Estimation of an Autoregressive Parameter when the Innovations are Heavy Tailed

Ronald C. Pruitt
University of Minnesota
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Abstract

Assume an autorgression $Y_i = \beta Y_{i-1} + V_i$, $i=1,\ldots,n$, where the V_i are i.i.d. with an unknown distribution lacking second moment. Some large sample properties of two estimates of β are compared: a nonparametric estimate given as $\hat{\beta} = \text{med}\ \{(Y_i - Y_j)/(Y_{i-1} - Y_{j-1}) : 2 \le i < j \le n\}$, and the least squares estimate. Under these conditions both estimators converge to β at a rate exceeding $n^{1/2}$, and the least squares estimate has a better rate of convergence than the nonparametric estimator.

1. Introduction. We consider estimates for the first order autoregressive parameter when the underlying innovation distribution may lack second moment. In the classical case where second moments exist, the asymptotic distribution of a wide variety of estimators is known. We consider two specific estimates which have been adapted from regression models: the least squares estimate, and a nonparametric estimate proposed by Theil (1950) which is the median of the slopes between pairs of observations. The asymptotic distribution of the Theil-type estimate has been studied by Chanda and Kulp (1978) who find an asymptotically normal distribution for the estimate with norming constant $n^{1/2}$. They also find the asymptotic relative efficiency of the Theil-type estimate relative to the least squares estimate for several innovation distributions which do have second moment; here we compare the estimates when second moments do not exist.

Part of the motivation for looking at nonparametric estimators in regression-type problems is their improved relative performance compared to least squares when the innovation distribution is heavy tailed. In an autogregression the tail character of the "dependent" and "independent" variable is constrained to be the same. Increased spread in the dependent variable hurts estimation of the slope parameter, and increased spread in the independent variable helps estimation. The autoregression model allows us to assess whether spread in the "dependent" or "independent" variable plays a greater role in convergence of the parameter by increasing the spread in both at the same time. Here we examine the behavior of the estimates when the innovation distribution belongs to the class of distributions attracted to a stable law with index α (see for example Feller (1971), or Ibragimov and Linnik (1971)). The stable laws are the only possible non-degenerate limits of normalized sums from strongly mixing strictly stationary processes (Ibragimov and Linnik (1971)) so they are mathematically convenient since the asymptotic theory is well understood, and it is also the case that the family considered provides a rich class of distributions for modelling heavy tails. The weight of the tail is indexed by a parameter α in the range $(0,2]$, where smaller α correspond to heavier tails; ignoring the boundary case $\alpha = 2$, the distributions with index α have absolute moments up to but not including α. Well known examples include the normal (α=2), Cauchy(α=1), and Pareto(various α) distributions. Stable distributions have been used, especially in economics, to describe such variables as stock price changes and size of cities. For a list of references see Press (1975) or DuMouchel (1983).

In some case we have been unable to find an asymptotic distribution of a parameter estimate, but instead have only been able to bound the rate of convergence. The following definition covers this case.

Definition 1.1. For an estimator $\hat{\beta}_n$ of β, from a sample of size n, if $n^{\gamma-\delta}\ (\hat{\beta}_n - \beta) \xrightarrow{P} 0$ and $n^{\gamma+\delta}|\hat{\beta}_n - \beta| \xrightarrow{P} \infty$ for any $\delta > 0$ we say the *weak rate of convergence of* $\hat{\beta}_n$ *to* β *is* n^{γ}. □

If an asymptotic distribution does exist then the rate of convergence provides the normalizing constant up to a function which varies more slowly than any power of n.

Hannan and Kanter (1977) have obtained a lower bound for the weak rate of convergence of the least squares estimate in an autoregression model with known intercept as $n^{1/\alpha}$. Here we show that for the model with unknown intercept the corresponding rate is $\min(n^{1/\alpha}, n^{1})$. The rate truncation occurs because of the inability to estimate the intercept. For the Theil-type estimate, which shows good results for distributions with tails heavier than normal but having second moment, the weak rate of convergence is $n^{1/2\alpha}$ which is slower than that for the least squares estimate.

Under these conditions, if the model is true, it is advantageous, at least in very large samples, to use the least squares estimate as oppposed to the more robust Theil-type estimate. Rather than just losing a small percentage in asymptotic efficiency, the asymptotic efficiency may be zero, so protection against observations which are not generated by the assumed model (outliers) comes at a greater cost in these models than in the case when second moments exist, at least when using the Theil-type estimate. Outliers are also more difficult to detect because the distributions are heavy tailed, and in fact the observations far in the tails which are generated by the model are those which cause the imporved rates of convergence. The LSE is also presumably not optimal in these situations. This raises the question whether a robust type estimator can achieve the same convergence rate as least squares for both heavy tailed and non-heavy tailed error distributions, which remains an open problem. Finally, the results of a small simulation study are presented in Section 4.

2. Asymptotic distribution of the estimate.

Throughout this paper assume a model of the form

$$Y_i = \beta Y_{i-1} + V_i, \quad i = \ldots, -2, -1, 0, 1, 2, \ldots, \qquad (2.1)$$

where the V_i are i.i.d. with an unknown distribution, β is an unknown parameter with $|\beta| < 1$, and we observe Y_i for $i=1,2,\ldots,n$. The model is often reparametrized as $Y_i = \beta_0 + \beta Y_{i-1} + V_i^*$, where β_0 is a location parameter for the distribution of V_1. The methods of this paper do not depend on knowledge of β_0, so they are properly compared to estimates where the mean is assumed unknown.

To define the estimate of β, denote the residual if b is the autoregressive parameter by

$$V_i(b) = Y_i - bY_{i-1} = V_i + (\beta-b)Y_{i-1}.$$

Now let

$$U_n(b) = \binom{n-1}{2}^{-1} \sum_{2\le i<j\le n}\sum \mathrm{sgn}(Y_{i-1}-Y_{j-1})\,\mathrm{sgn}(V_i(b)-V_j(b))$$

which is a Kendall's tau statistic between $\{Y_{i-1}\}$ and $\{V_i(b)\}$. Let $\hat{\beta}_n$ denote a zero crossing of U_n, that is a point such that $U_n(\hat{\beta}_n-\delta)U_n(\hat{\beta}_n+\delta)\le 0$, $\forall\delta>0$. Then $\hat{\beta}_n$ can be equivalently defined by $\hat{\beta}_n = \mathrm{med}\{(Y_i-Y_j)/(Y_{i-1}-Y_{j-1}) : 2\le i<j\le n\}$ (see Sen (1968)), this is of the estimates we study. The random function U_n is montonically decreasing in b, a fact we will make use of later. The LSE is the zero of a similar statistic where the sign functions are replaced by identity functions. Here as in the simple regression, a zero crossing of U_n provides a more robust, that is, less affected by outlying values, estimate of β than least squares.

In this section we derive the asymptotic distribution of $\hat{\beta}_n$ assuming the first absolute moment of V_1 is finite: this result (Theorem 2.1) has been previously proved by Chanda and Kulp (1978). We present the proof since we will make use of Lemma 2.2 in Section 3.

Before we state the theorem covering the asymptotic distribution of $\hat{\beta}$, we develop some notation. Let $Z_i=(Y_{i-1}, V_i)^t$ so that

$$\begin{aligned} U_n(\beta+s) &= \binom{n-1}{2}^{-1} \sum_{2\le i<j\le n}\sum \operatorname{sgn}(Y_{i-1}-Y_{j-1})\operatorname{sgn}(V_i-V_j-s(Y_{i-1}-Y_{j-1})) \\ &= \binom{n-1}{2}^{-1} \sum_{2\le i<j\le n}\sum h(s,Z_i, Z_j), \end{aligned} \tag{2.2}$$

where $h(s,\cdot,\cdot)$ is symmetric. Let Z^0 be independent of the process $\{Z_n\}$, but with the same distribution as Z_1. Then define

$$\begin{aligned} h_1(s,Z_i) &= E[h(s,Z^0,Z_i)|Z_i], \\ \mu(s) &= E[h(s,Z^0,Z_i)] = E[h_1(s,Z_i)], \end{aligned} \tag{2.3}$$

$$\text{and } \sigma^2(s) = 4\{\operatorname{Var}(h_1(s,Z_2)) + 2\sum_{j=3}^{\infty} \operatorname{Cov}(h_1(s,Z_2),h_1(s,Z_j))\}.$$

It will be shown in Theorem A1.2 that $n^{1/2}(U_n(\beta+s)-\mu(s)) \xrightarrow{d} N(0,\sigma^2(s))$, but for the moment we only need the definitions of μ and σ^2.

The expression for $\sigma^2(0)$ may be simplified as follows with F(G) the distribution function of $Y_1(V_1)$, and U_1 and U_2 and independent uniform (0,1) random variables

$$
\begin{aligned}
\sigma^2(0) &= 4\{\mathrm{Var}(h_1(0,Z_2)) + 2\sum_{j=3}^{\infty} \mathrm{Cov}(h_1(0,Z_2),h_1(0,Z_j))\} \\
&= 4\{E[h_1^2(0,Z_2)] + 2\sum_{j=3}^{\infty} E[h_1(0,Z_2)h_1(0,Z_j)]\} \\
&= 4\ E[(1-2F(Y_1))^2(1-2G(V_2))^2] \\
&= 64\ \mathrm{Var}(U_1U_2) \qquad\qquad (2.4) \\
&= 4/9.
\end{aligned}
$$

Theorem 2.1. Assume the model (2.1) holds and the following conditions are satisfied:

A1) V_1 has a bounded continuous density $g(v)$,

A2) $\int|g(v)-g(v-\theta)|dv = O(|\theta|^{\gamma})$ for some $\gamma > 0$ as $\theta \to 0$,

and A3) $E[|V_1-V_2|] < \infty$.

Then $n^{1/2}(\hat{\beta}_n-\beta) \xrightarrow{d} N(0,\lambda^2)$ where $\hat{\beta}_n$ is any zero crossing of U_n, $\lambda^2 = \sigma^2(0)\mu'(0)^{-2}$, $\sigma^2(0)$ is given in (2.4), and $\mu'(0)$ is given in (2.5). □

Conditions A1), A2), and $E[|V_1|^r] < \infty$ for some $r > 0$ are required in order to show a weak mixing condition holds for the first order autoregressive process. Condition A2) is a smoothness condition on $g(v)$ which is satisfied for instance with $\gamma = 1$ by any continuously differentiable g with a finite number of maxima or bounded derivative. Condition A3) implies that $E[|V_1|^r] < \infty$ for any $0 < r \leq 1$, and is needed in proving asymptotic normality. Some results when A3) does not hold are given in Section 3.

Similar techniques to these used in proving Theorem 2.1 can be used to prove an asymptotic distribution theorem for an estimate $\tilde{\beta}_n$ given as a zero crossing of

$$\tilde{U}_n(b) = \binom{n-1}{2}^{-1} \sum_{2 \le i < j \le n} \sum (Y_{i-1} - Y_{j-1}) \operatorname{sgn}(V_i(b) - V_j(b)).$$

The details are given in Pruitt (1987). This estimate was proposed by Adichie (1967) for the regression case, and for that model it seems to combine the good features of both $\hat{\beta}_n$ and the least squares estimate when dealing with heavy tails.

Proof of Theorem 2.1. To prove Theorem 2.1, it suffices to show that

$$\lim_{n \to \infty} P[n^{1/2}(\hat{\beta}_n - \beta) < t] = \Phi(\lambda^{-1} t),$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, and λ is given in the statement of Theorem 2.1. Now, due to the monotonicity of U_n,

$$\begin{aligned} P[n^{1/2}(\hat{\beta}_n - \beta) < t] &= P[\hat{\beta}_n < \beta + n^{-1/2} t] \\ &\le P[U_n(\beta + n^{-1/2} t) \le 0] \quad \{ \ge P[U_n(\beta + n^{-1/2} t) < 0] \} \\ &= P[T_n(n^{-1/2} t) \le -n^{-1/2} \mu(n^{-1/2} t)] \\ &\qquad \{ = P[T_n(n^{-1/2} t) < -n^{-1/2} \mu(n^{-1/2} t)] \}, \end{aligned}$$

where $T_n(s) = n^{1/2}[U_n(\beta + s) - \mu(s)]$. Hence it suffices to show the following: 1) For each fixed t, $T_n(n^{-1/2} t)$ converges in distribution to a normal variable with mean zero and variance $\sigma^2(0)$, and 2) $-n^{1/2} \sigma^{-1}(0) \mu(n^{-1/2} t) \to \lambda^{-1} t$.

To show 1) we prove Lemma 2.2 which is more general than is needed here, but the strengthened version is used in Section 3. The proof of Lemma 2.2 is somewhat long and is given in the Appendix.

Lemma 2.2 $T_n(n^{-\gamma}t)$ converges in distribution to a normal random variable with mean zero and variance $\sigma^2(0)$ for any t and any $\gamma > 0$. □

To finish the proof of Theorem 2.1, we need to show $-n^{1/2}\sigma^{-1}(0)\mu(n^{-1/2}t) \to \lambda^{-1}t$. Recall $\lambda^2 = \sigma^2(0)\mu'(0)^{-2}$, and note $\mu(0) = E[\mathrm{sgn}(Y_1-Y^0)\mathrm{sgn}(V_2-V^0)] = 0$ by symmetry, so that it suffices to show $-n^{1/2}\mu(n^{-1/2}t) \to |\mu'(0)|t$. To show this we need only show that $\mu'(r)$ is continuous and $\mu'(0) < 0$. Now

$$\begin{aligned}\mu(r) &= \mu(r) - \mu(0)\\ &= E[\mathrm{sgn}(Y^0-Y_{i-1})\{\mathrm{sgn}(V^0-V_i-r(Y^0-Y_{i-1})) - \mathrm{sgn}(V^0-V_i)\}]\\ &= -2\ E[\mathrm{sgn}(Y^0-Y_{i-1})\{G(V_i+r(Y^0-Y_{i-1})) - G(V_i)\}].\end{aligned}$$

Hence

$$\mu'(r) = -2\ E[|Y^0-Y_{i-1}|g(V_i+r(Y^0-Y_{i-1})]$$

by using A1) and A3) which show $\mu'(r)$ is finite. In particular

$$\mu'(0) = -2\ E[|Y^0-Y_{i-1}|] \cdot \int g^2(v)dv < 0, \qquad (2.5)$$

and $\mu'(r)$ is continuous by A1). This completes the proof of Theorem 2.1. □

3. Behavior of the estimates with heavy tailed errors. Theorem 2.1 does not hold if $E[|V_1|]$ is not finite. In autoregression cases with heavy tailed errors, estimation of β is generally improved as the tails get heavier (Hannan and Kanter (1977)). The class of heavy tailed distributions we consider is

those that are in the domain of attraction of a stable law which is defined as follows. Let $\{X_n\}$ be an independent identically distributed sequence of random variables. Let F be the distribution function of X_1. F belongs to the domain of attraction of a stable law with index α, $0 < \alpha < 2$ (we denote this as $F \in \mathcal{D}(\alpha)$), if and only if, as $|x| \to \infty$

$$F(x) \begin{cases} (c_1 + o(1))\, L(-x)\, |x|^{-\alpha}, & x < 0 \\ 1 - (c_2 + o(1))\, L(x)\, x^{-\alpha}, & x > 0 \end{cases},$$

where L(x) is slowly varying at ∞, $c_1, c_2 \geq 0$, and $c_1 + c_2 > 0$ (see Ibragimov and Linnik (1971), Sec. 2.6).

We will see that the weak rate of convergence (see Definition 1.1) for the least squares estimate β_n^* is at least $n^{1/\alpha}$ if $G_v \in \mathcal{D}(\alpha)$ for $1 < \alpha < 2$. The rate of convergence for $\hat{\beta}_n$ is not as good as that of β_n^* if $G_v \in \mathcal{D}(\alpha)$ for $1 < \alpha < 2$ since in this case Theorem 2.1 holds and the weak rate of convergence of $\hat{\beta}_n$ is $n^{1/2}$. If $0 < \alpha < 1$, Theorem 2.1 does not hold but Theorem 3.1 shows that the weak convergence rate of $\hat{\beta}_n$ is $n^{1/2\alpha}$. For β_n^* the weak convergence rate is n^1. The LSE has a better convergence rate if $0.5 < \alpha < 1$ and a worse rate if $0 < \alpha < 0.5$. The MPS estimate gets better as the tails of the error get heavier, but for distributions with no mean this is not true for the LSE, where the convergence rate is limited by the inability to estimate the location of the error distribution. Hence the convergence rate for the MPS estimate surpasses that of the LSE when $\alpha < 0.5$. But for $0.5 < \alpha < 2$ the convergence rate for β_n^* is strictly greater than that of $\hat{\beta}_n$. So to use $\hat{\beta}_n$ to provide protection against outliers in this autoregression model comes at a greater cost than when an error variance exists(α=2), at least in very large samples. When an error variance exists, the convergence rate for $\hat{\beta}_n$ is the same as for β_n^* although the asymptotic relative efficiency can be less than one. But in cases when $\alpha < 2$, the convergence rate itself may be worse. Whether this is true of robust methods in general remains an open question. The convergence rate of $\hat{\beta}_n$ is given in the following theorem.

Theorem 3.1. Assume the model (2.1) holds along with the following conditions:

B1) V_1 has a bounded continuous density $g(v)$,

B2) $\int|g(v)-g(v-\theta)|dv = O(|\theta|^{\gamma})$ for some $\gamma > 0$ as $\theta \to 0$,

and B3) $G_v \in \mathcal{D}(\alpha)$, $0 < \alpha \le 1$.

Then

$$n^{1/2\alpha-\delta}(\hat{\beta}_n-\beta) \xrightarrow{P} 0 \quad \text{and} \quad n^{1/2\alpha+\delta}|\hat{\beta}_n-\beta| \xrightarrow{P} \infty$$

for any $\delta > 0$, where $\hat{\beta}_n$ is any zero crossing of U_n. □

Proof of Theorem 3.1. Assume for the moment that it can be shown that $C_1|s|^{\alpha+z} \le |\mu(s)| \le C_2|s|^{\alpha-z}$ for any $z > 0$, and given z for s in a neighborhood of zero. Using these facts and the monotonicity of U_n,

$$\begin{aligned} P[n^{1/2\alpha-\delta}|\hat{\beta}_n-\beta| < \epsilon] &= P[\hat{\beta}_n<\beta+\epsilon n^{-1/2\alpha+\delta}]-P[\hat{\beta}_n < \beta-\epsilon n^{-1/2\alpha+\delta}] \\ &\ge P[U_n(\beta+\epsilon n^{-1/2\alpha+\delta})<0]-P[U_n(\beta-\epsilon n^{-1/2\alpha+\delta})\le 0] \\ &\to \lim_{n\to\infty} \{\Phi(-n^{1/2}\sigma^{-1}(0)\mu(\epsilon n^{-1/2\alpha+\delta})) \\ &\quad - \Phi(-n^{1/2}\sigma^{-1}(0)\mu(-\epsilon n^{-1/2\alpha+\delta}))\} \end{aligned}$$

$$\geq \lim \{\Phi(\sigma^{-1}(0)C_1\epsilon^{\alpha}n^{\alpha\delta}) - \Phi(-\sigma^{-1}(0)C_1\epsilon^{\alpha}n^{\alpha\delta})\} = 1$$

for small enough z using Lemma 2.2 Similarly $P(n^{1/2\alpha+\delta}|\hat{\beta}_n-\beta| > \epsilon) \to 1$ for any $\epsilon > 0$. Hence it suffices to show $|\mu(s)| \geq C_1|s|^{\alpha+z}$ to prove the first claim, and to show $|\mu(s)| \leq C_2|s|^{\alpha-z}$ to prove the second claim. Now for $|s| < 1$, without loss of generality assuming $s > 0$, we have

$$\begin{aligned}|\mu(s)| &= 2 \iiint \operatorname{sgn}(y-y_{i-1})[G(v_i+s(y-y_{i-1}))-G(v_i)]dF(y)dG(v_i)dF(y_{i-1}) \\ &\geq 2 \int_{-1}^{1}\int_{-1}^{1}\int_{4s^{-1}}^{\infty} [G(2)-G(1)]dF(y)dG(v_i)dF(y_{i-1}) \\ &\geq C\, s^{\alpha}4^{-\alpha}L(4s^{-1}) \\ &\geq C_1(z)\ |s|^{\alpha+z},\end{aligned}$$

where L is a slowly varying function. From properties of slowly varying functions (see Ibragimov and Linnik (1971), App. 1), $s^{-z}L(4s^{-1}) \to \infty$ as $s \to 0$ and hence the final inequality holds for s in some neighborhood of zero.

Also

$$\begin{aligned}|\mu(s)| &\leq 4P[|Y|>s^{-1}] + 2 \int_{-s^{-1}}^{s^{-1}}\int_{-s^{-1}}^{s^{-1}} s|y-y_{i-1}|\ \sup_v g(v)\ dF(y)dF(y_{i-1}) \\ &\leq 4s^{\alpha}L_F(s^{-1}) + 2\ \sup_v g(v)\ s \int_{-s^{-1}}^{s^{-1}} \int_{-s^{-1}}^{s^{-1}} |y|+|y_{i-1}|dF(y)dF(y_{i-1}) \\ &\leq C_2(z)|s|^{\alpha-z},\end{aligned}$$

since $s^{z}L_F(s^{-1}) \to 0$ as $s \to 0$, and $\int_{-t}^{t}|y|dF(y) = o(t^{1-\alpha+z})$ for all $z > 0$ since

$$\int_{d_1}^{t} y\,dF(y) = -t(1-F(t)) + d_2 + \int_{d_1}^{t} (1-F(y)dy$$
$$= (C_2 + o(1))L(t)\ t^{1-\alpha} + d_2 + \int_{d_1}^{t} (C_2+o(1))\ L(y)y^{-\alpha}dy$$
$$= o(t^{1-\alpha+z}). \qquad \square$$

For the least squares estimate, we prove the following results concerning the convergence rate of the LSE in model (2.1) for errors in the domain of attraction of a stable law. The ideas are very similar to Hannan and Kanter (1977) where results are proved if a location parameter for the distribution of $\bar{V}_1$ is known.

Theorem 3.2a. Assume the model (2.1) holds, and $G_v \in \mathcal{D}(\alpha)$, $1 \le \alpha < 2$. Let β_n^* be the LSE given by

$$\beta_n^* - \beta = \frac{\sum_{i=2}^{n} Y_{i-1}V_i - (n-1)^{-1} \sum_{i=2}^{n} Y_{i-1} \sum_{i=2}^{n} V_i}{\sum_{i=2}^{n} Y_{i-1}^2 - (n-1)\ \ (\sum_{i=2}^{n} Y_{i-1})^2} .$$

Then $n^{1/\alpha-\delta}(\beta_n^* - \beta) \to 0$ a.s. for any $\delta > 0$. $\square$

Proof: Hannan and Kanter (1977) show $n^{-1/\alpha-\epsilon}\sum_{i=2}^{n} Y_{i-1}V_i \to 0$ a.s. and $n^{-2/\alpha+\epsilon}\sum_{i=2}^{n} Y_{i-1}^2 \to \infty$ a.s. for any $\epsilon > 0$, hence it suffices to show $n^{-1-1/\alpha-\epsilon}\sum_{i=2}^{n} Y_{i-1} \sum_{i=2}^{n} V_i \to 0$ a.s. for any ϵ 0 and $n^{-1-2/\alpha}(\sum_{i=2}^{n} Y_{i-1})^2 \to 0$ a.s. Both of these follow from the Theorem of Chatterji (1969), part of which is given as Lemma 3.3.

Lemma 3.3. Chatterji (1969). Let X_n, $n \geq 1$, and X be random variables such that $X \in L^p$, $0 < p < 2$, $p \neq 1$ and $P[|X_n| \geq x] \leq P[|X| \geq x]$ for $0 \leq x < \infty$. Then

$$n^{-1/p}\sum_{i=1}^{n} (X_i - m_i) \to 0 \text{ a.s.},$$

where $m_i = 0$ if $0 < p < 1$, and $m_i = E[X_i|X_{i-1},\ldots, X_1]$ if $1 \leq p < 2$. □

This lemma shows $n^{-1/\alpha-\delta}\sum_{i=2}^{n} V_i \to 0$ a.s. for any $\delta > 0$ by taking $p = a/(1+\delta\alpha) < \alpha$ since $V_1 \in L^{\alpha-\gamma}$ for all $\gamma > 0$. By taking $p = (1+\delta)^{-1} < 1$, $n^{-1-\delta}\sum_{i=2}^{n} Y_{i-1} \to 0$ a.s. for any $\delta > 0$ since $\{Y_n\}$ is stationary and $Y_1 \in L^{\alpha-\gamma}$ for all $\gamma > 0$. By taking $p = 2\alpha/(\alpha+2) < 1$, $n^{-1-2/\alpha}(\sum_{i=2}^{n} Y_{i-1})^2 \to 0$ a.s. □

We have been unable to show $n^{1/\alpha+\delta}|\beta_n^* - \beta| \overset{P}{\to} \infty$ to completely determine the rate of convergence of β_n^*. The rate of convergence is at least $n^{1/\alpha}$, if it exists.

If the errors are in the domain of attraction of a stable variable with tails heavier than the Cauchy then we can determine the weak rate of convergence.

<u>Theorem 3.2b.</u> Assume the model (2.1) holds, and $G_v \in \mathcal{D}(\alpha)$, $0 < \alpha < 1$. Let β_n^* be as in Theorem 3.2a. Then $n^{1-\delta}(\beta_n^* - \beta) \to 0$ a.s. and $n^{1+\delta}|\beta_n^* - \beta| \overset{P}{\to} \infty$ for any $\delta > 0$. □

<u>Proof</u>: For the first claim, as before $n^{-2/\alpha+\delta} \sum_{i=2}^{n} Y_{i-1}^2 \to \infty$ a.s. and $n^{-1-2/\alpha+\delta}(\sum_{i=2}^{n} Y_{i-1})^2 \to 0$ a.s. Now $1-2/\alpha-\delta < -1/\alpha-\delta$ so that $n^{1-2/\alpha-\delta} \sum_{i=2}^{n} Y_{i-1}V_i \to 0$ a.s. from Hannan and Kanter's (1977) results, and also $n^{-2/\alpha-\delta} \sum_{i=2}^{n} Y_{i-1} \sum_{i=2}^{n} V_i \to 0$ a.s. by Lemma 3.3. This proves the first claim.

We need only prove the second claim for δ in a neighborhood of zero. For such δ, $n^{-1-2/\alpha-\delta}(\sum_{i=2}^{n} Y_{i-1})^2 \to 0$ a.s. and $n^{1-2/\alpha+\delta} \sum_{i=2}^{n} Y_{i-1}V_i \to 0$ a.s. Also note that $Y_i^2 \in L^{\alpha/2-\gamma}$ for positive γ since $Y_i^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \beta^{j+k} V_{i-j} V_{i-k}$ and $V_r^2 \in L^{\alpha/2-\gamma}$, $V_r V_m \in L^{\alpha-\gamma}$, and

$|a+b|^\epsilon \le |a|^\epsilon + |b|^\epsilon$ for $0 < \epsilon < 1$. By Lemma 3.3 this gives $n^{-2/\alpha-\delta} \sum_{i=2}^{n} Y_{i-1}^2 \to 0$ a.s. It remains to show

$$n^{-2/\alpha+\delta} \left| \sum_{i=2}^{n} Y_{i-1} \sum_{i=2}^{n} V_i \right| \overset{P}{\to} \infty. \tag{3.1}$$

Now

$$\sum_{i=2}^{n} Y_{i-1} = \sum_{s=2}^{n} \left(\sum_{j=0}^{s-2} \beta^j V_{s-1-j} + \beta^{s-1} Y_0 \right)$$

$$= \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \beta^{s-r-1} V_r + (\beta-\beta^n)\,(1-\beta)^{-1} Y_0$$

$$= (1-\beta)^{-1} \sum_{r=1}^{n-1} V_r - (1-\beta)^{-1} \sum_{r=1}^{n-1} \beta^{n-r} V_r + (\beta-\beta^n)(1-\beta)^{-1} Y_0.$$

The second and third terms converge in distribution as $n \to \infty$ without normalization and hence

$$n^{-2/\alpha+\delta} \left(\left| \sum_{i=2}^{n} Y_{i-1} \sum_{i=2}^{n} V_i - |(1-\beta)^{-1}| \left(\sum_{i=2}^{n} V_i \right)^2 \right| \right) \xrightarrow{P} 0,$$

so to show (3.1), it suffices to show $n^{-1/\alpha+\delta} \left| \sum_{i=2}^{n} V_i \right| \xrightarrow{P} \infty$. This follows since $n^{-1/\alpha} L(n) \sum_{i=2}^{n} V_i$ converges in distribution, where $L(n)$ is a slowly varying function and $n^{\delta} L^{-1}(n) \to \infty$. □

4. Simulation results. The results of a small simulation study are given in Tables 4.1 and 4.2. Values were generated according to model (2.1) with five different distributions for V_1: standard normal, and from the Pareto-like family with distribution function

$$G_{\alpha,c}(y) = F_\alpha(x) = \begin{cases} (-x)^{-\alpha}\,[2(\alpha+1)]^{-1} & x < -1 \\ [\alpha(x+1)+1]\,[2(\alpha+1)]^{-1}, & -1 \le x < 1, \\ 1 - F_\alpha(x) & , \quad 1 \le x \end{cases} \qquad (4.1)$$

where $x = y/c$ for $\alpha = 2.0$, 1.5, 1.0, and 0.5. The scale constant c has the value one. The stationary distribution for Y was approximated by generating 60 values from the series prior to the values used.

In an autoregression the estimates are not unbiased and the bias accounts for a significant proportion of the error. The MPS estimate appears to be less biased than the LSE, especially as the tails get heavier. With $\alpha = 0.5$, the interquartile range did not contain the true value when the LSE was used for estimation. The variability of the estimates was similar to a regression model (see Pruitt (1987) for the regression model results). The MPS estimate appeared comparable to or better than the LSE has a better asymptotic convergence rate for heavy tails. The asymptotics seem to take effect at larger sample sizes for heavier tails, also for larger samples than in the regression case. At $\alpha = 1$, one can see the LSE improving compared to the MPS estimate for larger sample sizes. As with the regression, and even more clearly at the heavier tails used in this simulation, the LSE has a sampling distribution with much heavier tails than the MPS estimate. The MPS estimate compares favorably over the range of conditions examined.

To examine the sensitivity of the estimator to outliers a similar study was done using the additive outliers model of Denby and Martin (1979). First a time series $\{Y_i\}$ is generated according to model (2.1). Then let $Y_i^* = Y_i + W_i$ where the W_i are i.i.d., and W_i is equal to zero with probability $1-\epsilon$ and otherwise conditionally has distribution function $G_{\alpha,c}$ given by (4.1). Here we take $\epsilon = 0.05$, $c = 3.0$, and α the same as that used to generate the original process. The time series Y_i^* is then analyzed. This series has isolated observation outliers which are not perpetuated in the series. The results are similar to Table 4.1. The contamination seems to have more effect for heavier tailed distributions and also seems to have more effect on the least squares estimate, especially at intermediate values of α. The study is not large enough to draw any firm conclusions.

Appendix

<u>A.1 Proof of Lemma 2.2</u> Fix γ and t and let $s_m = m^{-\gamma}t$. $T_n(s_m)$ is a triangular array of random variables, and we wish to show the diagonal elements converge in distribution. We first show that for each fixed m, $T_n(s_m) \xrightarrow{d} T(s_m) \sim N(0,\sigma^2(s_m))$ as $n \to \infty$, then show $T(s_m) \xrightarrow{d} T \sim N(0,\sigma^2(0))$ as $m \to \infty$, and finally verify the condition in Theorem 4.2 of Billingsley(1968) to show the same convergence holds for the diagonal elements, $T_n(n^{-\gamma}t)$, of the array. The subscript m on s_m will be suppressed unless m varies.

To determine the asymptotic distribution of $U_n(\beta+s)$ for fixed s, recall $Z_i=(Y_{i-1}, V_i)^t$ and note $U_n(\beta+s)$ is a U-statistic on the process $\{Z_n\}$. To show $T_n(s_m) \xrightarrow{d} T(s_m)$, we will show $\{Z_n\}$ satisfies a weak dependence condition, and then use a Central Limit Theorem for U-statistics on weakly dependent processes to conclude the argument.

We first show that the process $\{Z_n\}$ satisfies the weak dependence property of absolute regularity. A process $\{Z_n\}$ is called *absolutely regular* if

$$\alpha_2(n) = E[\sup_{A\in F_n^{\infty}} \{|P(A|F^0_{-\infty})-P(A)|\}]$$

decreases montonically to zero as $n \to \infty$. Here $F_a^b = \sigma(Z_a,\ldots,Z_b)$ is the sigma field generated by the random variables $Z_a,\ldots,Z_b$. To show $\{Z_n\}$ is absolutely regular write

$$Z_i = \begin{pmatrix} Y_{i-1} \\ V_i \end{pmatrix} = \begin{pmatrix} \beta & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y_{i-2} \\ V_{i-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_i \\ V_i \end{pmatrix}$$
$$= FZ_{i-1} + Ge_i,$$

where $e_i = (V_i, V_i)^t$. Note that g(v) is the density of e_i along the $v_1 = v_2$ diagonal in the (v_1, v_2)-plane. Theorem 3.1 of Pham and Tran(1985) is now applicable and a special case of it is given as Lemma A1.1.

Lemma A1.1 Pham and Tran (1985). Suppose that the eigenvalues of F are of modulus strictly less than 1, $E[\|e_i\|^\delta] < \infty$ for some $\delta > 0$, and $\int|g(v)-g(v-\theta)|dv = O(|\theta|^\gamma)$ for some $\gamma > 0$ as $\theta \to 0$. Then $\alpha_2(n) \to 0$ at an exponentia l rate, that is, $\alpha_2(n) = O(\rho^n)$ for some $\rho < 1$. □

Using assumptions A1)-A3), Lemma A1.1 may be applied to the process$\{Z_n\}$ to show it is absolutely regular. Note in particular that

$\sum_{n=1}^{\infty} \alpha_2(n)^{\delta/(2+\delta)} < \infty$ for any $\delta > 0$ since $\alpha_2(n)$ decreases at an exponential rate.

To determine the convergence of $U_n(\beta+s)$ for fixed s, we now need a Central Limit Theorem for U-statitics on absolutely regular processes. The result we need has been proved by Yoshihara (1976), and again by Denker and Keller (1983). Here we use Theorem 1(c) of Denker and Keller which is given as Theorem A1.2.

Theorem A1.2 Denker and Keller (1983). Let

$$U_n = \binom{n-1}{2}^{-1} \sum_{2 \le i<j \le n} \sum h(Z_i, Z_j),$$

where $\{Z_n\}$ is a strictly stationary process which is absolutely regular with coefficients $\alpha_2(n)$ satisfying $\sum_{n=1}^{\infty} \alpha_2(n)^{\delta/(2+\delta)} < \infty$ for some $\delta > 0$,

and $h:\Re^2 \to \Re$ is a non-degenerate kernel, that is $E[h(z,Z)] \neq 0$ for some z. Let $h_1(Z_i) = E[h(Z^0, Z_i)|Z_i]$ where Z^0 is independent of $\{Z_n\}$ and has the same distribution as Z_1. Let $\mu = E[h(Z^0, Z_i)] = E[h_1(Z_i)]$. Assume

$$0 < \sigma^2 = 4\{Var(h_1(Z_2)) + 2 \sum_{j=3}^{\infty} Cov(h_1(Z_2), h_1(Z_j))\} < \infty, \quad (A1.1)$$

and

$$\sup_{2 \le i \le j} E[|h(Z_i, Z_j)|^{2+\delta}] < \infty. \quad (A1.2)$$

Then $n^{1/2}(U_n - \mu) \xrightarrow{d} N(0, \sigma^2)$. □

In Theorem A1.2, take $h(Z_i, Z_j) = h(s, Z_i, Z_j)$ of (2.2), and note that μ is $\mu(s)$ and σ^2 is $\sigma^2(s)$ of (2.3). To check that Theorem A1.2 applies to $U_n(\beta+s)$, we need to show that (A2.2) and (A1.2) hold. Clearly $|h| \le 1$ so (A1.2) holds. To show (A1.1), we use Lemma 7 of Denker and Keller(1983) which we state as Lemma (A1.3). In this lemma, $(\mathcal{X}, \mathcal{G})$ is a countably generated measurable space, $\mathcal{A}$ and $\mathcal{B}$ are sub-σ-algebras of $\mathcal{G}$, $\mathcal{A} \vee \mathcal{B}$ is the σ-field generated by $\mathcal{A} \cup \mathcal{B}$, and f_1 and f_2 are real valued functions.

Lemma A1.3. Denker and Keller (1983). Let f_1 be an $\mathcal{A}$-measurable function, f_2 be an $\mathcal{A} \vee \mathcal{B}$ measurable function, $\delta > 0$, and let P and Q be probability measures on G coinciding on $\mathcal{a}$. Let

$$\hat{d}\,(P,Q;\mathcal{B}|\mathcal{a}) = E[\sup_{B\in\mathcal{B}}|P(B|\mathcal{a})-Q(B|\mathcal{a})|].$$

Then

$$\left| E_P[f_1 f_2] - E_P(E_Q[f_1 f_2|\mathcal{a}]] \right| \le$$

$$4\,\hat{d}^{\delta/2+\delta}(P,Q;\mathcal{B}|\mathcal{a})\ E[|f_1|^{2+\delta}]^{1/2+\delta}\ \max(E_P[|f_2|^{2+\delta}],\ E_Q[|f_2|^{2+\delta}])^{1/2+\delta}.\square$$

It now follows that

$$\begin{aligned}\sigma^2(s) &= 4\{\mathrm{Var}(h_1(s,Z_2)) + 2\sum_{j=3}^{\infty}\mathrm{Cov}(h_1(s,Z_2),\ h_1(s,Z_j))\}\\ &\le 4\{\ 1 + 2\sum_{j=3}^{\infty} 4\alpha_2^{\delta/2+\delta}(j-2)\} \qquad\qquad (A1.3)\\ &< \infty,\end{aligned}$$

after observing $|h_1| \le 1$ and applying Lemma A1.3 with $\mathcal{a} = \mathcal{F}_{\infty}^{2}$, $\mathcal{B} = \mathcal{F}_{j}^{\infty}$, $f_1 = h_1(s,Z_2)$, $f_2 = h_1(s,Z_j)$, P the probability measure of the process $\{Z_n\}$, and Q the probability measure of fa process consisting of i.i.d. random vectors each with distribution Z_1.

We have now shown that Theorem A1.2 applies to $U_n(\beta+s)$, hence for any s,

$$n^{1/2}(U_n(\beta+s)-\mu(s)) = T_n(s) \xrightarrow{d} T(s)\sim N(0,\sigma^2(s)) \text{ as } n \to \infty.$$

Note $T(s_m) \xrightarrow{d} T\sim N(0,\sigma^2(0))$ as $m \to \infty$, since $\sigma^2(s)$ is a continuous function of s, which follows from (A1.3) and the fact that g is continuous.

By Theorem 4.2 of Billingsley (1968), to show $T_n(n^{-\gamma}t) \xrightarrow{d} N(0,\sigma^2(0))$ it now suffices to show

$$\lim_{m\to\infty} \limsup_{n\to\infty} P[|T_n(m^{-\gamma}t)-T_n(n^{-\gamma}t)\ | > \epsilon] = 0 \qquad (A1.4)$$

for any $\epsilon > 0$. Now

$$P[T_n(m^{-\gamma}t)-T_n(n^{-\gamma}t)|>\epsilon] \le P[S_n(m^{-\gamma}t)|>\epsilon/2] + P[S_n(n^{-\gamma}t)|>\epsilon/2],$$

where $S_n(z)= T_n(z)-T_n(0)$. Then $P(|S_n(m^{-\gamma}t)|\epsilon/2) \le 4\epsilon^{-2}E[S_n^2(m^{-\gamma}t)]$ and $P(S_n(n^{-\gamma}t)| > \epsilon/2) \le 4\epsilon^{-2}E[S_n^2(m^{-\gamma}t)]$. Now

$$E[S_n^2(m^{-\gamma}t)] = nE[\{\binom{n-1}{2}^{-1} \sum\sum_{2\le i<j\le n} k(s,Z_i,Z_j)\}^2] = nE[W_n^2(m^{-\gamma}t)], \qquad (A1.5)$$

where $k(s,Z_i,Z_j)= h(s,Z_iZ_j) - h(0,Z_i,Z_j) - \mu(s)$. Using Hoeffding's (1948) projection method, we can decompose the U-statistic W_n as

$$W_n(m^{-\gamma}t) = 2(n-1)^{-1} \sum_{i=2}^{n} k_1(s,Z_i) + R_n(s),$$

where $k_1(s,Z_i) = E[k(s,Z_i,Z^0)|Z_i]$. Now to bound $E[W_n^2(m^{-\gamma}t)]$, by the Cauchy-Schwarz inequality it suffices to bound the expected value of the square of each of the two terms in this decomposition. The first term is handled by similar techniques to those used to bound $\sigma^2(s)$, and Proposition 2 of Denker and Keller(1983), given here as Lemma A1.4, gives a bound for the remainder term.

Lemma A1.4 Denker and Keller (1983). Let $\{Z_n\}$ be absolutely regular with mixing coefficient $\alpha_2(n)$ satisfying $\alpha_2(n)^{\delta/2+\delta} = O(n^{-2+\epsilon})$ for some δ, $\epsilon > 0$. Let

$$U_n = \binom{n-1}{2}^{-1} \sum_{2 \le i < j \le n} \sum h(Z_i, Z_j),$$

$h_1(Z_i) = E[h(Z^0, Z_i) | Z_i]$, where Z^0 is independent of $\{Z_n\}$ and has the same distribution as Z_1, and $\mu = E[h(Z^0, Z_i)] = E[h_1(Z_i)]$. Then if

$$R_n = U_n - \mu - 2(n-1)^{-1} \sum_{i=2}^{n} (h_1(Z_i) - \mu),$$

there exists a constant Γ_ϵ such that for any kernel h

$$E[R_n^2] \le \Gamma_\epsilon^2 \, n^{-2+\epsilon} \{ \sup_{2 \le i \le j} E[|h(Z_i, Z_j)|^{2+\delta}] \}^{2/2+\delta}. \quad \square$$

This can be applied to R_n (s) since

$$\sup_s \sup_{2 \le i \le j} E[|k(s, Z_i, Z_j)|^{2+\delta}] < \infty, \qquad \text{(A1.6)}$$

and $\alpha_2(n)^{\delta/2+\delta} = O(n^{-2+\epsilon})$ for any $\epsilon > 0$ by recalling that $\alpha_2(n)$ decreases exponentially. Hence using the Cauchy-Schwarz inequality and returning to (A1.5),

$$E[S_n^2(m^{-\gamma} t)] \le 2n\{E[\{2(n-1)^{-1} \sum_{i=2}^{n} k_1(s, Z_i)\}^2] + E[R_n^2(s)]\}$$

$$\le C_1 E[k_1^2(s, Z_2)] + C_2 \sum_{j=3}^{\infty} E[|k_1(s, Z_2) k_1(s, Z_j)|] + C_3 n^{-1+\epsilon}$$

$= C(m,n)$.

Finally note

$$\lim_{m\to\infty} \lim_{n\to\infty} C(m,n) = 0 \text{ and } \lim C(n,n) = 0$$

by the Dominated Convergence Theorem since, as with (A1.3), $C(m,n)$ is finite, and $\lim_{s\to 0} k_1(s,z) = 0$ for all z. Hence the limit in (A1.4) is zero, and $T_n(n^{-1/2}t) \xrightarrow{d} N(0,\sigma^2(0))$. This finishes the proof of Lemma 2.2.□

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TABLE 4.1

Simulation for estimation of β from $Y_i = \beta Y_{i-1} + V_i$ for a sample of size n, V_1 having a stable law with index α, and $\beta = 0.5$. The observed series is uncontaminated.

		least squares β_n^*				median of pairwise slopes $\hat{\beta}_n$			
α	n	bias	s.d.	median bias	iqr	bias	s.d.	median bias	iqr
normal	20	-.131	.218	-.111	.300	-.129	.235	-.105	.307
	70	-.036	.111	-.032	.163	-.036	.119	-.030	.172
	250	-.010	.053	-.004	.078	-.011	.054	-.004	.075
	800	-.003	.032	-.002	.045				
2.0	20	-.128	.214	-.101	.281	-.113	.217	-.078	.284
	70	-.032	.098	-.024	.122	-.029	.097	-.018	.120
	800	-.001	.030	-.001	.032				
1.5	20	-.112	.201	-.093	.245	-.095	.194	-.062	.235
	70	-.032	.099	-.023	.095	-.023	.081	-.014	.095
	800	-.001	.031	-.000	.024				
1.0	20	-.092	.611	-.084	.181	-.067	.150	-.030	.140
	70	-.029	.075	-.024	.062	-.015	.048	-.006	.049
	250	-.007	.032	-.006	.020	-.003	.017	-.002	.018
	800	-.001	.027	-.002	.008				
0.5	20	3.53	86.9	-.089	.109	-.018	.067	-.010	.019
	70	-.026	.077	-.022	.024	-.002	.009	-.000	.002
	800	-.001	.021	-.002	.002				

TABLE 4.2

Simulation for estimation of β from $Y_i = \beta Y_{i-1} + V_i$ for a sample of size n, V_i having a stable law with index α, and $\beta = 0.5$. The observed series is contaminated(see text).

		least squares				median of pairwise slopes			
		β_n^*				$\hat{\beta}_n$			
α	n	bias	s.d.	median bias	iqr	bias	s.d.	median bias	iqr
normal	20	-.149	.221	-.131	.311	-.138	.241	-.115	.339
	70	-.055	.110	-.051	.156	-.052	.115	-.048	.157
	250	-.025	.060	-.013	.077	-.024	.062	-.020	.079
	800	-.020	.033	-.020	.048				
2.0	20	-.182	.239	-.160	.298	-.153	.219	-.121	.302
	70	-.134	.140	-.112	.199	-.078	.107	-.063	.150
	800	-.105	.078	-.095	.090				
1.5	20	-.167	.265	-.139	.268	-.125	.195	-.095	.248
	70	-.118	.142	-.088	.184	-.062	.091	-.046	.124
	800	-.084	.094	-.060	.103				
1.0	20	-.232	2.44	-.115	.234	-.093	.167	-.047	.163
	70	-.092	.143	-.047	.113	-.034	.058	-.021	.063
	250	-.065	.107	-.024	.062	-.017	.024	-.010	.030
	800	-.055	.102	-.014	.057				
0.5	20	.880	32.9	-.092	.117	-.032	.087	-.003	.031
	70	-.072	.188	-.023	.029	-.004	.010	-.000	.004
	800	-.031	.100	-.003	.004				