AXISYMMETRIC SOLUTIONS TO A COUPLED NAVIER-STOKES/ALLEN-CAHN EQUATIONS

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Axisymmetric Solutions to a coupled Navier-Stokes/Allen-Cahn Equations

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Abstract

We investigate a family of axisymmetric solutions to a coupling of Navier-Stokes and Allen-Cahn equations in $\mathbb{R}^3$. Firstly, a $1D$ system of equations is derived from the method of separation of variables, which approximates the $3D$ system along its symmetry axis. Then based on them, by adding perturbation terms, we construct finite energy solutions to the $3D$ system. We prove the global regularity of the constructed solutions in both large viscosity and small initial data cases. These solutions can be considered as perturbations near infinite-energy solutions.

Keywords: axisymmetric solution, Navier-Stokes, Allen-Cahn, classical solution

Mathematics subject classification: 35Q35; 35K55; 76D05

1 Introduction

In this paper,we shall study the following coupled Navier-Stokes/Allen-Cahn equations in $\mathbb{R}^3 \times (0, +\infty)$:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla \phi \otimes \nabla \phi), \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

$$\phi_t + (\mathbf{u} \cdot \nabla)\phi = \gamma(\Delta \phi - f(\phi)). \tag{1.3}$$

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We assume $\mathbf{u}$ and $\nabla\phi$ decay sufficiently fast in the infinity. Here $\mathbf{u}$ is a vector function, ϕ and p are scalar functions, and $f(\phi) = \frac{1}{\eta^2}(\phi^3 - \phi)$, ν, λ, γ, η are positive constants. In addition, $\nabla\phi \otimes \nabla\phi$ is a tensor product–e.g., $(\nabla\phi \otimes \nabla\phi)_{ij} = (\nabla\phi)_i(\nabla\phi)_j$, $1 \le i,\ j \le 3$.

Multiplying (1.1) by $\mathbf{u}$, (1.3) by $\lambda(-\Delta\phi + f(\phi))$, then adding them up, and using integration by parts combined with (1.2), we get the basic energy law

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \lambda\|\nabla\phi\|^2 + \frac{\lambda}{2\eta^2}\|(\phi^2-1)\|^2\right) = -(\nu\|\nabla\mathbf{u}\|^2 + \lambda\gamma\|\Delta\phi - f(\phi)\|^2), \tag{1.4}$$

where $\|\cdot\|$ denotes the L^2 norm in $3D$ space $(\int_{\mathbb{R}^3} |\cdot|^2 dx)^{\frac{1}{2}}$.

The system (1.1)-(1.3) can be viewed as a phase field model describing the motion of a mixture of two incompressible viscous fluids (see [30]). The fluids are separated by a thin interface of width η. The velocity vector of the mixture is represented by $\mathbf{u}$, the pressure by p, the fluid kinetic viscosity by ν, and the phase of the fluid components by ϕ. The phase ϕ takes the value 1 in one bulk phase and -1 in the other. In the interfacial region, it undergoes rapid but smooth variation. It is assumed that the interface possesses a free energy $E_\eta = \int_\Omega \frac{1}{4\eta^2}(\phi^2-1)^2 + \frac{1}{2}|\nabla\phi|^2 dx$ caused by the mixing of fluids. Motion of the interface is caused by energy dissipation, which is given by $\phi_t = -\delta E_\eta/\delta\phi$. The term $\nabla\phi \otimes \nabla\phi$ in the momentum equation is the induced elastic stress due to the mixing of fluids. Finally, λ corresponds to the surface tension and γ the elastic relaxation time.

In another point of view, the above system (1.1)-(1.3) is closely related to the liquid crystal model, Magnetohydrodynamics (MHD) equations, and the viscoelastic system with finite Weissenberg number. All of these are shown in the appendix.

1.1 Navier-Stokes Equations

The system (1.1)-(1.3) possesses many essential properties of the Navier-Stokes equations. We note that the system includes the Navier-Stokes equations as a subsystem,

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\Delta\mathbf{u}, \tag{1.5}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.6}$$

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0. \tag{1.7}$$

Multiplying (1.5) by $\mathbf{u}$, and noting (1.6), (1.7), the basic energy law for Navier-Stokes equations can be written as

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\Omega)} = -\nu\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}. \tag{1.8}$$

It is well known that the weak solution is unique and regular in $2D$ (see [28]). The situation in $3D$, however, is more complicated. We recall that the Leray-Hopf weak solution of the Navier-Stokes equations is defined as a vector field $\mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ satisfying

$div\ \mathbf{u} = 0$ in the distribution sense, the energy inequality

$$\|\mathbf{u}(t)\|_{L^2(\Omega)}^2 + 2\int_0^t \|\nabla \mathbf{u}(s)\|_{L^2(\Omega)}^2 ds \le \|\mathbf{u}_0\|_{L^2(\Omega)}^2 \quad \text{a.e. } t \in [0,T]$$

and

$$\int_0^T \int_\Omega (-\mathbf{u}\cdot\boldsymbol{\psi}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\psi} + \nabla\mathbf{u}\cdot\nabla\boldsymbol{\psi})dxdt = \int_\Omega \mathbf{u}_0(x)\cdot\boldsymbol{\psi}(x,0)dx,$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega\times[0,T),\mathbb{R}^3)$ with $div\ \boldsymbol{\psi} = 0$. The Leray-Hopf weak solution was constructed in [15] and [10], and the regularity of weak solutions has always been an interesting problem. In $3D$, either a large viscosity constant ν (depending on u_0 and f) or small initial data u_0 (depending on ν), say, $\|\mathbf{u}_0\|_{H^1(\Omega)} < \frac{\nu}{C}$, are required to ensure the existence of a global classical solution ([28]). Later, as an approach to the regularity problem, J. Serrin in [26] studied the regularity criterion of the Leray-Hopf weak solutions, and obtained that if in $3D$ a weak solution $u \in L^p(0,T;L^q(\Omega))$, where $\frac{2}{p}+\frac{3}{q} < 1$, $2 < p \le \infty$, $3 < q \le \infty$, then u is regular, and becomes smooth in space variables on $(0,T]$. After Serrin's work, there are many improvements and developments regarding the study of regularity criterion (see [7]). It is found, in particular, the Leray-Hopf weak solution becomes smooth in (x,t) if $u \in L^p(0,T;L^q(\Omega))$, with $\frac{2}{p}+\frac{3}{q} \le 1$, $2 \le p \le \infty$, $3 < q \le \infty$. Some results about the partial regularity properties of suitable weak solutions to the Navier-Stokes equations were developed in [1], where it was proved that the one-dimensional Hausdorff measure of the set of singularities of the suitable weak solution is zero. Later, a simplified proof of this result was provided in [18]. The partial regularity result provides an important characterization of the nature of possible singularities of the $3D$ Navier-Stokes equations.

1.2 Axisymmetric solutions

In this paper, we study only axisymmetric solutions to (1.1)-(1.3). There have been some interesting developments in the study of axisymmetric solutions to the $3D$ Navier-Stokes equations, see for example [2], [3], [24], and [27]. The $2D$ Boussinesq equations are closely related to the $3D$ Navier-Stokes equations with swirl (away from the symmetry axis). Recently in [2], [11], the authors have independently proved the existence of solutions to the $2D$ global viscous Boussinesq equations with viscosity entering only in the fluid equation. And most interestingly, in [12], the authors constructed a smooth solution of (1.5)-(1.7), with initial conditions $\mathbf{u}_0 = \mathbf{u}(r,z,0)$ satisfying

$$\|\mathbf{u}_0\|_{L^2(\Omega)} \approx \frac{A}{\sqrt{M}}, \qquad \|\nabla\mathbf{u}_0\|_{L^2(\Omega)} \approx A\sqrt{M},$$

where A and M are constants to be determined. Since $\|\mathbf{u}_0\|_{L^2(\Omega)}\|\nabla\mathbf{u}_0\|_{L^2(\Omega)} \approx A^2$, by choosing A large enough, $\|\mathbf{u}_0\|_{L^2(\Omega)}\|\nabla\mathbf{u}_0\|_{L^2(\Omega)}$ can be made arbitrarily large. Thus, it violates the smallness condition that guarantees the existence of global classical solutions to $3D$ Navier-Stokes equations.

Motivated by these results, it seems natural to study the properties of $3D$ axisymmetric solutions to our system (1.1)-(1.3). For this system, we construct a family of global classical solutions with finite energy, which can also be regarded as perturbations of near infinite-energy solutions.

In contrast to the asymptotic expansion method in [12], we use the much more straightforward method of separation of variables to derive a system of $1D$ equations. Then, based on the solutions to these equations, using cutoff functions, we construct a family of finite energy solutions to the $3D$ system (1.1)-(1.3). After that, through a detailed study of weighted norm inequalities, we prove the global regularity of the solutions we construct in the case of large viscosity and small initial data.

1.3 Basic settings and 1D special configurations

Let

$$\mathbf{e_r} = (\frac{x}{r}, \frac{y}{r}, 0), \ \mathbf{e}_\theta = (-\frac{y}{r}, \frac{x}{r}, 0), \ \mathbf{e_z} = (0,0,1) \tag{1.9}$$

be three unit vectors, where $r = \sqrt{x^2+y^2}$. We can decompose the velocity field as

$$\mathbf{u} = v^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + v^z(r,z,t)e_z.$$

The vorticity field is expressed similarly as

$$\boldsymbol{\omega} = -(u^\theta)(r,z,t)e_r + \omega^\theta(r,z,t)e_\theta + \frac{1}{r}(ru^\theta)_r(r,z,t)e_z,$$

where $\omega^\theta = (v^r)_z - (v^z)_r$. To simplify our notation, we will use u and ω to denote u^θ and ω^θ in the rest of our paper.

Throughout the paper, ∇^2, Δ, and ∇ will stand for the Laplace, modified Laplace, and gradient operators, respectively in cylindrical coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial_r}{r} + \frac{\partial^2}{\partial z^2}, \tag{1.10}$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{3\partial_r}{r} + \frac{\partial^2}{\partial z^2} \equiv \Delta_r + \frac{\partial^2}{\partial z^2}, \tag{1.11}$$

$$\nabla = \partial_r \mathbf{e_r} + \partial_z \mathbf{e_z}. \tag{1.12}$$

Rewriting (1.1)–(1.3) into cylindrical coordinates, we obtain the equivalent system

$$u_t + v^r u_r + v^z u_z = \nu(\nabla^2 - \frac{1}{r^2})u - \frac{1}{r}v^r u, \tag{1.13}$$

$$\begin{aligned} \omega_t + v^r\omega_r + v^z\omega_z &= \nu(\nabla^2 - \frac{1}{r^2})\omega + \frac{1}{r}(u^2)_z - \frac{1}{r}v^r\omega \\ &\quad + \lambda(\phi_z\nabla^2\phi_r - \phi_r\nabla^2\phi_z - \frac{1}{r^2}\phi_r\phi_z), \end{aligned} \tag{1.14}$$

$$-(\nabla^2 - \frac{1}{r^2})\psi = \omega, \tag{1.15}$$

$$(v^r)_r + \frac{v^r}{r} + (v^z)_z = 0, \tag{1.16}$$

$$\phi_t + v^r\phi_r + v^z\phi_z = \gamma(\nabla^2\phi - \frac{1}{\eta^2}\phi^3 + \frac{1}{\eta^2}\phi). \tag{1.17}$$

Here u and ω stand for θ components of velocity $\mathbf{u}$ and vorticity $\boldsymbol{\omega}$ respectively, and v^r and v^z are the other two components of $\mathbf{u}$. ψ is the angular stream function, which is related to v^r and v^z as follows:

$$v^r = -\frac{\partial\psi}{\partial z}, \quad v^z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi). \tag{1.18}$$

One can alternatively derive the following $1D$ equations :

$$(u_1^*)_t + 2\psi_1^*(u_1^*)_z = \nu(u_1^*)_{zz} + 2(\psi_1^*)_z u_1^*, \tag{1.19}$$

$$(\omega_1^*)_t + 2\psi_1^*(\omega_1^*)_z = \nu(\omega_1^*)_{zz} + (u_1^{*2})_z, \tag{1.20}$$

$$-(\psi_1^*)_{zz} = \omega_1^*, \tag{1.21}$$

$$(\phi_0^*)_t + 2\psi_1^*(\phi_0^*)_z = \gamma(\phi_0^*)_{zz} - \frac{\gamma}{\eta^2}(\phi_0^*)^3 + \frac{\gamma}{\eta^2}\phi_0^*. \tag{1.22}$$

Here u_1^*, ω_1^*, ψ_1^* and ϕ_0^* are functions of only z and t.

We will consider solutions with periodic boundary conditions in the z direction with period 1, hence in the rest of the paper we set

$$\begin{aligned}
\Omega &= [0,\infty)\times[0,1],\ 0\le r\le\infty,\ 0\le z\le 1. \\
\|\cdot\| &= \|\cdot\|_{L^2(\Omega)} = (\int_0^1\int_0^\infty |\cdot|^2 r dr dz)^{\frac{1}{2}}. \\
\|\cdot\|_{L^4} &= \|\cdot\|_{L^4(\Omega)} = (\int_0^1\int_0^\infty |\cdot|^4 r dr dz)^{\frac{1}{4}}. \\
L^\infty(0,\infty;\mathbf{X}) &= \{x(t)\in\mathbf{X} \text{ for a.e. t}|\ sup_{t\in(0,\infty)}\|x\|_{\mathbf{X}} < \infty\}.
\end{aligned}$$

1.4 Construction of solutions to the $3D$ system and main results

By the $1D$ equations to (1.19)-(1.22), we can construct a family of exact solutions to the $3D$ system. If (u_1^*, ω_1^*, ψ_1^*, ϕ_0^*) is a solution to the $1D$ equations, then ($ru_1^*(z,t)$, $r\omega_1^*(z,t)$, $r\psi_1^*(z,t)$, $\phi_0^*(z,t)$) is an exact solution to the $3D$ system. Therefore, it is reasonable to think that the $1D$ equations retain some essential nonlinear features of the $3D$ system. However, ($ru_1^*(z,t)$, $r\omega_1^*(z,t)$, $r\psi_1^*(z,t)$, $\phi_0^*(z,t)$) is an exact solution with infinite energy. Thus, we want to look for global classical solutions to (1.13)-(1.17) with finite energy. To this end, we study solutions of the following form :

$$\tilde{u}(r,z,t) = r(u_1^*(z,t)\chi(r) + u_1(r,z,t)), \tag{1.23}$$

$$\tilde{\omega}(r,z,t) = r(\omega_1^*(z,t)\chi(r) + \omega_1(r,z,t)), \tag{1.24}$$

$$\tilde{\psi}(r,z,t) \;=\; r(\psi_1^*(z,t)\chi(r)+\psi_1(r,z,t)), \tag{1.25}$$
$$\tilde{\phi}(r,z,t) \;=\; \phi_0^*(z,t)\chi(r)+\phi_1(r,z,t), \tag{1.26}$$

where $\tilde{u}(r,z,t)$, $\tilde{\omega}(r,z,t)$, $\tilde{\psi}(r,z,t)$ are the θ components of velocity, vorticity and stream function, respectively, and $\chi(r)$ is a cut-off function, which ensures the solution has finite energy. Here, u_1, ω_1, ψ_1 and ϕ_1 are considered as perturbation terms.

Using a priori estimates of solutions to the $1D$ equations and delicate energy estimates, we prove that if the viscosity ν is large enough, then there exists a family of global classical functions $u_1(r,z,t)$, $\omega_1(r,z,t)$, $\psi_1(r,z,t)$ and $\phi_1(r,z,t)$ such that $\tilde{u}$, $\tilde{\omega}$, $\tilde{\psi}$ and $\tilde{\phi}$ are global classical solutions to the $3D$ system.

Since our system contains the $3D$ axisymmetric Navier-Stokes equation as a sub-system, one can not expect better results. In fact, we get theorems in both the large viscosity and small initial data cases. Our main theorems are stated as follows.

Theorem 1.1. *For the* $3D$ *system* (1.1)-(1.3)*, assume* $u_1^*(z,0)$*,* $\psi_1^*(z,0)$*,* $\omega_1^*(z,0)$*, and* $\phi_0^*(z,0)$ *are smooth functions which are periodic in* z *with period* 1*. Then there exists a global classical solution in the form of* (1.23)-(1.26)*, if initial conditions* $\tilde{\mathbf{u}}_{\mathbf{0}} \triangleq \tilde{\mathbf{u}}(r,z,0) \in H^1(\Omega)$*,* $\tilde{\phi}_0 \triangleq \tilde{\phi}(r,z,0) \in H^2(\Omega)$ *and* $\nu \geq \nu_0(\gamma,\lambda,\tilde{\mathbf{u}}_{\mathbf{0}},\tilde{\phi}_0)$*.*

In addition, without the assumption of large viscosity ν, if we assume $u_1^*(z,0)$, $\psi_1^*(z,0)$, $\omega_1^*(z,0)$, $\phi_0^*(z,0)$ are odd, periodic functions in the z direction with period 1, after some delicate analysis, we can also get a global smooth solution, provided the initial data is small enough.

Theorem 1.2. *Suppose the initial conditions for* u_1*,* ω_1*,* ψ_1*, and* ϕ_1 *are smooth functions with compact support and odd in* z*. Moreover, assume that* $\eta > 1$*, and* $\|\tilde{\mathbf{u}}(0)\|^2 + \lambda\|\nabla\tilde{\phi}(0)\|^2 + \frac{\lambda}{2\eta^2}\|\tilde{\phi}(0)^2-1\|^2 \leq \frac{C}{\sqrt{M}}$*. For any given* $\nu > 0$*, there exists* $C(\nu) > 0$*, such that if* $M \geq C(\nu)$ *and* $H(0) \leq 1$ *where* $H^2(t) = \|r\nabla u_1\|^2 + \|r\triangle\psi_1\|^2 + \|\nabla^2\phi_1\|^2$*. Then, solutions to the 3D system* (1.1)-(1.3) *in the form of* (1.23)-(1.26) *are globally smooth.*

The paper is organized as follows: a system of $1D$ equations is derived in Section 2 by separation of variables. Some useful lemmas and estimates are prepared in Section 3, in order to prove the regularity of perturbation terms later. The proof of global regularity of the solutions to the $3D$ system in the case of large viscosity is provided in Section 4, while the corresponding proof for small initial data is given in Section 5. Some related models and our future work are discussed briefly in Section 6.

2 Derivation of the 1D system of equations

In this section, we use the method of separation of variables to derive the 1D equations. Moreover, the regularity of solutions to the 1D equations is investigated. In the end, we present a key observation of the connection between solutions to the 1D equations and those to the $3D$ axisymmetric system.

Assume

$$\begin{aligned}
u(r,z,t) &= \bar{u}(r)u_1^*(z,t),\\
v^r(r,z,t) &= \bar{v}^r(r)a(z,t),\\
v^z(r,z,t) &= \bar{v}^z(r)b(z,t),\\
\psi(r,z,t) &= \bar{\psi}^r(r)\psi_1^*(z,t),\\
\omega(r,z,t) &= \bar{\omega}(r)\omega_1^*(z,t),\\
\phi(r,z,t) &= \phi_0^*(z,t).
\end{aligned}$$

Then (1.16) gives

$$\left[(\bar{v}^r)_r + \frac{\bar{v}^r}{r}\right] a(z,t) + \bar{v}^z b'(z,t) = 0,$$

implying

$$a(z,t) = b'(z,t), \tag{2.1}$$

$$(\bar{v}^r)_r + \frac{\bar{v}^r}{r} + \bar{v}^z = 0. \tag{2.2}$$

Since $v^r = -\psi_z$, $v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$ and $\omega = (v^r)_z - (v^z)_r$, by (2.1) we get

$$\bar{\psi}^r(r) = \bar{v}^r(r), \tag{2.3}$$

$$a(z,t) = -\psi_{1z}^*(z,t), \tag{2.4}$$

$$b(z,t) = -\psi_1^*(z,t), \tag{2.5}$$

$$w = -\bar{v}^r\psi_{1zz}^* + (\bar{v}^z)_r\psi_1^*, \tag{2.6}$$

then plugging (2.4) and (2.5) into (1.13) and (1.14), one arrives at

$$u_{1t}^* - \nu u_{1zz}^* = \bar{v}^z\psi_1^*u_{1z}^* + \nu\left[\frac{(\bar{u})_{rr}}{\bar{u}} + \frac{(\bar{u})_r}{r\bar{u}} - \frac{1}{r^2}\right]u_1^* + \left(\frac{1}{r} + \frac{(\bar{u})_r}{\bar{u}}\right)\bar{v}^r\psi_{1z}^*u_1^*, \tag{2.7}$$

$$\begin{aligned}
\omega_{1t}^* - \nu\omega_{1zz}^* &= \left(\frac{\bar{v}^r(\bar{\omega})_r}{\bar{\omega}} - \frac{\bar{v}^r}{r}\right)\psi_{1z}^*\omega_1^* + \bar{v}^z\psi_1^*\omega_{1z}^* + \frac{2}{r}\frac{(\bar{u})^2}{\bar{\omega}}u_1^*u_{1z}^*\\
&\quad +\nu\left[\frac{(\bar{\omega})_{rr}}{\bar{\omega}} + \frac{(\bar{\omega})_r}{r\bar{\omega}} - \frac{1}{r^2}\right]\omega_1^*.
\end{aligned} \tag{2.8}$$

Comparing the r and z components in (2.8), we know immediately that $\bar{v}^z$ is a constant. From (2.6), we have

$$\bar{\omega} = \bar{v}^r, \quad \omega_1^* = -\psi_{1zz}^*,$$

Comparing the r and z components in (2.8) again, it follows that

$$\frac{\bar{v}^r(\bar{\omega})_r}{\bar{\omega}} - \frac{\bar{v}^r}{r} = (\bar{\omega})_r - \frac{\bar{\omega}^r}{r}, \quad \frac{2}{r}\frac{(\bar{u})^2}{\bar{\omega}}, \quad \frac{(\bar{\omega})_{rr}}{\bar{\omega}} + \frac{(\bar{\omega})_r}{r\bar{\omega}} - \frac{1}{r^2}$$

are all constants. As a result,

$$\bar{\omega} = \bar{v}^r = \bar{u} = r,$$

together with (2.2) we obtain

$$\bar{v}^z = -2.$$

Consequently,

$$\begin{aligned} u(r,z,t) &= ru_1^*(z,t), && (2.9)\\ v^r(r,z,t) &= -r\psi_{1z}^*(z,t), && (2.10)\\ v^z(r,z,t) &= 2\psi_1^*(z,t), && (2.11)\\ \psi(r,z,t) &= r\psi_1^*(z,t), && (2.12)\\ \omega(r,z,t) &= r\omega_1^*(z,t). && (2.13) \end{aligned}$$

Plugging these into equations (1.13)-(1.17), one derives (1.19)-(1.22) of the $1D$ system.

Let $v_1^* = -(\psi_1^*)_z$. Integrating the ω_1^* equation with respect to z and using $-(\psi_1^*)_{zz} = \omega_1^*$, an equation for v_1^* is derived,

$$(v_1^*)_t + 2\psi_1^*(v_1^*)_z = \nu(v_1^*)_{zz} + u_1^{*2} - v_1^{*2} - c(t), \tag{2.14}$$

where $c(t)$ is an integration constant, which enforces that the mean value of v_1^* be zero. For instance, if ψ_1^* is periodic with period 1 in z, then $c(t) = 3\int_0^1 v_1^{*2}dz - \int_0^1 u_1^{*2}dz$. We point out that the equation for ω_1^* is equivalent to that for v_1^*. Using (1.19), (2.14) and the result in [12], we can get some regularity results for the $1D$ equations in the following lemmas.

Lemma 2.1. *Assume $u_1^*(z,0)$, $\psi_1^*(z,0)$, $\omega_1^*(z,0)$ are smooth and periodic functions with period 1, then $\psi_1^*(z,t)$, $\psi_{1z}^*(z,t)$, $u_1^*(z,t)$, $u_{1z}^*(z,t)$, and $\omega_1^*(z,t)$ are uniformly bounded.*

Lemma 2.2. *Assume $\phi^*(z,0)$ is a smooth and periodic function with period 1, then $\phi_0^*(z,t)$ and its derivatives are uniformly bounded.*

Proof. Multiplying (1.22) by ϕ_0^*, then integrating with respect to z over $[0,1]$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\phi_0^*\|_{L^2(0,1)}^2 + \frac{\gamma}{\eta^2}\int_0^1 (\phi_0^*)^4 dz + \gamma\|\phi_{0z}^*\|_{L^2(0,1)}^2 \\ = {}& -2\int_0^1 \psi_1^*\phi_{0z}^*\phi_0^* dz + \frac{\gamma}{\eta^2}\|\phi_0^*\|_{L^2(0,1)}^2 \le \frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + (\frac{C}{\gamma} + \frac{\gamma}{\eta^2})\|\phi_0^*\|_{L^2(0,1)}^2 \\ \le {}& \frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + (\frac{C}{\gamma} + \frac{\gamma}{\eta^2})\|\phi_0^*\|_{L^4(0,1)}^2 \\ \le {}& \frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + (\frac{C}{\gamma} + \frac{\gamma}{2\eta^2})\varepsilon\int_0^1 (\phi_0^*)^4 dz + C(\varepsilon). \end{aligned} \tag{2.15}$$

Multiplying (1.22) by ϕ_{0t}^*, and integrating with respect to z over $[0,1]$, it follows that

$$\|\phi_{0t}^*\|_{L^2(0,1)}^2 + \frac{d}{dt}\left[\frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + \frac{1}{4}\int_0^1 (\phi_0^*)^4 dz - \frac{\gamma}{2\eta^2}\|\phi_0^*\|_{L^2(0,1)}^2\right]$$

$$= \quad -2\int_0^1 \psi_1^*\phi_{0z}^*\phi_{0t}^* dz \le \|\phi_{0t}^*\|_{L^2(0,1)}^2 + C\|\phi_{0z}^*\|_{L^2(0,1)}^2 . \tag{2.16}$$

Multiplying (2.15) by a constant $\tilde{C}$, then summing up with (2.16), one arrives at

$$\frac{d}{dt}\left[\frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + \frac{1}{4}\int_0^1 (\phi_0^*)^4 dz + (\frac{\tilde{C}}{2} - \frac{\gamma}{2\eta^2})\|\phi_0^*\|_{L^2(0,1)}^2\right]$$
$$+(\frac{\tilde{C}\gamma}{2} - C)\|\phi_{0z}^*\|_{L^2(0,1)}^2 + \tilde{C}\left[\frac{\gamma}{\eta^2} - (\frac{C}{\gamma} + \frac{\gamma}{2\eta^2})\varepsilon\right]\int_0^1 (\phi_0^*)^4 dz \le \tilde{C}C(\varepsilon). \tag{2.17}$$

Choosing $\tilde{C}$ large enough and ε small enough that $\frac{\tilde{C}\gamma}{2} - C > 0$, $\frac{\gamma}{\eta^2} - (\frac{C}{\gamma} + \frac{\gamma}{2\eta^2})\varepsilon > 0$, we get

$$\frac{d}{dt}\left[\|\phi_0^*\|_{H^1(0,1)}^2 + \int_0^1 (\phi_0^*)^4 dz\right] + \|\phi_0^*\|_{H^1(0,1)}^2 + \int_0^1 (\phi_0^*)^4 dz \le C.$$

Applying Gronwall's lemma, we have that

$$\phi_0^*(z,t) \in L^\infty\left(0,\infty; H^1(0,1)\right). \tag{2.18}$$

□

Theorem 2.1. *Let u_1^*, ω_1^*, ψ_1^*, ϕ_0^* be the solution to the $1D$ equations* (1.19)-(1.22). *Define*

$$\begin{aligned} u(r,z,t) &= ru_1^*(z,t), \\ \omega(r,z,t) &= r\omega_1^*(z,t), \\ \psi(r,z,t) &= r\psi_1^*(z,t), \\ \phi(r,z,t) &= \phi_0^*(z,t). \end{aligned} \tag{2.19}$$

Then $(u(r,z,t)$, $\omega(r,z,t)$, $\psi(r,z,t)$, $\phi(r,z,t))$ is an exact solution to the $3D$ system (1.1)-(1.3).

Remark 2.1. *The exact solution $(u(r,z,t)$, $\omega(r,z,t)$, $\psi(r,z,t)$, $\phi(r,z,t))$ in Theorem 2.1 has infinite energy in $\mathbb{R}^3$.*

3 Preliminary work for energy estimates

In this section, using the solution to the $1D$ equations (1.19)-(1.22), we construct a global classical solution to the $3D$ axisymmetric system with finite energy . To do this, some preliminary work is necessary.

Denoting (u_1^*(z,t), ω_1^*(z,t), ψ_1^*(z,t), $\phi_0^*(z,t)$) as the solution to the $1D$ equations, ($\tilde{u}$(r,z,t), $\tilde{\omega}$(r,z,t), $\tilde{\psi}$(r,z,t), $\tilde{\phi}(r,z,t)$) as the solution to the 3D system. Further, we define

$$\tilde{u}_1 = \frac{\tilde{u}}{r}, \;\; \tilde{\omega}_1 = \frac{\tilde{\omega}}{r}, \;\; \tilde{\psi}_1 = \frac{\tilde{\psi}}{r}.$$

Let $\chi(r) = \chi_0(\frac{r}{R_0})$ be a smooth cut-off function, where $\chi_0(r) = 1$, if $0 \le r \le \frac{1}{2}$, and $\chi_0(r) = 0$, if $r \ge 1$. Our idea is to construct a global classical function $(ru_1, r\omega_1, r\psi_1, \phi_1)$, which is periodic in z direction with periods 1, such that

$$\begin{aligned}
\tilde{u}(r,z,t) &= r(u_1^*(z,t)\chi(r) + u_1(r,z,t)) = ru_1^*\chi + u, && (3.1)\\
\tilde{\omega}(r,z,t) &= r(\omega_1^*(z,t)\chi(r) + \omega_1(r,z,t)) = r\omega_1^*\chi + \omega, && (3.2)\\
\tilde{\psi}(r,z,t) &= r(\psi_1^*(z,t)\chi(r) + \psi_1(r,z,t)) = r\psi_1^*\chi + \psi, && (3.3)\\
\tilde{\phi}(r,z,t) &= \phi_0^*(z,t)\chi(r) + \phi_1(r,z,t), && (3.4)
\end{aligned}$$

is a global classical solution to the $3D$ axisymmetric system. From (3.1)-(3.4), we also know

$$\tilde{v}^r = -\tilde{\psi}_z = -r\psi_{1z}^*\chi - r\psi_{1z} = -r\psi_{1z}^*\chi + v^r, \tag{3.5}$$

$$\tilde{v}^z = \frac{(r\tilde{\psi})_r}{r} = r\psi_1^*\chi_r + 2\psi_1^*\chi + 2\psi_1 + r\psi_{1r} = r\psi_1^*\chi_r + 2\psi_1^*\chi + v^z. \tag{3.6}$$

Here v^r, v^z are considered as perturbation terms of radial and z-axis velocity components respectively.

From (1.19)-(1.22) of the $1D$ equations about $(u_1^*, \omega_1^*, \psi_1^*, \phi_0^*)$, and equations (1.13)-(1.17) of the $3D$ system on $(\tilde{u}, \tilde{\omega}, \tilde{\psi}, \tilde{\phi})$, one can derive the equations for $(u_1, \omega_1, \psi_1, \phi_1)$ as follows :

$$\begin{aligned}
u_{1t} + \tilde{v}^r u_{1r} + \tilde{v}^z u_{1z} &= \nu\Delta u_1 + 2\left(\tilde{\psi}_{1z}\tilde{u}_1 - \chi\psi_{1z}^* u_1^*\right) - \tilde{v}^r u_1^*\chi_r \\
&\quad -\chi\left(\left[r\chi_r + 2(\chi-1)\right]\psi_1^* + v^z\right)u_{1z}^* + \nu u_1^*\Delta_r\chi, && (3.7)\\
\omega_{1t} + \tilde{v}^r\omega_{1r} + \tilde{v}^z\omega_{1z} &= \nu\Delta\omega_1 + \left[(u_1^*\chi + u_1)^2\right]_z - (u_1^{*2})_z\chi - \tilde{v}^r\omega_1^*\chi_r \\
&\quad -\chi([r\chi_r + 2(\chi-1)]\psi_1^* + v^z)\omega_{1z}^* + \nu\omega_1^*\Delta_r\chi \\
&\quad +\frac{\lambda}{r}\left(\phi_{0z}^*\chi + \phi_{1z}\right)\left[(\nabla^2(\phi_0^*\chi))_r + (\nabla^2\phi_1)_r\right] \\
&\quad -\frac{\lambda}{r}(\phi_0^*\chi_r + \phi_{1r})\left[(\nabla^2(\phi_0^*\chi))_z + (\nabla^2\phi_1)_z\right], && (3.8)\\
\phi_{1t} + \tilde{v}^r\phi_{1r} + \tilde{v}^z\phi_{1z} &= \gamma\nabla^2\phi_1 - \frac{\gamma}{\eta^2}\left(\phi_1^3 + 3\phi_0^*\phi_1^2\chi + 3\phi_0^{*2}\phi_1\chi^2 - \phi_1\right) \\
&\quad +\gamma\phi_0^*\left(\chi_{rr} + \frac{\chi_r}{r}\right) - \frac{\gamma}{\eta^2}\phi_0^{*3}\left(\chi^3 - \chi\right) - \phi_0^*\tilde{v}^r\chi_r \\
&\quad +2\psi_1^*\phi_{0z}^*\chi - \phi_{0z}^*\tilde{v}^z\chi. && (3.9)
\end{aligned}$$

From the basic energy law (1.4), we know actually

$$\mathbf{u} \in L^\infty(0,\infty; L^2(\Omega)), \quad \phi_1 \in L^\infty(0,\infty; H^1(\Omega)). \tag{3.10}$$

Therefore, from the result in [26] and standard bootstrap arguments, all we need is to prove

$$\mathbf{u} \in L^\infty(0,\infty; H^1(\Omega)), \quad \phi_1 \in L^\infty(0,\infty; H^2(\Omega)). \tag{3.11}$$

We assume $\mathbf{u}$ and $\nabla\tilde{\phi}$ decay sufficiently fast in the infinity $r = \infty$, and has periodic boundary conditions in z direction, with period 1, then so do the perturbation terms u_1, ω_1, ψ_1 and ϕ_1. Due to the periodicity of our boundary conditions in z. Using the boundary conditions, we can get the following useful lemmas.

Lemma 3.1.

$$\|u_1\| \leq \|ru_{1r}\|.$$

Proof.

$$\begin{aligned}\int_0^1\int_0^\infty (u_1)^2 rdrdz &= \int_0^1\int_0^\infty (u_1)^2 d(\frac{r^2}{2})dz = -\int_0^1\int_0^\infty \frac{r^2}{2} 2u_1u_{1r}drdz \\ &= -\int_0^1\int_0^\infty u_1(u_{1r}r)rdrdz \leq \|u_1\|\|ru_{1r}\|,\end{aligned}$$

hence $\|u_1\| \leq \|ru_{1r}\|$. □

Lemma 3.2.

$$\|r\nabla u_1\| \leq \|r\Delta u_1\|.$$

Proof.

$$\begin{aligned}-\int_0^1\int_0^\infty u_1 \triangle u_1 r^2 rdrdz &= -\int_0^1\int_0^\infty u_1u_{1rr}r^2rdrdz - 3\int_0^1\int_0^\infty u_1u_{1r}r^2drdz \\ &\quad -\int_0^1\int_0^\infty u_1u_{1zz}r^2rdrdz \\ &= \int_0^1\int_0^\infty (u_{1r}r)^2rdrdz + \int_0^1\int_0^\infty (u_{1z}r)^2rdrdz.\end{aligned}$$

On the other hand,

$$-\int_0^1\int_0^\infty u_1\Delta u_1 r^2 rdrdz \leq \|ru_1\|\|r\Delta u_1\|,$$

hence

$$\|r\nabla u_1\|^2 \leq \|ru_1\|\|r\Delta u_1\| \leq \|r\nabla u_1\|\|r\Delta u_1\|.$$

□

Lemma 3.3.

$$\|ru_{1zz}\| + \|ru_{1zr}\| + \|ru_{1rr}\| + 3\|u_{1r}\| \leq \|r\Delta u_1\|.$$

Proof.

$$\begin{aligned}\int_0^1\int_0^\infty \Delta u_1 u_{1zz} r^2 rdrdz &= \|ru_{1zz}\|^2 + 3\int_0^1\int_0^\infty u_{1r}u_{1zz}r^2drdz + \int_0^1\int_0^\infty u_{1rr}u_{1zz}r^2rdrdz \\ &= \|ru_{1zz}\|^2 - 3\int_0^1\int_0^\infty u_{1zr}u_{1z}r^2drdz + \|ru_{1zr}\|^2 \\ &\quad +3\int_0^1\int_0^\infty u_{1zr}u_{1z}r^2drdz \\ &= \|ru_{1zz}\|^2 + \|ru_{1zr}\|^2.\end{aligned}$$

On the other hand,

$$\int_0^1\int_0^\infty \Delta u_1 u_{1zz} r^2 rdrdz \leq \|r\Delta u_1\|\|ru_{1zz}\| \leq \frac{1}{2}\|ru_{1zz}\|^2 + \frac{1}{2}\|r\Delta u_1\|^2,$$

hence

$$\|ru_{1zz}\|^2 + \|ru_{1zr}\|^2 \leq \frac{1}{2}\|ru_{1zz}\|^2 + \frac{1}{2}\|r\Delta u_1\|^2,$$

which implies

$$\|ru_{1zz}\|^2 + \|ru_{1zr}\|^2 \leq \|r\Delta u_1\|^2.$$

Similarly,

$$\begin{aligned}\int_0^1\int_0^\infty \Delta u_1 \Delta_r u_1 r^2 r dr dz &= \|r\Delta_r u_1\|^2 + \int_0^1\int_0^\infty u_{1zz}u_{1rr}r^2 r dr dz + 3\int_0^1\int_0^\infty u_{1r}u_{1zz}r^2 dr dz \\ &= \|r\Delta_r u_1\|^2 + \|ru_{1zr}\|^2,\end{aligned}$$

therefore

$$\|\Delta_r u_1 r\|^2 + \|u_{1zr}r\|^2 \leq \|\Delta u_1 r\|\|\Delta_r u_1 r\| \leq \frac{1}{2}\|\Delta u_1 r\|^2 + \frac{1}{2}\|\Delta_r u_1 r\|^2,$$

which tells us

$$\|r\Delta_r u_1\|^2 + \|ru_{1zr}\|^2 \leq \|r\Delta u_1\|^2.$$

Since

$$\begin{aligned}\|r\Delta_r u_1\|^2 &= \int_0^1\int_0^\infty (\Delta_r u_1)^2 r^2 r dr dz \\ &= \|ru_{1rr}\|^2 + 9\int_0^1\int_0^\infty (u_{1r})^2 r dr dz + 6\int_0^1\int_0^\infty u_{1rr}u_{1r}r^2 dr dz \\ &= \|ru_{1rr}\|^2 + 3\|ru_{1r}\|^2 + 3\int_0^1\int_0^\infty [(ru_{1r})^2]_r dr dz \\ &= \|ru_{1rr}\|^2 + 3\|ru_{1r}\|^2,\end{aligned}$$

we finish the proof. □

Analogously, we can get

Lemma 3.4.

$$\|\psi_{1z}\| \leq \|r\psi_{1zr}\|, \ \ \|\psi_{1r}\| \leq \|r\psi_{1rr}\|, \ \ \|\Delta\psi_1\| \leq \|r\nabla(\Delta\psi_1)\|.$$

Lemma 3.5.

$$\|r\psi_{1zz}\| + \|r\psi_{1zr}\| + \|r\Delta_r\psi_1\| \leq 2\|r\Delta\psi_1\|.$$

Lemma 3.6.

$$\|r\psi_{1rrz}\| + \|\psi_{1rz}\| + \|r\psi_{1zzr}\| + \|r\psi_{1rrr}\| + \|r\psi_{1zzz}\| \leq 3\|r\nabla(\Delta\psi_1)\|.$$

Lemma 3.7.

$$\|\phi_{1zz}\| + \|\phi_{1zr}\| + \|\phi_{1rr}\| \leq \|\nabla^2\phi_1\|.$$

Lemma 3.8.

$$\|\phi_{1rrz}\| + \|\phi_{1zzr}\| + \|\phi_{1rrr}\| + \|\phi_{1zzz}\| \leq 3\|\nabla(\nabla^2\phi_1)\|.$$

In all, we conclude from the lemmas above, that to prove (3.11), it is sufficient to prove

$$r\nabla u_1, \ r\Delta\psi_1, \nabla^2\phi_1 \in L^\infty\left(0,\infty, L^2(\Omega)\right) \tag{3.12}$$

4 Result for large viscosity case

In this section we are giving the proof of Theorem 1.1.

Proof. We begin to do estimates term by term, where Hölder inequality and Sobolev interpolation inequalities are used at times.

Multiplying (3.7) with $-r^2\Delta u_1$, then integrating over Ω, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left[(u_{1z})^2+(u_{1r})^2\right]r^2rdrdz\\
=\;& -\nu\int(\Delta u_1)^2r^2rdrdz+\int\tilde{v}^r u_{1r}\Delta u_1 r^2rdrdz+\int\tilde{v}^z u_{1z}\Delta u_1 r^2rdrdz\\
&-2\int\tilde{v}^r u_1\Delta u_1 r^2rdrdz-2\int\chi u_1^*\psi_{1z}\Delta u_1r^2rdrdz-2\int(\chi^2-\chi)\psi_{1z}^*u_1^*\Delta u_1r^2rdrdz\\
&+\int\tilde{v}^r u_1^*\chi_r\Delta u_1r^2rdrdz+\int\chi\left([r\chi_r+2(\chi-1)]\psi_1^*+v^z\right)u_{1z}^*\Delta u_1r^2rdrdz\\
&-\nu\int u_1^*\Delta_r\chi\Delta u_1r^2rdrdz\\
\equiv\;& -\nu\int(\Delta u_1)^2r^2rdrdz+I_a+I_b+I_c+I_d+I_e+I_f+I_g+I_h. \qquad (4.1)
\end{aligned}$$

Estimates for u_1 equation

From (3.5), Lemma 2.1 and 3.2, we know

$$\begin{aligned}
|I_a| &= \left|\int\left(-r\psi_{1z}^*\chi-r\psi_{1z}\right)u_{1r}\Delta u_1r^2rdrdz\right|\\
&\le C\left\|\psi_{1z}^*\right\|_{L^\infty}\|ru_{1r}\|\|r\Delta u_1\|+\left|\int\psi_{1z}u_{1r}\Delta u_1r^3rdrdz\right|\\
&\le C\|r\Delta u_1\|^2+\left|\int\psi_{1z}u_{1r}\Delta u_1r^3rdrdz\right|,
\end{aligned}$$

where the second term

$$\begin{aligned}
&\left|\int\psi_{1z}u_{1r}\Delta u_1r^3rdrdz\right|\\
\le\;& \|r\psi_{1z}\|_{L^4}\|ru_{1r}\|_{L^4}\|r\Delta u_1\| \;\le\; \|r\psi_{1z}\|^{\frac14}\|\nabla(r\psi_{1z})\|^{\frac34}\|ru_{1r}\|^{\frac14}\|\nabla(ru_{1r})\|^{\frac34}\|r\Delta u_1\|\\
\le\;& \|r\psi_{1z}\|^{\frac14}\left(\|r\psi_{1zz}\|+\|r\psi_{1zr}+\psi_{1z}\|\right)^{\frac34}\|ru_{1r}\|^{\frac14}\left(\|ru_{1rz}\|+\|ru_{1rr}+u_{1r}\|\right)^{\frac34}\|r\Delta u_1\|\\
\le\;& C\left(\|r\psi_{1zz}\|+\|r\psi_{1zr}\|\right)^{\frac34}\|ru_{1r}\|^{\frac14}\left(\|ru_{1rz}\|+\|ru_{1rr}\|\right)^{\frac34}\|r\Delta u_1\|\\
\le\;& C\|r\Delta\psi_1\|^{\frac34}\|ru_{1r}\|^{\frac14}\|r\Delta u_1\|^{\frac74} \;\le\; C\|r\Delta\psi_1\|^{\frac67}\|r\Delta u_1\|^2+C\|ru_{1r}\|^2\\
\le\;& C\left(\|r\Delta\psi_1\|^2+1\right)\|r\Delta u_1\|^2+C\|ru_{1r}\|^2\\
\le\;& C\left(\|r\Delta\psi_1\|^2+1\right)\|r\Delta u_1\|^2,
\end{aligned}$$

here we used (3.10) and Lemma 3.2, Lemma 3.3, Lemma 3.5 and Young's Inequality. As a result,

$$|I_a| \le C\left(\|r\Delta\psi_1\|^2+1\right)\|r\Delta u_1\|^2. \qquad (4.2)$$

Similar to I_a, we get

$$|I_b| = \left| \int \tilde{v}^z u_{1z} \Delta u_1 r^2 r dr dz \right| \leq C \left(\|r\Delta\psi_1\|^2 + 1 \right) \|r\Delta u_1\|^2. \tag{4.3}$$

Due to (3.5), (3.10), we have

$$|I_c| = 2 \left| \int \tilde{v}^r u_1 \Delta u_1 r^2 r dr dz \right| \leq C \left(\|r\Delta u_1\|^2 + 1 \right) + 2\|r\psi_{1z}\|_{L^4} \|ru_1\|_{L^4} \|r\Delta u_1\|,$$

Similar to I_a,

$$|I_c| \leq C \left(\|r\Delta\psi_1\|^2 + 1 \right) \|r\Delta u_1\|^2 + C. \tag{4.4}$$

For estimates from I_d to I_h, with the help of (3.5), (1.4), (3.10) and Lemma 2.1, we obtain

$$|I_d| = \left| -2 \int \chi u_1^* \psi_{1z} \Delta u_1 r^2 r dr dz \right| \leq 2\|u_1^*\|_{L^\infty} \|r\psi_{1z}\| \|r\Delta u_1\| \leq C\|r\Delta u_1\|^2 + C. \tag{4.5}$$

$$\begin{aligned} |I_e| &= \left| -2 \int (\chi^2 - \chi) \psi_{1z}^* u_1^* \Delta u_1 r^2 r dr dz \right| \leq C\|\psi_{1z}^*\|_{L^\infty} \|u_1^*\|_{L^\infty} \|r\Delta u_1\| \\ &\leq C\|r\Delta u_1\|^2 + C. \end{aligned} \tag{4.6}$$

$$|I_f| = \left| \int \tilde{v}^r u_1^* \chi_r \Delta u_1 r^2 r dr dz \right| \leq C\|u_1^*\|_{L^\infty} \|\tilde{v}^r\| \|r\Delta u_1\| \leq C\|r\Delta u_1\|^2 + C. \tag{4.7}$$

$$\begin{aligned} |I_g| &= \left| \int \chi \left([r\chi_r + 2(\chi - 1)]\psi_1^* + v^z \right) u_{1z}^* \Delta u_1 r^2 r dr dz \right| \\ &\leq C\|\psi_1^*\|_{L^\infty} \|u_{1z}^*\|_{L^\infty} \|r\Delta u_1\| + C\|u_{1z}^*\|_{L^\infty} \|v^z\| \|r\Delta u_1\| \\ &\leq C\|r\Delta u_1\|^2 + C. \end{aligned} \tag{4.8}$$

$$|I_h| = \left| -\nu \int u_1^* \Delta_r \chi \Delta u_1 r^2 r dr dz \right| \leq \frac{\nu}{2} \|r\Delta u_1\|^2 + C.$$

Therefore,

$$\frac{1}{2} \frac{d}{dt} \|r\nabla u_1\|^2 \leq - \left[\frac{\nu}{2} - C \left(\|r\Delta\psi_1\|^2 + 1 \right) \right] \|r\Delta u_1\|^2 + C. \tag{4.9}$$

Multiplying (3.8) with $-r^2\Delta\psi_1$, and integrating over Ω, since $-\omega_1 = \Delta\psi_1 + (\Delta_r\phi)\psi_1^*$, by (1.16), we know the fact that

$$\begin{aligned} & - \int \tilde{v}^r (\Delta\psi_1)_r \Delta\psi_1 r^2 r dr dz - \int \tilde{v}^z (\Delta\psi_1)_z \Delta\psi_1 r^2 r dr dz \\ = \; & - \frac{1}{2} \int \tilde{v}^r \left[(\Delta\psi_1)^2 \right]_r r^3 dr dz - \frac{1}{2} \int \tilde{v}^z \left[(\Delta\psi_1)^2 \right]_z r^3 dr dz \\ = \; & \frac{1}{2} \int (\Delta\psi_1)^2 \left[\left(r^3 \tilde{v}^r \right)_r + (r^3 \tilde{v}^z)_z \right] dr dz \\ = \; & \frac{1}{2} \int (\Delta\psi_1)^2 \left[(r\tilde{v}^r)_z + (r\tilde{v}^z)_z + 2\tilde{v}^r \right] r^2 dr dz \\ = \; & \int (\Delta\psi_1)^2 \tilde{v}^r r^2 dr dz, \end{aligned}$$

consequently, one arrives at

$$\frac{1}{2} \frac{d}{dt} \int (\Delta\psi_1)^2 r^2 r dr dz$$

$$
\begin{aligned}
= \;& -\nu \int \left[((\Delta\psi_1)_r)^2 + ((\Delta\psi_1)_z)^2\right] r^3 drdz - \int \Delta_r\chi\psi_{1t}^*\Delta\psi_1 r^3 drdz + \int (\Delta\psi_1)^2 \tilde{v}^r r^2 drdz \\
& - \int \tilde{v}^r(\Delta_r\chi)_r\psi_1^*\Delta\psi_1 r^3 drdz - \int \tilde{v}^z\Delta_r\chi\psi_{1z}^*\Delta\psi_1 r^3 drdz + \nu \int \Delta\left(\Delta_r\chi\psi_1^*\right)\Delta\psi_1 r^3 drdz \\
& -2\int \left[u_1^*u_{1z}^*(\chi^2-\chi) + u_{1z}^*\chi u_1 + u_1^*\chi u_{1z}\right]\Delta\psi_1 r^3 drdz - 2\int u_1u_{1z}\Delta\psi_1 r^3 drdz \\
& + \int \tilde{v}^r\omega_1^*\chi_r\Delta\psi_1 r^3 drdz + \int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right)\omega_{1z}^*\Delta\psi_1 r^3 drdz \\
& -\nu\int \omega_1^*\Delta_r\chi\Delta\psi_1 r^3 drdz - \lambda\int (\phi_{0z}^*\chi + \phi_{1z})\left[\left(\nabla^2(\phi_0^*\chi)\right)_r + (\nabla^2\phi_1)_r\right]\Delta\psi_1 r^2 drdz \\
& +\lambda\int (\phi_0^*\chi_r + \phi_{1r})\left[\left(\nabla^2(\phi_0^*\chi)\right)_z + (\nabla^2\phi_1)_z\right]\Delta\psi_1 r^2 drdz \\
\equiv \;& -\nu\int \left[\nabla(\Delta\psi_1)\right]^2 r^3 drdz + J_a + J_b + J_c + J_d + J_e + J_f + J_g \\
& + J_h + J_i + J_j + J_k + J_l. \qquad (4.10)
\end{aligned}
$$

Estimates for ω_1 equation

(1.21) infers $(\psi_{1t}^*)_{zz} = -\omega_{1t}^*$, By (1.20) and Lemma 2.1, we conclude $\|\psi_{1t}^*\|_{L^\infty} \le C$. As a result,

$$
|J_a| = \left|\int \Delta_r\chi\psi_{1t}^*\Delta\psi_1 r^3 drdz\right| \le C\|r\Delta\psi_1\| \le C\left(\|r\Delta\psi_1\|^2 + 1\right). \qquad (4.11)
$$

By (3.5), Lemma 2.1, 3.4, 3.5, 3.6, and Young's Inequality, it follows that

$$
\begin{aligned}
|J_b| &= \left|\int (\Delta\psi_1)^2\tilde{v}^r r^2 drdz\right| \qquad (4.12) \\
&= \left|\int -\psi_{1z}^*\chi(\Delta\psi_1)^2 r^3 drdz - \int \psi_{1z}(\Delta\psi_1)^2 r^3 drdz\right| \\
&\le C\|r\Delta\psi_1\|^2 + \|\psi_{1z}\|_{L^4}\|r\Delta\psi_1\|_{L^4}\|r\Delta\psi_1\| \\
&\le C\|r\Delta\psi_1\|^2 + C\|\psi_{1z}\|^{\frac14}\|\nabla\psi_{1z}\|^{\frac34}\|r\Delta\psi_1\|^{\frac14}\|\nabla(r\Delta\psi_1)\|^{\frac34}\|r\Delta\psi_1\| \\
&\le C\|r\Delta\psi_1\|^2 + C\|r\psi_{1rz}\|^{\frac14}\left(\|r\psi_{1zzr}\| + \|r\psi_{1rzz}\|\right)^{\frac34}\|r\Delta\psi_1\|^{\frac54}\|r\nabla(\Delta\psi_1)\|^{\frac34} \\
&\le C\|r\Delta\psi_1\|^2 + C\|r\Delta\psi_1\|^{\frac32}\|r\nabla(\Delta\psi_1)\|^{\frac32} \\
&\le C\|r\Delta\psi_1\|^2 + C\|r\Delta\psi_1\|\|r\nabla(\Delta\psi_1)\|^2 \\
&\le C\|r\Delta\psi_1\|^2 + C\left(\|r\Delta\psi_1\|^2 + 1\right)\|r\nabla(\Delta\psi_1)\|^2. \qquad (4.13)
\end{aligned}
$$

Using basic energy law (1.4) and Lemma 2.1, 3.2, one can get estimates of J_c to J_f as

$$
|J_c| = \left|-\int \tilde{v}^r(\Delta_r\chi)_r\psi_1^*\Delta\psi_1 r^3 drdz\right| \le C\|\tilde{v}^r\|\|r\Delta\psi_1\| \le C\left(\|r\Delta\psi_1\|^2 + 1\right). \qquad (4.14)
$$

$$
|J_d| = \left|-\int \tilde{v}^z\Delta_r\chi\psi_{1z}^*\Delta\psi_1 r^3 drdz\right| \le C\|\tilde{v}^z\|\|r\Delta\psi_1\| \le C\left(\|r\Delta\psi_1\|^2 + 1\right). \qquad (4.15)
$$

$$
\begin{aligned}
|J_e| &= \left|\nu\int \Delta(\Delta_r\chi\psi_1^*)\Delta\psi_1 r^3 drdz\right| \\
&\le \nu\left|\int \left[(\Delta_r\chi\psi_1^*)_{rr} + (\Delta_r\chi\psi_1^*)_{zz}\right](\Delta\psi_1) r^3 drdz + 3\int (\Delta_r\chi\psi_1^*)_r(\Delta\psi_1) r^2 drdz\right| \\
&= \left|-\nu\int \Delta_r\chi\psi_{1z}^*(\Delta\psi_1)_z r^3 drdz - \nu\int (\Delta_r\chi)_r\psi_1^*(\Delta\psi_1)_r r^3 drdz\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \nu(\|(r^2\Delta_r\chi)\psi_{1z}^*\|_{L^\infty} + \|r(\Delta_r\chi)_r\psi_1^*\|_{L^\infty})\|r\nabla\Delta\psi_1\| \\
&\leq \frac{\nu}{2}\|r\nabla\Delta\psi_1\|^2 + C. \qquad (4.16)
\end{aligned}
$$

$$
\begin{aligned}
|J_f| &= \left|-2\int \left[u_1^*u_{1z}^*(\chi^2-\chi) + u_{1z}^*\chi u_1 + u_1^*\chi u_{1z}\right]\Delta\psi_1 r^3 drdz\right| \\
&\leq C\|r\Delta\psi_1\| + C\|ru_1\|\|r\Delta\psi_1\| + C\|ru_{1z}\|\|r\Delta\psi_1\| \\
&\leq C\|ru_{1z}\|^2 + C\|r\Delta\psi_1\|^2 + C \\
&\leq C\|r\Delta u_1\|^2 + C\|r\Delta\psi_1\|^2 + C. \qquad (4.17)
\end{aligned}
$$

By Lemma 3.1, 3.2, 3.5, we have

$$
\begin{aligned}
|J_g| &= \left|-2\int u_1u_{1z}\Delta\psi_1 r^3 drdz\right| \\
&\leq C\|u_1\|_{L^4}\|ru_{1z}\|_{L^4}\|r\Delta\psi_1\| \leq C\|u_1\|^{\frac{1}{4}}\|\nabla u_1\|^{\frac{3}{4}}\|ru_{1z}\|^{\frac{1}{4}}\|\nabla(ru_{1z})\|^{\frac{3}{4}}\|r\Delta\psi_1\| \\
&\leq C\|ru_{1r}\|^{\frac{1}{4}}\left(\|ru_{1rr}\| + \|ru_{1zz}\|\right)^{\frac{3}{4}}\|ru_{1z}\|^{\frac{1}{4}}\left(\|ru_{1rz}\| + \|ru_{1zz}\|\right)^{\frac{3}{4}}\|r\Delta\psi_1\| \\
&\leq C\|ru_{1r}\|^{\frac{1}{4}}\|ru_{1z}\|^{\frac{1}{4}}\|r\Delta u_1\|^{\frac{3}{2}}\|r\Delta\psi_1\| \\
&\leq C\|ru_{1r}\|^{\frac{1}{2}}\|ru_{1z}\|^{\frac{1}{2}}\|r\Delta u_1\| + C\|r\Delta u_1\|^2\|r\Delta\psi_1\|^2 \\
&\leq C\left(\|ru_{1r}\|^2 + \|ru_{1z}\|^2\right) + C\|r\Delta u_1\|^2\left(\|r\Delta\psi_1\|^2 + 1\right) \\
&\leq C\|r\Delta u_1\|^2\left(\|r\Delta\psi_1\|^2 + 1\right). \qquad (4.18)
\end{aligned}
$$

Lemma 2.1 and basic energy law (1.4) tell us

$$
|J_h| = \left|\int \tilde{v}^r\omega_1^*\chi_r\Delta\psi_1 r^3 drdz\right| \leq C\|\tilde{v}^r\|\|r\Delta\psi_1\| \leq C\|r\Delta\psi_1\|^2 + C. \qquad (4.19)
$$

$$
\begin{aligned}
|J_i| &= \left|\int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right)\omega_{1z}^*\Delta\psi_1 r^3 drdz\right| \\
&\leq \left|\int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v_z^z\right)\omega_1^*\Delta\psi_1 r^3 drdz\right| \\
&\quad + \left|\int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right)\omega_1^*(\Delta\psi_1)_z r^3 drdz\right| \\
&\leq C\|r\Delta\psi_1\| + C\|r\nabla(\Delta\psi_1)\| + \left|\int \chi v_z^z\omega_1^*\Delta\psi_1 r^3 drdz\right|, \qquad (4.20)
\end{aligned}
$$

where the estimate of the third term can be derived from (3.6) and Lemma 3.5.

$$
\begin{aligned}
\left|\int \chi v_z^z\omega_1^*\Delta\psi_1 r^3 drdz\right| &= \left|\int \chi\left(2\psi_{1z} + r\psi_{1rz}\right)\omega_1^*\Delta\psi_1 r^3 drdz\right| \\
&\leq C\|r\Delta\psi_1\| + C\|r\psi_{1rz}\|\|r\Delta\psi_1\| \leq C\|r\Delta\psi_1\| + C\|r\Delta\psi_1\|^2,
\end{aligned}
$$

consequently,

$$
|J_i| \leq C\|r\Delta\psi_1\|^2 + C\|r\nabla(\Delta\psi_1)\|^2 + C. \qquad (4.21)
$$

Lemma 2.1 infers that

$$
|J_j| = \left|\nu\int \omega_1^*\Delta_r\chi\Delta\psi_1 r^3 drdz\right| \leq C\nu\|r\Delta\psi_1\| \leq \frac{\nu}{2}\|r\Delta\psi_1\|^2 + C. \qquad (4.22)
$$

For the estimate of J_k, using basic energy law (1.4) and Lemma 2.2, one derives

$$
\begin{aligned}
|J_k| &= \left| -\lambda \int (\phi_{0z}^*\chi + \phi_{1z}) \left[\left(\nabla^2(\phi_0^*\chi)\right)_r + (\nabla^2\phi_1)_r \right] \Delta\psi_1 r^2 drdz \right| \\
&\leq \lambda \int |\phi_{0z}^*\chi \left(\nabla^2(\phi_0^*\chi)\right)_r \Delta\psi_1| r^2 drdz + \lambda \int |\phi_{1z} \left(\nabla^2(\phi_0^*\chi)\right)_r \Delta\psi_1| r^2 drdz \\
&\quad + \lambda \int |\phi_{0z}^*\chi (\nabla^2\phi_1)_r \Delta\psi_1| r^2 drdz + \lambda \int |\phi_{1z} \left(\nabla^2\phi_1\right)_r \Delta\psi_1| r^2 drdz \\
&\leq C\left(1 + \|\phi_{1z}\| + \|(\nabla^2\phi_1)_r\|\right) \|r\Delta\psi_1\| + C\|\phi_{1z}\|_{L^4} \|(\nabla^2\phi_1)_r\| \|r\Delta\psi_1\|_{L^4} \\
&\leq C\left(\|r\Delta\psi_1\|^2 + 1\right) + C\|\phi_{1z}\|_{L^4} \|(\nabla^2\phi_1)_r\| \|r\Delta\psi_1\|_{L^4},
\end{aligned}
$$

where the estimate of the second term is obtained by using (1.4), Lemma 3.6, 3.7, and Young's Inequality,

$$
\begin{aligned}
&\|\phi_{1z}\|_{L^4} \|(\nabla^2\phi_1)_r\| \|r\Delta\psi_1\|_{L^4} \\
\leq\ & C\|\nabla^2\phi_1\|^{\frac{3}{4}} \|\nabla(\nabla^2\phi_1)\| \|r\Delta\psi_1\|^{\frac{1}{4}} \|\nabla(r\Delta\psi_1)\|^{\frac{3}{4}} \\
\leq\ & C\left(\nu^{\frac{3}{16}} \|\nabla^2\phi_1\|^{\frac{3}{4}} \|r\nabla(\Delta\psi_1)\|^{\frac{3}{4}}\right) \left(\nu^{\frac{1}{16}} \|r\Delta\psi_1\|^{\frac{1}{4}}\right) \left(\frac{1}{\nu^{\frac{1}{4}}} \|\nabla(\nabla^2\phi_1)\|\right) \\
\leq\ & C\nu^{\frac{1}{2}} \|r\Delta\psi_1\|^2 + C\nu^{\frac{1}{2}} \|\nabla^2\phi_1\|^2 \|r\nabla(\Delta\psi_1)\|^2 + \frac{C}{\nu^{\frac{1}{2}}} \|\nabla(\nabla^2\phi_1)\|^2,
\end{aligned}
$$

therefore,

$$
|J_k| \leq C\nu^{\frac{1}{2}} \left(1 + \|\nabla^2\phi_1\|^2\right) \|r\nabla(\Delta\psi_1)\|^2 + \frac{C}{\nu^{\frac{1}{2}}} \|\nabla(\nabla^2\phi_1)\|^2 + C\left(\|r\Delta\psi_1\|^2 + 1\right). \tag{4.23}
$$

Similarly, we have

$$
\begin{aligned}
|J_l| &= \left| \lambda \int (\phi_0^*\chi_r + \phi_{1r}) \left[\left(\nabla^2(\phi_0^*\chi)\right)_z + (\nabla^2\phi_1)_z \right] \Delta\psi_1 r^2 drdz \right| \\
&\leq C\nu^{\frac{1}{2}} \|\nabla^2\phi_1\|^2 \|r\nabla(\Delta\psi_1)\|^2 + C\nu^{-\frac{1}{2}} \|\nabla^2\phi_1\|^2 \|\nabla(\nabla^2\phi_1)\|^2 \\
&\quad + C\left(\|r\Delta\psi_1\|^2 + \|\nabla(\nabla^2\phi_1)\|^2 + 1\right).
\end{aligned} \tag{4.24}
$$

In sum,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|r\Delta\psi_1\|^2 \leq\ & -\left[\frac{\nu}{2} - \nu^{\frac{1}{2}} \|\nabla^2\phi_1\|^2 - C\left(\|r\Delta\psi_1\|^2 + 1\right)\right] \|r\nabla(\Delta\psi_1)\|^2 \\
& + C\nu^{-\frac{1}{2}} \|\nabla^2\phi_1\|^2 \|\nabla(\nabla^2\phi_1)\|^2 + C\left(\|r\Delta\psi_1\|^2 + 1\right) \|r\Delta u_1\|^2 \\
& + C\left(\|r\Delta\psi_1\|^2 + \|\nabla(\nabla^2\phi_1)\|^2 + \|r\Delta u_1\|^2 + 1\right).
\end{aligned} \tag{4.25}
$$

Multiplying (3.9) with $\nabla^2(\nabla^2\phi_1)$, then integrating over Ω, one arrives at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla^2\phi_1|^2 rdrdz \\
=\ & -\gamma \int \left[\nabla(\nabla^2\phi_1)\right]^2 rdrdz - \int \tilde{v}^r \phi_{1r} \nabla^2(\nabla^2\phi_1) rdrdz - \int \tilde{v}^z \phi_{1z} \nabla^2(\nabla^2\phi_1) rdrdz \\
& - \frac{\gamma}{\eta^2} \int \left(\phi_1^3 + 3\phi_0^*\phi_1^2\chi + 3\phi_0^{*2}\phi_1\chi^2 - \phi_1\right) \nabla^2(\nabla^2\phi_1) rdrdz
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma \int \phi_0^* \left(\chi_{rr} + \frac{\chi_r}{r}\right) \nabla^2(\nabla^2\phi_1) rdrdz - \int \frac{\gamma}{\eta^2}\phi_0^{*3}\left(\chi^3 - \chi\right)\nabla^2(\nabla^2\phi_1) rdrdz \\
&- \int \phi_0^* \tilde{v}^r \chi_r \nabla^2(\nabla^2\phi_1) rdrdz + 2\int \psi_1^* \phi_{0z}^* \chi \nabla^2(\nabla^2\phi_1) rdrdz - \int \phi_{0z}^* \tilde{v}^z \chi \nabla^2(\nabla^2\phi_1) rdrdz. \\
\equiv\ & -\gamma \int \left[\nabla(\nabla^2\phi_1)\right]^2 rdrdz + K_a + K_b + K_c + K_d + K_e + K_f + K_g + K_h. \qquad (4.26)
\end{aligned}
$$

Estimates for ϕ_1 equation

$$
\begin{aligned}
K_a &= \int \left[(\tilde{v}^r)_r \phi_{1r} (\nabla^2\phi_1)_r + \tilde{v}^r \phi_{1rr}(\nabla^2\phi_1)_r + (\tilde{v}^r)_z \phi_{1r}(\nabla^2\phi_1)_z + \tilde{v}^r \phi_{1rz}(\nabla^2\phi_1)_z\right] rdrdz \\
&\equiv a_1 + b_1 + c_1 + d_1,
\end{aligned}
$$

where estimates of a_1 to d_1 can be derived through (3.5), basic energy law (1.4), Lemma 2.1, 3.4, 3.5, 3.7, 3.8, and Young's Inequality.

$$
\begin{aligned}
|a_1| &= \left| \int (\tilde{v}^r)_r \phi_{1r} (\nabla^2\phi_1)_r rdrdz \right| \\
&\le C\|\phi_{1r}\|\|\nabla(\nabla^2\phi_1)\| + C\|\psi_{1z} + r\psi_{1rz}\|_{L^4}\|\phi_{1r}\|_{L^4}\|\nabla(\nabla^2\phi_1)\| \\
&\le C\|\nabla(\nabla^2\phi_1)\| + C\|\psi_{1z} + r\psi_{1rz}\|^{\frac{1}{4}}\|\nabla(\psi_{1z} + r\psi_{1rz})\|^{\frac{3}{4}}\|\phi_{1r}\|^{\frac{1}{4}}\|\nabla(\phi_{1r})\|^{\frac{3}{4}}\|\nabla(\nabla^2\phi_1)\| \\
&\le C\|\nabla(\nabla^2\phi_1)\| + C\|r\Delta\psi_1\|^{\frac{1}{4}}\|r\nabla(\Delta\psi_1)\|^{\frac{3}{4}}\|\nabla^2\phi_1\|^{\frac{3}{4}}\|\nabla(\nabla^2\phi_1)\| \\
&\le C\|\nabla(\nabla^2\phi_1)\| + C\left(\nu^{\frac{1}{16}}\|r\Delta\psi_1\|^{\frac{1}{4}}\right)\left(\nu^{\frac{3}{16}}\|r\nabla(\Delta\psi_1)\|^{\frac{3}{4}}\|\nabla^2\phi_1\|^{\frac{3}{4}}\right)\left(\frac{1}{\nu^{\frac{1}{4}}}\|\nabla(\nabla^2\phi_1)\|\right) \\
&\le \frac{C}{\sqrt{\nu}}\|\nabla(\nabla^2\phi_1)\|^2 + \sqrt{\nu}\|r\nabla(\Delta\psi_1)\|^2\|\nabla^2\phi_1\|^2 + C\left(\|r\Delta\psi_1\|^2 + 1\right).
\end{aligned}
$$

And

$$
\begin{aligned}
|b_1| &= \left| \int (\tilde{v}^r) \phi_{1rr} (\nabla^2\phi_1)_r rdrdz \right| \\
&\le C\|\phi_{1rr}\|\|\nabla(\nabla^2\phi_1)\| + C\|r\psi_{1z}\|_4\|\phi_{1rr}\|_4\|(\nabla^2\phi_1)_r\| \\
&\le C\|\nabla^2\phi_1\|\|\nabla(\nabla^2\phi_1)\| + C\|r\psi_{1z}\|^{\frac{1}{4}}\|\nabla(r\psi_{1z})\|^{\frac{3}{4}}\|\phi_{1rr}\|^{\frac{1}{4}}\|\nabla(\phi_{1rr})\|^{\frac{3}{4}}\|\nabla(\nabla^2\phi_1)\| \\
&\le C\|\nabla^2\phi_1\|\|\nabla(\nabla^2\phi_1)\| + C\|\Delta\psi_1\|^{\frac{3}{4}}\|\nabla^2\phi_1\|^{\frac{1}{4}}\|\nabla(\nabla^2\phi_1)\|^{\frac{7}{4}} \\
&\le \frac{C}{\sqrt{\nu}}(\|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2 + 1)\|\nabla(\nabla^2\phi_1)\|^2 + C.
\end{aligned}
$$

Similar to a_1,

$$
\begin{aligned}
|c_1| &= \left| \int (\tilde{v}^r)_z \phi_{1r} (\nabla^2\phi_1)_z rdrdz \right| \\
&\le \frac{C}{\sqrt{\nu}}\|\nabla(\nabla^2\phi_1)\|^2 + \sqrt{\nu}\|r\nabla(\Delta\psi_1)\|^2\|\nabla^2\phi_1\|^2 + C\left(\|r\Delta\psi_1\|^2 + 1\right).
\end{aligned}
$$

Similar to b_1,

$$
\begin{aligned}
|d_1| &= \left| \int (\tilde{v}^r) \phi_{1rz} (\nabla^2\phi_1)_z rdrdz \right| \le \frac{C}{\sqrt{\nu}}(\|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2 + 1)\|\nabla(\nabla^2\phi_1)\|^2 \\
&\le \frac{C}{\sqrt{\nu}}(\|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2 + 1)\|\nabla(\nabla^2\phi_1)\|^2 + C.
\end{aligned}
$$

To sum up, we conclude

$$
\begin{aligned}
|K_a| \ \le\ & \frac{C}{\sqrt{\nu}}(\|r\Delta\psi_1\|^2+\|\nabla^2\phi_1\|^2+1)\|\nabla(\nabla^2\phi_1)\|^2 \\
&+\sqrt{\nu}\|r\nabla(\Delta\psi_1)\|^2\|\nabla^2\phi_1\|^2+C\left(\|r\Delta\psi_1\|^2+1\right). \qquad (4.27)
\end{aligned}
$$

For K_b, we can get the same estimate like K_a. After expanding K_c, we get

$$
\begin{aligned}
K_c &= -\frac{\gamma}{\eta^2}\int\left(\phi_1^3+3\phi_0^*\phi_1^2\chi+3\phi_0^{*2}\phi_1\chi^2-\phi_1\right)\nabla^2(\nabla^2\phi_1)rdrdz \\
&= \frac{\gamma}{\eta^2}\int\nabla\left(\phi_1^3+3\phi_0^*\phi_1^2\chi+3\phi_0^{*2}\phi_1\chi^2-\phi_1\right)\nabla(\nabla^2\phi_1)rdrdz \\
&= \frac{3\gamma}{\eta^2}\int(3\phi_1^2-1)\nabla\phi_1\nabla(\nabla^2\phi_1)rdrdz+\frac{3\gamma}{\eta^2}\int\phi_1^2\nabla(\phi_0^*\chi)\nabla(\nabla^2\phi_1)rdrdz \\
&\quad+\frac{3\gamma}{\eta^2}\int\phi_1\nabla(\phi_0^{*2}\chi^2)\nabla(\nabla^2\phi_1)rdrdz+\frac{6\gamma}{\eta^2}\int\phi_0^*\chi\phi_1\nabla\phi_1\nabla(\nabla^2\phi_1)rdrdz \\
&\quad+\frac{3\gamma}{\eta^2}\int\phi_0^{*2}\chi^2\nabla\phi_1\nabla(\nabla^2\phi_1)\,rdrdz,
\end{aligned}
$$

using basic energy law (1.4) and Lemma 2.2, it is easy to obtain

$$
|K_c| \ \le\ \frac{1}{\sqrt{\nu}}\|\nabla(\nabla^2\phi_1)\|^2+C. \qquad (4.28)
$$

Similarly,

$$
|K_d| \ \le\ \frac{1}{\sqrt{\nu}}\|\nabla(\nabla^2\phi_1)\|^2+C. \qquad (4.29)
$$

$$
|K_e| \ \le\ \frac{1}{\sqrt{\nu}}\|\nabla(\nabla^2\phi_1)\|^2+C. \qquad (4.30)
$$

$$
|K_g| \ \le\ \frac{1}{\sqrt{\nu}}\|\nabla(\nabla^2\phi_1)\|^2+C. \qquad (4.31)
$$

For K_f, we use (3.5) and Lemma 3.4, 3.5,

$$
\begin{aligned}
|K_f| &= \left|-\int(\phi_0^*\tilde{v}^r\chi_r)\nabla^2(\nabla^2\phi_1)rdrdz\right| \\
&\le \left|\int\tilde{v}^r\nabla(\chi_r\phi_0^*)\cdot\nabla(\nabla^2\phi_1)rdrdz\right|+\left|\int\phi_0^*\chi_r\nabla(\tilde{v}^r)\cdot\nabla(\nabla^2\phi_1)rdrdz\right| \\
&\le C\|\tilde{v}^r\|\|\nabla(\nabla^2\phi_1)\|+C\|\nabla(\tilde{v}^r)\|\|\nabla(\nabla^2\phi_1)\| \\
&\le C\|\nabla(\nabla^2\phi_1)\|+C\|\nabla(r\psi_{1z})\|\|\nabla(\nabla^2\phi_1)\| \\
&\le C\|\nabla(\nabla^2\phi_1)\|+C\|r\Delta\psi_1\|\|\nabla(\nabla^2\phi_1)\| \\
&\le \frac{1}{\sqrt{\nu}}\|\nabla(\nabla^2\phi_1)\|^2+\frac{1}{\sqrt{\nu}}\|r\Delta\psi_1\|^2+C. \qquad (4.32)
\end{aligned}
$$

Similarly for K_h,

$$
|K_h| \ \le\ \frac{1}{\sqrt{\nu}}\|\nabla(\nabla^2\phi_1)\|^2+\frac{1}{\sqrt{\nu}}\|r\Delta\psi_1\|^2+C. \qquad (4.33)
$$

Thus,

$$
\frac{1}{2}\frac{d}{dt}\left\|\nabla^2\phi_1\right\|^2 \ \le\ -\left[\gamma-\frac{C}{\sqrt{\nu}}(\|r\Delta\psi_1\|^2+\|\nabla^2\phi_1\|^2+1)\right]\|\nabla(\nabla^2\phi_1)\|^2
$$

$$+C\nu^{\frac{1}{2}}\left(\|r\Delta\psi_1\|^2+\|\nabla^2\phi_1\|^2\right)\|r\nabla(\Delta\psi_1)\|^2+C\|r\Delta\psi_1\|^2+C. \quad (4.34)$$

Adding up estimates (4.9), (4.25) and (4.34), and denoting

$$H^2(t) = \|r\nabla u_1\|^2+\|r\Delta\psi_1\|^2+\|\nabla^2\phi_1\|^2, \quad (4.35)$$

$$E^2(t) = \|r\Delta u_1\|^2+\|r\nabla(\Delta\psi_1)\|^2+\|\nabla(\nabla^2\phi_1)\|^2. \quad (4.36)$$

Then we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}H^2(t) \leq & -\left[\nu-CH^2(t)-C\right]\|r\Delta u_1\|^2\\ & -\left[\nu-\nu^{\frac{1}{2}}(H^2(t)+C)\right]\|r\nabla(\Delta\psi_1)\|^2\\ & -\left[\gamma-\frac{C}{\sqrt{\nu}}H^2(t)-\frac{C}{\sqrt{\nu}}\right]\|\nabla(\nabla^2\phi_1)\|^2+CH^2(t)+C. \quad (4.37)\end{aligned}$$

Following the steps in [19], we can prove when ν is large enough, H(t) is uniformly bounded for all $t>0$. It follows that

$$\mathbf{u}\in L^\infty(0,\infty;H^1(\Omega)), \quad \phi\in L^\infty(0,\infty;H^2(\Omega)),$$

which is actually a classical solution. □

5 Result for small initial data case

In this section, we choose the initial data for the $1D$ system as:

$$\begin{aligned}\psi_1^*(z,0)=\frac{1}{M^2}\overline{\psi_1}(zM), \quad u_1^*(z,0)=\frac{1}{M}\overline{U_1}(zM),\\ \omega_1^*(z,0)=\overline{W_1}(zM), \quad \phi_0^*(z,0)=\frac{1}{M^3}\overline{\phi}(zM), \quad (5.1)\end{aligned}$$

where M is a positive constant to be determined, $\overline{\psi_1}$, $\overline{U_1}$, $\overline{W_1}$, $\overline{\phi}$ are smooth, periodic functions in y with period 1. Moreover, we assume $\overline{\psi_1}$, $\overline{U_1}$, $\overline{\phi}$ are odd functions in y. By (1.21), $\overline{W_1}=-\overline{\psi_1}_{zz}$, hence it is also a smooth, periodic, and odd function in y. In particular, $u_1^*(z,t)$, $\psi_1^*(z,t)$, $\omega_1^*(z,t)$ and $\phi_0^*(z,t)$ are periodic functions in z with period $\frac{1}{M}$ and odd in z within each period. Therefore, a priori estimates for the solutions to the $1D$ equations are modified from Lemma 2.1 as follows

$$\|\psi_1^*\|_{L^\infty}\leq\frac{C_0}{M^2}, \quad (5.2)$$

$$\|\psi_{1z}^*\|_{L^\infty}\leq\frac{C_0}{M}, \quad \|u_1^*\|_{L^\infty}\leq\frac{C_0}{M}, \quad (5.3)$$

$$\|\omega_1^*\|_{L^\infty}\leq C_0, \quad \|u_{1z}^*\|_{L^\infty}\leq C_0. \quad (5.4)$$

Let $R_0=M^{\frac{1}{4}}$, from the above inequalities (5.3), (5.4), we know

$$\|ru_1^*\|\leq\frac{C}{\sqrt{M}}, \quad \|\nabla(ru_1^*)\|\leq C\sqrt{M}, \quad \|r\psi_{1z}^*\|\leq\frac{C}{\sqrt{M}}. \quad (5.5)$$

As long as $\eta > \eta_0 > 1$, the right hand side of (2.15) can be refined as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\phi_0^*\right\|^2+\frac{\gamma}{\eta^2}\int_0^1(\phi_0^*)^4dz+\gamma\left\|\phi_{0z}^*\right\|^2\\
=\;& -2\int_0^1\psi_1^*\phi_{0z}^*\phi_0^*dz+\frac{\gamma}{\eta^2}\left\|\phi_0^*\right\|^2\\
\leq\;& \left(1-\frac{1}{\eta_0^2}\right)\gamma\left\|\phi_{0z}^*\right\|^2+\left(\frac{C(\eta_0)}{\gamma M^2}+\frac{\gamma}{\eta^2}\right)\left\|\phi_0^*\right\|^2\\
\leq\;& \left(\gamma-\frac{\gamma}{\eta_0^2}+\frac{C}{\gamma M^2}+\frac{\gamma}{\eta^2}\right)\left\|\phi_{0z}^*\right\|^2. \qquad (5.6)
\end{aligned}$$

Also, the right hand side of (2.16) is refined as

$$\begin{aligned}
&\left\|\phi_{0t}^*\right\|^2+\frac{d}{dt}\left[\frac{\gamma}{2}\left\|\phi_{0z}^*\right\|^2+\frac{1}{4}\int_0^1(\phi_0^*)^4dz-\frac{\gamma}{2\eta^2}\left\|\phi_0^*\right\|^2\right]\\
=\;& -2\int_0^1\psi_1^*\phi_{0z}^*\phi_{0t}^*dz\\
\leq\;& \left\|\phi_{0t}^*\right\|^2+\frac{C}{M^4}\left\|\phi_{0z}^*\right\|^2. \qquad (5.7)
\end{aligned}$$

Multiplying (5.6) by $\frac{\gamma}{\eta^2}$, then adding the resultant with (5.7), it infers that

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\gamma}{2}\left\|\phi_{0z}^*\right\|^2+\frac{1}{4}\int_0^1(\phi_0^*)^4dz\right]\\
&\leq-\left[\frac{\gamma^2}{\eta^2\eta_0^2}-\frac{\gamma^2}{\eta^4}-\frac{C}{\eta^2M^2}-\frac{C}{M^4}\right]\left\|\phi_{0z}^*\right\|^2. \qquad (5.8)
\end{aligned}$$

Since $\eta > \eta_0 > 1$, if M is chosen large enough, it follows that

$$\frac{d}{dt}\left[\left\|\phi_{0z}^*\right\|^2+\int_0^1(\phi_0^*)^4dz\right]+\left\|\phi_{0z}^*\right\|^2\leq 0. \qquad (5.9)$$

Hence we have the uniform bound

$$\|\phi_0^*\|_{H^1[0,1]}(t)\leq\|\phi_0^*\|_{H^1[0,1]}(0)\leq\frac{C}{M^2}. \qquad (5.10)$$

Similarly, one can derive the uniform bound of ϕ_{0z}^* in H^1 norm,

$$\|\phi_{0zz}^*\|^2_{[0,1]}(t)\leq\|\phi_{0zz}^*\|^2_{[0,1]}(0)\leq\frac{C}{M}. \qquad (5.11)$$

From (5.10), (5.11) and Morrey's inequality, the uniform L^∞ bounds for ϕ_0^* are

$$\|\phi_0^*\|_{L^\infty[0,1]}\leq\|\phi_0^*\|_{H^1[0,1]}(t)\leq\frac{C}{M^2},\qquad\|\phi_{0z}^*\|_{L^\infty[0,1]}\leq\|\phi_0^*\|_{H^2[0,1]}(t)\leq\frac{C}{M}. \qquad (5.12)$$

On the other hand, we assume the initial conditions of the $3D$ velocity vector $\tilde{\mathbf{u}}$, and the phase function ϕ as

$$\|\tilde{\mathbf{u}}(0)\|^2+\lambda\|\nabla\phi(0)\|^2+\frac{\lambda}{2\eta^2}\|\phi(0)^2-1\|^2\leq\frac{C}{\sqrt{M}}. \qquad (5.13)$$

From the basic energy law (1.4),

$$\|\tilde{\mathbf{u}}(t)\|\leq\frac{C}{\sqrt{M}},\quad\|\nabla\phi(t)\|\leq\frac{C}{\sqrt{M}}. \qquad (5.14)$$

By (5.5), (5.10) and (5.14), we get a priori bounds for the perturbed velocity and the phase function in L^2 norm :

$$\|ru_1\| \le \frac{C}{\sqrt{M}}, \quad \|v^r\| \le \frac{C}{\sqrt{M}}, \quad \|v^z\| \le \frac{C}{\sqrt{M}}, \quad \|\nabla\phi_1\| \le \frac{C}{\sqrt{M}}. \tag{5.15}$$

Now we begin to prove Theorem 1.2.

Proof. Under these conditions (5.2)-(5.4), (5.10)-(5.15), we shall refine some estimates in the proof of Theorem 1.1. Throughout the proof, β denotes a positive constant.

Refinement of estimates for u_1 equation

Using (3.5), (5.3) and (5.15), Lemma 3.3 to 3.5, one arrives at

$$\begin{aligned}
|I_a| &= \left|\int \tilde{v}^r u_{1r}\Delta u_1 r^2 rdrdz\right| \\
&= \left|\int \left(-r\psi_{1z}^*\chi - r\psi_{1z}\right) u_{1r}\Delta u_1 r^2 rdrdz\right| \\
&\le CR_0 \|\psi_{1z}^*\|_{L^\infty} \|ru_{1r}\|\|r\Delta u_1\| + \left|\int \psi_{1z}u_{1r}\Delta u_1 r^3 rdrdz\right| \\
&\le \frac{CR_0}{M}\|r\Delta u_1\|^2 + \|ru_{1r}\|_{L^4}\|r\psi_{1z}\|_{L^4}\|r\Delta u_1\| \\
&\le \frac{C}{M^{\frac{3}{4}}}\|r\Delta u_1\|^2 + \|ru_{1r}\|^{\frac{1}{4}}\|\nabla(ru_{1r})\|^{\frac{3}{4}}\|r\psi_{1z}\|^{\frac{1}{4}}\|\nabla(r\psi_{1z})\|^{\frac{3}{4}}\|r\Delta u_1\| \\
&\le \frac{C}{M^{\frac{3}{4}}}\|r\Delta u_1\|^2 + \frac{C}{M^{\frac{1}{8}}}\|ru_{1r}\|^{\frac{1}{4}}\|r\Delta\psi_1\|^{\frac{3}{4}}\|r\Delta u_1\|^{\frac{7}{4}}.
\end{aligned}$$

Note that

$$\|ru_{1r}\|^2 = \int r^3 u_{1r}u_{1r}drdz = -\int (r^3u_{1r})_r u_1 drdz = -\int u_{1rr}u_1 r^2 rdrdz - 3\int u_{1r}ru_1 rdrdz,$$

by (5.15), and Lemma 3.1 to 3.3, we have

$$\|ru_{1r}\|^2 \le \|ru_1\|\|ru_{1rr}\| + 3\|ru_1\|\|u_{1r}\| \le \frac{C}{\sqrt{M}}\|r\Delta u_1\|,$$

Combining the above estimates, we get

$$|I_a| \le \frac{C}{M^{\frac{3}{4}}}\|r\Delta u_1\|^2 + \frac{C}{M^{\frac{3}{16}}}\left(\|r\Delta\psi_1\| + \|r\Delta u_1\|^2 + 1\right). \tag{5.16}$$

By (3.6), (5.2), (5.14), Lemma 3.1 to 3.5, we can obtain

$$\begin{aligned}
|I_b| &= \left|\int \tilde{v}^z u_{1z}\Delta u_1 r^2 rdrdz\right| \\
&= \left|\int \left(2\psi_1^*\chi + r\psi_1^*\chi_r + 2\psi_1 + r\psi_{1r}\right)(ru_{1z})(r\Delta u_1)rdrdz\right| \\
&\le \left|\int (2\psi_1^*\chi + r\psi_1^*\chi_r)(ru_{1z})(r\Delta u_1)rdrdz\right| + \left|\int (2\psi_1 + r\psi_{1r})(ru_{1z})(r\Delta u_1)rdrdz\right| \\
&\le \frac{C}{M^2}\|r\Delta u_1\|^2 + \|2\psi_1 + r\psi_{1r}\|_{L^4}\|ru_{1z}\|_{L^4}\|r\Delta u_1\|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{M^2}\|r\Delta u_1\|^2 + \|2\psi_1 + r\psi_{1r}\|^{\frac14}\|\nabla(2\psi_1 + r\psi_{1r})\|^{\frac34}\|ru_{1z}\|^{\frac14}\|\nabla(ru_{1z})\|^{\frac34}\|r\Delta u_1\| \\
&\le \frac{C}{M^2}\|r\Delta u_1\|^2 + \frac{C}{M^{\frac18}}\|3\psi_{1r} + 2\psi_{1z} + r\psi_{1rz} + r\psi_{1rr}\|^{\frac34}\|r\nabla u_1\|^{\frac14}\|r\Delta u_1\|^{\frac74} \\
&\le \frac{C}{M^2}\|r\Delta u_1\|^2 + \frac{C}{M^{\frac18}}\|r\Delta\psi_1\|^{\frac34}\|r\nabla u_1\|^{\frac14}\|r\Delta u_1\|^{\frac74} \\
&\le \left(\frac{C}{M^2} + \frac{C}{M^{\frac18}}\right)\|r\Delta u_1\|^2 + \frac{C}{M^{\frac18}}\|r\Delta\psi_1\|^6\|r\nabla u_1\|^2. \qquad (5.17)
\end{aligned}
$$

The estimate of I_c comes from (3.5), (5.5), (5.14), (5.15) and Lemma 3.1, 3.4 and 3.5.

$$
\begin{aligned}
|I_c| &= \left|-2\int \tilde{v}^r u_1\Delta u_1 r^2 rdrdz\right| \\
&= 2\left|\int (r\psi_1^*\chi + r\psi_{1z})(ru_1)(r\Delta u_1)rdrdz\right| \\
&\le \frac{CR_0}{M^2}\|ru_1\|\|r\Delta u_1\| + C\|ru_1\|_{L^4}\|r\psi_{1z}\|_{L^4}\|r\Delta u_1\| \\
&\le \frac{C}{M^2}\|r\Delta u_1\| + C\|ru_1\|^{\frac14}\|\nabla(ru_1)\|^{\frac34}\|r\psi_{1z}\|^{\frac14}\|\nabla(r\psi_{1z})\|^{\frac34}\|r\Delta u_1\| \\
&\le \frac{C}{M^2}\|r\Delta u_1\| + \frac{C}{M^{\frac14}}\|r\nabla u_1\|^{\frac34}\|r\Delta\psi_1\|^{\frac34}\|r\Delta u_1\| \\
&\le \left(\frac{C}{M^2} + \frac{C}{M^{\frac14}}\right)\|r\Delta u_1\|^2 + \frac{C}{M^{\frac14}}\|r\nabla u_1\|^{\frac32}\|r\Delta\psi_1\|^{\frac32} + \frac{C}{M^2}. \qquad (5.18)
\end{aligned}
$$

Using (5.2), (5.3), (5.4), (5.14), (5.15), one can derive all the estimates of I_d to I_h.

$$
\begin{aligned}
|I_d| &= \left|-2\int \chi u_1^*\psi_{1z}\Delta u_1 r^2 rdrdz\right| \le \frac{C}{M}\|r\psi_{1z}\|\|r\Delta u_1\| \le \frac{C}{M^{\frac32}}\|r\Delta u_1\| \\
&\le \frac{C}{M^{\frac32}}(\|r\Delta u_1\|^2 + 1). \qquad (5.19) \\
|I_e| &= \left|-2\int (\chi^2 - \chi)\psi_{1z}^* u_1^*\Delta u_1 r^2 rdrdz\right| \le \frac{CR_0}{M^2}\|r\Delta u_1\| \\
&\le \frac{C}{M^{\frac74}}(\|r\Delta u_1\|^2 + 1). \qquad (5.20) \\
|I_f| &= \left|-2\int \tilde{v}^r u_1^*\chi_r\Delta u_1 r^2 rdrdz\right| \le \|u_1^*\|_{L^\infty}\|\tilde{v}^r\|\|r\Delta u_1\| \\
&\le \frac{C}{M^{\frac32}}\|r\Delta u_1\|^2 + \frac{C}{M^{\frac32}}. \qquad (5.21) \\
|I_g| &= \left|\int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right)u_{1z}^*\Delta u_1 r^2 rdrdz\right| \\
&\le C\|\psi_1^*\|_{L^\infty}\|u_{1z}^*\|_{L^\infty}\|r\Delta u_1\|R_0 + C\|u_{1z}^*\|_{L^\infty}\|v^z\|\|r\Delta u_1\|R_0 \\
&\le \left(\frac{C}{M^{\frac74}} + \frac{C}{M^{\frac14}}\right)\|r\Delta u_1\| \\
&\le \left(\frac{C}{M^{\frac74}} + \frac{C}{M^{\frac14}}\right)\|r\Delta u_1\|^2 + \frac{C}{M^{\frac74}} + \frac{C}{M^{\frac14}}. \qquad (5.22) \\
|I_h| &= \left|-\nu\int u_1^*(\Delta_r\chi)\Delta u_1 r^2 rdrdz\right| \\
&\le C\|u_1^*\|_{L^\infty}\left|\int r^2(\Delta_r\chi)\Delta u_1 rdrdz\right| \le \frac{C}{M}\left|\int_{\frac{R_0}{2}}^{R_0}\Delta u_1 rdrdz\right|
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{C}{MR_0}\left|\int_{\frac{R_0}{2}}^{R_0}(r\Delta u_1)rdrdz\right| \le \frac{C}{MR_0}\|r\Delta u_1\|R_0 \\
&\le \frac{C}{M}\|r\Delta u_1\|^2 + \frac{C}{M}. \qquad (5.23)
\end{aligned}$$

Refinement of estimates for ω_1 equation

Since $(\psi_{1t}^*)_{zz} = -\omega_{1t}^*$, and periodic in $[0, \frac{1}{M}]$, using (1.20) we have

$$\|\psi_{1t}^*\|_{L^\infty} \le \frac{1}{M^2}\|\omega_{1t}^*\|_{L^\infty} \le \frac{1}{M^2}\left(\|\psi_1^*\|_{L^\infty}\|\omega_{1z}^*\|_{L^\infty} + \|\omega_{1zz}^*\|_{L^\infty} + 2\|u_1^*\|_{L^\infty}\|u_{1z}^*\|_{L^\infty}\right) \le C,$$

using the above inequality, together with (1.20), (1.22), (5.2) to (5.4), and Lemma 3.6, we have

$$\begin{aligned}
|J_a| &= \left|\int \Delta_r\chi\psi_{1t}^*\Delta\psi_1 r^3 drdz\right| \\
&= \left|\int \Delta_r\chi\psi_{1t}^*(\psi_{1rr} + \frac{3\psi_{1r}}{r} + \psi_{1zz})r^3 drdz\right| \\
&\le \left|\int \Delta_r\chi\psi_{1t}^*(\psi_{1rr} + \frac{3\psi_{1r}}{r})r^3 drdz\right| + \left|\int \Delta_r\chi\psi_{1t}^*\psi_{1zz}r^3 drdz\right| \\
&\le \left|\int \psi_{1t}^*(\Delta_r\chi)_r\psi_{1r}r^3 drdz\right| + \left|\int \psi_{1zzt}^*\Delta_r\chi\psi_1 r^3 drdz\right| \\
&\le \frac{C}{R_0^2}\int_0^1\int_{\frac{R_0}{2}}^{R_0}|r\psi_{1r}|\, rdrdz + \left|\int \omega_{1t}^*\Delta_r\chi\psi_1 r^3 drdz\right| \\
&\le \frac{C}{M^{\frac14}}\|r\psi_{1r}\| + \left|\int \psi_1^*(\omega_1^*)_z\Delta_r\chi\psi_1 r^3 drdz\right| + \left|\int (u_1^{*2})_z\Delta_r\chi\psi_1 r^3 drdz\right| \\
&\quad + \left|\nu\int \omega_{1zz}^*\Delta_r\chi\psi_1 r^3 drdz\right| \\
&\le \frac{C}{M^{\frac14}}\|r\psi_{1r}\| + \frac{C}{MR_0}\int |r^2\Delta_r\chi(r\psi_1)|rdrdz + \frac{C}{M^2R_0}\int |r^2\Delta_r\chi(r\psi_{1z})|rdrdz \\
&\quad + \left|\nu\int \omega_1^* r^2\Delta_r\chi\psi_{1zz}rdrdz\right| \\
&\le \frac{C}{M^\beta}\|r\Delta\psi_1\| + \frac{C}{M^\beta}\|r\psi_{1z}\| + \left|\int \psi_{1zz}^* r^2\Delta_r\chi\psi_{1zz}rdrdz\right| \\
&\le \frac{C}{M^\beta}(\|r\Delta\psi_1\| + 1) + \|\psi_{1z}^*\|_{L^\infty}\left|\int r^2\Delta_r\chi\psi_{1zzz}rdrdz\right| \\
&\le \frac{C}{M^\beta}(\|r\nabla(\Delta\psi_1)\|^2 + \|r\Delta\psi_1\|^2 + 1). \qquad (5.24)
\end{aligned}$$

For J_b, we can make use of (3.5), (5.5), (5.14), (5.15) and Lemma 3.4 to 3.6.

$$\begin{aligned}
|J_b| &= \left|\int (\Delta\psi_1)^2\tilde{v}^r r^2 drdz\right| \\
&= \left|\int -\psi_{1z}^*\chi(\Delta\psi_1)^2 r^3 drdz - \int \psi_{1z}(\Delta\psi_1)^2 r^3 drdz\right| \\
&\le \|\psi_{1z}^*\|_{L^\infty}\|r\Delta\psi_1\|^2 + \|r\psi_{1z}\|_4\|r\Delta\psi_1\|_4\|\Delta\psi_1\|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{M}\|r\Delta\psi_1\|^2 + C\|r\psi_{1z}\|^{\frac14}\|\nabla(r\psi_{1z})\|^{\frac34}\|r\Delta\psi_1\|^{\frac14}\|\nabla(r\Delta\psi_1)\|^{\frac34}\|r\nabla\Delta\psi_1\| \\
&\le \frac{C}{M}\|r\Delta\psi_1\|^2 + \frac{C}{M^{\frac18}}\|r\Delta\psi_1\|\|r\nabla(\Delta\psi_1)\|^{\frac74} \\
&\le \frac{C}{M}\|r\Delta\psi_1\|^2 + \frac{C}{M^{\frac18}}\|r\Delta\psi_1\|^8 + \frac{C}{M^{\frac18}}\|r\nabla\Delta\psi_1\|^2. \qquad (5.25)
\end{aligned}
$$

From (5.2), (5.14), one arrives at

$$
\begin{aligned}
|J_c| &= \left|-\int \tilde{v}^r(\Delta_r\chi)_r\psi_1^*\Delta\psi_1 r^3 drdz\right| \le \frac{1}{R_0^2}\left|-\int \tilde{v}^r r^3(\Delta_r\chi)_r\psi_1^*(r\Delta\psi_1)rdrdz\right| \\
&\le \frac{1}{R_0^2}\|\psi_1^*\|_{L^\infty}\|\tilde{v}^r\|\|r\Delta\psi_1\| \le \frac{C}{M^3}\left(\|r\Delta\psi_1\|^2+1\right). \qquad (5.26)
\end{aligned}
$$

Similarly, for J_d, using (5.3) and (5.14), we obtain

$$
\begin{aligned}
|J_d| &= \left|-\int \tilde{v}^z(\Delta_r\chi)\psi_{1z}^*\Delta\psi_1 r^3 drdz\right| \le \frac{1}{R_0}\left|-\int \tilde{v}^z r^2(\Delta_r\chi)\psi_{1z}^*(r\Delta\psi_1)rdrdz\right| \\
&\le \frac{\|\psi_{1z}^*\|_{L^\infty}}{R_0}\|\tilde{v}^z\|\|r\Delta\psi_1\| \le \frac{C}{M^{\frac74}}\left(\|r\Delta\psi_1\|^2+1\right). \qquad (5.27)
\end{aligned}
$$

By (5.2), (5.3) and Lemma 3.6,

$$
\begin{aligned}
|J_e| &= \left|\nu\int \Delta(\Delta_r\chi\psi_1^*)\Delta\psi_1 r^3 drdz\right| \\
&\le \left|\nu\int \Delta_r\chi\psi_{1z}^*(\Delta\psi_1)_z r^3 drdz\right| + \left|\nu\int (\Delta_r\chi)_r\psi_1^*(\Delta\psi_1)_r r^3 drdz\right| \\
&\le \frac{C}{MR_0}\int |r(\Delta\psi_1)_z(r^2\Delta_r\chi)|rdrdz + \frac{C}{M^2R_0^2}\int |r(\Delta\psi_1)_r r^3(\Delta_r\chi)_r|rdrdz \\
&\le \frac{C}{M^{\frac54}}\|r\nabla(\Delta\psi_1)\|^2 + \frac{C}{M^{\frac54}}. \qquad (5.28)
\end{aligned}
$$

By (5.3), (5.4), (5.15), we have

$$
\begin{aligned}
|J_f| &= \left|-2\int \left[u_1^*u_{1z}^*(\chi^2-\chi) + u_{1z}^*\chi u_1 + u_1^*\chi u_{1z}\right]\Delta\psi_1 r^3 drdz\right| \\
&\le \frac{C}{\sqrt{M}}\|r\Delta\psi_1\| + C\|ru_1\|\|r\Delta\psi_1\| + \frac{C}{M}\|ru_{1z}\|\|r\Delta\psi_1\| \\
&\le \frac{C}{\sqrt{M}}(\|r\Delta\psi_1\|^2 + \|r\Delta u_1\|^2 + 1). \qquad (5.29)
\end{aligned}
$$

From (5.15), Lemma 3.1, 3.2, 3.3, and 3.6, it follows that

$$
\begin{aligned}
|J_g| &= \left|-2\int u_1u_{1z}\Delta\psi_1 r^3 drdz\right| = \left|-2\int u_1(ru_1)r(\Delta\psi_1)_z rdrdz\right| \\
&\le C\|u_1\|_{L^4}\|ru_1\|_{L^4}\|r\nabla(\Delta\psi_1)\| \\
&\le C\|ru_1\|^{\frac14}\|\nabla(ru_1)\|^{\frac34}\|u_1\|^{\frac14}\|\nabla u_1\|^{\frac34}\|r\nabla(\Delta\psi_1)\| \\
&\le \frac{C}{M^{\frac18}}\|r\nabla u_1\|\|r\Delta u_1\|^{\frac34}\|r\nabla(\Delta\psi_1)\| \\
&\le \frac{C}{M^\beta}\|r\nabla u_1\|^8 + \frac{C}{M^\beta}\|r\Delta u_1\|^2 + \frac{C}{M^\beta}\|r\nabla(\Delta\psi_1)\|^2. \qquad (5.30)
\end{aligned}
$$

By (5.3), (5.14) and (5.15), we get

$$|J_h| = \left|\int \tilde{v}^r \omega_1^* \chi_r \Delta\psi_1 r^3 drdz\right| \le C\|\omega_1^*\|_{L^\infty}\|\tilde{v}^r\|\|r\Delta\psi_1\| \le \frac{C}{\sqrt{M}}\|r\Delta\psi_1\|^2 + \frac{C}{\sqrt{M}}. \qquad (5.31)$$

By (1.21), (3.6), (5.3), (5.4), together with Lemma 3.5 and 3.6, one can derive

$$\begin{aligned}
|J_i| &= \left|\int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right)\omega_{1z}^* \Delta\psi_1 r^3 drdz\right| \\
&\le \frac{C}{M^{\frac{3}{4}}}\|r\Delta\psi_1\| + \frac{C}{M^{\frac{1}{4}}}\|r\nabla\Delta\psi_1\| + \left|\int \chi(v^z)_z \omega_1^* \Delta\psi_1 r^3 drdz\right| \\
&\le \frac{C}{M^{\frac{3}{4}}}\|r\Delta\psi_1\| + \frac{C}{M^{\frac{1}{4}}}\|r\nabla\Delta\psi_1\| + \left|\int \left[\chi(v^z)_{zz}\psi_{1z}^*\Delta\psi_1 + \chi(v^z)_z\psi_{1z}^*(\Delta\psi_1)_z\right] r^3 drdz\right| \\
&\le \frac{C}{M^{\frac{3}{4}}}\|r\Delta\psi_1\| + \frac{C}{M^{\frac{1}{4}}}\|r\nabla\Delta\psi_1\| + \frac{C}{M}\int |\chi\psi_{1zz}\Delta\psi_1| r^3 drdz + \frac{C}{M}\int |\chi\psi_{1rzz}\Delta\psi_1| r^4 drdz \\
&\quad + \frac{C}{M}\int |\chi\psi_{1z}(\Delta\psi_1)_z| r^3 drdz + \frac{C}{M}\int |\chi\psi_{1rz}\Delta(\psi_1)_z| r^4 drdz \\
&\le \frac{C}{M^{\frac{3}{4}}}\|r\Delta\psi_1\| + \frac{C}{M^{\frac{1}{4}}}\|r\nabla\Delta\psi_1\| + \frac{C}{M}\|r\Delta\psi_1\|^2 + \frac{C}{M}\|r\Delta\psi_1\|\|r\nabla(\Delta\psi_1)\| \\
&\quad + \frac{C}{M}\|r\psi_{1z}\|\|r\nabla(\Delta\psi_1)\| \\
&\le \frac{C}{M^\beta}\|r\Delta\psi_1\|^2 + \frac{C}{M^\beta}(\|r\nabla(\Delta\psi_1)\|^2 + 1). \qquad (5.32)
\end{aligned}$$

(1.21) and (5.3) provide us with the estimate of J_j as

$$\begin{aligned}
|J_j| &= \left|\nu\int \omega_1^* \Delta_r\chi \Delta\psi_1 r^3 drdz\right| = \left|-\nu\int \psi_{1zz}^* \Delta_r\chi\Delta\psi_1 r^3 drdz\right| \\
&= \left|\nu\int \psi_{1z}^*\Delta_r\chi(\Delta\psi_1)_z r^3 drdz\right| \le \frac{C}{M^{\frac{5}{4}}}\left(\|r\nabla(\Delta\psi_1)\|^2 + 1\right). \qquad (5.33)
\end{aligned}$$

Expanding J_k, we have

$$\begin{aligned}
J_k &= -\lambda\int (\phi_{0z}^*\chi + \phi_{1z})\left[\left(\nabla^2(\phi_0^*\chi)\right)_r + (\nabla^2\phi_1)_r\right]\Delta\psi_1 r^2 drdz \\
&= -\lambda\int \phi_{0z}^*\chi\left(\nabla^2(\phi_0^*\chi)\right)_r \Delta\psi_1 r^2 drdz - \lambda\int \phi_{1z}\left(\nabla^2(\phi_0^*\chi)\right)_r \Delta\psi_1 r^2 drdz \\
&\quad -\lambda\int \phi_{0z}^*\chi(\nabla^2\phi_1)_r\Delta\psi_1 r^2 drdz - \lambda\int \phi_{1z}\left(\nabla^2\phi_1\right)_r \Delta\psi_1 r^2 drdz \\
&\equiv a_3 + b_3 + c_3 + d_3,
\end{aligned}$$

For a_3, (5.10), (5.11) and (5.12) tell us

$$\begin{aligned}
|a_3| &\le \frac{C}{R_0}\int |(\phi_{0z}^*\chi)\phi_{0zz}^*(r\chi_r)(r\Delta\psi_1)| rdrdz \\
&\quad + \frac{C}{R_0^3}\int |(\phi_{0z}^*\chi)\phi_0^* r^3(\Delta_r\chi)_r(r\Delta\psi_1)| rdrdz \\
&\le \frac{C}{R_0}\|\phi_{0z}^*\|_{L^\infty}\|\phi_{0zz}^*\|\|r\Delta\psi_1\| + \frac{C}{R_0^3}\|\phi_{0z}^*\chi\|\|r\Delta\psi_1\| \\
&\le \frac{C}{M^\beta}\left(\|r\Delta\psi_1\|^2 + 1\right).
\end{aligned}$$

For b_3 and c_3, by (5.11), (5.12) and (5.15), we obtain

$$\begin{aligned}
|b_3| &\leq \frac{C}{R_0}\int |\phi_{1z}\phi_{0zz}^*(r\chi_r)(r\Delta\psi_1)|rdrdz + \frac{C}{R_0^3}\int |\phi_{1z}\phi_0^* r^3(\Delta_r\chi)_r(r\Delta\psi_1)|rdrdz \\
&\leq \frac{C}{R_0}\|\phi_{0zz}^*\|_{L^\infty}\|\nabla\phi_1\|\|r\Delta\psi_1\| + \frac{C}{R_0^3}\|\phi_{1z}\|\|r\Delta\psi_1\| \\
&\leq \frac{C}{M^\beta}\left(\|r\Delta\psi_1\|^2+1\right). \\
|c_3| &\leq C\|\phi_{0z}^*\|_{L^\infty}\|r\Delta\psi_1\|\|\nabla(\nabla^2\phi_1)\| \leq \frac{C}{M}(\|r\Delta\psi_1\|^2 + \|\nabla(\nabla^2\phi_1)\|^2).
\end{aligned}$$

And the estimate for d_3 can be derived by (5.15), Lemma 3.7, and Young's Inequality.

$$\begin{aligned}
|d_3| &\leq C\|\phi_{1z}\|_{L^4}\|\nabla(\nabla^2\phi_1)\|\|r\Delta\psi_1\|_{L^4} \\
&\leq C\|\phi_{1z}\|^{\frac{1}{4}}\|\nabla\phi_{1z}\|^{\frac{3}{4}}\|\nabla(\nabla^2\phi_1)\|\|r\Delta\psi_1\|^{\frac{1}{4}}\|\|r\nabla(\Delta\psi_1)\|^{\frac{3}{4}} \\
&\leq \frac{C}{M^{\frac{1}{8}}}(\|r\nabla(\Delta\psi_1)\|^2 + \|\nabla(\nabla^2\phi_1)\|^2 + \|\nabla^2\phi_1\|^6\|r\Delta\psi_1\|^2).
\end{aligned}$$

In all, summing up a_3 to d_3,

$$|J_k| \leq \frac{C}{M^\beta}\left(\|r\Delta\psi_1\|^2 + \|r\nabla(\Delta\psi_1)\|^2 + \|\nabla(\nabla^2\phi_1)\|^2 + \|\nabla^2\phi_1\|^6\|r\Delta\psi_1\|^2 + 1\right).$$

Expanding J_l,

$$\begin{aligned}
J_l &= \lambda\int(\phi_0^*\chi_r + \phi_{1r})\left[\left(\nabla^2(\phi_0^*\chi)\right)_z + (\nabla^2\phi_1)_z\right]\Delta\psi_1 r^2 drdz \\
&= \lambda\int \phi_0^*\chi_r[\nabla^2(\phi_0^*\chi)]_z\Delta\psi_1 r^2 drdz + \lambda\int \phi_{1r}[\nabla^2(\phi_0^*\chi)]_z\Delta\psi_1 r^2 drdz \\
&\quad +\lambda\int \phi_0^*\chi_r(\nabla^2\phi_1)_z\Delta\psi_1 r^2 drdz + \lambda\int \phi_{1r}(\nabla^2\phi_1)_z\Delta\psi_1 r^2 drdz \\
&\equiv a_4 + b_4 + c_4 + d_4.
\end{aligned}$$

For J_l, we can derive similar estimates as J_k.

Refinement of estimates for ϕ_1 equation

Expanding K_a,

$$\begin{aligned}
K_a &= \int\left[(\tilde{v}^r)_r\phi_{1r}(\nabla^2\phi_1)_r + (\tilde{v}^r)_z\phi_{1r}(\nabla^2\phi_1)_z + \tilde{v}^r\nabla\phi_{1r}\cdot\nabla(\nabla^2\phi_1)\right]rdrdz \\
&\equiv a_5 + b_5 + c_5,
\end{aligned}$$

where the estimate of a_5 can be obtained from (3.5), (5.3), (5.15), and Lemma 3.4 to 3.7,

$$\begin{aligned}
|a_5| &= \int\left|(\tilde{v}^r)_r\phi_{1r}(\nabla^2\phi_1)_r\right|rdrdz \\
&\leq \int\left|(r\chi\psi_{1z}^* + r\psi_{1z})_r\phi_{1r}(\nabla^2\phi_1)_r\right|rdrdz \\
&\leq \|\psi_{1z}^*\|_{L^\infty}\|\phi_{1r}\|\|(\nabla^2\phi_1)_r\| + \|\psi_{1z} + r\psi_{1zr}\|_{L^4}\|\phi_{1r}\|_{L^4}\|(\nabla^2\phi_1)_r\| \\
&\leq \frac{C}{M^\beta}\|\nabla(\nabla^2\phi_1)\| + \frac{C}{M^\beta}\|r\Delta\psi_1\|^{\frac{1}{4}}\|r\nabla(\Delta\psi_1)\|^{\frac{3}{4}}\|\nabla^2\phi_1\|^{\frac{3}{4}}\|\nabla(\nabla^2\phi_1)\|
\end{aligned}$$

$$\leq \frac{C}{M^\beta}(\|r\nabla(\Delta\psi_1)\|^2 + \|\nabla(\nabla^2\phi_1)\|^2 + \|\nabla^2\phi_1\|^6\|r\Delta\psi_1\|^2 + C).$$

Using (5.15), Lemma 3.6 and 3.7, it follows that

$$\begin{aligned}
|b_5| &= \int \left|(r\chi\psi_{1zz}^* + r\psi_{1zz})\phi_{1r}(\nabla^2\phi_1)_z\right| rdrdz \\
&\leq R_0\|\psi_{1zz}^*\|_{L^\infty}\|\phi_{1r}\|\|(\nabla^2\phi_1)_z\| + \|r\psi_{1zz}\|_{L^4}\|\phi_{1r}\|_{L^4}\|(\nabla^2\phi_1)_z\| \\
&\leq \frac{C}{M^\beta}\|\nabla(\nabla^2\phi_1)\| + \frac{C}{M^\beta}\|r\Delta\psi_1\|^{\frac14}\|r\nabla(\Delta\psi_1)\|^{\frac34}\|\nabla^2\phi_1\|^{\frac34}\|\nabla(\nabla^2\phi_1)\| \\
&\leq \frac{C}{M^\beta}(\|r\nabla(\Delta\psi_1)\|^2 + \|\nabla(\nabla^2\phi_1)\|^2 + \|\nabla^2\phi_1\|^6\|r\Delta\psi_1\|^2 + C).
\end{aligned}$$

And using (3.5), (5.3), Lemma 3.5 to 3.7, we get

$$\begin{aligned}
|c_5| &= \int \left|\tilde{v}^r\nabla\phi_{1r}\cdot\nabla(\nabla^2\phi_1)\right| rdrdz \\
&= \int \left|(r\psi_{1z}^*\chi + r\psi_{1z})\nabla\phi_{1r}\cdot\nabla(\nabla^2\phi_1)\right| rdrdz \\
&\leq R_0\|\psi_{1z}^*\|_{L^\infty}\|\nabla\phi_{1r}\|\|\nabla(\nabla^2\phi_1)\| + \|r\psi_{1z}\|_{L^4}\|\nabla(\phi_{1r})\|_{L^4}\|\nabla(\nabla^2\phi_1)\| \\
&\leq \frac{C}{M^\beta}\|\nabla^2\phi_1\|\|\nabla(\nabla^2\phi_1)\| + \frac{C}{M^\beta}\|r\Delta\psi_1\|^{\frac34}\|\nabla^2\phi_1\|^{\frac14}\|\nabla(\nabla^2\phi_1)\|^{\frac74} \\
&\leq \frac{C}{M^\beta}(\|\nabla^2\phi_1\|^2 + \|\nabla(\nabla^2\phi_1)\|^2 + \|\nabla^2\phi_1\|^2\|r\Delta\psi_1\|^6).
\end{aligned}$$

To sum up, the estimate of K_a is,

$$\begin{aligned}
|K_a| \leq\ & \frac{C}{M^\beta}(\|r\nabla(\Delta\psi_1)\|^2 + \|\nabla(\nabla^2\phi_1)\|^2 + \|\nabla^2\phi_1\|^6\|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2\|r\Delta\psi_1\|^6 \\
& + \|\nabla^2\phi_1\|^2 + 1).
\end{aligned} \tag{5.34}$$

We can do similar estimates to K_b. Expanding K_c,

$$\begin{aligned}
K_c &= -\frac{\gamma}{\eta^2}\int\left(\phi_1^3 + 3\phi_0^*\phi_1^2\chi + 3\phi_0^{*2}\phi_1\chi^2 - \phi_1\right)\nabla^2(\nabla^2\phi_1)rdrdz \\
&= \frac{\gamma}{\eta^2}\int\nabla\left(\phi_1^3 + 3\phi_0^*\phi_1^2\chi + 3\phi_0^{*2}\phi_1\chi^2 - \phi_1\right)\nabla(\nabla^2\phi_1)rdrdz \\
&= \frac{3\gamma}{\eta^2}\int(3\phi_1^2 - 1)\nabla\phi_1\nabla(\nabla^2\phi_1)rdrdz + \frac{3\gamma}{\eta^2}\int\phi_1^2\nabla(\phi_0^*\chi)\nabla(\nabla^2\phi_1)rdrdz \\
&\quad + \frac{3\gamma}{\eta^2}\int\phi_1\nabla(\phi_0^{*2}\chi^2)\nabla(\nabla^2\phi_1)rdrdz + \frac{6\gamma}{\eta^2}\int\phi_0^*\chi\phi_1\nabla\phi_1\nabla(\nabla^2\phi_1)rdrdz \\
&\quad + \frac{3\gamma}{\eta^2}\int\phi_0^{*2}\chi^2\nabla\phi_1\nabla(\nabla^2\phi_1)rdrdz \\
&\equiv a_6 + b_6 + c_6 + d_6 + e_6,
\end{aligned}$$

where using (5.12) and (5.15), we can estimate them term by term,

$$\begin{aligned}
|a_6| &\leq \|3\phi_1^2 - 1\|_{L^\infty}\|\nabla\phi_1\|\|\nabla(\nabla^2\phi_1)\| \leq \frac{C}{\sqrt{M}}\left(\|\nabla(\nabla^2\phi_1)\|^2 + 1\right). \\
|b_6| &\leq C\left|\int\phi_1^2\phi_{0z}^*\chi(\nabla^2\phi_1)_z rdrdz\right| + C\left|\int\phi_1^2\phi_0^*\chi_r(\nabla^2\phi_1)_r rdrdz\right| \\
&\leq \frac{C}{M^\beta}\left(\|\nabla(\nabla^2\phi_1)\|^2 + 1\right).
\end{aligned}$$

$|c_6|$ is similar to $|b_6|$, and

$$|d_6| \le C\|\phi_0^*\|_{L^\infty}\|\nabla\phi_1\|\|\nabla(\nabla^2\phi_1)\| \le \frac{C}{M^\beta}\left(\|\nabla(\nabla^2\phi_1)\|^2+1\right).$$

$|e_6|$ is similar to $|d_6|$. In all,

$$|K_c| \;\le\; \frac{C}{M^\beta}\left(\|\nabla(\nabla^2\phi_1)\|^2+1\right). \tag{5.35}$$

To get the estimates of K_d and K_e, it is sufficient to use (5.10) and (5.12)

$$\begin{aligned}
|K_d| &= \gamma\left|\int \phi_0^*\left(\chi_{rr}+\frac{\chi_r}{r}\right)\nabla^2(\nabla^2\phi_1)rdrdz\right| \\
&\le \gamma\int\left|\phi_{0z}^*\left(\chi_{rr}+\frac{\chi_r}{r}\right)(\nabla^2\phi_1)_z\right|rdrdz+\gamma\int\left|\phi_0^*\left(\chi_{rr}+\frac{\chi_r}{r}\right)_r(\nabla^2\phi_1)_r\right|rdrdz \\
&\le \frac{C}{R_0^2}\|\phi_{0z}^*(r^2\chi_{rr}+r\chi_r)\|\|\nabla(\nabla^2\phi_1)\|+\frac{C}{R_0^3}\|\phi_0^*r^3(\chi_{rr}+\frac{\chi_r}{r})_r\|\|\nabla(\nabla^2\phi_1)\| \\
&\le \frac{C}{M^\beta}(\|\nabla(\nabla^2\phi_1)\|^2+1). \qquad (5.36)\\
|K_e| &= \frac{\gamma}{\eta^2}\left|\int \phi_0^{*3}(\chi^3-\chi)\nabla^2(\nabla^2\phi_1)rdrdz\right| \\
&\le C\int\left|\phi_0^{*2}\phi_{0z}^*(\chi^3-\chi)(\nabla^2\phi_1)_z\right|rdrdz+C\int\left|\phi_0^{*3}(\chi^3-\chi)_r(\nabla^2\phi_1)_r\right|rdrdz \\
&\le C\|\phi_0^*\|_{L^\infty}^2\|\phi_{0z}^*\chi\|\|\nabla(\nabla^2\phi_1)\|+\frac{C}{R_0}\|\phi_0^*\|_{L^\infty}^2\|\phi_0^*(r\chi_r)\|\|\nabla(\nabla^2\phi_1)\| \\
&\le \frac{C}{M^\beta}(\|\nabla(\nabla^2\phi_1)\|^2+1). \qquad (5.37)
\end{aligned}$$

The estimates of the three remaining terms, K_f to K_h, can be obtained, from (3.5), (3.6), basic energy law (1.4), (5.2) to (5.4), (5.12), and Lemma 3.4, 3.5.

$$\begin{aligned}
|K_f| &= \frac{\gamma}{\eta^2}\left|\int \phi_0^*\tilde{v}^r\chi_r\nabla^2(\nabla^2\phi_1)rdrdz\right| \\
&\le C\left|\int \tilde{v}^r\nabla(\phi_0^*\chi_r)\cdot\nabla(\nabla^2\phi_1)+\phi_0^*\chi_r\nabla\tilde{v}^r\cdot\nabla(\nabla^2\phi_1)rdrdz\right| \\
&\le \frac{C}{M^\beta}\|\tilde{v}^r\|\|\nabla(\nabla^2\phi_1)\|+\frac{C}{M^\beta}\|(\chi+r\chi_r)\psi_{1z}^*+\psi_{1z}+r\psi_{1zr}\|\|\nabla(\nabla^2\phi_1)\| \\
&\quad+\frac{C}{R_0}\|\phi_0^*\|_{L^\infty}\|r\psi_{1zz}\|\|(\nabla^2\phi_1)_z\|+\frac{C}{R_0}\|\psi_{1zz}^*\|_{L^\infty}\|\phi_0^*\chi\|\|(\nabla^2\phi_1)_z\| \\
&\le \frac{C}{M^\beta}(\|\nabla(\nabla^2\phi_1)\|^2+\|r\Delta\psi_1\|^2+1) \qquad (5.38)\\
|K_g| &= 2\left|\int \psi_1^*\phi_{0z}^*\chi_r\nabla^2(\nabla^2\phi_1)rdrdz\right| \\
&= 2\left|\int \nabla(\psi_1^*\phi_{0z}^*\chi_r)\nabla(\nabla^2\phi_1)rdrdz\right| \\
&\le \frac{C}{M^\beta}(\|\nabla(\nabla^2\phi_1)\|^2+1). \qquad (5.39)\\
|K_h| &= \left|-\int \phi_{0z}^*\tilde{v}^z\chi\nabla^2(\nabla^2\phi_1)rdrdz\right| \\
&= \left|\int \tilde{v}^z\nabla(\phi_{0z}^*\chi)\cdot\nabla(\nabla^2\phi_1)rdrdz\right|+\left|\int \phi_{0z}^*\chi\nabla(\tilde{v}^z)\cdot\nabla(\nabla^2\phi_1)rdrdz\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{M^\beta}\|\tilde{v}^r\|\|\nabla(\nabla^2\phi_1)\| + \int \left|\phi_{0z}^*\chi(r\psi_{1z}^*\chi_r + 2\psi_{1z}^*\chi)(\nabla^2\phi_1)_z\right| rdrdz \\
&\quad + \int \left|\phi_{0z}^*\chi(2\psi_{1z} + r\psi_{1rz})(\nabla^2\phi_1)_z\right| rdrdz \\
&\quad + \int \left|\phi_{0z}^*\chi(r\psi_1^*\chi_{rr} + 3\psi_1^*\chi_r)(\nabla^2\phi_1)_r\right| rdrdz \\
&\quad + \int \left|\phi_{0z}^*\chi(3\psi_{1r} + r\psi_{1rr})(\nabla^2\phi_1)_r\right| rdrdz \\
&\le \frac{C}{M^\beta}\|\nabla(\nabla^2\phi_1)\| + \frac{C}{M^\beta}\|\phi_{0z}^*\chi\|\|\nabla(\nabla^2\phi_1)\| + \|\phi_{0z}^*\chi\|_{L^\infty}\|r\Delta\psi_1\|\|\nabla(\nabla^2\phi_1)\| \\
&\le \frac{C}{M^\beta}(\|\nabla(\nabla^2\phi_1)\|^2 + \|r\Delta\psi_1\|^2 + 1). \qquad (5.40)
\end{aligned}
$$

By adding all the estimates about u_1, ψ_1 and ϕ_1 equations, we find except for dissipation terms, all other terms are bounded by

$$\frac{C}{M^\beta}E^2 + \frac{1}{M^\beta}g(H),$$

where $g(H)$ is a polynomial of H with positive exponents and coefficients. Choose M large enough, then

$$\frac{d}{dt}H^2 \le -\frac{\mu}{2}E^2 + \frac{1}{M^\beta}g(H) \le -\frac{\mu}{2}H^2 + \frac{1}{M^\beta}g(H) \qquad (5.41)$$

since $H \le E$, and here $\mu = min(\nu, \gamma)$.

We can choose M large enough such that

$$-\frac{\mu}{2} + \frac{1}{M^\beta}g(1) \le 0. \qquad (5.42)$$

Therefore, if the initial conditions are small so that $H(0) \le 1$, we will get the uniform bound of $H(t)$, such that

$$H(t) \le 1, \quad for\ all\ \ t > 0. \qquad (5.43)$$

which indicates (3.12) holds, hence the proof is complete. □

6 Related models and future work

Our system (1.1)-(1.3) is closely related to Magnetohydrodynamics (MHD) equations, the liquid crystal model, and the viscoelastic system with finite or infinite Weissenberg numbers (see [13]).

6.1 Magnetohydrodynamics

Magnetohydrodynamics (MHD) is the theory of the macroscopic interaction of electrically conducting fluids with magnetic fields (see [4]). The dynamic motion of the fluids and the magnetic field strongly interact each other, thus the hydrodynamic and electrodynamic effects are coupled. MHD flow is governed by the Navier-Stokes equations and the Maxwell equations of the magnetic field. The incompressible MHD equations are

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + (\nabla \times \mathbf{B}) \times \mathbf{B} \tag{6.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{6.2}$$

$$\mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}) = \gamma \Delta \mathbf{B} \tag{6.3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{6.4}$$

where $\mathbf{u}$ is the fluid velocity, $\mathbf{B}$ the magnetic field, $\nu = \frac{1}{Re}$ the inverse of hydrodynamic Reynolds number, and $\gamma = \frac{1}{Rm}$ the inverse of magnetic Reynolds number. Now the energy law reads

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|^2 + \|\mathbf{B}\|^2) = -\nu\|\nabla \mathbf{u}\|^2 - \gamma\|\nabla \mathbf{B}\|^2$$

Due to (6.4), we know in $2D$ case there exists a function ϕ, such that

$$\mathbf{B} = \nabla^{\mathbf{T}}\phi = \begin{pmatrix} -\nabla_2\phi \\ \nabla_1\phi \end{pmatrix},$$

Hence the system above in $2D$ is equivalent to the following equations (see [13])

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla\phi \otimes \nabla\phi), \tag{6.5}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{6.6}$$

$$\phi_t + (\mathbf{u} \cdot \nabla)\phi = \gamma \Delta \phi. \tag{6.7}$$

Compared to the system (1.1)-(1.3), the nonlinear term $f(\phi)$ doesn't appear in (6.7), hence for axisymmetric solutions in $3D$ case, we can get similar results as follows.

Theorem 6.1. *For the* $3D$ *system* (6.5)-(6.7)*, assume* $u_1^*(z,0)$*,* $\psi_1^*(z,0)$*,* $\omega_1^*(z,0)$*, and* $\phi_0^*(z,0)$ *are smooth functions which are periodic in* z *with period* 1*. Then there exists a global classical solution in the form of* (1.23)-(1.26)*, if initial conditions* $\tilde{\mathbf{u}}_{\mathbf{0}} \triangleq \tilde{\mathbf{u}}(r,z,0) \in H^1(R^3)$*,* $\tilde{\phi}_0 \triangleq \tilde{\phi}(r,z,0) \in H^2(R^3)$ *and* $\nu \geq \nu_0(\gamma, \lambda, \tilde{\mathbf{u}}_{\mathbf{0}}, \tilde{\phi}_0)$*.*

Theorem 6.2. *Suppose that initial conditions for* u_1*,* ω_1*,* ψ_1*, and* ϕ_1 *are smooth functions with compact support and odd in* z*. Moreover, assume* $\|\tilde{\mathbf{u}}(0)\|^2 + \lambda\|\nabla\tilde{\phi}(0)\|^2 \leq \frac{C}{\sqrt{M}}$*. For any given* $\nu > 0$*, there exists* $C(\nu) > 0$*, such that if* $M \geq C(\nu)$ *and* $H(0) \leq 1$ *where* $H^2(t) = \|r\nabla u_1\|^2 + \|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2$*. Then, solutions to the 3D system* (6.5)-(6.7) *in the form of* (1.23)-(1.26) *are globally smooth.*

6.2 Liquid crystal

If taking ϕ as a vector, say $\mathbf{d}$, then our system (1.1)-(1.3) can be used to model the motion of liquid crystal flows [16]:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla \mathbf{d} \otimes \nabla \mathbf{d}), \tag{6.8}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{6.9}$$

$$\mathbf{d}_t + (\mathbf{u} \cdot \nabla)\mathbf{d} = \gamma(\Delta \mathbf{d} - f(\mathbf{d})), \tag{6.10}$$

with the basic energy law

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \lambda\|\nabla \mathbf{d}\|^2 + \frac{\lambda}{2\eta^2}\||\mathbf{d}|^2 - 1\|^2\right) = -(\nu\|\nabla \mathbf{u}\|^2 + \lambda\gamma\|\Delta \mathbf{d} - f(\mathbf{d})\|^2).$$

Here $\mathbf{d}$ represents the director field of liquid crystal molecules. Equations for components of the director field are decoupled. The above system is a simplified version of the Ericksen-Leslie model for the hydrodynamics of liquid crystals (see [5], [6], [9], and [16]). Generally speaking, the system is a macroscopic continuum description of the time evolutions of liquid crystal materials influenced by both the flow field $\mathbf{u}(x,t)$, and the microscopic orientational configuration $\mathbf{d}(x,t)$. The first equation combines the usual equation describing the flow of fluid with an extra nonlinear coupling term, which is the induced elastic stress from the elastic energy through the transport, represented by the third equation. The induced elastic stress term $\lambda\nabla \cdot (\nabla \mathbf{d} \otimes \nabla \mathbf{d})$ can also be derived using Least Action Principle, similarly like what we dealt with viscoelastic fluids. The third equation is associated with conservation of angular momentum, with the left hand side standing for the kinematic transport by the flow field, while the right hand side representing the internal relaxation due to the elastic energy. The term $f(\mathbf{d}) = \frac{1}{\eta^2}(|\mathbf{d}|^2 - 1)\mathbf{d}$ may be seen as a penalty function to approximate the constraint $|\mathbf{d}| = 1$, which is due to the liquid crystal molecules being of similar size.

For the liquid crystal model above, the existence of global weak solutions was proved in [19], where the existence of $2D$ global classical solutions was also given. There were also some results for the existence of global classical solutions in $3D$, if the viscosity constant ν, dependent on initial data $\mathbf{u_0}$ and ϕ_0, is large enough. The corresponding results for small initial data were also given, where the initial data is near the absolute minimizer of an energy functional. After that, the authors proved in [20] the partial regularity result that the one dimensional space-time Hausdorff measure of the singular set of the suitable weak solutions is zero. The results of the existence and asymptotic stability of the solutions to the general Ericksen-Leslie system were provided in [21].

Since the only difference here is that $\mathbf{d} = (\tilde{d}_1, \tilde{d}_2, \tilde{d}_3)$ is a vector, in the framework of axisymmetry, we have the equivalent system to (6.8)-(6.10) as

$$u_t + v^r u_r + v^z u_z = \nu(\nabla^2 - \frac{1}{r^2})u - \frac{1}{r}v^r u, \tag{6.11}$$

$$\begin{aligned} \omega_t + v^r \omega_r + v^z \omega_z &= \nu(\nabla^2 - \frac{1}{r^2})\omega + \frac{1}{r}(u^2)_z - \frac{1}{r}v^r\omega \\ &\quad + \lambda\sum_{i=1}^{3}\{(\tilde{d}_i)_z\nabla^2(\tilde{d}_i)_r - (\tilde{d}_i)_r\nabla^2(\tilde{d}_i)_z - \frac{1}{r^2}(\tilde{d}_i)_r(\tilde{d}_i)_z\}, \end{aligned} \tag{6.12}$$

$$-(\nabla^2 - \frac{1}{r^2})\psi = \omega, \tag{6.13}$$

$$(v^r)_r + \frac{v^r}{r} + (v^z)_z = 0, \tag{6.14}$$

$$(\tilde{d}_i)_t + v^r(\tilde{d}_i)_r + v^z(\tilde{d}_i)_z = \gamma\{\nabla^2\tilde{d}_i - \frac{|\mathbf{d}|^2}{\eta^2}\tilde{d}_i + \frac{1}{\eta^2}\tilde{d}_i\}, \quad 1 \le i \le 3. \tag{6.15}$$

And one can also derive the following $1D$ equations :

$$(u_1^*)_t + 2\psi_1^*(u_1^*)_z = \nu(u_1^*)_{zz} + 2(\psi_1^*)_z u_1^*, \tag{6.16}$$

$$(\omega_1^*)_t + 2\psi_1^*(\omega_1^*)_z = \nu(\omega_1^*)_{zz} + (u_1^{*2})_z, \tag{6.17}$$

$$-(\psi_1^*)_{zz} = \omega_1^*, \tag{6.18}$$

$$(d_{i0}^*)_t + 2\psi_1^*(d_{i0}^*)_z = \gamma(d_{i0}^*)_{zz} - \frac{\gamma}{\eta^2}|\mathbf{d_0^*}|^2 d_{i0}^* + \frac{\gamma}{\eta^2}d_{i0}^*, \quad 1 \le i \le 3. \tag{6.19}$$

Here u_1^*, ω_1^*, ψ_1^* and d_{i0}^*, $1 \le i \le 3$ are functions of only z and t. $\mathbf{d_0^*} = (d_{10}^*, d_{20}^*, d_{30}^*)$. Similarly, from the $3D$ exact solution (ru_1^*, rω_1^*, rψ_1^*, d_{i0}^*) based on $1D$ solution to the above system (6.16) to (6.17), we can construct finite-energy solutions in the following form,

$$\tilde{u}(r,z,t) = r(u_1^*(z,t)\chi(r) + u_1(r,z,t)), \tag{6.20}$$

$$\tilde{\omega}(r,z,t) = r(\omega_1^*(z,t)\chi(r) + \omega_1(r,z,t)), \tag{6.21}$$

$$\tilde{\psi}(r,z,t) = r(\psi_1^*(z,t)\chi(r) + \psi_1(r,z,t)), \tag{6.22}$$

$$\tilde{d}_i(r,z,t) = d_{i0}^*(z,t)\chi(r) + d_i(r,z,t), \quad 1 \le i \le 3, \tag{6.23}$$

where $\chi(r)$ is defined in Section 1. Then we have the similar results to Theorem 1.1 and 1.2.

Theorem 6.3. *For the* $3D$ *system* (6.8)-(6.10)*, assume* $u_1^*(z,0)$*,* $\psi_1^*(z,0)$*,* $\omega_1^*(z,0)$ *and* $d_{i0}^*(z,0)$ $(1 \le i \le 3)$ *are smooth functions which are periodic in* z *with period* 1*. Then there exists a global classical solution in the form of* (6.20)-(6.23)*, if initial conditions* $\tilde{\mathbf{u}}_\mathbf{0} \triangleq \tilde{\mathbf{u}}(r,z,0) \in H^1(R^3)$*,* $\tilde{d}_{i0} \triangleq \tilde{d}_i(r,z,0) \in H^2(R^3)$ $(1 \le i \le 3)$ *and* $\nu \ge \nu_0(\gamma, \lambda, \tilde{\mathbf{u}}_\mathbf{0}, \tilde{d}_{i0})$*.*

Theorem 6.4. *Suppose that initial conditions for* u_1*,* ω_1*,* ψ_1*, and* d_i $(1 \le i \le 3)$ *are smooth functions with compact support and odd in* z*. Moreover, assume that* $\eta > 1$*, and* $\|\tilde{\mathbf{u}}(0)\|^2 + \lambda\|\nabla\tilde{\mathbf{d}}(0)\|^2 + \frac{\lambda}{2\eta^2}\int_\Omega(|\tilde{\mathbf{d}}|^2-1)^2 r\,dr\,dz \le \frac{C}{\sqrt{M}}$*. For any given* $\nu > 0$*, there exists* $C(\nu) > 0$*, such that if* $M \ge C(\nu)$ *and* $H(0) \le 1$ *where* $H^2(t) = \|r\nabla u_1\|^2 + \|r\triangle\psi_1\|^2 + \sum_{i=1}^3\|\nabla^2 d_i\|^2$*. Then, solutions to the 3D system* (6.8)-(6.8) *in the form of* (6.20)-(6.23) *are globally smooth.*

6.3 Viscoelasticity

The system (1.1)-(1.3) can be considered as a special type of relaxed system of the Oldroyd-B model to describe the motion of incompressible viscoelastic fluids (see [23] and [13]). When $\gamma = 0$ in the equation (1.3), the system (1.1)-(1.3) makes an equivalent version of the Oldroyd-B model with infinite Weissenberg number,

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u} - \lambda\nabla\cdot(\nabla\phi\otimes\nabla\phi), \tag{6.24}$$

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, && (6.25) \\ \phi_t + (\mathbf{u} \cdot \nabla)\phi &= 0, && (6.26) \end{aligned}$$

where the basic energy law is

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^{\mathbf{2}} + \lambda\|\nabla\phi\|^{\mathbf{2}}\right) = -\nu\|\nabla\mathbf{u}\|^{\mathbf{2}}. \tag{6.27}$$

The entire coupled hydrodynamical system of Oldroyd-B model contains a linear momentum equation, the incompressibility and a microscopic equation specifying the special transport of the deformation tensor (or ϕ in our formulation).

Viscoelastic materials include a wide range of fluids with elastic properties, as well as solids with fluid properties. And it is due to the interaction between these microscopic elastic properties and macroscopic fluid motions, which can be viewed as the competition between the elastic energy and the kinetic energy, that provide us with many interesting rheological and hydrodynamical phenomena of viscoelastic material.

Remark 6.1. *In the context of hydrodynamics, the basic variable is the flow map (particle trajectory) $x(X,t)$. X is the original labeling (Lagrangian coordinate) of the particle, which is also referred to as material coordinate. x is the current (Eulerian) coordinate and called reference coordinate. For a given velocity field $\mathbf{u}(x,t)$, the flow map is defined by the ODE :*

$$x_t = \mathbf{u}(x(X,t),t), \;\; x(X,0) = X.$$

To incorporate the elastic properties of the material, we need to introduce the deformation tensor,

$$\mathcal{F}(X,t) = \frac{\partial x}{\partial X},$$

where we use the notation $\mathcal{F}_{ij} = \frac{\partial x_i}{\partial X_j}$. In the Eulerian coordinate, we can define $\tilde{\mathcal{F}}(x,t) = \mathcal{F}(X,t)$. Without any ambiguity, we will not distinguish these two notations. Using chain rule, we know $\mathcal{F}$ satisfies the following transport equation :

$$\mathcal{F}_t + (\mathbf{u} \cdot \nabla)\mathcal{F} \;=\; \nabla\mathbf{u}\mathcal{F}. \tag{6.28}$$

For incompressible fluids, $\nabla \cdot \mathbf{u} = 0$ indicates after taking divergence of both sides of the equation above, it becomes the transport equation for $\nabla \cdot \mathcal{F}$ as

$$(\nabla \cdot \mathcal{F})_t + \mathbf{u} \cdot \nabla(\nabla \cdot \mathcal{F}) = 0, \tag{6.29}$$

Since $\mathcal{F}$ is simply the identity matrix at $t = 0$, $\nabla \cdot \mathcal{F}|_{t=0} = 0$. It infers from the equation (6.29) *that for all $t \geq 0$,*

$$\nabla \cdot \mathcal{F} = 0. \tag{6.30}$$

The Oldroyd-B system with infinite Weissenberg number can be derived in the energetic variational point of view. For simplicity, we assume the internal energy W simply depends on $\mathcal{F}$ and the density to be the constant 1. Let us consider the action functional

$$\mathscr{A}(x) = \int_0^T \int_{\Omega_0} \left[\tfrac{1}{2}|x_t|^2 - \lambda W(\mathcal{F})\right] \, dX \, dt$$

Choose one parameter volume-preserving diffeomorphism x^ε, such that $\frac{dx^\varepsilon}{d\varepsilon}\big|_{\varepsilon=0} = y$, where volume-preserving indicates $det(\frac{\partial x^\varepsilon}{\partial X}) = 1$, hence $\nabla_x \cdot y = 0$. By Least Action Principle, we have

$$\begin{aligned}
0 &= \delta \mathscr{A}/\delta x = \frac{d}{d\varepsilon}|_{\varepsilon=0}\mathscr{A}(x^\varepsilon) \\
&= \int_0^T\int_{\Omega_0} (x_t \cdot y_t - \lambda \frac{\partial W}{\partial \mathcal{F}} : \frac{d\mathcal{F}^\varepsilon}{d\varepsilon}|_{\varepsilon=0}) dX \, dt \\
&= \int_0^T\int_{\Omega_t} -\tfrac{D\mathbf{u}}{Dt} \cdot y + \lambda \tfrac{\partial W}{\partial \mathcal{F}} \mathcal{F}^T \cdot \nabla y dx \, dt \\
&= \int_0^T\int_{\Omega_t} -[\frac{D}{Dt}\mathbf{u} + \lambda \nabla \cdot (\frac{\partial W}{\partial \mathcal{F}} \mathcal{F}^T)] y \, dx \, dt
\end{aligned}$$

Since y is an arbitrary divergence-free function, we actually derive the following momentum equation:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = -\lambda \nabla \cdot (\frac{\partial W}{\partial \mathcal{F}} \mathcal{F}^T),$$

with the pressure term as the Lagrangian multiplier for the incompressibility. Adding the dissipative viscosity term $\nu \Delta \mathbf{u}$, together with the transport equation (6.28)*, we recover a closed system*

$$\begin{aligned}
&\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \lambda \nabla \cdot (\frac{\partial W}{\partial \mathcal{F}} \mathcal{F}^T), \\
&\nabla \cdot \mathbf{u} = 0. \\
&\mathcal{F}_t + \mathbf{u} \cdot \nabla \mathcal{F} = \nabla \mathbf{u} \mathcal{F},
\end{aligned}$$

such a derivation by variation with respect to domain, i.e., Least Action Principle, is equivalent to the Principle of Virtual Work.

Remark 6.2. *We want to point out that, The MHD equations* (6.5)-(6.5) *and Liquid Crystal system* (6.8)-(6.10) *can also be derived by the energetic variational approach.*

Remark 6.3. *The term $\frac{\partial W}{\partial \mathcal{F}}$ is called the Piola-Krichhoff tensor, while $\frac{\partial W}{\partial \mathcal{F}} \mathcal{F}^T$ is the Cauchy-Green tensor. If we restrict ourselves to the Hookean elasticity case, i.e., $W(\mathcal{F}) = \frac{1}{2}tr(\mathcal{F}\mathcal{F}^T)$, then one arrives at the following equations for the viscoelastic system,*

$$\begin{aligned}
&\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \lambda \nabla \cdot (\mathcal{F}\mathcal{F}^T), \\
&\nabla \cdot \mathbf{u} = 0, \\
&\mathcal{F}_t + \mathbf{u} \cdot \nabla F = \nabla \mathbf{u} \mathcal{F}.
\end{aligned}$$

(6.30) implies there exists a vector function ϕ, such that

$$\mathcal{F} = \nabla^T \phi = \left(\begin{array}{c} -\nabla_2 \phi \\ \nabla_1 \phi \end{array} \right),$$

hence $2D$ equations can be rewritten as (6.24) to (6.26).

For the Oldroyd-B model with infinite Weissenberg number, the existence of local weak solutions has been proved by [22], [31] in $2D$ and $3D$ respectively. The main difficulties for obtaining the existence of global classical solution for the Cauchy problem, or periodic initial-boundary value problem, lie in the fact that the Oldroyd-B system has only partial dissipation. By delicate analysis, the authors managed to find extra damping mechanism from the induced stress term, and proved the existence of global classical solution near equilibrium.

However, there is no result for the existence of global solutions under the assumption of large viscosity constant ν–this remains a challenging open problem. Some attempts to find singular structures in the Oldroyd-B model have been done, as in [25], [29], where the energy is nevertheless assumed to be infinite. To this end, we are attempting to study the following intermediate system:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla \phi \otimes \nabla \phi), \tag{6.31}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{6.32}$$

$$\phi_t + (\mathbf{u} \cdot \nabla)\phi = -\gamma \cdot \phi, \tag{6.33}$$

where (6.33) is viewed as an equation with linear decay, with the basic energy law

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^{\mathbf{2}} + \lambda \|\nabla \phi\|^{\mathbf{2}}\right) = -\nu \|\nabla \mathbf{u}\|^{\mathbf{2}} - \gamma \|\nabla \phi\|^{\mathbf{2}}. \tag{6.34}$$

Actually, this system (6.31) to (6.33) is considered to be a bridge connecting Oldroyd-B model in finite and infinite cases. We are going to investigate this system on the existence of global weak solutions, as well as global classical solutions in both small initial data and large viscosity case. In addition, the properties of axisymmetric solutions will also be studied.

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