Arithmetic of the moduli of fibered algebraic surfaces with heuristics for counting curves over global fields

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Dedication

I would like thank my family and friends for believing in me.

Abstract

The classical theory of algebraic surfaces is essential in both geometry and number theory. The study of *fibrations* lies at the heart of the Enriques-Kodaira classification of compact complex surfaces as well as the Mumford-Bombieri classification of algebraic surfaces in positive characteristic. In my work, I consider the moduli of fibered algebraic surfaces through the moduli of fibrations and produce its arithmetic invariants of motivic nature with the aspiration of finding relevant applications to number theory under the global fields analogy.

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Chapter 1

Introduction

I believe my research work could be looked upon as a journey from the differential geometry of smooth 4-manifolds to the number theory in function fields through arithmetic geometry. While these two opposite peaks offer very distinct sceneries with different stories, the path to go from one extreme to another is a captivating adventure passing through various milestones namely the classics on algebraic curves and complex surfaces, the theory of moduli, the language of algebraic stacks, the invariants of arithmetic or homotopic nature and finally the unifying principle of the global fields analogy that connects function fields (geometry & topology) with number fields (number theory). Let me share with you my journal so far and also explain some of my recent works along the way.

As a geometer, we are inclined to seek the complete classification of class of objects we would like to understand. While difficult, such endeavors lead us to deeper understanding of the structure of the invariants (the 'geometry' of invariants) which often renders unexpected discoveries. Ever since learning about differential geometry & algebraic topology of manifolds, I developed a long standing interest in 4-manifolds. In many ways, 4-manifolds are more complicated than either lower or higher dimensional manifolds. In low dimensions, there are simply not many possibilities for complexity. The Geometrization conjecture and the 3-dimensional Poincaré conjecture were resolved by G. Perelman in 2003, yielding a relatively complete understanding of 3-manifolds [MT]. In dimensions strictly greater than 4, there is more fluidity than in

4-dimensions. For example, the *Whitney Trick* allows one to *untangle* submanifolds by moving them past each other in the extra dimensions. S. Smale used this trick to resolve the Poincaré conjecture in dimension ≥ 5 in 1961 [Smale], showing that certain aspects of high dimensional topology are quite tractable.

The dimension four is rather special, not only because we live in a *spacetime* which is a smooth 4-manifold equipped with the Einstein's field equation, it is the dimension where all of our sophisticated techniques and elegant theory converge to show us the intriguing behavior that is unique in dimension 4.

At the same time, it is also the desert where all of our efforts proved futile in a way that when we were able to come up with a new technique that shed light on one area, this ended up destroying the best classification conjectures of the given period as the new theory due to its effectiveness revealed to us a totally different aspects of the problem that weren't even known to us in the past which is unfortunately intractable with the current technology. That is, the smooth Poincaré Conjecture in 4 dimensions remains open, and symbolizes the lack of understanding in this boundary case between the simplicity of low dimensions and the fluidity of high dimensions. The discovery of exotic smooth structures on 4-manifolds [DK] added further complexity to the study of 4-manifolds.

As a mathematician, we have learned to start working on the problem from an angle where we can make slow but definite progresses. In this regard, the definitive result in topological realm was the Freedman's work on classifying simply-connected topological 4-manifolds upto homeomorphism by their intersection forms and in the differentiable realm was the Donaldson $SU(2)$ and Seiberg-Witten $U(1)$ gauge theoretic moduli spaces which gives us the smooth invariants.

This was truly revolutionary progress (an explosion in the 80's) where mathematical physics came into the differential geometry picture notably by *Sir Simon Donaldson* who considered that given a moduli (the 'space' of invariants) associated to the geometric objects, one can investigate the moduli space as a geometric object itself which means one could consider doing the integrals over the moduli space, study the intersections of the subvarieties on the moduli space or even count the size of the moduli space. While it was incredible what this invariant could do, the way Donaldson got to this invariant was not only beautiful but also remarkably inspiring philosophy. That is, through the notion of moduli we were able to find unity of the mathematical physics and the pure mathematics namely geometry. Often this differential geometric perspective has an analog in the algebraic geometry.

The classical theory of algebraic surfaces is essential in both geometry and number theory. An algebraic surface is an algebraic variety of dimension two. In the case of geometry over the field of complex numbers, an algebraic surface has complex dimension two (as a complex manifold, when it is non-singular) and so of dimension four as a smooth manifold. The theory of algebraic surfaces is much more complicated than that of algebraic curves (including the compact Riemann surfaces, which are genuine surfaces of (real) dimension two). The Enriques-Kodaira classification of surfaces was extended to the case of an arbitrary algebraically closed base field by Bombieri and Mumford (some results having been obtained previously by Zariski). It follows from their work that the classification of surfaces in characteristics $\neq 2, 3$ is identical to that over $\mathbb{C}$; in characteristics 2 and 3 certain non-classical surfaces appear. We restrict ourselves to characteristics larger than 3 for the moduli of elliptic surfaces and larger than 5 for the moduli of hyperelliptic genus 2 fibrations in this work. The study of *fibrations* lies at the heart of the Enriques-Kodaira classification of compact complex surfaces as well as the Mumford-Bombieri classification of algebraic surfaces in positive characteristic. In my work, I consider the moduli of fibered algebraic surfaces through the moduli of fibrations and produce its arithmetic invariants of motivic nature with the aspiration of finding relevant applications to number theory under the global fields analogy.

This project could be considered as the generalization of the beautiful work done in [EVW] where Jordan S. Ellenberg, Akshay Venkatesh and Craig Westerland proved a function field analogue of the Cohen-Lenstra heuristics on distributions of class groups by point counting the *Hurwitz spaces* which are the moduli spaces of branched covers of the complex projective line. That is, the philosophy of considering the moduli of fibrations and producing its arithmetic invariant with the aspiration of finding relevant applications to number theory under the global fields analogy. As the branched covers of the $\mathbb{P}^1$ are the fibrations with 0-dimensional fibers, the moduli of fibrations $f : X \to \mathbb{P}^1$

on fibered surfaces X with 1-dimensional fibers is the next most natural case to work on. The counting technique in our project is driven largely by the inspiring work of Benson Farb and Jesse Wolfson [FW] which in turn was motivated by the ideas in Graeme Segal's classical paper [Segal].

Chapter 2

Arithmetic of the moduli of semistable elliptic surfaces

2.1 Introduction

The study of *fibrations* lies at the heart of the Enriques-Kodaira classification of compact complex surfaces as well as the Mumford-Bombieri classification of algebraic surfaces in positive characteristics. The Kodaira dimension $\kappa \in \{-\infty, 0, 1, 2\}$ plays a crucial role in both classifications. In this regard, every elliptic surface has $\kappa \leq 1$ and the main classification result for surfaces states that every algebraic surface with $\kappa = 1$ is *elliptic*.

We call an algebraic surface X to be an *elliptic surface*, if it admits an elliptic fibration $f : X \to C$ which is a flat and proper morphism f from a nonsingular surface X to C where C is a nonsingular curve, such that a generic fiber is a smooth curve of genus one. While this is the most general setup, it is natural to work with the case when the base curve is the projective line $\mathbb{P}^1$ and there exists a distinguished section $S : \mathbb{P}^1 \hookrightarrow X$ coming from the identity points on each of the elliptic fibers.

All possible types of singular fibers of elliptic surfaces have been classified by the classical work of [Kodaira, Néron]. Recall that the singular fiber I_1 (also known as the *fishtail* fiber) is an irreducible rational curve with one unique node. The I_k–fiber (also known as the *necklace* fiber) is a nodal k–cycle of $\mathbb{P}^1$'s of self–intersection -2. One could restrict the class of elliptic fibrations to only have at worst nodal singular fibers such that the image of distinguished section S does not meet any of the nodal singular points and this in turn makes the elliptic fibration *semistable*, meaning the only possible types of fibers are smooth elliptic curves, fishtails, or necklaces.

What we have just described is a nonsingular semistable elliptic surface X which has a relatively minimal, semistable elliptic fibration $f : X \to \mathbb{P}^1$ that comes with a distinguished section $S : \mathbb{P}^1 \hookrightarrow X$. Note that semistable elliptic fibrations are also known as *elliptic Lefschetz fibrations* in differential geometry. A *smooth Lefschetz fibration* over S^2 is a differentiable surjection $f : M \to S^2$ of a closed oriented smooth 4-manifold M with finitely many critical points of the form $f(z_1, z_2) = z_1^2 + z_2^2$. The structure of Lefschetz fibrations is essential in symplectic geometry as it allows the topological characterization of symplectic manifolds by the work of [Donaldson, Gompf]. Amazingly, the classification of elliptic Lefschetz fibrations by Moishezon and Livné [Moishezon] implies algebracity in all cases meaning every smooth elliptic Lefschetz fibration is indeed diffeomorphic to a nonsingular semistable elliptic surface over $\mathbb{P}^1$.

To review a few other aspects of elliptic surfaces, we recall that the general elliptic fibers of any semistable elliptic surface are all Calabi-Yau curves. Therefore one could consider this family of elliptic curves as a *Calabi-Yau fibration*. In fact, these are the only kinds of Calabi-Yau fibrations over $\mathbb{P}^1$ as there are no other Calabi-Yau curves other than elliptic curves. Also from arithemetic geometry perspective, these can be interpreted as a relative curve over a Dedekind scheme which is the central object in the theory of arithmetic surfaces. Thus, any nonsingular semistable elliptic surface can be characterized as a family of elliptic curves with squarefree conductor $\mathcal{N}$.

Our primary goal of the paper is to enumerate the $\mathbb{F}_q$–points of the moduli of nonsingular semistable elliptic surfaces with the discriminant degree $12n$ by considering the moduli stack $\mathcal{L}_{1,12n}$ of stable elliptic fibrations over $\mathbb{P}^1$ with $12n$ nodal singular fibers and a distinguished section. This is justified by showing the bijection of K–points between the two moduli spaces where K is any field of characteristic neither 2 nor 3 (see proposition 21 for the proof). We show that $\mathcal{L}_{1,12n} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ which is a Deligne–Mumford algebraic mapping stack of regular morphisms. For the purpose of point counting, we consider the coarse moduli space $L_{1,12n}$ of $\mathcal{L}_{1,12n}$ instead as to avoid taking account of the automorphisms of the points of $\mathcal{L}_{1,12n}$.

In order to acquire the $\mathbb{F}_q$–points of the coarse moduli space $L_{1,12n}$ of $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$, we consider the more general case of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ and its corresponding coarse moduli space $c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$. We provide the explicit stratification of $c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$ which allows us to obtain $[c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))]$, a class in the Grothendieck ring of K–varieties with $\mathrm{char}(K)$ not dividing a or b, expressed as a polynomial of the Lefschetz motive $\mathbb{L} := [\mathbb{A}^1]$.

Theorem 1 (Motive count of the moduli $c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$)**.** *If* $\mathrm{char}(K)$ *does not divide a or b, then the class* $[c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))]$ *in* $K_0(\mathrm{Var}_K)$ *for the coarse moduli space of* $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ *is equivalent to*

$$[c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))] = \mathbb{L}^{(a+b)n+1} - \mathbb{L}^{(a+b)n-1}$$

Then, by recognizing $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over any field K of characteristic $\neq 2,3$ and by using $\#_q : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$ (see section 2.6 for a discussion) to count $\mathbb{F}_q$–points when $\mathrm{char}(\mathbb{F}_q) \neq 2,3$, we acquire the point count of the moduli of nonsingular semistable elliptic surfaces over $\mathbb{F}_q$:

Corollary 2 (Motive/Point count of the moduli $L_{1,12n}$)**.** *If char*$(\mathbb{F}_q) \neq 2,3$*, then*

$$[L_{1,12n}(\mathbb{F}_q)] = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1} .$$

Consequently,

$$|L_{1,12n}(\mathbb{F}_q)| = q^{10n+1} - q^{10n-1} .$$

Lastly, we consider the *global fields analogy* which is the observation that for any finite field $\mathbb{F}_q$, the finite extensions of the function field $\mathbb{F}_q(t)$ have much in common with the finite extensions of $\mathbb{Q}$. In our case, the arithmetic invariant $|L_{1,12n}(\mathbb{F}_q)|$ in the arithmetic function field realm lets us explicitly compute the growth rate of $Z_{\mathbb{F}_q(t)}(B)$ which is the counting function of semistable elliptic fibrations through the notion of *bounded height* of discriminant $\Delta(X)$.

Theorem 3 (Computation of $Z_{\mathbb{F}_q(t)}(B)$)**.** *The counting of semistable elliptic fibrations over $\mathbb{F}_q(t)$ by $ht(\Delta(X)) = q^{12n} \leq B$ satisfies the following inequality:*

$$Z_{\mathbb{F}_q(t)}(B) \leq \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot \left(B^{\frac{5}{6}} - 1\right)$$

In other words, $Z_{\mathbb{F}_q(t)}(B) \sim \mathcal{O}\left(B^{\frac{5}{6}}\right)$.

An analogous object in the number field realm $Z_{\mathbb{Q}}(B)$ is the counting function of semistable elliptic curves over $\mathbb{Q}$ with bounded height of discriminant Δ. In the end, we formulate a conjecture that the asymptotic of $Z_K(B)$ will match for both of the global fields.

Conjecture 4 (Asymptotic of $Z_{\mathbb{Q}}(B)$)**.** The asymptotic growth rate of $Z_{\mathbb{Q}}(B)$, the counting of semistable elliptic curves over $\mathbb{Q}$ by $ht(\Delta) \leq B$, follows from the polynomial growth rate of $Z_{\mathbb{F}_q(t)}(B) \sim \mathcal{O}\left(B^{\frac{5}{6}}\right)$.

While the counting of the stable elliptic curves with squarefree Δ has been done in the past over $\mathbb{Q}$ by the work of [Baier], the counting $Z_{\mathbb{F}_q(t)}$ of the semistable elliptic curves with non-squarefree Δ over the (global) function field $\mathbb{F}_q(t)$ as well as the analogous heuristic $Z_{\mathbb{Q}}$ over the number field $\mathbb{Q}$ were previously unknown.

2.2 Kodaira dimension κ and classification of curves and surfaces

While the classification of algebraic surfaces is not yet complete like in the theory of algebraic curves, we are able to say that it is reasonably complete by the combined effort of algebraic geometers [BHPV]. Let us recall the definition of the Kodaira dimension κ.

Definition 5. For X a smooth projective variety and K_X a canonical divisor on X. There is a rational map $\phi_{nK} : X \to \mathbb{P}^N$ associated to the pluricanonical linear system $|nK_X|$. The maximum dimension of the image of ϕ_{nK} is called $\kappa(X)$ the *Kodaira dimension of* X.

One can interpret this definition for smooth projective curves C (1-folds) where the $\kappa(C) \in \{-\infty, 0, 1\}$, Hodge theory tells us the dimension of the space of global sections of the canonical line bundle K_C is

$$dim(H^0(C, K_C)) = h^1(\mathcal{O}_C) = genus(C) = g$$

where the g is the topological genus of C which is the main discrete invariant of compact Riemann surface Σ_g. Thus the canonical map takes the form $C \dashrightarrow \mathbb{P}^{g-1}$. It determines easily the Kodaira dimension, and the Enriques classification of curves is the subdivision

1. $\kappa(C) \Leftrightarrow g(C) = 0 \Leftrightarrow C \cong \mathbb{P}^1$,
2. $\kappa(C) = 0 \Leftrightarrow g(C) = 1 \Leftrightarrow C \cong \mathrm{c}/(\mathbb{Z} + \tau\mathbb{Z})$, with $\tau \in \mathrm{c}, Im(\tau) > 0 \Leftrightarrow C$ is elliptic,
3. $\kappa(C) = 1 \Leftrightarrow g(C) \geq 2 \Leftrightarrow C$ is of general type.

Note that the $Im(\phi_{nK}(\mathbb{P}^1))$ is an empty set as the canonical line bundle has no global sections which we set the Kodaira dimension of the projective line $\mathbb{P}^1$ to be $-\infty$.

Furthermore, $Im(\phi_{nK}(E))$ is a point as the canonical map simply collapses an elliptic curve E to a point $\mathbb{P}^0$ and thus the Kodaira dimension of E is 0.

For a curve C of genus higher than 1, the canonical map $C \to \mathbb{P}^{g-1}$ is interesting and at genus two the canonical map is $C \to \mathbb{P}^1$ which makes all genus 2 curves to be hyperelliptic.

As for the algebraic surfaces X (2-folds) where $\kappa(X) \in \{-\infty, 0, 1, 2\}$, before giving the Enriques-Kodaira-Mumford-Bombieri classification of projective surfaces over the complex numbers or over a field K with characteristic larger than 3, it is convenient to discuss further the birational invariants of surfaces [BHPV].

Remark 6. An important birational invariant of smooth varieties X is the fundamental group $\pi_1(X)$.

For surfaces, the most important invariants are :

- the **irregularity** $q := h^1(\mathcal{O}_X)$
- the **geometric genus** $p_g := P_1 := h^0(X, K_X)$, which for surfaces combines with the irregularity to give the **holomorphic Euler-Poincaré characteristic** $\chi(X) := \chi(\mathcal{O}_S) := 1 - q + p_g$
- the **bigenus** $P_2 := h^0(X, 2K_X)$ and especially the **twelfth plurigenus** $P_{12} := h^0(X, 12K_X)$.

If X is a non ruled minimal surface, then also the following are birational invariants:

- the selfintersection of a canonical divisor K_X^2, equal to $c_1(X)^2$,
- the **topological Euler number** $e(X)$, equal to $c_2(X)$ by the Poincaré Hopf theorem, and which by **Noether's theorem** can also be expressed as

$$e(X) = 12\chi(X) - K_S^2 = 12(1 - q + p_g) - K_S^2,$$

- the **topological index** $\sigma(X)$ (the index of the quadratic form

$q_S : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z})$, which, by the **Hodge index theorem**, satisfies the equality

$$\sigma(X) = \frac{1}{3}(K_S^2 - 2e(X)),$$

- in particular, all the Betti numbers $b_i(X)$ and
- the positivity $b^+(X)$ and the negativity $b^-(X)$ of q_S (recall that $b^+(X)+b^-(X) = b_2(X)$).

The Enriques-Kodaira-Mumford-Bombieri classification of (complex) algebraic surfaces gives a very simple description of the surfaces with nonpositive Kodaira dimension:

- X is a ruled surface of irregularity $g \iff :$

  $\iff :$ X is birational to a product $C_g \times \mathbb{P}^1$, where C_g has genus $g \iff$

  $\iff P_{12}(X) = 0,\ q(X) = g \iff$

  $\iff \kappa(X) = -\infty,\ q(X) = g.$

- X has $\kappa(X) = 0 \iff P_{12}(X) = 1.$

There are four classes of such surfaces with $\kappa(X) = 0$:

- Tori $\iff P_1(X) = 1, q(X) = 2,$
- K3 surfaces $\iff P_1(X) = 1, q(X) = 0,$
- Enriques surfaces $\iff P_1(X) = 0, q(X) = 0,\ P_2(X) = 1,$
- Hyperelliptic surfaces $\iff P_{12}(X) = 1, q(X) = 1.$

Next come the surfaces with strictly positive Kodaira dimension:

- X is a properly elliptic surface $\iff :$

  $\iff : P_{12}(X) > 1$, and $H^0(12K_X)$ yields a map to a curve with fibres elliptic curves $\iff$

  $\iff$ X has $\kappa(X) = 1 \iff$

  $\iff$ assuming that X is minimal: $P_{12}(X) > 1$ and $K_X^2 = 0$.

- X is a surface of general type $\iff$:

  $\iff$: X has $\kappa(X) = 2$ $\iff$

  $\iff$ $P_{12}(X) > 1$, and $H^0(12K_X)$ yields a birational map onto its image Σ_{12} $\iff$

  $\iff$ assuming that X is minimal: $P_{12}(X) > 1$ and $K_X^2 \geq 1$.

Note that since the Kodaira dimension is defined for all surfaces, not necessarily minimal with respect to blow-downs, the above theorem does not require preparatory blow-downs. Indeed, $\kappa(X)$ is invariant with respect to blow-ups and blow-downs. Also, the Kodaira dimension can be $-\infty$, and, as stated above, that happens exactly for rational and ruled surfaces.

2.3 Semistable elliptic fibrations over $\mathbb{P}^1$

In this section, we define the central object of our investigation the semistable elliptic fibrations over $\mathbb{P}^1$ and review the classification, related invariants together with examples. For detailed references on elliptic curves and surfaces, we refer the reader to [Silverman, Miranda] respectively.

Let us first define the semistable fibrations, let $f : X \to \mathbb{P}^1$ be a fibration (a flat, proper morphism) over the projective line with $g > 0$, where g is the genus of the general fiber X_t for general geometric point t of $\mathbb{P}^1$.

Definition 7. A fiber X_t is *semistable*, if it has the following properties:

1. X_t is reduced,
2. The only singularities of X_t are nodes,
3. X_t contains no (-1)-curves.

The fibration f is semistable, if all fibers X_t are semistable.

By the semistable reduction theorem one can always reduce the study of general fibrations to the study of semistable fibrations which are much easier to handle. Note that the stable reduction theorem turns all fibers of the fibration into stable curves (unique nodal singularity) while producing singularities on the resulting surface as we would contract (-2)-curves.

In this note, we work with nonsingular semistable elliptic fibrations where the fiber genus is 1. The only semistable fibers with $g(X_t) = 1$ are of the following type I_k as in [Kodaira, Néron]

1. I_0 : nonsingular elliptic (generic smooth fiber),
2. I_1 : irreducible rational with one node (fishtail singular fiber),
3. $I_{k\geq 2}$: k–cycle of (-2)-curves (necklace singular fiber).

Definition 8. A nonsingular semistable elliptic surface X is a nonsingular surface equipped with a relatively minimal, semistable elliptic fibration $f : X \to \mathbb{P}^1$ that comes with a distinguished section $s : \mathbb{P}^1 \hookrightarrow X$ such that the image of s does not intersect nodal singular points of each fiber.

From now on, all semistable elliptic surfaces are assumed to be nonsingular. These are exactly the nonsingular semistable elliptic fibrations over $\mathbb{P}^1$. Semistable elliptic surfaces contain only nodal singular fibers of fishtail and necklace types I_k $(k \geq 1)$ such that for a given semistable elliptic fibration it has $12n$ nodal singular fibers distributed over μ distinct singular fibers that are $I_{k_1}, \cdots, I_{k_i}, \cdots, I_{k_\mu}$ with $\sum_{i=1}^{\mu} k_i = 12n$ as we allow each of the singular fiber to contain multiple nodal singular points but no cuspidal singularities.

The holomorphic Euler characteristic χ of the semistable elliptic surface X defined over $\mathbb{C}$ determines the number of nodal singular fibers which in turn determines all the other invariants of the smooth semistable elliptic surface X.

Example 9. Here we list some of the properties of a semistable elliptic surface X with $\chi = n$. This also works for any field K with Char $K \neq 2, 3$.

1. When $n = 1$, X is a rational elliptic surface with the Kodaira dimension $\kappa = -\infty$ which has 12 nodal singular fibers. It is acquired from a pencil of cubic curves in $\mathbb{P}^2$ by blowing up a base locus of nine points coming from the intersection of two general cubic curves.

2. When $n = 2$, X is a $K3$ surface with an elliptic fibration which has the Kodaira dimension $\kappa = 0$ that has 24 nodal singular fibers. Note that X is a minimal surface.

3. When $n \geq 3$, X is called a properly elliptic surface with Kodaira dimension $\kappa = 1$ that has $12n$ nodal singular fibers. Note that X is also a minimal surface.

The semistable elliptic fibrations over $\mathbb{P}^1$ in algebraic geometry are also known as elliptic Lefschetz fibrations over $\mathbb{P}^1$ in differential geometry. The smooth Lefschetz fibration where the fibration map f is a differentiable surjection $f : M \to S^2$ of a closed oriented smooth 4-manifold M to a 2-sphere $S^2 = \mathbb{P}^1$. We will first define the mapping class group Γ_g as Lefschetz fibrations are described by the monodromy factorizations induced by the singular fibers of the fibration. After basic properties of genus g Lefschetz fibration has been established, we will review the elliptic Lefschetz fibrations and discuss the possible singular fibers which are the fishtails I_1 and the necklaces I_n as well as the classification which shows smooth elliptic Lefschetz fibrations are all isomorphic to holomorphic elliptic Lefschetz fibrations. For more information on mapping class groups we refer the readers to a book written by Farb and Margalit [FM] and the proper reference for the geometry and topology of Lefschetz fibrations on 4-manifolds is a book by Gompf and Stipsicz [GS].

The definition of mapping class groups

The study of the mapping class group Γ_g of Riemann surface was initiated by Max Dehn and Jakob Nielsen in the twenties.

Definition 10. Let $Diff^{+}(\Sigma_g)$ denote the group of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$ and $Diff_0^{+}(\Sigma_g)$ be the canonical subgroup of $Diff^{+}(\Sigma_g)$ consisting of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$ that are isotopic to the identity. *The mapping class group* Γ_g of Σ_g is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g, i.e.,

$$\Gamma_g = Diff^{+}(\Sigma_g) / Diff_0^{+}(\Sigma_g).$$

Definition 11. Let α be a simple closed curve on Σ_g. A *right handed Dehn twist* t_α about α is the isotopy class of a self-diffeomophism of Σ_g obtained by cutting the surface Σ_g along α and gluing the ends back after rotating one of the ends 2π to the right.

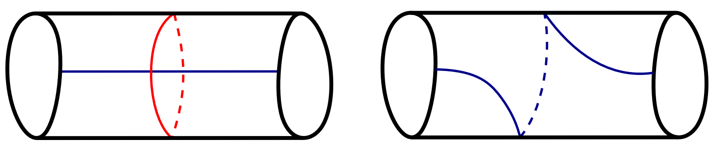


Figure 2.1: A positive Dehn twist along α to a cylinder

The mapping class group Γ_g is finitely generated by $3g-1$ Dehn twists which was proven by the work of Dehn and Lickorish (cf. [FM]). It follows that the conjugate of a Dehn twist is again a Dehn twist. That is, if $f : \Sigma_g \to \Sigma_g$ is an orientation-preserving diffeomorphism, then it is easy to check that $f \circ t_\alpha \circ f^{-1} = t_{f(\alpha)}$. The result below is well known [FM].

Theorem 12. *The mapping class group* Γ_1 *of the torus* T^2 *is isomorphic to* $SL(2,\mathbb{Z})$ *and can be presented as* $\Gamma_1 = \{A, B \mid ABA = BAB, (AB)^6 = 1\}$

By the above isomorphism, we have the Dehn twists t_α and t_β along the α meridian curve and the β parallel curve intersecting each other transversally in a unique point in T^2 map to the following matrices A and B respectively.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

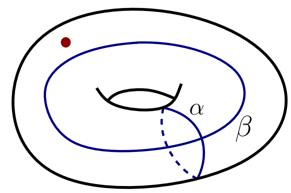


Figure 2.2: Meridian α and Parallel β curves on a torus

It is worth noting that A and B are in the same conjugacy class and the braid relation $ABA = BAB$ holds in Γ_1. More importantly, there is the positive relation $(AB)^6 = 1$. This will later manifest as the monodromy of the building block genus 1 Lefschetz fibration $f_1 : E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2 \to \mathbb{P}^1$. Accordingly, the monodromy of $E(n)$ is $(AB)^{6n} = 1$ for $f_n : E(n) \to \mathbb{P}^1$.

The definition of Lefschetz fibrations

The Lefschetz fibration is a generalization of the ideas in complex Morse theory applied to certain smooth and symplectic 4-manifolds.

Definition 13. A smooth map $f : X \to \Sigma$ from a closed connected oriented smooth 4-manifold X onto a closed connected oriented smooth 2-manifold Σ is said to be a *Lefschetz fibration*, if f admits finitely many critical points $C = p_1, p_2, \cdots, p_k$ on which there are orientation-preserving complex coordinate neighborhoods such that locally f takes the form $(z_1, z_2) \mapsto z_1^2 + z_2^2$. It is consequence of this definition that $X \setminus f^{-1}(C) \to \Sigma \setminus C$ is a smooth fiber bundle with fiber F a closed oriented 2-manifold. If the genus of a generic fiber F is g, we refer to f as a *genus g Lefschetz fibration.* Furthermore, we assume that f is relatively minimal, that is, there is no fiber containing a sphere of square -1 so in particular the fiber genus is always strictly positive.

We will restrict our investigation to Lefschetz fibrations over the sphere (i.e., the $\Sigma = S^2$ in $f : X \to \Sigma$). Restricting the class of Lefschetz fibrations to be over $\mathbb{P}^1$ is natural as an exact sequence $\pi_1(F) \to \pi_1(X) \to \pi_1(\Sigma) \to \pi_0(F) \to 0$ implies that a fiber F is always connected (cf. [GS]).

By f having only the non-degenerate and isolated critical points, each singular fiber of the Lefschetz fibration is a nodal curve with a unique nodal singularity and it is obtained by shrinking a simple closed curve (the *vanishing cycle*) in the regular fiber to the nodal point of the singular fiber. They fall into two classes: irreducible fibers, where we collapse a non-separating vanishing cycle in the Riemann surface and reducible fibers, where we collapse a separating vanishing cycle which gives the one-point union of smooth Riemann surfaces of smaller genera. The local monodromy around a singular fiber of a Lefschetz fibration $f : X \to S^2$ is a positive Dehn twist t_α along the corresponding vanishing cycle α. See Figure 2.1 on page 15. The product of all the local monodromies of f is trivial in the mapping class group Γ_g of genus g as $S^2 \setminus D$ where D is a disk big enough to contain all the critical values, has the trivial monodromy equal to the identity in the mapping class group. Such a relation in Γ_g is called a *positive relation*, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are vanishing cycles of f.

$$t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_n} = 1 \tag{2.1}$$

Given a *positive relation* it encodes the topology of X as the identity monodromy factorization determines the topology of X by a monodromy homomorphism $\psi_X : \pi_1(\mathbb{S}^2 \backslash \{f(p_i)\}) \to \Gamma_g$. The map ψ_X maps the generators of the fundamental group which encircle a single critical point once in an anticlockwise fashion to positive Dehn twists in the mapping class group.

Elliptic Lefschetz fibrations over $\mathbb{P}^1$

The Lefschetz fibrations over S^2 whose regular fibers are smooth tori and singular fibers are nodal elliptic curves are called elliptic Lefschetz fibrations over $\mathbb{P}^1$. We will provide in this section the classification of the elliptic Lefschetz fibrations over $\mathbb{P}^1$ as well as explain the possible singular fibers of elliptic Lefschetz fibrations which are either *fishtail* fiber I_1 or *necklace* fiber I_n $(n \geq 2)$. We will also describe the cyclic $n-$fold branched covering construction which can be applied to $E(1)$ along 2 regular fibers to construct a complex elliptic surface which is diffeomorphic to the 4-manifold $E(n)$ acquired by the fiber sum construction.

One has the building block genus 1 Lefschetz fibration $f_1 : E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2 \to S^2$ which can be obtained from a (Lefschetz) pencil of cubic curves by blowing up base locus of nine points coming from the intersection of two generic cubic curves. In terms of the mapping class group monodromy, the rational elliptic surface $E(1)$ has the positive relation $(t_\alpha \cdot t_\beta)^6 = 1$ in $\Gamma_1 \cong SL(2, \mathbb{Z})$ where α is the meridian curve of the torus and the β is the parallel curve of the torus.

For elliptic surfaces S_1, S_2, the fiber sum $S_1 \#_\phi S_2$ is defined in the following way: after deleting a neighborhood of a regular fiber in both surfaces, we glue the boundary T^3's via a fiber-preserving, orientation reversing diffeomorphism ϕ. The resulting surface which inherits a complex structure and an elliptic fibration if $\phi = id_{F=T^2} \times$ (*complex conjugation*) is called a untwisted fiber-sum of S_1 and S_2 and denoted by $S_1 \#_\phi S_2$. Note that $E(2)$ is the famous $K3$ surface and $E(n)$ is defined inductively as

$$E(n) = E(n-1) \#_{\phi=id} E(1)$$

Easy computation shows that $\pi_1(E(n)) = 1$, $e(E(n)) = 12n$, $\sigma(E(n)) = -8n$ and $b_2^+(E(n)) = 2n - 1$ with $(\chi_h(E(n)), c_1^2(E(n))) = (n, 0)$. $E(n)$ is spin iff n even.

Theorem 14. *[Moishezon]*
Let $f : M \to S^2$ be a genus 1 Lefschetz fibration and let $e(M)$ be the Euler characteristic of M. Then $e(M) > 0$, $e(M) \equiv 0$ (mod12) and $f : M \to S^2$ is isomorphic to $f \cong \# n f_1$ the fiber sum of $n = \frac{e(M)}{12}$ copies of $f_1 : \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2 \to S^2$. The 4–manifold M is Kähler and the Lefschetz fibration f is holomorphic.

The classification states that elliptic Lefschetz fibrations over $\mathbb{P}^1$ are classified to be $E(n) \cong \# nE(1) = \# n f_1 = \underbrace{f_1 \# \cdots \# f_1}_{n}$ the untwisted fiber-sum of n copies of genus 1 Lefschetz fibration $f_1 : E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2 \to S^2$. $E(n)$ has the positive relation $(t_\alpha \cdot t_\beta)^{6n} = 1$ in $\Gamma_1 \cong SL(2, \mathbb{Z})$. And the regular fibers are elliptic curves with marked point for the identity giving the distinguished section. One thing to note is that the number of irreducible singular fibers (nodal elliptic curves) n is always multiple of twelve. (i.e., $n \equiv 0$ (mod12))

For example, the standard elliptic fibration we get by blowing up nine base points of a generic elliptic pencil in $\mathbb{CP}^2$ results the monodromy factorization $(ab)^6$. Using the braid relation $aba = bab$ it can be shown that $(a^3b)^3$ also defines an elliptic fibration on $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$. Furthermore, it is easy to see that for any expression $x \in \Gamma_1$ the mapping class $a^x = xax^{-1}$ can be identified with the right–handed Dehn twist along the image of a under a map giving x. Note, for example, that the braid relation implies that $b = a^{ab}$.

The monodromy of a fishtail fiber can be shown to be equal to the right–handed Dehn twist along the vanishing cycle corresponding to the given singular fiber. An I_n–fiber can be created by collapsing n parallel (homologically essential) simple closed curves, therefore the monodromy of such a fiber is equal to the n^{th} power of the right–handed Dehn twist along one of the parallel curves.

In our constructions we will need the existence of a section, which can also be read off from the monodromy factorization. In general, a Lefschetz fibration admits a section if the monodromy factorization induced by it can be lifted from the mapping class group of its generic fiber to the mapping class group of the fiber with one marked point. In the case of a genus–1 Lefschetz fibration, however, the forgetful map $f\colon \Gamma_1^1 \to \Gamma_1$ mapping from the mapping class group Γ_1^1 of T^2 with one marked point to Γ_1 is an isomorphism, implying in particular

Lemma 15. *Any genus–1 Lefschetz fibration over S^2 admits a section.* □

The above (smooth) symplectic fiber sum construction of $E(n)$ can be made explicitly holomorphic by the cyclic $n-$fold branched covering construction,

$$\begin{array}{ccc} E(n) = \phi_n^* E(1) & \longrightarrow & E(1) \\ \downarrow f_n & & \downarrow f_1 \\ \mathbb{P}^1 & \xrightarrow{\phi_n} & \mathbb{P}^1 \end{array}$$

where one takes the $\phi_n : \mathbb{P}^1 \to \mathbb{P}^1$ such as $\phi_n(z) = z^n$ that has 0 and ∞ as fixed points. In the target Riemann sphere we arrange two regular elliptic fibers to be over the

0 and ∞ such that when we use the map ϕ_n to pull back holomorphic elliptic Lefschetz fibration $f_1 : E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2 \to \mathbb{P}^1$ it gives the holomorphic elliptic Lefschetz fibration over $\mathbb{P}^1$ isomorphic to the above $n-$fold untwisted fiber-sum $E(n) = \phi_n^* E(1)$ (cf. [GS]).

2.4 Deligne–Mumford moduli stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

In order to formulate the moduli $\mathcal{L}_{1,12n}$ of nonsingular semistable elliptic fibrations over $\mathbb{P}^1$, we are in need of the moduli of stable elliptic curves $\overline{\mathcal{M}}_{1,1}$. The motivation for moduli of curves is simple, the set of isomorphism classes of smooth curves of genus g over c are in bijection with the complex points of an irreducible variety $M_{g,n}$ called the coarse moduli space of smooth genus g curves with n markings.

One might notice that we need singular fibers in Lefschetz fibrations, this is accounted for by considering the *'compactification'* of $M_{g,n}$ by adding new points that correspond to the so-called *stable curves*. Stable curves are complex algebraic curves that are allowed to have exactly one type of singularity, namely, the simple nodal curve with a unqiue nodal point. The above is what we call the coarse moduli space of curves. Coarse moduli space is not enough for us, however, as we would like two points in the moduli to be close to each other if the objects that they represent are small deformations of each other (i.e, we want to be able to uniquely pullback families of (stable) curves). That is, we want the fine moduli space of curves. However, even at genus 1, the fine moduli 'space' $\overline{\mathcal{M}}_{1,1}$ is not an easy object to define nor to describe due to the extra automorphisms on objects we wish to parametrize (the special elliptic curves) which make the $\overline{\mathcal{M}}_{1,1}$ not a variety or even a scheme. As we will see, due to the simple fact that there exists the elliptic involutions for each and every curves of genus 1 (i.e., every point in the moduli has to have quotient singularity with $\mathbb{Z}_2$ isotropy), the moduli space of elliptic curves is the worst case for naïve hope that a moduli is just a variety. This means we must enlarge the category of varieties to include objects called 'stacks' which are even more general than schemes. Stacks often arise in nature as the quotient of a variety by an algebraic group action and these share many properties with true varieties.

We will review notions and collect results that are necessary to reach a point where we can present the following diagram over any fields K with characteristics not equal to 2 or 3.

$$
\begin{array}{ccc}
\overline{\mathcal{C}}_{1,1} & = & \overline{\mathcal{M}}_{1,2} \\
{\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\
\overline{\mathcal{M}}_{1,1} & = & \mathcal{P}(4,6)_{[a_4:a_6]} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\overline{M}_{1,1} & = & \mathbb{P}^1_{[j:1]}
\end{array}
$$

To explain briefly, the leading fine moduli is the Deligne–Mumford compactified stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves which comes with two wingmen, the universal family $\overline{\mathcal{C}}_{1,1}$ and the coarse moduli space $\overline{M}_{1,1}$. That is, as $\overline{\mathcal{M}}_{1,1}$ is a DM stack, we are able to say that the universal family $\overline{\mathcal{C}}_{1,1} \cong \overline{\mathcal{M}}_{1,2}$ exists over $\overline{\mathcal{M}}_{1,1}$ with $p : \overline{\mathcal{C}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ allowing the unique pullback of the stable family of elliptic curves. It also has the coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1_{[j:1]}$ (an honest variety which is the 1-point compactification $\overline{\mathbb{A}}_1 = \mathbb{P}^1_{[j:1]}$ of the j-line) which has a natural morphism $\pi : \overline{\mathcal{M}}_{1,1} \to \overline{M}_{1,1}$ giving the bijection on geometric points. Remaining question is why the fine moduli of stable elliptic curves is $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6)_{[a_4:a_6]}$ the weighted projective (line) stack with the weighted homogeneous coordinate $[a_4 : a_6]$. In order to answer this we need to go back to the idea of Weierstrass family of elliptic curves.

Let us start with the classification of elliptic curves over any fields K with prime characteristic not equal to 2 or 3. We will use the idea of the Weierstrass form of elliptic curves to talk about the family of elliptic curves as well as the Legendre form for thinking of elliptic curves as branched cover of $\mathbb{P}^1$ with 4 branch points. Then, we will introduce the unifying concept of a j-invariant and discuss the groups of extra automorphisms on special elliptic curves related to the special values of the j-invariant. Once one understands these elementary facts and connect it to the notion of the moduli of elliptic curves. It is not hard to see that $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6)$ over any fields K with characteristics not equal to 2 or 3. In our exposition, we will freely adopt the established literature for elliptic curves [Silverman] and Deligne-Mumford stacks [ACGH, Olsson2]. Since we are especially interested in presenting the Deligne–Mumford compactified stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves as weighted projective quotient stack $\mathcal{P}(4,6)$ over characteristic zero fields as well as any fields K with positive characteristics not equal to 2 and 3, we rely on [AOV] where the notion of 'tame stack' is developed carefully.

Elliptic curves E : Discriminant Δ and j-invariant

Let us define and collect properties of elliptic curves we will be needing later. Good reference for elliptic curves is a book written by [Silverman].

Definition 16. When a cubic equation called Weierstrass equation $y^2 = x^3 + a_4 x + a_6$ with $a_4,\ a_6 \in K$, $\mathrm{char}(K) \notin \{2,3\}$ has nonzero discriminant $\Delta = -16(4a_4^3 + 27a_6^2)$, it is called nonsingular and the set

$$E = \{(x,y) \in K^2 \mid E(x,y) = 0\} \cup \{\infty\}$$

is called an elliptic curve over K.

It is known that if E is a smooth projective algebraic curve of genus 1 over any fields K with prime characteristic not equal to 2 or 3 then E is isomorphic to a smooth cubic curve in $\mathbb{P}^2$ that can be presented in affine form by the Weierstrass equation. Coefficients a_4 and a_6 are defined not uniquely but only up to admissible transformations (which gives the same j-invariant and discriminant $\Delta \mapsto t^{12}\Delta$).

$$a_4 \mapsto t^4 a_4, \quad a_6 \mapsto t^6 a_6$$

This presentation is useful for thinking of the family of elliptic curves as the variation of the coefficients a_4 and a_6.

Also, there is another useful way to write down any elliptic curves called the Legendre form $y^2 = x(x-1)(x-\lambda)$ as any elliptic curves $E \in \overline{\mathcal{M}}_{1,1}$ can be expressed as a 2-fold branched covering of $\mathbb{P}^1$ branched at $0, 1, \lambda, \infty$ where $\lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\}$. The choice in normalizing the ramification points appear as S_3 symmetry which can be dealt with by associating to each curve its j-invariant which satisfies $j(\lambda) = j(\lambda')$ if and only if $\lambda' \in \{\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{\lambda-1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}\}$.

Now, we recall the formula for the j-invariant of elliptic curves which unifies the two presentations.

$$j = j(a_4, a_6) = 1728\frac{4a_4^3}{4a_4^3 + 27a_6^2} = j(\lambda) = 256\ \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

When two elliptic curves E_1 and E_2 are isomorphic, we know their j-invariants are the same $j(E_1) = j(E_2)$. The converse is not true in general due to the automorphisms on elliptic curves which makes two elliptic curves E_1 and E_2 with the identical j-invariants to be isomorphic after taking the finite field extensions of degree equal to the order of automorphism on E. To understand this better, let us work out the $Aut(E, P_0)$ where $P_0 \in E$ can be chosen to be the unity of the group structure on E. Regarding the automorphisms, every elliptic curves have at least one non-trivial automorphism of order $\mathbb{Z}_2$, induced by elliptic involution or an inversion in the presentation of Weierstrass form or Legendre form by sending $y \mapsto -y$. In the cases where $j = 1728$ or $j = 0$ there are extra automorphisms given as follows.

Corollary 17 (Groups of automorphisms on elliptic curves)**.** *Let X be an elliptic curve over K with char $K \neq 2, 3$. Let $P_0 \in X$, and let $G = Aut(X, P_0)$ be the group of automorphisms of X leaving P_0 fixed. Then G is a finite group of order*

$$\begin{array}{cc} 2 & \text{if } j \neq 0, 1728 \\ 4 & \text{if } j = 1728 \\ 6 & \text{if } j = 0 \end{array}$$

We may represent the second case with $j = 1728$ at $a_6 = 0$ by the curve

$$y^2 = x^3 + x,$$

which has an automorphism group $\mathbb{Z}_4$ generated by $x \mapsto -x$ and $y \mapsto iy$.

The third case with $j = 0$ at $a_4 = 0$ could be represented by the curve

$$y^2 = x^3 + 1$$

which has automorphism group $\mathbb{Z}_6$ generated by $x \mapsto \omega x$ and $y \mapsto -y$ where ω is a primitive cubic root of 1.

Weighted projective stacks

The weighted projective stack $\mathcal{P}(\lambda)$ is the quotient stack acquired from the affine space $\mathbb{A}^{n+1}$ under the action of $\mathbb{G}_m$ the multiplicative group of unity over the field K. That is, fix a nondecreasing sequence of positive integers called weights $\vec{\lambda}$

$$\vec{\lambda} = (\lambda_0, \cdots, \lambda_n)$$

and consider the associated linear action λ of $\mathbb{G}_m$ on affine space $\mathbb{A}^{n+1}$

$$\lambda \cdot (x_0, \cdots, x_n) = (t^{\lambda_0} x_0, \cdots, t^{\lambda_n} x_n) \text{ for any } t \in \mathbb{G}_m.$$

The *weighted projective space* associated to the action λ is defined as the quotient scheme

$$\mathbb{P}(\lambda) = \mathbb{P}(\lambda_0, \cdots, \lambda_n) = (\mathbb{A}^{n+1} \smallsetminus 0) \ / \ \mathbb{G}_m.$$

One defines the *weighted projective stack* as the quotient stack

$$\mathcal{P}(\lambda) = \mathcal{P}(\lambda_0, \cdots, \lambda_n) = [(\mathbb{A}^{n+1} \smallsetminus 0) \ / \ \mathbb{G}_m].$$

Each weighted projective stack has the corresponding weighted projective space as its coarse moduli space. Note for each positive integer d we have

$$\mathbb{P}(d\lambda_0, \cdots, d\lambda_n) \cong \mathbb{P}(\lambda_0, \cdots, \lambda_n)$$

However, the weighted projective stacks are not isomorphic unless $d = 1$. Note that the stabilizer group of a point in the moduli space of curves is isomorphic to a group of automorphisms of the corresponding curve. Thus, for us, it is important to note that $\mathbb{P}(4,6) \cong \mathbb{P}(2,3) \cong \mathbb{P}^1$ as a variety through 'rigidification' while as a stack $\mathcal{P}(4,6) \neq \mathcal{P}(2,3)$ as this would throw away the elliptic involution automorphism of order $\mathbb{Z}_2$ on every elliptic curves.

Deligne–Mumford compactified stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

We can now present the Deligne–Mumford compactified stack $\overline{\mathcal{M}}_{1,1}$ as the weighted projective stack $\mathcal{P}(4,6)$ in the category of schemes over a field K with characteristic different from 2 or 3.

$$\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)_{[a_4:a_6]} = [(Spec\ K[a_4,\ a_6] - (0,0))/\mathbb{G}_m]$$

with $\mathbb{G}_m$ acting with weights $(4,6)$

$$t \cdot (a_4,\ a_6) \mapsto (t^4 a_4,\ t^6 a_6)$$

The coarse moduli is

$$\overline{M}_{1,1} = \overline{\mathbb{A}}_1 \cong (Proj\ K[a_4,\ a_6] - (0,0)) \cong \mathbb{P}^1_{[j:1]}$$

It is a Deligne–Mumford stack over any fields K with characteristic not equal to 2 or 3, where all points have $\mathbb{Z}_2$ isotropy of elliptic involution. There are two special points with $\mathbb{Z}_4$ isotropy ($[a_4 : a_6] = [1 : 0]$ with $j = 1728$) and $\mathbb{Z}_6$ isotropy ($[a_4 : a_6] = [0 : 1]$ with $j = 0$) respectively. By making sure the prime characteristic of the field K does not divide the orders of the stabilizers, we have the tameness of $\overline{\mathcal{M}}_{1,1}$ allowing the formation of coarse moduli space $\overline{M}_{1,1}$ to commute with base field change [AOV].

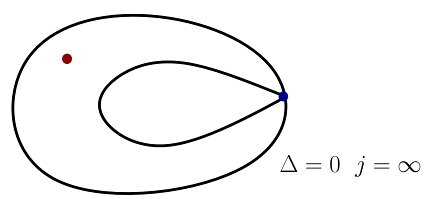


Figure 2.3: Nodal elliptic curve with $\Delta = 0$ and $j = \infty$

The Figure 2.3 is the geometric isomorphism class of a stable elliptic curve that corresponds to the nodal divisor which is a unique point $\{\infty\} = \overline{\mathcal{M}}_{1,1} \setminus 1,1$ with $\Delta = 0$ and $j = \infty$.

The below is well known result on singular Weierstrass equations.

Proposition 18. *Let E be a Weierstrass equation over K. Then*

1. *E is a smooth elliptic curve* $\iff \Delta \neq 0$,

2. *E is a nodal elliptic curve* $\iff \Delta = 0$ *and* $a_4 \neq 0$,

3. E is a cuspidal elliptic curve $\iff \Delta = 0$ *and* $a_4 = 0$*,*

Note that the cuspidal elliptic curve is $[a_4 : a_6] = [0 : 0] \notin \mathcal{P}(4,6) = \overline{\mathcal{M}}_{1,1}$.

2.5 DM moduli stack $\mathcal{L}_{1,12n}$ of stable elliptic fibrations over $\mathbb{P}^1$

In this section, we formulate the moduli stack $\mathcal{L}_{1,12n}$ of stable elliptic fibrations over $\mathbb{P}^1$ as the Deligne–Mumford algebraic mapping stack of regular morphisms $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ and establish the bijection between the isomorphism classes of semistable elliptic surfaces and that of stable elliptic fibrations over $\mathbb{P}^1$.

Let us first recall that a pair (E, p) is a stable elliptic curve if E is a nodal projective curve of arithmetic genus 1 and $p \in E$ is a smooth point. Then, it is well–known that $\overline{\mathcal{M}}_{1,1}$ is a proper Deligne-Mumford stack of stable elliptic curves with a coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1$. This $\mathbb{P}^1$ parametrizes the j–invariants of elliptic curves. When the characteristic of the field K is not equal to 2 or 3, [Hassett] shows that $(\overline{\mathcal{M}}_{1,1})_K \cong [(Spec\ K[a_4, a_6] - (0,0))/\mathbb{G}_m] =: \mathcal{P}_K(4,6)$ by using the Weierstrass equations. Note that this is no longer true if characteristic of K is 2 or 3, as the Weierstrass equations are more complicated.

Notice that $\overline{\mathcal{M}}_{1,1}$ comes equipped with a universal family $p : \overline{\mathcal{C}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$. Then, by the definition of p, any stable elliptic fibration $f : Y \to \mathbb{P}^1$ comes from a morphism $\varphi_f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ and vice versa. As this correspondence also works in families, we can formulate the moduli of stable elliptic fibrations as $\mathrm{Hom}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$. Since $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ with its coarse map $c : \overline{\mathcal{M}}_{1,1} \to \overline{M}_{1,1} \cong \mathbb{P}^1$, it is easy to see that $\deg(c \circ \varphi_f) = 12 \deg \varphi_f$ (cf. [RT, CCFK]). Note that the discriminant divisor Δ of f can be recovered by pulling back $\infty \in \mathbb{P}^1$ via $c \circ \varphi_f$, which implies the following proposition:

Proposition 19. *The moduli stack* $\mathcal{L}_{1,12n}$ *of stable elliptic fibrations over* $\mathbb{P}^1$ *with* $12n$ *nodal singular fibers and a distinguished section is the Deligne–Mumford mapping stack* $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$.

Proof. Above discussion shows that $\mathcal{L}_{1,12n} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$. To show that this is a Deligne–Mumford stack, notice that the target stack $\overline{\mathcal{M}}_{1,1}$ over K with char $K \neq 2, 3$ is a tame Deligne–Mumford stack by [AOV]. Thus, $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ is also a Deligne–Mumford stack by [Olsson]. □

We would like to relate $\mathcal{L}_{1,12n}$ to the moduli of semistable elliptic surfaces. To do so, consider any semistable elliptic surface $f : X \to \mathbb{P}^1$ with a distinguished section $S : \mathbb{P}^1 \hookrightarrow X$ that has the discriminant degree $12n$. Denote this semistable family of elliptic curves over $\mathbb{P}^1$ as (X, f, S). Note that by contracting all components of the fibers of f not meeting the distinguished section S, we obtain $g : Y \to \mathbb{P}^1$ with a distinguished section $S' : \mathbb{P}^1 \hookrightarrow Y$. This process is called the stable reduction of the family of elliptic curves (X, f, S). On the other hand, (X, f, S) can be recovered from (Y, g, S') by taking minimal resolution of singularities of Y and taking proper transform of the morphism S'. This is summarized as the following diagram, where $\nu : X \to Y$ is the minimal resolution of singularities:

$$\begin{array}{ccccc} X & \xrightarrow{\nu} & Y = \varphi_f^*(\overline{\mathcal{C}}_{1,1}) & \longrightarrow & \overline{\mathcal{C}}_{1,1} \\ \downarrow f & & \downarrow g & & \downarrow p \\ \mathbb{P}^1 & = & \mathbb{P}^1 & \xrightarrow{\varphi_f} & \overline{\mathcal{M}}_{1,1} \end{array} \tag{2.2}$$

Note that the contraction morphism ν introduces singularities of type A_k on Y. Conversely, given any regular morphism $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$, the total space $Y = \varphi_f^*(\overline{\mathcal{C}}_{1,1})$ of the stable family $g : \varphi_f^*(\overline{\mathcal{C}}_{1,1}) \to \mathbb{P}^1$ can have A_k singularities because any one parameter deformation of nodal singularities cannot induce other types of singularities (cf. [AB]). In fact, there is a criterion for when (Y, g, S') coming from $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ has A_k singularities:

Proposition 20. *The necklace singular fiber I_k with monodromy A^k or B^k which is a nodal cycle of k smooth rational curves with self-intersections -2 corresponds to the regular morphism $\varphi_g \in \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ being ramified over the nodal divisor point $[\infty] = \overline{\mathcal{M}}_{1,1} \setminus 1, 1$ of order $k - 1$.*

Proof. This follows from the [AB, Lemma 4.1]. Indeed étale locally the coordinate for the universal family $\overline{\mathcal{C}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ around the node of the fiber at infinity is $xy = s$ where

s is an étale local parameter at $[\infty] \in \overline{\mathcal{M}}_{1,1}$. Thus if the moduli map $\varphi_g : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ is ramified over $[\infty]$ to order $k-1$, the equation is $t^k = s$ where t is a local parameter at the ramified point on $\mathbb{P}^1$ which implies that the Weierstrass surface Y obtained by pulling back $\overline{\mathcal{C}}_{1,1}$ has local equation $xy = t^k$, i.e. an A_{k-1} singularity. Resolving this produces an I_k singular fiber at that point. □

For any field K of characteristic neither 2 nor 3, We are now able to establish the bijection between $\mathcal{L}_{1,12n}(K)$, the K–points of the moduli stack $\mathcal{L}_{1,12n}$ of stable elliptic fibrations over $\mathbb{P}^1$, and the moduli stack of semistable elliptic surfaces over K with the discriminant degree $12n$. In fact, we have a stronger result:

Proposition 21. *Fix any field K of characteristic $\neq 2, 3$. Then there is a canonical equivalence of groupoids between $\mathcal{L}_{1,12n}(K)$ and the groupoid of nonsingular semistable elliptic surfaces over K with the discriminant degree $12n$.*

Proof. The minimal resolution of singularities and the stable reduction discussed in diagram 2.2 gives the expected equivalences of groupoids. As it is easy to see that they are inverses to each other on the level of objects, it suffices to show that these are well-defined as morphisms of groupoids.

Given an isomorphism $h : X_1 \to X_2$ between semistable elliptic surfaces, notice that (-2)–curves in X_1 maps to (-2)–curves in X_2. This shows that h induces a morphism $s(h) : Y_1 \to Y_2$ between the corresponding stable elliptic fibrations. Similarly, singular points of Y_1 map to those of Y_2, so that any isomorphism $\alpha : Y_1 \to Y_2$ lifts to an isomorphism $r(\alpha) : X_1 \to X_2$ between the minimal resolutions. Since s and r are inverses of each other, the minimal resolutions and the stable reductions are well-defined and are inverse to each other. □

More concretely, note that any semistable elliptic surface (X, f, S) is associated uniquely to a stable elliptic fibration (Y, g, S') which is associated to a *Weierstrass model* $y^2 = x^3 + a_4 x + a_6$ where $a_4 \in H^0(\mathbb{P}^1, \mathcal{O}(4n))$ and $a_6 \in H^0(\mathbb{P}^1, \mathcal{O}(6n))$. Choice of such models does not change the discriminant divisor $\Delta(X) = -16(4a_4^3 + 27a_6^2) \in \Gamma(\mathbb{P}^1, \mathcal{O}(12n))$ as the discriminant of $g : Y \to \mathbb{P}^1$ will have k^{th} order of vanishing at a point in $\mathbb{P}^1$ where there used to be an I_k necklace. One can say that the total

$Deg(\Delta(X)) = \sum_{i=1}^{\mu} k_i = 12n$ while the order of the vanishing of the discriminant for a given point $t \in \mathbb{P}^1$ depends on the arrangement of the semistable elliptic singular fibers. That is, if the discriminant divisor $\Delta(X)$ has μ number of zeroes which corresponds to the μ number of distinct singular fibers then the arrangement of the semistable elliptic singular fibers is the same question as the partition of $12n$ nodal singular points into μ distinct singular fibers $I_{k_1}, \cdots, I_{k_i}, \cdots, I_{k_\mu}$ that comes from the minimal resolutions of $A_{k_1}, \cdots, A_{k_i}, \cdots, A_{k_\mu}$ singularities which in turn gives $k_i{}^{\text{th}}$ vanishing for the $i-$th zero of the discriminant divisor $\Delta(X)$.

Remark 22. A very important consequence of Proposition 21 is that counting points of $\mathcal{L}_{1,12n}$ gives the same number as counting points of the moduli of nonsingular semistable elliptic surfaces. Since the former has a more concrete description through the algebraic mapping stack, we will focus on acquiring the arithmetic invariants of $\mathcal{L}_{1,12n}$.

2.6 Motive/Point count of $c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$ over finite fields

In this section, we enumerate the moduli stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ over finite fields $\mathbb{F}_q$ for q prime power with characteristic not dividing a or b. Since the point counting is done on the level of schemes, we work with the coarse moduli space $c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$ over $\mathbb{F}_q$ instead.

To count the $\mathbb{F}_q$–points on $c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$, we will use the idea of cut-and-paste by Grothendieck:

Definition 23. Fix a field K. Then the *Grothendieck ring* $K_0(\mathrm{Var}_K)$ *of* K*-varieties* is a group generated by isomorphism classes of K-varieties $[X]$, modulo *scissor* relations $[X] = [Z] + [X - Z]$ for $Z \subset X$ a closed subvariety. Multiplication on $K_0(\mathrm{Var}_K)$ is induced by $[X][Y] := [X \times_K Y]$. There is a distinguished element $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Var}_K)$, called the *Lefschetz motive*.

It is easy to see that when $K = \mathbb{F}_q$, the assignment $[X] \mapsto |X(\mathbb{F}_q)|$ gives a well-defined ring homomorphism $\#_q : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$. Thus, if we can express $[c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))]$

as the linear combination of classes of other varieties with their point counts, then we can deduce $|c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))(\mathbb{F}_q)|$. Therefore, Corollary 2 follows from Theorem 1 and Proposition 21. Now we are ready to prove Theorem 1.

2.7 Proof of Theorem 1

Observe that the regular morphism φ_f from $\mathbb{P}^1 \to \mathcal{P}(a,b)$ is equivalent to considering the line bundles on $\mathbb{P}^1$ which are $\mathcal{L} \simeq \varphi_f^*\mathcal{O}_{\mathcal{P}(a,b)}(1)$ of degree n together with sections $u \in H^0(\mathbb{P}^1, \mathcal{L}^{\otimes a})$ and $v \in H^0(\mathbb{P}^1, \mathcal{L}^{\otimes b})$ such that the global sections u, v are not simultaneously vanishing at any points of $\mathbb{P}^1$ (cf. [RT, CCFK]). Moreover, such pairs (u,v) and (u',v') are equivalent when there exists $\lambda \in \mathbb{G}_m(\overline{K})$ so that $u' = \lambda^a u$ and $v' = \lambda^b v$. Hence, $\mathcal{L}_{1,12n}$ can be thought of as an open substack of $\mathcal{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(an)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(bn)))$, where the defining $\mathbb{G}_m$-action is as above.

However, we would like to work with the coarse moduli $c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$ of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ instead. Denote $U := H^0(\mathcal{O}_{\mathbb{P}^1}(an))$ and $V := H^0(\mathcal{O}_{\mathbb{P}^1}(bn))$. Then, we have a closed embedding

$$\begin{array}{ccc} \mathcal{P}(U \oplus V) & \hookrightarrow & \mathcal{P}(\mathrm{Sym}^b U \oplus \mathrm{Sym}^a V) \\ (u,v) & \mapsto & (u^b, v^a) \end{array}$$

where $\mathbb{G}_m$ acts on $\mathcal{P}(\mathrm{Sym}^b U \oplus \mathrm{Sym}^a V)$ by $\lambda \cdot (u',v') = (\lambda^{ab}u', \lambda^{ab}v')$. Since the coarse moduli space for $\mathcal{P}(\mathrm{Sym}^b U \oplus \mathrm{Sym}^a V)$ is just a usual projective space $\mathbb{P}(\mathrm{Sym}^b U \oplus \mathrm{Sym}^a V)$ where $\mathbb{G}_m$ acts by $\lambda \cdot (u',v') = (\lambda u', \lambda v')$, the coarse moduli space $\mathbb{P}(U \oplus V)$ parametrizes sections $u', v' \in H^0(\mathcal{O}_{\mathbb{P}^1}(abn))$ such that $\exists\, u \in H^0(\mathcal{O}_{\mathbb{P}^1}(an))$ and $\exists\, v \in H^0(\mathcal{O}_{\mathbb{P}^1}(bn))$ (not necessarily unique) with $u^b = u'$ and $v^a = v'$. This shows that $c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$ parametrizes such (u',v') as above, with an additional condition that u' and v' do not simultaneously vanish on any points of $\mathbb{P}^1$.

Now fix a chart $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ with $x \mapsto [1:x]$, and call $0 = [1:0]$ and $\infty = [0:1]$. Then, u and v become polynomials of x with degrees at most an and bn respectively. For instance, $\deg u < an$ if and only if u vanishes at ∞. Denoting $\deg u := k$ and $\deg v := l$, then $\varphi_f \in \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is characterized by equivalence classes of polynomials (u,v)

where either $k = an$ or $l = bn$ (so that they do not simultaneously vanish at ∞) and u, v have no common roots. $\varphi_f \in c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$ has analogous descriptions in terms of (u', v') where $u' = u^b$, $v' = v^a$. Define $F_{k,l} := \{(u', v') \in c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) : \deg u' = bk,\ \deg v' = al\}$. Since polynomials of degree k limits to degree $k-1$ by sending a root to the point of infinity, we obtain the following stratification:

$$c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) = F_{an,bn} \sqcup \left(\bigsqcup_{k=0}^{an-1} F_{k,bn}\right) \sqcup \left(\bigsqcup_{l=0}^{bn-1} F_{an,l}\right)$$
$$c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) = \overline{F_{an,bn}} \supsetneq \overline{F_{an-1,bn}} \supsetneq \cdots \supsetneq \overline{F_{0,bn}} = F_{0,bn}$$
$$c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) = \overline{F_{an,bn}} \supsetneq \overline{F_{an,bn-1}} \supsetneq \cdots \supsetneq \overline{F_{an,0}} = F_{an,0}$$

$$\overline{F_{an-k,bn}} \cap \overline{F_{an,bn-l}} = \emptyset \quad \forall k, l > 0$$

Then,

$$[c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))] = [F_{an,bn}] + \sum_{k=0}^{an-1} [F_{k,bn}] + \sum_{l=0}^{bn-1} [F_{an,l}] \tag{2.3}$$

Define

$$F'_{k,l} := \{(u,v) \in c(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) : \deg(u', v') = (bk, al),\ u', v' \text{ are monic}\}\ .$$

Then, $F'_{k,l} \hookrightarrow F_{k,l}$ is a section of the projection morphism $F_{k,l} \to F'_{k,l}$ (induced by making (u', v') to be a monic pair), which has $\mathbb{G}_m$–fibers. Hence, $F_{k,l}$ is a trivial $\mathbb{G}_m$–bundle over $F'_{k,l}$, so that $[F_{k,l}] = [\mathbb{G}_m][F'_{k,l}]$. Moreover, given such $(u', v') \in F'_{k,l}$, there is a unique pair (u, v) of monic polynomials of degree k and l respectively, so that $u^b = u'$ and $v^a = v'$, as $\mathrm{char}(K)$ does not divide a or b. This gives an alternative description of $F'_{k,l}$ as below (inspired by [FW]):

Definition 24. Fix a field K with algebraic closure $\overline{K}$. Fix $k, l \geq 0$. Define $\mathrm{Poly}_1^{(k,l)}$ to be the set of pairs (u, v) of monic polynomials in $K[z]$ so that:

1. $\deg u = k$ and $\deg v = l$.

2. u and v have no common root in $\overline{K}$.

Therefore, $F'_{k,l} \cong \mathrm{Poly}_1^{(k,l)}$. To finish the proof, it suffices to find a description of $\left[\mathrm{Poly}_1^{(k,l)}\right]$ as a polynomial of $\mathbb{L}$. Farb and Wolfson [FW] found such expression when $k = l$, and we claim that $\left[\mathrm{Poly}_1^{(k,l)}\right]$ has a similar description, as below:

Proposition 25. *Fix $d_1, d_2 \geq 0$. Then,*

$$\left[\mathrm{Poly}_1^{(d_1,d_2)}\right] = \begin{cases} \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1+d_2-1}, & \text{if } d_1, d_2 > 0 \\ \mathbb{L}^{d_1+d_2}, & \text{if } d_1 = 0 \text{ or } d_2 = 0 \end{cases}$$

Proof. The proof for this is analogous to [FW], Theorem 1.2 (1). Here, we only state the differences to their work.

Step 1: The space of (u, v) monic polynomials of degree d_1, d_2 is instead the quotient $\mathbb{A}^{d_1} \times \mathbb{A}^{d_2}/(S_{d_1} \times S_{d_2}) \cong \mathbb{A}^{d_1+d_2}$. We have the same filtration of $\mathbb{A}^{d_1+d_2}$ by $R_{1,k}^{(d_1,d_2)}$, which is the space of (u, v) monic polynomials of degree d_1, d_2 respectively for which there exists a monic $h \in K[z]$ with $\deg(h) \geq k$ and monic polynomials $g_i \in K[z]$ so that $u = g_1 h$ and $v = g_2 h$. The rest of the arguments follow analogously, keeping in mind that the group action is via $S_{d_1} \times S_{d_2}$.

Step 2: Here, we prove that $R_{1,k}^{(d_1,d_2)} - R_{1,k+1}^{(d_1,d_2)} \cong \mathrm{Poly}_1^{(d_1-k,d_2-k)} \times \mathbb{A}^k$. Just as in [FW], the base case of $k = 0$ follows from the definition. For $k \geq 1$, the rest of the arguments follow analogously just as in Step 1 of loc. cit.

Step 3: By combining Step 1 and 2 as in [FW], we obtain

$$\left[\mathrm{Poly}_1^{(d_1,d_2)}\right] = \mathbb{L}^{d_1+d_2} - \sum_{k \geq 1} \left[\mathrm{Poly}_1^{(d_1-k,d_2-k)}\right] \mathbb{L}^k$$

For the induction on the class $\left[\mathrm{Poly}_1^{(d_1,d_2)}\right]$, we use lexicographic induction on the pair (d_1, d_2). Since the order of d_1, d_2 does not matter for Grothendieck class, we assume that $d_1 \geq d_2$. For the base cases, consider when $d_2 = 0$. Then the monic polynomial of degree 0 is nowhere vanishing, so that any polynomial of degree d_1 constitutes a member of $\mathrm{Poly}_1^{(d_1,0)}$, so that $\mathrm{Poly}_1^{(d_1,0)} \cong \mathbb{L}^{d_1}$. Similarly, $d_1 = 0$ is taken care of. Then for $d_1, d_2 > 0$, we obtain

$$
\begin{aligned}
&\left[\mathrm{Poly}_1^{(d_1,d_2)}\right] \\
&= \mathbb{L}^{(d_1+d_2)} - \sum_{k\geq 1}\left[\mathrm{Poly}_1^{(d_1-k,d_2-k)}\right]\mathbb{L}^k \\
&= \mathbb{L}^{d_1+d_2} - \left(\sum_{k=1}^{d_2-1}(\mathbb{L}^{(d_1-k)+(d_2-k)} - \mathbb{L}^{(d_1-k)+(d_2-k)-1})\mathbb{L}^k + \mathbb{L}^{d_1-d_2}\mathbb{L}^{d_2}\right) \\
&= \mathbb{L}^{d_1+d_2} - \left(\sum_{k=1}^{d_2-1}(\mathbb{L}^{d_1+d_2-k} - \mathbb{L}^{d_1+d_2-k-1}) + \mathbb{L}^{d_1}\right) \\
&= \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1+d_2-1}
\end{aligned}
$$

□

Applying the Proposition 25 to the equation (2.3), we get the motive count:

$$
\begin{aligned}
&[c(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b)))] \\
&= [F_{an,bn}] + \sum_{k=0}^{an-1}[F_{k,bn}] + \sum_{l=0}^{bn-1}[F_{an,l}] \\
&= [\mathbb{G}_m]\left(\left[\mathrm{Poly}_1^{(an,bn)}\right] + \sum_{k=0}^{an-1}\left[\mathrm{Poly}_1^{(k,bn)}\right] + \sum_{l=0}^{bn-1}\left[\mathrm{Poly}_1^{(an,l)}\right]\right) \\
&= (\mathbb{L}-1)\left((\mathbb{L}^{(a+b)n} - \mathbb{L}^{(a+b)n-1}) + \mathbb{L}^{bn} + \sum_{k=1}^{an-1}(\mathbb{L}^{bn+k} - \mathbb{L}^{bn+k-1})\right) \\
&\quad + (\mathbb{L}-1)\left(\mathbb{L}^{an} + \sum_{l=1}^{bn-1}(\mathbb{L}^{an+l} - \mathbb{L}^{an+l-1})\right) \\
&= (\mathbb{L}-1)(\mathbb{L}^{(a+b)n} - \mathbb{L}^{(a+b)n-1} + \mathbb{L}^{bn} + \mathbb{L}^{(a+b)n-1} - \mathbb{L}^{bn} + \mathbb{L}^{an} + \mathbb{L}^{(a+b)n-1} - \mathbb{L}^{an}) \\
&= \mathbb{L}^{(a+b)n+1} - \mathbb{L}^{(a+b)n-1}
\end{aligned}
$$

This finishes the proof of Theorem 1.

2.8 Heuristic for counting semistable elliptic curves over global fields

In this section, we would like to consider the connection between the number fields and the function fields called the *global fields analogy*. That is, when one studies the questions in number theory involving the number field $\mathbb{Q}$ and the algebraic extensions of $\mathbb{Q}$ there is an observation that the questions can be brought to the geometry of curves over the finite fields involving the (global) function field $\mathbb{F}_q(t)$ and the algebraic extensions of $\mathbb{F}_q(t)$. While it was well known in the past that the elliptic surfaces over the complex numbers have some strict analogies with elliptic curves over the number fields, we take this analogy further by passing the arithmetic invariant of the moduli of semistable elliptic surfaces over the finite fields through the global fields analogy (see Remark 22).

Through the notion of *bounded height*, we will consider $Z_{\mathbb{F}_q(t)}(B)$ which is the counting of semistable elliptic surfaces (Definition 8) with $12n$ nodal singular fibers and a distinguished section. The growth rate of $Z_{\mathbb{F}_q(t)}(B)$ can be computed by the arithmetic invariant $|L_{1,12n}(\mathbb{F}_q)|$ in the function field setting. An analogous object in the number field setting is $Z_{\mathbb{Q}}(B)$ which is the counting of semistable elliptic curves over $\mathbb{Q}$. In the end, we formulate a heuristic that for both of the global fields the asymptotic of $Z_K(B)$ will match with one another.

Let K be a global field and $\mathcal{O}_K$ be its ring of integers such as

1. The function field $K = \mathbb{F}_q(t)$ with $\mathcal{O}_K = \mathbb{F}_q[t]$
2. The number field $K = \mathbb{Q}$ with $\mathcal{O}_K = \mathbb{Z}$

As the function field of $\mathbb{P}^1_{\mathbb{F}_q}$ (the base of semistable elliptic fibrations) is indeed a rational function field of one variable t over $\mathbb{F}_q$, one could think of a semistable elliptic surface X over $\mathbb{P}^1$ as the choice of a model for semistable elliptic curves E over $K = \mathbb{F}_q(t)$ or equivalently over $\mathcal{O}_K = \mathbb{F}_q[t]$ by clearing the denominators. On the number field, the analogy would be the semistable elliptic curves E with the squarefree conductors

$\mathcal{N} = p_1 \cdot \cdots \cdot p_\mu$ over $\mathbb{Q}$ or equivalently over $\mathcal{O}_K = \mathbb{Z}$ by the minimal integral Weierstrass model of an elliptic curve. In order to draw the analogy, we need to fix an affine chart $\mathbb{A}^1_{\mathbb{F}_q} \subset \mathbb{P}^1_{\mathbb{F}_q}$ and its corresponding ring of functions $\mathbb{F}_q[t]$, since $\mathbb{F}_q[t]$ could come from any affine chart of $\mathbb{P}^1_{\mathbb{F}_q}$, whereas the ring of integers for the number field K is canonically determined. We denote $\infty \in \mathbb{P}^1_{\mathbb{F}_q}$ to be the unique point not in the affine chart.

Note that for a maximal ideal $\mathfrak{p}$ in $\mathcal{O}_K$, the residue field $\mathcal{O}_K/\mathfrak{p}$ is finite for both of our global fields: In the function field if $\mathfrak{p} = (p(t))$ for a monic irreducible polynomial $p(t) \in \mathbb{F}_q[t]$ of degree k, then $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_{q^k}$ which is the splitting field of $p(t)$ over $\mathbb{F}_q$ whereas for the number field if $\mathfrak{p} = (p)$ for p prime integer, then $\mathcal{O}_K/\mathfrak{p} = \mathbb{Z}_p \cong \mathbb{F}_p$. One could think of $\mathfrak{p}$ as a point in *Spec* $\mathcal{O}_K$ and define the *height* of a point $\mathfrak{p}$ which connects the global fields together.

Definition 26. Define the height of a point $\mathfrak{p}$ to be $ht(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$ the cardinality of the residue field $\mathcal{O}_K/\mathfrak{p}$.

We now introduce the notion of *bad reduction* & *good reduction*.

Definition 27. Let E be an elliptic curve given by an Weierstrass equation $y^2 = x^3 + a_4 x + a_6$, with $a_4,\ a_6 \in \mathcal{O}_K$. Then E has bad reduction at $\mathfrak{p}$ if through the base change from $\mathcal{O}_K$ to $\mathcal{O}_K/\mathfrak{p}$ on E, the resulting curve $E_\mathfrak{p}$ is a singular cubic. The prime $\mathfrak{p}$ is said to be of good reduction if $E_\mathfrak{p}$ is a smooth elliptic curve.

For simplicity, assume that X does not have a singular fiber over $\infty \in \mathbb{P}^1_{\mathbb{F}_q}$. Note that the primes of bad reductions are precisely the divisors of the discriminant Δ which in the function field $K = \mathbb{F}_q(t)$ we have $\Delta(X) \in H^0(\mathbb{P}^1, \mathcal{O}(12n))$ that has the following factorization for pairwise distinct maximal ideals $\mathfrak{p}_i \subset \mathbb{F}_q[t]$ and $\alpha \in \mathbb{F}_q^*$ over the affine chart:

$$\Delta(X) = -16(4a_4^3 + 27a_6^2) = \alpha \prod_{i=1}^{\mu} \mathfrak{p}_i^{k_i}$$

There are two ways in which the bad reductions can occur: E can become nodal which is called a multiplicative reduction at $\mathfrak{p}$ or E can become cuspidal which is called an additive reduction at $\mathfrak{p}$. For our consideration, we only have multiplicative reductions as possible bad reductions since semistable elliptic fibrations contain only singular fibers of fishtail I_1 or necklace I_k ($k \geq 2$) types such that for a given semistable elliptic fibration it has $12n$ nodal points distributed over μ distinct singular fibers that are $I_{k_1}, \cdots, I_{k_i}, \cdots, I_{k_\mu}$ with $\sum_{i=1}^{\mu} k_i = 12n$ as we allow each of the singular fiber to contain multiple nodal singular points but no cuspidal singularities.

As the discriminant divisor $\Delta(X)$ is an invariant of the choice of semistable model $f : X \to \mathbb{P}^1$, we count the number of isomorphism classes of nonsingular semistable elliptic fibrations on the function field $\mathbb{F}_q(t)$ by the bounded height of $\Delta(X)$.

$$ht(\Delta(X)) = \prod_{i=1}^{\mu} |\mathbb{F}_q|^{k_i} = q^{k_1} \cdots q^{k_i} \cdots q^{k_\mu} = q^{k_1 + \cdots + k_\mu} = q^{12n}$$

In general, the height of a discriminant $\Delta(X)$ of any X (without nonsingular fiber assumption over ∞) is defined as q^{12n} where $Deg(\Delta(X)) = 12n$. Now we are ready to define a function $Z_K(B)$ which in the function field realm is $Z_{\mathbb{F}_q(t)}(B)$.

$Z_{\mathbb{F}_q(t)}(B) :=$
{ # Nonsingular semistable elliptic fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ with $12n$ nodal singular fibers and a distinguished section counted by $0 < ht(\Delta(X)) = \prod_{\mathfrak{p}} ht(\mathfrak{p}) = q^{12n} \leq B$ }

This counting is equivalent to counting semistable elliptic surfaces over $\mathbb{F}_q$. We now compute the $Z_{\mathbb{F}_q(t)}(B)$ by the arithmetic invariant $|L_{1,12n}(\mathbb{F}_q)|$.

Theorem 28 (Computation of $Z_{\mathbb{F}_q(t)}(B)$)**.** *The counting of semistable elliptic fibrations over $\mathbb{F}_q(t)$ by $ht(\Delta(X)) = q^{12n} \leq B$ satisfies the following inequality:*

$$Z_{\mathbb{F}_q(t)}(B) \leq \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot \left(B^{\frac{5}{6}} - 1\right)$$

In other words, $Z_{\mathbb{F}_q(t)}(B) \sim \mathcal{O}\left(B^{\frac{5}{6}}\right)$.

Proof. Knowing the exact point count of the coarse moduli space for semistable elliptic surfaces over $\mathbb{P}^1_{\mathbb{F}_q}$ to be $|L_{1,12n}(\mathbb{F}_q)| = q^{10n+1} - q^{10n-1}$ by Remark 22 and Theorem 1, we can explicitly compute the bound for $Z_{\mathbb{F}_q(t)}(B)$ as the following,

$$
\begin{aligned}
Z_{\mathbb{F}_q(t)}(B) &= \sum_{n=1}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} |L_{1,12n}(\mathbb{F}_q)| = \sum_{n=1}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} (q^{10n+1} - q^{10n-1}) \\
&= (q^1 - q^{-1}) \sum_{n=1}^{\left\lfloor \frac{log_q B}{12} \right\rfloor} q^{10n} \leq (q^1 - q^{-1}) \left(q^{10} + \cdots + q^{10 \cdot (\frac{log_q B}{12})} \right) \\
&= (q^1 - q^{-1}) \frac{q^{10}(B^{\frac{5}{6}} - 1)}{(q^{10} - 1)} = \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot (B^{\frac{5}{6}} - 1)
\end{aligned}
\tag{2.4}
$$

□

Note that we have an equality $Z_{\mathbb{F}_q(t)}(B) = \frac{(q^{11}-q^9)}{(q^{10}-1)} \cdot (B^{\frac{5}{6}} - 1)$ when $\frac{log_q B}{12}$ is a positive integer.

Switching to the number field realm with $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, one could choose the minimal integral Weierstrass model of an elliptic curve which has the discriminant divisor Δ that is already a number.

In order to match the counting with the function field, we define the $ht(\Delta)$ to be the cardinality of ring of functions on subscheme $Spec(\mathbb{Z}/(\Delta))$. And this leads to the following analog of $Z_K(B)$ over $\mathbb{Q}$ which is $Z_{\mathbb{Q}}(B)$.

$$Z_{\mathbb{Q}}(B) = \{\# \text{ Semistable elliptic curves } E \text{ over } Spec\ \mathbb{Z} \text{ with } 0 < ht(\Delta) \leq B\ \}$$

Our arithmetic invariant $|L_{1,12n}(\mathbb{F}_q)|$ for the moduli of nonsingular semistable elliptic fibrations over $\mathbb{P}^1_{\mathbb{F}_q}$ renders the following number theoretic heuristic by the global fields analogy on $Z_K(B)$.

Conjecture 29 (Asymptotic of $Z_{\mathbb{Q}}(B)$)**.** The asymptotic growth rate of $Z_{\mathbb{Q}}(B)$, the counting of semistable elliptic curves over $\mathbb{Q}$ by $ht(\Delta) \leq B$, follows from the polynomial growth rate of $Z_{\mathbb{F}_q(t)}(B) \sim \mathcal{O}\left(B^{\frac{5}{6}}\right)$.

It would be interesting if one could show this analogy holds at least asymptotically. Note that the asymptotic of our counting $Z_{\mathbb{F}_q(t)}(B) \sim \mathcal{O}\left(B^{\frac{5}{6}}\right)$ for the semistable elliptic curves over $\mathbb{F}_q(t)$ matches with the asymptotic of the counting done by [Baier] for the stable elliptic curves over $\mathbb{Q}$. The semistable elliptic curves were used by Andrew Wiles to prove *Taniyama-Shimura-Weil conjecture* now known as *modularity theorem* which was enough to establish *Fermat's last theorem* as a true theorem [Wiles].

Chapter 3

Arithmetic of the moduli of semistable hyperelliptic fibrations

3.1 Introduction

Naturally, we wish to acquire similar arithmetic invariants for $\mathcal{L}_{g,\Delta}$ the moduli of higher genus $g \geq 2$ fibrations over $\mathbb{P}^1$. We are indeed able to do so by changing the target stack to be $\overline{\mathcal{H}}_g$ the moduli of genus g stable hyperelliptic curves. The reason for focusing upon the hyperelliptic curves of genus $g \geq 2$ is that the moduli of stable hyperelliptic curves are known to be rational.

3.2 Rationality of $\overline{\mathcal{H}}_g$

The unirationality of the target stack $\mathfrak{X}$ in $\mathrm{Hom}_n(\mathbb{P}^1, \mathfrak{X})$ is crucial for us to acquire arithmetic invariants the way we did. In higher genus, $\overline{\mathcal{M}}_g$ is known to be rational for $2 \leq g \leq 6$ and unirational for $7 \leq g \leq 14$ while for $g \geq 14$, $\overline{\mathcal{M}}_g$ is known to be of general type.

Surprisingly, rationality is known to continue to the moduli of higher genus curves as long as they are hyperelliptic.

Theorem 30 (Rationality of $\overline{\mathcal{H}}_g$)**.** *For any genus g, there is a regular morphism $\overline{\mathcal{H}}_g \to \mathbb{P}^{2g+2}//SL_2$.*

Proof. This follows from the [AL, Section 2]. Avritzer and Lange show in Proposition 2.4 that there is canonical isomorphism of moduli spaces $\overline{\mathcal{H}}_g$ of stable hyperelliptic curves of genus g onto the moduli spaces $\mathbb{H}_{2,g}$ of admissible double covers of $(2g+2)-$marked curves of genus 0 and in Corollary 2.5 they show that $\overline{\mathcal{H}}_g$ is isomorphic onto the moduli space $\overline{\mathcal{M}}_{0,2g+2}$ of stable $(2g+2)-$marked curves of genus 0. We note that the moduli of rational curves with n marked points $\overline{\mathcal{M}}_{0,n}$ is birational to $\mathbb{P}^{n-3}$ for every $n \geq 3$. □

This result stems from the fact that $\overline{\mathcal{H}}_g \cong \overline{\mathcal{M}}_{0,2g+2}//S_{2g+2}$ where S_{2g+2} is the symmetric group of $2g+2$ letters. While we know that $\overline{\mathcal{H}}_g$ is a $(2g-1)$-fold moduli stack that is birational to $\mathbb{P}^{2g+2}//SL_2$, it is not a simple matter to identify it as a weighted projective stack. For genus 2, $\mathbb{P}^6//SL_2$ which is a 3-fold moduli is a hypersurface in $\mathcal{P}(2,4,6,10,15)$ since there is an additional invariant of degree 15. But [Dolgachev] shows that this hypersurface is isomorphic to $\mathcal{P}(2,4,6,10)$. For genus 3, $\mathbb{P}^8//SL_2$ is 5-dimensional subvariety of $\mathcal{P}(2,3,4,5,6,7,8,9,10)$. As in the case of $g=2$, this embedding is not an optimal one. While one may find an embedding into a smaller weighted projective space we do not know a general algorithm to find a better embedding. And this is indeed a very difficult invariant theoretic problems with long history.

Using the definition of projective GIT quotient which says any projective GIT quotient is the Proj of the invariant ring, we see that for genus 4 the $\mathbb{P}^{10}//SL_2$ is 7-dimensional subvariety of $\mathcal{P}(\vec{\lambda}) = \mathcal{P}(\lambda_0, \cdots, \lambda_n)$ where the weights for the ambient weighted projective stack would come from the invariant theory of binary forms.

3.3 Semistable hyperelliptic genus 2 fibrations over $\mathbb{P}^1$

In this section, we give the presentation of the hyperelliptic mapping class group Γ_g^{hyp} and define the hyperelliptic genus g Lefschetz fibration $f : X \to \mathbb{P}^1$ where the basespace is a parametrized $\mathbb{P}^1$ and the generic fiber is a smooth hyperelliptic genus g curve. Similarly, every general type surface with $\kappa = 2$ admits a pencil of genus $g \geq 2$ curves which can be blown up at the base locus to provide the structure of a genus $g \geq 2$ fibration.

We will focus on the genus 2 case as all closed, orientable, Riemann surfaces of genus 2 are hyperelliptic which induces isomorphism between the hyperelliptic and the generic mapping class groups $\Gamma_2^{hyp} \cong \Gamma_2$ which in turn makes all genus 2 Lefschetz fibrations hyperelliptic whereas for higher genus $g \geq 3$ Lefschetz fibrations we must start to distinguish between the hyperelliptic and non-hyperelliptic Lefschetz fibrations.

Contrary to the elliptic fibrations, there are many smooth, symplectic genus 2 Lefschetz fibrations which are non-holomorphic and thus the total space X is non-complex (non-Kähler) mainly due to the existence of the reducible nodal singularities in the fibration.

We will end our review of the genus 2 hyperelliptic Lefschetz fibrations by providing the classification result of holomorphic genus 2 Lefschetz fibrations in terms of (n, s) type and its corresponding fibersum decompositions which are unique decompositions into fibersum of 3 types of holomorphic building block genus 2 Lefschetz fibrations which are $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$, $K3\#2\overline{\mathbb{CP}}^2$, and HK the Horikawa surface that all lie on (extended) Noether-Horikawa line for the fibrations with only irreducible nodal singularities by the work of Siebert, Tian, Auroux, Smith and Chakiris. For fibrations with reducible nodal singularities, there is yet to be a complete classfication and thus we will provide the stabilization that shows all smooth genus 2 Lefschetz fibration with irreducible as well as reducible nodal singularities become holomorphic after sufficient number of fibersum with the rational genus 2 Lefschetz fibrations where the stabilized genus 2 fibration decompses uniquely into fibersum of holomorphic building blocks that are $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$, $K3\#2\overline{\mathbb{CP}}^2$, and the Auroux's fibration by the work of Endo, Kamada and Auroux.

Hyperelliptic mapping class groups

The presentation of Γ_g^{hyp} the hyperelliptic mapping class groups was originally obtained by Birman [Birman, FM]. Let t_i $(i = 1, \ldots, 2g + 1)$ be positive Dehn twists along the loops c_i illustrated in Figure 3.1. The mapping class group Γ_g^{hyp} of a hyperelliptic genus-g Riemann surface is generated by $t_1, \ldots, t_{2g+1}$ with the following defining relations.

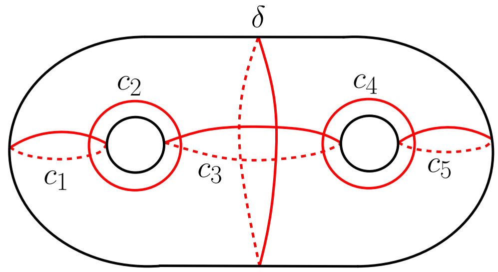


Figure 3.1: Curves c_1, c_2, c_3, c_4, and c_5

$$t_i t_j = t_j t_i \quad \text{if } |i - j| \geq 2, \tag{3.1}$$

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \quad \text{for } i = 1, \ldots, 2g, \tag{3.2}$$

$$\tau\, t_i = t_i\, \tau \quad \text{for } i = 1, \ldots, 2g + 1. \tag{3.3}$$

$$\tau^2 = 1 \quad \text{where } \tau = t_1 t_2 \ldots t_{2g} t_{2g+1}^2 t_{2g} \ldots t_2 t_1, \tag{3.4}$$

$$(t_1 t_2 \ldots t_{2g} t_{2g+1})^{2g+1} = 1, \tag{3.5}$$

$$(t_1 t_2 \ldots t_{2g-1} t_{2g})^{2(2g+1)} = 1, \tag{3.6}$$

$$(t_h = (t_1 \cdots t_{2h})^{4h+2}, \tag{3.7}$$

The first relation is the *commutativity relation* for two disjoint cycles followed by the *braid relation* for three joint cycles. The *hyperelliptic relation* τ is central and involutive. One could succinctly define the hyperelliptic mapping class group as the centralizer of the hyperelliptic relation τ that corresponds to the hyperelliptic involution automorphism element in the mapping class group (i.e., $\Gamma_g^{hyp} := Z(\tau) \subseteq \Gamma_g$). The rests are various *chain relations* which consists of chain of Dehn twists along the sequence of joint cycles. It is important to note the last chain relation for $h = 1, \ldots, [g/2]$, we let t_h be a positive Dehn twist along the loop h (the separating closed curve) illustrated in Figure 3.1.

Hyperelliptic Lefschetz fibrations over $\mathbb{P}^1$

A *smooth* Lefschetz fibration is a differentiable surjection $f : M \to S^2$ of a closed oriented smooth 4-manifold M with finitely many critical points of the form $w \circ f(z_1, z_2) = z_1^2 + z_2^2$. Here z_1, z_2 and w are complex coordinates on M and S^2 respectively that are compatible with fixed global orientations on M and S^2. Hyperelliptic Lefschetz fibrations are Lefschetz fibrations for which the image of the monodromy is included in the hyperelliptic mapping class group. From complex algebraic surfaces point of view, genus 2 is where we begin to have the total spaces 4-manifolds that have the Kodaira dimension 2 (in symplectic category we also have the notion of symplectic Kodaira dimension and this notion matches with the holomorphic Kodaira dimension when they are both defined).

If we restrict the genus 2 fibration over the small circle C, the $X|_C$ is fibration of genus 2 curve $\Sigma_2 \times [0,1]$ where $\Sigma_2 \times 0$ is glued to $\Sigma_2 \times 1$. The isotopy class of this gluing of the singular fiber can be encoded in the mapping class group of the fiber namely Γ_2. This extends globally for the Lefschetz fibrations in the way that given an arbitrary Lefschetz fibration $f : X \to \mathbb{P}^1$ we can consider the positive relation as the global monodromy factorization in the mapping class group that determines the arrangement of the singular fibers.

They are considered to be a natural generalization of elliptic surfaces because several properties are common to these two kinds of fibrations. For instance, many of fibrations can be obtained by branched covering construction, the signature of a fibration localizes on the singular fibers, typical fibrations are used as building blocks for constructions of more complicated fibrations and 4–manifolds, etc.

Lefschetz fibrations on a closed oriented smooth 4-manifolds have crucial role in 4-manifolds theory as they form the natural stage for investigating the boundaries among the three different structures on 4-manifolds namely differentiable, symplectic and holomorphic structures. In the late nineties, Simon Donaldson established a remarkable result on the existence of symplectic Lefschetz pencils on arbitrary symplectic manifolds [Donaldson]. This implies that after appropriately blowing up basepoints of the symplectic Lefschetz pencil one acquires symplectic Lefschetz fibrations on any given symplectic manifolds. Conversely, by the observation of Robert Gompf the total space of differentiable Lefschetz fibration has a symplectic structure that is unique up to isotopy [Gompf]. Therefore, the study of differentiable Lefschetz fibrations is essentially equivalent to the study of symplectic manifolds.

Naturally, one is curious about how far smooth Lefschetz fibrations are from holomorphic ones. A Lefschetz fibration is called *holomorphic* if the total space M is a smooth complex surface, and for a suitable complex structure on the base sphere $\mathbb{P}^1 \cong S^2$ the fibration map $f : M \to \mathbb{P}^1$ is holomorphic.

The definitive result on holomorphicity of genus 2 Lefschetz fibrations is the work of Siebert and Tian where they consider genus 2 Lefschetz fibrations with only irreducible nodal singularities (equivalently has no reducible nodal singularities) that also has the transitive monodromy. We say that a monodromy factorization is transitive if the images of the factors under the morphism $\Gamma_2 \to S_6$ mapping t_i to the transposition $(i, i+1)$ generate the entire symmetric group S_6.

Theorem 31 (Siebert and Tian [ST])**.** *Let $f : X \to S^2$ be a genus 2 differentiable Lefschetz fibration with transitive monodromy. If all nodal singularities are irreducible then f is isomorphic to a holomorphic Lefschetz fibration.*

By the work of Auroux, holomorphicity result of Siebert and Tian can be reformulated terms of the mapping class group factorizations.

Theorem 32 ([Auroux])**.** *Let $f : X \to S^2$ be a genus 2 differentiable Lefschetz fibration with transitive monodromy. If all nodal singularities are irreducible then f is isomorphic (Hurwitz equivalence in Γ_2) to a holomorphic Lefschetz fibration which has the transitive monodromy factorization of the form $A^k \cdot B^\epsilon = 1$ for some integer $k \geq 0$ and $\epsilon \in \{0, 1\}$*

And finally there is the work of Chakiris and Smith which says we can reduce all complex genus two fibrations with no reducible fibres to fibre sums of three basic examples. If the Lefschetz fibration has no reducible nodal singularity and only has irreducible nodal singularities namely the type $(n, s) = (n, 0)$ then we are able to say that it is holomorphic and it must be decomposable as one of the following types Rationals, K3 blown up at 2pts, and the Horikawa.

Theorem 33 ([Chakiris, Smith]). *Let $f : X \to S^2$ be a genus 2 Lefschetz fibration that has no reducible nodal singularities and total space X is Kähler. Then it is a fibresum of the shape $A^m B^n = 1$ or $C^p = 1$ where $m, n, p \in \mathbb{Z} \geq 0$ and the basic words A, B, C are given by:*

1. A : $(t_1 t_2 t_3 t_4 t_5^2 t_4 t_3 t_2 t_1)^2 = 1$
2. B : $(t_1 t_2 t_3 t_4 t_5)^6 = 1$
3. C : $(t_1 t_2 t_3 t_4)^{10} = 1$

These basic holomorphic genus 2 Lefschetz fibrations building blocks are very important since from complex point of view they are Kähler surfaces that are on (the extension of) the Noether-Horikawa line $K^2 = 2p_g - 4$ and they are the generators of the genus 2 Lefschetz fibrations with no separating nodal singularities which are all holomorphic by the work of Chakiris and Smith.

Thus if we are given genus 2 Lefschetz fibration of type $(20, 0)$ then we know it is isomorphic to the $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}^2}$, and for type $(30, 0)$ then it is $K3 \# 2\overline{\mathbb{CP}^2}$, and $(40, 0)$ then it is the Horikawa surface. Similarly $(50, 0)$ then it is fibersum AB, $(60, 0)$ is A^3, $(70, 0)$ is $A^2 B$, $(80, 0)$ is either A^4 or C^2 and so on. Except for the ambiguity at the number of irreducible nodal singularities being the multiple of $40m$ we have a unique fibersum decomposition in terms of known holomorphic building blocks. And for the $40m$ ones we have two possible fibersum decompositions. For any $(n, s) = (10m, 0)$ with $m \neq 0 \pmod 4$ we have exactly one type of decomposition while at $(n, s) = (40m, 0)$ we have exactly two kinds of decompositions and they are not deformation equivalent by the work of [Horikawa].

It seems there is little prospect for a complete classification of isomorphism classes of hyperelliptic Lefschetz fibrations. In fact, there are infinitely many distinct Lefschetz fibrations of genus two with the same numbers of singular fibers of each type. Two Lefschetz fibrations of the same genus over a given base space are called stably isomorphic if they become isomorphic after fiber-summed with the same number of copies of a 'universal' Lefschetz fibration.

If the Lefschetz fibration has reducible nodal singularity and thus it is of type (n, s) then we do not know priori whether this smooth and symplectic Lefschetz fibration is holomorphic or not. While for small Euler characterstic ones we can determine given ones underlying 4-manifolds diffeomorphic type, the complete classification is not yet complete. We do have, however, a classfication in stable sense where one can consider fiber summing with large number of rational genus 2 holomorphic Lefschetz fibrations which makes the fiber summed one to be holomorphic as well as to admit explicit decomposition in terms of known types that are Rationals, K3 blown up at 2pts, and Auroux's fibration. This stabilization is in a way complexificaiton of a non-holomorphic genus 2 Lefschetz fibration in a sense that it remembers the number of reducible nodal singularities that the original non-holomorphic one had in stabilized copy.

Theorem 34 (Endo and Kamada [EK]). *Let $f : X \to S^2$ be a genus 2 Lefschetz fibration with (n, s). There exists a positive integer m_0 such that for any integer $m \geq m_0$,*

$f \# m f_0 \cong \#(a+m) f_0 \# b f_1 \# s f_{2,1}$

for a sufficiently large integer m

What this shows is that every genus 2 Lefschetz fibration with s number of reducible nodal singularities (which obstruct the holomorphicity of the fibration) will become holomorphic after it stablizes by fibersumming with sufficient number of rational genus 2 Lefschetz fibrations and it also decomposes in a way that it is fibersum determined number of $A^a B^b$ together with s number of Auroux's fibration which is known to be holomorphic by the work of Auroux.

3.4 Formulation of $\mathcal{L}_{2,10m}$ via $\overline{\mathcal{H}}_2 \to \mathcal{P}(2,4,6,10)$

Recall that all genus 2 curves are hyperelliptic which gives $\overline{\mathcal{M}}_2 = \overline{\mathcal{H}}_2$ and starting with genus 2, we are able to reach infinitely many surfaces of general type with $\kappa = 2$. Thus in terms of the geography lattice (χ, c_1^2), the general type surfaces have $K^2 = c_1^2 > 0$ meaning they are strictly above the elliptic line $(c_1^2 = 0)$. Also the genus 2 fibrations play a special role for surfaces of general type as the presence of a genus 2 fibration accounts for the standard exception to the birationality of the bicanonical map.

In order to formulate the moduli stack $\mathcal{L}_{2,10m}$ of fibered algebraic surfaces over $\mathbb{P}^1$ with genus 2 curves as fibers and discriminant divisor degree equal to $10m = n+2s$ with n number of irreducible and s number of reducible nodal singularities distributed over μ number of singular fibers, we are in need of $\overline{\mathcal{M}}_2$ the Deligne–Mumford compactified stack of stable genus 2 curves that has two distinct nodal divisors $\overline{\mathcal{M}}_2 \setminus \mathcal{M}_2 = \{\delta_0, \delta_1\}$ where δ_0 is the irreducible and δ_1 is the reducible stable nodal genus 2 curve respectively.

As $\overline{\mathcal{M}}_2$ is a 3-fold moduli, we bring Mori's minimal model program on $\overline{\mathcal{M}}_2$ [Hassett, Moon] and GIT quotient result [Dolgachev] as well as Igusa's invariant & moduli of genus 2 curves over $Spec(\mathbb{Z}[1/2])$ [Igusa] to arrive at the following birational geometric fact on the coarse moduli space (over c as well as over $\mathbb{F}_q$ with characteristic $\mathbb{F}_q \neq 2,3,5$) that the tame Deligne–Mumford compactified moduli stack $\overline{\mathcal{M}}_2$ of stable genus 2 curves has a minimal model $\mathbb{P}^6//SL_2$ acquired through the divisorial contraction of the nodal divisor δ_1 to a point. The diagonal maps are divisorial contractions.

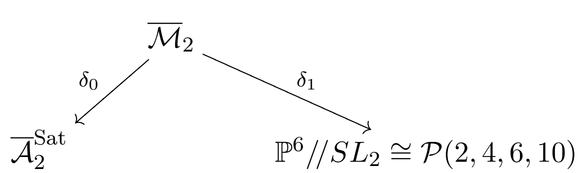
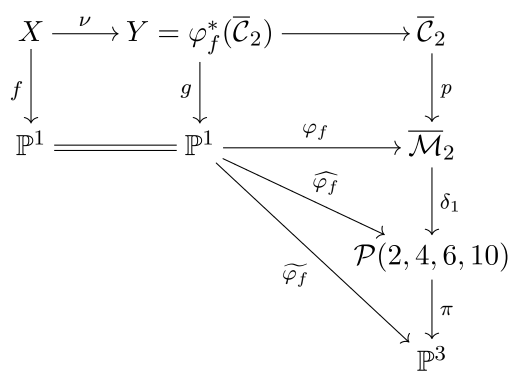


Figure 3.2: Mori's program for $\overline{\mathcal{M}}_2$

Let us designate the regular morphism $\varphi_f \in \mathrm{Hom}_m(\mathbb{P}^1, \overline{\mathcal{M}}_2)$ that induces the stable genus 2 fibrations $f : Y = \varphi_f^*(\overline{\mathcal{C}}_2) \to \mathbb{P}^1$ as the moduli map $\varphi_f : \mathbb{P}^1 \to \overline{\mathcal{M}}_2$ which becomes nonsingular semistable genus 2 fibrations $f : X \to \mathbb{P}^1$ after the resolution of singularity $\nu : X \to Y$



(3.8)

Proposition 35. *The moduli stack $\mathcal{L}_{2,10m}$ of stable genus 2 fibrations over $\mathbb{P}^1$ with degree of the discriminant divisor equal to $10m = n + 2s$ with (n, s) nodal singularities is the Deligne–Mumford mapping stack* $\mathrm{Hom}_m(\mathbb{P}^1, \overline{\mathcal{M}}_2)$*.*

Proof. Above discussion shows that $\mathcal{L}_{2,10m} \cong \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_2)$. To show that this is a Deligne–Mumford stack, notice that the target stack $\overline{\mathcal{M}}_2$ over K with char $K \neq 2, 3, 5$ is a tame Deligne–Mumford stack by [AOV]. Thus, $\mathrm{Hom}_m(\mathbb{P}^1, \overline{\mathcal{M}}_2)$ is also a Deligne–Mumford stack by [Olsson]. □

3.5 Motive count of $\mathrm{Hom}_m(\mathbb{P}^1, \mathcal{P}(2, 4, 6, 10))$

We first need to work out the motive counts for the class $\left[\mathrm{Poly}_1^{(d_1,\cdots,d_m)}\right]$ in $K_0(\mathrm{Var}_K)$.

Proposition 36. *Fix $0 \le d_1 \le d_2 \le \cdots \le d_m$. Then,*

$$\left[\mathrm{Poly}_1^{(d_1,\cdots,d_m)}\right] = \begin{cases} \mathbb{L}^{d_1+\cdots+d_m} - \mathbb{L}^{d_1+\cdots+d_m-m+1}, & \text{if } d_1 \neq 0 \\ \mathbb{L}^{d_1+\cdots+d_m}, & \text{if } d_1 = 0 \end{cases}$$

Proof. The proof for this is analogous to [FW], Theorem 1.2 (1), and is a direct generalization of Proposition 15 in [HP]. Here, we recall the differences to the work in [FW].

Step 1: The space of $(f_1, \ldots, f_m)$ monic polynomials of degree $d_1, \ldots, d_m$ is instead the quotient $\mathbb{A}^{d_1} \times \cdots \times \mathbb{A}^{d_m}/(S_{d_1} \times \cdots \times S_{d_m}) \cong \mathbb{A}^{d_1+\cdots+d_m}$. We have the same filtration of $\mathbb{A}^{\sum d_i}$ by $R_{1,k}^{(d_1,\ldots,d_m)}$: the space of monic polynomials $(f_1, \ldots, f_m)$ of degree $d_1, \ldots, d_m$ respectively for which there exists a monic $h \in K[z]$ with $\deg(h) \geq k$ and monic polynomials $g_i \in K[z]$ so that $f_i = g_i h$ for any i. The rest of the arguments follow analogously, keeping in mind that the group action is via $S_{d_1} \times \cdots \times S_{d_m}$.

Step 2: Here, we prove that $R_{1,k}^{(d_1,\ldots,d_m)} - R_{1,k+1}^{(d_1,\ldots,d_m)} \cong \mathrm{Poly}_1^{(d_1-k,\ldots,d_m-k)} \times \mathbb{A}^k$. Just as in [FW], the base case of $k = 0$ follows from the definition. For $k \geq 1$, the rest of the arguments follow analogously just as in Step 1 of loc. cit.

Step 3: By combining Step 1 and 2 as in [FW], we obtain

$$\left[\mathrm{Poly}_1^{(d_1,\ldots,d_m)}\right] = \mathbb{L}^{d_1+\cdots+d_m} - \sum_{k\geq 1} \left[\mathrm{Poly}_1^{(d_1-k,\ldots,d_m-k)}\right] \mathbb{L}^k$$

For the induction on the class $\left[\mathrm{Poly}_1^{(d_1,\ldots,d_m)}\right]$, we use lexicographic induction on the pair $(d_1, \ldots, d_m)$. For the base case, consider when $d_1 = 0$. Here the monic polynomial of degree 0 is nowhere vanishing, so that any polynomial of degree d_i for $i > 1$ constitutes a member of $\mathrm{Poly}_1^{(0,d_2,\ldots,d_m)}$, so that $\mathrm{Poly}_1^{(0,d_2,\ldots,d_m)} \cong \mathbb{A}^{d_2+\cdots+d_m}$.

Now assume that $d_1 > 0$. Then, we obtain

$$
\begin{aligned}
&\left[\mathrm{Poly}_1^{(d_1,\ldots,d_m)}\right] \\
&= \mathbb{L}^{(d_1+\cdots+d_m)} - \sum_{k\geq 1}\left[\mathrm{Poly}_1^{(d_1-k,\ldots,d_m-k)}\right]\mathbb{L}^k \\
&= \mathbb{L}^{d_1+\cdots+d_m} \\
&- \left(\sum_{k=1}^{d_1-1}(\mathbb{L}^{(d_1-k)+\cdots+(d_m-k)} - \mathbb{L}^{(d_1-k)+\cdots+(d_m-k)-m+1})\mathbb{L}^k + \mathbb{L}^{(d_2-d_1)+\cdots+(d_m-d_1)}\mathbb{L}^{d_1}\right) \\
&= \mathbb{L}^{d_1+\cdots+d_m} \\
&- \left(\sum_{k=1}^{d_1-1}(\mathbb{L}^{d_1+\cdots+d_m-(m-1)k} - \mathbb{L}^{d_1+\cdots+d_m-(m-1)(k+1)}) + \mathbb{L}^{d_1+\cdots+d_m-(m-1)d_1}\right) \\
&= \mathbb{L}^{d_1+\cdots+d_m} - \mathbb{L}^{d_1+\cdots+d_m-m+1}
\end{aligned}
$$

□

Note that whenever $d_i \neq 0$,

$$[\mathrm{Poly}_1^{(d_1,\cdots,d_m)}] = [\mathrm{Poly}_1^{(1,\cdots,1)}](\mathbb{L}^{(d_1+\cdots+d_m)-m}) = (\mathbb{L}^m - \mathbb{L})(\mathbb{L}^{(d_1+\cdots+d_m)-m})$$

And we set $[F_{(d_1,\cdots,d_m)}] = [\mathbb{G}_m]\cdot[\mathrm{Poly}_1^{(d_1,\cdots,d_m)}]$.

Here is the class $[\mathrm{Hom}_m(\mathbb{P}^1,\mathcal{P}(2,4,6,10))]$ in $K_0(\mathrm{Var}_K)$ for the coarse moduli space of $\mathrm{Hom}_m(\mathbb{P}^1,\mathcal{P}(2,4,6,10))$ over field K that $\mathrm{char}(K) \neq 2,3,5$.

We can easily see how below method would generalize to the motive count of $\mathcal{P}(\vec{\lambda})$

$$[c(\mathrm{Hom}_d(\mathbb{P}^1,\mathcal{P}(\vec{\lambda})))] \in K_0(\mathrm{V}ar_K)$$

$$
\begin{aligned}
&[c(\mathrm{Hom}_m(\mathbb{P}^1, \mathcal{P}(2,4,6,10)))] \\
&= [F_{2m,4m,6m,10m}] + \sum_{i=0}^{2m-1} [F_{i,4m,6m,10m}] + \sum_{j=0}^{4m-1} [F_{2m,j,6m,10m}] \\
&\quad + \sum_{k=0}^{6m-1} [F_{2m,4m,k,10m}] + \sum_{l=0}^{10m-1} [F_{2m,4m,6m,l}] \\
&= [\mathbb{G}_m] \left(\left[\mathrm{Poly}_1^{(2m,4m,6m,10m)}\right] + \sum_{i=0}^{2m-1} \left[\mathrm{Poly}_1^{(i,4m,6m,10m)}\right] \right) \\
&\quad + \left(\sum_{j=0}^{4m-1} \left[\mathrm{Poly}_1^{(2m,j,6m,10m)}\right] + \sum_{k=0}^{6m-1} \left[\mathrm{Poly}_1^{(2m,4m,k,10m)}\right] + \sum_{l=0}^{10m-1} \left[\mathrm{Poly}_1^{(2m,4m,6m,l)}\right] \right) \\
&= (\mathbb{L}-1)\left(\mathbb{L}^{22m} - \mathbb{L}^{22m-3}\right) \\
&\quad + (\mathbb{L}-1)\left(\mathbb{L}^{20m} + \sum_{i=1}^{2m-1} (\mathbb{L}^{20m+i} - \mathbb{L}^{20m+i-3})\right) \\
&\quad + (\mathbb{L}-1)\left(\mathbb{L}^{18m} + \sum_{j=1}^{4m-1} (\mathbb{L}^{18m+j} - \mathbb{L}^{18m+j-3})\right) \\
&\quad + (\mathbb{L}-1)\left(\mathbb{L}^{16m} + \sum_{k=1}^{6m-1} (\mathbb{L}^{16m+i} - \mathbb{L}^{16m+k-3})\right) \\
&\quad + (\mathbb{L}-1)\left(\mathbb{L}^{12m} + \sum_{l=1}^{10m-1} (\mathbb{L}^{12m+l} - \mathbb{L}^{12m+l-3})\right) \\
&= (\mathbb{L}-1)\left(\mathbb{L}^{22m} - \mathbb{L}^{22m-3}\right) \\
&\quad + (\mathbb{L}-1)\left(-\mathbb{L}^{20m-2} - \mathbb{L}^{20m-1} + \mathbb{L}^{22m-3} + \mathbb{L}^{22m-2} + \mathbb{L}^{22m-1}\right) \\
&\quad + (\mathbb{L}-1)\left(-\mathbb{L}^{18m-2} - \mathbb{L}^{18m-1} + \mathbb{L}^{22m-3} + \mathbb{L}^{22m-2} + \mathbb{L}^{22m-1}\right) \\
&\quad + (\mathbb{L}-1)\left(-\mathbb{L}^{16m-2} - \mathbb{L}^{16m-1} + \mathbb{L}^{22m-3} + \mathbb{L}^{22m-2} + \mathbb{L}^{22m-1}\right) \\
&\quad + (\mathbb{L}-1)\left(-\mathbb{L}^{12m-2} - \mathbb{L}^{12m-1} + \mathbb{L}^{22m-3} + \mathbb{L}^{22m-2} + \mathbb{L}^{22m-1}\right) \\
&= (\mathbb{L}-1)\left(\mathbb{L}^{22m} + 4\cdot\mathbb{L}^{22m-1} + 4\cdot\mathbb{L}^{22m-2} + 3\cdot\mathbb{L}^{22m-3}\right) \\
&\quad + (\mathbb{L}-1)\left(-\mathbb{L}^{20m-2} - \mathbb{L}^{20m-1} - \mathbb{L}^{18m-2} - \mathbb{L}^{18m-1}\right) \\
&\quad + (\mathbb{L}-1)\left(-\mathbb{L}^{16m-2} - \mathbb{L}^{16m-1} - \mathbb{L}^{12m-2} - \mathbb{L}^{12m-1}\right)
\end{aligned}
$$

3.6 Point count of $\mathcal{L}_{2,10m}$ the moduli of semistable genus 2 hyperelliptic fibrations

Given the above motive count of $[c(\mathrm{Hom}_m(\mathbb{P}^1, \mathcal{P}(2,4,6,10)))]$, we are now able to give the upper bound for the point count of the moduli $L_{2,10m}$ of nonsingular semistable genus 2 fibrations over $\mathbb{P}^1$ with discriminant divisor degree equal to $10m = n + 2s$ with (n, s) nodal singularities distributed over μ distinct number of singular fibers. Taking the motivic measure to $\mathbb{Z}$ to acquire the $|c(\mathrm{Hom}_m(\mathbb{P}^1, \mathcal{P}(2,4,6,10)))(\mathbb{F}_q)|$, then we will subtract the point count of the boundary component $\mathbb{P}^1 \hookrightarrow \delta_0$ which is when the projective line $\mathbb{P}^1$ is embedded to be entirely inside the nodal divisor δ_0. While we can't get the exact point count of this, we are able to provide the upper bound point count through the identification $\delta_0 = \overline{\mathcal{M}}_{1,2}/\mathbb{Z}_2$ which implies $|c(\mathrm{Hom}_m(\mathbb{P}^1, \delta_0 = \overline{\mathcal{M}}_{1,2}/\mathbb{Z}_2))(\mathbb{F}_q)|$ and apply the Tate-Shioda formula for the upperbound of the rank of $MW(X)$ the Mordell-Weil group of X.

As usual, we let $X \to \mathbb{P}^1$ be an elliptic surface over k with generic fibre E over $k(\mathbb{P}^1)$. The K-rational points $E(K)$ form a group which is traditionally called $MW(X)$ the Mordell-Weil group of X. The $MW(X)$ is a group of sections on elliptic surface X.

Theorem 37 (The Shioda-Tate formula)**.** *Let X be a nonsingular elliptic surface over $\mathbb{P}^1$ with a distinguished section and μ number of distinct singular fibers. Denote by $\rho(X)$ the rank of $NS(X)$ which is the same as Picard number of X and r_i the number of irreducible components of a singular fiber together with $MW(X)$ the Mordell-Weil group of X.*

$$\rho(X) = 2 + \sum_{i=1}^{\mu} r_i + rk(MW(X))$$

The following relation is worth noticing:

$$\# \begin{pmatrix} \text{components of} \\ \text{singular fibre} \end{pmatrix} = \begin{cases} v(\Delta) & \text{in the multiplicative case;} \\ v(\Delta) - 1 & \text{in the additive case;} \end{cases} \tag{3.9}$$

Note that if X is semistable (only I_{k_i} singular fibers) then $r_i = k_i - 1,\ \forall i$. Thus, we have

$$\rho(X) = 2 + \sum_{i=1}^{\mu}(k_i - 1) + rk(MW(X))$$
$$\rho(X) = 2 + \sum_{i=1}^{\mu}(k_i) - \mu + rk(MW(X))$$
$$\rho(X) = 2 + 12n - \mu + rk(MW(X))$$

Note by the Igusa bound $0 \leq \rho \leq b_2$ and by $b_2 = 12n - 2$

$$0 \leq \rho \leq 12n - 2$$

which leads us to

$$4 \leq \mu \leq 12n$$

by

$\mu_{min} = 2 + 12n - \rho + rk(MW(X)) = 2 + 12n - (12n - 2) + 0 = 4$

and $\mu_{max} = 12n$ when X is stable.

Thus

$rk(MW(X))_{max} = \rho_{max} + \mu_{max} - (2 + 12n) = (12n - 2) + (12n) - (2 + 12n) = 12n - 4$

(Note that the $rk(MW(X))_{max}$ is achieved when X has the maximal Picard number as well as being a stable elliptic surface.) Finally,

$$0 \leq rk(MW(X)) \leq 12n - 4$$

Theorem 38 (Point count of the moduli $L_{2,10m}$)**.** *If char*$(\mathbb{F}_q) \neq 2, 3, 5$*, then*

$$\begin{aligned}
&||L_{2,10m}(\mathbb{F}_q)| \\
&\leq (q-1)\left(q^{22m} + 4 \cdot q^{22m-1} + 4 \cdot q^{22m-2} + 3 \cdot q^{22m-3}\right) \\
&\quad + (q-1)\left(-q^{20m-2} - q^{20m-1} - q^{18m-2} - q^{18m-1}\right) \\
&\quad + (q-1)\left(-q^{16m-2} - q^{16m-1} - q^{12m-2} - q^{12m-1}\right) \\
&\quad - \left(\frac{(12m-4)}{2}(q^{10m+1} - q^{10m-1})\right)
\end{aligned}$$

Proof. Note that the point count of the Igusa data space is

$$\begin{aligned}
&|c(\mathrm{Hom}_m(\mathbb{P}^1, \mathcal{P}(2,4,6,10)))(\mathbb{F}_q)| \\
&= (q-1)\left(q^{22m} + 4 \cdot q^{22m-1} + 4 \cdot q^{22m-2} + 3 \cdot q^{22m-3}\right) \\
&\quad + (q-1)\left(-q^{20m-2} - q^{20m-1} - q^{18m-2} - q^{18m-1} - q^{16m-2}\right) \\
&\quad + (q-1)\left(-q^{16m-1} - q^{12m-2} - q^{12m-1}\right)
\end{aligned}$$

From this, we substract the point count of the boundary component which is $|c(\mathrm{Hom}_m(\mathbb{P}^1, \delta_0 = \overline{\mathcal{M}}_{1,2}/\mathbb{Z}_2))(\mathbb{F}_q)|$.

$$\left(\frac{(12m-4)}{2}(q^{10m+1} - q^{10m-1})\right)$$

Since $\mathbb{P}^1 \to \overline{\mathcal{M}}_{1,2}$ would correspond to the semistable elliptic surfaces with a distinguished section for the zero in the Mordell-Weil group of elliptic fiber together with another point in Mordell-Weil lattice, this is equivalent to the rank of the Mordell-Weil group of X which is bounded above by $(12m-4)$ for a degree m semistable elliptic surface which we divide by 2 and multiply by $(q^{10m+1} - q^{10m-1})$ since that is exactly how many degree m semistable elliptic surfaces there are.

□

3.7 Upper bound point counts of $c(\mathrm{Hom}_m(\mathbb{P}^1, \overline{\mathcal{H}}_g))$ for $g \geq 2$

Begin with the observation that,

$$\left|c\Big(\operatorname{Hom}_m(\mathbb{P}^1,\overline{\mathcal{H}}_g)\Big)(\mathbb{F}_q)\right| \le \left|c\Big(\operatorname{Hom}_m(\mathbb{P}^1,\mathcal{P}(\vec{\lambda_g}))\Big)(\mathbb{F}_q)\right|$$

which follows from the minimal model of $\overline{\mathcal{H}}_g \to \mathbb{P}^{2g+2}//SL_2 \to \mathcal{P}(\vec{\lambda_g})$ being a proper, birational map.

We are now able to state how the upper bound point counts are for $c\Big(\operatorname{Hom}_m(\mathbb{P}^1,\overline{\mathcal{H}}_g)\Big)$ for $g \ge 2$ by using $\Big[c\Big(\operatorname{Hom}_m(\mathbb{P}^1,\mathcal{P}(\vec{\lambda_g}))\Big)\Big] \in K_0(\mathrm{V}ar_K)$ for $\vec{\lambda_g} = (d_1, \cdots, d_\eta)$ over any field K with $\mathrm{char}(K) \nmid d_i$ for $1 \le i \le \eta$

We apply to the following,

$\mathcal{L}_{g,(4g+2)m} \cong \operatorname{Hom}_m(\mathbb{P}^1,\overline{\mathcal{H}}_g)$ for g even.

$\mathcal{L}_{g,2(4g+2)m} \cong \operatorname{Hom}_m(\mathbb{P}^1,\overline{\mathcal{H}}_g)$ for g odd.

Begin with $g = 2$, note $\overline{\mathcal{M}}_2 = \overline{\mathcal{H}}_2 \to \mathbb{P}^6//SL_2 \to \mathcal{P}(2,4,6,10)$

$$\Big[c\Big(\operatorname{Hom}_m(\mathbb{P}^1,\mathcal{P}(2,4,6,10))\Big)\Big] \sim \mathcal{O}\left(q^{22m+1}\right)$$

This is by the above.

$g = 3$, note $\overline{\mathcal{H}}_3 \to \mathbb{P}^8//SL_2 \to \mathcal{P}(2,3,4,5,6,7,8,9,10)$

$$\Big[c\Big(\operatorname{Hom}_m(\mathbb{P}^1,\mathcal{P}(2,3,4,5,6,7,8,9,10))\Big)\Big] \sim \mathcal{O}\left(q^{54m+1}\right)$$

The proof follows directly from the above $g = 2$ case.

$g = 4$, note that $\overline{\mathcal{H}}_4 \to \mathbb{P}^{10}//SL_2 \to \mathcal{P}(\vec{\lambda_4})$ where $\vec{\lambda_4}$ is

$\vec{\lambda_4} = (2,4,6,6,6,6,8,8,8,8,8,9,9,9,9,9,10,10,10,10,10,10,10,10,-$

$-11,11,11,11,11,11,11,11,12,12,12,12,12,12,12,12,12,12,12,12,13,13,13,13,-$

$-13,13,13,13,13,13,13,13,13,13,13,14,14,14,14,14,14,14,14,14,14,14,14,-$

$-15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,16,16,16,16,16,-$

$-17,17,17,17,17,18,19,19,21,21)$

There are 106 invariants [BP] and $\sum_{i=1}^{\eta=106} d_i = 1352$

$$\left[c\Big(\operatorname{Hom}_m(\mathbb{P}^1, \mathcal{P}(\vec{\lambda_4}))\Big)\right] \sim \mathcal{O}\left(q^{1352m+1}\right)$$

Starting with $g = 4$ there is an explosion of the invariants of binary forms, and yet, using the above we can show that the motive count can be computed given that we have the weights $\vec{\lambda_g}$ related to the minimal model of $\overline{\mathcal{H}}_g$. The weights come from the invariant theory of binary forms.

$g \geq 4$, note that $\overline{\mathcal{H}}_g \to \mathbb{P}^{2g+2}//SL_2 \to \mathcal{P}(\vec{\lambda_g})$

$$\left[c\Big(\operatorname{Hom}_m(\mathbb{P}^1, \mathcal{P}(\vec{\lambda_g}))\Big)\right] \sim \mathcal{O}\left(q^{(\sum_{i=1}^{\eta} d_i)m+1}\right)$$

Theorem 39 (Point count of the moduli $L_{g,\Delta}$)**.** *For $\vec{\lambda_g} = (d_1, \cdots, d_\eta)$ over any field K with* $\operatorname{char}(K) \nmid d_i$ *for $1 \leq i \leq \eta$, then $|L_{g,\Delta}(\mathbb{F}_q)|$ is bounded above by the polynomial in q.*

Proof. Let $\mathcal{O}(V_n)^{SL_2}$ denote the algebra of invariants of binary forms (forms in two variables) of degree n with complex coefficients. $\mathcal{O}(V_n)^{SL_2}$ has a finite basis for all n by the Hilbert's theorem. And thus we see that the point count will be a polynomial in q. □

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