Well-posedness and Blow-up Solutions for an Integrable Nonlinearly Dispersive Model Wave Equation

Yi A. Li
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
yili@math.umn.edu

Peter J. Olver†
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
olver@ima.umn.edu
http://www.math.umn.edu/~olver

Abstract. We establish local well-posedness in the Sobolev space H^s with any $s > \frac{3}{2}$ for an integrable nonlinearly dispersive wave equation arising as a model for shallow water waves known as the Camassa–Holm equation. However, unlike the more familiar Korteweg-deVries model, we demonstrate conditions on the initial data that lead to finite time blow-up of certain solutions.

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1. Introduction.

Scott–Russell's observation of solitary water waves, [**55**], which are not predicted by purely linear models, served to motivate the development of nonlinear partial differential equations for the modeling of wave phenomena in fluids, plasmas, elastic bodies, etc. In the case of the free boundary problem for incompressible, irrational water waves, the fundamental perturbation expansion was developed by Boussinesq, [**10, 11**]; see Whitham, [**59**], for a modern presentation. Expanding to the first order in the small parameter representing the ratio of wave amplitude to undisturbed fluid depth and the square of the ratio of fluid depth to wave length, Boussinesq derived two models for the unidirectional propagation of one-dimensional waves. The first is now known as the Boussinesq equation, [**9**; p. 258],

$$u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, \tag{1.1}$$

which, although it admits waves traveling in both directions, is a valid water wave model only for waves moving to the right. (We will use the familiar form for the models, where a suitable rescaling has eliminated the physical parameters.) Less well known is the fact that in the 1870's Boussinesq also wrote down the celebrated Korteweg-de Vries (KdV) equation

$$u_t + u_x + u u_x + u_{xxx} = 0, \tag{1.2}$$

named after its rediscovery in the seminal 1895 paper of Korteweg and de Vries, [**33**]. The equation appears in [**10**; eq. (30), p. 77], [**11**; eqs. (283, 291)], and the subsequent discussion also includes the derivation of the first three conservation laws of the KdV equation and its one-soliton and periodic traveling wave solutions. An alternative model, having better analytical properties, is the BBM or regularized long wave equation,

$$u_t + u_x + u u_x - u_{xxt} = 0, \tag{1.3}$$

which was originally proposed by Benjamin, Bona and Mahony, [**5**].

The remarkable discovery of the soliton by Gardner, Green, Kruskal and Miura, [**27**], led to the realization that many well-known model wave equations are, in fact, integrable. Hallmarks of integrability include the existence of infinitely many symmetries and conservation laws, cleanly interacting soliton solutions, linearization of the equation by the method of inverse scattering, and many other remarkable properties. Both the Boussinesq and Korteweg-de Vries equations are integrable in this sense; the BBM is not since it only admits three conservation laws, [**41, 20**]. A particularly powerful method for proving the integrability of a nonlinear evolution equation was the discovery by Magri, [**38**], that all soliton systems admit two distinct, but compatible Hamiltonian structures, making them into a "biHamiltonian system". As discussed in [**38, 45**], the two Hamiltonian operators are combined to form a recursion operator that recursively constructs the infinite hierarchies of symmetries and conservation laws.

The reason why integrable models arise so often in physical systems remains a mystery. In an attempt to understand this phenomenon, the second author conducted a careful investigation of how the Hamiltonian structure and conservation laws enter into the Boussinesq perturbation expansion, [**42, 43**]. As shown first by Zakharov, [**60**], the free boundary problem for water waves admits a Hamiltonian structure. (This was later applied in [**6**] to determine a complete system of symmetries and conservation laws.) Remarkably, neither of the Hamiltonian structures of the Korteweg-deVries model comes directly from the water wave Hamiltonian structure; since the perturbation expansion is not canonical, the Hamiltonian operator for the water wave problem expands to a certain linear combination of the two KdV Hamiltonian operators. This fact — that the non-canonical perturbation expansion of a physical system reduces to a Hamiltonian model — provides a possible explanation of the previously mentioned observation.

All of the classical models of nonlinear wave phenomena are only valid in the weakly nonlinear regime. However, many of the most interesting physical phenomena, such as wave breaking, waves of maximal height, etc., [**57, 4**], require a transition to full nonlinearity. In this direction, motivated in part by the Hamiltonian perturbation theory of [**42, 43**], Camassa and Holm, [**12, 13**], proposed the following model equation for water waves:

$$u_t + \kappa u_x - u_{xxt} = -3uu_x + uu_{xxx} + 2u_x u_{xx}. \tag{1.4}$$

Note that the linear terms on the left hand side mirror the linear terms in the BBM model. However, the term uu_{xxx} makes (1.4) nonlinear in its highest order derivatives, and so it lies in the class of "nonlinearly dispersive" wave models. As shown in [**12**], the Camassa–Holm equation is biHamiltonian, and hence admits an infinite hierarchy of symmetries and conservation laws. Indeed, the equation (1.4) and its biHamiltonian structure were written down earlier (albeit with an error in one of the coefficients) by Fuchssteiner, [**24**; (5.3)]. The basic method of "Hamiltonian duality", introduced by Fokas and Fuchssteiner, [**26**], and extensively developed in [**46, 22, 25**], is used to produce nonlinearly dispersive integrable dual biHamiltonian systems for most of the classical soliton models. However, in contrast to the KdV and Boussinesq models, the Magri recursion scheme now leads to nonlocal higher order symmetries and conservation laws. An inverse scattering problem for (1.4) has been proposed, [**12**]; see also [**25, 53**], and [**21**] for an associated Riemann–Hilbert problem. However, the full details of the inverse scattering linearization of (1.4) remains undeveloped. Schiff, [**54**], describes Bäcklund transformations based on a loop group approach to the equation. The periodic problem for (1.4) has been extensively analyzed by Constantin and McKean, [**18**].

As emphasized by Rosenau, [**47, 48, 49, 50**], the passage to the fully nonlinearly dispersive regime leads to the appearance of new types of solutions not predicted by the classical weakly nonlinearly theory. The solitons for the KdV and Boussinesq model are nice analytic solutions; in contrast, the Camassa-Holm model admits non-analytic waves with corners — peakons — as solutions; changing the sign of the u_{xxt} term leads to compacton solutions. General multi-peakon solutions were constructed in [**1, 2**]. Our earlier work, [**35, 36**], investigated in what sense these compactons and peakons are weak solutions, and how they appear as the limits of analytic solitary wave solutions. More recent work, [**49, 34**], has uncovered further inhabitants in a vast menagerie of different species of non-analytic solutions, including cuspons, tipons, ramptons, mesaons, etc., admitted by nonlinearly dispersive models, both integrable and nonintegrable.

These earlier studies have helped us understand the distinct differences between the two systems (1.4) and (1.2). The present paper continues the analytical study of the

Camassa–Holm equation, focusing on well-posedness and singularities of solutions. For the KdV equation (1.2), all solitary wave solutions are real analytic functions, having unique analytic extensions to the complex plane except countably many poles. The equation itself has a smoothing effect on the initial data, and solutions of the Cauchy problem of (1.2) gain in regularity due to the effect of linear dispersion. On the other hand, the smooth solitary wave solutions to the Camassa–Holm equation (1.4) have non-unique analytic extensions to the complex plane with countably many branch points and branch lines. Moreover, it admits non-analytic solitary wave solutions having singularities on the real line that propagate in time. Moreover, the cuspon solutions of (1.4) have branch points singularities of order three, which implies that its derivative is square integrable but not cubic integrable. These facts strongly indicate that the nonlinear dispersion term uu_{xxx} has diminished the smoothing effect of the linear dispersion terms u_{xxt} and u_{xxx}.

In this paper, we establish local well-posedness in the Sobolev space H^s with any $s > 3/2$ for the following model wave equation:

$$u_t - \nu u_{xxt} = \alpha u_x + \beta u_{xxx} + 3\gamma u u_x - \gamma\nu(u u_{xxx} + 2u_x u_{xx}), \tag{1.5}$$

where $\alpha, \beta, \gamma, \nu$ are constants, and we take $\nu > 0$. Although (1.5) looks more general, the transformation

$$u(x,t) \longmapsto -\frac{1}{\gamma}\, u\left(\frac{x - \beta\, t/\nu}{\sqrt{\nu}}, \frac{t}{\sqrt{\nu}}\right)$$

will simplify (1.5) to the Camassa–Holm equation (1.4). The basic technique to be used is to regularized this equation and obtain a solution of the equation as the limit of solutions to the regularized equations. This method was developed for the Korteweg-deVries equation by several authors, including Bona and Smith, [**8**], Dushane [**19**], Masayoshi and Mukasa, [**39**], and Saut and Temam [**52, 56**]. In Section 3, we derive *a priori* estimates for solutions to the regularized equation, which is used in Section 4 to prove the local well-posedness for (1.5) in the Sobolev space H^s for any $s > 3/2$. We show that when $s > 3/2$, a necessary and sufficient condition for a global solution u to exist in the space H^s is that the L^∞-norm of its derivative u_x remains bounded. In Section 5, we study existence of solutions to the Cauchy problem of (1.5) in H^s for $1 < s \le 3/2$, which includes the nonsmooth weak peakon solutions. If $1 < s \le 3/2$, then there is still a local solution in H^s corresponding to the Cauchy data u_0, provided $u_{0x} \in H^{s-1}$ is essentially bounded. In the last section, we shall discuss the conditions under which solutions of (1.5) blow up in H^s norm in finite time. These results are related to, but different from those of Constantin and Escher [**15, 16, 17**], who also considered the blow up of solutions to the Camassa–Holm equation. They showed that its solutions $u(x,t)$ whose initial data $u(x,0) \in H^3$ is odd has $u_x(0,t)$ become infinite in finite time; these solutions have apparently developed singularities at $x = 0$.

The fact that the solutions of the integrable equation (1.5) can develop singularities in finite time is perhaps surprising, when compared with the familiar Korteweg-deVries theory. One explanation is that the KdV equation has infinitely many local conserved quantities which can be used to demonstrate the boundedness of the Sobolev s-norm of its solutions independent of time, and thus are important quantities to show global well-posedness of the equation. Thus far, only three local conserved quantities of the equation

(1.4) have been found. One shows that the H^1-norm of solutions remains constant, i.e., $\|u\|_{H^1} = \text{const}$, and the other two are

$$\int_{\mathbb{R}} u\,dx = \text{const.} \quad \text{and} \quad \int_{\mathbb{R}} (u^3 + \kappa u^2 + u u_x^2)\,dx = \text{const.}$$

Although our blow-up results indicates that (1.5) does not have the required local conserved quantities, this is not the complete story. Other examples of classical integrable systems having infinite hierarchies of local conservation laws do have solutions which develop singularities in finite time. In [**51**], it was shown that the completely integrable Boussinesq equation (1.1) does have solutions that blow up when their initial data is not small. We should also mention the work of Nutku and the second author, [**40, 44**], that produced multi-Hamiltonian structures for a wide variety of nonlinearly hyperbolic systems, including the equations of gas dynamics, leading to several infinite hierarchies of symmetries and conservation laws despite the fact that their solutions develop shocks in finite time.

The precise mode of blow up and singularity formation for the Camassa–Holm equation remains unclear, and awaits a detailed numerical investigation, which we hope to report on in a later paper. We include some preliminary numerical solutions obtained using a pseudo-spectral code at the end of the paper, but defer drawing definitive conclusions until further investigation has been completed. The complete solution of the equation by inverse scattering could help shed further light on the singularity formation. Extensions of these results to other classes of nonlinearly dispersive equations, both integrable and nonintegrable, are under investigation.

2. Notation.

We begin by summarizing our basic notation. The space of all infinitely differentiable functions $\phi(x,t)$ with compact support in $\mathbb{R} \times [0,\infty)$, is denoted by C_c^∞. Let p be any constant with $1 \le p < \infty$ and denote $L^p = L^p(\mathbb{R})$ to be the space of all measurable functions f such that $\|f\|_{L^p}^p = \int_{\mathbb{R}} |f(x)|^p\,dx < \infty$. The space $L^\infty = L^\infty(\mathbb{R})$ consists of all essentially bounded, Lebesgue measurable functions f with the standard norm

$$\|f\|_{L^\infty} = \inf_{m(e)=0} \sup\nolimits_{x\in\mathbb{R}\setminus e} |f(x)|.$$

For any real number s, we let $H^s = H^s(\mathbb{R})$ denote the Sobolev space consisting of all tempered distributions f such that

$$\|f\|_{H^s} = \left(\int_{-\infty}^{\infty} (1 + |\zeta|^2)^s |\widehat{f}(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} < \infty.$$

For any function $u = u(x,t) : \mathbb{R} \times [0,T) \to \mathbb{R}$ of two variables with $T > 0$, denote its Fourier transform, L^p-norm and Sobolev norm with respect to x by $\widehat{u} = \widehat{u}(\xi,t)$, $\|u\|_{L^p} = \|u(\cdot,t)\|_{L^p}$ and $\|u\|_{H^s} = \|u(\cdot,t)\|_{H^s}$, respectively.

The integral operator $\Lambda = (I - \partial_x^2)^{1/2}$ will play a key role. For later estimation of Sobolev norms of solutions, we will require a few basic inequalities.

Lemma 2.1. *For any $\xi, \eta \in \mathbb{R}$, there exists a constant c such that the inequalities*

$$(1+\xi^2)^q \le c\Big[(1+(\xi-\eta)^2)^q + (1+\eta^2)^q\Big], \qquad q>0 \tag{2.1}$$

and

$$|(1+\xi^2)^{q/2} - (1+\eta^2)^{q/2}| \le \begin{cases} c|\xi-\eta|\big((1+\xi^2)^{\frac{q-1}{2}} + (1+\eta^2)^{\frac{q-1}{2}}\big), & q>1, \\ c|\xi-\eta|(1+\eta^2)^{\frac{q-1}{2}}, & 0<q\le 1, \end{cases} \tag{2.2}$$

hold.

Proof: The first inequality is elementary. The inequality (2.2) follows directly from the mean value theorem when $q>1$ and the estimate

$$\begin{aligned} |(1+\xi^2)^{q/2} - (1+\eta^2)^{q/2}| &= \frac{q|\xi^2-\eta^2|}{2}\int_0^1 \frac{d\theta}{\big((1-\theta)(1+\eta^2)+\theta(1+\xi^2)\big)^{1-\frac{q}{2}}} \\ &\le \frac{q|\xi-\eta|}{2}\left(\frac{|\xi|}{(1+\eta^2)^{\frac{1-q}{2}}(1+\xi^2)^{\frac{1}{2}}}\int_0^1 \frac{d\theta}{(1-\theta)^{\frac{1-q}{2}}\theta^{\frac{1}{2}}} + \frac{|\eta|}{(1+\eta^2)^{1-\frac{q}{2}}}\int_0^1\frac{d\theta}{(1-\theta)^{1-\frac{q}{2}}}\right) \\ &\le \frac{q|\xi-\eta|}{2(1+\eta^2)^{\frac{1-q}{2}}}\left(\int_0^1 \frac{d\theta}{(1-\theta)^{\frac{1-q}{2}}\theta^{\frac{1}{2}}} + \int_0^1 \frac{d\theta}{(1-\theta)^{1-\frac{q}{2}}}\right) \end{aligned}$$

if $0<q\le 1$. *Q.E.D.*

Lemma 2.2. *Given $q\ge 0$, let $u=u(x)\in H^q$ be any function such that $\|u_x\|_{L^\infty} < \infty\}$. Then there is a constant c_q depending only on q such that the following inequalities hold:*

$$\left|\int_{\mathbb{R}} \Lambda^q u\, \Lambda^q(uu_x)dx\right| \le c_q\|u_x\|_{L^\infty}\|u\|^2_{H^q}, \tag{2.3}$$

$$\left|\int_{\mathbb{R}} \Lambda^q u\, \Lambda^q(u^2)\, dx\right| \le c_q\|u\|_{L^\infty}\|u\|^2_{H^q}. \tag{2.4}$$

Moreover, if u and f are functions in $H^{q+1}\cap\{\|u_x\|_{L^\infty}<\infty\}$, then

$$\left|\int_{\mathbb{R}} \Lambda^q u\, \Lambda^q(uf)_x dx\right| \le \begin{cases} c_q\|f\|_{H^{q+1}}\|u\|^2_{H^q}, & q\in(1/2,1], \\ c_q(\|f\|_{H^{q+1}}\|u\|_{L^\infty}\|u\|_{H^q} + \\ \quad +\|f\|_{H^q}\|u_x\|_{L^\infty}\|u\|_{H^q} + \|f_x\|_{L^\infty}\|u\|^2_{H^q}), & q\in(0,\infty), \end{cases} \tag{2.5}$$

Proof: Inequalities (2.3) and (2.4) are direct consequences of Lemmas X1 and X4 in [**30**]. For any fixed $q\in(1/2,1]$, one may rewrite the integral

$$\int_{\mathbb{R}} \Lambda^q u\, \Lambda^q(uf)_x dx = \int_{\mathbb{R}} \big(\Lambda^q u\, \Lambda^q(uf_x) + f\Lambda^q u\, \Lambda^q u_x + \Lambda^q u\,(\Lambda^q(fu_x) - f\Lambda^q u_x)\big)\, dx. \tag{2.6}$$

We then use the Schwarz and Young inequalities to obtain

$$\begin{aligned}\left|\int_{\mathbb{R}} \Lambda^q u\, \Lambda^q(u f_x) dx\right| &\le \|u\|_{H^q} \left(\int_{\mathbb{R}} (1+\xi^2)^q \left(\int_{\mathbb{R}} \widehat{u}(\xi-\eta)\widehat{f}_x(\eta)d\eta\right)^2 d\xi\right)^{1/2} \\ &\le \|u\|_{H^q}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}[(1+(\xi-\eta)^2)^{q/2}+(1+\eta^2)^{q/2}]|\widehat{u}(\xi-\eta)\widehat{f}_x(\eta)|d\eta\right)^2 d\xi\right)^{1/2} \\ &\le \|u\|_{H^q}^2 \|\widehat{f}_x\|_{L^1} + \|u\|_{H^q}\|\widehat{u}\|_{L^1}\|f\|_{H^{q+1}} \le c\|f\|_{H^{q+1}}\|u\|_{H^q}^2\end{aligned}$$

for some constant $c > 0$. On the other hand, one may estimate the following integral using integration by parts

$$\left|\int_{\mathbb{R}} f\Lambda^q u\,\Lambda^q u_x dx\right| = \frac{1}{2}\left|\int_{\mathbb{R}} f_x(\Lambda^q u)^2 dx\right| \le \frac{1}{2}\|f_x\|_{L^\infty}\|u\|_{H^q}^2.$$

We evaluate the next integral using the Schwarz inequality and (2.2) as follows:

$$\begin{aligned}&\left|\int_{\mathbb{R}} \Lambda^q u\,(\Lambda^q(f u_x) - f\Lambda^q u_x) dx\right| \\ &\quad\le \|u\|_{H^q}\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\left((1+\xi^2)^{q/2}-(1+\eta^2)^{q/2}\right)\widehat{f}(\xi-\eta)\widehat{u_x}(\eta)d\eta\right|^2 d\xi\right)^{1/2} \\ &\quad\le \alpha\|u\|_{H^q}\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}}|(\xi-\eta)\widehat{f}(\xi-\eta)|\frac{|\eta\widehat{u}(\eta)|}{(1+\eta^2)^{\frac{1-q}{2}}}d\eta\right|^2 d\xi\right)^{1/2} \\ &\quad\le \alpha\|\widehat{f}_x\|_{L^\infty}\|u\|_{H^q}^2 \le c_q\|f\|_{H^{q+1}}\|u\|_{H^q}^2.\end{aligned}$$

Applying the above three inequalities to the integral (2.6) yields the first part of (2.5). The second part of (2.5) also follows from Lemmas X1 and X4 in [**30**]. *Q.E.D.*

3. A Priori Estimates.

In this section, we look at the Cauchy problem for a regularized version of the Camassa–Holm equation (1.4). Consider the initial value problem

$$\begin{aligned} u_t - u_{xxt} + \epsilon u_{xxxxt} = \alpha u_x + \beta u u_x + u u_{xxx} + 2u_x u_{xx}, \quad t>0,\ x\in\mathbb{R}, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}), \quad s \ge 1.\end{aligned} \tag{3.1}$$

Here α, β, ϵ are constant, and $0 < \epsilon < 1/4$. We begin by inverting the linear differential operator on the left hand side.

Lemma 3.1. *For any $0 < \epsilon < 1/4$ and any s, the integral operator*

$$\mathcal{D} = (I - \partial_x^2 + \epsilon\partial_x^4)^{-1} \colon H^s \longrightarrow H^{s+4} \tag{3.2}$$

defines a bounded linear operator on the indicated Sobolev spaces. Moreover,

$$\mathcal{D}(f) = \big(G_\epsilon * f\big)(x) = \int_{\mathbb{R}} G_\epsilon(x-y)f(y)dy, \qquad f\in H^s,$$

can be expressed as a convolution with respect to

$$G_\epsilon(x) = \frac{1}{2\sqrt{1-4\epsilon}} \left(\frac{\sqrt{1+\sqrt{1-4\epsilon}}}{\sqrt{2}} e^{-\sqrt{\frac{2}{1+\sqrt{1-4\epsilon}}}|x|} - \frac{\sqrt{2\epsilon}}{\sqrt{1+\sqrt{1-4\epsilon}}} e^{-\frac{\sqrt{1+\sqrt{1-4\epsilon}}}{\sqrt{2\epsilon}}|x|} \right).$$

To show the existence of a solution to the problem (3.1), we apply the operator (3.2) to both sides of the equation (3.1) and then integrate the resulting equation with respect to time t. This leads to the following equation

$$u(x,t) = u_0(x) + \frac{1}{2}\int_0^t \mathcal{D}\Big[2\alpha u_x + \beta\partial_x(u^2) + \partial_x^3(u^2) - \partial_x(u_x^2)\Big](x,\tau)d\tau. \tag{3.3}$$

A standard application of the contraction mapping theorem leads to the following existence result.

Theorem 3.2. *For each initial data $u_0 \in H^s$ with $s \geq 1$, there is a $T > 0$ depending only on the norm of u_0 in H^s such that there corresponds a unique solution $u(x,t) \in C([0,T); H^s)$ of the equation (3.1) in the sense of distribution. If $s \geq 2$, the solution $u \in C^\infty\big([0,\infty); H^s\big)$ exists for all time. In particular, when $s \geq 4$, the corresponding solution is a classical globally defined solution of* (3.1).

The global existence result follows from using the conservation law

$$\int_{\mathbb{R}} (u^2 + u_x^2 + \epsilon u_{xx}^2)dx = \int_{\mathbb{R}} (u_0^2 + u_{0x}^2 + \epsilon u_{0xx}^2)dx, \tag{3.4}$$

admitted by (3.1) in its integral form (3.3).

Now we study norms of solutions of (3.1) using energy estimates.

Theorem 3.3. *Suppose that for some $s \geq 4$, the function $u(x,t)$ is a solution of the equation* (3.1) *corresponding to the initial data $u_0 \in H^s$. Then the following inequalities hold*:

$$\|u\|_{H^1}^2 \leq \int_{\mathbb{R}} (u^2 + u_x^2 + \epsilon u_{xx}^2)\, dx = \int_{\mathbb{R}} (u_0^2 + u_{0x}^2 + \epsilon u_{0xx}^2)\, dx. \tag{3.5}$$

For any real number $q \in (0, s-1]$, there exists a constant c depending only on q, such that

$$\int_{\mathbb{R}} (\Lambda^{q+1}u)^2\, dx \leq \int_{\mathbb{R}} \big((\Lambda^{q+1}u_0)^2 + \epsilon(\Lambda^q u_{0xx})^2\big)\, dx + c\int_0^t \|u_x\|_{L^\infty} \int_{\mathbb{R}} (\Lambda^{q+1}u)^2 dx\, d\tau\,. \tag{3.6}$$

For any $q \in (1/2, s-1]$ and any $r \in (1/2, q]$, there is a constant c depending only on r and q such that

$$\begin{aligned} \int_{\mathbb{R}} (\Lambda^{q+1}u)^2 dx \leq & \int_{\mathbb{R}} \big((\Lambda^{q+1}u_0)^2 + \epsilon(\Lambda^q u_{0xx})^2\big)\, dx + \\ & + c\int_0^t \left(\int_{\mathbb{R}} (\Lambda^{r+1}u)^2 dx\right)^{1/2} \left(\int_{\mathbb{R}} (\Lambda^{q+1}u)^2 dx\right) d\tau. \end{aligned} \tag{3.7}$$

For any $q \in [0, s-1]$, there is a constant c such that

$$(1-2\epsilon)\|u_t\|_{H^q} \le c(\|u\|_{H^1}+1)\|u\|_{H^{q+1}}. \tag{3.8}$$

Proof: Multiplying both sides of equation (3.1) by u and integrating with respect to x leads to the equation

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(u^2+u_x^2+\epsilon u_{xx}^2)dx = 0$$

which implies the inequality (3.5). For any $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ to both sides of the equation (3.1) and integrating with respect to x again, one obtains the equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\big((\Lambda^q u)^2+(\Lambda^q u_x)^2+\epsilon(\Lambda^q u_{xx})^2\big)dx \\
&\quad = \beta\int_{\mathbb{R}}\Lambda^q u\,\Lambda^q(uu_x)dx+\frac{1}{2}\int_{\mathbb{R}}\big((\Lambda^{2q}u)\,(u^2)_{xxx}+\Lambda^q u_x\,\Lambda^q(u_x^2)\big)dx \\
&\quad = \int_{\mathbb{R}}\left[(\beta+1)\Lambda^q u\,\Lambda^q(uu_x)-\Lambda^{q+1}u\,\Lambda^{q+1}(uu_x)+\frac{1}{2}\Lambda^q u_x\,\Lambda^q(u_x^2)\right]dx,
\end{aligned} \tag{3.9}$$

using integration by parts. It follows from the inequalities (2.3) and (2.4) that there is a constant c_q such that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left[(\Lambda^{q+1}u)^2+\epsilon(\Lambda^q u_{xx})^2\right]dx \le c_q\|u_x\|_{L^\infty}\|u\|_{H^{q+1}}^2.$$

Integrating with respect to t on both sides of the above inequality leads to the inequality (3.6). Applying the inequality $\|u_x\|_{L^\infty} \le c_r\|u\|_{H^{r+1}}$, for $r > 1/2$, to the right-hand side of (3.6) yields the estimate

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left[(\Lambda^{q+1}u)^2+\epsilon(\Lambda^q u_{xx})^2\right]dx \le c\,\|u\|_{H^{r+1}}\|u\|_{H^{q+1}}^2$$

for some constant c. Integration with respect to t results in the inequality (3.7). To estimate the norm of u_t, one may apply the operator $(I-\partial_x^2)^{-1}$ to both sides of the equation in (3.1) to obtain the equation

$$(1-\epsilon)u_t-\epsilon u_{xxt}+uu_x = (I-\partial_x^2)^{-1}\left[-\epsilon u_t+\partial_x(\alpha u+\frac{1+\beta}{2}u^2-\frac{u_x^2}{2})\right]. \tag{3.10}$$

Then applying $\Lambda^q u_t\,\Lambda^q$ to both sides of (3.10) for some $q \in [0, s-1]$, one obtains the equation

$$\begin{aligned}
&(1-\epsilon)\int_{\mathbb{R}}(\Lambda^q u_t)^2dx+\epsilon\int_{\mathbb{R}}(\Lambda^q u_{xt})^2+\int_{\mathbb{R}}\Lambda^q u_t\Lambda^q(uu_x)dx \\
&\quad = \int_{\mathbb{R}}\Lambda^q u_t(I-\partial_x^2)^{-1}\Lambda^q\big(-\epsilon u_t+\partial_x(\alpha u+\frac{\beta+1}{2}u^2-\frac{u_x^2}{2})\big)dx.
\end{aligned} \tag{3.11}$$

Since

$$\left|\int_{\mathbb{R}}\Lambda^q u_t\Lambda^q(uu_x)dx\right| \le \|\Lambda^q u_t\|_{L^2}\left(\int_{\mathbb{R}}(1+\xi^2)^{q+1}d\xi\left(\int_{\mathbb{R}}\widehat{u}(\xi-\eta)\widehat{u}(\eta)d\eta\right)^2\right)^{1/2},$$

it follows from Young's inequality and (2.1) that the estimate

$$\int_{\mathbb{R}} \Lambda^q u_t \Lambda^q (u u_x) dx \le \sqrt{2c}\, \|u_t\|_{H^q} \|\widehat{u}\|_{L^1} \|u\|_{H^{q+1}} \le c_1 \|u_t\|_{H^q} \|u\|_{H^1} \|u\|_{H^{q+1}}$$

holds for some constant c_1. On the other hand, the inequalities

$$\left| \int_{\mathbb{R}} \Lambda^q u_t (I - \partial_x^2)^{-1} \Lambda^q \big(-\epsilon u_t + \alpha u_x \big) dx \right| \le \epsilon \|u_t\|_{H^q}^2 + |\alpha| \|u_t\|_{H^q} \|u\|_{H^q}$$

and

$$\begin{aligned} &\left| \int_{\mathbb{R}} \Lambda^q u_t (I - \partial_x^2)^{-1} \Lambda^q \partial_x \Big(\frac{\beta+1}{2} u^2 - \frac{u_x^2}{2} \Big) dx \right| \\ &\quad \le \|u_t\|_{H^q} \left(\int_{\mathbb{R}} (1+\xi^2)^{q-1} d\xi \left(\int_{\mathbb{R}} \Big(\frac{\beta+1}{2} \widehat{u}(\xi-\eta)\widehat{u}(\eta) - \frac{1}{2} \widehat{u_x}(\xi-\eta)\widehat{u_x}(\eta) \Big) d\eta \right)^2 \right)^{1/2} \\ &\quad \le \|u_t\|_{H^q} \left(\int_{\mathbb{R}} \frac{c\,(\|u\|_{L^2}^2 \|u\|_{H^q}^2 + \|u_x\|_{L^2}^2 \|u_x\|_{H^q}^2)\, d\xi}{1+\xi^2} \right)^{1/2} \le c_2 \|u_t\|_{H^q} \|u\|_{H^1} \|u\|_{H^{q+1}} \end{aligned}$$

hold for some constant c_2. Applying the above three estimates to (3.11) yields the inequality

$$\begin{aligned} (1-\epsilon) \|u_t\|_{H^q}^2 &\le (1-\epsilon) \|u_t\|_{H^q}^2 + \epsilon \|u_{xt}\|_{H^q}^2 \\ &\le c_1 \|u_t\|_{H^q} \|u\|_{H^1} \|u\|_{H^{q+1}} + \epsilon \|u_t\|_{H^q}^2 + |\alpha| \|u_t\|_{H^q} \|u\|_{H^q} + c_2 \|u_t\|_{H^q} \|u\|_{H^1} \|u\|_{H^{q+1}}, \end{aligned}$$

or,

$$(1-2\epsilon) \|u_t\|_{H^q} \le c\,(1 + \|u\|_{H^1}) \|u\|_{H^{q+1}}$$

for some constant c. *Q.E.D.*

Remark: In the next section, we will show that for any $u_0 \in H^s$ with $s > 3/2$, there is a $T > 0$ depending on $\|u_0\|_{H^s}$ such that the Cauchy problem

$$\begin{aligned} u_t - u_{xxt} &= \alpha u_x + \beta u u_x + 2 u_x u_{xx} + u u_{xxx}, \\ u(x,0) &= u_0(x), \end{aligned} \tag{3.12}$$

has a unique solution $u(x,t) \in C([0,T), H^s)$ in the sense of distribution and u is obtained as the limit of solutions of (3.1) as $\epsilon \to 0$. Then the estimates (3.6) and Gronwall's inequality imply that if $\|u_x\|_{L^\infty}$ is bounded whenever u exists, then u can be extended to a solution in the space $C([0,\infty); H^s)$.

4. Local Well-posedness.

Roughly speaking, local well-posedness includes existence, uniqueness and persistence of a solution of the specified problem for finite time, and the continuous dependence of its solutions on the corresponding initial data. To show existence of a solution to problem (3.12), we regularize its initial data u_0 and the equation as follows. For any fixed real number s with $s > 3/2$, suppose that the function u_0 is in $H^s(\mathbb{R})$, and let $u_{\epsilon 0}$ be the convolution $u_{\epsilon 0} = \phi_\epsilon * u_0$ of the functions $\phi_\epsilon(x) = \epsilon^{-1/4}\phi(\epsilon^{-1/4}x)$ and u_0 such that the Fourier transform $\widehat{\phi}$ of ϕ satisfies $\widehat{\phi} \in C_c^\infty$, $\widehat{\phi}(\xi) \geq 0$ and $\widehat{\phi}(\xi) = 1$ for any $\xi \in (-1,1)$. Then it follows from Section 3 that for each ϵ with $0 < \epsilon < 1/4$, the Cauchy problem

$$\begin{gathered} u_t - u_{xxt} + \epsilon u_{xxxxt} = \alpha u_x + \beta u u_x + 2u_x u_{xx} + u u_{xxx}, \quad t > 0,\, x \in \mathbb{R}, \\ u(x,0) = u_{\epsilon 0}(x), \quad x \in \mathbb{R} \end{gathered} \tag{4.1}$$

has a unique solution $u_\epsilon(x,t) \in C^\infty([0,\infty); H^\infty)$. To show that u_ϵ is convergent to a solution of the problem (3.12), we first demonstrate the properties of the initial data $u_{\epsilon 0}$ in the following theorem. The proof is similar to that of Lemma 5 in [**8**].

Theorem 4.1. *Under the above assumptions, the following estimates hold for any ϵ with $0 < \epsilon < 1/4$:*

$$\|u_{\epsilon 0}\|_{H^q} \leq c, \qquad \text{if} \quad q \leq s, \tag{4.2}$$

$$\|u_{\epsilon 0}\|_{H^q} \leq c\epsilon^{\frac{s-q}{4}}, \qquad \text{if} \quad q > s, \tag{4.3}$$

$$\|u_{\epsilon 0} - u_0\|_{H^q} \leq c\epsilon^{\frac{s-q}{4}}, \qquad \text{if} \quad q \leq s, \tag{4.4}$$

$$\|u_{\epsilon 0} - u_0\|_{H^s} = o(1) \tag{4.5}$$

Here c is a constant independent of ϵ.

Combining the estimates in Theorem 4.1 and Section 3, we shall evaluate norms of the function u_ϵ in the following theorem, which will be used to show the convergence of $\{u_\epsilon\}$.

Theorem 4.2. *There exist constants c_1 and c_2 such that the following inequalities*

$$\|u_\epsilon\|_{H^s} \leq \frac{c_1}{(2-Mt)^{c_2}}, \tag{4.6}$$

$$\|u_\epsilon\|_{H^{s+p}} \leq \frac{c_1 \epsilon^{-\frac{p}{4}}}{(2-Mt)^{c_2}}, \qquad p > 0, \tag{4.7}$$

$$\|u_{\epsilon t}\|_{H^{s+p}} \leq \frac{c_1 \epsilon^{-\frac{p+1}{4}}}{(2-Mt)^{c_2}}, \qquad p > -1, \tag{4.8}$$

hold for any ϵ sufficiently small.

Proof: Choose a fixed number r with $3/2 < r < s$. It follows from (3.7) that

$$\int_{\mathbb{R}} (\Lambda^{r+1} u_\epsilon)^2 dx \leq \int_{\mathbb{R}} \left((\Lambda^{r+1} u_{\epsilon 0})^2 + \epsilon(\Lambda^r u_{\epsilon 0 xx})^2\right) dx + c\int_0^t \left(\int_{\mathbb{R}} (\Lambda^{r+1} u_\epsilon)^2 dx\right)^{3/2} d\tau.$$

Then the inequality

$$\|u_\epsilon\|_{H^r}^2 = \int_{\mathbb{R}} (\Lambda^r u_\epsilon)^2 dx \leq \frac{4M_r}{(2 - cM_r^{1/2}\, t)^2} \leq \frac{M}{(2-Mt)^2} \tag{4.9}$$

holds for any $t \in \big[0, 2/M\big)$, where

$$M_r = \int_{\mathbb{R}} \big((\Lambda^r u_{\epsilon 0})^2 + \epsilon(\Lambda^{r-1} u_{\epsilon 0xx})^2\big)\, dx \qquad \text{and} \qquad M = \max\{4M_r, cM_r^{1/2}\}.$$

Substituting the above inequality into (3.7) with $q+1 = s$ and $u = u_\epsilon$, one obtains the estimate

$$\int_{\mathbb{R}} (\Lambda^s u_\epsilon)^2\, dx \le \int_{\mathbb{R}} \big((\Lambda^s u_{\epsilon 0})^2 + \epsilon(\Lambda^{s+1} u_{\epsilon 0})^2\big)\, dx + c\int_0^t \frac{\sqrt{M}}{2 - Mt}\left(\int_{\mathbb{R}} (\Lambda^s u_\epsilon)^2 dx\right) d\tau$$

for any $t \in [0, 2/M)$. It follows from Gronwall's inequality and (4.2) that there is a constant, still denoted by c for simplicity, such that

$$\|u_\epsilon\|_{H^s} = \int_{\mathbb{R}} (\Lambda^s u_\epsilon)^2\, dx \le \frac{2^c c}{(2 - Mt)^c}.$$

In a similar way, applying inequalities (4.3) and (4.9) to (3.7) for $q+1 = s+p$ for any real number $p > 0$, one may obtain the inequality

$$\|u_\epsilon\|_{H^{s+p}} \le \frac{2^c c\, (\epsilon^{-\frac{p}{4}} + \epsilon^{1-(p+1)/4})}{(2 - Mt)^c}$$

for some constant c. Then the inequality (4.8) is just a direct consequence of the inequalities (3.8), (4.2) and (4.7). *Q.E.D.*

We shall next demonstrate that $\{u_\epsilon\}$ is a Cauchy sequence. Let u_ϵ and u_δ be solutions of (4.1), corresponding to the parameters ϵ and δ, respectively, with $0 < \epsilon < \delta < 1/4$, and let $w = u_\epsilon - u_\delta$ and $f = u_\epsilon + u_\delta$. Then w satisfies the problem

$$\begin{aligned} &(1-\epsilon)w_t - \epsilon w_{xxt} + (\delta - \epsilon)(u_{\delta t} + u_{\delta xxt}) + \frac{1}{2}(wf)_x = && (4.10)\\ &= (I - \partial_x^2)^{-1}\left[-\epsilon w_t + (\delta - \epsilon)u_{\delta t} + \alpha w_x + \frac{1+\beta}{2}(wf)_x - \frac{1}{2}(w_x f_x)_x\right], && t > 0,\ x \in \mathbb{R},\\ &\qquad w(x, 0) = w_0(x) = u_{\epsilon 0}(x) - u_{\delta 0}(x), && x \in \mathbb{R}. \end{aligned}$$

Theorem 4.3. *There exists $T > 0$, such that $\{u_\epsilon\}$ is a Cauchy sequence in the space $C([0, T); H^s(\mathbb{R}))$.*

Proof: For a constant q with $1/2 < q < \min\{1, s-1\}$, multiplying $\Lambda^{2q} w$ to both sides of the equation (4.10) and then integrating with respect to x, we obtain the equation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \Big[(1-\epsilon)(\Lambda^q w)^2 + \epsilon(\Lambda^q w_x)^2\Big]\, dx \\ &\quad = (\epsilon - \delta)\int_{\mathbb{R}} (\Lambda^q w)\Big[(\Lambda^q u_{\delta t}) + (\Lambda^q u_{\delta xxt})\Big]\, dx - \frac{1}{2}\int_{\mathbb{R}} (\Lambda^q w)\Lambda^q (wf)_x dx + \\ &\quad + \int_{\mathbb{R}} (\Lambda^q w)\Lambda^{q-2}\left[-\epsilon w_t + (\delta - \epsilon)u_{\delta t} + \alpha w_x + \frac{1+\beta}{2}(wf)_x - \frac{1}{2}(w_x f_x)_x\right] dx \end{aligned}$$

using integration by parts. It follows from the Schwarz inequality and (2.5) that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\Big[(1-\epsilon)(\Lambda^q w)^2+\epsilon(\Lambda^q w_x)^2\Big]dx \qquad (4.11)\\
&\quad\le 2\delta\|\Lambda^q w\|_{L^2}(\|\Lambda^q u_{\delta t}\|_{L^2}+\|\Lambda^q u_{\delta xxt}\|_{L^2}+\|\Lambda^{q-2}w_t\|_{L^2})+c\,\|f\|_{H^{q+1}}\|w\|^2_{H^q}+\\
&\qquad+|\beta+1|\left|\int_{\mathbb{R}}(\Lambda^q w)\Lambda^{q-2}(wf)_x dx\right|+\left|\int_{\mathbb{R}}(\Lambda^q w)\Lambda^{q-2}(w_x f_x)_x dx\right|.
\end{aligned}$$

On the other hand, the inequality

$$\begin{aligned}
&\left|\int_{\mathbb{R}}(\Lambda^q w)\Lambda^{q-2}(wf)_x dx\right| \qquad (4.12)\\
&\quad\le c\int_{\mathbb{R}}(1+\xi^2)^{\frac{q-1}{2}}|\widehat{w}(\xi)|d\xi\int_{\mathbb{R}}\big[(1+(\xi-\eta)^2)^{q/2}+(1+(\eta)^2)^{q/2}\big]|\widehat{w}(\xi-\eta)\widehat{f}(\eta)|d\eta\\
&\quad\le c\,\|w\|_{H^q}(\|f\|_{L^2}\|w\|_{H^q}+\|f\|_{H^q}\|u\|_{L^2})\le 2c\,\|f\|_{H^q}\|w\|^2_{H^q}
\end{aligned}$$

holds for some constant c. Moreover, one may obtain the estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}}(\Lambda^q w)\Lambda^{q-2}(w_x f_x)_x dx\right|\\
&\quad=\left|\int_{\mathbb{R}}(1+\xi^2)^{q-1}\xi\widehat{w}(\xi)d\xi\int_{\mathbb{R}}(\xi-\eta)\widehat{w}(\xi-\eta)\eta\widehat{f}(\eta)d\eta\right|\\
&\quad\le\int_{\mathbb{R}}(1+\xi^2)^{q-1}|\xi\widehat{w}(\xi)|d\xi\left(\int_{\mathbb{R}}\frac{|\xi-\eta|^2|\widehat{w}(\xi-\eta)|^2}{(1+\eta^2)^{s-1}}d\eta\cdot\int_{\mathbb{R}}|\eta|^2(1+\eta^2)^{s-1}|\widehat{f}(\eta)|^2d\eta\right)^{1/2}\\
&\quad\le\|f\|_{H^s}\left(\int_{\mathbb{R}}(1+\xi^2)^q|\widehat{w}(\xi)|^2d\xi\right)^{1/2}\left(\int_{\mathbb{R}}\frac{|\xi|^2}{(1+\xi^2)^{2-q}}\int_{\mathbb{R}}\frac{|\eta|^2|\widehat{w}(\eta)|^2}{(1+(\xi-\eta)^2)^{s-1}}d\eta\right)^{1/2}\\
&\quad\le\|f\|_{H^s}\|w\|_{H^q}\left(\int_{\mathbb{R}}|\eta|^2|\widehat{w}(\eta)|^2d\eta\int_{\mathbb{R}}\frac{d\xi}{(1+(\xi-\eta)^2)^{s-1}(1+\xi^2)^{1-q}}\right)^{1/2}.
\end{aligned}$$

It follows from Lemma 3.1.1 in [**7**] that there is a constant $B>0$ such that

$$\int_{\mathbb{R}}\frac{d\xi}{(1+(\xi-\eta)^2)^{s-1}(1+\xi^2)^{1-q}}\le\frac{B}{(1+\eta^2)^{1-q}}.$$

Hence, combining the last two inequalities, one obtains the estimate

$$\left|\int_{\mathbb{R}}(\Lambda^q w)\Lambda^{q-2}(w_x f_x)_x dx\right|\le\tilde{c}\|f\|_{H^s}\|w\|^2_{H^q} \qquad (4.13)$$

for some constant $\tilde{c}$. Applying the inequalities (4.6), (4.7), (4.8), (4.12) and (4.13) to (4.11), one concludes that for any $\tilde{T}\in(0,2/M)$, there is a constant c depending on $\tilde{T}$ such that the estimate

$$\frac{d}{dt}\int_{\mathbb{R}}[(1-\epsilon)(\Lambda^q w)^2+\epsilon(\Lambda^q w_x)^2]dx\le c(\delta^\gamma\|w\|_{H^q}+\|w\|^2_{H^q})$$

holds for any t with $t \in [0, \tilde{T})$, where $\gamma = 1$ if $s \geq 3 + q$, and $\gamma = \frac{1+s-q}{4}$ if $s < 3 + q$. Integrating the above inequality with respect to t, one obtains the estimate

$$\begin{aligned}
\frac{1}{2}\|w\|^2_{H^q} &= \frac{1}{2}\int_{\mathbb{R}}(\Lambda^q w)^2 dx \\
&\leq \int_{\mathbb{R}}((1-\epsilon)(\Lambda^q w)^2 + \epsilon(\Lambda^q w_x)^2)dx \\
&\leq \int_{\mathbb{R}}\big((\Lambda^q w_0)^2 + \epsilon(\Lambda^q w_{0x})^2\big)dx + c\int_0^t (\delta^\gamma \|w\|_{H^q} + \|w\|^2_{H^q})d\tau.
\end{aligned}$$

Then applying Gronwall's inequality and (4.4) to the above estimate yields the inequality

$$\begin{aligned}
\|w\|_{H^q} &\leq \big(2\int_{\mathbb{R}}\big((\Lambda^q w_0)^2 + \epsilon(\Lambda^q w_{0x})^2\big)dx\big)^{1/2} e^{ct} + \delta^\gamma(e^{ct}-1) \\
&\leq c_1 \delta^{\frac{s-q}{4}} e^{ct} + \delta^\gamma(e^{ct}-1)
\end{aligned} \tag{4.14}$$

for some constants c_1 and any $t \in [0, \tilde{T})$.

Next, multiplying both sides of the equation (4.10) by $\Lambda^{2s} w$ and integrating the resulting expression with respect to x, one obtains the following equality by using integration by parts,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}((1-\epsilon)(\Lambda^s w)^2 + \epsilon(\Lambda^s w_x)^2)dx \\
&\quad = (\epsilon - \delta)\int_{\mathbb{R}}(\Lambda^s w)\Lambda^s(u_{\delta t} + u_{\delta xxt})dx - \frac{1}{2}\int_{\mathbb{R}}(\Lambda^s w)\Lambda^s(wf)_x dx + \\
&\quad\quad + \int_{\mathbb{R}}(\Lambda^{2s-2} w)(-\epsilon w_t + (\delta - \epsilon)u_{\delta t} + \alpha w_x + \frac{1+\beta}{2}(wf)_x - \frac{1}{2}(w_x f_x)_x)dx
\end{aligned}$$

Because $wf = w^2 + 2wu_\delta$, it follows from (2.3), (2.5), (4.12) that the following estimates

$$\begin{aligned}
&\left|\int_{\mathbb{R}}(\Lambda^s w)\Lambda^s(wf)_x dx\right| \leq c_3(\|w_x\|_{L^\infty} + \|u_\delta\|_{H^s})\|w\|^2_{H^s} + c_3\|u_\delta\|_{H^{s+1}}\|w\|_{H^q}\|w\|_{H^s}, \\
&\left|\int_{\mathbb{R}}(\Lambda^s w)\Lambda^{s-2}\big(\frac{\beta+1}{2}wf\big)_x dx\right| \leq c_3\|f\|_{H^s}\|w\|^2_{H^s}
\end{aligned}$$

hold, where c_3 is a constant depending only on s. In addition,

$$\begin{aligned}
&\left|\int_{\mathbb{R}}(\Lambda^s w)\Lambda^{s-2}(w_x f_x)_x dx\right| \\
&= \left|\int_{\mathbb{R}}(1+\xi^2)^{s-1}\xi\widehat{w}(\xi)d\xi\int_{\mathbb{R}}(\xi-\eta)\widehat{w}(\xi-\eta)\eta\widehat{f}(\eta)d\eta\right| \\
&\leq c_4\int_{\mathbb{R}}(1+\xi^2)^{\frac{s}{2}}|\widehat{w}(\xi)|d\xi\int_{\mathbb{R}}[(1+(\xi-\eta)^2)^{\frac{s-1}{2}} + (1+\eta^2)^{\frac{s-1}{2}}]|(\xi-\eta)\widehat{w}(\xi-\eta)\eta\widehat{f}(\eta)|d\eta \\
&\leq c_4\|w\|_{H^s}(\|\widehat{f}_x\|_{L^1}\|w\|_{H^s} + \|f\|_{H^s}\|\widehat{w}_x\|_{L^1}) \leq c_4\|f\|_{H^s}\|w\|^2_{H^s}
\end{aligned}$$

is valid for some constant $c_4 > 0$. Then it follows from the above estimates, and the inequalities (4.6), (4.7), (4.8) and (4.14) that there exists a constant c depending on the real number $\tilde{T} \in (0, 2/M)$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left((1-\epsilon)(\Lambda^s w)^2 + \epsilon(\Lambda^s w_x)^2\right)dx \\
&\qquad \le 2\delta(\|u_{\delta t}\|_{H^s} + \|u_{\delta xxt}\|_{H^s} + \|\Lambda^{s-2} w_t\|_{L^2} + \|\Lambda^{s-2} u_{\delta t}\|_{L^2})\,\|w\|_{H^s} + \\
&\qquad\qquad + c\left(\|w\|^2_{H^s} + \|u_\delta\|_{H^{s+1}}\|w\|_{H^q}\|w\|_{H^s}\right) \\
&\qquad \le c\Big(\delta^m \|w\|_{H^s} + \|w\|^2_{H^s}\Big),
\end{aligned}$$

where $m = \min\{1/4, (s-q-1)/4\} > 0$. Therefore, integrating the above inequality with respect to t leads to the estimate

$$\begin{aligned}
\frac{1}{2}\|w\|^2_{H^s} &\le \int_{\mathbb{R}}\left((1-\epsilon)(\Lambda^s w)^2 + \epsilon(\Lambda^s w_x)^2\right)dx \\
&\le \int_{\mathbb{R}}\left((\Lambda^s w_0)^2 + \epsilon(\Lambda^s w_{0x})^2\right)dx + c\int_0^t (\delta^m\|w\|_{H^s} + \|w\|^2_{H^s})d\tau.
\end{aligned}$$

It follows from Gronwall's inequality and (4.3) that

$$\begin{aligned}
\|w\|_{H^s} &\le \left(2\int_{\mathbb{R}}\left((\Lambda^s w_0)^2 + \epsilon(\Lambda^s w_{0x})^2\right)dx\right)^{1/2} e^{ct} + \delta^m(e^{ct}-1) \\
&\le c_1(\|w_0\|_{H^s} + \delta^{\frac{3}{4}})e^{ct} + \delta^m(e^{ct}-1)
\end{aligned}$$

Then (4.5) and the above inequality show that $\|w\|_{H^s} \to 0$ as $\epsilon, \delta \to 0$.

Next, we consider convergence of the sequence $\{u_{\epsilon t}\}$. Multiplying both sides of the equation (4.10) by $\Lambda^{2s-2} w_t$ and integrating the resulting equation with respect to x, one obtains the equation using integration by parts

$$\begin{aligned}
(1-\epsilon)\|w_t\|^2_{H^{s-1}} &+ \int_{\mathbb{R}}\Big(-\epsilon(\Lambda^{s-1}w_t)(\Lambda^{s-1}w_{xxt}) + \\
&+ (\delta-\epsilon)(\Lambda^{s-1}w_t)\Lambda^{s-1}(u_{\delta t} + u_{\delta xxt}) + \frac{1}{2}(\Lambda^{s-1}w_t)\Lambda^{s-1}(wf)_x\Big)\,dx \\
= \int_{\mathbb{R}}&(\Lambda^{s-1}w_t)\Lambda^{s-3}\Big(-\epsilon w_t + (\delta-\epsilon)u_{\delta t} + \alpha w_x + \frac{1+\beta}{2}(wf)_x - \frac{1}{2}(w_x f_x)_x\Big)dx.
\end{aligned}$$

It follows from the inequalities (4.6), (4.7) and (4.8), as well as Schwarz inequality that there is a constant c depending on $\tilde{T}$ such that

$$(1-\epsilon)\|w_t\|^2_{H^{s-1}} \le c(\delta^{1/2} + \|w\|_{H^s} + \|w\|_{H^{s-1}})\|w_t\|_{H^{s-1}} + \epsilon\|w_t\|^2_{H^{s-1}}.$$

Hence,

$$\frac{1}{2}\|w_t\|_{H^{s-1}} \le (1-2\epsilon)\|w_t\|_{H^{s-1}} \le c(\delta^{1/2} + \|w\|_{H^s} + \|w\|_{H^{s-1}}),$$

and $w_t \to 0$ as ϵ, $\delta \to 0$ in H^{s-1}-norm. This implies that both $\{u_\epsilon\}$ and $\{u_{\epsilon t}\}$ are Cauchy sequences in the spaces $C([0,\tilde{T}); H^s)$ and $C([0,\tilde{T}); H^{s-1})$, respectively. Let $u(x,t)$ be the limit of the sequence $\{u_\epsilon\}$. Taking the limit on both sides of the equation (3.10) as $\epsilon \to 0$, one shows that u is a solution of the problem

$$\begin{aligned} u_t + uu_x &= (I-\partial_x^2)^{-1}\partial_x\left[\alpha u + \frac{1+\beta}{2}u^2 - \frac{u_x^2}{2}\right] \quad t>0,\ x\in\mathbb{R}, \\ u(x,t) &= u_0(x), \quad x\in\mathbb{R}, \end{aligned} \tag{4.15}$$

and hence u is a solution of (3.12) in the sense of distribution. In particular, if $s \geq 3$, u is also a classical solution of the Cauchy problem (3.12). *Q.E.D.*

The verification for the uniqueness of the solution u follows the technique to obtain the norm $\|w\|_{H^q}$ in Theorem 4.3.

Theorem 4.4. *Suppose that $u_0 \in H^s$ for some constant $s > 3/2$. Then there is a $T > 0$, such that the problem (4.15) has a unique solution $u(x,t)$ in $C([0,T);H^s)$.*

Proof: Suppose that u and v are two solutions of the problem (4.15) corresponding to the same initial data u_0 such that $u, v \in L^2([0,T);H^s)$. Then $w = u - v$ satisfies the Cauchy problem

$$\begin{aligned} w_t + \frac{1}{2}(wf)_x &= (I-\partial_x^2)^{-1}\partial_x\left[\alpha w + \frac{1+\beta}{2}wf - \frac{1}{2}w_x f_x\right], \quad t>0,\ x\in\mathbb{R}, \\ w(x,0) &= 0, \quad x\in\mathbb{R}, \end{aligned}$$

where $f = u + v$. For any $1/2 < q < \min\{1, s-1\}$, applying the operator Λ^q to both sides of the above equation and then multiplying the resulting expression by $\Lambda^q w$ to integrate with respect x, one obtains the equality

$$\frac{1}{2}\frac{d}{dt}\|w\|_{H^q}^2 + \frac{1}{2}\int_{\mathbb{R}}(\Lambda^q w)\Lambda^q(wf)_x dx = \int_{\mathbb{R}}\Lambda^q w \Lambda^{q-2}\partial_x\left[\alpha w + \frac{1+\beta}{2}wf - \frac{1}{2}w_x f_x\right] dx.$$

It follows from (4.12), (4.13) and (2.5) that there is a constant c such that

$$\frac{d}{dt}\|w\|_{H^q}^2 \leq c\|f\|_{H^s}\|w\|_{H^q}^2,$$

Then Gronwall's inequality and boundedness of $\|f\|_{H^s}$ lead to the conclusion that

$$\|w\|_{H^q} \leq \|w_0\|_{H^q} e^{\tilde{c}t} = 0,$$

for some constant $\tilde{c}$ and any $t \in (0,T)$. Hence, $w = 0$. *Q.E.D.*

The last issue on well-posedness is the continuous dependency of solutions on initial data. One may verify it by using a similar technique used for the KdV equation by Bona and Smith, [**8**]. We summarize the main conclusions of this section in the next theorem.

Theorem 4.5. *Suppose that the function $u_0(x)$ belongs to the Sobolev space H^s for some $s > 3/2$. Then there is a $T > 0$, which depends only on $\|u_0\|_{H^s}$, such that there exists a unique function $u(x,t)$ solving the Cauchy problem (3.12) in the sense of distribution with $u \in C([0,\tilde{T});H^s)$ and $u(x,t) = u_0(x)$. When $s \geq 3$, u is also a classical solution of (3.12). Moreover, the solution u depends continuously on the initial data u_0 in the sense that the mapping of the initial data to the solution is continuous from the Sobolev space H^s to the space $C([0,\tilde{T});H^s)$.*

5. Existence of solutions in lower order Sobolev spaces.

As remarked in the introduction, the KdV equation and many of its generalizations have a smoothing effect on their solutions. Because of this effect, solutions gain more regularity than the corresponding initial data, cf. [**29, 31, 32**]. This regularizing effect became an important fact used to show well-posedness of these equations in lower order Sobolev spaces. On the other hand, the peakon solution of the Camassa–Holm equation demonstrates that, in general, its solutions do not gain more regularity as time evolves. Therefore, one may expect to use different techniques dealing with well-posedness of the Camassa–Holm equation in the lower order Sobolev spaces. In this section, we shall give a sufficient condition for a solution of the Camassa–Holm equation to exist in the Sobolev space H^s for some $1 < s \leq 3/2$. First, we still use the regularized equation (4.1) to estimate norms of its solutions, showing that they are bounded when ϵ is sufficiently small, which leads to weak convergence of these solutions to a solution of the Camassa–Holm equation.

Theorem 5.1. *Suppose that $u_0(x)$ is a function in the Sobolev space H^s for some $s \in [1, 3/2]$ such that $\|u_{0x}\|_{L^\infty} < \infty$. Let $u_{\epsilon 0}$ be defined the same as in Section 4. Then there are constants $T > 0$ and $c > 0$ independent of ϵ such that the corresponding solution u_ϵ of (4.1) satisfies the inequality $\|u_{\epsilon x}\|_{L^\infty} \leq c$.*

Proof: We start from the equation (3.10) with $u = u_\epsilon$. Differentiating with respect to x on both sides of (3.10), we obtain

$$(1-\epsilon)u_{xt} - \epsilon u_{xxxt} + uu_{xx} + \frac{u_x^2}{2} = -\alpha u - \frac{\beta+1}{2}u^2 + \Lambda^{-2}\left(-\epsilon u_{xt} + \alpha u + \frac{\beta+1}{2}u^2 - \frac{u_x^2}{2}\right).$$

Let $n > 0$ be an integer. Then multiplying the above equation by $(u_x)^{2n+1}$ to integrate with respect to x yields the equality

$$\begin{aligned}\frac{1-\epsilon}{2n+2}\frac{d}{dt}\int_{\mathbb{R}}(u_x)^{2n+2}dx - \epsilon\int_{\mathbb{R}}(u_x)^{2n+1}u_{xxxt}\,dx + \frac{n}{2n+2}\int_{\mathbb{R}}(u_x)^{2n+3}dx \\ = -\int_{\mathbb{R}}(u_x)^{2n+1}(\alpha u + \frac{\beta+1}{2}u^2)\,dx + \int_{\mathbb{R}}(u_x)^{2n+1}\Lambda^{-2}(-\epsilon u_{xt} + \alpha u + \frac{\beta+1}{2}u^2 - \frac{u_x^2}{2})\,dx.\end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned}\frac{1-\epsilon}{2n+2}\frac{d}{dt}\int_{\mathbb{R}}(u_x)^{2n+2}dx \leq \Bigg[\epsilon\left(\int_{\mathbb{R}}|u_{xxxt}|^{2n+2}dx\right)^{\frac{1}{2n+2}} + |\alpha|\left(\int_{\mathbb{R}}|u|^{2n+2}dx\right)^{\frac{1}{2n+2}} + \\ + \left|\frac{\beta+1}{2}\right|\left(\int_{\mathbb{R}}|u|^{4n+4}dx\right)^{\frac{1}{2n+2}} + \left(\int_{\mathbb{R}}|g|^{2n+2}dx\right)^{\frac{1}{2n+2}}\Bigg]\left(\int_{\mathbb{R}}|u_x|^{2n+2}dx\right)^{\frac{2n+1}{2n+2}} + \\ + \frac{n}{2n+2}\|u_x\|_{L^\infty}\int_{\mathbb{R}}|u_x|^{2n+2}dx,\end{aligned}$$

or

$$(1-\epsilon)\frac{d}{dt}\left(\int_{\mathbb{R}}|u_x|^{2n+2}dx\right)^{\frac{1}{2n+2}} \le \epsilon\left(\int_{\mathbb{R}}|u_{xxxt}|^{2n+2}dx\right)^{\frac{1}{2n+2}} + |\alpha|\left(\int_{\mathbb{R}}|u|^{2n+2}dx\right)^{\frac{1}{2n+2}} +$$
$$+\left|\frac{\beta+1}{2}\right|\left(\int_{\mathbb{R}}|u|^{4n+4}dx\right)^{\frac{1}{2n+2}} + \left(\int_{\mathbb{R}}|g|^{2n+2}dx\right)^{\frac{1}{2n+2}} + \frac{n\|u_x\|_{L^\infty}}{2n+2}\left(\int_{\mathbb{R}}|u_x|^{2n+2}dx\right)^{\frac{1}{2n+2}},$$

where

$$g = \Lambda^{-2}\left(-\epsilon u_{xt} + \alpha u + \frac{\beta+1}{2}u^2 - \frac{u_x^2}{2}\right).$$

Because $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \cap L^2$, integration with respect to t and taking the limit as $n \to \infty$ on both sides of the above inequality leads to the estimate

$$\begin{aligned}(1-\epsilon)\|u_x\|_{L^\infty} &\le (1-\epsilon)\|u_{0x}\|_{L^\infty} + \\ &+ \int_0^t \left[\epsilon\,\|u_{xxxt}\|_{L^\infty} + c\,(\|u\|_{L^\infty} + \|u^2\|_{L^\infty} + \|g\|_{L^\infty}) + \frac{1}{2}\|u_x\|^2_{L^\infty}\right]d\tau.\end{aligned} \tag{5.1}$$

Because

$$\|g\|_{L^\infty} \le \tilde{c}\,(\|u_t\|_{L^2} + \|u\|_{L^2} + \|u\|^2_{L^2} + \|u_x\|^2_{L^2})$$

for some constant $\tilde{c}$ depending only on Λ^{-2}, it follows from (3.5), (3.8) and (4.2) that

$$\|g\|_{L^\infty} \le c_1\,(\|u_{\epsilon 0}\|_{H^1} + 1)^2 \le c_2,$$

where c_1 and c_2 are constants independent of ϵ when ϵ is sufficiently small. Moreover, for any fixed $r \in (1/2, 1)$, there is a constant c_r such that $\|u_{xxxt}\|_{L^\infty} \le c_r\,\|u_{xxxt}\|_{H^r} \le c_r\,\|u_t\|_{H^{r+3}}$, which combined with (3.8), (3.5) yields

$$\|u_{xxxt}\|_{L^\infty} \le c\,\|u\|_{H^{r+4}}. \tag{5.2}$$

Applying Gronwall's inequality to (3.6) with $q = r+3$ and $u = u_\epsilon$, one has

$$\|u\|^2_{H^{r+4}} \le \left(\int_{\mathbb{R}}\left[(\Lambda^{r+4}u_0)^2 + \epsilon(\Lambda^{r+3}u_{0xx})^2\right]dx\right)\exp\left(c\int_0^t \|u_x\|_{L^\infty}\,d\tau\right).$$

Then it follows from (4.3) and (5.2) that

$$\|u_{xxxt}\|_{L^\infty} \le c\,\epsilon^{\frac{s-r-4}{4}}\exp\left(c\int_0^t \|u_x\|_{L^\infty}\,d\tau\right) \tag{5.3}$$

for some constant $c > 0$. Therefore, as $\epsilon < 1/4$, one obtains the inequality

$$\begin{aligned}\|u_x\|_{L^\infty} &\le \|u_{0x}\|_{L^\infty} + \frac{c}{1-\epsilon}\int_0^t\left[\epsilon^{\frac{s-r}{4}}\exp\left(c\int_0^\tau \|u_x\|_{L^\infty}ds\right) + \frac{1}{2}\|u_x\|^2_{L^\infty} + 1\right]d\tau \\ &\le \|u_{0x}\|_{L^\infty} + \frac{4c}{3}\int_0^t\left[\epsilon^{\frac{s-r}{4}}\exp\left(c\int_0^\tau \|u_x\|_{L^\infty}ds\right) + \frac{1}{2}\|u_x\|^2_{L^\infty} + 1\right]d\tau\end{aligned}$$

by combining (5.1), (5.3) and (3.5). It follows from the contraction mapping theorem that there is a $T > 0$ such that the equation

$$f(t) = \|u_{0x}\|_{L^\infty} + \frac{4c}{3} \int_0^t \left[\exp\left(c \int_0^\tau f(s) ds \right) + \frac{1}{2} f^2(\tau) + 1 \right] d\tau$$

has a unique solution $f(t) \in C[0, T]$. Theorem II in [**58**; §I.1] shows that $\|u_x\|_{L^\infty} \le f(t)$ for any $t \in [0, T]$, which implies the conclusion of Theorem 5.1. *Q.E.D.*

As a direct result of Theorem 5.1, one may estimate norms of $u = u_\epsilon$ by using (3.6), (3.8), (4.2), (4.3) and Gronwall's inequality to show that there is a constant $c > 0$ such that the inequalities

$$\|u_\epsilon\|_{H^q} = \|u\|_{H^q} \le c \exp c \int_0^t \|u_x\|_{L^\infty} d\tau \le c \exp c \int_0^t f(\tau) d\tau,$$

and

$$\|u_{\epsilon t}\|_{H^r} = \|u_t\|_{H^r} \le c \exp c \int_0^t f(\tau) d\tau$$

hold for any $q \in (0, s]$, $r \in (0, s-1]$ and any $t \in [0, T]$. Then it follows from Aubin's compactness theorem, cf. [**37**], that there is a subsequence of $\{u_\epsilon\}$, denoted by $\{u_{\epsilon_n}\}$, such that $\{u_{\epsilon_n}\}$ and their temporal derivatives $\{u_{\epsilon_n t}\}$ are weakly convergent to a function $u(x, t)$ and its temporal derivative u_t in $L^2([0, T], H^s)$ and $L^2([0, T], H^{s-1})$, respectively. Moreover, for any real number $R > 0$, $\{u_{\epsilon_n}\}$ is convergent to the function u strongly in the space $L^2([0, T], H^q(-R, R))$ for any $q \in [0, s)$ and $\{u_{\epsilon_n t}\}$ converges to u_t strongly in the space $L^2([0, T], H^r(-R, R))$ for any $r \in [0, s-1)$. Therefore, one obtains the existence of a weak solution to the Cauchy problem (3.12) as follows.

Theorem 5.2. *Let $u_0(x)$ be a function in the Sobolev space H^s for some $s \in (1, 3/2]$, satisfying $\|u_{0x}\|_{L^\infty} < \infty$. Then there is a $T > 0$ such that the Cauchy problem (3.12) with the initial data u_0 has a solution $u(x, t) \in L^2([0, T], H^s)$ in the sense of distribution, and $u_x \in L^\infty([0, T] \times \mathbb{R})$.*

Proof: It follows from Theorem 5.1 that $\{u_{\epsilon_n x}\}$ is bounded in the space L^∞. Hence, the sequences $\{u^2_{\epsilon_n}\}$ and $\{u^2_{\epsilon_n x}\}$ are also weakly convergent to u^2 and u_x^2 in $L^2\big([0, T], H^r(-R, R)\big)$ for any $r \in [0, s-1)$, respectively. Therefore, u satisfies the equation

$$\int_0^T \int_{\mathbb{R}} u(f_t - f_{xxt}) \, dx \, dt = \int_0^T \int_{\mathbb{R}} \left[(\alpha u + \tfrac{1}{2}\beta u^2 - \tfrac{1}{2} u_x^2) f_x + \tfrac{1}{2} u^2 f_{xxx} \right] \, dx \, dt,$$

with $u(x, 0) = u_0(x)$ and any $f \in C_c^\infty$. Moreover, since $X = L^1([0, T] \times \mathbb{R})$ is a separable Banach space and $\{u_{\epsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0, T] \times \mathbb{R})$ of X, there is a subsequence of $\{u_{\epsilon_n x}\}$, still denoted by $\{u_{\epsilon_n x}\}$, weakly star convergent to a function v in $L^\infty([0, T] \times \mathbb{R})$. Because $\{u_{\epsilon_n x}\}$ is also weakly convergent to u_x in $L^2([0, T] \times \mathbb{R})$, it follows that $u_x = v$ almost everywhere. Hence, $u_x \in L^\infty([0, T] \times \mathbb{R})$. *Q.E.D.*

6. Blowing-up of solutions.

Even though the Camassa–Holm equation also has a bi-Hamiltonian structure, unlike the KdV equation, it has no conserved quantities providing boundedness of H^s-norms independent of time for its solutions with any $s \geq 2$. In this section, we shall verify this fact by showing that there are solutions of the Camassa–Holm equation, whose H^q-norms blow up in finite time for any $q > 3/2$. This phenomena also implies that in general, one can not obtain global well-posedness of the Camassa–Holm equation in H^s for $s > 3/2$ unconditionally. Moreover, in contrast to using conserved quantities to prove global existence of solutions for the KdV equation, in the next theorem, we shall use the conserved quantity $\|u\|_{H^1} = \|u_0\|_{H^1}$ of the Camassa–Holm equation to show that some of its solutions exist only in finite time.

Theorem 6.1. *Let $s \in [2, \infty)$ be any real number. If the initial data u_0 of the Cauchy problem (3.12) satisfies the conditions*

$$u_0 \in H^s, \qquad \int_{\mathbb{R}} u_{0x}^3 dx < 0 \qquad \text{and} \qquad 8b\|u_0\|_{H^1}^2 < \left(\int_{\mathbb{R}} (u_{0x})^3 dx \right)^2$$

where $b = c\,(\|u_0\|_{H^1}^3 + \|u_0\|_{H^1}^4)$ and c is a constant to be specified in the proof, then there is a $0 < T^* \leq -4\|u_0\|_{H^1}^2 / \int_{\mathbb{R}} (u_{0x})^3 dx$ such that the corresponding solution $u \in C([0, T^*); H^s)$ and u ceases to exist in H^s at the time T^* in the sense that

$$\limsup_{t \to T^*} \|u_x\|_{L^\infty} = \infty \qquad \text{and} \qquad \lim_{t \to T^*} \|u\|_{H^q} = \infty$$

for any $q \in (3/2, s]$.

Proof: It follows from Theorems 4.3 and 4.5 that there is a $T_0 > 0$ such that the Cauchy problem (3.12) has a unique solution $u(x,t) \in C([0, T_0); H^s)$, satisfying the equation (4.15). Applying $u_x^2 \partial_x$ to both sides of (4.15) and integrating with respect to x, one obtains the equality

$$\begin{aligned} \frac{1}{3}\frac{d}{dt} \int_{\mathbb{R}} (u_x)^3 dx + \frac{1}{6} \int_{\mathbb{R}} (u_x)^4 dx = - \int_{\mathbb{R}} (u_x)^2 \left(\alpha u + \frac{\beta+1}{2} u^2 \right) dx \, + \\ + \int_{\mathbb{R}} (u_x)^2 \Lambda^{-2} \left(\alpha u + \frac{\beta+1}{2} u^2 - \frac{u_x^2}{2} \right) dx. \end{aligned} \tag{6.1}$$

Because

$$\left| \int_{\mathbb{R}} (u_x)^3 dx \right| \leq \left(\int_{\mathbb{R}} |u_x|^4 dx \right)^{1/2} \left(\int_{\mathbb{R}} |u_x|^2 dx \right)^{1/2},$$

it follows that

$$\int_{\mathbb{R}} |u_x|^4 dx \geq \frac{1}{\|u\|_{H^1}^2} \left(\int_{\mathbb{R}} (u_x)^3 dx \right)^2. \tag{6.2}$$

In addition, since the inequalities $\|f\|_{L^\infty} \le \|f\|_{H^1}$ and

$$|\Lambda^{-2}f(x)| = \left|\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}f(y)\,dy\right| \le \begin{cases} \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} |f(y)|\,dy \\ \dfrac{1}{2}\left(\displaystyle\int_{\mathbb{R}} e^{-2|x|}dx\right)^{1/2} \|f\|_{L^2} \le \dfrac{1}{2}\|f\|_{H^1} \end{cases}$$

hold for any $f \in L^1 \cap H^1$, the estimate

$$\begin{aligned} &\left|\int_{\mathbb{R}} 3(u_x)^2\big(\alpha u + \frac{\beta+1}{2}u^2 - \Lambda^{-2}\big(\alpha u + \frac{\beta+1}{2}u^2 - \frac{u_x^2}{2}\big)\big)\,dx\right| \\ &\le 3\|u\|^2_{H^1}\big(|\alpha|\|u\|_{L^\infty} + \frac{|\beta+1|}{2}\|u\|^2_{L^\infty} + \big|\Lambda^{-2}\big(\alpha u + \frac{\beta+1}{2}u^2 - \frac{u_x^2}{2}\big)\big|_{L^\infty}\big) \\ &\le 3\|u\|^2_{H^1}\big(\frac{3|\alpha|}{2}\|u\|_{H^1} + \frac{3|\beta+1|+1}{4}\|u\|^2_{H^1}\big) \le c\,(\|u\|^3_{H^1} + \|u\|^4_{H^1}) \end{aligned} \tag{6.3}$$

holds for some constant c with $c \le 3\max\{\frac{3|\alpha|}{2}, \frac{3|\beta+1|+1}{4}\}$. Applying (6.2), (6.3) and the equality $\|u\|_{H^1} = \|u_0\|_{H^1}$ to (6.1), and then integrating with respect to t lead to the estimate

$$\int_{\mathbb{R}}(u_x)^3\,dx + \frac{1}{2\|u_0\|^2_{H^1}}\int_0^t \left(\int_{\mathbb{R}}(u_x)^3\,dx\right)^2 d\tau \le \int_{\mathbb{R}}(u_{0x})^3\,dx + c\,(\|u_0\|^3_{H^1} + \|u_0\|^4_{H^1})t.$$

Let $b = c\,(\|u_0\|^3_{H^1} + \|u_0\|^4_{H^1})$. When $t < t_0 = \min\big\{T_0, -\int_{\mathbb{R}}(u_{0x})^3\,dx/(2b)\big\}$, the inequality

$$\int_{\mathbb{R}}(u_x)^3\,dx + \frac{1}{2\|u_0\|^2_{H^1}}\int_0^t \left(\int_{\mathbb{R}}(u_x)^3\,dx\right)^2 d\tau \le \frac{1}{2}\int_{\mathbb{R}}(u_{0x})^3\,dx$$

holds, which leads to the estimate

$$\int_{\mathbb{R}}(u_x)^3\,dx \le \frac{\frac{1}{2}\int_{\mathbb{R}}(u_{0x})^3\,dx}{1 + \frac{t}{4\|u_0\|^2_{H^1}}\int_{\mathbb{R}}(u_{0x})^3\,dx} < 0 \tag{6.4}$$

for any $t < \min\{t_0, -4\|u_0\|^2_{H^1}/\int_{\mathbb{R}}(u_{0x})^3\,dx\}$. This implies that

$$t_0 \le t_1 = \frac{-4\|u_0\|^2_{H^1}}{\int_{\mathbb{R}}(u_{0x})^3\,dx}.$$

Because if $t_0 > t_1 = -4\|u_0\|^2_{H^1}/\int_{\mathbb{R}}(u_{0x})^3\,dx$, then

$$\lim_{t\to t_1}\int_{\mathbb{R}}(u_x)^3\,dx \le \lim_{t\to t_1}\frac{\frac{1}{2}\int_{\mathbb{R}}(u_{0x})^3\,dx}{1 + \frac{t}{4\|u_0\|^2_{H^1}}\int_{\mathbb{R}}(u_{0x})^3\,dx} = -\infty$$

and the inequality

$$c\,\|u_0\|^2_{H^1}\|u\|_{H^2} \ge \left|\int_{\mathbb{R}}(u_x)^3\,dx\right| \ge \frac{-\frac{1}{2}\int_{\mathbb{R}}(u_{0x})^3\,dx}{1 + \frac{t}{4\|u_0\|^2_{H^1}}\int_{\mathbb{R}}(u_{0x})^3\,dx}$$

would show that the H^2-norm of the solution u blows up at the time $t = t_1 < t_0 \le T_0$, contrary to the condition $u \in C([0,T_0), H^s)$ for some $s \ge 2$. On the other hand, since

$$t_0 \le t_1 = \frac{-4\|u_0\|_{H^1}^2}{\int_{\mathbb{R}} (u_{0x})^3\, dx} < \frac{-1}{2b} \int_{\mathbb{R}} (u_{0x})^3\, dx,$$

it follows that $t_0 = T_0$, which combined with (6.4) and the estimate

$$\left| \int_{\mathbb{R}} (u_x)^3\, dx \right| \le c_q \|u\|_{H^1}^2 \|u\|_{H^q} = c_q \|u_0\|_{H^1}^2 \|u\|_{H^q}$$

shows that $\lim_{t\to T_0} \|u\|_{H^q} = \infty$, where q is any real number with $q \in (3/2, \infty)$ and c_q is a constant independent of u.

To verify $\limsup_{t\to T_0} \|u\|_{L^\infty} = \infty$, one may use (3.6), (4.2), (4.3) and Theorem 4.3 to show that u, as the limit of the solutions $\{u_\epsilon\}$ of (4.1), satisfies the inequality

$$\|u\|_{H^q}^2 \le \|u_0\|_{H^q}^2 + c \int_0^t \|u_x\|_{L^\infty} \|u\|_{H^q}^2 d\tau,$$

for any $q \in (3/2, s]$. It follows from Gronwall's inequality that

$$\|u\|_{H^q}^2 \le \|u_0\|_{H^q}^2 \exp \int_0^t \|u_x\|_{L^\infty} d\tau. \tag{6.5}$$

Therefore, if $\limsup_{t\to T_0} \|u\|_{L^\infty} < \infty$, then it would lead to the boundedness of $\|u\|_{H^q}^2$, *i.e.*

$$\limsup_{t\to T_0} \|u\|_{H^q}^2 < \infty$$

which is contrary to $\lim_{t\to T_0} \|u_0\|_{H^q} = \infty$. Hence, $T_0 = T^*$ is the finite time for u_x and u to cease existing in L^∞ and H^q for any $q \in (3/2, s]$, respectively. *Q.E.D.*

In general, the Cauchy problem (3.12) does not necessarily have a global solution. But one might have realized from the proof of last theorem that a necessary condition for a global solution u to exist is the boundedness of L^∞-norm of u_x. This result is in contrast to that of the generalized KdV equation

$$u_t + f(u)_x + u_{xxx} = 0.$$

The singularities of its blowing-up solutions are caused by the nonlinear term $f(u)$ when f becomes too strong compared with the linear dispersion term u_{xxx}, and these solutions become unbounded in their L^∞-norm in finite time [**3**]. Whereas, the nonlinearly dispersive term uu_{xxx} of the equation (1.4) has weakened the smoothing effect of the linear dispersion term u_{xxt}, causing some of its solutions to form singularities and their first derivatives to blow up in finite time, but their own L^∞-norms are always bounded because of the conserved quantity $\|u(\cdot,t)\|_{H^1} = \text{const}$. Now we state this result in the next theorem.

Theorem 6.2. *Suppose that $u_0 \in H^s$ for some $s > 3/2$ and that T_0 is the maximum time for the corresponding solution u of (3.12) to exist in the space $C([0,T_0), H^s)$. If $T_0 < \infty$, then $\sup_{0\le t<T_0} \|u_x(\cdot,t)\|_{L^\infty} = \infty$.*

Proof: Assume that $T_0 < \infty$ and $\sup_{0 \le t < T_0} \|u_x(\cdot, t)\|_{L^\infty} < \infty$. Then it follows from (6.5) that $\sup_{0 \le t < T_0} \|u(\cdot, t)\|_{H^q} < \infty$ for any $q \in (3/2, s]$. Hence, one may use an argument similar to that in the proof of Theorem 4.3 to show that u has a unique extension as a solution of (3.12) in the space $C([0, T_1), H^s)$ for some $T_1 > T_0$ which contradicts the condition that T_0 is maximum. Hence, $T_0 = \infty$. *Q.E.D.*

Remark: The technique we have used to show well-posedness, and the existence of blow-up solutions of (3.12) for the initial data $u_0 \in H^s(\mathbb{R})$ also applies to the initial value problem (3.12) with periodic boundary conditions, *i.e.* its solutions satisfy the condition $u(x, t) = u(x + 2\pi, t)$ and $u_0 \in H^s(\mathbb{T})$ for some $s > 3/2$, where $\mathbb{T}$ is the unit circle. Therefore, there also exist periodic solutions of (3.12), which develop singularities in finite time. A related study was recently conducted by Constantin and Escher, [**15, 16**], who proved that the Camassa–Holm equation (1.5) has solutions u whose initial data $u_0(x, 0) \in H^3$ are odd and $u_x(0, t)$ become infinite in finite time. These solutions have apparently developed singularities at $x = 0$. It will also be interesting to investigate whether the spatial derivatives u_x of the blowing-up solutions given in Theorem 6.1 also develop singularities in finite time. We have planned to study this problem both theoretically and numerically.

Now, we show an example of the initial data u_0 of (3.12), which will generates a solution existing only in the finite time $t = T^*$ and T^* can be chosen as small as possible. Then we shall conclude this section by our preliminary, numerical computation report.

Example 6.3. For a fixed $\epsilon \in (0, 1)$, define the function

$$u_0(x) = \begin{cases} \dfrac{4e^x}{(1-\epsilon^2)^2} + \dfrac{(\epsilon^2 - 2\epsilon + x - \epsilon x)e^{x/\epsilon}}{\epsilon(1-\epsilon)^2} & x < 0 \\ \dfrac{\left((1+\epsilon)x + \epsilon^2 + 2\epsilon\right)e^{-x/\epsilon}}{\epsilon(1+\epsilon)^2} & x \ge 0. \end{cases}$$

Then $u_0 \in H^s(\mathbb{R})$ for any $s < 9/2$. Since

$$\int_{-\infty}^{\infty} \left(u_0'(x)\right)^3 dx = -\frac{8(4\epsilon^3 + 44\epsilon^2 + 89\epsilon + 52)}{27\epsilon^2(1+\epsilon)^4(1+2\epsilon)^2(2+\epsilon)^3},$$

and

$$\|u_0\|_{H^1}^2 = \int_{-\infty}^{\infty} \left[\left(u_0(x)\right)^2 + \left(u_0'(x)\right)^2\right] dx = \frac{5}{2\epsilon},$$

it follows that

$$8b\|u_0\|_{H^1}^2 = 8c(\|u_0\|_{H^1}^3 + \|u_0\|_{H^1}^4)\|u_0\|_{H^1}^2 = 125c(1 + \sqrt{2\epsilon/5}\,)/\epsilon^3$$

and

$$\frac{8b\|u_0\|_{H^1}^2}{\left(\int_{-\infty}^{\infty} \left(u_0'(x)\right)^3 dx\right)^2} = \frac{125(27)^2\epsilon c(1 + \sqrt{2\epsilon/5}\,)(1+\epsilon)^8(1+2\epsilon)^4(2+\epsilon)^6}{64(4\epsilon^3 + 44\epsilon^2 + 89\epsilon + 52)^2} \longrightarrow 0$$

as $\epsilon \to 0$. Therefore, when ϵ is sufficiently small, u_0 satisfies all conditions stated in Theorem 6.1. It follows that the corresponding solution $u(x, t)$ blows up in finite time in the Sobolev space H^r for any $r > 3/2$. In addition, since $\|u_0\|_{H^1}^2 / \int_{-\infty}^{\infty} (u_0')^3 dx \to 0$ as $\epsilon \to 0$, for any $T > 0$, there is also an $\epsilon_1 > 0$, whenever $0 < \epsilon < \epsilon_1$,

$$0 < t_1 = \frac{-4\|u_0\|_{H^1}^2}{\int_{-\infty}^{\infty} (u_0')^3 dx} < T.$$

As we have pointed out in Theorem 6.1 that $u(x,t)$ blows up at some time $T_0 \le t_1$. This shows that one can always find some initial data for which the corresponding solution blows up in any short, designated time.

As a matter of fact, one may construct a smooth initial function u_0 by regularizing the function

$$f(x) = \begin{cases} e^x, & x < 0, \\ 0, & x \ge 0. \end{cases}$$

i.e. define u_0 to be the convolution of f and ϕ_ϵ, where $\phi_\epsilon(x) = \frac{1}{\epsilon}\phi(x/\epsilon)$ such that $\phi \in H^s$ for some $s > 1$ and $0 < \int_{\mathbb{R}} \phi dx \le \int_{\mathbb{R}} |\phi| dx < \infty$. Then $f * \phi_\epsilon \in H^{s+1}$ and as ϵ is sufficiently small, $u_0 = f * \phi_\epsilon$ satisfies conditions in Theorem 6.1. We have used this method to obtain the initial data u_0 by choosing $\phi(x) = (1 + |x|)e^{-|x|} \in H^s$ for any $s < 7/2$.

The figures included at the end of the paper illustrate the finite time blow-up of the first and second derivatives of a solution whose initial data satisfies the conditions of Constantin and Escher, [**15**, **17**]. (Unfortunately, we were not able to construct initial data for the periodic problem that satisfies our blow-up conditions, and yet blows up in a sufficiently short time before periodic effects — the front of the disturbance catching up with the end of the wave — are manifest. We are hoping to implement these in a later, more extensive numerical computation.) The constants in the Camassa-Holm model (1.5) have been taken to be

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = -1, \qquad \nu = 1. \tag{6.6}$$

The initial data is

$$u(x,0) = -\tanh\left(\tfrac{3}{2}x\right) \operatorname{sech}\left(\tfrac{3}{2}x\right). \tag{6.7}$$

In the first figure, we plot the initial data and its first and second derivatives. The numerical solution of the equation is obtained using a standard pseudo-spectral code, cf. [**23**], using $N = 1024$ mesh points with the uniform spatial step size $\Delta x = 20/N$ on the interval $(-10, 10)$, using periodic boundary conditions. The size of the interval was chosen so that no significant signal propagation across the periodic boundary was detected during the time interval of solution. The time step was taken to be $0.0157\Delta x$. Initially, the numerical solution is well behaved. There is a noticeable steepening of the profile between the crest and the trough, as well as a sharpening of the crest and trough. A typical plot is shown in the second Figure, at time $t_1 = 0.552$, after 1800 time steps. The top plot gives the solution $u(t_1, x)$ for $-10 \le x \le 10$, and the left hand graphs show its first and second spatial derivatives on the interval $-5 \le x \le 5$; the right hand graphs zoom in on parts of their left hand counterparts and show the absence of numerical noise at this time. Notice particularly how the first derivative has become much larger negative between the peak and the dip; the blowing up of the second derivative is even more pronounced. At a time between $t_1 = 0.552$ and $t_2 = 0.828$, as shown in the final Figure, the numerical integration method has broken down, and numerical instabilities are now in evidence according to the noise appearing in the two derivative plots, even though the plot of $u(t_2, x)$ looks fairly smooth. This is strong evidence that the solution has experienced a blow-up in its first two derivatives before t_2, and the numerical solution is no longer valid.

We are now conducting a more detailed investigation into the blow-up mechanism. The pseudo-spectral approach is not so directly applicable, and one must resort, either to a finite difference scheme with mesh refinement, or, in a more speculative direction, to some form of pseudo-spectral wavelets, [**14**, **28**], which will allow focusing in of localized small-scale phenomena.

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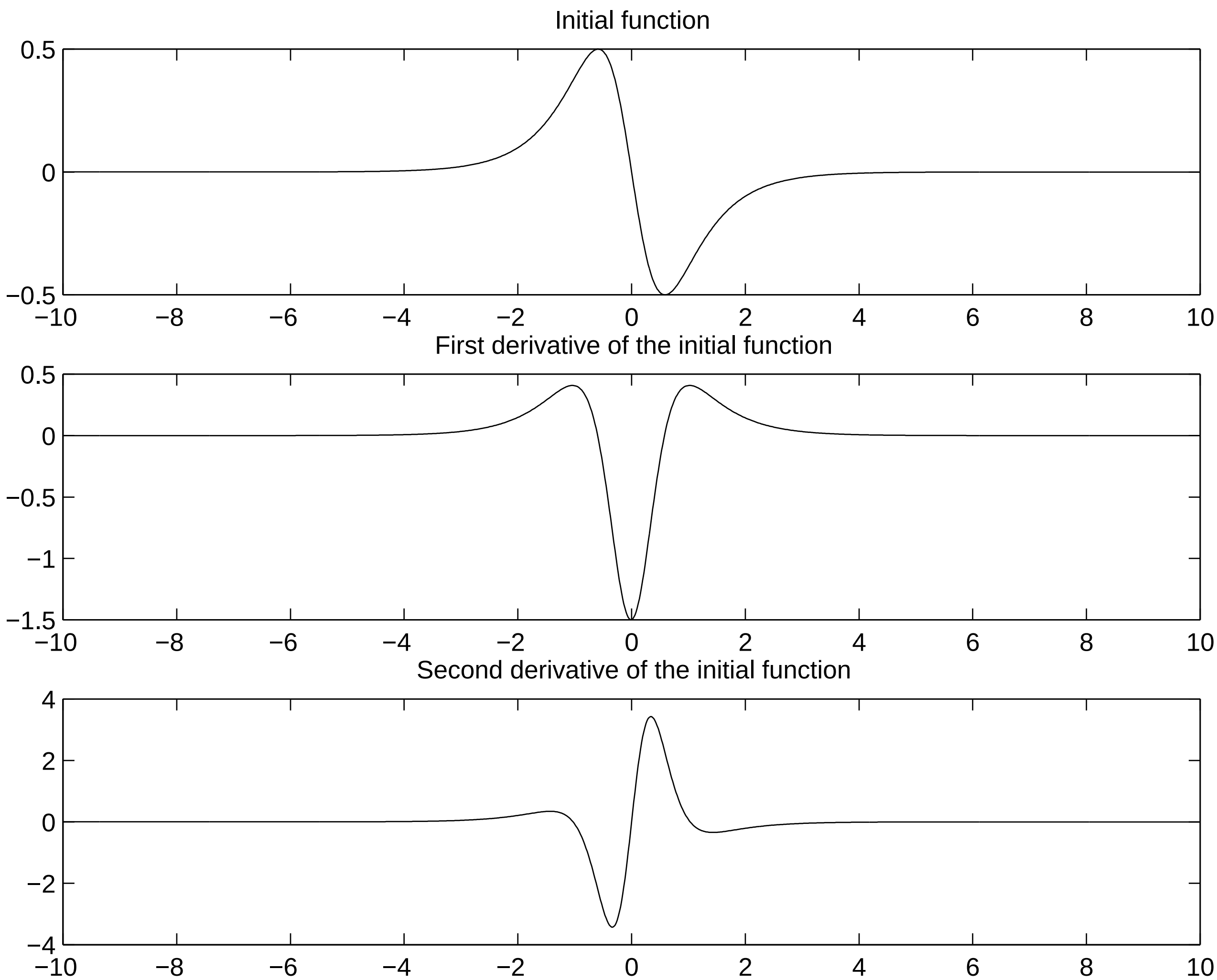
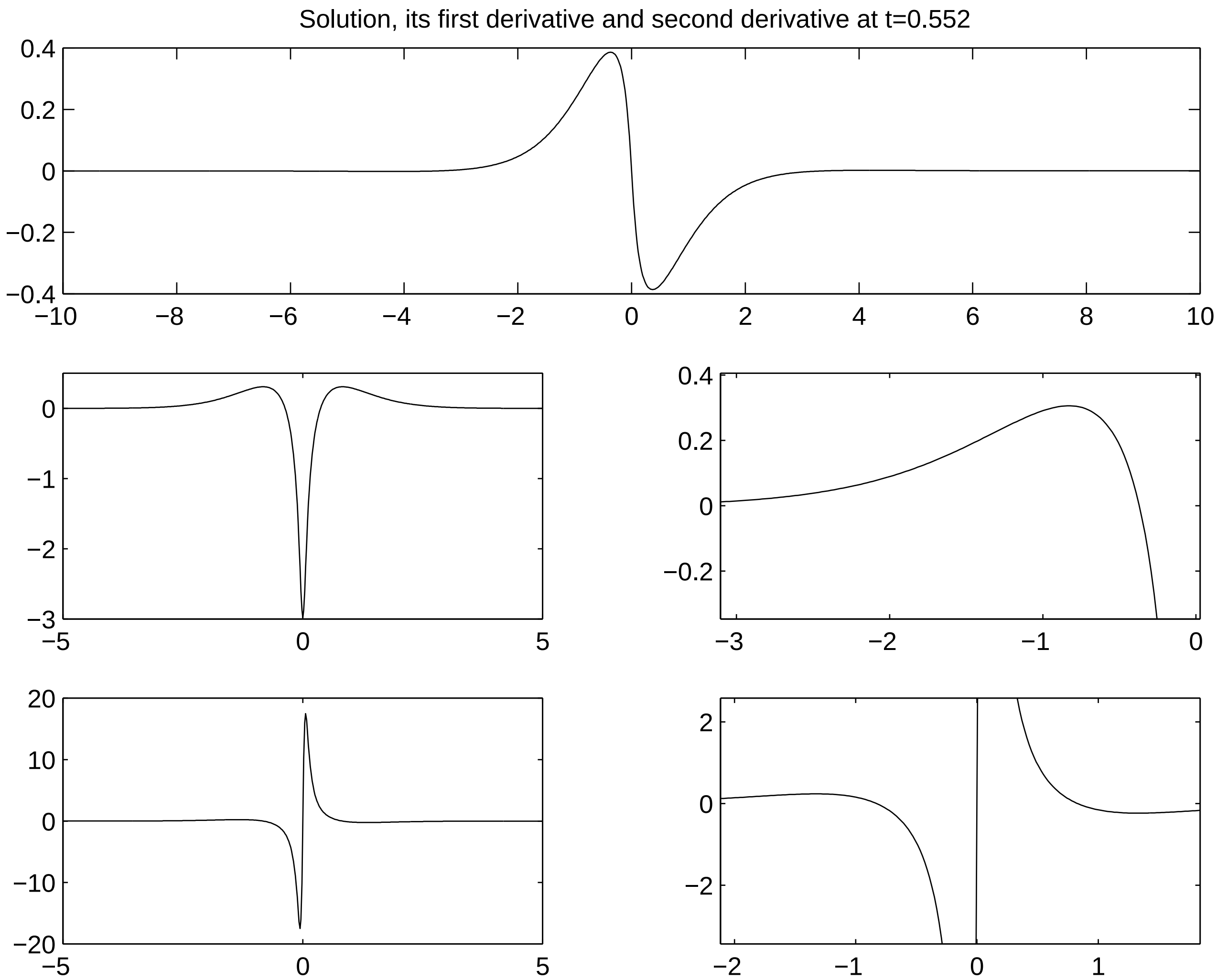

Initial function
0.5
0
−0.5
−10
−8
−6
−4
−2
0
2
4
6
8
10
First derivative of the initial function
0.5
0
−0.5
−1
−1.5
Second derivative of the initial function
4
2
0
−2
−4

Figure #1



Solution, its first derivative and second derivative at t=0.552
0.4
0.2
0
−0.2
−0.4
−10
−8
−6
−4
−2
0
2
4
6
8
10
0
−1
−2
−3
−5
0
5
0.4
0.2
0
−0.2
−3
−2
−1
0
20
10
0
−10
−20
−5
0
5
2
0
−2
−2
−1
0
1

Figure #2

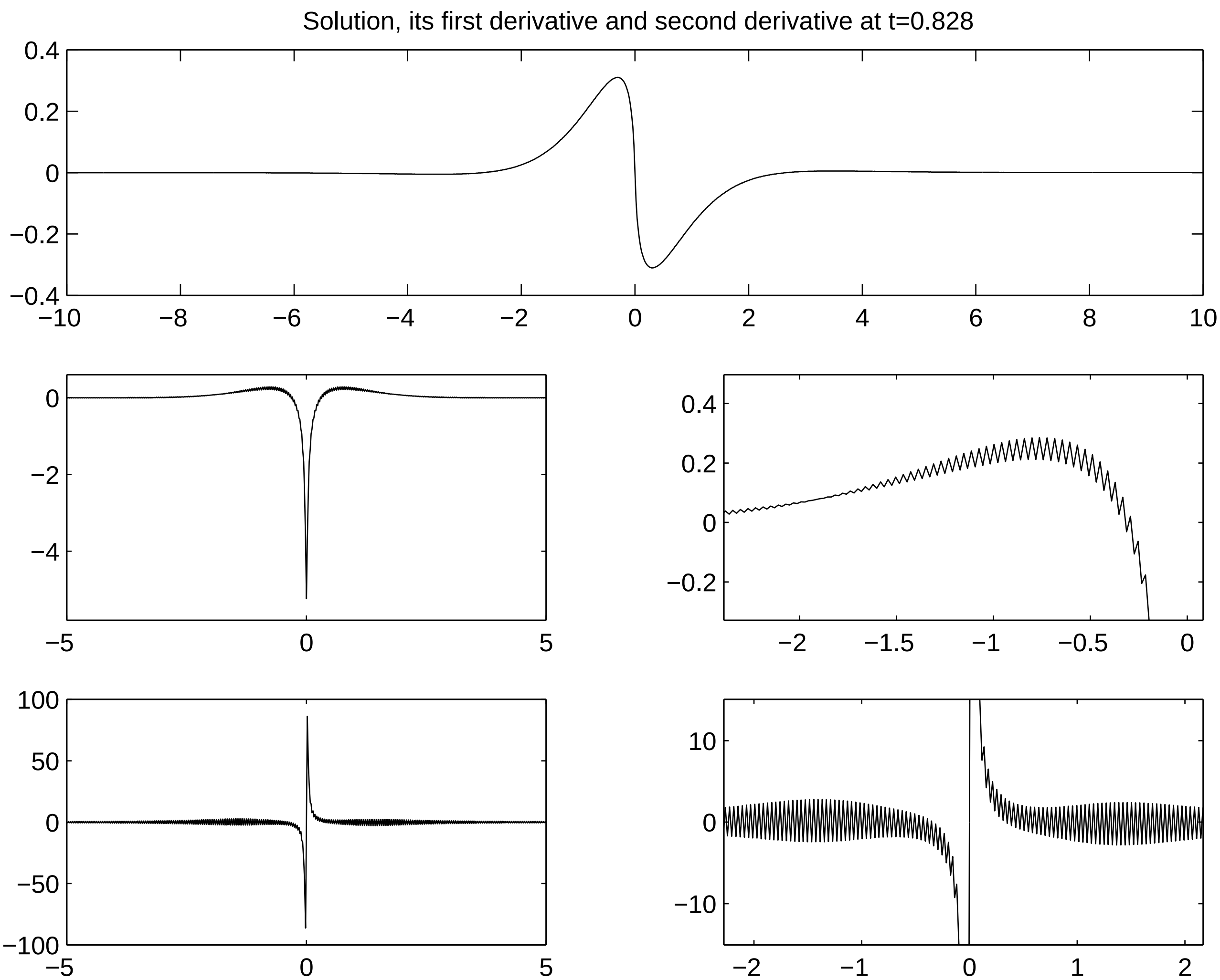

Solution, its first derivative and second derivative at t=0.828

Figure #3