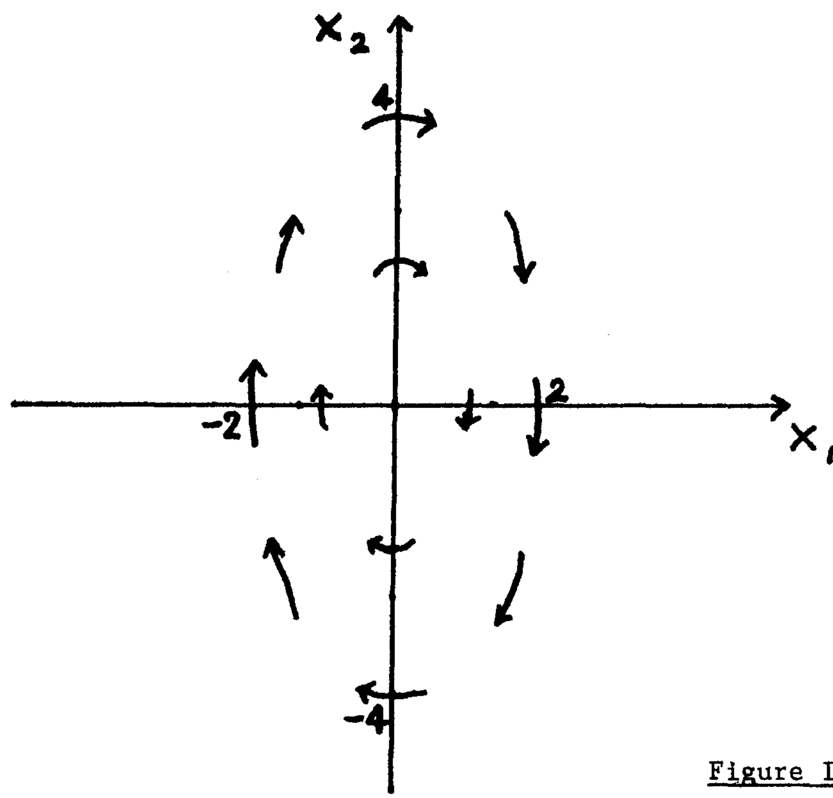


Figure I.2

("glued" or "patched up") system have two values. However, a trajectory can be "connected in a natural manner," which we will explain later. In mathematical terms, the overall system (†) is defined as follows,

$$\dot{x} = \begin{cases} (*), & \text{if } x \in R_* \subset \mathbb{R}^2 \\ (**), & \text{if } x \in R_{**} \subset \mathbb{R}^2 \end{cases} \qquad (\dagger)$$

where

$$R_* \equiv \{(x_1, x_2) \mid x_1 \leq 0 \text{ and/or } x_2 \geq 0\}$$

and

$$R_{**} \equiv \{(x_1, x_2) \mid x_1 \geq 0 \text{ and } x_2 \leq 0\} .$$

The motion of a trajectory starting at (-2, 0) for (†) is illustrated in Figure I.3. Three remarks on existence, uniqueness, and stability of a trajectory of (†) are in order.

Remark 1

At the boundaries, i.e., $\{x_1 > 0, x_2 = 0\}$ and $\{x_1 = 0, x_2 < 0\}$, the right-hand sides of differential equations of (†) are multivalued. For example, at (2, 0),

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ -2 \end{pmatrix} \text{ by } (*) ; \begin{pmatrix} 0 \\ -4 \end{pmatrix} \text{ by } (**) \right\} .$$

Therefore any standard theorems of existence and uniqueness of a solution do not apply to this system at the boundary points. However, we may consider some rules to connect the trajectories at the boundary of two regimes. First we introduce notations. Denote by $\Phi\{t \mid (\bar{x}_1, \bar{x}_2)\}$ the trajectory of the system (*) in R_* with an initial point $(\bar{x}_1, \bar{x}_2) \in R_*$; and by $\Psi\{t \mid (\bar{\bar{x}}_1, \bar{\bar{x}}_2)\}$ the trajectory of the system

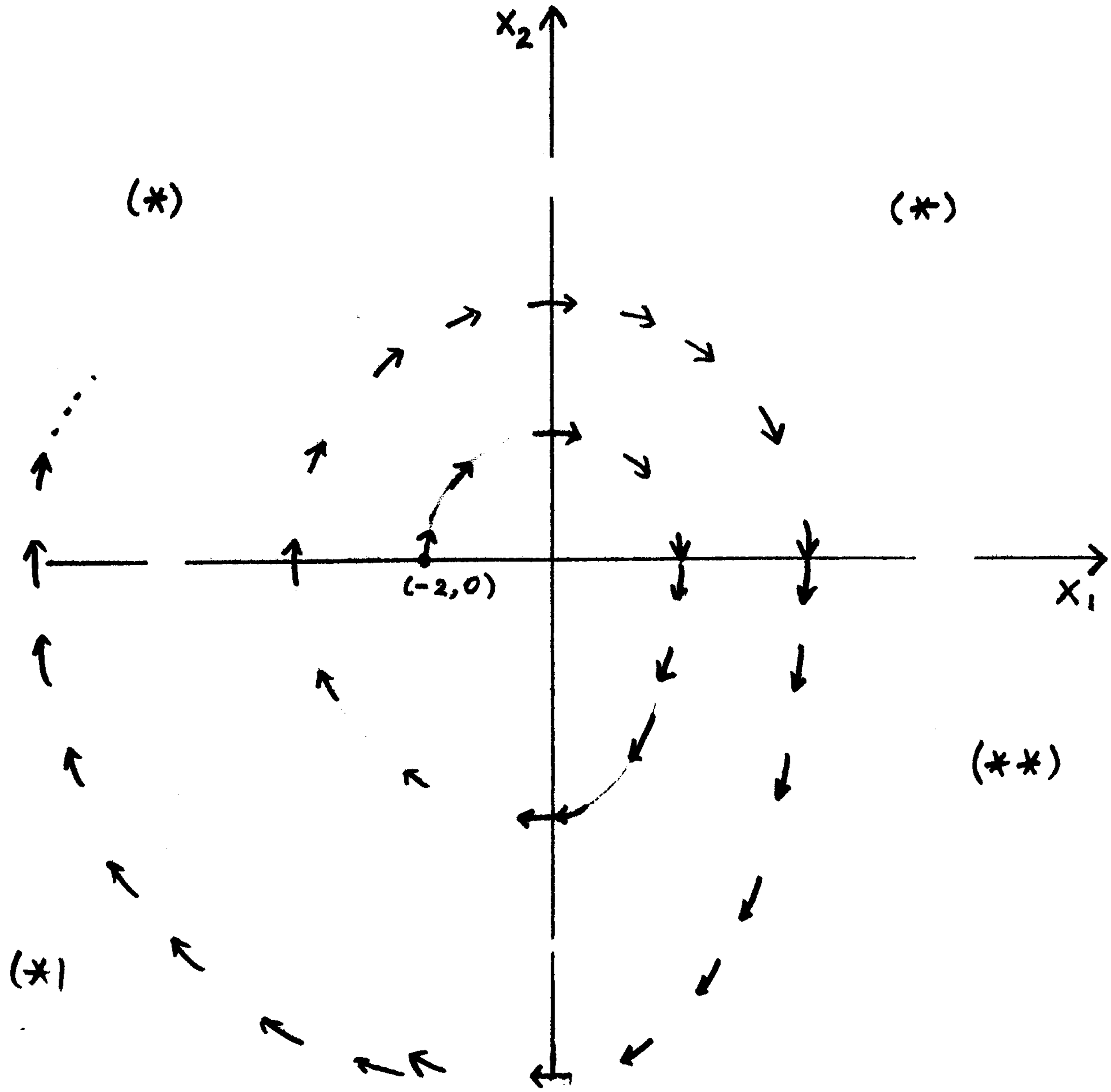


Figure I.3

in R_{**} with an initial point $(\bar{\bar{x}}_1, \bar{\bar{x}}_2) \in R_{**}$. Consider the case $(\bar{x}_1, \bar{x}_2) \in \text{int } R_*$. The trajectory of (†) coincides with the trajectory of (*). At some finite time, the trajectory $\Phi\{t \mid (\bar{x}_1, \bar{x}_2)\}$ hits a point on the horizontal boundary, say, $\Phi\{\hat{t} \mid (\bar{x}_1, \bar{x}_2)\} = (\hat{x}, 0)$. Then "switch" the system from (*) to (**) at time $\hat{t}$, i.e., take $\Psi\{t - \hat{t} \mid (\hat{x}, 0)\}$ as a trajectory of (†). By construction $\Phi\{t \mid (\bar{x}_1, \bar{x}_2)\} = \Psi\{t - \hat{t} \mid (\hat{x}, 0)\} = (\hat{x}, 0)$ at $t = \hat{t}$. Therefore a trajectory of (†) is absolutely continuous. Apply a similar switching procedure when $\Psi\{t - \hat{t} \mid (\hat{x}, 0)\}$ hits the vertical boundary. By solving (**), we know when and where it will happen, namely, $\Psi\{\pi/2 \mid (\hat{x}, 0)\} = (0, -2\hat{x})$. Then an overall (connected) trajectory, $\Gamma\{t \mid (\bar{x}_1, \bar{x}_2)\}$ for the system (†) is defined as follows:

$$\Gamma\{t \mid (\bar{x}_1,\bar{x}_2)\} = \begin{cases} \Phi\{t \mid (\bar{x}_1,\bar{x}_2)\}, & 0 \le t \le \hat{t} \\ \Psi\{t-\hat{t} \mid (\hat{x},0)\}, & \hat{t} \le t \le \hat{t}+\pi/2 \\ \Phi\{t-\hat{t}-\pi/2 \mid (0,-2\hat{x})\}, & \hat{t}+\pi/2 \le t \le \hat{t}+2\pi \\ \vdots & \\ \Psi\{t-\hat{t}-2n\pi \mid (2n\hat{x},0)\}, & \hat{t}+2n\pi \le t \le \hat{t}+2n\pi+\pi/2 \\ \Phi\{t-\hat{t}-2n\pi-\pi/2 \mid (0,2(n+1)\hat{x})\}, & \hat{t}+2n\pi+\pi/2 \le t \le \hat{t}+2(n+1)\pi \\ \quad n = 1, 2, \ldots. & \end{cases}$$

For example, when the initial point, $(\bar{x}_1, \bar{x}_2)$ is given at $(-2, 0)$ as illustrated in Figure I.3, the trajectory is explicitly solved as follows:

$$(*) \begin{cases} x_1 = 2 \quad \sin(t - \pi/2) \\ x_2 = 2 \quad \cos(t - \pi/2) \end{cases} \qquad \text{for } \pi \ge t \ge 0 \quad ,$$

$$(**) \begin{cases} x_1 = 2^{n+1} \quad \sin(t - \pi/2) \\ x_2 = 2^{n+2} \quad \cos(t - \pi/2) \end{cases} \quad ,$$

for

$$\{2n + 3/2\}\pi \geq t \geq \{2n + 1\}\pi \quad ,$$
$$n = 0, 1, 2, \ldots \quad ,$$

and

$$(*) \begin{cases} x_1 = 2^{n+2} \sin (t - \pi/2) \\ x_2 = 2^{n+2} \cos (t - \pi/2) \end{cases} ,$$

for

$$\{2(n + 1) + 1\}\pi \geq t \geq \{2n + 3/2\}\pi \quad ;$$
$$n = 0, 1, 2, \ldots \quad .$$

<u>Remark 2</u>

Note that $\Gamma\{t| \cdot \}$ is continuous and unique, despite the multivaluedness of the differential equations at the boundaries. The multivaluedness does not make any difference on the trajectory since both sets of vectors of differential equations point toward the same regime and the multivaluedness happens at a point of (Lebesgue) measure zero.

<u>Remark 3</u>

Although both original systems, $(*)$ and $(**)$, are stable (not asymptotically but in the Lyapunov sense), the overall system $(\dagger)$ is unstable. It is easy to construct an example of two asymptotically stable systems glued into an unstable overall system. For example, consider $(\dagger)$ with small modification on the coefficient matrices:

$$\left.\begin{array}{l} \begin{pmatrix} -\frac{1}{100\pi} & 1 \\ -1 & -\frac{1}{100\pi} \end{pmatrix} \text{ replacing } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ of } (*) \\ \begin{pmatrix} -\frac{1}{100\pi} & 1 \\ -2 & -\frac{1}{100\pi} \end{pmatrix} \text{ replacing } \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \text{ of } (**) \end{array}\right\} \quad (\dagger\dagger)$$

This example shows that a class C^1 cannot be dispensed with in Olech's theorem. (See Olech (1963) and Ito (1978a) for Olech's theorem.)

Remark 4

By a similar observation, it is obvious that it is possible to have overall stability from patching up two unstable regimes. For example, reverse all the signs of the elements in (††) .

Now we are ready to address the questions we shall investigate in the following sections. First we shall develop a solution concept for a discontinuous system. We have devised a rule to connect trajectories crossing boundaries. However, we might immediately face a problem of connecting trajectories when two trajectories point toward different regimes. In that case we have to define another rule. Then one might wonder whether it is possible to define a generalized solution. What might be an existence condition for a generalized solution of a "patched up" system? Under what circumstances is a solution unique?

Secondly, we shall develop several stability conditions for an overall system. As we saw above, showing the stability for each regime separately is neither necessary nor sufficient for the stability of an overall system.

II. Existence and Uniqueness

Consider an autonomous system of differential equations of the following form.

$$\dot{x} = g(x) \qquad \begin{array}{l} x = (x_1, x_2, \ldots, x_n) \\ g = (g_1, g_2, \ldots, g_n) \end{array} \tag{2.1}$$

where $g(x)$ is a real bounded measurable function, defined almost everywhere in a domain $Q \subset \mathbb{R}^n$.

With $g(x)$ associate the convex set

$$K\{g(x)\} = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{co}\,\{g[B(x, \delta) - N]\}$$

where $\overline{co}$ denotes closed convex hull, $B(x, \delta)$ is a closed δ neighborhood of x, N an arbitrary set in $\mathbb{R}^n$ and μ is n dimensional Lebesgue measure. In very intuitive terms, $K\{g(x)\}$ is an "average" set of vector directions g which are collected near x but excluding points in a set of measure zero.[2] To put it another way, take limits of vector directions as you shrink a diameter of a neighborhood ball to zero; however ignore a set of peculiar vector directions which are defined only on a set of measure zero. A set $K\{\cdot\}$ is the convex hull of such limits.

<u>Definition II.1 [Filippov solution]</u>

<u>An absolutely continuous vector valued function</u> $\varphi(t)$ <u>defined on</u> $[0, T]$, <u>is called a solution in the sense of Filippov of</u> $\dot{x} = g(x)$ <u>if for almost all</u> t,

$$\dot{\varphi}(t) \in K\{g(\varphi(t))\},$$

<u>and if it satisfies the initial condition</u> $\varphi(0) = x_0$.

Filippov (1960) showed such a solution always exists. Moreover, if g is continuous, then $K\{g(x)\} = g(x)$. Examples shall serve for illustrating the force of this definition. The first example shows how to determine the direction and velocity of a solution when vectors are directed toward the boundary from both regimes. The second example examines the possibilities of additional rest points and nonunique solution, which will later lead to the uniqueness conditions. The third example explains why we want to eliminate vectors on a set of measure zero in defining the Filippov solution.

Example II.1 [Filippov (1960)]

Every solution of the system

$$\begin{cases} \dot{x}_1 = 4 + 2\,\mathrm{sgn}\,x_2 \\ \dot{x}_2 = 2 - 4\,\mathrm{sgn}\,x_2 \end{cases}$$

hits the horizontal axis sooner or later and then cannot leave it. The closed convex hull $K\{\cdot\}$ in the definition of a Filippov solution to which the vector direction must belong is defined as,

$$K\{g(x)\} = \overline{co}\left\{\begin{pmatrix}2\\6\end{pmatrix}, \begin{pmatrix}6\\-2\end{pmatrix}\right\}, \qquad \text{for} \quad x = (x_1, 0) \quad .$$

At the same time the vector direction, $\dot{\varphi}$, of a solution must lie on the x_1 - axis, since the solution cannot depart from it either above or below. The first requirement is

$$\dot{\varphi}(\cdot) \in K\{\cdot\} = \left\{\begin{pmatrix}a\\10 - 2a\end{pmatrix}\right\} \qquad 2 \leq a \leq 6 \quad .$$

The second requirement says $10 - 2a = 0$. Thus $\dot{\varphi}(x) = \begin{pmatrix}5\\0\end{pmatrix}$ for $x = (x_1, 0)$. Thus a Filippov solution is uniquely defined for the system of this example.[3]

Since vector directions on any set of measure zero cannot be counted in construction of the Filippov solution, the vector directions on a boundary between regimes are irrelevant. This observation immediately leads us to a conjecture that an overall system which is composed of several regimes separated by boundaries of measure zero has a unique solution, given that each regime consists of a system of continuous Lipschitzian differential equations. This conjecture holds true (i) if a boundary is twofold smooth and (ii) if both vectors are not directed away from the boundary or if they are nor parallel to the tangent hyperplane at any point on the boundary. In order to see the latter requirement we will give another example, and proceed to state a formal uniqueness theorem.

Example II.2

Consider our introductory example again, with a modification that the signs of the coefficient matrix of (**) are reversed, i.e.,

$$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \quad \text{instead of} \quad \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} .$$

An overall system is defined in a similar manner to obtain trajectories illustrated in Figure II.1. At a point on the lower vertical axis the vectors point toward the interior of both regimens. Therefore Filippov solutions exist but not uniquely determined. That is

$$\dot{\phi}(x) \in K\{g(x)\} = \overline{co}\left\{\begin{pmatrix} -b \\ 0 \end{pmatrix}\begin{pmatrix} b \\ 0 \end{pmatrix}\right\} \qquad \text{at} \quad x = (0, -b), \; b > 0 .$$

Then the axis becomes a "knife-edge." Both a solution with $\dot{\phi} > 0$ which leads a solution in the fourth quadrant and one with $\dot{\phi} < 0$ which leads to the third quadrant are Filippov solutions. Therefore a solution is not uniquely determined. Now, sooner or later a solution path comes to a point on the right horizontal axis, where

$$\dot{\phi}(x) \in K\{g(x)\} = \overline{co}\left\{\begin{pmatrix} 0 \\ -c \end{pmatrix}\begin{pmatrix} 0 \\ 2c \end{pmatrix}\right\} \qquad \text{at} \quad x = (c, 0), \; c > 0$$

At this point, a Filippov solution comes to rest, since it does not have any momentum in the horizontal direction, while the vertical forces "counter-balance" each other. Therefore the right half of the horizontal axis becomes additional equilibrium points.

Example II.3

One might wonder whether it is possible to define the third set of differential equations on the boundary, leaving two regimes as open sets. Suppose we define,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{if } \begin{cases} x_1 \geq 0 \\ x_2 = 0 \end{cases}$$

in addition to

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{array}{l} \text{if } \; x_2 > 0 \\ \text{or if } \; (x_1 \leq 0 \text{, and } x_2 \leq 0) \end{array} \quad .$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{array}{l} \text{if } \; x_1 > 0 \\ \text{and } \; x_2 < 0 \end{array} \quad .$$

This is the case where horizontal vectors toward the origin are added on the right half of the horizontal axis on Figure II.1. It seems, according to the "natural connection" explained in section I, that any connected solution path is asymptotically stable in the sense it converges to the origin sooner or later. However, the Filippov solution ignores the force of vectors defined only on a set of measure zero, because of the way $K\{\cdot\}$ is defined. Therefore the Filippov solution of this example is the same with the Filippov solution of Example II.2. This argument leads us to an observation that it does not matter for a Filippov solution whether a regime is open or closed along the boundary, or whether there is the third set of differential equations defined on the boundary. If we are interested in a solution concept which is robust against a little perturbation to a system, the Filippov solution is a much more appealing concept. Although we do not

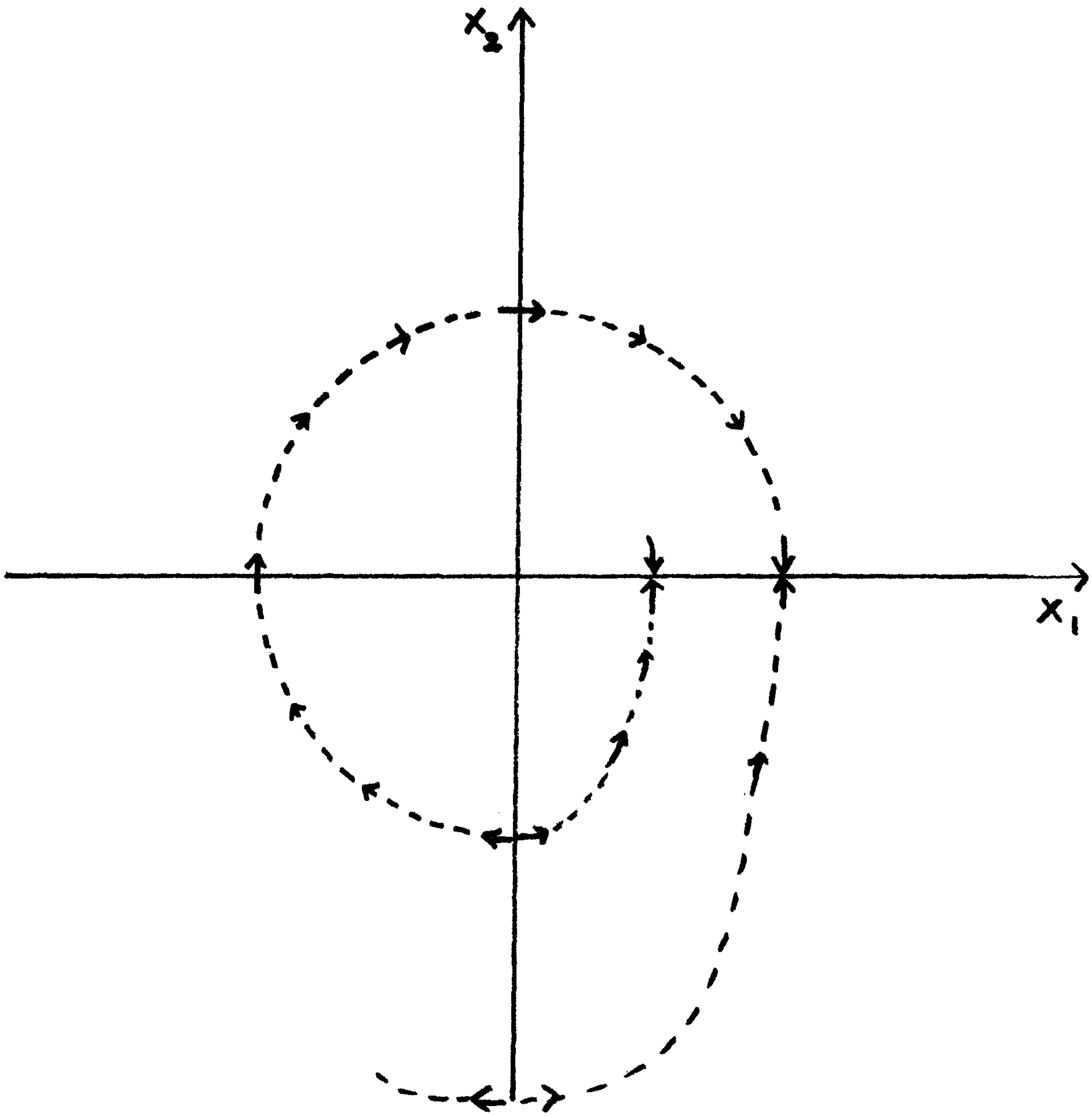


Figure II.1

explicitly introduce constant perturbation to a system, or measurement error in observing a system under control, those factors are the justification for employing the Filippov solution. This idea is formalized as a notion of "stability with respect to measurement." (See Brunovský (1965) (1976), Brunovský and Mirică (1975), and Hermes (1967).) Hermes (1967: Theorem 1) showed that if a system is "stable with respect to measurement," then every classical solution is a Filippov solution, where a classical (or Carathéodory) solution means that it satisfies the initial condition; is absolutely continuous with respect to time; and its time derivative coincides with the vector direction of differential equations at that point almost everywhere. Hermes' theorem implies that a Filippov solution is a class of classical solutions which is robust against small perturbations to the system. Note that Henry (1972) (1973) employs a concept of classical solutions, but not necessarily a Filippov solution. Once the Filippov solution is employed in a system of piecewise continuous differential equations, then the "discontinuity" and "multivaluedness" at the boundary (or "switching space" by W. Hahn (1967, page 371)) become the same phenomenon, since it does not matter for the Filippov solution how the right-hand side is "defined" on the boundary of measure zero.

Now we consider system (2.1) in the beginning of this section with additional assumptions of a smooth boundary and piecewise continuity of differential equations.

Assumption U

(i) Let a bounded region $Q \subset \mathbb{R}^n$ be divided by a twofold smooth surface S into the domains Q^- and Q^+.

$$S = \{x \mid s(x) = 0\}$$
$$Q^+ = \{x \mid s(x) > 0\}$$
$$Q^- = \{x \mid s(x) < 0\}$$

where $s(x) = 0$ can be solved for one of the coordinates,

$$x_i = \eta(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

where the function η has continuous second derivatives.

(ii) g_i is continuously differentiable in x in Q^+ and Q^-, and let there exist bounded limiting values of the functions g under approach to an arbitrary point x of the surface S from Q^+ and Q^-. Denote them by $g^+(x)$ and $g^-(x)$, respectively.

(iii) At least one of the following inequalities is satisfied:

$$\text{(A)} \quad \langle \frac{\partial s}{\partial x}(x), \; g^+(x) \rangle < 0$$

$$\text{(B)} \quad \langle \frac{\partial s}{\partial x}(x), \; g^-(x) \rangle > 0$$

where $\langle \cdot \rangle$ represents an inner product of vectors.

A couple of remarks on Assumption U-(iii) are in order. If both $\langle \cdot , g^{+}(x)\rangle$ and $\langle \cdot , g^{-}(x)\rangle$ are negative (resp., positive) then both vectors, g^{+} and g^{-}, are directed toward the Q^{-} (resp., Q^{+}) regime. Therefore we expect a solution path to cross over the boundary (i.e., the surface S). If both (A) and (B) are satisfied then this trajectory is called "sliding" or "chattering" trajectory and the Filippov solution is continued in a completely determined way along the boundary. This was the case in an example II.1.[4] The possibility of multiple solutions on the vertical axis and additional rest points on the horizontal axis in example II.2 is eliminated by Assumption U-(iii).

Theorem II.1 [Uniqueness]

If Assumptions U: (i) - (iii) are satisfied for system (2.1) in the domain Q , *then we have a unique Filippov solution path of system (2.1) given an arbitrary initial point in the domain* Q . *Moreover, the solution continuously depends on the initial point.*

See Filippov (1960: Theorem 14) for the rigorous proof. Heuristically speaking, the solution is unique unless Assumption U-(iii) is violated, i.e., a vector in the neighborhood of a point on the boundary is directed toward its own regime leaving the boundary as a "knife-edge," or vectors near the boundary are parallel to the boundary in the contradicting directions according to different regimes. In the following, we focus on a system of piecewise continuous differential equations satisfying assumption U, so that there exists a unique Filippov solution. We call it simply "a trajectory."

III. Stability

III.1 Definition

In this section, we study the stability property of a system of piecewise continuous differential equations. Suppose that such a system has a unique equilibrium at the origin $\{0\}$. Denote by $\Gamma(t|x_o)$ a trajectory starting at x_o at $t = 0$. By stability, we mean the asymptotic stability with respect to displacement of an initial point from the equilibrium, which is common terminology in mathematical economics.[5]

Definition III.1 [Global asymptotic stability][6]

A system of piecewise continuous differential equations, defined in $Q \subset \mathbb{R}^n$ with only one equilibrium at $\{0\}$, is said to be globally asymptotically stable, if there exists a Filippov solution $\Gamma(t|x_o)$ initiating at x_o

$$\lim_{t\to\infty} \Gamma(t|x_o) = \{0\},$$

for all $x_o \in Q$.

In this paper, "overall stability" is used for asymptotic stability defined above. A regime is said to be stable when the equilibrium is asymptotically stable if a system of differential equations corresponding to the regime was defined for an entire Q as well as that regime. (Recall Remarks 3 and 4 in Introduction.)

Definition III.2 [Local asymptotic stability]

An equilibrium of a system of piecewise continuous differential equations is said to be locally asymptotically stable, if exists a neighborhood of the equilibrium, N, such that a Filippov solution $\Gamma(t|x_o)$ initiating at $x_o \in N$ exists and

$$\lim_{t \to \infty} \Gamma(t | x_o \in N) = \{0\} .$$

<u>for all $x_o \in N$.</u>

When a regime is said to be locally stable, it means that an equilibrium is locally stable if a system of differential equations corresponding to the regime was defined for an entire N as well as that regime. The "overall stability" as defined in theorems and the "regime stability" should be carefully differentiated.

If one can calculate a trajectory directly as demonstrated in the introductory example, then we can trace a Filippov solution of a system to check stability. Calculating a trajectory may be a feasible way only when a system is given numerically. Even so, it may not be easy. The purpose of this paper is to develop a simpler condition for the overall stability in relation with conditions for stable regimes.

As we showed in the introductory examples, it is possible that two stable regimes are glued together into an unstable system. Therefore, in general it is not enough for the stability of an overall system to establish the stability for each regime separately. Suppose that each regime is stable. What kind of additional assumptions do we need for the overall stability? The possibilities that two stable regimes are patched into an unstable system arise when (i) a solution path crosses boundaries at a series of points which become farther and farther from an equilibrium (the origin); or (ii) a solution path "slides" along a boundary in the direction divergent from the origin. (Recall example II.1). Thus we look for additional conditions to eliminate these possibilities.

III.2 Two Regimes in $\mathbb{R}^2$

Let us start from the simplest case: two stable regimes in $\mathbb{R}^2$ which share a linear boundary. Suppose that two sets of linear differential equations are defined on $\mathbb{R}^2$, assuming that the origin is the equilibrium point without loss of generality:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad (x,\ y) \in R_I \subset \mathbb{R}^2 \tag{3.1}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad (x,\ y) \in R_{II} \subset \mathbb{R}^2\ , \tag{3.2}$$

where $R_I \cap R_{II} = \emptyset$ and $R_I \cup R_{II} = \mathbb{R}^2$.

Conditions of stability for each regime are obtained by the Routh-Hurwitz condition.[7]

$$a_i + d_i < 0\ , \qquad i = 1,\ 2\ ,$$

$$a_i d_i - b_i c_i > 0\ , \quad i = 1,\ 2\ .$$

A "patched-up" system with a linear boundary is defined by the following:

$$\begin{aligned} &\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \quad \begin{matrix} i = 1 \text{ if } (x,\ y) \in R_I \\ i = 2 \text{ if } (x,\ y) \in R_{II}\end{matrix} \\ &R_I = \{(x,\ y) \in \mathbb{R}^2 \mid hx + ky \geq 0\} \\ &R_{II} = \{(x,\ y) \in \mathbb{R}^2 \mid hx + ky < 0\}\ . \end{aligned} \tag{3.3}$$

As we shall show in the following, the linear boundary eliminates a possibility of switching regimes farther and farther from an equilibrium point. First, we give a condition of regime switching, i.e., every time a solution path hits the boundary, a trajectory at a point on the boundary is directed into one of the regimes.

In mathematical notation

$$\text{(E)} \qquad (h\ k)\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} k \\ -h \end{pmatrix} \cdot (h\ k)\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} k \\ -h \end{pmatrix} > 0 \quad .$$

Since the boundary is linear through the origin, a case of sliding trajectory on a half line of the boundary implies that the uniqueness condition is violated on the other half line. This can be shown easily. A set of the boundary is

$$(x_1, x_2) = \lambda(k, -h), \quad -\infty < \lambda < \infty \quad .$$

Suppose for $(x_1, x_2) = (\bar{\lambda} k, -\bar{\lambda} h)$, it is a case of a sliding trajectory: that is,

$$\langle \frac{\partial s}{\partial x}(x), g^+(x) \rangle = (h\,k)\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} \bar{\lambda} k \\ -\bar{\lambda} h \end{pmatrix} < 0 \quad ,$$

and

$$\langle \frac{\partial s}{\partial x}(x), g^-(x) \rangle = (h\,k)\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} \bar{\lambda} k \\ -\bar{\lambda} h \end{pmatrix} > 0 \quad ,$$

then at $(x_1, x_2) = (-\bar{\lambda} k, \bar{\lambda} h)$,

$$\langle \frac{\partial s}{\partial x}(x), g^+(x) \rangle > 0$$

$$\langle \frac{\partial s}{\partial x}(x), g^-(x) \rangle < 0 \quad .$$

It violates Assumption U - (iii).

Now we are ready to state a theorem.

Theorem III.1 [Ito (1979) and Picard (1979)][8]

Suppose a piecewise linear differential equation system defined by (3.3) satisfies condition (E). If each regime is stable, i.e.,

$$a_i + d_i < 0 ,$$

$$a_i d_i - b_i c_i > 0 , \qquad i = 1, 2 ,$$

then the "patched-up" system with a linear boundary, (3.3), has a unique stable solution path, i.e.,

$$\lim_{t \to \infty} x(t | (x_o, y_o)) = 0$$

$$\lim_{t \to \infty} y(t | (x_o, y_o)) = 0 \qquad \underline{\text{for}} \quad (x_o, y_o) \in \mathbb{R}^2 \quad .$$

Existence and uniqueness are guaranteed by the assumptions of boundary with measure zero and of condition (E). In a case of linear differential equations defined on the two-dimensional space with a linear boundary, a stable regime implies that a trajectory starting at a boundary toward its interior must either converge to the equilibrium point without regime switching or switch regime at a point on a portion of the boundary with opposite signs but closer to the origin in Euclidean distance. See Ito (1979: Appendix III) and Picard (1979: Appendix 3) for a rigorous proof. Therefore regime switching does not upset the stability.

Next we consider a system of nonlinear piecewise differential equations with two regimes in $\mathbb{R}^2$. The local stability will be shown if each Jacobian matrix satisfies the Routh-Hurwitz condition. First, let us define such a nonlinear system.

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\phi^i(x, y)\\ \psi^i(x, y)\end{pmatrix} \qquad \begin{matrix} i = 1 & \text{if} & (x, y) \in R_I \\ i - 2 & \text{if} & (x, y) \in R_{II}\end{matrix} \tag{3.4}$$

where

$$R_I = \{(x, y) \in \mathbb{R}^2 \mid h(x, y) \geq 0\}$$
$$R_{II} = \{(x, y) \in \mathbb{R}^2 \mid h(x, y) < 0\} \quad ,$$

and

$$\phi^i(0, 0) = \psi^i(0, 0) = 0 , \qquad h(0, 0) = 0 \quad .$$

Functions ϕ^i and ψ^i are assumed to be continuously differentiable in a neighborhood of the equilibrium, i.e., the origin, except on the boundary. The boundary is assumed to be two-fold smooth. (Recall Assumption U in Section II.)

Taking Taylor expansions about the origin, (3.4) is rewritten as [9]

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}f^i(x, y)\\ g^i(x, y)\end{pmatrix} \qquad \begin{matrix} i = 1, & \text{if} & (x, y) \in R_I \\ i = 2, & \text{if} & (x, y) \in R_{II}\end{matrix} \tag{3.5}$$

where

$$\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix} = \begin{pmatrix}\phi^i_x\ (0, 0) & \phi^i_y\ (0, 0)\\ \psi^i_x\ (0, 0) & \psi^i_y\ (0, 0)\end{pmatrix}$$

and $\|f^i(x, y)\|/\|(x, y)\| \to 0$ as $\|(x, y)\| \to 0$ and $\|g^i(x, y)\|/\|(x, y)\| \to 0$ as $\|(x, y)\| \to 0$. As in the linear case, the existence of a unique Filippov solution should be assumed. In other words, the following condition should be satisfied,

$$\text{(E2)} \qquad (h_x\ h_y)\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \cdot (h_x\ h_y)\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} > 0 \quad ,$$

where all the partial derivatives are evaluated at the origin.

If a trajectory is not a spiral, that is, if a trajectory never comes back to the same regime repeatedly, then the usual linearization and local stability result can be applied.[10] Therefore, in the following, we concentrate on a case of two spiral regimes, or regime switching. We are going to show that if each linearized system is stable, then the overall system is stable.

First, we prove the stability theorem for the case that the boundary, $h(x, y) = 0$, is linear in x and y. Then we show that the nonlinearity of h does not upset the local stability.

Assume that the set of boundary, L, is a line through the origin:

$$L = \{(x, y) | hx + ky = 0\} \quad , \tag{3.6}$$

where h and k are scalars. Let us assume the initial point (x_0, y_0) on the boundary without loss of generality. Then condition (E2) guarantees that the unique Filippov solution $\Gamma(t|(x_0, y_0))$ of system (3.5) is directed into one of the two regimes, say, the I-st regime, R_I. Denote by $\tilde{\Gamma}(t|(x_0, y_0))$ the trajectory of <u>linearized</u> system in regime I, i.e., equation (3.5) with $i = 1$ and without $f^1(\cdot)$ and $g^1(\cdot)$. Since $f^1(\cdot)$ and $g^1(\cdot)$ are at least of the second order, $\tilde{\Gamma}(t|(x_0, y_0))$ is also directed into R_I. Within a regime, the linearized and non-linear systems share a common characteristic of the equilibrium. Therefore for (x_0, y_0) sufficiently small, if $\tilde{\Gamma}(\cdot)$, switch regimes at a point on a part of the boundary opposite of the equilibrium, then so does $\Gamma(\cdot)$. Denote by t_1 and t_2 time that $\Gamma(t|\cdot)$ and $\tilde{\Gamma}(t|\cdot)$ hit the boundary after traveling in R_I. Mathematically, for $(x_0, y_0) \in L$

$$\Gamma(t|\cdot) \in R_I \quad \text{for} \quad 0 < t < t_1 \quad \text{and} \quad \Gamma(t_1|\cdot) \in L$$

and

$$\tilde{\Gamma}(t|\cdot) \in R_I \quad \text{for} \quad 0 < t < t_2 \quad \text{and} \quad \tilde{\Gamma}(t_2|\cdot) \in L \quad .$$

Figure III.1 illustrates the above definitions for the case $t_2 < t_1$. Points A, B, C, D and E denote (x_o, y_o), $\tilde{\Gamma}(t_2|\cdot)$, $\tilde{\Gamma}(t_1|\cdot)$, $\Gamma(t_1|\cdot)$ and $(0, 0)$. For the case $t_2 > t_1$, point C lies in the interior of R_I. However, all the following arguments also apply. We will show that for the initial point sufficiently close to the equilibrium, the distance from the origin to the switching point is smaller than the distance to the initial point. Then a parallel argument holds for the trajectory in the other regime starting from $\Gamma(t_1|\cdot)$. Mathematically, it is sufficient for the local stability to show that for some δ, $\|(x_o, y_o)\| < \delta$,

$$\|\Gamma(t_1|\cdot)\| / \|(x_o, y_o)\| < 1 \quad . \tag{3.7}$$

In order to prove (3.7), we use the following results:

(i) $\|\tilde{\Gamma}(t_2|\cdot)\| / \|(x_o, y_o)\| = \theta < 1$, where θ is a constant. This is a result in Theorem III.1, the stability of regime switching with a linear boundary.

(ii) As (x_o, y_o) approaches, a deviation in timings that Γ and $\tilde{\Gamma}$ hit the boundary becomes very small.

The second claim is formally proved as follows.

<u>Lemma: For a given $\varepsilon > 0$, there exists δ such that for all (x_o, y_o), $\|(x_o, y_o)\| < \delta$, $|t_2 - t_1| < \varepsilon$.</u>

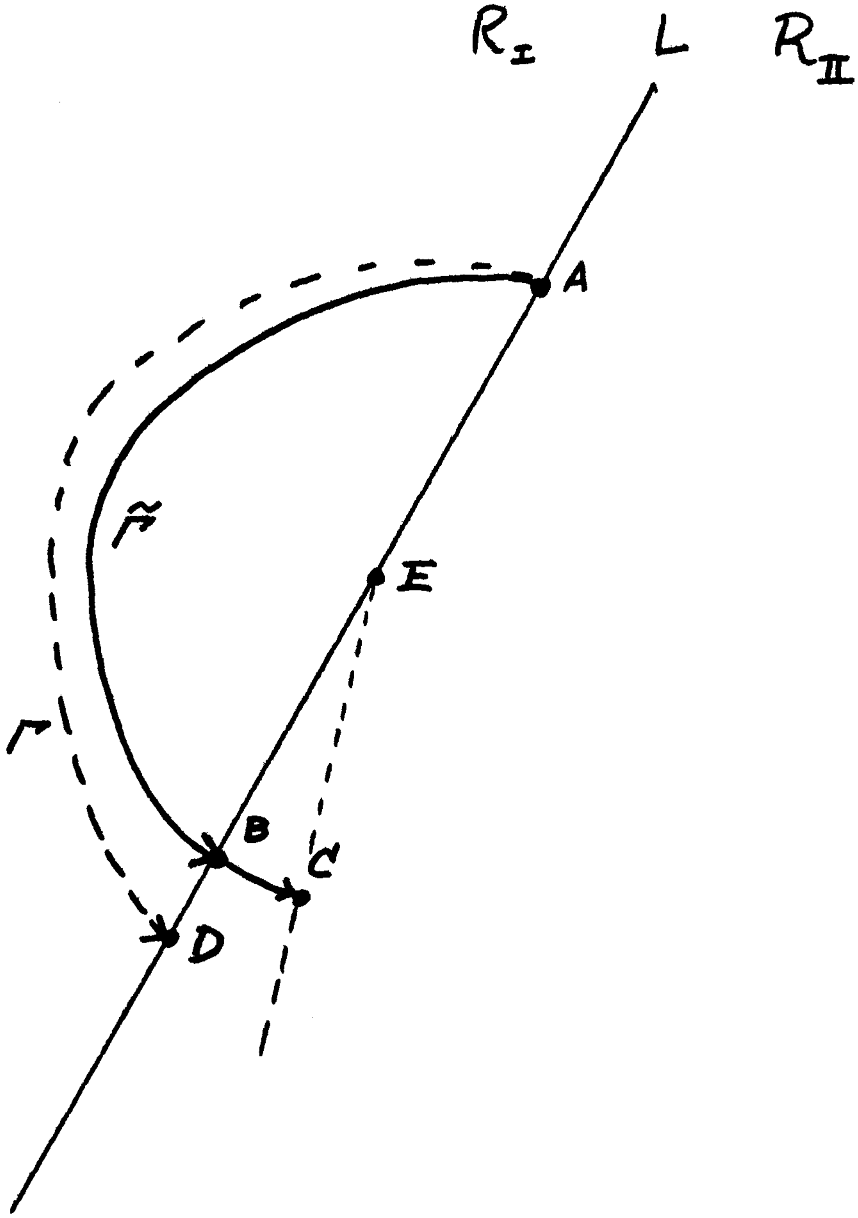


Figure III.1

Legend

A = (x_o, y_o)
B = $\tilde{\Gamma}(t_2 \mid A)$
C = $\tilde{\Gamma}(t_1 \mid A)$
D = $\Gamma(t_1 \mid A)$
E = $(0, 0)$

<u>Proof</u>: Suppose the contrary and $t_2 < t_1$ without loss of generality. That is, there exists $\varepsilon > 0$, such that $|t_2 - t_1| \geq \varepsilon$ for all $(x_o, y_o) \in L \setminus \{0\}$. Then there exists a nondegenerate angle between L and a line connecting $\tilde{\Gamma}(t_2 + \varepsilon | \cdot)$ and $\{0\}$. Therefore if we project $\tilde{\Gamma}(t_2 + \varepsilon | \cdot)$ on L, then the length is strictly positive, for all $(x_o, y_o) \in L \setminus \{0\}$.

$$\|\tilde{\Gamma}(t_2 + \varepsilon | \cdot) - P[(\tilde{\Gamma}(t_2 + \varepsilon | \cdot), L]/\|(x_o, y_o)\| = \delta > 0 \quad \text{for all } (x_o, y_o) \in L \setminus \{0\}$$

where δ is a constant, independent of (x_o, y_o) so long as $(x_o, y_o) \in L$, and $P[\cdot, L]$ is a projection of $\tilde{\Gamma}(t_1 + \varepsilon | \cdot)$ on a line, L. Considering both $P[\cdot, L]$ and $\tilde{\Gamma}(t_2 | \cdot)$ belong to L, the following inequality holds because of the definition of projection.

$$\|\Gamma(t_1 | \cdot) - \tilde{\Gamma}(t_1 | \cdot)\| \geq \|P[\tilde{\Gamma}(t_1 | \cdot), L] - \tilde{\Gamma}(t_1 | \cdot)\|$$

$$\geq \|P[\tilde{\Gamma}(t_2 + \varepsilon | \cdot), L] - \tilde{\Gamma}(t_2 + \varepsilon | \cdot)\| .$$

Therefore

$$\lim_{\substack{(x_o, y_o) \to \{0\} \\ (x_o, y_o) \in L}} \frac{\|\Gamma(t_1 | \cdot) - \tilde{\Gamma}(t_1 | \cdot)\|}{\|(x_o, y_o)\|} \geq \frac{\|P[\tilde{\Gamma}(t_1 | \cdot), L) - \tilde{\Gamma}(t_1 | \cdot)\|}{\|(x_o, y_o)\|} > \delta > 0 .$$

Since $\|\tilde{\Gamma}(t_1 | \cdot)\| / \|(x_o, y_o)\|$ stays constant as $\|(x_o, y_o)\| \to 0$, the above inequality contradicts the fact that (f^i, g^i) which is equal to $(\Gamma(t_1 | \cdot) - \tilde{\Gamma}(t_1 | \cdot))$ is at least of second order.

Q.E.D.

Now we are ready to state a theorem and prove it.

<u>Theorem III.2:</u>[11]

<u>Suppose a piecewise nonlinear differential equation defined by (3.4) or equivalently (3.5) satisfying condition (E2) with the linear boundary (3.6). If each regime is stable</u>, i.e.,

$$a_i + d_i < 0, \qquad i = 1, 2$$

<u>and</u>

$$a_i d_i - b_i c_i > 0, \qquad i = 1, 2$$

<u>then the origin is locally asymptotically stable</u>.

<u>Proof</u>: In order to prove (3.7) which implies the result of this theorem, it is sufficient to show $\| \Gamma(t_1|\cdot) - \tilde{\Gamma}(t_2|\cdot)\| / \|(x_o, y_o)\| \to \bar{\theta}$, $\bar{\theta} < 1 - \theta$, as $(x_o, y_o) \to \{0\}$, $(x_o, y_o) \in L$. A scalar θ is defined as $\|\tilde{\Gamma}(t_2|\cdot)\| / \|(x_o, y_o)\|$. This can be seen as follows:

$$\text{L.H.S.}(3.7) = \frac{\|\Gamma(t_1|\cdot)\|}{\|(x_o,y_o)\|} = \frac{\|\Gamma(t_1|\cdot) - \tilde{\Gamma}(t_2|\cdot) + \tilde{\Gamma}(t_2|\cdot)\|}{\|(x_o,y_o)\|}$$

$$\le \frac{\|\Gamma(t_1|\cdot) - \tilde{\Gamma}(t_2|\cdot)\|}{\|(x_o,y_o)\|} + \frac{\|\tilde{\Gamma}(t_2|\cdot)\|}{\|(x_o,y_o)\|}$$

$$\le \frac{\|\Gamma(t_1|\cdot) - \tilde{\Gamma}(t_2|\cdot)\|}{\|(x_o,y_o)\|} + \theta .$$

Arranging the first term

$$\frac{\|\Gamma(t_1|\cdot) - \tilde{\Gamma}(t_2|\cdot)\|}{\|(x_o, y_o)\|} = \frac{\|\Gamma(t_1|\cdot) - \tilde{\Gamma}(t_1|\cdot) + \tilde{\Gamma}(t_1|\cdot) - \tilde{\Gamma}(t_2|\cdot)\|}{\|\tilde{\Gamma}(t_2|\cdot)\|} \cdot \frac{\|\tilde{\Gamma}(t_2|\cdot)\|}{\|(x_o,y_o)\|}$$

$$\le \left\{ \frac{\|\Gamma(t_1|\cdot) - \tilde{\Gamma}(t_1|\cdot)\|}{\|\tilde{\Gamma}(t_1|\cdot)\|} \; \frac{\|\tilde{\Gamma}(t_1|\cdot)\|}{\|\tilde{\Gamma}(t_2|\cdot)\|} + \frac{\|\tilde{\Gamma}(t_1|\cdot) - \tilde{\Gamma}(t_2|\cdot)\|}{\|\tilde{\Gamma}(t_2|\cdot)\|} \right\} \theta .$$

The first term in the brace goes to zero as $\|(x_o, y_o)\| \to 0$, because of the definition of the second order term. Therefore there exists some δ_1, such that for $\|(x_o, y_o)\| < \delta_1$, the first term is less than $(1 - \theta)/2\theta$ Since the second term in the brace is defined only by the linearized system, its value depends only on $|t_1 - t_2|$ From the lemma above, we know this value can be as small as we desire. Choose $|t_1 - t_2|$, such that $\|\tilde{\Gamma}(t_1|\cdot) - \tilde{\Gamma}(t_2|\cdot)\| / \|\tilde{\Gamma}(t_2|\cdot)\| < (1 - \theta)/2\theta$. Denote by δ_2 the distance of (x_o, y_o) determined by such $|t_1 - t_2|$. For all (x_o, y_o) such that $(x_o, y_o) < \min(\delta_1, \delta_2)$, L.H.S.(3.7) < 1 .

Q.E.D.

In the case of a nonlinear but smooth boundary, the argument proceeds in a parallel manner. That is, given the fact that a trajectory of the linearized system hit a switching point on the linearized boundary which is closer to the origin than the initial point is, the deviation of nonlinearity does not upset the result for the initial point sufficiently close to the origin. The nonlinear boundary, $H = \{(x, y) | h(x, y) = 0\}$, is assumed to be two-fold smooth at the origin. Therefore there exists the linearized boundary around the origin $L = \{(x, y) | h_x(0) \cdot x + h_y(0) \cdot y = 0\}$. Take an initial point, A , on a linearized boundary. Define points D, G, and F as a trajectory of nonlinear system (3.5) hits the linearized boundary with $t > 0$; the nonlinear boundary with $t > 0$, and the nonlinear boundary with $t < 0$, for the first time, respectively. Therefore, define the timings, $t_3 > 0$ and $t_4 < 0$ as the trajectory, Γ, comes to G and F . Define also $\tilde{F} = \tilde{\Gamma}(t_4|\cdot)$. Then $G = \Gamma(t_3|(x_o, y_o))$ and $F = \Gamma(t_4|(x_o, y_o))$. Figure III.2 shows a figure which adds these points to Figure III.1. In addition to the results of the preceding theorem, we have to show the deviations between A and F , and D and G relative to the length of A become negligible as A approached the origin.

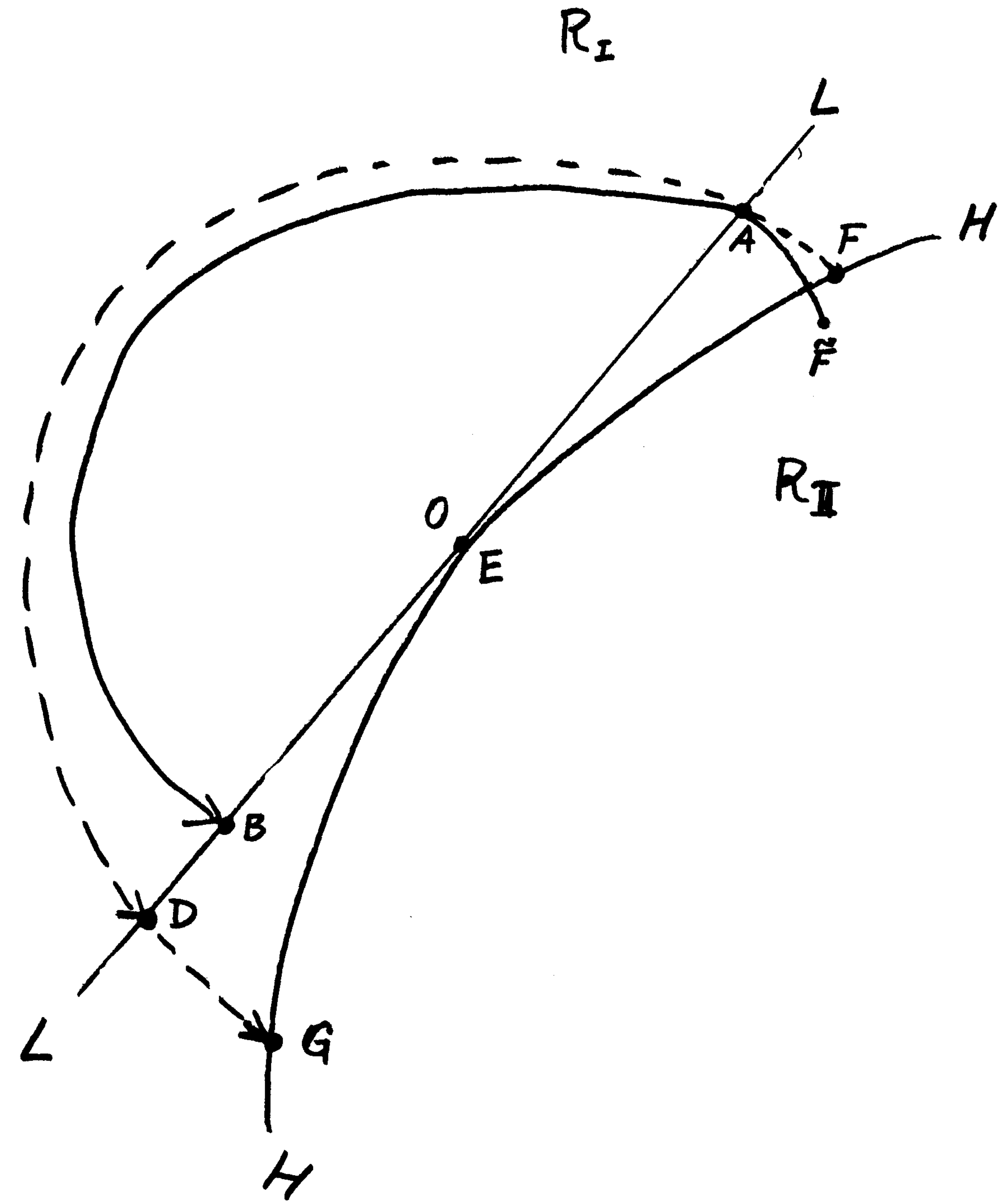


Figure III.2

Legend

A = (x_o, y_o)
B = $\tilde{\Gamma}(t_2|A)$
C = $\tilde{\Gamma}(t_1|A)$
D = $\Gamma(t_1|A)$
E = (0, 0)
F = $\Gamma(t_4|A)$
$\tilde{F}$ = $\tilde{\Gamma}(t_4|A)$
G = $\Gamma(t_3|A)$

Theorem III.3

The results of Theorem III.2 hold true for the case of a nonlinear smooth boundary, $H = \{(x, y) | h(x, y) = 0\}$ instead of a linear boundary.

Proof: It suffices to show that (i) for all $\sigma_1 > 0$, there exists δ_3 such that $\|\Gamma(t_3|\cdot) - \Gamma(t_1|\cdot)\|/\|(x_o, y_o)\| < \sigma_1$ for $\|(x_o, y_o)\| < \delta_3$, $(x_o, y_o) \in L\backslash\{0\}$; (ii) for all $\sigma_2 > 0$, there exists δ_4 such that $\|\Gamma(t_4|\cdot) - (x_o, y_o)\|/\|(x_o, y_o)\| < \sigma_2$ for $\|(x_o, y_o)\| < \delta_4$, $(x_o, y_o) \in L \backslash \{0\}$.

In order to show (ii), suppose contrary. That is, $\|\Gamma(t_4|\cdot) - (x_o, y_o)\|/\|(x_o, y_o)\| \geq \sigma_1$ for all $(x_o, y_o) \in L\backslash\{0\}$. Recall also condition (E2) which implies the angle OAF is strictly less than 180°. Therefore the angle AOF is nondegenerate as $A \rightarrow 0$. On the other hand, since H is tangent to L at the origin, condition (E2) implies $t_4 \rightarrow 0$ as $A \rightarrow 0$. However, this contradicts to an earlier claim that the angle AOF remains nondegenerate.

A parallel argument applies to the distance between D and G. Combining results from Theorem III.2 we can choose (x_o, y_o) sufficiently small so that $\|\Gamma(t_3|\cdot)\|/\|\Gamma(t_4|\cdot)\| < 1$.

Q.E.D.

The catch of these theorems is that it is sufficient to check the stability for each regime separately given condition (E) which ensures that when a solution path hits a boundary it crosses the boundary into another regime. The force of these theorems comes from the linearity or smoothness of the boundary as well as that of differential equations. It was shown by the introductory example that this theorem cannot be generalized to a case of piecewise linear boundaries (two half lines from the origin with different slopes).[12] It is also not applicable in more than three dimensions with two regimes because a trajectory in general does not cross boundaries at symmetrical points any more.

III.3 Many Regimes in $\mathbb{R}^n$ with Nonlinear Boundaries of Measure Zero

We can extend our framework to a general case with many regimes in $\mathbb{R}^2$ and with nonlinear smooth boundaries (but still boundaries of measure zero). However, stability conditions become very restrictive. We maintain the condition for uniqueness of a trajectory. If we can find a Lyapunov function which is continuous with respect to x (but not necessarily differentiable) and which is decreasing with respect to time with a trajectory defining a law of motion x, then the well-known Lyapunov's direct method of stability is applicable.[13]

We now extend our system to a case with more than two regimes. Consider the following system of nonlinear differential equations defined in a bounded domain, Q, which is partitioned into m regimes: R_1, R_2, ..., R_m. That is,

$$\bigcup_{j=1}^{m} R_j = Q$$

$$\mu(R_i \cap R_j) = 0 \qquad i \neq j$$

where μ is a Lebesgue measure of n dimension. Different continuous differential equations are defined for different regimes. Therefore, a (overall) system of differential equations is only piecewise continuous. We assume that the origin $\{0\}$ is the common equilibrium. That is,

$$\dot{x} = g(x) \qquad x \in Q \subset R^n \qquad \text{(N)}$$

$$g = g^j \qquad \text{if } x \in R_j, \quad j = 1, \ldots, m$$

with

$$g^j(0) = \{0\} \qquad \text{for all } j$$

$$x = (x_1, x_2, \ldots, x_n)$$

$$g^j = (f_1^j, f_2^j, \ldots, f_n^j)$$

$$\bigcap_{j=1}^{m} R_j = \{0\} \quad ,$$

where g^j is continuously differentiable and bounded in R_j. In this system we will apply the uniqueness theorem to each trajectory hitting any boundary, except the origin, between two regimes. Therefore we will have the uniqueness of a trajectory even if there are more than two regimes. To this end, denote by S_k, a boundary between R_i and R_j which share more than the origin,

$$S_k = R_i \cap R_j \qquad i \neq j$$

and

$$S_k \setminus \{0\} \neq \phi \quad .$$

Then we assume, for each S_k, g_k, R_i and R_j, Assumptions U: (i) - (iii) in section II. Then where exists a unique trajectory (a Filippov solution path) $\Gamma(t|x_o)$ for (N) in Q with an initial point $x_o \in Q$. Moreover, the trajectory continuously depends on the initial point. We also assume that $\Gamma(t|x_o)$ is contained in a compact set in Q.

Even if each regime is described by a system of linear differential equations with linear boundaries, it is not true any more that the stability condition for each regime implies the overall stability: $\Gamma(t|x_o) \to \{0\}$.

Even in the case of many regimes in a more-than-two dimensional space, we can apply the Lyapunov's direct method. Two important remarks are (i) that a Lyapunov function continuous with respect to x should be defined for an entire domain, and (ii) that a Lyapunov function does not have to be differentiable.[14]

Definition [Lyapunov Function]

A function $\nu(x)$ defined on $x \in Q$ is called a Lyapunov function for a trajectory $\Gamma(t|x_o)$ for all $x_o \in Q$, if

$$\varphi(t) \equiv \nu(\Gamma(t|x_o))$$

is a strictly decreasing function with respect to t, unless $\Gamma(t|x_o)$ is an equilibrium.

Theorem III.4 [Asymptotic Stability]

Consider a system of piecewise continuous differential equations, (N). Assume that the uniqueness condition, (U), is satisfied. Therefore for any initial position $x_o \in Q$, a trajectory $\Gamma(t|x_o)$ exists for $t \geq 0$, which is uniquely determined by x_o and is continuous with respect to x_o. Suppose also that for any initial position $x_o \in Q$, the trajectory $\Gamma(t|x_o)$ is contained in a compact set in Q. If there exists a Lyapunov function $\nu(x)$, then a system (N) is asymptotically stable.

The proof of Uzawa (1961) is applicable. Although Uzawa's system consists of continuous differential equations, his proof only needs a unique trajectory which is continuous with respect to x_o.

Remark

A Lyapunov function does not have to be differentiable as long as it is continuous with respect to x. This includes the possibility that a differentiable Lyapunov function is defined regime by regime with a continuous switching between regimes. That is,

$$\nu(x) = \nu^j(x) \qquad \text{if} \quad x \in R_j \qquad \forall_j$$

such that

$$\nu^j(x) = \nu^i(x) \qquad \text{if} \quad x \in \{R_j \cap R_i\}, \quad \text{for all } i, j, \quad i \neq j$$

where $\nu^j(x)$'s are differentiable functions.

First, we investigate a simple condition for the global asymptotic stability for piecewise linear differential equations. Secondly, we establish the local asymptotic stability for nonlinear differential equations with regime switching.

Consider the following linear system:

$$\dot{x} = Mx \qquad x \in Q \subset R^n$$

$$M = M^j \text{, a } (n \times n) \text{ real matrix, if} \quad x \in R_j, \quad j = 1, \ldots, m. \tag{L}$$

$$\bigcap_{j=1}^{m} R_j = \{0\}, \qquad \bigcup_{j=1}^{m} R_j = Q$$

with boundaries $S_k = R_i \cap R_j$, $i \neq j$; $S_k \setminus \{0\} \neq \emptyset$.

It is obvious that a Filippov solution exists for such a system.

Theorem III.5 [Global Stability for (L)]

A system of differential equation, (L), is globally asymptotically stable, if there exists an $(n \times n)$ positive-definite matrix B such that for all j

$$B M^j + (M^j)^T B$$

is negative-definite, where $(M^j)^T$ denotes the transpose of M^j.

Proof: Define a Lyapunov function as

$$V(x) = \frac{1}{2} x^T B x \quad .$$

Since B is positive definite, $V(x) > 0 \;\; \forall x \in Q \setminus \{0\}$ and $V(0) = 0$. $V(x)$ is continuous with respect to x because a matrix B is "common" to all regimes. When $\Gamma(t|x_o) \in \text{int } R_j$,

$$\frac{d}{dt} V(\Gamma(t|x_o)) = x^T [B M^j + (M^j)^T B] \, x < 0 \quad .$$

When a trajectory is at a point on the boundary, a derivative of a Lyapunov function with respect to time may not exist from the left-hand side, but always exists from the right-hand side. Let us denote the dx/dt on the right as $\dot{x}^\theta(x)$. That is

$$\dot{x}^\theta(x) = \lim_{t \downarrow \bar{t}} \dot{x} \quad , \quad \text{where} \quad x = \Gamma(\bar{t}|x_o) \quad .$$

By definition of the Filippov solution $\dot{x}^\theta(x)$ belongs to a convex closure of $M^i x$ and $M^j x$ at a boundary point, $x \in S_k = R_i \cap R_j$. Therefore

$$\dot{x}^\theta(x) \in \overline{co} \; \{M^i x \, , M^j x\} \quad .$$

That is, there exist some $\theta \in [0, 1]$, such that

$$\dot{x}^\theta(x) = \theta M^i x + (1 - \theta) \, M^j x \, , \quad .$$

Then for any $\theta \in [0, 1]$ and $\Gamma(\bar{t}|x_o) \in S_k = R_i \cap R_j$,

$$\begin{aligned} \lim_{t \downarrow \bar{t}} \frac{d}{dt} V(\Gamma(t|x_o)) &= (\dot{x}^\theta)^T \, Bx + (x)^T B \dot{x}^\theta \quad , \\ &= \theta \, x^T (M^i)^T Bx + (1 - \theta) \, x^T (M^j)^T Bx + \theta (x)^T BM^i x + (1 - \theta)(x)^T BM^j x \\ &= \theta [x^T \{BM^i + (M^i)^T B\} \, x] + (1 - \theta) [x^T \{BM^j + (M^j)^T B\} \, x] \\ &< 0 \quad . \end{aligned}$$

The last inequality comes from the assumption $BM^k + (M^k)^T B$ is negative definite for all $k = 1, 2, \ldots, i, j, \ldots, m$. Therefore $V(x)$ satisfies all the assumptions in Theorem III.3.

Q.E.D.

Corollary

Assume all the M^j's in system (L) are quasi-negative definite, i.e., $M^j + (M^j)^T$ is negative definite for all j, then (L) is asymptotically stable. This is the case where $B = I$, the identity matrix, in Theorem III.5.

Remark 1 [15]

Theorem III.5 and corollary suggest that we may not need the uniqueness of a Filippov solution. It implies that there exists at least one Filippov solution, and possibly many Filippov solutions and that all of them converge to the unique equilibrium. We may do so, by modifying the definition of stability and Theorem III.4. For the present purpose, we have followed the traditional definition of the stability.

Remark 2

The stability condition in Theorem III.5 may seem very restrictive, since the obtained Lyapunov function is a norm on $\mathbb{R}^n$. Nevertheless the theorem is useful considering the use of Sylvester inequalities to check the positive definiteness.[16] This will be demonstrated in the next section.

Finally we shall provide a theorem of linearization to show that the local stability of a nonlinear system is available when the Jacobian matrices satisfy the stability condition of Theorem III.5.

First, rewriting system (N), consider a system of nonlinear differential equations:

$$\dot{x} = M_0^j x + f^j x \qquad x \in R_j, \quad j = 1, 2, \ldots, n \tag{N2}$$

where $M_0^j = [\partial g/\partial x]_{x=0}^j$, the Jacobian matrix evaluated at the equilibrium where the regimes are defined as before in (N), and

$$f^j(0) = 0 \; ; \quad \|f^j(x)\| / \|x\| \to 0 \quad \text{as} \quad \|x\| \to 0 \quad .$$

Theorem III.6

A system of differential equations, (N), is locally asymptotically stable if there exists an $(n \times m)$ positive-definite matrix B such that for all j

$$B M_o^j + (M_o^j)^T B$$

is negative-definite, where M_o^j is defined in (N2) and $(M_o^j)^T$ denotes the transpose of M_o^j .

Proof: Define a Lyapunov function and $\dot{x}^\theta(x)$ as those in the proof of Theorem III.4. Suppose that a trajectory of (N) is at a point x on the boundary between regimes, R_i and R_j, at time $\bar{t}$,

$$\dot{x}^\theta \in \overline{co}\,\{M_o^j x + f^j(x) \,,\, M_o^j x + f^i(x)\} \,, \quad \Gamma(\bar{t}|x_o) = x \in R_j \cap R_i \quad .$$

That is, there exists some $\theta \in [0, 1]$ such that,

$$\dot{x}^\theta = \theta[M_o^j x + f^j(x)] + (1 - \theta)\,[M_o^i x + f^i(x)] \,.$$

$$\begin{aligned}\lim_{t \downarrow \bar{t}} \frac{d}{dt} V(\Gamma(\bar{t}|x_o)) &= (\dot{x}^\theta)^T B x + (x)^T B \dot{x}^\theta \\ &= \theta[x^T\{B M_o^i + (M_o^i)^T B\}x] + (1 - \theta)[x^T\{B M^j + (M^j)^T B\}x] \\ &\quad + \theta[(f^j(x))^T B x + (x)^T B f^j(x)] \\ &\quad + (1 - \theta)[(f^j(x))^T B x + (x)^T B f^j(x)] \quad .\end{aligned}$$

The first two terms are negative by the assumption. The last two terms are of at least third order, since $f^j(x)$ are of the second order. Therefore

$$\lim_{t \downarrow \bar{t}} \frac{d}{dt} V(\Gamma(\bar{t}|x_o)) < 0$$

for $\|x_o\|$ sufficiently small. At an interior point of a regime, the above proof holds true with θ being unity.

Q.E.D.

IV. Applications to Economic Models

IV.1 Veendorp model[17]

Veendorp (1975) faces a problem of stability with regime switching in analyzing the price - cum - quantity adjustment around the Walrasian equilibrium. There are two commodities exchanged.[18] By combination of signs of excess effective demand in both markets, there are four regimes around the general equilibrium.

The prices of commodities are assumed to respond to excess effective demands. Let the deviations from the equilibrium level of the prices be denoted by p_1 and p_2. Deriving the effective demand from the maximizing behavior, Veendorp has the following system:

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = M^j \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \qquad j = 1, 2, 3, 4 \qquad \text{(V)}$$

$$M^I = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \qquad M^{IV} = k\begin{bmatrix} m_{11} & m_{12} \\ n_{21} & n_{22} \end{bmatrix}$$

$$M^{II} = h\begin{bmatrix} n_{11} & n_{12} \\ m_{21} & m_{22} \end{bmatrix} \qquad M^{III} = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} .$$

The boundaries of these four regimes are given by

$$\left.\begin{array}{lll} \dot{p}_2 = m_{21} p_1 + m_{22} p_2 = 0 & \text{between I and II} \\ \dot{p}_1 = n_{11} p_1 + n_{12} p_2 = 0 & \text{between II and III} \\ \dot{p}_2 = n_{21} p_1 + n_{22} p_2 = 0 & \text{between III and IV} \\ \dot{p}_1 = m_{11} p_1 + m_{12} p_2 = 0 & \text{between IV and I} \end{array}\right\} \quad (B) \quad .$$

There are two results Veendorp tries to prove: (i) the stability of the spillover process under gross substitutability, and (ii) (in the absence of gross substitutability) spilling over may actually stabilize an otherwise unstable system. According to the graphical method of proof by Veendorp, the former result needs an additional condition, and the latter result is not proved in a general framework.[19] He shows a numerical example (which may have been shown stable by successive approximation), but a general stability condition for such a system (in the absence of gross substitutability) is not presented by Veendorp. This paper fills this gap, showing a sufficient condition of stability in the absence of gross substitutability.

First, we review the case of gross substitutability.

There is a sign condition derived from the assumption of gross substitutability:

$$m_{ii} < 0 , \quad n_{ii} < 0 , \qquad i = 1, 2 \tag{A.1}$$

$$\det M^j > 0 , \qquad j = 1, 2, 3, 4 \quad . \tag{A.2}$$

Notice that (A1) and (A2) ensure that the system for each regime is stable if there is no switching between them. However, one needs an additional condition to ensure the stability for an overall system (V) , since regime switching may upset the overall stability as demonstrated in the previous sections.

Case I: sign m_{12} = sign $n_{12} \neq$ sign m_{21} = sign n_{21}

In this case each quadrant contains exactly one of the four boundaries (B) . The phase diagram for this case is given by Figure IV.1, which corresponds to a figure given by Veendorp (1975; Figure 1).

Theorem IV.1 (Veendorp)

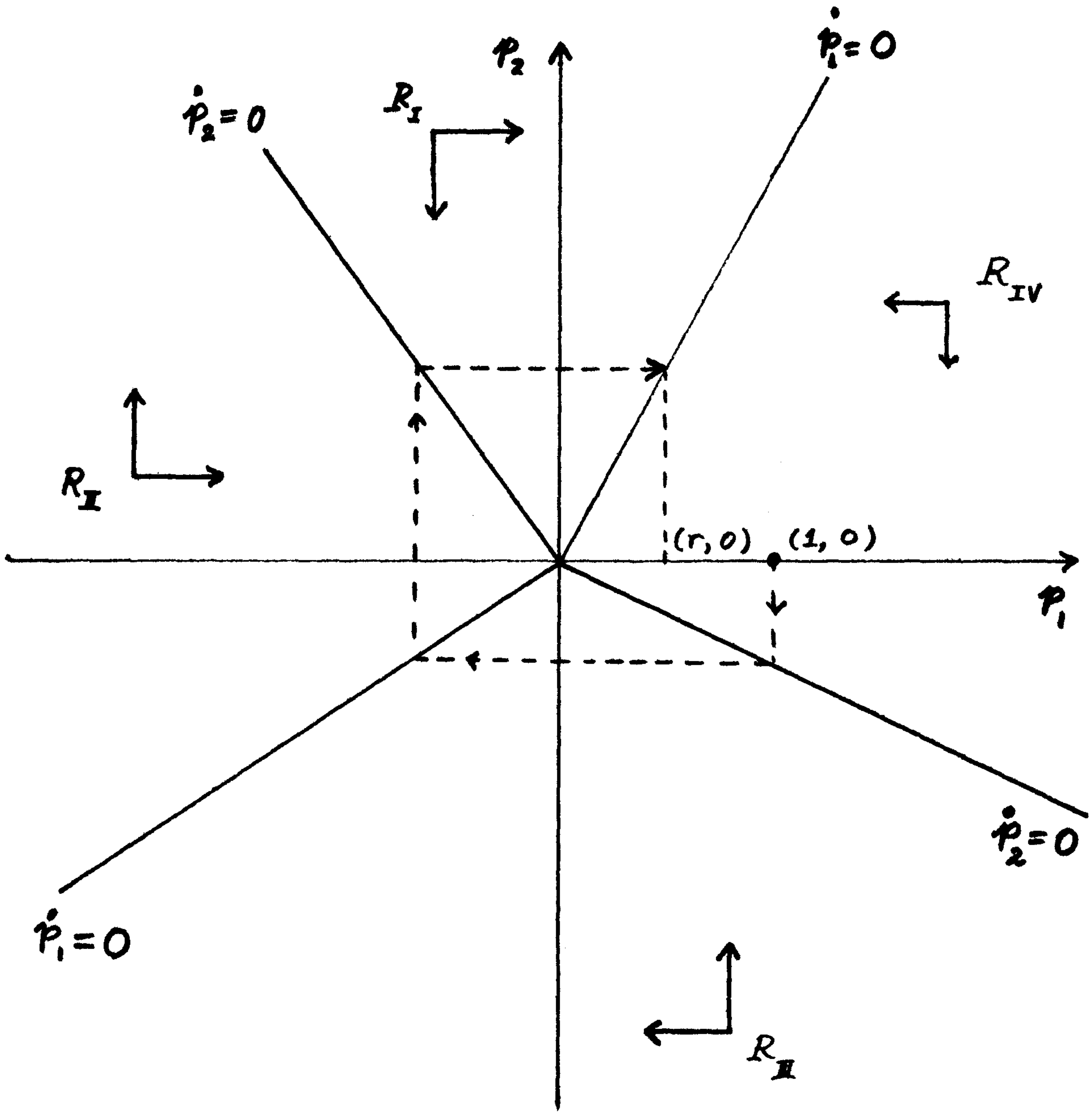
It is sufficient for the stability of (V) in Case I to show

$$r \equiv \frac{m_{12}\, m_{21}\, n_{12}\, n_{21}}{m_{11}\, m_{22}\, n_{11}\, n_{22}} < 1 \quad .$$

Proof: Take the "worst" case in the sense of destabilizing switching in Figure IV.1. Starting from point (1, 0), trace a possible Suppose that the worst solution path comes back to a point nearer to the origin on the positive horizontal axis, say (r, 0). Then by the proportionality of linear differential equations and piecewise linear boundaries, the second round of the oscillation will give an intercept $(r^2, 0)$. In the limit, the worst solution path converges to the origin. Therefore any solution path converges to the origin.

Case II: Any case other than Case I.

We still maintain Assumptions (A1) and (A2). As Veendorp asserts, this case turns out to be stable regime switching, since some of the regimes



sign m_{12} = sign $n_{12} > 0$

sign m_{21} = sign $n_{21} < 0$

Corresponding to Figure 1 of Veendorp (1975; page 452)

<u>Figure IV.1</u>

have a "lock-in" effect. That is, at the boundary of such a regime, a Filippov solution is oriented toward inside of the regime.

General Case

We can apply our theorem in the preceding section to the Veendorp model. If we can find a positive definite matrix, B, that is

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} ,$$

$$a > 0 , \quad c > 0 , \quad a c - b^2 > 0$$

such that, for all j,

$$C \equiv B M^j + (M^j)^T B$$

is negative definite, then (V) is globally asymptotically stable. This is quite "vague" in the sense that there is no specific way to find a, b, and c. The following theorem gives a specific way to check a sufficient condition, by using the so-called Sylvester inequalities.

First let us define two scalars, assuming (A.1) and (A.2).

$$\alpha(j; +) = \{(m^j_{11}\ m^j_{22})^{1/2} + (\det M^j)^{1/2}\}^2 / (m^j_{12})^2$$

$$\alpha(j; -) = \{(m^j_{11}\ m^j_{22})^{1/2} - (\det M^j)^{1/2}\}^2 / (m^j_{12})^2$$

where M^j_{kh} is the k-th row, h-th column entity of a matrix M^j in system (V).

Theorem IV.2

Consider a system, (V). Assume conditions (A.1) and (A.2). If the following conditions are met, then it is globally asymptotically stable.

$$\min_{j=I,\ II,\ III,\ IV} [\alpha(j\,;+)] > \max_{j=I,\ II,\ III,\ IV} [\alpha(j\,;-)] .$$

Proof: Define

$$B = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \qquad \alpha > 0 \quad .$$

Then B is positive definite. Define

$$B\,M^j + (M^j)^T B \equiv C^j \equiv \begin{bmatrix} c^j_{11} & c^j_{12} \\ c^j_{21} & c^j_{22} \end{bmatrix} \quad .$$

$$c^j_{11} = 2\alpha\, m^j_{11} < 0 ,$$
$$c^j_{22} = 2\; m^j_{22} < 0 , \qquad \text{by Assumption (A1)} .$$

Therefore it is sufficient for negative definiteness of C to show that, for all j ,

$$\det\; C^j = c^j_{11}\, c^j_{22} - c^j_{12}\, c^j_{21} > 0 \quad .$$

Now

$$\det\; C^j = -\,(m^j_{12})^2\, \alpha^2 + (4m^j_{11}\, m^j_{22} - 2m^j_{21}\, m^j_{12})\, \alpha - (m^j_{21})^2$$

$\det\; C^j > 0$, if and only if

$$\alpha(j\,;+) > \alpha > \alpha(j\,;-) \quad .$$

If condition in the theorem is met, there exists a positive constant α such that for all j

$$\alpha(j\,;+) > \alpha > \alpha(j\,;-) \quad .$$

Therefore choosing such an α which is positive, C^j is negative definite everywhere.

Q.E.D.

Several remarks are in order. First, one advantage of having Theorems III.5 and IV.2 is that we do not have to rely on drawing pictures for all the possible cases. Whether Theorem III.1, IV.1 or IV.2 gives a desired outcome depends on specific numbers of elements of M^j 's .

Secondly, note that system (V) has a special property that two matrices and their common boundary share the same row vector of coefficients, Laroque (1979) showed that (i) it is possible to find a multiplier to each matrix which would change the speed of adjustment but not a <u>geographical</u> trajectory so that the system of differential equations becomes continuous across the boundary; and (ii) for a two-dimensional continuous system, it is enough to prove the stability of each regime. The latter result is also obtained for a different model in Eckalbar (1980). Given the special property of system (V), Laroque's result requires a stability condition for system (V) which is weaker than ours.

Thirdly, another strong advantage is that Theorem III.5 is applicable even in the case of non-gross substitutability. This is illustrated in the following example taken from Veendorp (1975). The four matrices for corresponding four regimes in system (V) are given as follows:

<u>Veendorp's Numerical Example</u>

$$M^{I} = \begin{bmatrix} -1992 & 276 \\ -3234 & 357 \end{bmatrix} \qquad M^{IV} = \begin{bmatrix} -1992 & 276 \\ -3332 & 406 \end{bmatrix}$$

$$M^{II} = \begin{bmatrix} -2296 & 308 \\ -3234 & 357 \end{bmatrix} \qquad M^{III} = \begin{bmatrix} -2296 & 308 \\ -3332 & 406 \end{bmatrix}$$

The boundaries, (B) , are defined as explained above, using rows of these matrices.

The phase diagram corresponding to this example is shown as Figure IV.2. Since $m_{22} > 0$ and $n_{22} > 0$, Assumption (A.1) for system (V) is violated, so that neither Theorem IV.1 or Theorem IV.2 is applicable. Veendorp's assertion that his specific numerical example is overall stable seems proved not by his theorem (i.e., Theorem IV.1 of this paper) but by successive approximations of a trajectory. However, Theorem III.5 of this paper is still applicable, only if we can find an appropriate positive-definite matrix B.

Proposition

In Veendorp's numerical example, the equilibrium is asymptotically globally stable.

Proof: Take a positive definite matrix

$$B = \begin{pmatrix} 3.84 & -1 \\ -1 & 0.46 \end{pmatrix} .$$

Confirm that

$$BM^i + (M^i)^T B \equiv \begin{pmatrix} c^i_{11} & c^i_{12} \\ c^i_{21} & c^i_{22} \end{pmatrix} , \qquad i = I, II, III, IV$$

is a negative definite matrix. That is,

$$c^i_{11} < 0 ; \qquad c^i_{22} < 0$$

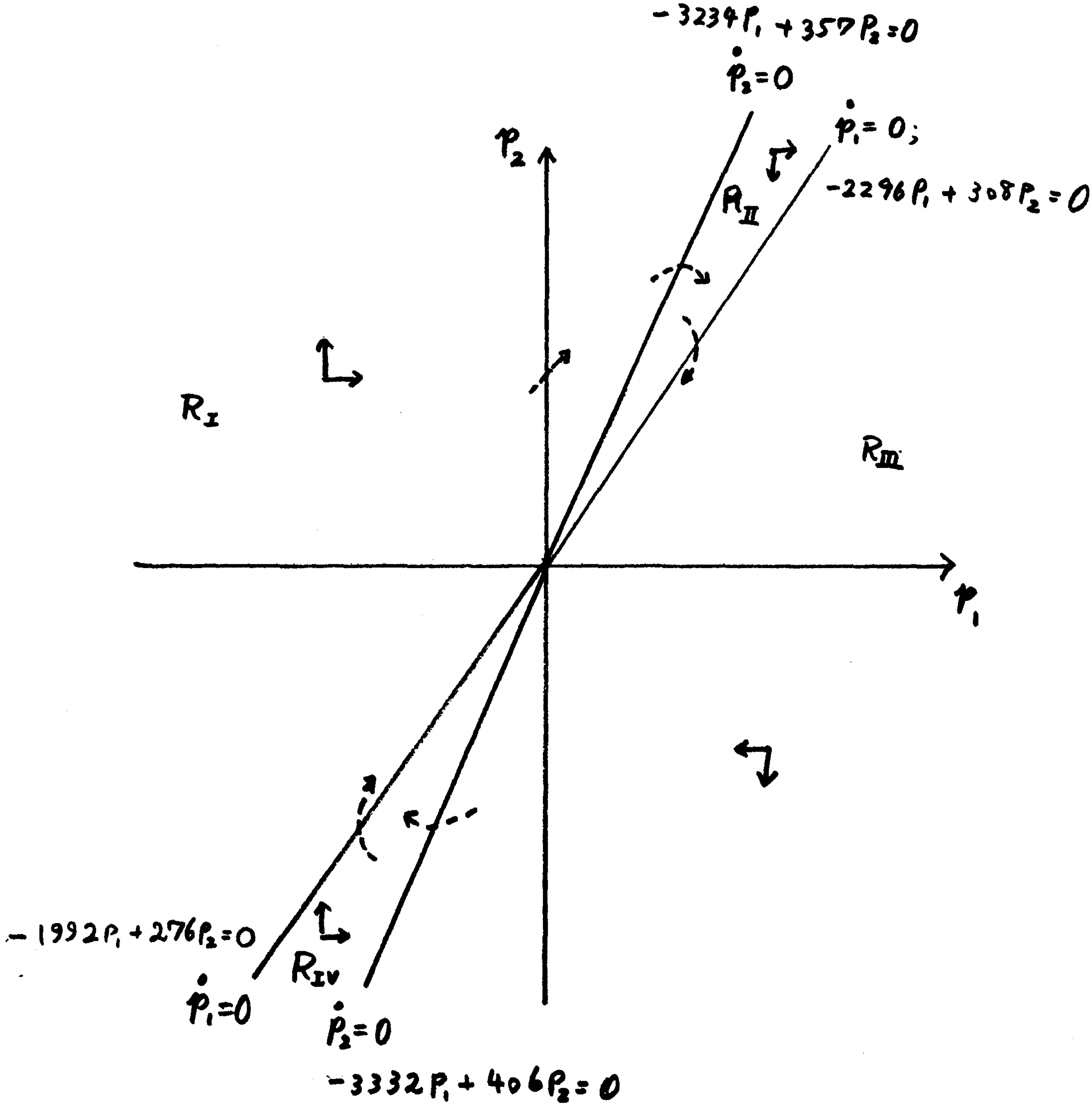
$$c^i_{11}\, c^i_{22} - c^i_{12}\, c^i_{21} > 0 , \qquad i = I, II, III, IV$$

Apply Theorem III.5.

Q.E.D.

IV. Dynamics of a Macroeconomic Disequilibrium Model

In the disequilibrium macroeconomic model literature, the price and wage are fixed in the short run, but flexible in the intermediate run. Transaction takes place at the minimum of effective demand and effective



$m_{11} < 0, \quad n_{11} < 0$

$m_{22} > 0, \quad n_{22} > 0$

Figure IV.2

supply in a market. An economy is classified in different types of regimes according to the signs of excess demands in both markets. Static disequilibrium macroeconomic models have been developed by Barro and Grossman (1976; Chapter 2), Benassy (1978) and Malinvaud (1977). Recent studies in this area are directed toward a dynamic analysis, either with capital stock or with the government budget balance constraint. See Ito (1978b) (1979), Malinvaud (1980), and Picard (1979) for the former type, and Böhm (1978) and Honkapohja (1979) (1980) for the latter. In the following, a model of the evolution of the short-run quantity-constrained equilibria in line with Böhm and Honkapohja will be presented and the stability theorems in the previous sections will be applied to the model.[20]

There are three different regimes: Keynesian unemployment K, Repressed Inflation R and Classical unemployment C. These are represented in Figure IV.3, in terms of real balances m and real wages w which are fixed in the short run since the corresponding nominal variables are held constant and behavioral relationships depend only on the real variables. Following Honkapohja (1979) it is postulated that wages and prices start to respond to excess demands or supplies prevailing in the markets after the temporary equilibrium with rationing has been reached. Moreover household savings and the government budget constraint may alter the nominal money stock over time and this provides an additional dynamic adjustment over time. These considerations result in an autonomous system of differential equations for real balances m and real wages w of the following form (see Honkapohja (1979) for all details):

$$\dot{m} = T(m, w)$$

$$\dot{w} = V(m, w) \quad .$$

Where the right-hand sides are defined regime by regime:

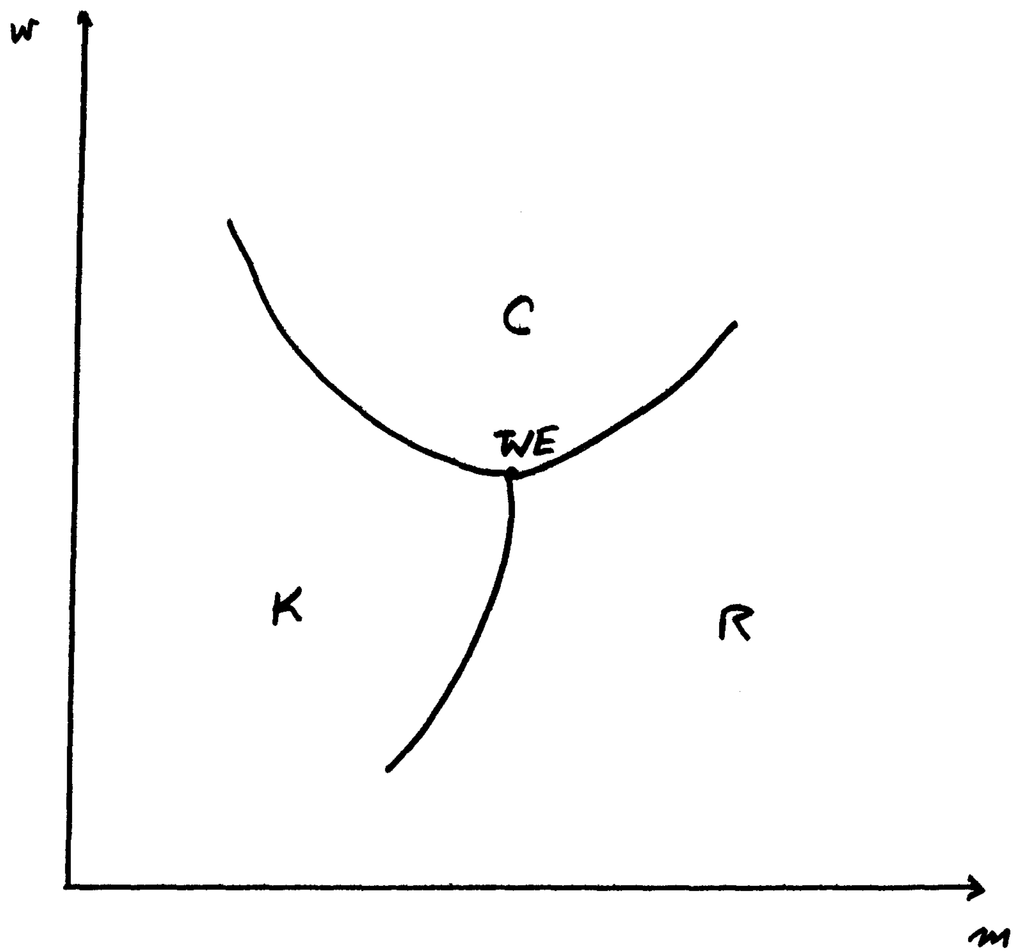


Figure IV.3

$$T(m, w) = T^K(m, w) \quad , \quad V(m, w) = V^K(m, w) \quad \text{if} \quad (m, w) \in K$$
$$T(m, w) = T^R(m, w) \quad , \quad V(m, w) = V^R(m, w) \quad \text{if} \quad (m, w) \in R$$
$$T(m, w) = T^C(m, w) \quad , \quad V(m, w) = V^C(m, w) \quad \text{if} \quad (m, w) \in C \quad .$$

Here T^K, T^R, T^C, V^K, V^R, V^C are differentiable in the interior of the respective regimes. The whole system is continuous on the boundaries $K \cap C$ and $R \cap C$, but discontinuous on the third boundary $K \cap R$.[21] Note that it is impossible to transform this system into a continuous one as Laroque (1979) did for the Veendorp model. In general the locus of stationary values of the real wage, i.e., $V(m, w) = 0$ lies in the regions K and R, and it goes through the Walrasian equilibrium WE which is at the intersection of the three regimes. On the other hand, the locus of the stationary values of the real balances $T(m, w) = 0$ can take different positions depending on the circumstances. Therefore the steady state may lie in the regimes K or R or at their intersection with C. In the last case the long-run quantity-constrained steady state coincides with the Walrasian equilibrium. The local stability of the first two cases was examined in detail in Honkapohja (1979) by using the standard method of linearization, but the case of a stationary Walrasian equilibrium cannot be handled with the standard method since it is at the boundary of the regimes. To develop stability conditions for that case Theorems III.5, III.6 and IV.2 can be applied.

<u>Theorem IV.4</u>

<u>Stationary Walras equilibrium in the Honkapohja (1979) model is locally stable, if the matrices</u>

$$M^K = \begin{bmatrix} T^K_m & T^K_w \\ V^K_m & V^K_w \end{bmatrix}, \quad M^R = \begin{bmatrix} T^R_m & T^R_w \\ V^R_m & V^R_w \end{bmatrix}, \quad M^C = \begin{bmatrix} T^C_m & T^C_w \\ V^C_m & V^C_w \end{bmatrix}$$

are all negative quasi-definite, or, as a special case, the following condition is satisfied,

$$\min_{i=K,R,C} \alpha(i, +) > \max_{i=K,R,C} \alpha(i, -),$$

where $\alpha(i, +)$ and $\alpha(i, -)$ are given in Theorem IV.2 above. Here the matrices M^K, M^R, M^C are all evaluated at the Walrasian equilibrium.

In general either of these conditions is somewhat cumbersome. For example, signs of the different partial derivatives T^K_m, etc. cannot alone be used to determine stability. Some information about the quantitative magnitudes of the partial derivatives is needed to ensure stability through these conditions. However, it is worth pointing out that our techniques are useful to prove stability even if some other methods fail. In the present context in a recent work by van den Heuvel (1980), the Veendorp method in the proof of Theorem IV.1 is used to develop sufficient conditions for the stability of a stationary Walrasian equilibrium. Let us adopt the following sign conditions (circumstances for their validity are discussed in detail in Honkapohja (1979)).

$$M^K,\ M^R,\ M^C = \begin{bmatrix} - & - \\ + & - \end{bmatrix} \quad . \tag{4.1}$$

Van den Heuvel shows that under (4.1) the stability condition using the Veendorp method is

$$(V^R_m/V^R_w)(V^K_m/V^K_m) < (T^R_m/T^R_w)(T^K_m/T^K_w) \quad . \tag{4.2}$$

It is, however, easy to give examples where this condition is not satisfied and yet our Theorem IV.3 can be applied. Here is a simple numerical example.

Example: Let

$$M^K = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}, \quad M^R = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}, \quad M^C = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} .$$

Letting $A^K = M^K + (M^K)^T$ etc., where T denotes transposition, we have det $A^K = 16$, det $A^R = 5$, det $A^C = 18$ so that M^K, M^R and M^C are all seen to be negative quasi-definite. Therefore the stationary Walras equilibrium would be locally stable, though clearly the Veendorp condition (4.2) fails. In general though the conditions of Theorem IV.3 and condition (4.2) are independent so that our techniques do not necessarily subsume the Veendorp method.

V. Concluding Remarks

In this paper we have shown several stability results for a system of piecewise continuous differential equations. First, we investigated the relation between the overall stability for such a system and the regime stability, i.e., each system of continuous differential equations for a regime being stable if it were defined for an entire space. We showed in numerical examples that the regime stability is in general neither necessary nor sufficient for the overall stability. Secondly, we sought some additional assumptions which make it possible to prove the overall stability by checking only the regime stability. We showed that in the case of two regimes in $\mathbb{R}^2$ partitioned by a smooth boundary, it is enough for the local stability to show that the linearized systems of each regime satisfies the (Routh-Hurwitz) stability condition. This is a generalization of Ito's earlier result where both the boundary and differential equations are linear.

Thirdly, in a general case of many regimes in $\mathbb{R}^n$ if we find a Lyapunov function which is <u>continuous across boundaries</u> of regimes, then it is sufficient to assert the overall stability. We provided with several ways to find such a Lyapunov function. Finally we showed two economic models where our results are applicable.

There are many directions to extend this study. Our results in $\mathbb{R}^2$ largely depend on the smooth boundary. Instead, it is conceivable to derive some conditions on the relation between the two systems of differential equations given a boundary with a kink at the origin. Sufficient conditions in $\mathbb{R}^n$ are still very restrictive and an attempt of its relaxation is needed.

Footnotes

1. See W. Hahn (1967: pp. 61, 89, 369 - 383), and Hermes (1967) and Brunovský (1974)(1976).

2. Notation, $K\{g(x)\}$, is not quite precise in the sense that the set K depends on not only $g(x)$ but also g of $B(x, \delta)$.

3. The resulting third set of differential equations which dictates a sliding trajectory along the boundary can be interpreted from the underlying model. See Hahn's (1967: p. 370) discussion of an example in physics.

4. See Utkin (1978) for an extensive discussion of the use of sliding trajectories in control theory. See also Brunovský (1974: Figure 2) for a simple illustration.

5. As will be seen in the next footnote, there are different kinds of the stability concept of a system. We do not discuss "stability with respect to measurement" (Hermes) in the following.

6. This is the definition of "global asymptotic stability," which implies that the equilibrium is "attractive." It does not necessarily mean that one equilibrium is "stable in the sense of Lyapunov." See W. Hahn (1967; especially sections 26 and 40) for the subtle distinctions. For the purpose of this paper, it is not beneficial to go into this discussion. For the <u>linear</u> differential equations, these concepts become identical.

7. See Quirk and Saposnik (1968: section 5-4).

8. Papers by Ito (1979) and Picard (1979) were presented independently in the Econometric Society European Meeting, Athens, Greece, 1979. Picard's (1979) system consists of continuous Lipschitzian differential equations. Therefore it automatically satisfies condition (E). Ito (1979) explicitly mentioned condition (E) for a "solution" to be well defined.

9. For the linearization and its behavioral properties, see W. Hahn (1967: pp. 120 - 127).

10. For example, if two improper nodes are patched up, then it is necessary and sufficient for the overall stability that both nodes are stable. If one regime is an improper node and the other is a center or a spiral, then the necessary and sufficient condition for the overall stability is that the improper node is stable.

11. Ito (1979) conjectured this theorem without the proof.

12. Laroque (1979) considers a similar problem and reaches a different conclusion. His system of linear differential equations is <u>continuous</u> across boundaries even though there exist different coefficient matrices for different regimes. He proves that the overall system is stable if each system of a regime is stable, and boundaries are linear. His assumption implies that coefficients of differential equations and boundaries have to satisfy certain conditions. Our discussion above is different from Laroque's in the sense that we deal with a discontinuous system of differential equations without restricting coefficients.

13. See, for example, W. Hahn (1967; Chapter IV) and Uzawa (1961).

14. When we need a Lyapunov function to be "crawling down" a hill, it does not matter whether a road is smooth or it has a "kink." Recently, Eckalbar (1980) also discusses the use of the Lyapunov function in the stability analysis with regime switching. However, he assumes a system to be continuous across boundaries and the Lyapunov function to be differentiable. This is a very strong restriction and often violated in disequilibrium models.

15. We owe this observation to Truman Bewley.

16. The Sylvester inequalities are explained in W. Hahn, (1967: p. 100).

17. Laroque (1979) also considers the Veendorp model and shows the stability by a different method from ours. The differences will be explained below. We are grateful to him for his kindness to make his discussion paper available to us.

18. Keep in mind there is the third commodity which absorbs the remains in the budget. In other words these two markets in concern are two "independent" markets after the use of the Walras law.

19. Laroque has correctly pointed out that it is Veendorp's slip to say that the additional condition for the first claim is implied by the gross-substitutability.

20. For an application of a Filippov solution to the Malinvaud model, see Ito (1980).

21. There is a slight inaccuracy in Honkapohja (1979, diagram 4) because of the discontinuity, but it does not alter the conclusions in that paper.

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