Fox–Neuwirth cells, quantum shuffle algebras, and applications in arithmetic statistics

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Anh Trong Nam Hoang

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Craig Westerland

June, 2024

Acknowledgements

> "Immortality" may be a silly word, but probably a mathematician has the best chance of whatever it may mean.
> – G. H. Hardy, *A Mathematician's Apology.*

I am fairly sure there were numerous reasons that I decided to pursue a career in mathematics, but this quote was certainly one of them. What this journey has taught me, however, was that every piece of my mathematics that may or may not survive with time was evidence of my own intellectual mortality and, more importantly, of how much support and blessing I have received from all of those in my life. Of course, the same holds especially true for this dissertation.

First of all, I am indebted to my advisor, Craig Westerland, for the profound insight, the boundless patience, and the delightful demeanor with which he has guided me during my time in graduate school. Craig, every meeting with you I came in not knowing what I did not know and left with more knowledge and belief in my work than ever before. Thank you, also, for reminding me time and time again that there is so much more in life than math.

I owe many thanks to other faculty at the University of Minnesota that I had the pleasure of interacting with, especially Sasha Voronov, Tyler Lawson, Maru Sarazola, Peter Webb, Michelle Chu, and Vic Reiner. Special thanks to Sasha, Maru, Tyler, and Peter for serving on my committees, and the first two for being wonderful co-organizers of the Topology Seminar. And Dick McGehee, you always have a special place in my career as a mathematician as the one who made all of this possible.

I would definitely enjoy this time a lot less without my delightful fellow graduate students, in particular those in the Topology group: Liam Keenan, David Demark, Eli

Schlossberg, Marshall Smith, Zane Huttinga, Mathieu de Langis, Jacob Hegna, and Toby Ramaswamy. I am especially truly grateful for the constant companionship of Shengkai Mao and Peihang Wu ever since my very first days at Minnesota, and the rest of my cohort for being a part of this memorable time.

Finally, there are no words that can properly convey my gratitude to my parents for giving me all they have and allowing me to follow my heart and passion. I am also truly grateful to my in-laws for, well, handing me their daughter, but also for their support during the time this dissertation was written. To my beautiful wife, Trang, for your love, dedication, and belief in me, I owe you everything I have and will ever achieve. And to my little boy, Huy, you have astonished us in more ways than I could imagine ever since you came into our life, and I cannot wait to teach you some topology (fine, or number theory) someday.

The author was supported by the 2023–2024 University of Minnesota Doctoral Dissertation Fellowship. Parts of this thesis first appeared in the author's published article [55] and are reproduced with permission from Springer Nature.

Dedication

For my wonderful wife and cheeky little boy.

Abstract

In this thesis, we develop a topological framework for studying the twisted homology of certain families of braid subgroups, which arise naturally as the fundamental groups of various configuration spaces. Our construction relies on a new cellular stratification of configuration spaces of the plane with punctures, based on the classical Fox–Neuwirth stratification of configuration spaces of the plane.

Using this new tool, we identify the homology of braid groups on punctured genus-0 surfaces with exponential coefficients arising from braided vector spaces, with the cohomology of certain bimodules over a quantum shuffle algebra. This structural theorem has several consequences. First, we give a complete characterization of the homology of Artin groups of type B with one-dimensional twisted coefficients over a field of characteristic 0. Second, we compute the homology of genus-0 surface braid groups and prove a vanishing range for the homology of mixed braid groups with certain one-dimensional twisted coefficients. Third, we prove an upper bound on the Betti numbers of Hurwitz spaces over punctured curves of genus 0.

Our topological findings have applications in number theory. We give an upper bound on character sums of the resultant over pairs of monic squarefree polynomials of given degrees, answering and generalizing a question of Ellenberg and Shusterman. Finally, we sketch a blueprint for proving the upper bound in a version of the weak Malle's conjecture on the enumeration of finite extensions of $\mathbb{F}_q(t)$ with specified Galois group, bounded discriminant, and prescribed ramification at finitely many primes, refining a result of Ellenberg–Tran–Westerland.

Contents

List of Figures

Chapter 1

Introduction

1.1 Homology of braid groups and generalizations

Braid groups and their generalizations have been a historically rich object of study from a homological standpoint. The cohomology of braid groups with trivial coefficients was established in the 1970s by Arnol'd [4], Fuks [48], F. Cohen [29, 30], and Vainshtein [90]. Callegaro [20] computed their homology with coefficients in $R[q, q^{-1}]$, the ring of Laurent polynomials over a ring R, when $R = \mathbb{Z}$ or R is a generic field. For specific local coefficients, the (co)homology of braid groups with coefficients in the (reduced) Burau representation was determined by De Concini–Procesi–Salvetti [34] and Chen [26], while the case with coefficients in the Tong–Yang–Ma representation was treated by Callegaro–Moroni–Salvetti [23]. For arbitrary polynomial coefficients, Randal-Williams and Wahl [78] showed that the twisted homology of braid groups stabilizes when the number of strands is large relative to the homological degree, a phenomenon commonly known as *homological stability.* Notable examples of polynomial braid representations include the Burau representations, the Tong–Yang–Ma representations [88], and the Lawrence–Krammer–Bigelow representations [63, 62, 15] whose polynomiality was newly established by Palmer and Soulié [77]. More recently, Ellenberg–Tran–Westerland [40] studied the homology of braid groups with exponential coefficients arising from braided vector spaces, and produced explicit computations in some cases and upper bounds in others. Finally, Bergström–Diaconu–Petersen–Westerland [9] computed the stable homology and Miller–Patzt–Petersen–Randal-Williams [74] proved a uniform range of homological stability in the case when the coefficients come from an irreducible representation of symplectic groups, in particular relevant to the integral Burau representation of braid groups.

Braid groups are famously known to be the fundamental groups of *configuration spaces* of the plane. Naturally, a generalization of braid groups in topology arises from fundamental groups of configuration spaces of surfaces, called the *surface braid groups.* Fewer homological computations are known for these groups. The rational homology of surface braid groups on an arbitrary surface of finite type was computed by Drummond-Cole and Knudsen [36], subsuming calculations for specific surfaces by Arnol'd [3], Bödigheimer–Cohen [17], Salvatore [80], Wang [95], and Knudsen [60]. In the case of once-punctured orientable surfaces, the homology of surface braid groups with mod p coefficients was studied using different approaches by Bianchi–Stavrou [14], Brantner–Hahn–Knudsen [19], and Chen–Zhang [25]. For coefficients in polynomial representations such as the An–Ko representations [2], the twisted homology of surface braid groups again exhibits stability by the work of Randal-Williams and Wahl [78].

Another important generalization of braid groups in geometric group theory and combinatorics is the *Artin groups*, a variant of the well-studied *Coxeter groups.* For other Artin and Coxeter groups, again few homological computations are established. The integral cohomology of Artin groups of types B and D was computed by Gorjunov [52], and those of Artin groups associated with exceptional Coxeter groups were determined by Salvetti [82]. De Concini–Procesi–Salvetti–Stumbo [35] calculated the cohomology of all finite-type Artin groups with coefficients in $\mathbb{Q}[q, q^{-1}]$, while the homology of Artin groups of type B with coefficients in $\mathbb{Q}[q^{\pm1}, t^{\pm1}]$ was studied by Callegaro–Moroni–Salvetti [22, 23]. A general approach to the computation of the cohomology of Artin and Coxeter groups was detailed in [81, 82, 83]. More recently, Boyd [18] determined the second and third integral homology of an arbitrary finitely generated Coxeter group, while the second mod 2 homology of an arbitrary Artin group was computed by Akita and Liu [1]. For a review of the (co)homology of braid groups and relatives, as well as other results on generalized braid groups, see [47, 92, 93, 21, 71].

1.2 Main topological results

Recall that a *braided vector space* over a field $\mathbf{k}$ is a finite dimensional vector space V equipped with an automorphism $\sigma : V \otimes V \to V \otimes V$ that satisfies the braid equation $(\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma)$ on $V^{\otimes 3}$. There is a natural representation of the braid group B_n on $V^{\otimes n}$, where the standard generator σ_i of B_n acts by $\mathrm{id}^{\otimes i-1} \otimes \sigma \otimes \mathrm{id}^{\otimes n-i-1}$. Furthermore, we may define a braided, graded Hopf algebra named the *quantum shuffle algebra* $\mathfrak{A}(V)$, whose underlying coalgebra is the cofree coalgebra on V and multiplication is given by a shuffle product involving the braiding σ (Definition 2.2.3). Shuffle algebras have previously been used to express the cellular homology of configuration spaces and consequently the homology of braid groups with certain twisted coefficients [48, 90, 72, 20, 56]. However, very little is known about the quantum shuffle algebra and its cohomology. Some notable characterizations of this algebra can be found in [37, 79, 64].

The topological framework developed in this dissertation is inspired by the work of Ellenberg–Tran–Westerland [40] on the homology of braid groups with coefficients arising from braided vector spaces. In particular, they identified the homology of the braid group B_n with coefficients in $V^{\otimes n}$ with the cohomology of the quantum shuffle algebra $\mathfrak{A} = \mathfrak{A}(V_\epsilon^*)$, where V_ϵ^* is the dual vector space V^* with the braiding dual to that of V and twisted by a sign.

Theorem 1.2.1 (Ellenberg–Tran–Westerland [40])**.** *There is an isomorphism*

$$H_q(B_n; V^{\otimes n}) \cong \mathrm{Ext}_{\mathfrak{A}}^{n-q,n}(\mathbf{k}, \mathbf{k})$$

where the first index in the bigrading on Ext *is the homological degree, and the second the internal degree. Furthermore, the natural multiplication on the braid homology is carried to the Yoneda product on* Ext*; that is,*

$$\bigoplus_{n=0}^{\infty} H_*(B_n; V^{\otimes n}) \cong \bigoplus_n \mathrm{Ext}_{\mathfrak{A}}^{n-*,n}(\mathbf{k}, \mathbf{k})$$

is an isomorphism of bigraded rings.

In Chapters 3–6, we will develop and prove analogs of Theorem 1.2.1 for certain braid subgroups that arise naturally as the fundamental groups of various configuration spaces. Given a topological space X, the n^{th} *(unordered) configuration space* of X, $\mathrm{Conf}_n(X)$, parameterizes sets of n distinct unlabelled points in X. The classical configuration spaces of the plane are famously known to be the classifying spaces of the braid groups. Several variations of configuration spaces will be considered throughout this dissertation, including:

- *Ordered configuration space* $\mathrm{PConf}_n(\mathbb{C})$ parameterizing sets of n distinct labelled points on the plane, whose fundamental group is the *pure braid group* PB_n;

- *Bicolor configuration space* $\mathrm{Conf}_{n,m}(\mathbb{C})$ parameterizing sets of n unlabelled blue points and m unlabelled red points (all distinct) on the plane, whose fundamental group is the *mixed braid group* $B_{n,m}$; and

- *Configuration space of surface* $\mathrm{Conf}_n(\Sigma_{g,m})$ where $\Sigma_{g,m}$ denotes a surface of genus g with m punctures, whose fundamental group is the *surface braid group* $B_n(\Sigma_{g,m})$. In particular, we are concerned with the case when the surface is $\mathbb{C}_m := \mathbb{C} \setminus \{z_1, \dots, z_m\}$, the plane with m punctures for any $m \geq 1$, where there is a choice of inclusion $B_n(\mathbb{C}_m) \subset B_{n,m}$ as subgroups of the braid group B_{n+m}.

We remark that the latter two families of fundamental groups overlap precisely at $B_{n,1} \cong B_n(\mathbb{C}_1)$, which coincidentally is isomorphic to the Artin group of type B_n. For this reason, this case enjoys several nice properties and is often treated separately in this document.

Given a braided vector space V and another (possibly braided) vector space W, under mild assumptions there is a nice representation of the mixed braid group $B_{n,m}$ on $V^{\otimes n} \otimes W^{\otimes m}$, analogous to the representation of B_n on $V^{\otimes n}$. This restricts to a representation of $B_n(\mathbb{C}_m)$ on the same vector space. Our main topological theorem expresses the homology of $B_n(\mathbb{C}_m)$ with coefficients in this representation as the cohomology of certain bimodules over the quantum shuffle algebra $\mathfrak{A}$.

Theorem 1.2.2. *Let $\mathfrak{M}_1 = \mathfrak{M}_2 = \cdots = \mathfrak{M}_m := \mathfrak{M}(V_\epsilon^*, W_\epsilon^*)$, a certain $\mathfrak{A}$-bimodule depending on the pair (V, W) (Definition 5.1.1). Then, there is an isomorphism*

$$H_*(B_n(\mathbb{C}_m); V^{\otimes n} \otimes W^{\otimes m}) \cong \mathrm{Ext}_{\mathfrak{A}^e}^{n-*,n+m}\left(\mathfrak{M}_1 \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_2 \underset{\mathfrak{A}}{\otimes} \dots \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_m, \mathbf{k}\right)$$

where $\mathfrak{A}^{\mathrm{e}} = \mathfrak{A} \otimes \mathfrak{A}^{\mathrm{op}}$ *is the enveloping algebra of* $\mathfrak{A}$.

Bigrading indices on Ext follow the notation in Theorem 1.2.1. This result is proved in Theorem 5.2.1 for the special case $m = 1$, where there are less restrictions on the pair (V, W), and Theorem 6.3.4 in full generality.

More generally, in Sections 5.4 and 6.1, we develop a framework for studying the homology of $B_n(\mathbb{C}_m)$ with arbitrary coefficients. Our construction relies on a new cellular stratification of $\mathrm{Conf}_n(\mathbb{C}_m)$, based on the classical Fox–Neuwirth stratification of configuration spaces of the plane introduced by Fox–Neuwirth [46] and Fuks [48]. We are also inspired by the work of Ellenberg–Tran–Westerland [40] who introduced a framework for computing the twisted homology of configuration spaces of the plane with arbitrary coefficients. We remark that for the special case $m = 1$, we give an alternate short algebraic proof for Theorem 1.2.2 by utilizing their framework and the theory of induced representation of braid subgroups. For the general case, our argument involves a delicate interplay between the geometric approach using explicit cellular models and the algebraic approach using induced representations.

Theorem 1.2.2 has several computational consequences. In Section 5.3, by specializing V and W to be one-dimensional vector spaces over a field of characteristic 0, we give a complete characterization of the homology of Artin groups of type B with one-dimensional twisted coefficients (Theorem 5.3.9). In Section 6.4, with the same specialization, we compute the homology of genus-0 surface braid groups (Theorems 6.4.1 and 6.4.3) and prove a vanishing range for the homology of mixed braid groups (Corollaries 6.4.2 and 6.4.4) with certain one-dimensional twisted coefficients relevant to the arithmetic application in Chapter 7. Finally, in Chapter 8, we prove an upper bound on the Betti numbers of certain finite covers of $\mathrm{Conf}_n(\mathbb{C}_m)$ and discuss an application in arithmetic statistics.

The translations between homological study of spaces and number theory, especially arithmetic statistics, typically employ the Grothendieck–Lefschetz trace formula (possibly with twisted coefficients), Weil conjectures, and various comparison theorems in étale cohomology theory (see, e.g., [27, 45]). Notable examples that involve configuration spaces include work of Ellenberg–Venkatesh–Westerland [42] on the Cohen–Lenstra conjecture over function fields, Liu–Wood–Zureick-Brown [66] on the distribution of Galois groups of maximal unramified extensions of Γ-extensions of $\mathbb{F}_q(t)$, Ellenberg–Landesman

[39] on the distribution of Selmer groups of quadratic twist families of abelian varieties over function fields, and Bergström–Diaconu–Petersen–Westerland [9] and Miller–Patzt–Petersen–Randal-Williams [74] on the asymptotics of moments of quadratic L-functions.

1.3 Character sums of the resultant

A beautiful classical fact in number theory is that two monic polynomials f and g in one variable of positive degrees over a field $\mathbf{k}$ share a common root if and only if an integer-coefficient polynomial expression of their coefficients, named the *resultant* $\mathcal{R}(f,g)$, vanishes. If

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

then $\mathcal{R}(f,g)$ can be computed by taking the determinant of the polynomials' Sylvester matrix:

$$\mathcal{R}(f,g) = \begin{vmatrix} a_0 & a_1 & \dots & a_{n-1} & 1 & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{n-1} & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & a_0 & a_1 & \dots & a_{n-1} & 1 \\ b_0 & b_1 & \dots & b_{m-1} & 1 & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_{m-1} & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & b_0 & b_1 & \dots & b_{m-1} & 1 \end{vmatrix}.$$

Alternatively, the resultant is given by the product formula

$$\mathcal{R}(f,g) = \prod_{i,j}(x_i - y_j)$$

where $x_1, \dots, x_n$ are roots of f and $y_1, \dots, y_m$ are roots of g in any algebraically closed extension of $\mathbf{k}$.

This classical concept was first introduced by Cayley [24] in the context of elimination theory and has since found abundant applications in generalizing the notion of discriminants, solving systems of polynomial equations, and proving important theorems such as Bezout's Theorem and Hilbert's Nullstellensatz. A beautiful survey on the general theory of resultants from this perspective can be found in [49]. Over a field $\mathbf{k}$, the resultant can be interpreted as a map $\mathcal{R} : \mathbb{A}^n_{\mathbf{k}} \times \mathbb{A}^m_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$, where the affine spaces $\mathbb{A}^n_{\mathbf{k}}$ and $\mathbb{A}^m_{\mathbf{k}}$ may be identified with the spaces of monic degree-n and degree-m polynomials over $\mathbf{k}$. The resultant locus $\mathbb{A}^n_{\mathbf{k}} \times \mathbb{A}^m_{\mathbf{k}} \setminus \mathcal{R}^{-1}(0)$, namely the space of pairs of monic coprime polynomials of degrees n and m, is a classically studied object, whereas the topology and arithmetic of the hypersurface $\mathcal{R}^{-1}(1)$ were recently studied by Farb and Wolfson [44] when $n = m$.

Fix a finite field $\mathbb{F}_q$, a prime ℓ invertible in $\mathbb{F}_q$, and a choice of inclusion $\overline{\mathbb{Q}}_\ell \subseteq \mathbb{C}$. Let $\chi : \mathbb{F}_q \to \mathbb{C}$ be a nontrivial character. In Chapter 7, we are concerned with the *character sum*

$$F_\chi(n, m, q) := \sum_{f,g} \chi(\mathcal{R}(f, g))$$

where f and g range over monic *squarefree* polynomials of degrees n and m over $\mathbb{F}_q$. This sum was first considered by Ellenberg and Shusterman [38] in the case when χ is the quadratic character. Characters of the resultant over finite fields are analogs of residue symbols for polynomials (e.g., the Jacobi symbol when the associated character is quadratic) and therefore play a central role in recent work on quadratic reciprocity [28] as well as Dirichlet series and Möbius functions over function fields [84, 85, 86].

Our main result is an upper bound on the character sum $F_\chi(n, m, q)$ as a function of q:

Theorem 1.3.1. *Let χ be a nontrivial character of $\mathbb{F}_q$. Then*

$$|F_\chi(n, m, q)| \le 2^{2n+2m-1} \frac{q^{n+m+1-\max(n,m)/2} - 1}{\sqrt{q} - 1}$$

if χ is quadratic, and

$$|F_\chi(n, m, q)| \le 2^{2n+2m-1} \frac{q^{n+m+(1-\max(n,m))/2} - 1}{\sqrt{q} - 1}$$

otherwise.

This result is proved in Theorem 7.0.1. As a consequence, we make the following observation about the asymptotic behavior of character averages of the resultant (Corollary 7.0.2).

Corollary 1.3.2. *For a sufficiently large q, the asymptotic average of a nontrivial character of the resultant over pairs of monic squarefree polynomials over $\mathbb{F}_q$ approaches* 0 *as the degree of either or both polynomials grows indefinitely.*

We translate this problem to topology. Let $\mathrm{Conf}_{n,m}$ denote the space of pairs of monic squarefree *coprime* polynomials of degrees n and m. Observe that the sum $F_\chi(n,m,q)$ is the same as

$$F_\chi(n,m,q) = \sum_{(f,g)\in \mathrm{Conf}_{n,m}(\mathbb{F}_q)} \chi(\mathcal{R}(f,g))$$

since $\mathcal{R}(f,g) = 0$ whenever f and g share a common root. By pulling back the Kummer sheaf $\mathcal{L}_\chi$ associated to the multiplicative character χ along the resultant map $\mathcal{R} : \mathrm{Conf}_{n,m} \to \mathbb{A}^1 \setminus \{0\}$, we obtain a rank-1 local system $\mathcal{R}^*\mathcal{L}_\chi$ on $\mathrm{Conf}_{n,m}$ with the property that the trace of Frobenius acting on the stalk at $(f,g) \in \mathrm{Conf}_{n,m}(\mathbb{F}_q)$ equals $\chi(\mathcal{R}(f,g))$. By the standard machinery of arithmetic topology described above, $F_\chi(n,m,q)$ can then be approached by studying the homology of the bicolor configuration space $\mathrm{Conf}_{n,m}(\mathbb{C})$ with coefficients in $\mathcal{R}^*\mathcal{L}_\chi$. Theorem 1.3.1 then follows from our homological computations in Corollaries 6.4.2 and 6.4.4.

1.4 Malle's conjecture for function fields

Another fundamental problem in number theory is the enumeration of global fields, which include number fields (finite algebraic extensions of $\mathbb{Q}$) and function fields (finite algebraic extensions of $\mathbb{F}_q(t)$). Given a global field K and a finite extension L/K of degree d, the two basic invariants of L/K are the *Galois group* $\mathrm{Gal}(L/K)$, a transitive subgroup of S_d, and the *discriminant* $\Delta(L/K)$, a real number measuring the extent to which L/K is ramified. Given $G \subseteq S_d$, define $a(G) := [\min_{g\in G\setminus\{1\}} \mathrm{ind}(g)]^{-1}$ where $\mathrm{ind}(g) = d - \#(\{1,\ldots,d\}/\langle g\rangle)$ is the *index* of an element $g \in S_d$.

Conjecture 1.4.1 (Malle [68, 69]). *Let $N_G(K,X)$ be the number of isomorphism classes of degree-d extensions L of a global field K with Galois group G and discriminant $|\Delta(L/K)| < X$.*

1. *(Weak) For $\epsilon > 0$, there exist positive constants $c_1(K,G)$ and $c_2(K,G,\epsilon)$ such that*

$$c_1 X^{a(G)} < N_G(K,X) < c_2 X^{a(G)+\epsilon};$$

2. *(Strong) There exists a positive constant $c(K,G)$ such that*

$$N_G(K,X) \sim cX^{a(G)}(\log X)^{b-1}$$

for a specific constant $b(K,G)$.

The strong Malle's conjecture was shown to hold when G is abelian by Wright [98], when $G = S_3$ by Davenport–Heilbronn [33] and Datskovsky–Wright [32], and when $G = S_4, S_5$ by Bhargava and collaborators [10, 11, 12]. Klüners [58] gave a counterexample to Malle's conjectured formula for $b(K,G)$, and subsequently Türkelli [89] proposed a slight modification of Conjecture 1.4.1.2 which accommodates this result. In the number field setting, there is recent work of Klüners [59] and Koymans–Pagano [61] when G is nilpotent, and of Wang and collaborators [94, 73] when G is a direct product of a small finite group with an abelian group. Switching to the function field setting, Ellenberg–Venkatesh–Westerland [42] proved Conjecture 1.4.1.2 when $K = \mathbb{F}_q(t)$ and G is a generalized dihedral group of order prime to q. For general G, beyond bounds when $G = S_d$ which are far from Malle's prediction [41, 13, 65], very little was known until recent work of Ellenberg–Tran–Westerland [40] which established the upper bound in the weak Malle's conjecture over $\mathbb{F}_q(t)$ for all choices of the Galois group G and all sufficiently large q. In fact, they proved the upper bound in a more general version of Conjecture 1.4.1.1 where the local monodromy at the ramified places of $\mathbb{F}_q(t)$ is contained in a specified conjugacy invariant subset c of G. For any such subset c, let $\mathrm{ind}(c) = \min_{g\in c}\mathrm{ind}(g)$ and $a(G,c) := \mathrm{ind}(c)^{-1}$.

Theorem 1.4.2 (Ellenberg–Tran–Westerland [40]). *Let $N^c_G(\mathbb{F}_q(t), X)$ be the number of isomorphism classes of degree-d extensions L of $\mathbb{F}_q(t)$ with Galois group G, discriminant*

$|\Delta(L/\mathbb{F}_q(t))| < X$, and all local monodromy elements contained in a conjugacy invariant subset c of G. Then, there exist constants $e(G,c)$, $C(G,c)$, and $Q(G,c)$ such that for all $q > Q$ coprime to $|G|$ and all $X > 0$,

$$N_G^c(\mathbb{F}_q(t), X) \leq CX^{a(G,c)}(\log X)^{e-1}.$$

When $c = G \setminus \{1\}$, Theorem 1.4.2 implies the upper bound in Conjecture 1.4.1.1 for $K = \mathbb{F}_q(t)$. It was noted in [40] that the lower bound can be obtained in a few cases without significant improvements of their methods, and in general the constant $e(G,c)$ differs from the constant $b(K,G)$ in Conjecture 1.4.1.2 (see Remark 7.17 of [40]).

Their proof of Theorem 1.4.2 relies on a homological computation of configuration spaces. Since $\mathbb{F}_q(t)$ is the function field of the affine line $\mathbb{A}^1_{\mathbb{F}_q}$ (or the projective line $\mathbb{P}^1_{\mathbb{F}_q}$), a degree-$d$ extension L of $\mathbb{F}_q(t)$ is the function field of a degree-d branched cover $f : \mathrm{Spec}\mathcal{O}_L \to \mathrm{Spec}\mathbb{F}_q[t] = \mathbb{A}^1$. Thus, to count G-extensions of $\mathbb{F}_q(t)$ with bounded discriminant and local monodromy in c, it suffices to count the number of $\mathbb{F}_q$-rational points on *Hurwitz moduli stacks* $\mathrm{Hn}^c_{G,n}$ parameterizing branched G-covers of $\mathbb{A}^1$ with the number n of branched points bounded and local monodromy in c. Via the standard machinery of arithmetic topology, this problem can be approached by studying the homology of the complex points of $\mathrm{Hn}^c_{G,n}$. These homology groups can be approximated by the homology of the *topological Hurwitz spaces* $\mathrm{Hur}^c_{G,n}$ (cf. Ellenberg–Venkatesh–Westerland [42]) defined as follows. Observe that there is an action of the braid group B_n on the finite set $c^{\times n}$, called the *Hurwitz action*, such that the standard generator of B_n acts by

$$\sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \ldots, g_n).$$

Then, $\mathrm{Hur}^c_{G,n}$ is the space

$$\mathrm{Hur}^c_{G,n} := \widetilde{\mathrm{Conf}}_n(\mathbb{C}) \times_{B_n} c^{\times n}$$

where $\widetilde{\mathrm{Conf}}_n(\mathbb{C})$ is the universal cover of $\mathrm{Conf}_n(\mathbb{C})$ and the action of the braid group is diagonal. It follows that

$$H_*(\mathrm{Hur}^c_{G,n}(\mathbb{C});\mathbf{k}) \cong H_*(B_n;V^{\otimes n})$$

where $V = \mathbf{k}c$ forms a braided vector space with the braiding given by the Hurwitz action and extended linearly. Theorem 1.2.1 coupled with the following upper bound on the cohomology of the quantum shuffle algebra $\mathfrak{A}$ then provides the necessary topological input for proving Theorem 1.4.2.

Theorem 1.4.3 (Ellenberg–Tran–Westerland [40])**.** *For some constants $B(G,c)$, $C(G,c)$, and the constant $e(G,c)$ appearing in Theorem 1.4.2,*

$$\mathrm{rk}\ \mathrm{Ext}^{n-j,n}_{\mathfrak{A}}(\mathbf{k},\mathbf{k}) \leq Cn^{e-1}B^j.$$

The constant $e(G,c)$ in Theorem 1.4.2 is the Gelfand–Kirillov dimension of the ring R of components of the Hurwitz spaces $\mathrm{Hur}^c_{G,n}$:

$$R = \bigoplus_{n=0}^{\infty} H_0(\mathrm{Hur}^c_{G,n};\mathbf{k}) \cong \bigoplus_{n=0}^{\infty} H_0(B_n;V^{\otimes n}) \cong \bigoplus_{n=0}^{\infty} \mathrm{Ext}^{n,n}_{\mathfrak{A}}(\mathbf{k},\mathbf{k}).$$

It can be described combinatorially: e is the maximum of the number of H-conjugacy classes making up $c \cap H$ over all subgroups $H < G$. In particular, $e = 1$ if and only if c consists of a single conjugacy class and (G,c) satisfies the non-splitting condition of [42] (see Remark 7.17 of [40]). Typically, $e \ll |G|$.

In Chapter 8, we sketch a proof of a refined version of Theorem 1.4.2 with additional specification of the ramification at finitely many primes. The enumeration of global fields with specified local conditions is the subject of the Malle–Bhargava principle (see, e.g., [97]). This heuristic predicts, for example, that specifying the ramification at a finite number of ramified primes does not change the order of growth of the number of G-extensions of a global field K with bounded discriminant, provided that there is at least one G-extension of K with this local behavior (see Remark 1.2 of [69]).

Fix a finite set $\mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_k)$ of primes in $\mathbb{F}_q[t]$. For every $1 \leq i \leq k$, let m_i be the degree of $\mathfrak{p}_i$, and let $m := \sum_{i=1}^k m_i$. Fix a finite group $G \subseteq S_d$ and a finite set $\mathbf{c} = (c_0, c_1, \ldots, c_k)$ of conjugacy invariant subsets of G, i.e., unions of conjugacy classes in G. The main result of Chapter 8 is the following:

Theorem 1.4.4. *Let $N_G^{\mathfrak{p},\mathbf{c}}(\mathbb{F}_q(t), X)$ be the number of isomorphism classes of degree-d extensions L of $\mathbb{F}_q(t)$ with Galois group G, discriminant $|\Delta(L/\mathbb{F}_q(t))| < X$, local monodromy at the prime $\mathfrak{p}_i$ contained in c_i for every $1 \leq i \leq k$, and all other local monodromy elements contained in c_0. Then, there exist constants $C(G, \mathfrak{p}, \mathbf{c})$ and $Q(G, c_0)$ such that for all $q > Q$ coprime to $|G|$ and all $X > 0$,*

$$N_G^{\mathfrak{p},\mathbf{c}}(\mathbb{F}_q(t), X) \leq CX^{a(G,c_0)}(\log X)^{e(G,c_0)+m-1}$$

where $a(G, c_0)$ is the constant predicted by Malle's conjecture and $e(G, c_0)$ is the constant appearing in Theorem 1.4.2.

This statement recovers the power term in Theorem 1.4.2 when $c_i \subseteq c_0$ for all $1 \leq i \leq k$. More importantly, when $c_0 \subsetneq \cup_i c_i =: c$, oftentimes $a(G, c_0) < a(G, c)$. Thus, Theorem 1.4.4 provides a finer upper bound than Theorem 1.4.2 when, for example, most of the local monodromy elements except at a finite number of ramified primes belong to a conjugacy invariant subset c_0 of large index.

As before, our proof strategy involves reducing the counting problem to a homological computation of configuration spaces with twisted coefficients. Theorem 1.4.4 then follows from an upper bound on the Betti numbers of a variant of Hurwitz spaces over punctured curves of genus 0, denoted by $\mathrm{Hur}_{G,n}^{\mathfrak{p},\mathbf{c}}$.

Theorem 1.4.5. *For some constants $C(G, \mathfrak{p}, \mathbf{c})$, $B(G, c_0)$, and the constant $e(G, c_0)$ appearing in Theorem 1.4.2,*

$$\mathrm{rk}\, H_j(\mathrm{Hur}_{G,n}^{\mathfrak{p},\mathbf{c}}; \mathbf{k}) \leq Cn^{e+m-1}B^j.$$

This statement is proved in Theorem 8.0.2. The homology of $\mathrm{Hur}_{G,n}^{\mathfrak{p},\mathbf{c}}$ is precisely the homology of $\mathrm{Conf}_n(\mathbb{C}_m)$ with coefficients arising from the braided vector spaces $V_i = \mathbf{k}c_i$. Theorem 1.4.5 then results from a variant of Theorem 1.2.2 (Proposition 8.0.3) coupled with the upper bound on the cohomology of $\mathfrak{A}$ in Theorem 1.4.3.

Chapter 2

Preliminaries

In this chapter, we will give a brief review of concepts in topology and quantum algebra that are central to the discussion in this dissertation, such as configuration spaces, braid groups, braided vector spaces, and quantum shuffle algebras. The content of this chapter is extracted from Sections 3.1 of [54], 2.1 of [55], and 2.1 and 2.5 of [40].

2.1 Configuration spaces and fundamental groups

The central subject of study in this dissertation is *configuration spaces* and their variations. Let X be a topological space.

Definition 2.1.1. The n^{th} *(unordered) configuration space* of X is the space of unordered sets of n points in X, i.e.,

$$\mathrm{Conf}_n(X) := \{\{x_1, \ldots, x_n\} \subset X : x_i \neq x_j \text{ if } i \neq j\}.$$

Another important variation of configuration spaces is the following:

Definition 2.1.2. The n^{th} *ordered configuration space* of X is the space of n-tuples of points in X, i.e.,

$$\mathrm{PConf}_n(X) := \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}.$$

There is a natural action of the symmetric group S_n on $\mathrm{PConf}_n(X)$ by permuting the

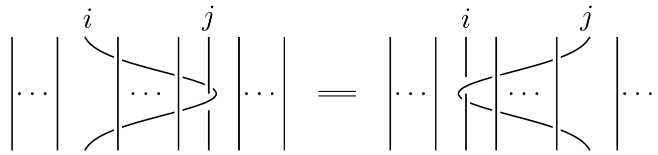

Figure 2.1: Pure braid generator θ_{ij}.

coordinates. The quotient space $\mathrm{PConf}_n(X)/S_n$ can be identified with the unordered configuration space $\mathrm{Conf}_n(X)$. The topology on $\mathrm{PConf}_n(X) \subset X^n$ is the subspace topology, whereas $\mathrm{Conf}_n(X)$ is equipped with the quotient topology.

The fundamental group of $\mathrm{Conf}_n(X)$ is called the *braid group on n strands* on X, denoted by $B_n(X)$. When $X = \mathbb{C}$, $\mathrm{Conf}_n(\mathbb{C})$ and $\mathrm{PConf}_n(\mathbb{C})$ are the well-studied unordered and ordered configuration spaces of the plane. From an algebro-geometric point of view, $\mathrm{Conf}_n(\mathbb{C})$ can also be interpreted as the complex points of the space Conf_n of monic squarefree polynomials, via the identification which sends such a polynomial to its complex roots. The fundamental group of $\mathrm{PConf}_n(\mathbb{C})$ is the *pure braid group on n strands PB_n*, whereas the braid group on $\mathbb{C}$ is precisely the classical *Artin's braid group* B_n (cf. [5, 6]), which may be presented as

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle.$$

There is an exact sequence of groups

$$1 \to PB_n \to B_n \to S_n \to 1,$$

where the map $B_n \to S_n$ sends a braid b to its underlying permutation $\underline{b}$ [46]. The pure braid group PB_n can be presented as a subgroup of B_n generated by the elements

$$\theta_{ij} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1}\sigma_{j-1}^2\sigma_{j-2} \cdots \sigma_i$$

for $1 \le i < j \le n$ [16][1]; geometrically, the generator θ_{ij} is represented by the braid that wraps the i^{th} strand around the j^{th} strand (see Figure 2.1).

[1]Strictly speaking, our formula of θ_{ij} represents a mirror image of the pure braid generators given by [16]. This alternate choice of generators proves to be more compatible with our topological constructions in this paper.

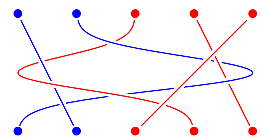

Figure 2.2: A $(2, 3)$-mixed braid.

Observe that there is an action of the group $S_n \times S_m$ as a subgroup of S_{n+m} on $\mathrm{PConf}_{n+m}(\mathbb{C})$. The quotient space $\mathrm{PConf}_{n+m}(\mathbb{C})/(S_n \times S_m)$ can be identified with the following space:

Definition 2.1.3. The (n, m)-*bicolor configuration space* $\mathrm{Conf}_{n,m}(\mathbb{C})$ is the space of all configurations of n blue points and m red points, all distinct, in $\mathbb{C}$. That is,

$$\mathrm{Conf}_{n,m}(\mathbb{C}) := \{\{b_1, \ldots, b_n, r_1, \ldots, r_m\} : b_i \neq b_j, r_i \neq r_j \text{ if } i \neq j; b_i \neq r_j \text{ for all } i, j\}.$$

Its fundamental group is the (n, m)-*mixed braid group* $B_{n,m}$, the preimage of $S_n \times S_m$ under the projection $B_{n+m} \to S_{n+m}$. Equivalently, $B_{n,m}$ may be identified with the subgroup of B_{n+m} consisting of braids that preserve the partition (n, m) on the endpoints. By convention, we color the strands starting from the first n endpoints blue and the others red (see Figure 2.2). Manfredini [70] gave a presentation of this group, in terms of the braid generators $\{\sigma_i\}_{i\neq n}$ of B_{n+m} and $\tau_n = \sigma_n^2$, with the relations

$$\begin{aligned}
&\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1;\\
&\sigma_i\tau_n = \tau_n\sigma_i \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } i < n-1 \text{ or } i > n;\\
&\sigma_i\tau_n\sigma_i\tau_n = \tau_n\sigma_i\tau_n\sigma_i \text{ if } i = n-1, n+1; \text{ and}\\
&\sigma_{n-1}\tau_n\sigma_{n-1}^{-1}\sigma_{n+1}\tau_n\sigma_{n+1}^{-1} = \sigma_{n+1}\tau_n\sigma_{n+1}^{-1}\sigma_{n-1}\tau_n\sigma_{n-1}^{-1}.
\end{aligned}$$

In particular, the group $B_{n,1}$ has a shorter list of relations:

$$\begin{aligned}
&\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1;\\
&\sigma_i\tau_n = \tau_n\sigma_i \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } i < n-1; \text{ and}\\
&\sigma_i\tau_n\sigma_i\tau_n = \tau_n\sigma_i\tau_n\sigma_i \text{ if } i = n-1.
\end{aligned}$$

Pictorially, the generators $\sigma_{i<n}$, $\sigma_{i>n}$, and τ_n are represented by the crossings of two blue strands, two red strands, and the full twist of the n^{th} strand (blue) and the $n+1^{\text{st}}$

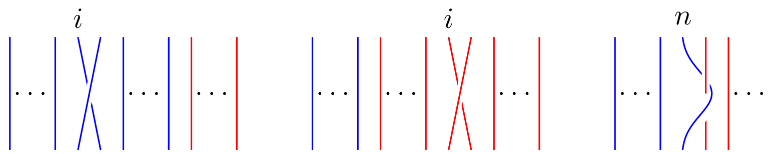

Figure 2.3: Mixed braid generators $\sigma_{i<n}$, $\sigma_{i>n}$, and τ_n.

strand (red), respectively (see Figure 2.3). For simplicity, in this document we will refer to these types of generators as the *blue*, *red*, and *mixed* generators, respectively.

Fadell and Neuwirth [43] showed that there is a fiber sequence

$$\mathrm{Conf}_n(\mathbb{C}_m) \to \mathrm{Conf}_{n,m}(\mathbb{C}) \to \mathrm{Conf}_m(\mathbb{C})$$

that results in an exact sequence of fundamental groups

$$1 \to B_n(\mathbb{C}_m) \to B_{n,m} \to B_m \to 1$$

where $B_n(\mathbb{C}_m)$ is the *surface braid group on n strands* on $\mathbb{C}_m \cong \Sigma_{0,m+1}$, a surface of genus 0 with $m+1$ punctures. This group is isomorphic to the subgroup of B_{n+m} consisting of braids whose last m strands are straight, or equivalently, (n,m)-mixed braids with only straight red strands. For $m = 1$, $B_n(\mathbb{C}^\times)$ is isomorphic to the $(n,1)$-mixed braid group $B_{n,1}$, due to the fact that it is always possible to "straighten" the last pure strand. In general, as evidenced by the exact sequence above, $B_n(\mathbb{C}_m)$ is a subgroup of $B_{n,m}$ of infinite index for $m \geq 2$. For this reason, the case of $B_{n,1}$ will be treated separately throughout our discussion.

Bellingeri and Godelle [8] gave a positive presentation of all surface braid groups; in particular, $B_n(\mathbb{C}_m)$ can be generated by the generators $\sigma_1, \ldots, \sigma_{n-1}$ of B_{n+m}, and the generators θ_{nj} for $n+1 \leq j \leq n+m$ of the pure braid group PB_{n+m}, subject to the relations

$\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$;
$\sigma_i\theta_{nj} = \theta_{nj}\sigma_i$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i < n-1$ and $n+1 \leq j \leq n+m$;
$\sigma_{n-1}\theta_{nj}\sigma_{n-1}\theta_{nj} = \theta_{nj}\sigma_{n-1}\theta_{nj}\sigma_{n-1}$ for $n+1 \leq j \leq n+m$; and
$\sigma_{n-1}\theta_{nj}\theta_{nk}\sigma_{n-1}\theta_{nk} = \theta_{nk}\sigma_{n-1}\theta_{nj}\theta_{nk}\sigma_{n-1}$ for $n+1 \leq j < k \leq n+m$.

We may rewrite the pure braid generator $\theta_{ij} = \sigma_i^{-1} \dots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \dots \sigma_i$ as $\theta_{ij} = \sigma_{j-1} \dots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1}$ (see Figure 2.1). It follows that there is a natural embedding of $B_n(\mathbb{C}_m)$ into $B_{n,m}$ that sends the generators σ_i to the corresponding σ_i in $B_{n,m}$ for $1 \le i \le n-1$, and θ_{nj} to $\sigma_{j-1} \dots \sigma_{n+1} \tau_n \sigma_{n+1}^{-1} \dots \sigma_{j-1}^{-1}$ for $n+1 \le j \le n+m$.

Finally, we remark that the groups B_n and $B_{n,1}$ also arise in group theory and combinatorics as the *Artin groups* of types A_{n-1} and B_n. Therefore, the subject of study in this paper could be of independent interests to researchers working in these fields.

2.2 Braided vector spaces and quantum shuffle algebras

Let $\mathbf{k}$ be a field; unless otherwise noted, all tensor products will be over $\mathbf{k}$.

Recall that the groupoid $\mathscr{B} = \sqcup_{n\ge 0}[*/B_n]$ of all braid groups has the structure of a braided monoidal category. The family of strictly monoidal functors $\Phi : \mathscr{B} \to \mathrm{FinVect}_{\mathbf{k}}$ forms the category of *monoidal braid representations*, where $\mathrm{FinVect}_{\mathbf{k}}$ denotes the category of finite dimensional $\mathbf{k}$-vector spaces.

Definition 2.2.1. A *braided vector space* V over $\mathbf{k}$ is a finite dimensional $\mathbf{k}$-vector space equipped with an invertible *braiding* $\sigma : V \otimes V \to V \otimes V$ such that it satisfies the braid equation on $V^{\otimes 3}$:

$$(\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma).$$

Braided vector spaces form a category where morphisms $(V_1, \sigma_1) \to (V_2, \sigma_2)$ are $\mathbf{k}$-linear maps $f : V_1 \to V_2$ that satisfy $(f \otimes f) \circ \sigma_1 = \sigma_2 \circ (f \otimes f)$. There is a natural action of B_n on $V^{\otimes n}$ defined by $\sigma_i \mapsto \mathrm{id}^{\otimes i-1} \otimes \sigma \otimes \mathrm{id}^{\otimes n-i-1}$.

Proposition 2.2.2 (Ellenberg–Tran–Westerland [40])**.** *There is a pair of inverse equivalences between the categories of monoidal braid representations and braided vector spaces that send Φ to $\Phi(1)$ and V to the braid representation on $V^{\otimes n}$ discussed above.*

Recall that an (n,m)-*shuffle* $\gamma : \{1, \dots, n\} \sqcup \{1, \dots, m\} \to \{1, \dots, n+m\}$ is a bijection that preserves the order on both $\{1, \dots, n\}$ and $\{1, \dots, m\}$. Alternatively, an (n,m)-shuffle is a permutation of S_{n+m} that preserves the order on the first n and the last

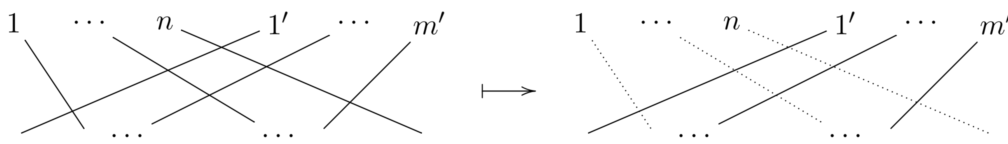

Figure 2.4: Lifting an (n, m)-shuffle to a braid.

m elements. Let $\mathrm{Sh}(n, m)$ denote the set of all (n, m)-shuffles. Given any shuffle γ, there is a choice of a lift $\widetilde{\gamma} \in B_{n+m}$ given by the braid that shuffles the endpoints according to γ by moving the right m strands in front of the left n strands (see Figure 2.4).

Let (V, σ) be a braided vector space. We will write elements of $V^{\otimes n}$ using bar complex notation, i.e., $[a_1|\ldots|a_n]$.

Definition 2.2.3. The *quantum shuffle algebra* $\mathfrak{A}(V)$ is a braided, graded bialgebra: its underlying coalgebra is the tensor coalgebra

$$T^{\mathrm{co}}(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

with the deconcatenation coproduct Δ, equipped with a multiplication given by the quantum shuffle product:

$$[a_1|\ldots|a_n] \star [b_1|\ldots|b_m] = \sum_{\gamma} \widetilde{\gamma}[a_1|\ldots|a_n|b_1|\ldots|b_m]$$

where the sum is over all (n, m)-shuffles γ.

The quantum shuffle algebra has the structure of a Hopf algebra in a braided monoidal category. For more specific properties of this algebra, see [75, 79, 64, 57, 40].

Chapter 3

Representations of mixed braid groups

In this chapter, we will explore the representation theory of mixed braid groups. In particular, we will construct representations of mixed braid groups from braided vector spaces and study their inductions. Recall that the families of mixed braid groups and surface braid groups on the plane with punctures overlap precisely at $B_{n,1} \cong B_n(\mathbb{C}^\times)$; in general, $B_n(\mathbb{C}_m)$ is isomorphic to a subgroup of $B_{n,m}$ of infinite index. It turns out that this distinction plays an important role in our topological arguments, for reasons that will become evident in Chapters 5 and 6. As a result, we proceed to address the case of $B_{n,1}$ separately in the next four chapters. The content of this chapter is extracted from Sections 2.2–2.3 of [55] and 3.2–3.3 of [54].

3.1 Left-braided and mixed-braided vector spaces

The purpose of this section is to develop a representation for mixed braid groups using analogs of braided vector spaces. We will first start with $B_{n,1}$, then address the more general case $B_{n,m}$.

Given a representation of B_{n+1}, we may obtain a representation of the subgroup $B_{n,1}$ by restriction. Let (V, σ) be a braided vector space. Recall that we have a B_{n+1}-representation on $V^{\otimes n+1}$, which we can restrict to a $B_{n,1}$-representation on the same vector space. The action of the generator σ_i of $B_{n,1}$ for all $1 \leq i \leq n-1$ is the same as that of the corresponding σ_i of B_{n+1}, while the last generator τ_n acts on $V^{\otimes n+1}$ by squaring the action of σ_n, i.e., $\tau_n \mapsto \mathrm{id}^{\otimes n-1} \otimes \sigma^2$. If we restrict this action on the last two tensor factors, it is clearly given by a mapping $\tau := \sigma^2 : V \otimes V \to V \otimes V$ that preserves the order of the factors; furthermore, in this restricted representation, there is no well-defined action of the group $B_{n,1}$ that applies the braiding σ individually to these two factors. In other words, we lose information about the action of the generator σ_n of B_{n+1} on $V^{\otimes n+1}$, which can only be recovered partially as the "square root" of the action of $\tau_n \in B_{n,1}$ on the same space. We will generalize this restricted representation into a family of $B_{n,1}$-representations based on the observations above.

Definition 3.1.1. A *left-braided vector space* (V, W) over $\mathbf{k}$ is a pair of finite dimensional $\mathbf{k}$-vector spaces V and W, where V is a braided vector space with a braiding σ, further equipped with another isomorphism $\tau : V \otimes W \to V \otimes W$ such that it satisfies an additional braid equation on $V^{\otimes 2} \otimes W$:

$$(\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) = (\mathrm{id} \otimes \tau) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\sigma \otimes \mathrm{id}).$$

We may define a morphism between left-braided vector spaces $(V_1, W_1, \sigma_1, \tau_1)$ and $(V_2, W_2, \sigma_2, \tau_2)$ to be a pair of $\mathbf{k}$-linear maps $f_V : V_1 \to V_2$ and $f_W : W_1 \to W_2$ where f_V is a morphism of braided vector spaces $(V_1, \sigma_1) \to (V_2, \sigma_2)$ and f_W satisfies the relation: $(f_V \otimes f_W) \circ \tau_1 = \tau_2 \circ (f_V \otimes f_W)$ on $V_1 \otimes W_1$. The collection of left-braided vector spaces then forms a category. Similar to the case of braided vector spaces, we may define an action of $B_{n,1}$ on $V^{\otimes n} \otimes W$ by $\sigma_i \mapsto \mathrm{id}^{\otimes i-1} \otimes \sigma \otimes \mathrm{id}^{n-i}$ for all $1 \leq i \leq n-1$ and $\tau_n \mapsto \mathrm{id}^{\otimes n-1} \otimes \tau$. From this identification, the following is straight-forward:

Proposition 3.1.2. *Given a left-braided vector space* (V, W, σ, τ)*,* $V^{\otimes n} \otimes W$ *provides a representation for the group* $B_{n,1}$*.*

Example 3.1.3. Given $V = W = \mathbf{k}$, we can define a left-braided vector space (V, W) with braidings σ and τ given by multiplications by q and p respectively, for some $p, q \in \mathbf{k}^\times$. The braid action of $B_{n,1}$ on the representation $V^{\otimes n} \otimes W \cong \mathbf{k}$ is therefore given by $\sigma_i \mapsto q$ for all $1 \leq i \leq n-1$ and $\tau_n \mapsto p$.

Example 3.1.4. If (V, σ) is a braided vector space, then (V, V, σ, σ^2) forms a left-braided vector space. In this case, there is an obvious choice for the "square root" of the braiding

$\tau = \sigma^2$; generally, this is not the case. We will discuss this matter in the next section.

The $B_{n,1}$-representation constructed from this left-braided vector space per Proposition 3.1.2 is precisely the restricted representation to $B_{n,1}$ of the previously described B_{n+1}-representation on $V^{\otimes n+1}$.

Recall that there is a bijection between the category of strictly monoidal functors $\Phi : \mathscr{B} \to \mathrm{FinVect}_{\mathbf{k}}$ and the category of braided vector spaces. There is a similar functorial description for the category of left-braided vector spaces. Define the category $\mathscr{C}$ of $(n,1)$-mixed braid groups to be the wide subcategory of the groupoid $\mathscr{B}$ of braid groups with morphisms given only by their $(n,1)$-mixed braid subgroups: the objects n of $\mathscr{C}$ are indexed by positive integers, and the morphisms in $\mathscr{C}$ are automorphisms of n given by the group $B_{n-1,1}$, i.e., $\mathrm{Hom}_{\mathscr{C}}(n,n) = B_{n-1,1}$. There is a tensor product $\mathscr{B} \times \mathscr{C} \to \mathscr{C}$ induced by the homomorphism $B_n \times B_{m,1} \to B_{n+m,1}$ that places braids side-by-side. It is easy to see that this tensor product must agree with the tensor product in the groupoid $\mathscr{B}$ via the inclusion map $\mathscr{C} \hookrightarrow \mathscr{B}$, i.e., the diagram

$$\begin{array}{ccc} \mathscr{B} \times \mathscr{C} & \hookrightarrow & \mathscr{B} \times \mathscr{B} \\ \otimes \downarrow & & \downarrow \otimes \\ \mathscr{C} & \hookrightarrow & \mathscr{B} \end{array}$$

commutes (up to natural isomorphism), i.e., $\mathscr{C}$ is a left tensor ideal in $\mathscr{B}$.

Given a left-braided vector space (V, W, σ, τ), while a monoidal functor $\Phi : \mathscr{B} \to \mathrm{FinVect}_{\mathbf{k}}$ is enough to capture all data of the braided vector space (V, σ), we need an additional functor $\Psi : \mathscr{C} \to \mathrm{FinVect}_{\mathbf{k}}$ to capture the information of the vector space W and the braiding τ. In addition, these functors must be compatible with the tensor product $\mathscr{B} \times \mathscr{C} \to \mathscr{C}$ discussed above. These observations lead to the following identification of the category of left-braided vector spaces.

Proposition 3.1.5. *There is an equivalence of categories between the category of left-braided vector spaces and the category $\mathscr{F}$ of pairs of functors $\Phi : \mathscr{B} \to \mathrm{FinVect}_{\mathbf{k}}$ and $\Psi : \mathscr{C} \to \mathrm{FinVect}_{\mathbf{k}}$ that satisfy the following conditions:*

1. *Φ is a monoidal functor; and*

2. The diagram

$$\begin{array}{ccc} \mathscr{B} \times \mathscr{C} & \xrightarrow{\Phi\times\Psi} & \mathrm{FinVect}_{\mathbf{k}} \times \mathrm{FinVect}_{\mathbf{k}} \\ {\scriptstyle\otimes}\downarrow & & \downarrow{\scriptstyle\otimes} \\ \mathscr{C} & \xrightarrow{\quad\Psi\quad} & \mathrm{FinVect}_{\mathbf{k}} \end{array}$$

commutes (up to natural isomorphism).

Proof. It is not hard to see that the collection of such pairs of functors (Φ, Ψ) forms a well-defined category when equipped with pairs of natural transformations as morphisms. Given a functor pair $(\Phi, \Psi) \in \mathscr{F}$, we obtain a left-braided vector space by setting $V = \Phi(1)$ and $W = \Psi(1)$. The braiding morphism σ is the image under Φ of the positive generator of $\mathrm{Hom}_{\mathscr{B}}(2, 2) = B_2 \cong \mathbb{Z}$, while τ is obtained by applying Ψ to the positive generator of $\mathrm{Hom}_{\mathscr{C}}(2, 2) = B_{1,1} \cong \mathbb{Z}$. Condition (1) enforces that (V, σ) is a braided vector space by Proposition 2.2.2; meanwhile, (2) maintains that the tensor product in the representation is compatible with the tensor product in the categories $\mathscr{B}$ and $\mathscr{C}$. Conversely, given an arbitrary left-braided vector space (V, W, σ, τ), we have constructed above a B_n-representation on $V^{\otimes n}$ and a $B_{m,1}$-representation on $V^{\otimes m} \otimes W$. It is straightforward to verify that these identifications are inverses (up to natural isomorphism) and hence form a pair of inverse equivalences between the category of left-braided vector spaces and the category $\mathscr{F}$ of functor pairs (Φ, Ψ) that satisfy (1–2). □

We now turn our attention to representations of arbitrary mixed braid groups $B_{n,m}$ coming from braided vector spaces. Observe that the presentation of $B_{n,m}$ for $m \geq 2$ given by Manfredini [70] is more complicated than that of $B_{n,1}$, specifically with the appearance of the red generators $\sigma_{i>n}$ and new relations involving them. Subsequently, we need a suitable alteration of the constructions above to produce a representation for $B_{n,m}$.

Definition 3.1.6. A *mixed-braided vector space* (V, W, τ) over $\mathbf{k}$ is a pair of braided vector spaces (V, σ_V) and (W, σ_W), equipped with an isomorphism $\tau : V \otimes W \to V \otimes W$ (called the *mixed braiding*) which satisfies braid equations:

(1) $(\sigma_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\sigma_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) = (\mathrm{id} \otimes \tau) \circ (\sigma_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\sigma_V \otimes \mathrm{id})$

on $V^{\otimes 2} \otimes W$;

(2) $(\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W) = (\mathrm{id} \otimes \sigma_W) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W) \circ (\tau \otimes \mathrm{id})$

on $V \otimes W^{\otimes 2}$; and

(3) $(\sigma_V \otimes \mathrm{id}^{\otimes 2}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\sigma_V^{-1} \otimes \mathrm{id}^{\otimes 2}) \circ (\mathrm{id}^{\otimes 2} \otimes \sigma_W) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 2} \otimes \sigma_W^{-1})$
$= (\mathrm{id}^{\otimes 2} \otimes \sigma_W) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\mathrm{id}^{\otimes 2} \otimes \sigma_W^{-1}) \circ (\sigma_V \otimes \mathrm{id}^{\otimes 2}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\sigma_V^{-1} \otimes \mathrm{id}^{\otimes 2})$

on $V^{\otimes 2} \otimes W^{\otimes 2}$.

Observe that a mixed-braided vector space (V, W, τ) is in fact a left-braided vector space where W is braided in such a way that is compatible with the braidings σ_V and τ. The mixed-braided vector spaces form a subcategory of the category of left-braided vector spaces, where the morphisms need to satisfy an additional condition that f_W is a morphism of braided vector spaces.

As in the previous constructions, we may define an action of the (n, m)-mixed braid group $B_{n,m}$ on $V^{\otimes n} \otimes W^{\otimes m}$ by mapping σ_i to $\mathrm{id}^{\otimes i-1} \otimes \sigma_V \otimes \mathrm{id}^{\otimes n+m-1-i}$ if $1 \leq i \leq n-1$, and $\mathrm{id}^{\otimes i-1} \otimes \sigma_W \otimes \mathrm{id}^{\otimes n+m-1-i}$ if $n+1 \leq i \leq n+m-1$, and mapping τ_n to $\mathrm{id}^{\otimes n-1} \otimes \tau \otimes \mathrm{id}^{\otimes m-1}$. From this identification, the following is straightforward:

Proposition 3.1.7. *Given a mixed-braided vector space (V, W, τ), $V^{\otimes n} \otimes W^{\otimes m}$ provides a representation for the (n, m)-mixed braid group $B_{n,m}$.*

While it is unclear to the author whether all left-braided vector spaces can be endowed with more structures to form mixed-braided vector spaces, conventional wisdom suggests a negative answer. However, for a few natural examples that are most relevant to the discussion in this dissertation, it is easy to make such modifications.

Example 3.1.8. Example 3.1.3 can be extended to produce a mixed-braided vector space. Given $V = W = \mathbf{k}$, we can endow the pair (V, W) with the braidings σ_V, σ_W, and τ given by multiplications by q, u, and p respectively, for some $p, q, u \in \mathbf{k}^\times$. The braid action of $B_{n,m}$ on the representation $V^{\otimes n} \otimes W^{\otimes m} \cong \mathbf{k}$ is therefore given by $\sigma_{i<n} \mapsto q$, $\sigma_{i>n} \mapsto u$, and $\tau_n \mapsto p$.

Example 3.1.9. Similarly, if (V, σ) is a braided vector space, then (V, V, σ^2) forms a mixed-braided vector space. The resulting $B_{n,m}$-representation per Proposition 3.1.7 is

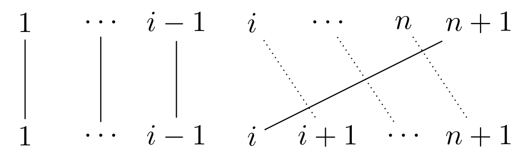

Figure 3.1: Coset representative $\widetilde{\alpha_{i,n+1}}$.

precisely the restricted representation to $B_{n,m}$ of the monoidal braid representation of B_{n+m} on $V^{\otimes n+m}$.

3.2 Induced representation of $B_{n,1}$

Recall that there is a natural inclusion of $B_{n,1}$ into the braid group B_{n+1} that identifies elements of $B_{n,1}$ with braids of $n+1$ strands whose last strand is pure. Consider the left cosets of $B_{n,1}$ in B_{n+1}.

Proposition 3.2.1. *The collection of left cosets of $B_{n,1}$ in B_{n+1} has the form*

$$B_{n+1}/B_{n,1} = \{\widetilde{\alpha_{i,n+1}} B_{n,1}\}_{i=1}^{n+1}$$

where $\alpha_{i,n+1}$ is the $(n,1)$-shuffle that sends $n+1$ to i, and $\widetilde{\alpha_{i,n+1}}$ is its lift to B_{n+1}. Explicitly, $\widetilde{\alpha_{i,n+1}} = \sigma_i \ldots \sigma_{n-1}\sigma_n$ for all $1 \le i \le n$ and $\widetilde{\alpha_{n+1,n+1}} = \mathrm{id}$.

Proof. We claim that the left cosets $aB_{n,1}$ are indexed by the image of the $n+1^{\text{st}}$ endpoint under the braid $a \in B_{n+1}$, and therefore $|B_{n+1} : B_{n,1}| = n+1$. This statement, particularly the second fact, was proved by Crisp [31] in slightly different language. For any $a_1, a_2 \in B_{n+1}$, $a_1 B_{n,1} = a_2 B_{n,1}$ as cosets iff $a_1^{-1} a_2 \in B_{n,1}$. Let $\underline{a}$ denote the underlying permutation of a braid a, then this is equivalent to $\underline{a_1}^{-1}\underline{a_2}(n+1) = n+1$, or $\underline{a_1}(n+1) = \underline{a_2}(n+1)$. So we have a simple characterization of the cosets of $B_{n,1}$ in B_{n+1}: two braid elements of B_{n+1} are in the same coset of $B_{n,1}$ if and only if their underlying permutations map $n+1$ to the same number. Since there are $n+1$ choices for the image, the index of $B_{n,1}$ in B_{n+1} is $n+1$. Furthermore, we may explicitly choose representatives for the cosets of $B_{n,1}$ to be the lift of all $(n,1)$-shuffles $\alpha_{i,n+1}$. It is a straightforward exercise to derive the stated formulae for these elements. □

The second index of a representative element $\widetilde{\alpha_{i,n+1}}$ records the number of strands in the braid; when this datum is unambiguous it is omitted from the notation.

Given a representation of any subgroup, we may define a representation of the parent group by means of the *induced representation*. Let L be a representation of $B_{n,1}$. The braid representation of $B_{n,1}$ on L induces a representation on

$$\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L) = \mathbf{k}[B_{n+1}] \otimes_{\mathbf{k}[B_{n,1}]} L$$

of the braid group B_{n+1}. We may give a more detailed description of this induced representation based on the cosets of the subgroup $B_{n,1}$ in B_{n+1} described above. Since the collection $\{\widetilde{\alpha_1}, \ldots, \widetilde{\alpha_{n+1}}\}$ gives a full set of representatives in B_{n+1} for the left cosets of $B_{n,1}$, as vector spaces, the induced representation can be identified as

$$\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L) \cong \bigoplus_{i=1}^{n+1} \widetilde{\alpha}_i L.$$

Here each $\widetilde{\alpha}_i L$ is an isomorphic copy of the vector space L whose elements are written as $\widetilde{\alpha}_i \ell$ where $\ell \in L$. We may give a concrete description of the action of the braid group on this induced representation.

Proposition 3.2.2. *The action of the braid group B_{n+1} on the induced representation $\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L)$ is given by*

$$a \sum_{i=1}^{n+1} \widetilde{\alpha}_i \ell_i = \sum_{i=1}^{n+1} \widetilde{\alpha_{\underline{a}(i)}} \left[(\widetilde{\alpha_{\underline{a}(i)}}^{-1} a \widetilde{\alpha}_i)(\ell_i) \right]$$

where $(\widetilde{\alpha_{\underline{a}(i)}}^{-1} a \widetilde{\alpha}_i)(\ell_i)$ is obtained by applying the action of $\widetilde{\alpha_{\underline{a}(i)}}^{-1} a \widetilde{\alpha}_i \in B_{n,1}$ on $\ell_i \in L$ for every $1 \leq i \leq n+1$.

Proof. Since the collection $\{\widetilde{\alpha}_i\}_{i=1}^{n+1}$ forms a full set of representatives, for each $a \in B_{n+1}$ and each $\widetilde{\alpha}_i$, there exist $b_i \in B_{n,1}$ and $\widetilde{\alpha_j}$ such that $a\widetilde{\alpha}_i = \widetilde{\alpha_j} b_i$. The action of a on an element in the induced representation is defined by

$$a \sum_{i=1}^{n+1} \widetilde{\alpha}_i \ell_i = \sum_{i=1}^{n+1} \widetilde{\alpha_j}[b_i(\ell_i)]$$

where the action of b_i on ℓ_i is defined by the $B_{n,1}$-representation L. In this case, we can make specific choices for the elements $\widetilde{\alpha_j}$ and b_i. Since the underlying permutation of $b_i = \widetilde{\alpha_j}^{-1} a \widetilde{\alpha_i} \in B_{n,1}$ fixes $n+1$, it follows that $\alpha_j(n+1) = \underline{a}[\alpha_i(n+1)] = \underline{a}(i)$, thus $j = \underline{a}(i)$. Hence the choices of $\widetilde{\alpha_j} = \widetilde{\alpha_{\underline{a}(i)}}$ and $b_i = \widetilde{\alpha_{\underline{a}(i)}}^{-1} a \widetilde{\alpha_i}$ for each summand in the element of the representation give the desired action of the braid group B_{n+1} on the induced representation $\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L)$. □

Corollary 3.2.3. *The action of the generators of B_{n+1} on $\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L)$ can be expressed in terms of the action of the generators of $B_{n,1}$ in the following way:*

$$\sigma_m(\widetilde{\alpha_i}\ell) = \begin{cases} \widetilde{\alpha_i}[\sigma_m(\ell)] & 1 \leq m \leq i-2 \\ \widetilde{\alpha_{i-1}}\ell & m = i-1 \\ \widetilde{\alpha_{i+1}}[(\widetilde{\alpha_i}\tau_n\widetilde{\alpha_i}^{-1})(\ell)] & m = i \\ \widetilde{\alpha_i}[\sigma_{m-1}(\ell)] & i+1 \leq m \leq n+1. \end{cases}$$

Strictly speaking, the element of $B_{n,1}$ presented in the formula when $m = i$ is not entirely in terms of the generators of $B_{n,1}$, since $\widetilde{\alpha_i}$ contains the braid element $\sigma_n \in B_{n+1} \backslash B_{n,1}$. However, it can be rewritten as $\widetilde{\alpha_i}\tau_n\widetilde{\alpha_i}^{-1} = \sigma_i \ldots \sigma_n \sigma_n^2 \sigma_n^{-1} \ldots \sigma_i^{-1} = \sigma_i \ldots \sigma_{n-1} \sigma_n^2 \sigma_{n-1}^{-1} \ldots \sigma_i^{-1} = \sigma_i \ldots \sigma_{n-1} \tau_n \sigma_{n-1}^{-1} \ldots \sigma_i^{-1} \in B_{n,1}$.

Proof. We will prove this statement case-by-case.

Case 1 ($1 \leq m \leq i-2$)*:* This case follows directly from the commutativity of non-adjacent braid generators.

Case 2 ($m = i-1$)*:* Observe that as braid elements, $\widetilde{\alpha_{i-1}} = \sigma_{i-1}\widetilde{\alpha_i}$. Hence $\sigma_{i-1}(\widetilde{\alpha_i}\ell) = \widetilde{\alpha_{i-1}}[(\widetilde{\alpha_{i-1}}^{-1}\sigma_{i-1}\widetilde{\alpha_i})(\ell)] = \widetilde{\alpha_{i-1}}\ell$.

Case 3 ($m = i$)*:* It is equivalent to prove that $\widetilde{\alpha_{i+1}}^{-1}\sigma_i\widetilde{\alpha_i} = \widetilde{\alpha_i}\sigma_n^2\widetilde{\alpha_i}^{-1}$ as braid elements for all i. We will prove this case by induction on i with the starting index $i = n$. The base case is simple: $\widetilde{\alpha_{n+1}}^{-1}\sigma_n\widetilde{\alpha_n} = (\mathrm{id})\sigma_n\sigma_n = \sigma_n^2$.

Suppose the hypothesis holds for all $k \geq i$, i.e., $\widetilde{\alpha_{k+1}}^{-1}\sigma_k\widetilde{\alpha_k} = \widetilde{\alpha_k}\sigma_n^2\widetilde{\alpha_k}^{-1}$. To show that it holds for $i-1$, first we apply the statement of case 2 to the right side:

$\widetilde{\alpha_{i-1}}\sigma_n^2\widetilde{\alpha_{i-1}}^{-1} = (\sigma_{i-1}\widetilde{\alpha_i})\sigma_n^2(\widetilde{\alpha_i}^{-1}\sigma_{i-1}^{-1})$. By the induction hypothesis, it follows that

$$\begin{aligned}
&\sigma_{i-1}(\widetilde{\alpha_i}\sigma_n^2\widetilde{\alpha_i}^{-1})\sigma_{i-1}^{-1} = \sigma_{i-1}(\widetilde{\alpha_{i+1}}^{-1}\sigma_i\widetilde{\alpha_i})\sigma_{i-1}^{-1} = \widetilde{\alpha_{i+1}}^{-1}\sigma_{i-1}\sigma_i(\sigma_i\widetilde{\alpha_{i+1}})\sigma_{i-1}^{-1} \\
&= \widetilde{\alpha_i}^{-1}(\sigma_i\sigma_{i-1}\sigma_i)\sigma_i\sigma_{i-1}^{-1}\widetilde{\alpha_{i+1}} = \widetilde{\alpha_i}^{-1}\sigma_{i-1}(\sigma_i\sigma_{i-1}\sigma_i)\sigma_{i-1}^{-1}\widetilde{\alpha_{i+1}} \\
&= \widetilde{\alpha_i}^{-1}\sigma_{i-1}\sigma_{i-1}\sigma_i\sigma_{i-1}\sigma_{i-1}^{-1}\widetilde{\alpha_{i+1}} = \widetilde{\alpha_i}^{-1}\sigma_{i-1}(\sigma_{i-1}\sigma_i\widetilde{\alpha_{i+1}}) = \widetilde{\alpha_i}^{-1}\sigma_{i-1}\widetilde{\alpha_{i-1}}.
\end{aligned}$$

Case 4 ($i + 1 \leq m \leq n + 1$): We will use another induction argument on i with the starting index $i = n$. The base case $i = n$ is vacuously true. For $i = n - 1$, we only need to check when $m = n$; this case follows directly from the braid relation in B_{n+1}: $\sigma_n(\widetilde{\alpha_{n-1}}\ell) = \widetilde{\alpha_{n-1}}[(\widetilde{\alpha_{n-1}}^{-1}\sigma_n\widetilde{\alpha_{n-1}})(\ell)] = \widetilde{\alpha_{n-1}}[(\sigma_n^{-1}\sigma_{n-1}^{-1}\sigma_n\sigma_{n-1}\sigma_n)(\ell)] = \widetilde{\alpha_{n-1}}[(\sigma_n^{-1}\sigma_{n-1}^{-1}\sigma_{n-1}\sigma_n\sigma_{n-1})(\ell)] = \widetilde{\alpha_{n-1}}[\sigma_{n-1}(\ell)]$.

Suppose the statement holds true for all $k \geq i$, i.e., $\sigma_m(\widetilde{\alpha_k}\ell) = \widetilde{\alpha_k}[\sigma_{m-1}(\ell)]$ for all $k + 1 \leq m \leq n + 1$. We will make use of the fact that for all k, $\widetilde{\alpha_{k-1}}\ell = \sigma_{k-1}(\widetilde{\alpha_k}\ell)$ as proved in case 2. For $m \geq i + 1$,

$$\sigma_m(\widetilde{\alpha_{i-1}}\ell) = \sigma_m\sigma_{i-1}(\widetilde{\alpha_i}\ell) = \sigma_{i-1}\sigma_m(\widetilde{\alpha_i}\ell) = \sigma_{i-1}\widetilde{\alpha_i}[\sigma_{m-1}(\ell)] = \widetilde{\alpha_{i-1}}[\sigma_{m-1}(\ell)].$$

For $m = i$,

$$\sigma_i(\widetilde{\alpha_{i-1}}\ell) = \sigma_i\sigma_{i-1}\sigma_i(\widetilde{\alpha_{i+1}}\ell) = \sigma_{i-1}\sigma_i\sigma_{i-1}(\widetilde{\alpha_{i+1}}\ell) = \sigma_{i-1}\sigma_i\widetilde{\alpha_{i+1}}[\sigma_{i-1}(\ell)] = \widetilde{\alpha_{i-1}}[\sigma_{i-1}(\ell)].$$

We have verified the action of the generators of B_{n+1} on $\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L)$ for all cases, hence the proof is finished. □

This corollary highlights a benefit of our choice of the coset representatives $\{\widetilde{\alpha_i}\}_{i=1}^{n+1}$. Since the composition of braid actions works well with the braid multiplication, the fact that we understand the action of the generators of B_{n+1} in terms of the action of the generators of $B_{n,1}$ implies that it is possible to decompose a general braid action on $\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L)$ into a series of actions of the generators of the subgroup $B_{n,1}$ on the original representation L.

Consider when $L = V^{\otimes n} \otimes W$, the representation of $B_{n,1}$ formed from a left-braided vector space (V, W, σ, τ) per Proposition 3.1.2. Recall that the induced representation $\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n} \otimes W)$ can be written as the direct sum $\bigoplus_{i=1}^{n+1} \widetilde{\alpha_{i,n+1}}(V^{\otimes n} \otimes W)$ where each

$\widetilde{\alpha_{i,n+1}}(V^{\otimes n} \otimes W)$ is isomorphic to $V^{\otimes n} \otimes W$. For all $1 \leq i \leq n+1$, since $\alpha_{i,n+1}$ sends $n+1$ to i, it is natural to identify $\widetilde{\alpha_{i,n+1}}(V^{\otimes n} \otimes W)$ with $V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$ (as vector spaces), where W is the i^{th} tensor factor, via an isomorphism $\xi_{i,n+1} : \widetilde{\alpha_{i,n+1}}(V^{\otimes n} \otimes W) \xrightarrow{\cong} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$. This yields the following identification:

Proposition 3.2.4. *There is an isomorphism of vector spaces*

$$\operatorname{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n} \otimes W) \cong \bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}.$$

Moreover, given a choice of isomorphism $\xi_{i,n+1} : \widetilde{\alpha_{i,n+1}}(V^{\otimes n} \otimes W) \to V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$ for all $1 \leq i \leq n+1$, there is a B_{n+1}-action on $\bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$ defined by $a \mapsto \xi_{\underline{a}(i),n+1} a \xi_{i,n+1}^{-1}$, such that this is an isomorphism of B_{n+1}-representations.

Proof. The isomorphism of B_{n+1}-representations follows immediately from the definition of the action of B_{n+1} on $\bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$. □

By convention, $\xi_{n+1,n+1}$ is always the identity map. The second index of the map $\xi_{i,n+1}$ again denotes the total degree of the domain and is omitted from the notation if there is no ambiguity. Observe that on the right hand side, we can apply braids that "move" the factor W, an operation that is forbidden in the $B_{n,1}$-representation on $V^{\otimes n} \otimes W$. This allows for a more intuitive framework to study the action of B_{n+1} on the induced representation, analogous to its action in the monoidal braid representation on $V^{\otimes n+1}$.

As in the quantum shuffle algebra, we will also denote elements of $V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$ by the bar complex notation, i.e., $[v_1|\dots|v_{i-1}|w|v_{i+1}|\dots|v_{n+1}]$. In a specific case of composition of braid actions, if we apply the braid element σ_n to an element in $\widetilde{\alpha_n}(V^{\otimes n} \otimes W)$, we obtain

$$\sigma_n(\widetilde{\alpha_n}[v_1|\dots|v_n|w]) = (\sigma_n^2)[v_1|\dots|v_n|w] = (\mathrm{id}^{\otimes n-1} \otimes \tau)[v_1|\dots|v_n|w].$$

Hence on the vector space $V^{\otimes n} \otimes W$, σ_n acts as the "left square root" of the braid action of τ_n. Formally, we may define maps $s_1 : V^{\otimes n} \otimes W \to V^{\otimes n-1} \otimes W \otimes V$ by

$$[v_1|\dots|v_n|w] \mapsto \xi_n(\sigma_n[v_1|\dots|v_n|w])$$

$$
\begin{array}{ccc}
V^{\otimes p} \otimes (V^{\otimes i-1} \otimes W \otimes V^{q-i+1}) \otimes V^{\otimes n-p-q} & \xrightarrow{\xi^{-1}_{p+i,n+1}} & \widetilde{\alpha_{p+i,n+1}}(V^{\otimes n} \otimes W) \\
\Big\downarrow {\scriptstyle \mathrm{id}^{\otimes p} \otimes \xi^{-1}_{i,q+1} \otimes \mathrm{id}^{\otimes n-p-q}} & & \\
V^{\otimes p} \otimes \widetilde{\alpha_{i,q+1}}(V^{\otimes q} \otimes W) \otimes V^{\otimes n-p-q} & & \Big\downarrow {\scriptstyle a'} \\
\Big\downarrow {\scriptstyle \mathrm{id}^{\otimes p} \otimes a \otimes \mathrm{id}^{\otimes n-p-q}} & & \\
V^{\otimes p} \otimes \widetilde{\alpha_{\underline{a}(i),q+1}}(V^{\otimes q} \otimes W) \otimes V^{\otimes n-p-q} & & \\
\Big\downarrow {\scriptstyle \mathrm{id}^{\otimes p} \otimes \xi_{\underline{a}(i),q+1} \otimes \mathrm{id}^{\otimes n-p-q}} & & \\
V^{\otimes p} \otimes (V^{\otimes \underline{a}(i)-1} \otimes W \otimes V^{q-\underline{a}(i)+1}) \otimes V^{\otimes n-p-q} & \xleftarrow{\xi_{\underline{a'}(p+i),n+1}} & \widetilde{\alpha_{\underline{a'}(p+i),n+1}}(V^{\otimes n} \otimes W)
\end{array}
$$

Figure 3.2: Diagram $\mathscr{D}$.

and $s_2 : V^{\otimes n-1} \otimes W \otimes V \to V^{\otimes n} \otimes W$ by

$$[v_1|\dots|v_{n-1}|w|v_{n+1}] \mapsto \sigma_n(\xi_n^{-1}[v_1|\dots|v_{n-1}|w|v_{n+1}])$$

that can be treated as "left square roots" of the braiding map $\mathrm{id}^{\otimes n-1} \otimes \tau$ in the B_{n+1}-representation on $\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n} \otimes W)$.

Observe that in general, the B_{n+1}-action on $\bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$, while slightly more intuitive than the induced representation, does not have an explicit formula that can be used for computation. This issue is more tractable when the representation on $\bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$ behaves similarly to that on $V^{\otimes n+1}$, in the sense that the B_{q+1}-action on a tensor subfactor $V^{\otimes i-1} \otimes W \otimes V^{q-i+1}$ of a summand $V^{\otimes i'-1} \otimes W \otimes V^{n-i'+1}$ agrees with the B_{n+1}-action on the entire summand for all $1 \le q \le n$ and $1 \le i \le q+1$. That is, we desire the following property: for any $a \in B_{q+1}$, Diagram $\mathscr{D}$ (Figure 3.2) commutes, where the braid a' is the natural inclusion of a into the copy $B_{q+1} \le B_{n+1}$ consisting of braids that are only nontrivial on the $q+1$ strands starting with the $p+1^{\text{st}}$. The following proposition gives criteria to detect this property.

Proposition 3.2.5. *Let $\varphi_{i,n} : V^{\otimes n-1} \otimes W \to V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i}$ be defined by $\varphi_{i,n} := \xi_{i,n}\widetilde{\alpha_{i,n}}$ for all $n \ge 1$ and $1 \le i \le n$, and in particular denote $\varphi := \varphi_{1,2} = \xi_{1,2}\widetilde{\alpha_{1,2}}$. Then Diagram $\mathscr{D}$ always commutes if and only if*

1. $\varphi_{i,n} = (\mathrm{id}^{\otimes i-1} \otimes \varphi \otimes \mathrm{id}^{\otimes n-i-1}) \circ (\mathrm{id}^{\otimes i} \otimes \varphi \otimes \mathrm{id}^{\otimes n-i-2}) \circ \dots \circ (\mathrm{id}^{\otimes n-2} \otimes \varphi)$

and the following identities hold:

2. $(\mathrm{id}\otimes\sigma)\circ(\varphi\otimes\mathrm{id})\circ(\mathrm{id}\otimes\varphi)=(\varphi\otimes\mathrm{id})\circ(\mathrm{id}\otimes\varphi)\circ(\sigma\otimes\mathrm{id})$;

3. $(\tau\otimes\mathrm{id})\circ(\mathrm{id}\otimes\varphi)=(\mathrm{id}\otimes\varphi)\circ(\sigma\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\sigma^{-1}\otimes\mathrm{id})$.

Proof. Since every braid action is decomposable into those of the generators, it suffices to study the commutativity of Diagram $\mathscr{D}$ for all braid generators. By a straightforward yet arduous reduction, we can show that Diagram $\mathscr{D}$ commutes for all braid generators if and only if the following identities hold:

(a) $(\mathrm{id}^{\otimes p}\otimes\varphi_{i,q}\otimes\mathrm{id}^{\otimes n-p-q})\varphi_{p+q,n}=\varphi_{p+i,n}$;

(b) $\sigma_m\varphi_{i,n}=\begin{cases}\varphi_{i,n}\sigma_m & \text{if } m\leq i-2\\ \varphi_{i,n}\sigma_{m-1} & \text{if } m\geq i+1\end{cases}$ for $\sigma_m=\mathrm{id}^{\otimes m-1}\otimes\sigma\otimes\mathrm{id}^{\otimes n-m-1}$; and

(c) $\tau_{i-1}\varphi_{i,n}=\varphi_{i,n}(\sigma_{i-1}\dots\sigma_{n-2}\tau_{n-1}\sigma_{n-2}^{-1}\dots\sigma_{i-1}^{-1})$, for $\tau_j=\mathrm{id}^{\otimes j-1}\otimes\tau\otimes\mathrm{id}^{\otimes n-j-1}$.

We will show that these conditions are equivalent to (1–3).

First, we will prove that given a choice of $\varphi=\varphi_{1,2}$, (1) is equivalent to (a). It is easy to see that the formula given in (1) for $\varphi_{i,n}$ satisfies (a). Conversely, assuming (a), we will derive (1) by induction on n. The base case $n=3$ can be checked directly:

$$\varphi_{2,3}=(\mathrm{id}\otimes\varphi_{1,2})\varphi_{3,3}=\mathrm{id}\otimes\varphi$$

and

$$\varphi_{1,3}=(\varphi_{1,2}\otimes\mathrm{id})\varphi_{2,3}=(\varphi\otimes\mathrm{id})\circ(\mathrm{id}\otimes\varphi).$$

Suppose the formula holds for n. Consider $\varphi_{i,n+1}$. For all $1\leq i\leq n$, we have

$$\begin{aligned}\varphi_{i+1,n+1}&=(\mathrm{id}\otimes\varphi_{i,n})\varphi_{n+1,n+1}\\ &=\mathrm{id}\otimes\left[(\mathrm{id}^{\otimes i-1}\otimes\varphi\otimes\mathrm{id}^{\otimes n-i-1})\circ(\mathrm{id}^{\otimes i}\otimes\varphi\otimes\mathrm{id}^{\otimes n-i-2})\circ\cdots\circ(\mathrm{id}^{\otimes n-2}\otimes\varphi)\right]\\ &=(\mathrm{id}^{\otimes i}\otimes\varphi\otimes\mathrm{id}^{\otimes n-i-1})\circ(\mathrm{id}^{\otimes i+1}\otimes\varphi\otimes\mathrm{id}^{\otimes n-i-2})\circ\cdots\circ(\mathrm{id}^{\otimes n-1}\otimes\varphi).\end{aligned}$$

Finally, apply the above formula for $\varphi_{2,n+1}$ to $\varphi_{1,n+1}=(\varphi_{1,2}\otimes\mathrm{id}^{\otimes n-1})\varphi_{2,n+1}$ to complete the remaining case. With this identification, observe that (b) gives $\sigma_2\varphi_{1,3}=\varphi_{1,3}\sigma_1$, which is (2), and (c) gives $\tau_1\varphi_{2,3} = \varphi_{2,3}(\sigma_1\tau_2\sigma_1^{-1})$, which is (3). So the forward direction of the statement holds.

Conversely, assume (1–3). For simplicity, let $\varphi_i := \mathrm{id}^{\otimes i-1} \otimes \varphi \otimes \mathrm{id}^{\otimes n-i-1}$, then we may rewrite these formulae as

1. $\varphi_{i,n} = \varphi_i\varphi_{i+1}\dots\varphi_{n-1}$;
2. $\sigma_{i+1}\varphi_i\varphi_{i+1} = \varphi_i\varphi_{i+1}\sigma_i$;
3. $\tau_i\varphi_{i+1} = \varphi_{i+1}\sigma_i\tau_{i+1}\sigma_i^{-1}$.

In addition to these identities, note that $\sigma_m\varphi_i = \varphi_i\sigma_m$ whenever $|m - i| \geq 2$. To prove (b), observe that if $m \leq i - 2$, $\sigma_m\varphi_{i,n} = \sigma_m\varphi_i \dots \varphi_{n-1} = \varphi_i \dots \varphi_{n-1}\sigma_m = \varphi_{i,n}\sigma_m$, since σ_m commutes with each φ_j. If $m \geq i + 1$, by applying (1) and (2), we have $\sigma_m\varphi_{i,n} = \varphi_i \dots \varphi_{m-2}\sigma_m\varphi_{m-1}\varphi_m \dots \varphi_{n-1} = \varphi_i \dots \varphi_{m-1}\varphi_m\sigma_{m-1}\varphi_{m+1} \dots \varphi_{n-1} = \varphi_i \dots \varphi_{n-1}\sigma_{m-1} = \varphi_{i,n}\sigma_{m-1}$, so (b) is satisfied. We will prove (c) by backward induction on i for a fixed n. The base case $i = n - 1$ is precisely (3). Suppose that (c) holds for i. We then have

$$\begin{aligned}
\tau_{i-2}\varphi_{i-1,n} &= \tau_{i-2}\varphi_{i-1}\varphi_{i,n} = \varphi_{i-1}\sigma_{i-2}\tau_{i-1}\sigma_{i-2}^{-1}\varphi_{i,n} = \varphi_{i-1}\sigma_{i-2}\tau_{i-1}\varphi_{i,n}\sigma_{i-2}^{-1} \\
&= \varphi_{i-1}\sigma_{i-2}\varphi_{i,n}(\sigma_{i-1}\dots\sigma_{n-2}\tau_{n-1}\sigma_{n-2}^{-1}\dots\sigma_{i-1}^{-1})\sigma_{i-2}^{-1} \\
&= \varphi_{i-1}\varphi_{i,n}(\sigma_{i-2}\sigma_{i-1}\dots\sigma_{n-2}\tau_{i-1}\sigma_{n-2}^{-1}\dots\sigma_{i-1}^{-1}\sigma_{i-2}^{-1}) \\
&= \varphi_{i-1,n}(\sigma_{i-2}\dots\sigma_{n-2}\tau_{i-1}\sigma_{n-2}^{-1}\dots\sigma_{i-2}^{-1}).
\end{aligned}$$

This concludes our induction. □

It follows that our desired property for the action of B_{n+1} on $\bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$ can be detected by the existence of an isomorphism $\varphi : V \otimes W \to W \otimes V$ satisfying (2–3) in Proposition 3.2.5. In principle, the choices of maps φ and $\xi_{1,2} : \widetilde{\alpha_{1,2}}(V \otimes W) \to W \otimes V$ are equivalent via the relation $\varphi = \xi_{1,2}\widetilde{\alpha_{1,2}}$; in practice however, given an explicit left-braided vector space, it is often more convenient to construct a map φ, due to the fact that $\widetilde{\alpha_{1,2}}(V \otimes W)$ is abstract. Observe that these identities directly involve the braiding maps σ and τ of the left-braided vector space (V, W), while $\varphi : V \otimes W \to W \otimes V$ plays the role of the "left square root" of τ. These criteria therefore are strictly internal to the structure of left-braided vector spaces, as encapsulated in the following definition:

Definition 3.2.6. A left-braided vector space (V, W, σ, τ) is *separable* if there exists an isomorphism $\varphi : V \otimes W \to W \otimes V$ (called the *separated braiding*) that satisfies the following braid equations on $V^{\otimes 2} \otimes W$:

1. $(\mathrm{id} \otimes \sigma) \circ (\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) = (\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) \circ (\sigma \otimes \mathrm{id})$;

2. $(\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) = (\mathrm{id} \otimes \varphi) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\sigma^{-1} \otimes \mathrm{id})$.

A separable left-braided vector space (V, W, σ, τ) with the choice of separated braiding φ is denoted by $(V, W, \sigma, \tau, \varphi)$.

Example 3.2.7. When $V = W = \mathbf{k}$, the left-braided vector space in Example 3.1.3 is separable with the separated braiding φ given simply by permutation of tensor factors.

Generally, this choice of φ does not satisfy (2). One such example is (V, V, σ, σ^2) for a given braided vector space (V, σ) (see Example 3.1.4). In this case, however, the left-braided vector space is separable with the obvious choice $\varphi = \sigma$.

Separability of a left-braided vector space (V, W) is integrally connected to the existence of a braid structure on the direct sum $V \oplus W$.

Proposition 3.2.8. *Let V and W be finite dimensional $\mathbf{k}$-vector spaces, and let $X = V \oplus W$. Suppose there is an automorphism σ_X of $(V \oplus W)^{\otimes 2} \cong V^{\otimes 2} \oplus (V \otimes W) \oplus (W \otimes V) \oplus W^{\otimes 2}$ defined summand-wise by isomorphisms $\sigma_V : V^{\otimes 2} \to V^{\otimes 2}$, $\varphi : V \otimes W \to W \otimes V$, $\psi : W \otimes V \to V \otimes W$, and $\sigma_W : W^{\otimes 2} \to W^{\otimes 2}$.*

1. *If (X, σ_X) is a braided vector space, then $(V, W, \sigma_V, \tau, \varphi)$ is a separable left-braided vector space where $\tau = \psi\varphi$;*

2. *A weak version of the converse holds: if the assumption on σ_X is relaxed by setting $\sigma_W = 0$ (in particular, σ_X is no longer an isomorphism) and $(V, W, \sigma_V, \tau, \varphi)$ is a separable left-braided vector space, then (X, σ_X) is a lax braided vector space, i.e., the map σ_X satisfies the braid equation on $X^{\otimes 3}$:*

$$(\sigma_X \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_X) \circ (\sigma_X \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma_X) \circ (\sigma_X \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_X).$$

Proof. This proof rests on the following key observation: (X, σ_X) is a (lax) braided vector space if and only if the braid equation holds on each of the eight summands of $(V \oplus W)^{\otimes 3}$, i.e.,

(a) $(\sigma_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma_V) \circ (\sigma_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_V)$;

(b) $(\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) \circ (\sigma_V \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma_V) \circ (\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi)$;

(c) $(\psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_V) \circ (\varphi \otimes \mathrm{id}) = (\mathrm{id} \otimes \varphi) \circ (\sigma_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi)$;

(d) $(\sigma_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi) \circ (\psi \otimes \mathrm{id}) = (\mathrm{id} \otimes \psi) \circ (\psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_V)$;

(e) $(\sigma_W \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) \circ (\varphi \otimes \mathrm{id}) = (\mathrm{id} \otimes \varphi) \circ (\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W)$;

(f) $(\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W) \circ (\psi \otimes \mathrm{id}) = (\mathrm{id} \otimes \psi) \circ (\sigma_W \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi)$;

(g) $(\psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi) \circ (\sigma_W \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma_W) \circ (\psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi)$;

(h) $(\sigma_W \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W) \circ (\sigma_W \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma_W) \circ (\sigma_W \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W)$.

When $\sigma_W = 0$, (e–h) are automatically satisfied, so it suffices to show that (a–d) are the necessary and sufficient conditions for the braid structure and separability of $(V, W, \sigma_V, \tau, \varphi)$. The arguments for both directions are very similar; here we will only show the proof of the first part of the statement. For the rest of this proof, we will use the notation $\sigma_1 := \sigma_V \otimes \mathrm{id}$ and $\sigma_2 := \mathrm{id} \otimes \sigma_V$, and analogous notations for τ, φ, and ψ.

Assume (a–d). First we show that (V, W, σ_V, τ) is a left-braided vector space where $\tau = \psi\varphi$. Condition (a) implies that (V, σ_V) is a braided vector space. For the additional braid equation, we have

$$\begin{aligned} \tau_2\sigma_1\tau_2\sigma_1 &= \psi_2(\varphi_2\sigma_1\psi_2)\varphi_2\sigma_1 = (\psi_2\psi_1\sigma_2)(\varphi_1\varphi_2\sigma_1) \\ &= \sigma_1\psi_2(\psi_1\sigma_2\varphi_1)\varphi_2 = \sigma_1(\psi_2\varphi_2)\sigma_1(\psi_2\varphi_2) = \sigma_1\tau_2\sigma_1\tau_2. \end{aligned}$$

So indeed (V, W, σ_V, τ) forms a left-braided vector space. Condition (b) is the same as (1) in Definition 3.2.6, while for (2) we have:

$$\tau_1\varphi_2 = \psi_1(\varphi_1\varphi_2\sigma_1)\sigma_1^{-1} = (\psi_1\sigma_2\varphi_1)\varphi_2\sigma_1^{-1} = \varphi_2\sigma_1\psi_2\varphi_2\sigma_1^{-1} = \varphi_2\sigma_1\tau_2\sigma_1^{-1}.$$

Therefore (V, W, σ_V, τ) is separable with the separated braiding φ. $\square$

Roughly speaking, Proposition 3.2.8 states that the separability of a left-braided vector space (V, W) is equivalent to half of the data of a braid structure on $V \oplus W$. This fact again exhibits an asymmetry in the structure of left-braided vector spaces, which stems from the nature of $(n, 1)$-mixed braid groups. Finally, when (V, W) is a separable left-braided vector space, Propositions 3.2.5 and 3.2.8 together imply that there is a simple description of the action of B_{n+1} in the braid representation on $\bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$ defined in Proposition 3.2.4:

Corollary 3.2.9. *Let $(V, W, \sigma, \tau, \varphi)$ be a separable left-braided vector space. Then there is an isomorphism of B_{n+1}-representations*

$$\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n} \otimes W) \cong \bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$$

where the action of B_{n+1} on $\bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$ is defined by

$$\sigma_m[v_1|\dots|w_i|\dots|v_{n+1}] = \begin{cases} (\mathrm{id}^{\otimes m-1} \otimes \sigma \otimes \mathrm{id}^{\otimes n-m})[v_1|\dots|w_i|\dots|v_{n+1}] & m \neq i-1, i \\ (\mathrm{id}^{\otimes i-2} \otimes \varphi \otimes \mathrm{id}^{\otimes n-i+1})[v_1|\dots|w_i|\dots|v_{n+1}] & m = i-1 \\ (\mathrm{id}^{\otimes i-1} \otimes \tau\varphi^{-1} \otimes \mathrm{id}^{\otimes n-i})[v_1|\dots|w_i|\dots|v_{n+1}] & m = i. \end{cases}$$

3.3 Induced representations of other mixed braid groups

Recall that an (n, m)-shuffle is a permutation of $n+m$ elements that preserves the order on the first n and the last m elements. An (n, m)-shuffle γ can be completely determined by an indexing set $\mathcal{I} = \{i_1, \dots, i_m\}$ where $i_k = \gamma(n+k)$ $(1 \leq i_1 < \dots < i_m \leq n+m)$. We denote the (n, m)-shuffle defined by the indexing set $\mathcal{I}$ by $\gamma_{\mathcal{I},n+m}$; the second index will usually be omitted when there is no ambiguity. If $\mathcal{I} = (i_1, \dots, i_m)$ is an m-tuple whose elements are distinct but not necessarily increasing, we let the associated shuffle $\gamma_{\mathcal{I},n+m}$ be the (n, m)-shuffle determined by the underlying set of $\mathcal{I}$.

Consider the left cosets of $B_{n,m}$ in B_{n+m}.

Proposition 3.3.1. *The collection of left cosets of $B_{n,m}$ in B_{n+m} has the form*

$$B_{n+m}/B_{n,m} = \{\widetilde{\alpha} B_{n,m} : \alpha \in \mathrm{Sh}(n, m)\}.$$

Proof. This is a simple generalization of Proposition 3.2.1. We claim that the left cosets $aB_{n,m}$ are indexed by the image of the integer interval $[\![n{+}1, n{+}m]\!]$ under the underlying permutation of the braid $a \in B_{n+m}$. For any $a_1, a_2 \in B_{n+m}$, $a_1 B_{n,m} = a_2 B_{n,m}$ as cosets iff $a_1^{-1}a_2 \in B_{n,m}$. By definition, this is equivalent to $\underline{a_1}^{-1}\underline{a_2}([\![n+1, n+m]\!]) = [\![n+1, n+m]\!]$, or $\underline{a_1}([\![n+1, n+m]\!]) = \underline{a_2}([\![n+1, n+m]\!])$. So we have a simple characterization of the cosets of $B_{n,m}$ in B_{n+m}: two braid elements of B_{n+m} are in the same coset of $B_{n,m}$ if and only if the images of the interval $[\![n+1, n+m]\!]$ under their underlying permutations coincide. It follows that the index of $B_{n,m}$ in B_{n+m} is $\binom{n+m}{m}$. Furthermore, if we impose that the underlying permutations of the representative braids preserve the order on $[\![n{+}1, n{+}m]\!]$, observe that an explicit choice for the representatives of the cosets of $B_{n,m}$ is the collection of the lifts of all (n, m)-shuffles α, as desired. □

Since $\{\widetilde{\alpha} : \alpha \in \mathrm{Sh}(n, m)\}$ forms a full set of representatives, for each $a \in B_{n+m}$ and each $\widetilde{\alpha}$, there exist uniquely elements $b \in B_{n,m}$ and $\widetilde{\alpha}'$ such that $a\widetilde{\alpha} = \widetilde{\alpha}' b$. We may give a concrete description of these elements. For any (n, m)-shuffle $\alpha_{\mathcal{I}}$ and $p \in S_{n+m}$, let $p_{\alpha_{\mathcal{I}}}$ denote the (n, m)-shuffle associated to $p(\mathcal{I}) = (p(i_1), \ldots, p(i_m))$. Note that in general, $p_{\alpha_{\mathcal{I}}} = p\alpha_{\mathcal{I}}$ if and only if p preserves the order on $\mathcal{I}$ and $[n + m] \setminus \mathcal{I}$. Since the underlying permutation of $b = (\widetilde{\alpha}')^{-1} a \widetilde{\alpha_{\mathcal{I}}} \in B_{n,m}$ preserves $[\![n + 1, n + m]\!]$, it follows that $\alpha'([\![n + 1, n + m]\!]) = \underline{a}\alpha_{\mathcal{I}}([\![n + 1, n + m]\!]) = \underline{a}(\mathcal{I})$ as unordered sets. By the above definition, $\alpha' = \underline{a}_{\alpha_{\mathcal{I}}}$, so $b = \widetilde{\underline{a}_\alpha}^{-1} a\widetilde{\alpha}$.

In the topological setup of this paper (see Sections 4.3 and 6.1), braids often arise from lifting shuffles on $n + m$ elements that preserve the order on the set $\mathcal{I}$ of overall positions of the fixed points in a configuration. As a result, we are particularly interested in the case when the i_j^{th} and i_k^{th} strands of the braid a are pairwise parallel for all distinct $i_j, i_k \in \mathcal{I}$. In this case, for all $n + 1 \leq j < k \leq n + m$, the j^{th} and k^{th} strands of b are always parallel in each successive component $\widetilde{\alpha}$, a, and $\widetilde{\underline{a}_\alpha}^{-1}$. This property allows us to straighten each of the last m strands in the braid, which implies that $b \in B_n(\mathbb{C}_m)$. Moreover, since $\underline{a}$ preserves the order on $\mathcal{I}$, the tuple $\underline{a}(\mathcal{I}) = (\underline{a}(i_1), \ldots, \underline{a}(i_m))$ already has the desired increasing order defining the shuffle $\underline{a}_\alpha$.

As in the previous section, given a $B_{n,m}$-representation L, we may study the induced representation

$$\mathrm{Ind}_{B_{n,m}}^{B_{n+m}}(L) = \mathbf{k}[B_{n+m}] \otimes_{\mathbf{k}[B_{n,m}]} L$$

explicitly using the cosets of the subgroup $B_{n,m}$ in B_{n+m} described above. Since the collection $\{\widetilde{\alpha} : \alpha \in \mathrm{Sh}(n,m)\}$ gives a full set of representatives in B_{n+m} for the left cosets of $B_{n,m}$, as vector spaces, the induced representation can be identified as

$$\mathrm{Ind}_{B_{n,m}}^{B_{n+m}}(L) \cong \bigoplus_{\alpha \in \mathrm{Sh}(n,m)} \widetilde{\alpha} L.$$

Here each $\widetilde{\alpha} L$ is an isomorphic copy of the vector space L whose elements are written as $\widetilde{\alpha}\ell$ where $\ell \in L$. An explicit formula for the action of the braid group on this induced representation follows immediately from the previous paragraphs.

Proposition 3.3.2. *The action of the braid group B_{n+m} on the induced representation $\mathrm{Ind}_{B_{n,m}}^{B_{n+m}}(L)$ is given by*

$$a \sum_{\alpha \in \mathrm{Sh}(n,m)} \widetilde{\alpha}\ell_\alpha = \sum_{\alpha \in \mathrm{Sh}(n,m)} \widetilde{\underline{a}_\alpha}\big[(\widetilde{\underline{a}_\alpha}^{-1} a \widetilde{\alpha})(\ell_\alpha)\big].$$

By a case-by-case analysis similar to the proof of Corollary 3.2.3, we obtain the following computation:

Corollary 3.3.3. *Let $\mathcal{I} = (i_1, \dots, i_m)$ be the indexing set of an (n,m)-shuffle α, and $\mathcal{I}_{k,h} = (i_1, \dots, i_k + h, \dots, i_m)$. The action of the generators of B_{n+m} on $\mathrm{Ind}_{B_{n,m}}^{B_{n+m}}(L)$ can be expressed in terms of the action of the generators of $B_{n,m}$ in the following way:*

$$\sigma_i(\widetilde{\alpha}\ell) = \begin{cases} \widetilde{\alpha}[\sigma_i(\ell)] & 1 \le i < i_1 - 1 \\ \widetilde{\alpha}[\sigma_{i-k}(\ell)] & 1 \le k \le m-1, i_k < i < i_{k+1} - 1 \\ \widetilde{\alpha}[\sigma_{i-m}(\ell)] & i_m < i < n+m \\ \widetilde{\alpha}[\sigma_{n+k}(\ell)] & 1 \le k \le m-1, i = i_k = i_{k+1} - 1 \\ \widetilde{\alpha_{\mathcal{I}_{k,-1}}}\ell & 1 \le k \le m, i = i_k - 1 > i_{k-1} \\ \widetilde{\alpha_{\mathcal{I}_{k,1}}}[(\sigma_{i_k-k+1} \dots \sigma_{n-1}\theta_{n,n+k}\sigma_{n-1}^{-1} \dots \sigma_{i_k-k+1}^{-1})(\ell)] & \\ & 1 \le k \le m, i = i_k < i_{k+1} - 1, \end{cases}$$

where $\theta_{n,n+k} = \sigma_{n+k-1} \dots \sigma_{n+1}\tau_n\sigma_{n+1}^{-1} \dots \sigma_{n+k-1}^{-1}$.

Let (V, W, τ) be a mixed-braided vector space. Recall that there is a representation of $B_{n,m}$ on $V^{\otimes n} \otimes W^{\otimes m}$ by Proposition 3.1.7. Consider the induced representation $\operatorname{Ind}_{B_{n,m}}^{B_{n+m}}(V^{\otimes n} \otimes W^{\otimes m})$. Each summand $\widetilde{\alpha}(V^{\otimes n} \otimes W^{\otimes m})$ of this induced representation is isomorphic to $V^{\otimes n} \otimes W^{\otimes m}$. Since there is a one-to-one correspondence between the set of (n,m)-shuffles and tuples $\mathcal{I} = (i_1, \dots, i_m)$ of m strictly increasing integers in $[\![1, n+m]\!]$, it is natural to identify $\widetilde{\alpha_{\mathcal{I},n+m}}(V^{\otimes n} \otimes W^{\otimes m})$ with

$$V^{\otimes i_1 - 1} \otimes W \otimes V^{\otimes i_2 - i_1 - 1} \otimes W \otimes \cdots \otimes W \otimes V^{\otimes n+m-i_m},$$

where the i_k^{th} tensor factor is W for all $1 \leq k \leq m$, via an isomorphism $\xi_{\mathcal{I},n+m} : \widetilde{\alpha_{\mathcal{I},n+m}}(V^{\otimes n} \otimes W^{\otimes m}) \xrightarrow{\cong} V^{\otimes i_1 - 1} \otimes W \otimes V^{\otimes i_2 - i_1 - 1} \otimes W \otimes \cdots \otimes W \otimes V^{\otimes n+m-i_m}$. The following mirrors Proposition 3.2.4:

Proposition 3.3.4. *There is an isomorphism of vector spaces*

$$\operatorname{Ind}_{B_{n,m}}^{B_{n+m}}(V^{\otimes n} \otimes W^{\otimes m}) \cong \bigoplus_{\mathcal{I}} V^{\otimes i_1 - 1} \otimes W \otimes V^{\otimes i_2 - i_1 - 1} \otimes W \otimes \cdots \otimes W \otimes V^{\otimes n+m-i_m}$$

where $\mathcal{I}$ runs over all tuples $(i_1, \dots, i_m)$ with $1 \leq i_1 < \cdots < i_m \leq n+m$.

Moreover, given a choice of isomorphisms $\xi_{\mathcal{I},n+m}$, there is a B_{n+m}-action on the right hand side defined by $a \mapsto \xi_{\underline{a}(\mathcal{I})} a \xi_{\mathcal{I}}^{-1}$, such that the above is an isomorphism of B_{n+m}-representations.

Proof. The first statement results directly from the previous paragraph. The second follows immediately from the definition of the action. □

By convention, $\xi_{\mathcal{I},n+m}$ is always the identity if $\mathcal{I} = (n+1, \dots, n+m)$. The second index of the map $\xi_{\mathcal{I},n+m}$ again denotes the total degree of the domain and is omitted from the notation if there is no ambiguity. Observe that on the right hand side, we can apply braids that "swap" V and W, an operation that is forbidden in the $B_{n,m}$-representation on $V^{\otimes n} \otimes W^{\otimes m}$. This allows for a more intuitive framework to study the action of B_{n+m} on the induced representation, more analogous to its action in the monoidal braid representation on $V^{\otimes n+m}$. As in the quantum shuffle algebra, we will also denote elements of $V^{\otimes i_1 - 1} \otimes W \otimes V^{\otimes i_2 - i_1 - 1} \otimes W \otimes \cdots \otimes W \otimes V^{\otimes n+m-i_m}$ by the bar complex notation, i.e., $[v_1 | \dots | v_{i_k - 1} | w_k | v_{i_k + 1} | \dots | v_{n+m}]$.

Recall that for the purpose of this paper, we are interested the action of braids a

$$\begin{array}{ccc}
Y^{\otimes p} \otimes (V^{\otimes j-1} \otimes W_k \otimes V^{q-j}) \otimes Y^{\otimes n+m-p-q} & \xrightarrow{\xi^{-1}_{\mathcal{I},n+m}} & \widetilde{\alpha_{\mathcal{I},n+m}}(V^{\otimes n} \otimes W^{\otimes m}) \\
\Big\downarrow {\scriptstyle \mathrm{id}^{\otimes p} \otimes \xi^{-1}_{j,q} \otimes \mathrm{id}^{\otimes n+m-p-q}} & & \\
Y^{\otimes p} \otimes \widetilde{\alpha_{j,q}}(V^{\otimes q-1} \otimes W) \otimes Y^{\otimes n+m-p-q} & & \Big\downarrow {\scriptstyle a'} \\
\Big\downarrow {\scriptstyle \mathrm{id}^{\otimes p} \otimes a \otimes \mathrm{id}^{\otimes n+m-p-q}} & & \\
Y^{\otimes p} \otimes \widetilde{\alpha_{\underline{a}(j),q}}(V^{\otimes q-1} \otimes W) \otimes Y^{\otimes n+m-p-q} & & \\
\Big\downarrow {\scriptstyle \mathrm{id}^{\otimes p} \otimes \xi_{\underline{a}(j),q} \otimes \mathrm{id}^{\otimes n+m-p-q}} & & \\
Y^{\otimes p} \otimes (V^{\otimes \underline{a}(j)-1} \otimes W_k \otimes V^{q-\underline{a}(j)}) \otimes Y^{\otimes n+m-p-q} & \xleftarrow{\xi_{\underline{a'}(\mathcal{I}),n+m}} & \widetilde{\alpha_{\underline{a'}(\mathcal{I}),n+m}}(V^{\otimes n} \otimes W^{\otimes m})
\end{array}$$

Figure 3.3: Diagram $\mathscr{E}$.

whose i_j^{th} and i_k^{th} strands are pairwise parallel for all distinct $i_j, i_k \in \mathcal{I}$. In particular, the order on $\mathcal{I}$ is preserved throughout a. It follows that generators in the standard decomposition of a, where each crossing in a corresponds to a braid generator or its inverse, never swap two copies of W on the right hand side of Proposition 3.3.4. It is therefore suggestive to denote the k^{th} occurrence of W by W_k; we will adopt this convention whenever this case applies.

Recall from the previous section that the separability of (V, W) as a left-braided vector space is the necessary and sufficient condition for the B_{n+1}-representation on $\bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$ to behave analogously to that on $V^{\otimes n+1}$, i.e., Diagram $\mathscr{D}$ (Figure 3.2) commutes in all applicable cases. We desire a similar property for the B_{n+m}-action on $\bigoplus_{\mathcal{I}} V^{\otimes i_1-1} \otimes W \otimes V^{\otimes i_2-i_1-1} \otimes W \otimes \cdots \otimes W \otimes V^{\otimes n+m-i_m}$. That is, for any $n, m \geq 1$, $1 \leq j \leq q \leq n+m$, and $a \in B_q$, Diagram $\mathscr{E}$ (Figure 3.3) commutes where $p = i_k - j$, Y denotes a copy of either V or W, and the braid a' is the natural inclusion of a into the copy $B_q \leq B_{n+m}$ consisting of braids that are only nontrivial on the q strands starting with the $p+1^{\text{st}}$. Roughly speaking, we want the inclusion of the subspace $\bigoplus_{j=1}^{q} V^{\otimes j-1} \otimes W_k \otimes V^{\otimes q-j}$ into $\bigoplus_{\mathcal{I}} V^{\otimes i_1-1} \otimes W_1 \otimes V^{\otimes i_2-i_1-1} \otimes W_2 \otimes \cdots \otimes W_m \otimes V^{\otimes n+m-i_m}$ to be equivariant with respect to the braid action. The following proposition gives criteria to detect this property.

Proposition 3.3.5. *Let (V, W, τ) be a mixed-braided vector space. For $\mathcal{I} = (i_1, \ldots, i_m)$, let $\varphi_{\mathcal{I},n+m} : V^{\otimes n} \otimes W^{\otimes m} \to V^{\otimes i_1-1} \otimes W_1 \otimes V^{\otimes i_2-i_1-1} \otimes W_2 \otimes \cdots \otimes W_m \otimes V^{\otimes n+m-i_m}$ be* *defined by* $\varphi_{\mathcal{I},n+m} := \xi_{\mathcal{I},n+m}\widetilde{\alpha_{\mathcal{I},n+m}}$ *for all* $n \geq 1$*. Then Diagram* $\mathscr{E}$ *always commutes if and only if* (V, W) *is separable as a left-braided vector space with a separated braiding* φ *and the following identities hold:*

1. $\varphi_{\mathcal{I},n+m} = \varphi_{i_m,n+m} \circ \cdots \circ (\varphi_{i_2,n+2} \otimes \mathrm{id}^{\otimes m-2}) \circ (\varphi_{i_1,n+1} \otimes \mathrm{id}^{\otimes m-1})$*;*

2. $(\mathrm{id} \otimes \tau) \circ (\varphi \otimes \mathrm{id}) = (\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W^{-1})$*.*

Proof. The separability of (V, W) means the following hold:

3. $(\mathrm{id} \otimes \sigma) \circ (\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) = (\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) \circ (\sigma \otimes \mathrm{id})$;

4. $(\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) = (\mathrm{id} \otimes \varphi) \circ (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\sigma^{-1} \otimes \mathrm{id})$.

Since every braid action is decomposable into those of the generators, it suffices to study the commutativity of Diagram $\mathscr{E}$ for all braid generators. By brute force, we observe that Diagram $\mathscr{E}$ commutes for all braid generators if and only if the following hold:

(a) $(\mathrm{id}^{\otimes p} \otimes \varphi_{j-p,i_k-p} \otimes \mathrm{id}^{\otimes n+m-i_k})\varphi_{\mathcal{I},n+m} = \varphi_{\mathcal{I}_{k,j-i_k},n+m}$
for $0 \leq p < j$ and $i_{k-1} < j \leq i_k$;

(b) $$\sigma_i\varphi_{\mathcal{I},n+m} = \begin{cases} \varphi_{\mathcal{I},n+m}\sigma_i & \text{if } 1 \leq i \leq i_1 - 2 \\ \varphi_{\mathcal{I},n+m}\sigma_{i-k} & \text{if } i_k + 1 \leq i \leq i_k - 2 \\ \varphi_{\mathcal{I},n+m}\sigma_{i-m} & \text{if } i_m + 1 \leq i < n + m \end{cases}$$
for $\sigma_i = \mathrm{id}^{\otimes i-1} \otimes \sigma \otimes \mathrm{id}^{\otimes n+m-i-1}$;

(c) $\tau_{i_k-1}\varphi_{\mathcal{I},n+m} = \varphi_{\mathcal{I},n+m}(\sigma_{i_k-k} \ldots \sigma_{n-1}\theta_{n,n+k}\sigma_{n-1}^{-1} \ldots \sigma_{i_k-k}^{-1})$
for $\tau_i = \mathrm{id}^{\otimes i-1} \otimes \tau \otimes \mathrm{id}^{\otimes n+m-i-1}$.

We will show that these conditions are equivalent to (1–4).

First, we will prove that (a) is equivalent to (1) given the formula (cf. Proposition 3.2.5):

$$\varphi_{i,n} = (\mathrm{id}^{\otimes i-1} \otimes \varphi \otimes \mathrm{id}^{\otimes n-i-1}) \circ (\mathrm{id}^{\otimes i} \otimes \varphi \otimes \mathrm{id}^{\otimes n-i-2}) \circ \cdots \circ (\mathrm{id}^{\otimes n-2} \otimes \varphi).$$

It is easy to verify that the formula given in (1) for $\varphi_{\mathcal{I},n+m}$ satisfies (a). Conversely, we will derive (1) by induction on the smallest $k \geq 1$ such that $i_k = n + k$ (which implies that $i_r = n+r$ for all $k \le r \le m$). The base case $k=1$ is trivial, as both sides are the identity. Suppose for all $\mathcal{I} = (i_1, \dots, i_{k-1}, n+k, \dots, n+m)$,

$$\begin{aligned}\varphi_{\mathcal{I},n+m} &= \varphi_{i_m,n+m} \circ \dots \circ (\varphi_{i_k,n+k} \otimes \mathrm{id}^{\otimes m-k}) \circ (\varphi_{i_{k-1},n+k-1} \otimes \mathrm{id}^{\otimes m-k+1}) \circ \dots \\ &\quad \circ (\varphi_{i_1,n+1} \otimes \mathrm{id}^{\otimes m-1}) \\ &= (\varphi_{i_{k-1},n+k-1} \otimes \mathrm{id}^{\otimes m-k+1}) \circ \dots \circ (\varphi_{i_1,n+1} \otimes \mathrm{id}^{\otimes m-1}).\end{aligned}$$

Observe that any $\mathcal{I}' = (i_1, \dots, i_{k-1}, i_k, n+k+1, \dots, n+m)$ can be written as $\mathcal{I}_{k,i_k-(n+k)}$, then by (a) we have

$$\begin{aligned}\varphi_{\mathcal{I}',n+m} &= \varphi_{\mathcal{I}_{k,i_k-(n+k)},n+m} = (\varphi_{i_k,n+k} \otimes \mathrm{id}^{\otimes n+m-(n+k)})\varphi_{\mathcal{I},n+m} \\ &= (\varphi_{i_k,n+k} \otimes \mathrm{id}^{\otimes m-k}) \circ (\varphi_{i_{k-1},n+k-1} \otimes \mathrm{id}^{\otimes m-k+1}) \circ \dots \circ (\varphi_{i_1,n+1} \otimes \mathrm{id}^{\otimes m-1})\end{aligned}$$

which proves the claim for case $k+1$. With this identification, observe that (c) gives $\tau_2\varphi_{(1,3,\dots,m+1),m+1} = \varphi_{(1,3,\dots,m+1),m+1}\theta_{1,3}$, which is (2). Observe that (b–c) in the proof of Proposition 3.2.5 are special cases of (b–c) above, hence (3–4) follows. We have thus proved the forward direction.

The converse can be verified using a similar induction argument. Condition (b) is implied by (1) and the first three cases of Corollary 3.3.3. Meanwhile, (c) follows from (2–4) and the last case of the same corollary. □

As before, we package this desired property into the following definition:

Definition 3.3.6. A mixed-braided vector space (V, W, τ) is *left-separable* if it is separable as a left-braided vector space with a separated braiding φ, and

$$(\mathrm{id} \otimes \tau) \circ (\varphi \otimes \mathrm{id}) = (\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W^{-1}).$$

The technical requirement of separability is essential for our topological arguments in this paper, particularly the proof of Proposition 6.3.1. It is unclear to us whether all mixed-braided vector spaces are left-separable; however, for the few natural examples that are most relevant to the applications discussed in this dissertation, it is easy to detect a suitable separated braiding (see Example 3.2.7).

As for left-braided vector spaces, the left-separability of a mixed-braided vector space

(V, W) is integrally connected to the existence of a braid structure on the direct sum $V \oplus W$. The following mirrors Proposition 3.2.8:

Proposition 3.3.7. *Let V and W be finite dimensional $\mathbf{k}$-vector spaces, and let $X = V \oplus W$. Suppose there is an automorphism σ_X of $(V \oplus W)^{\otimes 2} \cong V^{\otimes 2} \oplus (V \otimes W) \oplus (W \otimes V) \oplus W^{\otimes 2}$ defined summand-wise by isomorphisms $\sigma_V : V^{\otimes 2} \to V^{\otimes 2}$, $\varphi : V \otimes W \to W \otimes V$, $\psi : W \otimes V \to V \otimes W$, and $\sigma_W : W^{\otimes 2} \to W^{\otimes 2}$.*

1. *If (X, σ_X) is a braided vector space, then $((V, \sigma_V), (W, \sigma_W), \tau, \varphi)$ is a left-separable mixed-braided vector space where $\tau = \psi\varphi$;*

2. *A weak version of the converse holds: if (V, W, τ, φ) is a left-separable mixed-braided vector space such that*

$$(\sigma_W \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi) \circ (\varphi \otimes \mathrm{id}) = (\mathrm{id} \otimes \varphi) \circ (\varphi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_W)$$

   then (X, σ_X) is a braided vector space.

Proof. The argument is a simple modification of the proof of Proposition 3.2.8 and thus is omitted. For interested readers, see Proposition 3.16 of [54]. □

Chapter 4

Stratifications of configuration spaces

In this chapter, we first recall the Fox–Neuwirth stratification of $\mathrm{Conf}_n(\mathbb{C})$ by Euclidean spaces, which provides a CW-complex structure for the 1-point compactification of $\mathrm{Conf}_n(\mathbb{C})$. This construction was first demonstrated by Fox–Neuwirth [46] and Fuks [48], and further studied in [91, 51, 40]; the treatment detailed here is taken primarily from [40]. We then extend it to produce a cellular stratification for configuration spaces of the plane with punctures. For simplicity, we will first demonstrate a stratification of $\mathrm{Conf}_n(\mathbb{C}^\times)$, before generalizing to the case with an arbitrary number of punctures. The content of this chapter is extracted from Sections 3.1 and 4.1 of [55], and Section 2.2 of [54].

4.1 Fox–Neuwirth cellular stratification of $\mathrm{Conf}_n(\mathbb{C})$

A *composition* λ of n is an ordered partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n where $\sum \lambda_i = n$. The number of parts k is called the *length* of λ, denoted by $l(\lambda)$. Recall that the n^{th} symmetric product $\mathrm{Sym}_n(\mathbb{R})$ has a stratification given by these partitions $\mathrm{Sym}_n(\mathbb{R}) = \coprod_{\lambda \vdash n} \mathrm{Sym}_\lambda(\mathbb{R})$. Elements of $\mathrm{Sym}_\lambda(\mathbb{R})$ are unordered subsets of $l(\lambda)$ distinct points $x_1, \ldots, x_{l(\lambda)}$, where the multiplicity of x_i is λ_i. We further assume that $x_1 < \cdots < x_{l(\lambda)}$ where the ordering is that of the real line. Define a map $\pi : \mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Sym}_n(\mathbb{R})$ by projecting each coordinate onto the real line, i.e., $\pi(z_1, \ldots, z_n) = (\mathrm{Re}(z_1), \ldots, \mathrm{Re}(z_n))$. The

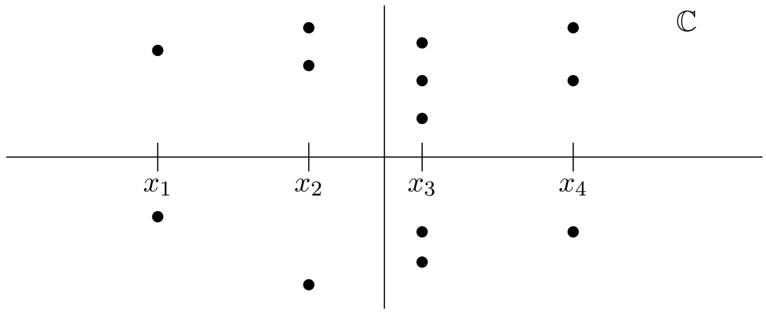

Figure 4.1: A configuration in $\mathrm{Conf}_{(2,3,5,3)}(\mathbb{C}) \subset \mathrm{Conf}_{13}(\mathbb{C})$.

preimage of $\mathrm{Sym}_\lambda(\mathbb{R})$ under π, $\mathrm{Conf}_\lambda(\mathbb{C})$ is homeomorphic to $\mathrm{Sym}_\lambda(\mathbb{R}) \times \prod_{i=1}^{l(\lambda)} \mathrm{Conf}_{\lambda_i}(\mathbb{R})$, where each configuration factor $\mathrm{Conf}_{\lambda_i}(\mathbb{R})$ keeps track of the imaginary parts of points with the same real part. For example, Figure 4.1 shows a configuration in the cell $\mathrm{Conf}_{(2,3,5,3)}(\mathbb{C})$ of $\mathrm{Conf}_{13}(\mathbb{C})$. This configuration is mapped by π to

$$(x_1, x_1, x_2, x_2, x_2, x_3, x_3, x_3, x_3, x_3, x_4, x_4, x_4) \in \mathrm{Sym}_{(2,3,5,3)}(\mathbb{R}) \subset \mathrm{Sym}_{13}(\mathbb{R}).$$

Geometrically, the points in the configuration are arranged into columns based on their (ordered) real coordinates, while the configuration factors $\mathrm{Conf}_{\lambda_i}(\mathbb{R})$ record the imaginary parts of the points in respective columns. Since $\mathrm{Sym}_k(\mathbb{R}) \cong \mathrm{Conf}_k(\mathbb{R}) \cong \mathbb{R}^k$, that implies $\mathrm{Conf}_\lambda(\mathbb{C}) \cong \mathbb{R}^{n+l(\lambda)}$.

The collection of the spaces $\mathrm{Conf}_\lambda(\mathbb{C})$ forms a cellular decomposition for the 1-point compactification of $\mathrm{Conf}_n(\mathbb{C})$. Each configuration space $\mathrm{Conf}_\lambda(\mathbb{C})$ gives a cell of dimension $n + l(\lambda)$. Loosely speaking, a cell can be constructed by first picking a composition λ of n, placing the points in the configuration into columns according to the partition, and finally letting the columns to move horizontally (the symmetric part of the product) and the points on each column to move vertically without colliding (the configuration part of the product). The *lexicographic order* of points in such a configuration is defined by labelling the lowest point on the left most column with 1, increasing the indices as we move up, and continuing the process for all subsequent columns on the right. The index of a point in this order is called the *overall position* of that point in the configuration.

The boundary of a cell is obtained in two ways. The first type of boundary occurs by moving a point in a configuration to approach either the point at infinity or another point on the same vertical line, i.e., with the same real part. In this case, the boundary is the point at infinity, since the number of points in the configuration decreases to $n-1$ and hence the configuration is no longer an element in $\mathrm{Conf}_n(\mathbb{C})$. The second type of boundary occurs by horizontally joining two neighboring vertical columns of the configuration without colliding the points. The boundary cell of $\mathrm{Conf}_\lambda(\mathbb{C})$ obtained by joining the i^{th} and $i+1^{\mathrm{st}}$ columns has the form $\mathrm{Conf}_\rho(\mathbb{C})$, where $\rho^i = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \lambda_{i+1}, \ldots, \lambda_k)$ is the *coarsening* of λ obtained by summing λ_i and λ_{i+1} ($1 \leq i < l(\lambda)$).

Proposition 4.1.1 (Fox–Neuwirth [46], Fuks [48])**.** *The space* $\mathrm{Conf}_n(\mathbb{C}) \cup \{\infty\}$ *has a CW-complex decomposition where the positive dimension cells are given by* $\mathrm{Conf}_\lambda(\mathbb{C})$ *(of dimension* $n + l(\lambda)$*) with indices* λ *coming from compositions of* n*. The boundary of* $\mathrm{Conf}_\lambda(\mathbb{C})$ *is the union of* $\mathrm{Conf}_\rho(\mathbb{C})$ *where* λ *is a refinement of* ρ*.*

Fox–Neuwirth and Fuks provided an explicit cellular chain complex for $\mathrm{Conf}_n(\mathbb{C}) \cup \{\infty\}$ based on this decomposition:

Definition 4.1.2 (The Fox–Neuwirth complex)**.** For integers i and j, let $c_{i,j} = \sum_\gamma (-1)^{|\gamma|}$ be the sum of the signs of all (i,j)-shuffles $\gamma : \{1, \ldots, i\} \sqcup \{1, \ldots, j\} \to \{1, \ldots, i+j\}$. Let $C(n)_*$ denote the chain complex which in degree q is generated over $\mathbb{Z}$ by the set of ordered partitions $\lambda = (\lambda_1, \ldots, \lambda_{q-n})$ of n with $q-n$ parts. The differential $d : C(n)_q \to C(n)_{q-1}$ is given by the formula

$$d(\lambda_1, \ldots, \lambda_{q-n}) = \sum_{i=1}^{q-n-1} (-1)^{i-1} c_{\lambda_i, \lambda_{i+1}} (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \lambda_{i+1}, \ldots, \lambda_{q-n}).$$

The constant $c_{\lambda_i, \lambda_{i+1}}$ results from the formation of the boundary cells of $\mathrm{Conf}_\lambda(\mathbb{C})$ by shuffling the points in the i^{th} and $i+1^{\mathrm{st}}$ columns into a single vertical line. The signs in the formula arise from the induced orientations on the boundary strata, as detailed by Giusti and Sinha [51]. There is a simple formula to compute this constant using the quantum binomial coefficient (see, e.g., Proposition 1.7.1 of [87]):

$$c_{p,q} = \binom{p+q}{q}_{-1}.$$

The complex $C(n)_*$ is isomorphic to the cellular chain complex of $\mathrm{Conf}_n(\mathbb{C}) \cup \{\infty\}$, relative to the point at infinity. For more about this complex, consult [48, 90, 91].

4.2 Cellular stratification of $\mathrm{Conf}_n(\mathbb{C}^\times)$

First, we observe that there is a canonical embedding $\mathrm{Conf}_n(\mathbb{C}^\times) \hookrightarrow \mathrm{Conf}_{n+1}(\mathbb{C})$ by inserting the removed origin z_1. This gives a homeomorphic image of $\mathrm{Conf}_n(\mathbb{C}^\times)$ as a subspace of $\mathrm{Conf}_{n+1}(\mathbb{C})$ consisting of all configurations where the point z_1 is always fixed. We will give a stratification of this subspace based on the Fox–Neuwirth cellular stratification of $\mathrm{Conf}_{n+1}(\mathbb{C})$ developed above. For the rest of this paper, we will indiscriminately use the notation $\mathrm{Conf}_n(\mathbb{C}^\times)$ for both the original configuration space of the punctured complex plane and its homeomorphic image in $\mathrm{Conf}_{n+1}(\mathbb{C})$.

Given a composition λ of $n+1$, we consider the intersection of the cell $\mathrm{Conf}_\lambda(\mathbb{C})$ and the subspace $\mathrm{Conf}_n(\mathbb{C}^\times)$ of $\mathrm{Conf}_{n+1}(\mathbb{C})$. Starting with a configuration in $\mathrm{Conf}_\lambda(\mathbb{C})$, we insist that one of the points must be the fixed point z_1 at the origin. This requirement restricts the configuration in two ways. First, the vertical column that contains z_1 must coincide with the imaginary axis, i.e., the real part of all points on that column must be 0. This column plays a special role in our stratification and will be recorded by the index i. Secondly, as we let the points in a configuration move along the vertical line without colliding in a cell, fixing z_1 implies that it is impossible for points on the imaginary axis to move past it. The number of points on this vertical line with a negative imaginary part is hence fixed and denoted by the index j. Therefore, the connected components in the above intersection can be denoted by $e_{(\lambda,i,j)} = \mathrm{Conf}_{(\lambda,i,j)}(\mathbb{C})$ where λ is a composition of $n+1$, i is the index of the vertical column that contains z_1, i.e., the imaginary axis ($1 \leq i \leq l(\lambda)$), and j is the number of points lying below z_1 on the imaginary axis ($0 \leq j \leq \lambda_i - 1$). For example, Figure 4.2 shows a configuration in $e_{((2,3,4,5,3),3,1)} \subset \mathrm{Conf}_{16}(\mathbb{C}^\times)$. The removed origin z_1 (mapped to a fixed point in the embedded image of $\mathrm{Conf}_{16}(\mathbb{C}^\times)$ in $\mathrm{Conf}_{17}(\mathbb{C})$) lies on the third vertical column from the left with one point below. The lexicographic order of a configuration in the embedded image of $\mathrm{Conf}_n(\mathbb{C}^\times)$ is inherited from the parent space $\mathrm{Conf}_{n+1}(\mathbb{C})$; in particular, the overall position of z_1 in a configuration in $e_{(\lambda,i,j)}$ is $\iota = j + 1 + \sum_{m=1}^{i-1} \lambda_m$.

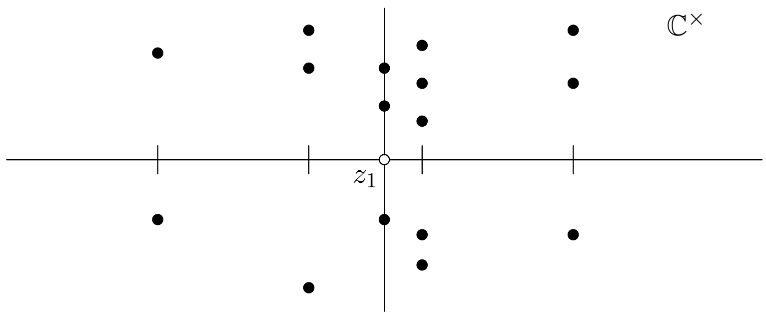

Figure 4.2: A configuration in $e_{((2,3,4,5,3),3,1)} \subset \mathrm{Conf}_{16}(\mathbb{C}^\times)$.

The spaces $e_{(\lambda,i,j)}$ then provide the positive dimension cells for our cellular decomposition of $\mathrm{Conf}_n(\mathbb{C}^\times) \cup \{\infty\}$. Each cell $e_{(\lambda,i,j)}$ is homeomorphic to the product

$$\Big[\mathrm{Conf}_{i-1}(\mathbb{R}) \times \mathrm{Conf}_{l(\lambda)-i}(\mathbb{R})\Big] \times \prod_{k=1,k\neq i}^{l(\lambda)} \mathrm{Conf}_{\lambda_k}(\mathbb{R}) \times \Big[\mathrm{Conf}_j(\mathbb{R}) \times \mathrm{Conf}_{\lambda_i-j-1}(\mathbb{R})\Big].$$

The first bracket represents the configurations of the vertical columns on the left and right of the imaginary axis, i.e., recording the real parts of the points. The latter bracket keeps track of the imaginary parts of points below and above z_1 on the imaginary axis, while the middle product records the same information for those on all other vertical lines. By applying the isomorphism $\mathrm{Conf}_k(\mathbb{R}) \cong \mathbb{R}^k$, we see that the cell $e_{(\lambda,i,j)}$ has dimension $n + l(\lambda) - 1$; loosely speaking, compared to the classical Fox–Neuwirth cells indexed by the same composition λ, we lost two dimensions due to fixing the real and imaginary parts of the point z_1.

As in the Fox–Neuwirth cellular decomposition of $\mathrm{Conf}_n(\mathbb{C})$, the boundary of a cell is obtained in two ways. For the first type, besides letting points in a configuration approach another or infinity, we also allow moving points towards the punctured origin; in this case, the boundary is still the point at infinity for the same reason. The second type of boundary again occurs by horizontally joining two adjacent vertical columns of the configuration without colliding the points. However, due to the second restriction on a configuration in $e_{(\lambda,i,j)}$, namely points below the fixed point z_1 cannot move across it on the imaginary axis, the boundary cells obtained this way have four general forms, depending on the positions of the columns relative to the imaginary axis and on whether this axis itself is among those combined. In particular, when combining an adjacent vertical column with the imaginary axis, we must keep track of the number of points going below z_1, i.e., adding to the index j. In summary:

Proposition 4.2.1. *The space* $\mathrm{Conf}_n(\mathbb{C}^\times) \cup \{\infty\}$ *may be presented as a CW complex whose positive dimension cells* $e_{(\lambda,i,j)} = \mathrm{Conf}_{(\lambda,i,j)}(\mathbb{C})$ *(of dimension* $n + l(\lambda) - 1$*) are indexed by triples* (λ, i, j)*, where* λ *is an ordered partition of* $n+1$*,* i *is the index of the imaginary axis in the configuration* $(1 \leq i \leq l(\lambda))$*, i.e., there are* $i-1$ *vertical columns to the left of the imaginary axis, and* j *is the number of points with zero real parts and negative imaginary parts* $(0 \leq j \leq \lambda_i - 1)$*.*

Let $\rho^m = (\lambda_1, \ldots, \lambda_m + \lambda_{m+1}, \ldots, \lambda_{l(\lambda)})$ *be the coarsening of* λ *obtained by summing* λ_m *and* λ_{m+1} $(1 \leq m < l(\lambda))$*. The codimension-*1 *boundary cells of* $e_{(\lambda,i,j)}$ *have four general forms:*

1. $e_{(\rho^m,i-1,j)}$ $\qquad 1 \leq m < i-1,$
2. $e_{(\rho^m,i,j)}$ $\qquad i < m < l(\lambda),$
3. $e_{(\rho^{i-1},i-1,j+h)}$ $\qquad 0 \leq h \leq \lambda_{i-1},$
4. $e_{(\rho^i,i,j+h)}$ $\qquad 0 \leq h \leq \lambda_{i+1},$

where h *denotes the number of points going below the origin* z_1 *when combining the imaginary axis with the column on the left (3) or right (4).*

From this we can write down an explicit cellular chain complex for $\mathrm{Conf}_n(\mathbb{C}^\times)\cup\{\infty\}$. First, we will develop some combinatorial concepts needed to state the definition of this complex.

Recall that a (p, q)-shuffle $\gamma : \{1, \ldots, p\} \sqcup \{1, \ldots, q\} \to \{1, \ldots, p+q\}$ is a bijection that preserves orders on both $\{1, \ldots, p\}$ and $\{1, \ldots, q\}$. Alternatively, a (p, q)-shuffle can be treated as a permutation in S_{p+q} that preserves orders on the first p and the last q elements. In the discussion below, we will primarily refer to them by the latter definition.

Definition 4.2.2. For $0 \leq h \leq q$ and $0 \leq j \leq p-1$, a $(p,(q,h),j)$-*shuffle* is defined to be a (p, q)-shuffle that sends $j+1$ to $j+h+1$. Similarly, for $0 \leq h \leq p$ and $0 \leq j \leq q-1$, a $((p, h), q, j)$-*shuffle* is a (p, q)-shuffle that sends $p+j+1$ to $h+j+1$.

For convenience, we will refer to the designated elements in the above definition as the *marked elements*; they will later correspond to the fixed point z_1 in a configuration in $\mathrm{Conf}_n(\mathbb{C}^\times)$. The naming conventions of these shuffles are geometrically motivated: for example, the $(p,(q,h),j)$-shuffles arise when we combine the imaginary axis with p points in total and j points below the fixed point z_1 from the right with a column containing q points, sending h out of q points below z_1 in the process. Similarly, the $((p,h),q,j)$-shuffles occur when the imaginary axis with q points in total and j points below z_1 is joined from the left by a column with p points, sending h out of p points below z_1.

Consider a $(p,(q,h),j)$-shuffle γ. Since γ as a (p,q)-shuffle preserves orders on the first p and the last q elements, and γ sends the marked element from $j+1$ to $j+h+1$, it must follows that

$$\gamma(p+1) < \gamma(p+2) < \cdots < \gamma(p+h) < j+h+1;$$

loosely speaking, the elements in the integer interval $[\![p+1,p+h]\!]$ must "move left" on the number line to fill in the h holes left behind by the move of the marked element. This observation leads to a useful decomposition of the $(p,(q,h),j)$-shuffles.

Proposition 4.2.3. *There is a unique decomposition of a $(p,(q,h),j)$-shuffle into a sequence of three permutations: a fixed (p,q)-shuffle that maps $[\![p+1,p+h]\!]$ onto $[\![j+1,j+h]\!]$, followed by a (j,h)-shuffle on $[\![1,j+h]\!]$ and a $(p-j-1,q-h)$-shuffle on the last $[\![j+h+2,p+q]\!]$ (see Figure 4.3). As a consequence, there is a bijection*

$$\mathrm{Sh}(p,(q,h),j) \cong \mathrm{Sh}(j,h) \times \mathrm{Sh}(p-j-1,q-h)$$

where $\mathrm{Sh}(p,(q,h),j)$, $\mathrm{Sh}(j,h)$ and $\mathrm{Sh}(p-j-1,q-h)$ are the sets of $(p,(q,h),j)$-, (j,h)- and $(p-j-1,q-h)$-shuffles, respectively.

Proof. Let γ be a $(p,(q,h),j)$-shuffle. Let ω be the permutation of $p+q$ elements defined

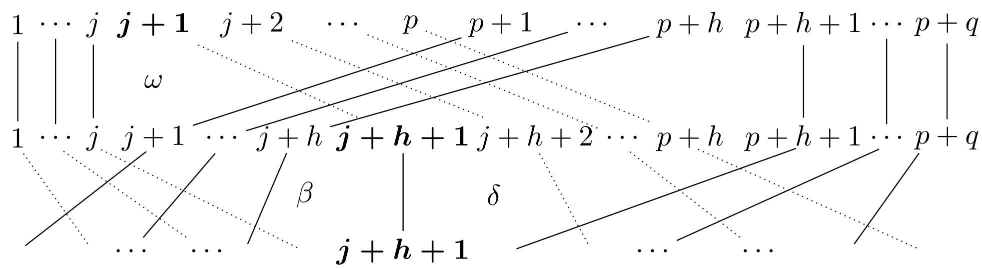

Figure 4.3: Decomposition of a $(p,(q,h),j)$-shuffle.

by

$$\omega(m) = \begin{cases} m & 1 \leq m \leq j \\ m + h & j + 1 \leq m \leq p \\ m - p + j & p + 1 \leq m \leq p + h \\ m & p + h + 1 \leq m \leq p + q. \end{cases}$$

One can verify that ω is a (p, q)-shuffle. Furthermore, observe that ω maps the integer interval $[\![p + 1, p + h]\!]$ to $[\![j + 1, j + h]\!]$, thus sending the first h elements in the second set (while maintaining their order) left past the marked element now at $j + h + 1$.

Consider γ as a permutation of $p + q$ elements. Since γ preserves order on the first p elements, $\gamma(m) \geq \gamma(j + 1) = j + h + 1$ for all $j + 1 \leq m \leq p$. On the other hand, $\gamma(p + h)$ must be smaller than $j + h + 1$, since otherwise $\gamma(m) \geq \gamma(p + h) \geq j + h + 1$ for all $p + h \leq m \leq p + q$ and hence γ maps at least $p + q - j - h + 1$ elements bijectively onto $[\![j + h + 1, p + q]\!]$, a contradiction. Since γ also preserves order on the last q elements, $\gamma(m) \leq \gamma(p + h) < j + h + 1$ for all $p + 1 \leq m \leq p + h$. It follows that $\gamma([\![1, j]\!] \sqcup [\![p+1, p+h]\!]) = [\![1, j+h]\!]$ and $\gamma([\![j+1, p]\!] \sqcup [\![p+h+1, p+q]\!]) = [\![j+h+1, p+q]\!]$; moreover, γ is order-preserving on each of the four component intervals in these two disjoint unions. Observe that ω^{-1} restricts to order-preserving bijections:

$$\llbracket 1, j\rrbracket \xrightarrow{\text{id}} \llbracket 1, j\rrbracket,$$
$$\llbracket j+1, j+h\rrbracket \xrightarrow{+(p-j)} \llbracket p+1, p+h\rrbracket,$$
$$\llbracket j+h+1, p+h\rrbracket \xrightarrow{-h} \llbracket j+1, p\rrbracket, \text{ and}$$
$$\llbracket p+h+1, p+q\rrbracket \xrightarrow{\text{id}} \llbracket p+h+1, p+q\rrbracket.$$

Hence $\gamma\omega^{-1}$ maps each of the intervals $\llbracket 1, j+h\rrbracket$ and $\llbracket j+h+1, p+q\rrbracket$ bijectively onto itself, while preserving orders on the subintervals $\llbracket 1, j\rrbracket$, $\llbracket j+1, j+h\rrbracket$, $\llbracket j+h+1, p+h\rrbracket$, and $\llbracket p+h+1, p+q\rrbracket$. Since $\gamma\omega^{-1}(j+h+1) = j+h+1$, the restrictions of $\gamma\omega^{-1}$ on $\llbracket 1, j+h\rrbracket$ and $\llbracket j+h+2, p+q\rrbracket$ then form a (j,h)-shuffle and a $(p-j-1, q-h)$-shuffle, respectively. Thus we may write $\gamma\omega^{-1} = \delta\beta$, where β nontrivially only on $\llbracket 1, j+h\rrbracket$ by a (j,h)-shuffle, and δ acts nontrivially only on $\llbracket j+h+2, p+q\rrbracket$ by a $(p-j-1, q-h)$-shuffle. This gives a decomposition $\gamma = \delta\beta\omega$ as desired. Since ω is fixed, this decomposition is unique if so is the decomposition of $\gamma\omega^{-1}$ into the shuffles β and δ for any $(p,(q,h),j)$-shuffle γ, which is evident.

The existence and uniqueness of the decomposition of $(p,(q,h),j)$-shuffles above give an injection

$$\mathrm{Sh}(p,(q,h),j) \hookrightarrow \mathrm{Sh}(j,h) \times \mathrm{Sh}(p-j-1, q-h).$$

On the other hand, given a (j,h)-shuffle β and a $(p-j-1, q-h)$-shuffle δ, from the previous argument it is easy to see that $\gamma = \delta'\beta'\omega$ is a $(p,(q,h),j)$-shuffle, where β' acts as β on $\llbracket 1, j+h\rrbracket$ and id else, while δ' acts as δ on $\llbracket j+h+2, p+q\rrbracket$ and id else. This shows surjectivity of the set identification and thus completes our proof. □

Observe that given a $(p,(q,h),j)$-shuffle $\gamma = \delta\beta\omega$, for every $m \in \llbracket 1, p\rrbracket$ we have $m \le \omega(m) \le \beta\omega(m) \le \delta\beta\omega(m)$. Thus in the product of lifts $\widetilde{\delta}\widetilde{\beta}\widetilde{\omega}$, the strand originally starting at m is always behind throughout all component lifts, and hence matches the strand starting at m in the lift of γ. Similar observation shows that the strand starting at each $m \in \llbracket p+1, p+q\rrbracket$ is always in the front for all component lifts. Therefore, the lift of the shuffle γ is in fact given by the product of the lifts of the component shuffles, i.e., $\widetilde{\gamma} = \widetilde{\delta}\widetilde{\beta}\widetilde{\omega}$.

A similar observation applies to the $((p,h),q,j)$-shuffles: since a $((p,h),q,j)$-shuffle

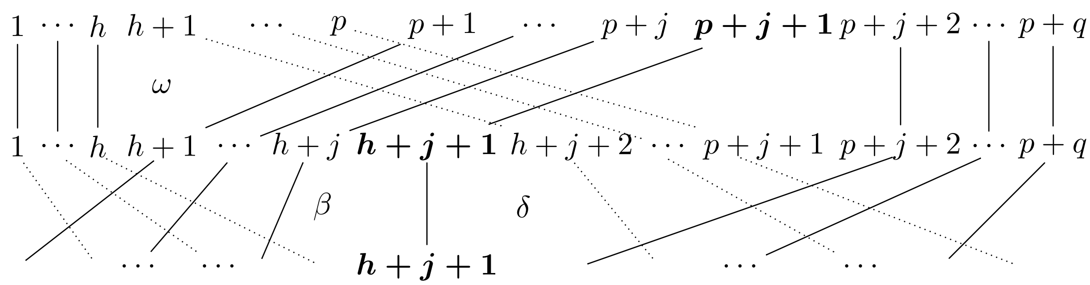

Figure 4.4: Decomposition of a $((p, h), q, j)$-shuffle.

γ sends the marked element from $p+j+1$ to $h+j+1$, it must follows that

$$j + h + 1 < \gamma(h + 1) < \gamma(h + 2) < \cdots < \gamma(p);$$

namely, the elements in $[\![h + 1, p]\!]$ must "move right" on the number line to fill in the $p-h$ holes left behind by the move of the marked element. The same argument as above proves the following decomposition theorem for the $((p, h), q, j)$-shuffles.

Proposition 4.2.4. *There is a unique decomposition of a $((p, h), q, j)$-shuffle into a sequence of three permutations: a fixed (p, q)-shuffle that sends $[\![h + 1, p]\!]$ to $[\![h + j + 2, p+j+1]\!]$, followed by an (h, j)-shuffle on $[\![1, h+j]\!]$ and a $(p-h, q-j-1)$-shuffle on $[\![h + j + 2, p + q]\!]$ (see Figure 4.4). As a consequence, there is a bijection*

$$\mathrm{Sh}((p, h), q, j) \cong \mathrm{Sh}(h, j) \times \mathrm{Sh}(p - h, q - j - 1)$$

where $\mathrm{Sh}((p, h), q, j)$ denotes the set of $((p, h), q, j)$-shuffles.

Similarly to $(p,(q, h), j)$-shuffles, if a $((p, h), q, j)$-shuffle γ is decomposed as $\gamma = \delta\beta\omega$, the lift $\widetilde{\gamma}$ is equivalent to the product of lifts $\widetilde{\delta}\widetilde{\beta}\widetilde{\omega}$.

Recall that $c_{p,q} = \sum_\gamma (-1)^{|\gamma|}$ denotes the sum of the signs of all (p, q)-shuffles γ. Since $(p,(q, h), j)$- and $((p, h), q, j)$-shuffles are essentially elements of the symmetric group, their signs are also well-defined. Let $c_{p,(q,h),j}$ and $c_{(p,h),q,j}$ be the sums of the signs of all $(p,(q, h), j)$- and $((p, h), q, j)$-shuffles, respectively. Based on the decompositions described above, we may express these constants in terms of the constants $c_{i,j}$

corresponding to the component shuffles.

Lemma 4.2.5. *The sums of the signs of all $(p,(q,h),j)$- and $((p,h),q,j)$-shuffles can be computed by the following formulae:*

1. $c_{p,(q,h),j} = (-1)^{h(p-j)} c_{j,h} c_{p-j-1,q-h}$
2. $c_{(p,h),q,j} = (-1)^{(j+1)(p-h)} c_{h,j} c_{p-h,q-j-1}$.

Proof. Let γ be a $(p,(q,h),j)$-shuffle, and suppose we have a decomposition $\gamma = \delta'\beta'\omega$ as described in the proof of Proposition 4.2.3 (here β' and δ' are respectively the previously defined lifts of the (j,h)-shuffle β and the $(p-j-1,q-h)$-shuffle δ to the symmetric group S_{p+q}). Observe that the shuffle ω swaps $p-j$ points in $[\![j+1,p]\!]$ across h points in $[\![p+1,p+h]\!]$ (order-preservingly on each segment), so the number of crossings in ω is $h(p-j)$; hence the sign of ω is $(-1)^{h(p-j)}$. Meanwhile, the signs of β' and δ' are equal to those of β and δ, respectively. It follows that

$$\begin{aligned} c_{p,(q,h),j} &= \sum_{\gamma} (-1)^{|\gamma|} = \sum_{\beta',\delta'} (-1)^{|\omega|}(-1)^{|\beta'|}(-1)^{|\delta'|} = \sum_{\beta,\delta} (-1)^{h(p-j)}(-1)^{|\beta|}(-1)^{|\delta|} \\ &= (-1)^{h(p-j)} \sum_{\beta} (-1)^{|\beta|} \sum_{\delta} (-1)^{|\delta|} = (-1)^{h(p-j)} c_{j,h} c_{p-j-1,q-h} \end{aligned}$$

thus the first identity holds. A similar argument proves the second claim. □

We now give the definition for the Fox–Neuwirth cellular chain complex of the 1-point compactification $\mathrm{Conf}_n(\mathbb{C}^\times) \cup \{\infty\}$.

Definition 4.2.6 (The Fox–Neuwirth complex for $\mathrm{Conf}_n(\mathbb{C}^\times) \cup \{\infty\}$)**.** Let $D(n,1)_*$ denote the chain complex which in degree q is generated over $\mathbb{Z}$ by the set of triples (λ, i, j) where $\lambda = (\lambda_1, \ldots, \lambda_{q-n+1})$ is a composition of $n+1$ of length $q-n+1$, $1 \le i \le l(\lambda)$ and $0 \le j \le \lambda_i - 1$. The differential $d : D(n,1)_q \to D(n,1)_{q-1}$ is given by the formula

$$\begin{aligned} d(\lambda,i,j) = &\sum_{m=1}^{i-2} (-1)^{m-1} c_{\lambda_m,\lambda_{m+1}} (\rho^m, i-1, j) + \sum_{m=i+1}^{q-n} (-1)^{m-1} c_{\lambda_m,\lambda_{m+1}} (\rho^m, i, j) \\ &+ (-1)^{i-2} \sum_{h=0}^{\lambda_{i-1}} c_{(\lambda_{i-1},h),\lambda_i,j} (\rho^{i-1}, i-1, j+h) + (-1)^{i-1} \sum_{h=0}^{\lambda_{i+1}} c_{\lambda_i,(\lambda_{i+1},h),j} (\rho^i, i, j+h). \end{aligned}$$

As in the original Fox–Neuwirth chain complex, the signs in the formula of the differential result from the induced orientations on the boundary strata, following the general scheme described in [51]. Notice that the differential is more complicated than that of the classical Fox–Neuwirth complex, which is a consequence of a larger collection of boundary cells. The proof that the chain complex $D(n,1)_*$ is well-defined is nontrivial and exhibits the usefulness of the combinatorial identities introduced above.

Proposition 4.2.7. $d^2 = 0$.

Proof. The general strategy is to enumerate all types of boundary cells in $d^2(\lambda, i, j)$ and show that their coefficients all vanish. Loosely speaking, cells in $d^2(\lambda, i, j)$ are formed by subsequently performing two column-combining operations on $e_{(\lambda,i,j)}$. In general, there are two types of results: either (1) two pairs of columns in $e_{(\lambda,i,j)}$ are combined separately, or (2) three adjacent columns are combined into a single column. If none of these columns is the imaginary axis, the coefficient of the boundary cell vanishes in the exact same way as in the classical Fox–Neuwirth complex (one may treat the imaginary axis as a normal column in this case). There are four different subtypes of (1) and three subtypes of (2) that involve the imaginary axis. We will exhibit the argument for a subtype of each case.

The representative boundary cell we choose for type (1) is

$$((\lambda_1, \dots, \lambda_{i-1} + \lambda_i, \dots, \lambda_m + \lambda_{m+1}, \dots, \lambda_{l(\lambda)}), i-1, j+h),$$

obtained by joining two pairs of columns indexed by $\{i-1, i\}$ and $\{m, m+1\}$ $(m \geq i+1)$. There are two orders to perform the operations: either (1a) combining the first pair then the second pair, or (1b) combining the second pair first. Thus the coefficient for the cell above in $d^2(\lambda, i, j)$ is

$$(-1)^{i-2} c_{(\lambda_{i-1},h),\lambda_i,j} \cdot (-1)^{m-2} c_{\lambda_m,\lambda_{m+1}}$$
$$+(-1)^{m-1} c_{\lambda_m,\lambda_{m+1}} \cdot (-1)^{i-2} c_{(\lambda_{i-1},h),\lambda_i,j} = 0.$$

A similar argument shows the same result for the other subtypes of case (1).

For an example of type (2), consider a boundary cell of the form

$$((\lambda_1, \ldots, \lambda_{i-2} + \lambda_{i-1} + \lambda_i, \ldots, \lambda_{l(\lambda)}), i-2, j+h),$$

obtained by joining three adjacent columns indexed by $\{i-2, i-1, i\}$. Similarly, there are two orders to perform the operations: either (2a) combining the first two columns then combining the joint column with the third, or (2b) combining the last two columns first. Particularly in case (2b), the imaginary axis involves in both column combinations, so the h points that move below the fixed point z_1 in the final configuration can be split into two steps: s points in the first operation followed by $h-s$ points in the second $(0 \leq s \leq h)$. The coefficient of the cell above in $d^2(\lambda, i, j)$ hence contains a sum over all s:

$$\begin{gathered}(-1)^{i-3} c_{\lambda_{i-2},\lambda_{i-1}} \cdot (-1)^{i-3} c_{(\lambda_{i-2}+\lambda_{i-1},h),\lambda_i,j} \\ + \sum_{s=0}^{h} (-1)^{i-2} c_{(\lambda_{i-1},s),\lambda_i,j} \cdot (-1)^{i-3} c_{(\lambda_{i-2},h-s),\lambda_{i-1}+\lambda_i,j+s}.\end{gathered}$$

We apply Lemma 4.2.5 to write this coefficient completely in terms of the constants $c_{p,q}$ and observe that in order for it to vanish, the identity

$$\begin{gathered}c_{\lambda_{i-2},\lambda_{i-1}} c_{h,j} c_{\lambda_{i-2}+\lambda_{i-1}-h,\lambda_i-j-1} = \\ \sum_{s=0}^{h} (-1)^{s(\lambda_{i-2}-h+s)} c_{s,j} c_{\lambda_{i-1}-s,\lambda_i-j-1} c_{h-s,j+s} c_{\lambda_{i-2}-h+s,\lambda_{i-1}+\lambda_i-j-s-1}\end{gathered}$$

must hold true. This can be proved using a series of arithmetic manipulations and application of the shuffles' properties.

First, to simplify the notation, set $p = \lambda_{i-2}$, $q = \lambda_{i-1}$ and $r = \lambda_i - j - 1$, then the identity becomes

$$c_{p,q} c_{h,j} c_{p+q-h,r} = \sum_{s=0}^{h} (-1)^{s(p-h+s)} c_{s,j} c_{q-s,r} c_{h-s,j+s} c_{p-h+s,q-s+r}.$$

Applying the identity $c_{p,q}c_{p+q,r} = c_{q,r}c_{p,q+r}$ (a consequence of the associativity of (p,q,r)-shuffles) to the equation above yields

$$\begin{aligned} c_{p,q}c_{h,j}c_{p+q-h,r} &= \sum_{s=0}^{h}(-1)^{s(p-h+s)}(c_{s,j}c_{h-s,j+s})(c_{q-s,r}c_{p-h+s,q-s+r}) \\ &= \sum_{s=0}^{h}(-1)^{s(p-h+s)}(c_{h-s,s}c_{h,j})(c_{p-h+s,q-s}c_{p+q-h,r}) \\ &= c_{h,j}c_{p+q-h,r}\sum_{s=0}^{h}(-1)^{s(p-h+s)}c_{h-s,s}c_{p-h+s,q-s}; \end{aligned}$$

therefore it suffices to show that $c_{p,q} = \sum_{s=0}^{h}(-1)^{s(p-h+s)}c_{h-s,s}c_{p-h+s,q-s}$. We will prove this identity by induction on h.

First, recall that there is a bijection of sets $\mathrm{Sh}(p,q) \cong \mathrm{Sh}(p,q-1) \sqcup \mathrm{Sh}(p-1,q)$ (see, e.g., [7, 67]), which results in the shuffle identity $c_{p,q} = (-1)^p c_{p,q-1} + c_{p-1,q} = (-1)^q c_{p-1,q} + c_{p,q-1}$. The base cases are straightforward: for $h = 0$, $c_{p,q} = (-1)^0 c_{0,0}c_{p,q}$, while for $h = 1$, we recover the identity above $c_{p,q} = c_{1,0}c_{p-1,q} + (-1)^p c_{0,1}c_{p,q-1} = (-1)^p c_{p,q-1} + c_{p-1,q}$. Suppose it holds for h, then we may apply the identity in the following way:

$$\begin{aligned} c_{p,q} &= \sum_{s=0}^{h}(-1)^{s(p-h+s)}c_{h-s,s}c_{p-h+s,q-s} \\ &= \sum_{s=0}^{h}(-1)^{s(p-h+s)}c_{h-s,s}[(-1)^{p-h+s}c_{p-h+s,q-s-1} + c_{p-h+s-1,q-s}] \\ &= \sum_{s=0}^{h}(-1)^{s(p-h+s)}c_{h-s,s}[(-1)^{p-h+s}c_{p-(h+1)+(s+1),q-(s+1)} + c_{p-(h+1)+s,q-s}]. \end{aligned}$$

Thus $c_{p,q}$ can be written as a linear combination of terms of the form $c_{p-(h+1)+s,q-s}$ where $0 \leq s \leq h+1$. In the sum above, each term $c_{p-(h+1)+s,q-s}$ is derived from two terms $c_{p-h+s,q-s}$ and $c_{p-h+(s-1),q-(s-1)}$, hence its coefficient can computed to be

$$\begin{gathered} (-1)^{s(p-h+s)}c_{h-s,s} + (-1)^{(s-1)(p-h+s-1)}(-1)^{p-h+s-1}c_{h-s+1,s-1} = \\ (-1)^{s(p-(h+1)+s)}[(-1)^s c_{h-s,s} + c_{h-s+1,s-1}] = (-1)^{s(p-(h+1)+s)}c_{(h+1)-s,s}. \end{gathered}$$

This completes our induction argument. □

By construction, the complex $D(n,1)_*$ is isomorphic to the relative cellular chain complex of $\mathrm{Conf}_n(\mathbb{C}^\times)\cup\{\infty\}$, relative to the point at infinity. In particular,

$$H_*(\mathrm{Conf}_n(\mathbb{C}^\times)\cup\{\infty\},\{\infty\})\cong H_*(D(n,1)_*).$$

4.3 Cellular stratification of $\mathrm{Conf}_n(\mathbb{C}_m)$

We generalize the construction in the previous section to produce a stratification of $\mathrm{Conf}_n(\mathbb{C}_m)$ for any integer $m\geq 2$. Similarly, observe that there is a canonical embedding $\mathrm{Conf}_n(\mathbb{C}_m)\hookrightarrow\mathrm{Conf}_{n+m}(\mathbb{C})$ by inserting the previously removed points $z_1,\ldots,z_m$. This gives a homeomorphic image of $\mathrm{Conf}_n(\mathbb{C}_m)$ as a subspace of $\mathrm{Conf}_{n+m}(\mathbb{C})$ consisting of all configurations where the points $z_1,\ldots,z_m$ are always fixed. We will give a stratification of this subspace based on the Fox–Neuwirth cellular stratification of $\mathrm{Conf}_{n+m}(\mathbb{C})$ introduced above. For the rest of this paper, we will indiscriminately use the notation $\mathrm{Conf}_n(\mathbb{C}_m)$ for both the original configuration space of the m-punctured complex plane and its homeomorphic image embedded in $\mathrm{Conf}_{n+m}(\mathbb{C})$.

Given a composition λ of $n+m$, we consider the intersection of the cell $\mathrm{Conf}_\lambda(\mathbb{C})$ and the subspace $\mathrm{Conf}_n(\mathbb{C}_m)$ of $\mathrm{Conf}_{n+m}(\mathbb{C})$. Starting with a configuration in $\mathrm{Conf}_\lambda(\mathbb{C})$, the restriction on the fixed points $z_1,\ldots,z_m$ results in two constraints. First, for all $1\leq k\leq m$, the vertical column that contains z_k (indexed by i_k) must be fixed, i.e., the real part of all points on that column must be $z_k=k-1$; we refer to these columns as the *fixed columns* in the cell, and others the *free columns*. Since the fixed points all have distinct real parts which keeps the fixed columns separate, the number of vertical columns in a cell must be at least m. Secondly, for every $1\leq k\leq m$, it is forbidden for points on the k^{th} fixed column (i_k^{th} overall) to move past the fixed point z_k. The number of points on this vertical line with a negative imaginary part is hence fixed and denoted by the index j_k. Therefore, the connected components in the above intersection can be denoted by $e_{(\lambda,I,J)}=\mathrm{Conf}_{(\lambda,I,J)}(\mathbb{C})$ where λ is a composition of $n+m$, $I=(i_1,\ldots,i_m)$ is the m-tuple of indices of the fixed columns ($1\leq i_1<\cdots<i_m\leq l(\lambda)$), and $J=(j_1,\ldots,j_m)$ where j_k is the number of points lying below z_k on the i_k^{th} column ($0\leq j_k\leq\lambda_{i_k}-1$); see Figure 4.5 for an example. Given a composition λ, the overall position ι of a point

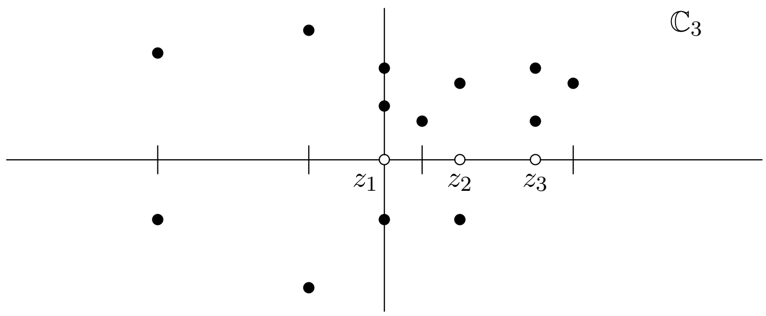

Figure 4.5: A configuration in $e_{(\lambda,I,J)} \subset \mathrm{Conf}_{13}(\mathbb{C}_3)$, where $\lambda = (2,2,4,1,3,3,1)$, $I = (3,5,6)$, and $J = (1,1,0)$.

z in a cell indexed by λ is the same information as the pair of indices (i,j) where i indexes the column containing z and j indexes the number of points lying below z on that column, via the identification $\iota = j + 1 + \sum_{k=1}^{i-1} \lambda_k$. Hence it is possible to re-index the subspace $e_{(\lambda,I,J)}$ by the composition λ and a tuple $\mathcal{I} = (\iota_1, \ldots, \iota_m)$ that contains the overall positions of the fixed points $z_1, \ldots, z_m$. For example, the cell containing the configuration in Figure 4.5 can be indexed by $\lambda = (2,2,4,1,3,3,1)$ and the tuple $\mathcal{I} = (6, 11, 13)$.

The spaces $e_{(\lambda,I,J)}$ then provide the positive dimension cells for our cellular decomposition of $\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}$. Each cell $e_{(\lambda,I,J)}$ is homeomorphic to the product

$$\left[\mathrm{Conf}_{i_1-1}((-\infty,0)) \times \left(\prod_{k=1}^{m-1} \mathrm{Conf}_{i_{k+1}-i_k-1}((k-1,k))\right) \times \mathrm{Conf}_{l(\lambda)-i_m}((m-1,\infty))\right]$$
$$\times \prod_{i=1, i\notin I}^{l(\lambda)} \mathrm{Conf}_{\lambda_i}(\mathbb{R}) \times \left[\prod_{k=1}^{m} \mathrm{Conf}_{j_k}((-\infty,0)) \times \mathrm{Conf}_{\lambda_{i_k}-j_k-1}((0,\infty))\right].$$

The first bracket represents the configurations of the free columns before the first fixed column, between each pair of fixed columns, and after the last fixed column, i.e., recording the real parts of the points. The last bracket keeps track of the imaginary parts of points below and above the fixed point on each fixed column, while the middle product records the same information for those on the free columns. By identifying each open interval with $\mathbb{R}$ and applying the homeomorphism $\mathrm{Conf}_k(\mathbb{R}) \cong \mathbb{R}^k$, we see that the cell

$e_{(\lambda,I,J)}$ has dimension $n + l(\lambda) - m$; loosely speaking, compared to the classical Fox–Neuwirth cell indexed by the same composition λ, we lost $2m$ dimensions due to fixing the real and imaginary parts of m points.

As in the classical Fox–Neuwirth cellular decomposition of $\mathrm{Conf}_n(\mathbb{C})$, the boundary of a cell in this stratification is obtained in two ways. For the first type, we let points in a configuration approach other (free or fixed) points or infinity; in this case, the boundary is still the point at infinity. The second type of boundary again occurs by horizontally joining two adjacent vertical columns of the configuration without colliding the points. Note that the fixed columns are not allowed to merge with one another. Due to the second constraint of the cell, namely points on a fixed column cannot move across the fixed point in that column, the boundary cells obtained this way have five general forms, depending on the types of columns (free or fixed) involved in the column combination and their relative positions. In particular, when combining a free column with the k^{th} fixed column, we must keep track of the number of points going below the fixed point, i.e., adding to the index j_k; in the alternate indexing system, this results in a change of the overall position of the fixed point z_k. In summary:

Proposition 4.3.1. *The space* $\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}$ *may be presented as a CW complex whose positive dimension cells* $e_{(\lambda,I,J)} = \mathrm{Conf}_{(\lambda,I,J)}(\mathbb{C})$ *(of dimension* $n + l(\lambda) - m$*) are indexed by triples* (λ, I, J)*, where* λ *is an ordered partition of* $n+m$ *of length* $q - n + k$*,* $I = (i_1, \ldots, i_m)$ *is the* m*-tuple of indices of the fixed columns* $(1 \le i_1 < \cdots < i_m \le l(\lambda))$*, and* $J = (j_1, \ldots, j_m)$ *where* j_k *is the number of points on the* i_k^{th} *column with negative imaginary parts* $(0 \le j_k \le \lambda_{i_k} - 1)$*.*

Let I_k *denote the* m*-tuple* $(i_1, \ldots, i_{k-1}, i_k - 1, \ldots, i_m - 1)$*, and* $J_{k,h}$ *denote* $(j_1, \ldots, j_k + h, \ldots, j_m)$*. The codimension-*1 *boundary cells of* $e_{(\lambda,I,J)}$ *have five general forms:*

1. $e_{(\rho^i, I_1, J)}$ $\qquad 1 \le i < i_1 - 1$
2. $e_{(\rho^i, I_{k+1}, J)}$ $\qquad 1 \le k \le m-1,\ i_k < i < i_{k+1} - 1$
3. $e_{(\rho^i, I, J)}$ $\qquad i_k < i < l(\lambda)$
4. $e_{(\rho^{i_k - 1}, I_k, J_{k,h})}$ $\qquad 1 \le k \le m,\ 0 \le h \le \lambda_{i_k - 1}$
5. $e_{(\rho^{i_k}, I_{k+1}, J_{k,h})}$ $\qquad 1 \le k \le m,\ 0 \le h \le \lambda_{i_k + 1}$

where $\rho^i = (\lambda_1, \dots, \lambda_i + \lambda_{i+1}, \dots, \lambda_{l(\lambda)})$ *is the coarsening of* λ *obtained by summing* λ_i *and* λ_{i+1} $(1 \leq i < l(\lambda))$*, and* h *denotes the number of points going below the fixed point when combining a fixed column with the free column on its left (4) or right (5).*

The overall positions of all fixed points are unchanged in the first three types of codimension-1 boundaries, whereas only the position of z_k changes to $\iota_k - \lambda_{i_k-1} + h$ (type 4) or $\iota_k + h$ (type 5). In particular, the relative order of the overall positions of the fixed points is always preserved. This property is very crucial to the argument of this paper and will be revisited frequently in later sections.

As before, from the stratification, we can write down an explicit cellular chain complex for the 1-point compactification $\mathrm{Conf}_n(\mathbb{C}_k) \cup \{\infty\}$.

Definition 4.3.2 (Fox–Neuwirth complex for $\mathrm{Conf}_n(\mathbb{C}_k) \cup \{\infty\}$)**.** Let $D(n,m)_*$ denote the chain complex which in degree q is generated over $\mathbb{Z}$ by the set of triples (λ, I, J) where $\lambda = (\lambda_1, \dots, \lambda_{q-n+m})$ is a composition of $n+m$ of length $q-n+m$, $I = (i_1, \dots, i_m)$ with $1 \leq i_1 < \cdots < i_m \leq l(\lambda)$, and $J = (j_1, \dots, j_m)$ with $0 \leq j_k \leq \lambda_{i_k} - 1$.

The differential $d : D(n,m)_q \to D(n,m)_{q-1}$ is given by the formula

$$\begin{aligned}
d(\lambda, I, J) = &\sum_{i=1}^{i_1-2} (-1)^{i-1} c_{\lambda_i, \lambda_{i+1}} (\rho^i, I_1, J) \\
&+ \sum_{k=1}^{m-1} \sum_{i=i_k+1}^{i_{k+1}-2} (-1)^{i-1} c_{\lambda_i, \lambda_{i+1}} (\rho^i, I_{k+1}, J) \\
&+ \sum_{i=i_m+1}^{q-n+m-1} (-1)^{i-1} c_{\lambda_i, \lambda_{i+1}} (\rho^i, I, J) \\
&+ \sum_{k=1}^{m} (-1)^{i_k-2} \sum_{h=0}^{\lambda_{i_k-1}} c_{(\lambda_{i_k-1}, h), \lambda_{i_k}, j_k} (\rho^{i_k-1}, I_k, J_{k,h}) \\
&+ \sum_{k=1}^{m} (-1)^{i_k-1} \sum_{h=0}^{\lambda_{i_k+1}} c_{\lambda_{i_k}, (\lambda_{i_k+1}, h), j_k} (\rho^{i_k}, I_{k+1}, J_{k,h}).
\end{aligned}$$

The proof that the chain complex $D(n,m)_*$ is well-defined rests on the following observation: all types of boundary cells in this chain complex arise in the exact same way independent of the number m of fixed points. It follows that parts of this proof can be reduced to Proposition 4.2.7.

Proposition 4.3.3. $d^2 = 0$.

Proof. As before, the general strategy is to enumerate all types of boundary cells in $d^2(\lambda, I, J)$ and show that their coefficients all vanish. Loosely speaking, cells in $d^2(\lambda, I, J)$ are formed by subsequently performing two column combinations in the cell $e_{(\lambda,I,J)}$. There are two outcomes: either (1) two pairs of columns in $e_{(\lambda,I,J)}$ are combined separately, or (2) three adjacent columns are combined into a single column. Within each of these types, the argument follows the same logic and only differs slightly in the exact details depending on the types of columns (free or fixed) involved in the combinations and their relative positions. Therefore, we will present the argument for a representative of each type.

For a representative of type (1), we consider the boundary cell

$$((\lambda_1, \ldots, \lambda_{i_r-1} + \lambda_{i_r}, \ldots, \lambda_i + \lambda_{i+1}, \ldots, \lambda_{l(\lambda)}), (I_r)_{s+1}, J_{r,h})$$

obtained by joining two pairs of columns indexed by $\{i_r - 1, i_r\}$ and $\{i, i+1\}$ ($i_r < i_s < i < i_{s+1} - 1$). Since $J_{r,h}$ is completely determined by the combination of the first pair of columns and $(I_r)_{s+1} = (I_{s+1})_r$, both of the following orders to perform the operations result in this boundary cell: either (1a) combining the first pair then the second pair, or (1b) combining the second pair first. Thus the coefficient of the cell above in $d^2(\lambda, I, J)$ is

$$(-1)^{i_r-2} c_{(\lambda_{i_r-1},h),\lambda_{i_r},j_r} \cdot (-1)^{i-2} c_{\lambda_i,\lambda_{i+1}} + (-1)^{i-1} c_{\lambda_i,\lambda_{i+1}} \cdot (-1)^{i_r-2} c_{(\lambda_{i_r-1},h),\lambda_{i_r},j_r} = 0.$$

For a representative of type (2), consider a boundary cell of the form

$$((\lambda_1, \ldots, \lambda_{i_r-2} + \lambda_{i_r-1} + \lambda_{i_r}, \ldots, \lambda_{l(\lambda)}), (I_r)_r, J_{r,h})$$

obtained by joining three adjacent columns indexed by $\{i_r - 2, i_r - 1, i_r\}$. Similarly, there are two orders to perform the operations: either (2a) combining the first two columns then combining the joint column with the third, or (2b) combining the last two columns first. Particularly in case (2b), the fixed column involves in both column combinations, so the h points that move below the fixed point z_r in the final configuration can be split into two steps: s points in the first operation followed by $h - s$ points in the second

$(0 \leq s \leq h)$. The coefficient of this representative cell in $d^2(\lambda, I, J)$ hence contains a sum over all s:

$$(-1)^{i_r-3} c_{\lambda_{i_r-2},\lambda_{i_r-1}} \cdot (-1)^{i_r-3} c_{(\lambda_{i_r-2}+\lambda_{i_r-1},h),\lambda_{i_r},j_r}$$
$$+ \sum_{s=0}^{h} (-1)^{i_r-2} c_{(\lambda_{i_r-1},s),\lambda_{i_r},j_r} \cdot (-1)^{i_r-3} c_{(\lambda_{i_r-2},h-s),\lambda_{i_r-1}+\lambda_{i_r},j_r+s}.$$

This expression vanishes as shown in the proof of Proposition 4.2.7.

□

By construction, the complex $D(n,m)_*$ is isomorphic to the relative cellular chain complex of $\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}$, relative to the point at infinity. In particular,

$$H_*(\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}, \{\infty\}) \cong H_*(D(n,m)_*).$$

This construction therefore provides an approach to compute the cellular homology of configuration spaces of multi-punctured complex planes with trivial coefficients.

Chapter 5

Twisted homology of Artin groups of type *B*

Let (V, W) be a separable left-braided vector space. In this chapter, we identify

$$H_*(B_{n,1}; V^{\otimes n} \otimes W) \cong \operatorname{Ext}^{n-*,n+1}_{\mathfrak{A}^e}(\mathfrak{M}, \mathbf{k})$$

where $\mathfrak{A}$ is a quantum shuffle algebra and $\mathfrak{M}$ is an $\mathfrak{A}$-bimodule defined momentarily. We will provide two proofs of this statement, a short algebraic argument and an elaborate geometric argument. Each approach offers a glimpse into a part of the proof of the general case in the next chapter. The content of this chapter is extracted from Sections 3.2–3.4 and 4.2–4.3 of [55].

5.1 Homological algebra prerequisites

The purpose of this section is to define and study two algebraic objects necessary for the discussion in this chapter.

Let $(V, W, \sigma, \tau, \varphi)$ be a separable left-braided vector space, and let $\mathfrak{A}(V)$ be the quantum shuffle algebra associated to (V, σ).

Definition 5.1.1. The *graded* $\mathfrak{A}(V)$*-bimodule* $\mathfrak{M} = \mathfrak{M}(V, W)$ is defined by

$$\mathfrak{M} = \bigoplus_{q \geq 1} \bigoplus_{0 \leq j \leq q-1} V^{\otimes j} \otimes W \otimes V^{\otimes q-j-1}.$$

Multiplication on both sides is given by the quantum shuffle product in the following sense: for the left multiplication we have

$$[a_1|\ldots|a_p] \star [b_1|\ldots|b_j|w|b_{j+2}|\ldots|b_q] = \sum_\gamma \widetilde{\gamma}[a_1|\ldots|a_p|b_1|\ldots|b_j|w|b_{j+2}|\ldots|b_q]$$

and for the right multiplication

$$[a_1|\ldots|a_j|w|a_{j+2}|\ldots|a_p] \star [b_1|\ldots|b_q] = \sum_\gamma \widetilde{\gamma}[a_1|\ldots|a_j|w|a_{j+2}|\ldots|a_p|b_1|\ldots|b_q]$$

where γ draws from all (p, q)-shuffles, and $\widetilde{\gamma}$ is the lift of γ to B_{p+q}.

The action of the braid $\widetilde{\gamma}$ on an element in $V^{\otimes j} \otimes W \otimes V^{\otimes q-j-1}$ is described in Corollary 3.2.9. The formulae of both the left and the right multiplications of $\mathfrak{M}$ resemble that of the quantum shuffle product in $\mathfrak{A}(V)$. In particular, the definition of $\mathfrak{M}$ satisfies the associative requirement for the multiplication in an $\mathfrak{A}(V)$-bimodule.

The second algebraic construction of interest is the following:

Definition 5.1.2. Given an associative $\mathbf{k}$-algebra A and an A-bimodule M, the *chain complex* $F_*(M, A)$ is defined at degree $q \geq 1$ by

$$F_q(M, A) = \bigoplus_{i=1}^{q} A^{\otimes i-1} \otimes M \otimes A^{\otimes q-i}$$

with face maps for $1 \leq m \leq q-1$:

$$d_m(a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu_i \otimes a_{i+1} \otimes \cdots \otimes a_q) = \begin{cases} a_1 \otimes \cdots \otimes a_m a_{m+1} \otimes \cdots \otimes a_q & m \neq i-1, i \\ a_1 \otimes \cdots \otimes a_{i-1}\mu_i \otimes \cdots \otimes a_q & m = i-1 \\ a_1 \otimes \cdots \otimes \mu_i a_{i+1} \otimes \cdots \otimes a_q & m = i \end{cases}$$

and differential $d = \sum_{m=1}^{q-1} (-1)^{m-1} d_m$.

The graded group structure and the differential of $F_*(M, A)$ are similar to that of the extended two-sided bar complex $B^{\mathrm{e}}_*(A, A, A)$ of the associative algebra A (see, e.g., [50, 40]). In fact, the chain complex $F_*(M, A)$ can be constructed by introducing modifications to $B^{\mathrm{e}}_*(A, A, A)$ in the following way: to construct $F_q(M, A)$, we start with $B^{\mathrm{e}}_{q-2}(A, A, A) \cong A^{\otimes q}$ and subsequently replace exactly one copy of A with a copy of M for every copy of A in the tensor product. This yields q tensor products of the form $A^{\otimes i-1} \otimes M \otimes A^{\otimes q-i}$ for $1 \leq i \leq q$, and their direct sum forms the $\mathbf{k}$-module $F_q(M, A)$. Any multiplication with this new factor M is replaced by either the left or the right multiplication of this A-bimodule. It is easy to see that $F_*(M, A)$ forms a well-defined chain complex in a similar manner as the bar complex $B^{\mathrm{e}}_*(A, A, A)$.

Proposition 5.1.3. *$F_*(M, A)$ is an exact chain complex.*

Proof. $F_*(M, A)$ has a simplicial structure X_* where $X_q = F_{q-2}(M, A)$. The face maps are given in the definition of the chain complex, while the degeneracy maps are defined by

$$s_m(a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu_i \otimes a_{i+1} \otimes \cdots \otimes a_q) =$$

$$a_1 \otimes \cdots \otimes a_m \otimes 1 \otimes a_{m+1} \otimes \cdots \otimes a_{i-1} \otimes \mu_i \otimes a_{i+1} \otimes \cdots \otimes a_q$$

for all $0 \leq m \leq q$. The extra degeneracy s_0 guarantees that the simplicial object is contractible, hence the chain complex $F_*(M, A)$ is exact. □

It follows that $F_*(M, A)$, upon omitting M in degree 1, gives a free resolution of M as an A-bimodule.

Let $I = A/\mathbf{k}$ be the *augmentation ideal* of A, consisting of elements of positive degree. We then have a decomposition $A \cong I \oplus \mathbf{k}$, hence given any $a \in A$ we may write $a = \overline{a} + \hat{a}$ where $\overline{a}$ and $\hat{a}$ are the projections of a onto the factors I and $\mathbf{k}$, respectively. Let $F_*(M, I)$ denote the chain complex obtained by replacing all copies of A in $F_*(M, A)$ with I (one may think of $F_*(M, I)$ as a reduced form of the chain complex $F_*(M, A)$).

Given an associative $\mathbf{k}$-algebra A, let A^{op} denote the opposite algebra, and $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ be the enveloping algebra of A. There is a canonical isomorphism $(A^{\mathrm{e}})^{\mathrm{op}} \cong A^{\mathrm{e}}$, thus an A-bimodule can be regarded as a left (or equivalently, right) A^{e}-module [50]. We compute the homology of the chain complex $F_*(M, I)$ in the following theorem.

Theorem 5.1.4. $H_*(F_*(M, I)) \cong \mathrm{Tor}^{A^{\mathrm{e}}}_{*-1}(M, \mathbf{k})$.

Proof. Recall that the Hochschild chain complex $CH_*(A, M)$ is defined degree-wise by $CH_q(A, M) = M \otimes A^{\otimes q}$, with face maps

$$d_m(\mu \otimes a_1 \otimes \cdots \otimes a_q) = \begin{cases} \mu a_1 \otimes a_2 \otimes \cdots \otimes a_q & m = 0 \\ \mu \otimes a_1 \otimes \cdots \otimes a_m a_{m+1} \otimes \cdots \otimes a_q & 1 \leq m \leq q-1 \\ a_q \mu \otimes a_1 \otimes \cdots \otimes a_{q-1} & m = q \end{cases}$$

and differential $d = \sum_{m=0}^{q} (-1)^m d_m$. When the A-bimodule M is treated as a left A^{e}-module, there is an identification of the $\mathbf{k}$-module $CH_q(A, M)$ as $M \otimes A^{\otimes q} \cong M \otimes_{A^{\mathrm{e}}} A^{\otimes q+2}$, which induces an isomorphism of chain complexes $CH_*(A, M) \cong M \otimes_{A^{\mathrm{e}}} B_*(A, A, A)$ where $B_*(A, A, A)$ is the two-sided bar complex [50].

Construct a chain map $f_* : F_*(M, I) \to CH_*(A, M)$ by mapping

$$a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu_i \otimes a_{i+1} \otimes \cdots \otimes a_q \mapsto (-1)^{q(i-1)} \mu_i \otimes a_{i+1} \otimes \cdots \otimes a_q \otimes 1 \otimes a_1 \otimes \cdots \otimes a_{i-1}.$$

Note that for all $m \neq i$, $\deg(a_m) > 0$ since each a_m is in the augmentation ideal I of A. It follows that there is precisely one occurrence of the unit in the image of an arbitrary element of $F_*(M, I)$, and its position is determined by the indices q and i. We deduce that this map is injective. The isomorphic image of $F_*(M, I)$ via this map forms a subcomplex of the Hochschild chain complex which in degree q has the form $\bigoplus_{i=1}^{q} M \otimes I^{\otimes q-i} \otimes \mathbf{k} \otimes I^{\otimes i-1}$. Via the identification described above, we have the following isomorphisms of $\mathbf{k}$-modules

$$F_q(M, I) \cong \bigoplus_{i=1}^{q} M \otimes I^{\otimes q-i} \otimes \mathbf{k} \otimes I^{\otimes i-1} \cong \bigoplus_{i=1}^{q} M \underset{A^{\mathrm{e}}}{\otimes} (A \otimes I^{\otimes q-i} \otimes \mathbf{k} \otimes I^{\otimes i-1} \otimes A)$$

and thus an isomorphism of chain complexes $F_*(M, I) \cong M \otimes_{A^{\mathrm{e}}} N_*(M, I)$, where $N_*(M, I)$ is the subcomplex of the two-sided bar complex $B_*(A, A, A)$ defined degree-wise by

$$N_q(M, I) = \bigoplus_{i=1}^{q} A \otimes I^{\otimes q-i} \otimes \mathbf{k} \otimes I^{\otimes i-1} \otimes A$$

for all $q \geq 1$. Observe that $N_*(M, I)$ is free as a graded right A^{e}-module, so to prove the statement of the theorem, it suffices to show that $N_*(M, I)$ is a resolution of $\mathbf{k}$.

We will show that $N_*(M,I)$ can be written as the direct sum of two subcomplexes of the two-sided bar complex $B_*(A,A,A)$ and compute their homologies. Let X_* and Y_* be the subcomplexes of $B_*(A,A,A)$ defined degree-wise respectively by

$$X_q = \mathbf{k} \otimes \mathbf{k} \otimes I^{\otimes q-1} \otimes A$$

and

$$Y_q = (I \otimes \mathbf{k} \otimes I^{\otimes q-1} \otimes A) \oplus \left(\bigoplus_{i=1}^{q-1} A \otimes I^{\otimes q-i} \otimes \mathbf{k} \otimes I^{\otimes i-1} \otimes A \right).$$

Observe that for every $q \geq 1$, $N_q(M,I) \cong X_q \oplus Y_q$, so as chain complexes (more specifically, subcomplexes of the bar complex) indeed we have $N_*(M,I) \cong X_* \oplus Y_*$.

Define collections of maps $g_* : X_* \to X_{*+1}$ by

$$g_q(1 \otimes 1 \otimes a_2 \otimes \cdots \otimes a_q \otimes a_{q+1}) = (-1)^{q-1} 1 \otimes 1 \otimes a_2 \otimes \cdots \otimes a_q \otimes \overline{a_{q+1}} \otimes 1$$

and $h_* : Y_* \to Y_{*+1}$ by

$$h_q(a_0 \otimes a_1 \otimes \cdots \otimes a_q \otimes a_{q+1}) = 1 \otimes \overline{a_0} \otimes a_1 \otimes \cdots \otimes a_q \otimes a_{q+1}$$

where $\overline{a}$ denotes the positive-degree part of a. We claim that g_* satisfies the null homotopy condition for all degrees except for degree 1, i.e., $\mathrm{id} = d_{q+1}g_q + g_{q-1}d_q$ for all $q \geq 2$. We compute:

$$\begin{aligned}
&d_{q+1}g_q(1 \otimes 1 \otimes a_2 \otimes \cdots \otimes a_q \otimes a_{q+1}) \\
&= d_{q+1}\left((-1)^{q-1} 1 \otimes 1 \otimes a_2 \otimes \cdots \otimes a_q \otimes \overline{a_{q+1}} \otimes 1\right) \\
&= (-1)^{q-1} \Bigg[\sum_{m=2}^{q-1} \left((-1)^m 1 \otimes 1 \otimes \cdots \otimes a_m a_{m+1} \otimes \cdots \otimes \overline{a_{q+1}} \otimes 1\right) \\
&\quad + (-1)^q 1 \otimes 1 \otimes \cdots \otimes a_q \overline{a_{q+1}} \otimes 1 + (-1)^{q+1} 1 \otimes 1 \otimes \cdots \otimes a_q \otimes \overline{a_{q+1}} \Bigg]
\end{aligned}$$

and

$$\begin{aligned}
&g_{q-1}d_q(1 \otimes 1 \otimes a_2 \otimes \cdots \otimes a_q \otimes a_{q+1}) \\
&= g_{q-1}\left[\sum_{m=2}^{q}(-1)^m 1 \otimes 1 \otimes \cdots \otimes a_m a_{m+1} \otimes \cdots \otimes a_{q+1} \otimes 1\right] \\
&= (-1)^q\left[\sum_{m=2}^{q-1}\left((-1)^m 1 \otimes 1 \otimes \cdots \otimes a_m a_{m+1} \otimes \cdots \otimes \overline{a_{q+1}} \otimes 1\right)\right. \\
&\quad \left. +(-1)^q 1 \otimes 1 \otimes \cdots \otimes \overline{a_q a_{q+1}} \otimes 1\right].
\end{aligned}$$

Observe that $\overline{a_q a_{q+1}} = a_q a_{q+1}$ since $\deg(a_q) > 0$. We can then simplify the sum of the above formulae to get

$$\begin{aligned}
&(d_{q+1}g_q + g_{q-1}d_q)(1 \otimes 1 \otimes a_2 \otimes \cdots \otimes a_q \otimes a_{q+1}) \\
&= 1 \otimes 1 \otimes \cdots \otimes a_q a_{q+1} \otimes 1 - 1 \otimes 1 \otimes \cdots \otimes a_q \overline{a_{q+1}} \otimes 1 + 1 \otimes 1 \otimes \cdots \otimes a_q \otimes \overline{a_{q+1}} \\
&= 1 \otimes 1 \otimes \cdots \otimes a_q \widehat{a_{q+1}} \otimes 1 + 1 \otimes 1 \otimes \cdots \otimes a_q \otimes \overline{a_{q+1}} \\
&= 1 \otimes 1 \otimes \cdots \otimes a_q \otimes \widehat{a_{q+1}} + 1 \otimes 1 \otimes \cdots \otimes a_q \otimes \overline{a_{q+1}} = 1 \otimes 1 \otimes \cdots \otimes a_q \otimes a_{q+1}.
\end{aligned}$$

The existence of the (almost) null homotopy g_* implies that $H_q(X_*)$ vanishes for all $q \geq 2$. The computation of $H_1(X_*)$ is straighforward:

$$H_1(X_*) = \frac{\ker(d_1)}{\operatorname{im}(d_2)} = \frac{\mathbf{k} \otimes \mathbf{k} \otimes A}{\mathbf{k} \otimes \mathbf{k} \otimes I} \cong \frac{A}{I} \cong \mathbf{k}.$$

Hence the chain complex X_* is in fact a resolution of $\mathbf{k}$. A similar computation shows that h_* is a null homotopy and thus the complex Y_* is exact. It follows from the direct sum decomposition above that $N_*(M, I)$ is a resolution of $\mathbf{k}$, concluding our proof. □

If M is graded, for an element $f = a_1 \otimes \cdots \otimes \mu_i \otimes \cdots \otimes a_q$ with a_m homogeneous elements of A of degree $\deg(a_m)$, we may define the degree of f to be $\deg(f) := \deg(\mu_i) + \sum_{m \neq i} \deg(a_m)$. The differential in $F_*(M, I)$ strictly preserves the degree of elements, hence we may define the split subcomplex generated by homogeneous elements of $F_*(M, I)$ of degree precisely n, denoted by $F_{*,n}(M, I)$.

Corollary 5.1.5. $H_*(F_{*,n}(M, I)) \cong \operatorname{Tor}^{A^e}_{*-1,n}(M, \mathbf{k})$.

Proof. Since the differentials of both the complex $F_*(M, I)$ and the Hochschild chain complex strictly preserve the degree of elements, this corollary follows directly from the previous theorem. □

5.2 Homology with coefficients arising from left-braided vector spaces

Write ϵ for the braided $\mathbf{k}$-module $\epsilon = \mathbf{k}$ with braiding on $\epsilon^{\otimes 2} \cong \mathbf{k}$ given by multiplication by -1. Given a general braided $\mathbf{k}$-module (V, σ), write $V_\epsilon = V \otimes \epsilon$ with braiding twisted by the sign on ϵ.

Let $(V, W, \sigma, \tau, \varphi)$ be a separable left-braided vector space, then its twisted dual $(V_\epsilon^*, W^*, \sigma_\epsilon^*, \tau^*, \varphi_\epsilon^*)$ also forms a separable left-braided vector space with $\varphi_\epsilon^* = -\varphi^*$. Let $\mathfrak{A} = \mathfrak{A}(V_\epsilon^*)$ be the quantum shuffle algebra generated by the twisted dual V_ϵ^*, and $\mathfrak{M} = \mathfrak{M}(V_\epsilon^*, W^*)$ be the $\mathfrak{A}$-bimodule defined from $(V_\epsilon^*, W^*, \sigma_\epsilon^*, \tau^*, \varphi_\epsilon^*)$ per Definition 5.1.1. We are now ready to state the main result of this chapter.

Theorem 5.2.1. *There is an isomorphism*

$$H_*(B_{n,1}; V^{\otimes n} \otimes W) \cong \mathrm{Ext}_{\mathfrak{A}^e}^{n-*,n+1}(\mathfrak{M}, \mathbf{k}).$$

As promised, we will give a short algebraic proof of this statement, before working out an elaborate example in the next section. We will return to this theorem in Section 5.4 with an alternate geometric proof.

First, we revisit a construction of Ellenberg–Tran–Westerland [40] that provides a framework to compute the homology of braid groups with twisted coefficients. Recall from Section 4.1 that the Fox–Neuwirth chain complex $C(n)_*$ computes the cellular homology of $\mathrm{Conf}_n(\mathbb{C}) \cup \{\infty\}$, relative to the point at infinity. Let T be a representation of B_n, and $\mathcal{T}$ the associated local system over $\mathrm{Conf}_n(\mathbb{C})$. Ellenberg, Tran, and Westerland [40] provided an explicit formula for the differential of the chain complex $C(n)_* \otimes T$ that incorporates the braid action on T:

$$d[(\lambda_1, \ldots, \lambda_{q-n}) \otimes t] = \sum_{i=1}^{q-n-1} (-1)^{i-1} \Big[(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i + \lambda_{i+1}, \ldots, \lambda_{q-n}) \otimes \sum_{\gamma} (-1)^{|\gamma|} \widetilde{\gamma}(t)\Big]$$

where γ is drawn from the $(\lambda_i, \lambda_{i+1})$-shuffles, and $\widetilde{\gamma}$ is its lift to the copy $B_{\lambda_i+\lambda_{i+1}} \leq B_n$

consisting of braids that are only nontrivial on the $\lambda_i + \lambda_{i+1}$ strands starting with the $\lambda_1 + \cdots + \lambda_{i-1} + 1^{\text{st}}$.

Theorem 5.2.2 (Ellenberg–Tran–Westerland [40]). *There is an isomorphism*

$$H_*(\mathrm{Conf}_n(\mathbb{C}) \cup \{\infty\}, \{\infty\}; \mathcal{T}) \cong H_*(C(n)_* \otimes T).$$

Let $\mathcal{T}^*$ denote the dual local system of $\mathcal{T}$, which is associated with the dual representation T^* of T. By applying the universal coefficient theorem and Poincaré duality to the dual over $\mathbf{k}$ of the left hand side with coefficients in $\mathcal{T}^*$, we have

$$\begin{aligned} H_*(\mathrm{Conf}_n(\mathbb{C}) \cup \{\infty\}, \{\infty\}; \mathcal{T}^*)^* &\cong H^*(\mathrm{Conf}_n(\mathbb{C}) \cup \{\infty\}, \{\infty\}; \mathcal{T}) \\ &\cong H^*_c(\mathrm{Conf}_n(\mathbb{C}); \mathcal{T}) \\ &\cong H_{2n-*}(\mathrm{Conf}_n(\mathbb{C}); \mathcal{T}). \end{aligned}$$

Since $\pi_1(\mathrm{Conf}_n(\mathbb{C})) = B_n$, we may rewrite the isomorphism in the previous theorem as

$$H_*(B_n; T) \cong H_{2n-*}(C(n)_* \otimes T^*)^*.$$

Recall that given any $B_{n,1}$-representation, the induced representation gives a representation for the parent group B_{n+1}. Combining this fact with the previous result, we obtain the following corollary:

Corollary 5.2.3. *For any $B_{n,1}$-representation L, there is an isomorphism*

$$H_*(B_{n,1}; L) \cong H_{2n+2-*}\Big(C(n+1)_* \otimes \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L^*)\Big)^*.$$

Proof. By Shapiro's Lemma (see, e.g., Lemma 6.3.2 of [96]), the homology of $B_{n,1}$ with local coefficients in L can be identified with the homology of the parent group B_{n+1} with coefficients in the induced representation, i.e.

$$H_*(B_{n,1}; L) \cong H_*\Big(B_{n+1}; \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L)\Big).$$

Applying the rewritten form of Theorem 5.2.2 to the right side completes the proof of the corollary. □

This homological algebra result provides a convenient machinery to compute the homology of $B_{n,1}$ (or similarly any other subgroup of the braid groups) with twisted coefficients, as long as we understand the action of B_{n+1} on the induced representation. However, it does not offer a geometrically intuitive explanation for how the cellular homology on the right side is related to the homology of the group $B_{n,1}$. We will resolve this issue in Section 5.4, by explicitly constructing a cellular chain complex that computes the homology of $B_{n,1}$ with coefficients in L and relating it to the complex $C(n+1)_* \otimes \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L^*)$.

Let $(V, W, \sigma, \tau, \varphi)$ be a separable left-braided vector space, then its twisted left-braided vector space $(V_\epsilon, W, \sigma_\epsilon, \tau, \varphi_\epsilon)$ is also separable with $\varphi_\epsilon = -\varphi$. Let $\mathfrak{A} = \mathfrak{A}(V_\epsilon)$ and $\mathfrak{M} = \mathfrak{M}(V_\epsilon, W)$, and let $\mathfrak{I}$ be the augmentation ideal of $\mathfrak{A}$, consisting of elements of positive degree. The following proposition shows the relationship between the algebraic structures we developed in the previous section and the cellular chain complex of configuration spaces with twisted coefficients.

Proposition 5.2.4. *There is an isomorphism of chain complexes*

$$F_{*,n+1}(\mathfrak{M}, \mathfrak{I}) \cong C(n+1)_{*+n+1} \otimes \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n} \otimes W).$$

Proof. Recall that the induced representation $\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n} \otimes W)$ is isomorphic to the direct sum $\bigoplus_{i=1}^{n+1} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i+1}$. Observe that $F_{q,n+1}(\mathfrak{M}, \mathfrak{I})$ consists of all spaces of the form

$$V_\epsilon^{\otimes \lambda_1} \otimes \cdots \otimes V_\epsilon^{\otimes \lambda_{i-1}} \otimes (V_\epsilon^{\otimes j} \otimes W \otimes V_\epsilon^{\otimes \lambda_i - j - 1}) \otimes V_\epsilon^{\otimes \lambda_{i+1}} \otimes \cdots \otimes V_\epsilon^{\lambda_q}$$

where $\sum \lambda_m = n+1$. This is an ordered partition of $n+1$ with q parts labelled by an element of $V_\epsilon^{\iota-1} \otimes W \otimes V_\epsilon^{\otimes n+1-\iota} \cong V^{\iota-1} \otimes W \otimes V^{\otimes n+1-\iota}$, where $\iota = j + 1 + \sum_{m=1}^{i-1} \lambda_m$ is the overall position of the factor W in the tensor product. Hence there is an isomorphism of $\mathbf{k}$-modules between $F_{q,n+1}(\mathfrak{M}, \mathfrak{I})$ and $C(n+1)_{q+n+1} \otimes \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n} \otimes W)$.

There are two main pieces of data in the boundary of an element $\lambda \otimes t$ in the chain complex $C(n+1)_* \otimes \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n} \otimes W)$ (stated in its general form prior to Theorem 5.2.2): the coarsening ρ^i of λ and the signed sum over all $(\lambda_i, \lambda_{i+1})$-shuffles of the actions of their lifts on t. Both are encapsulated in the differential of $F_{*,n+1}(\mathfrak{M}, \mathfrak{I})$: the coarsening ρ^i is encoded in the choice of two multiplied elements, and the sum of the braid actions is contained in the quantum shuffle product of $\mathfrak{A}$ or the multiplication of $\mathfrak{M}$ by $\mathfrak{A}$. Observe that in the differential of $F_{*,n+1}(\mathfrak{M},\mathfrak{I})$, the braids act only on a tensor subfactor of $V^{\otimes\iota-1}\otimes W\otimes V^{\otimes n+1-\iota}$, whereas the corresponding braids act on the isomorphic image $\alpha_\iota(V^{\otimes n}\otimes W)$ of this full factor in the differential of $C(n+1)_{*+n+1}\otimes \operatorname{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n}\otimes W)$. These braid actions match precisely due to the commutativity of Diagram $\mathscr{D}$ (Figure 3.2), which is equivalent to the separability of the left-braided vector space (V_ϵ, W) by Proposition 3.2.5. The signs coming from ϵ encode the boundary orientations on cells in the Fox–Neuwirth model. Via these identifications, the differentials of $F_{*,n+1}(\mathfrak{M},\mathfrak{I})$ and $C(n+1)_{*+n+1}\otimes \operatorname{Ind}_{B_{n,1}}^{B_{n+1}}(V^{\otimes n}\otimes W)$ are precisely the same formula, which shows their isomorphism as chain complexes. □

We are now ready to prove the main result of this chapter.

Proof of Theorem 5.2.1. We invoke Corollary 5.2.3 for the $B_{n,1}$-representation $L = V^{\otimes n}\otimes W$, Proposition 5.2.4, Corollary 5.1.5, and the duality of the Tor and Ext functors subsequently to get the desired isomorphism

$$\begin{aligned} H_*(B_{n,1}; V^{\otimes n}\otimes W) &\cong H_{2n+2-*}\Big(C(n+1)_*\otimes \operatorname{Ind}_{B_{n,1}}^{B_{n+1}}((V^*)^{\otimes n}\otimes W^*)\Big)^* \\ &\cong H_{n+1-*}(F_{*,n+1}(\mathfrak{M},\mathfrak{I}))^* \\ &\cong \operatorname{Tor}^{\mathfrak{A}^{\mathrm{e}}}_{n-*,n+1}(\mathfrak{M},\mathbf{k})^* \cong \operatorname{Ext}^{n-*,n+1}_{\mathfrak{A}^{\mathrm{e}}}(\mathfrak{M},\mathbf{k}) \end{aligned}$$

where $\mathfrak{A} = \mathfrak{A}(V_\epsilon^*)$ and $\mathfrak{M} = \mathfrak{M}(V_\epsilon^*, W^*)$. □

5.3 Homology with one-dimensional coefficients

We revisit Example 3.1.3 when the base field $\mathbf{k}$ has characteristic 0. Let the left-braided vector space (V, W, σ, τ) be composed of one-dimensional $\mathbf{k}$-vector spaces $V = \mathbf{k}$ and $W = \mathbf{k}\{w\}$, with braidings σ on $V\otimes V = \mathbf{k}$ and τ on $V\otimes W \cong \mathbf{k}$ given by multiplications by q and p respectively, for some $p, q \in \mathbf{k}^\times$. Thus the braid action of $B_{n,1}$ on the representation $V^{\otimes n}\otimes W \cong \mathbf{k}$ is given by $\sigma_i \mapsto q$ for all $1 \le i \le n-1$ and $\tau_n \mapsto p$. In this case, the quantum shuffle algebra $\mathfrak{A} = \mathfrak{A}(V)$ is generated (as a $\mathbf{k}$-module) by the classes $x_n = [1|\dots|1]_n$, where there are n occurrences of 1. It has been shown that the algebra $\mathfrak{A}$ is isomorphic as graded rings to the quantum divided power algebra $\Gamma_q[x]$ [40], whose structure has been previously studied by Callegaro [20].

Definition 5.3.1. The *quantum divided power algebra* $\Gamma_q[x]$ associated to $q \in \mathbf{k}^\times$ is additively generated by elements x_n in degree n, equipped with the product

$$x_n \star x_m = \binom{n+m}{m}_q x_{n+m}$$

where the quantum binomial coefficient is defined by

$$\binom{a}{b}_q = \frac{[a]_q[a-1]_q \cdots [a-b+1]_q}{[b]_q[b-1]_q \cdots [1]_q} \quad \text{with} \quad [r]_q = \frac{1-q^r}{1-q} = 1 + q + \cdots + q^{r-1}.$$

The isomorphism between $\Gamma_q[x]$ and $\mathfrak{A}$ sends the class x_n to $[1|\ldots|1]_n$ in $\mathfrak{A}$ [40]. The following identification of the algebra $\Gamma_q[x]$ is due to Callegaro.

Proposition 5.3.2 (Callegaro [20])**.** *If q is not a root of unity in $\mathbf{k}$, then there is an isomorphism $\Gamma_q[x] \cong \mathbf{k}[x_1]$. If q is a primitive m^{th} root of unity, then*

$$\Gamma_q[x] = \mathbf{k}[x_1]/x_1^m \otimes \Gamma[x_m].$$

As analyzed in Example 3.1.3, the left-braided vector space (V, W, σ, τ) is separable with the separated braiding φ' given simply by permuting tensor factors. Observe that this separated braiding is not unique: multiplying φ' by an arbitrary unit gives another separated braiding. For generality, we will choose the separated braiding to be $\varphi := u\varphi'$ for some $u \in \mathbf{k}^\times$.

Recall that the $\mathfrak{A}$-bimodule $\mathfrak{M}$ is defined by

$$\mathfrak{M} = \bigoplus_{n \geq 1} \bigoplus_{1 \leq i \leq n} V^{\otimes i-1} \otimes W \otimes V^{\otimes n-i}.$$

In this case, $\mathfrak{M}$ as a $\mathbf{k}$-module is generated by the classes $y_{i,n} := [1|\ldots|1|w^{(i)}|1|\ldots|1]_n$ (where i records the position of w) for all $n \geq 1$ and $1 \leq i \leq n$; in particular, denote $y_n := y_{n,n} = [1|\ldots|1|w]_n$. Corollary 3.2.9 specifies the action of $B_{n,1}$ on these generators as follows:

$$\sigma_m(y_{i,n}) = \begin{cases} qy_{i,n} & m \neq i-1, i \\ uy_{i-1,n} & m = i-1 \\ pu^{-1}y_{i+1,n} & m = i. \end{cases}$$

For the remainder of this section, we will study the structure of the module $\mathfrak{M}$ as an $\mathfrak{A}$-bimodule and compute its homology.

Consider the left multiplicative structure of $\mathfrak{M}$. Recall that $\{y_n, y_{n-1,n}, \ldots, y_{1,n}\}$ forms a basis for the degree-n homogeneous subspace $\mathfrak{M}_n$ of $\mathfrak{M}$.

Lemma 5.3.3. *For $n, m \geq 1$,*

$$x_m y_n = \sum_{h=0}^{m} u^{m-h} q^{(n-1)(m-h)} \binom{n-1+h}{h}_q y_{n+h,n+m}.$$

The proof of this lemma involves the concept of $((p,h),q,j)$-shuffles developed in Section 4.2, in particular Definition 4.2.2 and Proposition 4.2.4.

Proof. The left multiplication formula of $\mathfrak{M}$ gives

$$x_m y_n = [1|\ldots|1]_m \star [1|\ldots|1|w]_n = \sum_\gamma \widetilde{\gamma}[1|\ldots|1|w]_{n+m} = \sum_\gamma \widetilde{\gamma}(y_{n+m})$$

where γ draws from all (m,n)-shuffles. Given a shuffle γ, the summand of $\mathfrak{M}_{n+m}$ that $\widetilde{\gamma}(y_{n+m})$ belongs to is determined by the image of $n+m$ under the shuffle γ, i.e., $\widetilde{\gamma}(y_{n+m}) \in V^{\otimes i-1} \otimes W \otimes V^{\otimes n+m-i}$ where $i = \gamma(n+m)$. Such a shuffle γ belongs to a family of $((p,h),q,j)$-shuffles; in particular, γ is an $((m,h),n,n-1)$-shuffle, for $h = i-n$. Proposition 4.2.4 then states that γ can be uniquely decomposed into a sequence of three permutations: a permutation ω that sends the integer interval $[\![h+1,m]\!]$ to $[\![n+h+1,n+m]\!]$ (while preserving its order) and $n+m$ to $n+h$, followed by an $(h,n-1)$-shuffle β on $[\![1,h+n-1]\!]$ and an $(m-h,0)$-shuffle on $[\![n+h+1,n+m]\!]$ (i.e., the identity). Observe that the lift $\widetilde{\omega}$ can be written as the product of n braids, each of which represents the lift of the shuffle that moves $m-h$ points in the original interval $[\![h+1,m]\!]$ past the next point on the right (see Figure 5.1). The action of the first $n-1$ such braids each results in the multiplication by q^{m-h}, while the last braid element is

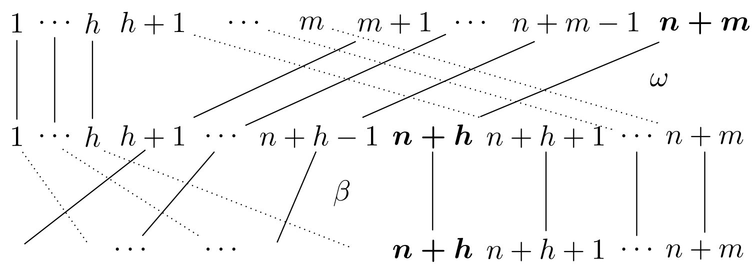

Figure 5.1: Decomposition of an $((m,h),n,n-1)$-shuffle.

precisely $\widetilde{\alpha_{n+h}}$ which acts by $\varphi_{n+h,n+m} = \varphi_{n+h} \dots \varphi_{n+m-1}$ on y_{n+m}. It follows that

$$x_m y_n = \sum_{\gamma} \widetilde{\gamma}(y_{n+m}) = \sum_{h=0}^{m} \sum_{\beta} q^{(n-1)(m-h)} \widetilde{\beta} \varphi_{n+h,n+m}(y_{n+m})$$
$$= \sum_{h=0}^{m} u^{m-h} q^{(n-1)(m-h)} \binom{n-1+h}{h}_q y_{n+h,n+m}$$

where β draws from all $(h,n-1)$-shuffles. The fact that the weighted sum over all $(h,n-1)$-shuffles β of $q^{cr(\beta)}$, where $cr(\beta)$ is the number of crossings in β, computes the quantum binomial coefficient is well established (see, e.g., Proposition 1.7.1 of [87]). □

Proposition 5.3.4. *The set $\{x_{n-k}y_k\}_{k=1}^n$ forms a basis for $\mathfrak{M}_n$. Consequently, $\mathfrak{M}$ is a free left $\mathfrak{A}$-module with respect to the basis $\mathcal{Y} = \{y_1, y_2, \dots\}$.*

Proof. Observe that for any given $1 \leq k \leq n$, the coefficient of $y_{i,n}$ in the expansion of $x_{n-k}y_k$ given by the previous lemma vanishes for all $1 \leq i \leq k-1$, and that of $y_{k,n}$ is $u^{n-k}q^{(k-1)(n-k)}$. Hence there is a change-of-basis matrix from the basis $\{y_{k,n}\}_{k=1}^n$ to $\{x_{n-k}y_k\}_{k=1}^n$

	y_n	x_1y_{n-1}	x_2y_{n-2}	$\cdots$	$x_{n-1}y_1$
y_n	1	$*$	$*$	$\cdots$	$*$
$y_{n-1,n}$	0	uq^{n-2}	$*$	$\cdots$	$*$
$y_{n-2,n}$	0	0	$u^2q^{2(n-3)}$	$\cdots$	$*$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$y_{1,n}$	0	0	0	$\cdots$	u^{n-1}

which is upper-triangular with all non-zero diagonal entries. This matrix is therefore invertible, which implies that $\{x_{n-k}y_k\}_{k=1}^n$ indeed forms a basis for $\mathfrak{M}_n$. It follows that the $\mathfrak{A}$-module $\mathfrak{M}$ is generated (as a $\mathbf{k}$-module) by the basis $\bigcup_{n=1}^{\infty}\{x_{n-k}y_k\}_{k=1}^n$. This basis can be freely generated from the set $\mathcal{Y}$ by the left multiplication by generators of the algebra $\mathfrak{A}$. $\square$

We now move onto the right multiplicative structure of $\mathfrak{M}$.

Lemma 5.3.5. *For $n, m \geq 1$,*

$$y_nx_m = \sum_{h=0}^{m} \left(\frac{p}{u}\right)^h \binom{n-1+h}{h}_q y_{n+h,n+m}.$$

Similar to that of Lemma 5.3.3, the proof of this lemma involves the $(p,(q,h),j)$-shuffles that were introduced in Section 4.2, particularly Definition 4.2.2 and Proposition 4.2.3.

Proof. The right multiplication formula of $\mathfrak{M}$ gives

$$y_nx_m = [1|\dots|1|w]_n \star [1|\dots|1]_m = \sum_{\gamma} \widetilde{\gamma}[1|\dots|1|w^{(n)}|1|\dots|1]_{n+m} = \sum_{\gamma} \widetilde{\gamma}(y_{n,n+m})$$

where γ draws from all (n,m)-shuffles. Similar to the case with left multiplication, a shuffle γ in this case belongs to a family of $(p,(q,h),j)$-shuffles; in particular, if γ sends n to i ($n \leq i \leq n+m$), it is classified as an $(n,(m,h),n-1)$-shuffle, for $h = i-n$. By applying Proposition 4.2.3, we observe that γ can be uniquely decomposed

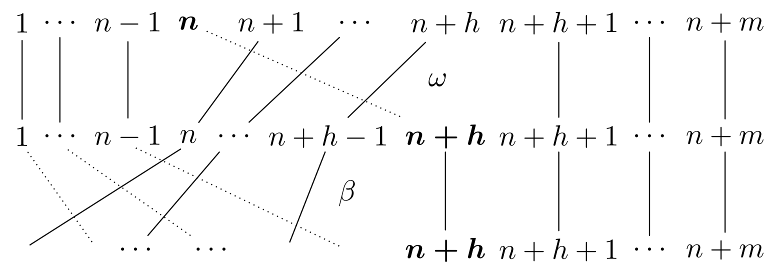

Figure 5.2: Decomposition of an $(n,(m,h),n-1)$-shuffle.

into a sequence of two permutations: a permutation ω that sends the integer interval $[\![n+1,n+h]\!]$ to $[\![n,n+h-1]\!]$ (while preserving its order) and n to $n+h$, followed by an $(n-1,h)$-shuffle β on $[\![1,n+h-1]\!]$ (see Figure 5.2). The lift $\widetilde{\omega}$ can be written as the product $\widetilde{\omega}=\sigma_{n+h-1}\sigma_{n+h-2}\ldots\sigma_n$. Observe that $\sigma_n(y_{n,n+m})=pu^{-1}y_{n+1,n+m}$, so by successive multiplication, it follows that $\widetilde{\omega}(y_{n,n+m})=\sigma_{n+h-1}\sigma_{n+h-2}\ldots\sigma_n(y_{n,n+m})=(pu^{-1})^h y_{n+h,n+m}$. Thus

$$
\begin{aligned}
y_n x_m &= \sum_{\gamma}\widetilde{\gamma}(y_{n,n+m}) = \sum_{h=0}^{m}\sum_{\beta}\left(\frac{p}{u}\right)^h \widetilde{\beta} y_{n+h,n+m} \\
&= \sum_{h=0}^{m}\left(\frac{p}{u}\right)^h \binom{n-1+h}{h}_q y_{n+h,n+m}
\end{aligned}
$$

where β draws from all $(n-1,h)$-shuffles. The proof is complete. $\square$

Proposition 5.3.6. *The set $\{y_k x_{n-k}\}_{k=1}^n$ forms a basis for $\mathfrak{M}_n$. Consequently, $\mathfrak{M}$ is a free right $\mathfrak{A}$-module with respect to the basis $\mathcal{Y}$.*

Proof. This proof follows the same logic as the proof of Proposition 5.3.4. Observe that for any given $1\le k\le n$, the coefficient of $y_{i,n}$ in the expansion of $y_k x_{n-k}$ given by Lemma 5.3.5 vanishes for all $1\le i\le k-1$, and that of $y_{k,n}$ is $(pu^{-1})^0=1$. Thus the change-of-basis matrix from the basis $\{y_{k,n}\}_{k=1}^n$ to $\{y_k x_{n-k}\}_{k=1}^n$ is given by

	y_n	$y_{n-1}x_1$	$y_{n-2}x_2$	$\cdots$	y_1x_{n-1}
y_n	1	*	*	$\cdots$	*
$y_{n-1,n}$	0	1	*	$\cdots$	*
$y_{n-2,n}$	0	0	1	$\cdots$	*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$y_{1,n}$	0	0	0	$\cdots$	1

which is upper-triangular with all non-zero diagonal entries and hence invertible. It follows that $\{y_k x_{n-k}\}_{k=1}^{n}$ forms a basis for $\mathfrak{M}_n$, and more generally the $\mathfrak{A}$-module $\mathfrak{M}$ is generated (as a **k**-module) by the basis $\bigcup_{n=1}^{\infty}\{y_k x_{n-k}\}_{k=1}^{n}$. This basis of $\mathfrak{M}$ can be freely generated from the set $\mathcal{Y}$ by the right multiplication by generators of the algebra $\mathfrak{A}$. □

So far in this discussion, we have explored three different bases for the module $\mathfrak{M}$ as a **k**-module: $\{y_{k,n}\}$, $\{x_{n-k}y_k\}$, and $\{y_k x_{n-k}\}$. The following lemma establishes a formula for the change of basis between the latter two.

Lemma 5.3.7. *For $n, m \geq 1$,*

$$y_n x_m = \frac{1}{u^m}\sum_{h=0}^{m}\left[\frac{1}{q^{(m-h)(n-1+h)}}\binom{n-1+h}{h}_q \prod_{k=0}^{h-1}\left(p - \frac{1}{q^{n-1+k}}\right)\right] x_{m-h}y_{n+h}.$$

Proof. We will prove this formula by induction on m. For $m = 1$, first we apply Lemma 5.3.3 to get

$$x_1 y_n = \sum_{h=0}^{1} u^{1-h} q^{(n-1)(1-h)} \binom{n-1+h}{h}_q y_{n+h,n+1} = \binom{n}{1}_q y_{n+1} + u q^{n-1} y_{n,n+1}$$

and apply Lemma 5.3.5 to get

$$y_n x_1 = \sum_{h=0}^{1}\left(\frac{p}{u}\right)^h \binom{n-1+h}{h}_q y_{n+h,n+1} = \frac{p}{u}\binom{n}{1}_q y_{n+1} + y_{n,n+1}.$$

By solving for $y_{n,n+1}$ from the first equation and substitute it into the second equation, we obtain

$$\begin{aligned} y_n x_1 &= \frac{p}{u}\binom{n}{1}_q y_{n+1} + \frac{1}{uq^{n-1}}\left(x_1 y_n - \binom{n}{1}_q y_{n+1}\right) \\ &= \frac{1}{u}\left[\binom{n}{1}_q \left(p - \frac{1}{q^{n-1}}\right) y_{n+1} + \frac{1}{q^{n-1}} x_1 y_n\right] \end{aligned}$$

which proves the base case.

Assume that the formula holds for all $1 \leq k \leq m$. The key observation here is that we can write $x_{m+1} = \frac{1}{[m+1]_q} x_m x_1$ in the quantum divided power algebra $\Gamma_q[x]$. By applying the induction hypothesis, we have

$$\begin{aligned} y_n x_{m+1} &= \frac{1}{[m+1]_q}(y_n x_m) x_1 \\ &= \frac{1}{[m+1]_q}\frac{1}{u^m}\sum_{h=0}^{m}\left[\frac{1}{q^{(m-h)(n-1+h)}}\binom{n-1+h}{h}_q \prod_{k=0}^{h-1}\left(p - \frac{1}{q^{n-1+k}}\right)\right] x_{m-h} y_{n+h} x_1 \\ &= \frac{1}{[m+1]_q}\frac{1}{u^m}\sum_{h=0}^{m}\left\{\left[\frac{1}{q^{(m-h)(n-1+h)}}\binom{n-1+h}{h}_q \prod_{k=0}^{h-1}\left(p - \frac{1}{q^{n-1+k}}\right)\right] x_{m-h}\right. \\ &\quad \left.\frac{1}{u}\left[\binom{n+h}{1}_q\left(p - \frac{1}{q^{n+h-1}}\right) y_{n+h+1} + \frac{1}{q^{n+h-1}} x_1 y_{n+h}\right]\right\} \\ &= \frac{1}{[m+1]_q}\frac{1}{u^{m+1}}\sum_{h=0}^{m}\left\{\left[\frac{[n+h]_q}{q^{(m-h)(n-1+h)}}\binom{n-1+h}{h}_q \prod_{k=0}^{h}\left(p - \frac{1}{q^{n-1+k}}\right)\right] x_{m-h}\right. \\ &\quad \left. y_{n+h+1} + \left[\frac{[m-h+1]_q}{q^{(m-h+1)(n-1+h)}}\binom{n-1+h}{h}_q \prod_{k=0}^{h-1}\left(p - \frac{1}{q^{n-1+k}}\right)\right] x_{m-h+1} y_{n+h}\right\}. \end{aligned}$$

This is an expansion of $y_n x_{m+1}$ in terms of the basis $\{x_{m+1-h} y_{n+h}\}_{h=0}^{m+1}$. In particular, the coefficient of $x_{m+1-h} y_{n+h}$ comes from those of the terms $x_{m-h-1} y_{n+h-1+1}$ and

$x_{m+1-h}y_{n+h}$:

$$\frac{1}{[m+1]_q}\frac{1}{u^{m+1}}\left\{\left[\frac{[n+h-1]_q}{q^{(m-(h-1))(n-1+h-1)}}\binom{n-1+h-1}{h-1}_q\prod_{k=0}^{h-1}\left(p-\frac{1}{q^{n-1+k}}\right)\right]\right.$$
$$\left.+\left[\frac{[m-h+1]_q}{q^{(m-h+1)(n-1+h)}}\binom{n-1+h}{h}_q\prod_{k=0}^{h-1}\left(p-\frac{1}{q^{n-1+k}}\right)\right]\right\}$$
$$=\frac{q^{m+1-h}[h]_q+[m-h+1]_q}{u^{m+1}[m+1]_q}\frac{1}{q^{(m+1-h)(n-1+h)}}\binom{n-1+h}{h}_q\prod_{k=0}^{h-1}\left(p-\frac{1}{q^{n-1+k}}\right).$$

Finally, observe that $q^{m+1-h}[h]_q+[m-h+1]_q = q^{m+1-h}(1+q+\cdots+q^{h-1})+(1+q+\cdots+q^{m-h}) = 1+q+\cdots+q^{m-h}+q^{m-h+1}+\cdots+q^m = [m+1]_q$, so the coefficient of $x_{m+1-h}y_{n+h}$ computed above matches the desired formula, completing our proof. □

Corollary 5.3.8. *If q is a primitive m^{th} root of unity, then*

$$y_n x_m = \frac{1}{u^m}\left[\left\lceil\frac{n}{m}\right\rceil\prod_{k=0}^{m-1}\left(p-\frac{1}{q^{n-1+k}}\right)\right]y_{n+m}+\langle x_h y_{n+m-h}\rangle$$

where $\lceil x\rceil$ denotes the ceiling function of x, and $\langle x_h y_{n+m-h}\rangle$ consists of all terms of the form $x_h y_{n+m-h}$ for $1\le h\le m$.

Proof. It suffices to show that $\binom{n-1+m}{m}_q=\lceil\frac{n}{m}\rceil$. By definition, we have

$$\binom{n-1+m}{m}_q=\frac{(1-q^{n+m-1})(1-q^{n+m-2})\dots(1-q^n)}{(1-q)(1-q^2)(1-q^m)}.$$

Among m consecutive numbers in $[\![n,n+m-1]\!]$, there exists exactly one number of the form $N=ma$ for some $a\ge 1$, for which we get

$$\frac{1-q^{ma}}{1-q^m}=\frac{(1-q^m)(1+q^m+\cdots+q^{m(a-1)})}{1-q^m}=a$$

since $q^m=1$. Each of the other terms in the numerator has the form $1-q^{mb+r}=1-q^r\ne 0$ for a unique $1\le r\le m-1$, which cancels out with the corresponding $1-q^r$ term in the denominator; here b is either a or $a-1$. It follows that $\binom{n-1+m}{m}_q=a$. Since N is the smallest multiple of m that is greater than or equal to n, we deduce that

$a = \lceil \frac{n}{m} \rceil$. □

We will now proceed to compute the homology of the $\mathfrak{A}$-bimodule $\mathfrak{M}$ using the description above. Including the sign twist ϵ and taking the dual of the braidings σ, τ, and φ is the same as replacing q, p, u with $-q^{-1}, p^{-1}, -u^{-1}$ respectively. Let $\mathfrak{A}(V_\epsilon^*) \cong \Gamma_{-q^{-1}}[x] =: \Gamma$ and $\mathfrak{M} = \mathfrak{M}(V_\epsilon^*, W^*)$. It follows from Proposition 5.3.4 that

$$\mathrm{Tor}^{\Gamma^e}_{*,*}(\mathfrak{M}, \mathbf{k}) = \mathfrak{M} \overset{L}{\underset{\Gamma\otimes\Gamma^{\mathrm{op}}}{\otimes}} \mathbf{k} = (\Gamma \otimes \mathbf{k}\mathcal{Y}) \overset{L}{\underset{\Gamma\otimes\Gamma^{\mathrm{op}}}{\otimes}} \mathbf{k} = \mathbf{k}\mathcal{Y} \overset{L}{\underset{\Gamma^{\mathrm{op}}}{\otimes}} \mathbf{k}$$

where $\mathcal{Y} = \{y_1, y_2, \dots\}$ is the chosen basis for $\mathfrak{M}$ as a free left Γ-module. We will continue this computation in the following cases:

Case 1: $-q$ is not a root of unity in $\mathbf{k}$. Recall from Proposition 5.3.2 that $\Gamma \cong \mathbf{k}[x_1]$, so it suffices to only look at the right multiplication of the generators y_n by x_1. Since all $x_m y_n$ terms vanish in $\mathbf{k}\mathcal{Y}$ for $m \geq 1$, Lemma 5.3.7 gives

$$\begin{aligned} y_n x_1 &= \frac{1}{-u^{-1}} \binom{n}{1}_{-q^{-1}} \left(p^{-1} - \frac{1}{(-q^{-1})^{n-1}} \right) y_{n+1} \\ &= \frac{1-(-q)^{-n}}{-u^{-1}(1+q^{-1})} \left(p^{-1} - \frac{1}{(-q^{-1})^{n-1}} \right) y_{n+1}. \end{aligned}$$

If p is not a power of $-q^{-1}$ (including $p = (-q^{-1})^0 = 1$), $\mathbf{k}\mathcal{Y}$ is freely generated as a Γ^{op}-module by a single generator y_1, and hence

$$\mathrm{Tor}^{\Gamma^e}_{*,*}(\mathfrak{M}, \mathbf{k}) = (\mathbf{k}\{y_1\} \otimes \Gamma^{\mathrm{op}}) \overset{L}{\underset{\Gamma^{\mathrm{op}}}{\otimes}} \mathbf{k} = \mathbf{k}\{y_1\} \overset{L}{\otimes} \mathbf{k} = \mathbf{k}\{y_1\} = \Sigma^1 \mathbf{k}$$

where $\Sigma^i M$ denotes the shift by i internal degrees of a graded module M, i.e.,

$$\mathrm{Tor}^{\Gamma^e}_{j,n}(\mathfrak{M}, \mathbf{k}) = \begin{cases} \mathbf{k} & \text{for } j = 0, n = 1 \\ 0 & \text{else.} \end{cases}$$

If $p = (-q)^{-(r-1)}$ for some $r \geq 1$, $y_r x_1 = 0$ whereas $y_n x_1 \neq 0$ for all $n \neq r$. It follows that the elements y_n for $1 \leq n \leq r$ can be generated from y_1 by multiplying with x_1^{n-1}, but y_{r+1} cannot. All elements $y_{\geq r+1}$ can be freely generated from y_{r+1} by multiplying with powers of x_1. Thus as a right Γ-module, $\mathbf{k}\mathcal{Y} \cong \mathbf{k}\{y_1\}[x_1]/x_1^r \oplus \mathbf{k}\{y_{r+1}\}[x_1] \cong$

$\Sigma^1\mathbf{k}[x_1]/x_1^r \oplus \Sigma^{r+1}\mathbf{k}[x_1]$. Since $\Gamma^{\text{op}} \cong \Gamma$, we have

$$\operatorname{Tor}_{*,*}^{\Gamma^e}(\mathfrak{M},\mathbf{k}) = \left(\Sigma^1\Gamma/x_1^r \oplus \Sigma^{r+1}\Gamma\right) \overset{L}{\underset{\Gamma}{\otimes}} \mathbf{k} = \Sigma^1\left(\Gamma/x_1^r \overset{L}{\underset{\Gamma}{\otimes}} \mathbf{k}\right) \oplus \Sigma^{r+1}\mathbf{k}.$$

To compute the first summand, we use the graded free resolution of $\mathbf{k}$ as a $\mathbf{k}[x_1]$-module

$$0 \to \Sigma^1\mathbf{k}[x_1] \xrightarrow{x_1} \mathbf{k}[x_1] \xrightarrow{\epsilon} \mathbf{k} \to 0$$

where ϵ is the augmentation map. Applying $- \otimes_{\mathbf{k}[x_1]} \Gamma/x_1^r$, we get

$$0 \to \Sigma^1\mathbf{k}[x_1]/x_1^r \xrightarrow{x_1} \mathbf{k}[x_1]/x_1^r \to 0.$$

Multiplication by x_1 has image (x_1) and kernel $\mathbf{k}\{x_1^{r-1}\}$, so

$$\operatorname{Tor}_{j,*}^{\Gamma}(\Gamma/x_1^r,\mathbf{k}) = \begin{cases} \mathbf{k} & \text{for } j=0 \\ \Sigma^1\mathbf{k}\{x_1^{r-1}\} \cong \Sigma^r\mathbf{k} & \text{for } j=1 \\ 0 & \text{else,} \end{cases}$$

or

$$\operatorname{Tor}_{j,n}^{\Gamma}(\Gamma/x_1^r,\mathbf{k}) = \begin{cases} \mathbf{k} & \text{for } j=0, n=0 \\ \mathbf{k} & \text{for } j=1, n=r \\ 0 & \text{else.} \end{cases}$$

Hence

$$\operatorname{Tor}_{j,n}^{\Gamma^e}(\mathfrak{M},\mathbf{k}) = \begin{cases} \mathbf{k} & \text{for } j=0, \text{ and } n=1,r+1 \\ \mathbf{k} & \text{for } j=1, \text{ and } n=r+1 \\ 0 & \text{else.} \end{cases}$$

Case 2: $-q$ is a primitive m^{th} root of unity in $\mathbf{k}$. By Proposition 5.3.2, $\Gamma = \mathbf{k}[x_1]/x_1^m \otimes \Gamma[x_m]$, so it suffices to study the right multiplication of the generators y_n by x_1 and x_m. Let $\Lambda_m := \mathbf{k}[x_1]/x_1^m$ denote the degree-m truncated polynomial algebra in variable x_1. If $\mathbf{k}$ has characteristic 0, there is an isomorphism $\Gamma[x_m] \cong \mathbf{k}[x_m]$. Consider the multiplication by x_m. In this case, Corollary 5.3.8 gives

$$y_n x_m = \frac{1}{(-u)^{-m}} \left[\left\lceil \frac{n}{m} \right\rceil \prod_{k=0}^{m-1} \left(p^{-1} - \frac{1}{(-q^{-1})^{n-1+k}} \right) \right] y_{n+m}.$$

Since the power of $-q$ in the product cycles through m consecutive values, we see that $y_n x_m = 0$ if and only if p is a power of $-q$.

If p is not a power of $-q$, observe that

$$y_n x_1 = \frac{1-(-q)^{-n}}{-u^{-1}(1+q^{-1})} \left(p^{-1} - \frac{1}{(-q^{-1})^{n-1}} \right) y_{n+1}$$

vanishes precisely when m is a divisor of n. Observe that for every $n \geq 1$, if we write $n = ma + r$ where $1 \leq r \leq m$, then the element y_n of $\mathcal{Y}$ can be generated uniquely as $y_n = C y_1 x_m^a x_1^{r-1}$ for some nonzero constant C. Thus we may identify

$$\mathbf{k}\mathcal{Y} \cong \mathbf{k}\{y_1\}(\mathbf{k}[x_1]/x_1^m \otimes \mathbf{k}[x_m]) \cong \Sigma^1(\Lambda_m \otimes \mathbf{k}[x_m]) = \Sigma^1\Gamma,$$

so

$$\mathrm{Tor}_{*,*}^{\Gamma^e}(\mathfrak{M}, \mathbf{k}) = \Sigma^1\Gamma \overset{L}{\underset{\Gamma}{\otimes}} \mathbf{k} = \Sigma^1\mathbf{k}.$$

On the other hand, if $p = (-q)^r$ for some $1 \leq r \leq m$, $y_n x_m = 0$ for all n, so $\mathbf{k}\mathcal{Y}$ is trivial as a $\mathbf{k}[x_m]$-module. In this case, $y_n x_1$ vanishes iff m divides n or $n + r - 1$. If $r = 1$, i.e., $p = -q$, these two conditions coincide. It follows that $\mathbf{k}\mathcal{Y}$ is freely generated by $\{y_1, y_{m+1}, \dots\}$ as a Λ_m-module, i.e.

$$\mathbf{k}\mathcal{Y} \cong \mathbf{k}\{y_1, y_{m+1}, \dots\}[x_1]/x_1^m \cong \bigoplus_{a=0}^{\infty} \Lambda_m\{y_{ma+1}\} \cong \bigoplus_{a=0}^{\infty} \Sigma^{ma+1}\Lambda_m.$$

We then have

$$\begin{aligned} \mathrm{Tor}_{*,*}^{\Gamma^e}(\mathfrak{M}, \mathbf{k}) &= \left(\bigoplus_{a=0}^{\infty} \Sigma^{ma+1}\Lambda_m \right) \overset{L}{\underset{\Lambda_m \otimes \mathbf{k}[x_m]}{\otimes}} \mathbf{k} \\ &= \bigoplus_{a=0}^{\infty} \Sigma^{ma+1}\mathbf{k} \overset{L}{\underset{\mathbf{k}[x_m]}{\otimes}} \mathbf{k} = \bigoplus_{a=0}^{\infty} \Sigma^{ma+1}\Lambda[z_m] \end{aligned}$$

for some $z_m \in \mathrm{Tor}_{1,m}$, i.e.

$$\mathrm{Tor}^{\Gamma^{\mathrm{e}}}_{j,n}(\mathfrak{M},\mathbf{k}) = \begin{cases} \mathbf{k} & \text{for } j=0, \text{ and } n = ma+1 \text{ for } a \geq 0 \\ \mathbf{k} & \text{for } j=1, \text{ and } n = ma+1 \text{ for } a \geq 1 \\ 0 & \text{else.} \end{cases}$$

Finally, if $r > 1$, we then have $y_n x_1 = 0$ whenever $n = ma$ or $n = ma - r + 1$. Following the same analysis above, we see that as a Λ_m-module,

$$\begin{aligned} \mathbf{k}\mathcal{Y} &\cong \mathbf{k}\{y_1\}[x_1]/x_1^{m-r+1} \oplus \mathbf{k}\{y_{m-r+2}\}[x_1]/x_1^{r-1} \oplus \mathbf{k}\{y_{m+1}\}[x_1]/x_1^{m-r+1} \oplus \dots \\ &= \left(\bigoplus_{a=0}^{\infty} \Lambda_{m-r+1}\{y_{ma+1}\}\right) \oplus \left(\bigoplus_{a=1}^{\infty} \Lambda_{r-1}\{y_{ma-r+2}\}\right) \\ &\cong \left(\bigoplus_{a=0}^{\infty} \Sigma^{ma+1}\Lambda_{m-r+1}\right) \oplus \left(\bigoplus_{a=1}^{\infty} \Sigma^{ma-r+2}\Lambda_{r-1}\right). \end{aligned}$$

Since $\mathbf{k}\mathcal{Y}$ is trivial as a $\mathbf{k}[x_m]$-module, by using the universal property of the tensor product (see, e.g., Proposition 13.104.1 in the ancillary file of [53]), we may write

$$\begin{aligned} \mathbf{k}\mathcal{Y} \overset{L}{\underset{\Lambda_m \otimes \mathbf{k}[x_m]}{\otimes}} \mathbf{k} &= (\mathbf{k}\mathcal{Y} \otimes \mathbf{k}) \overset{L}{\underset{\Lambda_m \otimes \mathbf{k}[x_m]}{\otimes}} (\mathbf{k} \otimes \mathbf{k}) \\ &= \left(\mathbf{k}\mathcal{Y} \overset{L}{\underset{\Lambda_m}{\otimes}} \mathbf{k}\right) \otimes \left(\mathbf{k} \overset{L}{\underset{\mathbf{k}[x_m]}{\otimes}} \mathbf{k}\right) = \left(\mathbf{k}\mathcal{Y} \overset{L}{\underset{\Lambda_m}{\otimes}} \mathbf{k}\right) \otimes \Lambda[z_m]. \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{Tor}^{\Gamma^{\mathrm{e}}}_{*,*}(\mathfrak{M},\mathbf{k}) &\cong \left[\left(\left(\bigoplus_{a=0}^{\infty} \Sigma^{ma+1}\Lambda_{m-r+1}\right) \oplus \left(\bigoplus_{a=1}^{\infty} \Sigma^{ma-r+2}\Lambda_{r-1}\right)\right) \overset{L}{\underset{\Lambda_m}{\otimes}} \mathbf{k}\right] \otimes \Lambda[z_m] \\ &\cong \left[\left(\bigoplus_{a=0}^{\infty} \Sigma^{ma+1}\left(\Lambda_{m-r+1} \overset{L}{\underset{\Lambda_m}{\otimes}} \mathbf{k}\right)\right) \oplus \left(\bigoplus_{a=1}^{\infty} \Sigma^{ma-r+2}\left(\Lambda_{r-1} \overset{L}{\underset{\Lambda_m}{\otimes}} \mathbf{k}\right)\right)\right] \otimes \Lambda[z_m]. \end{aligned}$$

It is left to compute $\Lambda_s \overset{L}{\underset{\Lambda_m}{\otimes}} \mathbf{k}$ for $1 \leq s \leq m-1$. We shall use the resolution

$$\dots \xrightarrow{x^s} \Sigma^{2m}\Lambda_m \xrightarrow{x^{m-s}} \Sigma^{m+s}\Lambda_m \xrightarrow{x^s} \Sigma^m\Lambda_m \xrightarrow{x^{m-s}} \Sigma^s\Lambda_m \xrightarrow{x^s} \Lambda_m \xrightarrow{\epsilon} \Lambda_s \to 0$$

where the augmentation map ϵ is the quotient map. Applying $- \overset{L}{\underset{\Lambda_m}{\otimes}} \mathbf{k}$, we get

$$\dots \xrightarrow{0} \Sigma^{2m}\mathbf{k} \xrightarrow{0} \Sigma^{m+s}\mathbf{k} \xrightarrow{0} \Sigma^{m}\mathbf{k} \xrightarrow{0} \Sigma^{s}\mathbf{k} \xrightarrow{0} \mathbf{k} \to 0.$$

Hence

$$\mathrm{Tor}^{\Lambda_m}_{j,n}(\Lambda_s, \mathbf{k}) = \begin{cases} \mathbf{k} & \text{for } j = 2a \text{ and } n = ma, \text{ for } a \geq 0 \\ \mathbf{k} & \text{for } j = 2a+1 \text{ and } n = ma+s, \text{ for } a \geq 0 \\ 0 & \text{else.} \end{cases}$$

By applying Theorem 5.2.1 and the duality of Tor and Ext functors to the computation of $\mathrm{Tor}^{\Gamma^e}_{*,*}(\mathfrak{M}, \mathbf{k})$ above, we obtain a characterization of the homology of $B_{n,1}$ with coefficients in one-dimensional braid representations.

Theorem 5.3.9. *Let $\mathbf{k}_{q,p}$ be the one-dimensional representation of $B_{n,1}$ over $\mathbf{k}$ where the generators σ_i $(1 \leq i \leq n-1)$ and τ_n act by multiplication by q and p respectively, for some $p, q \in \mathbf{k}^{\times}$. The homology of $B_{n,1}$ with coefficients in $\mathbf{k}_{q,p}$ is given in the following cases:*

1. *If $-q$ is not a root of unity in $\mathbf{k}$, and*

   (a) *p is not a power of $-q^{-1}$:*

   $$H_j(B_{n,1}; \mathbf{k}_{q,p}) = \begin{cases} \mathbf{k} & \textit{for } n = 0, j = 0 \\ 0 & \textit{else;} \end{cases}$$

   (b) *$p = (-q)^{-(r-1)}$ for some $r \geq 1$:*

   $$H_j(B_{n,1}; \mathbf{k}_{q,p}) = \begin{cases} \mathbf{k} & \textit{for } n = 0, j = 0 \\ \mathbf{k} & \textit{for } n = r, \textit{ and } j = r-1, r \\ 0 & \textit{else.} \end{cases}$$

2. *If $-q$ is a primitive m^{th} root of unity in $\mathbf{k}$, and*

(a) p is not a power of −q:

$$H_j(B_{n,1};\mathbf{k}_{q,p}) = \begin{cases} \mathbf{k} & \text{for } n=0, j=0 \\ 0 & \text{else;} \end{cases}$$

(b) p = −q:

$$H_j(B_{n,1};\mathbf{k}_{q,p}) = \begin{cases} \mathbf{k} & \text{for } n=0, j=0 \\ \mathbf{k} & \text{for } n=ma \ (a\geq 1), \text{ and } j=n-1,n \\ 0 & \text{else;} \end{cases}$$

(c) $p=(-q)^r$ *for* $2\leq r\leq m$*: for* $n=mk$ $(k\geq 0)$

$$H_j(B_{n,1};\mathbf{k}_{q,p}) = \begin{cases} \mathbf{k} & \text{for } j=n-2k,n \\ \mathbf{k}\oplus\mathbf{k} & \text{for } n-2k+1\leq j\leq n-1 \\ 0 & \text{else,} \end{cases}$$

and for $n=mk+m-r+1$ $(k\geq 0)$

$$H_j(B_{n,1};\mathbf{k}_{q,p}) = \begin{cases} \mathbf{k} & \text{for } j=n-2k-1,n \\ \mathbf{k}\oplus\mathbf{k} & \text{for } n-2k\leq j\leq n-1 \\ 0 & \text{else,} \end{cases}$$

while for all other n it vanishes.

Observe that in all cases, the homology of $B_{n,1}$ with coefficients in one-dimensional braid representations over $\mathbf{k}$ has very large vanishing ranges. Interesting phenomena happen when $-q$ is a primitive m^{th} root of unity and p is a power of $-q$. The following is a direct consequence of the previous theorem.

Corollary 5.3.10. *Let* $-q$ *be a primitive* m^{th} *root of unity. If* $p=-q$, $H_j(B_{n,1};\mathbf{k}_{q,p})$ *has a lower vanishing line* $j<n-1$*. If* $p=(-q)^r$ *for* $2\leq r\leq m$*, this vanishing line is* $j\leq \frac{m-2}{m}n-1$*.*

There are two specific cases of the computation in this section that are worth mentioning. When $q = 1$ and $p = 1$ (i.e., $m = r = 2$), we recover the homology of $B_{n,1}$ with trivial coefficients. In this case, $\Lambda_2 = \mathbf{k}[x_1]/x_1^2 \cong \Lambda[x_1]$ is the exterior algebra on a generator x_1, so we have

$$\mathbf{k}\mathcal{Y} \cong \left(\bigoplus_{a=0}^{\infty} \Sigma^{2a+1}\Lambda[x_1]/x_1\right) \oplus \left(\bigoplus_{a=1}^{\infty} \Sigma^{2a}\Lambda[x_1]/x_1\right) \cong \bigoplus_{a=1}^{\infty} \Sigma^{a}\mathbf{k}$$

and hence

$$\mathrm{Tor}^{\Gamma^e}_{*,*}(\mathfrak{M}, \mathbf{k}) \cong \bigoplus_{a=1}^{\infty} \Sigma^{a}\mathrm{Tor}^{\Gamma}_{*,*}(\mathbf{k}, \mathbf{k}) \cong \bigoplus_{a=1}^{\infty} \Sigma^{a}\mathbf{k}[y_1] \otimes \Lambda[z_2]$$

where $y_1 \in \mathrm{Tor}_{1,1}$ and $z_2 \in \mathrm{Tor}_{1,2}$. Observe that this is an infinite sum of the cohomologies of braid groups, which exhibits the same pattern as Gorjunov's classical computation over $\mathbb{Z}$ [52]. More explicitly, Theorem 5.3.9 gives

$$H_j(B_{n,1}; \mathbf{k}_{q,p}) = \begin{cases} \mathbf{k} & \text{for } j = 0, n \\ \mathbf{k} \oplus \mathbf{k} & \text{for } 1 \le j \le n-1 \\ 0 & \text{else.} \end{cases}$$

The second case of interest is when $q = 1$ and $p = -1$ (i.e., $m = 2, r = 1$); in this case, the representation $\mathbf{k}_{q,p}$ of $B_{n,1}$ tracks the parity of the number of generators τ_n in the decomposition of a braid, i.e., whether the number of times a braid wraps around its pure last strand is even or odd. Theorem 5.3.9 gives

$$H_j(B_{n,1}; \mathbf{k}_{q,p}) = \begin{cases} \mathbf{k} & \text{for } n = 0, j = 0 \\ \mathbf{k} & \text{for } n = 2a \ (a \ge 1), \text{ and } j = n-1, n \\ 0 & \text{else.} \end{cases}$$

In particular, the homology of all odd-numbered groups B_{2a+1} vanishes. When $n = 2a$, only the top two homology groups are nontrivial and have rank 1; the vanishing line of the homology of the even-numbered groups B_{2a} with these coefficients is $j < n - 1$, as deduced in Corollary 5.3.10.

5.4 Twisted cellular chain complex of $\mathrm{Conf}_n(\mathbb{C}^\times)$

In this section, we detail an alternate proof for Theorem 5.2.1, particularly a geometric proof of Corollary 5.2.3 based on the Fox–Neuwirth cellular stratification of $\mathrm{Conf}_n(\mathbb{C}^\times)$ developed in Section 4.2.

Let L be a representation of $B_{n,1}$, and $\mathcal{L}$ be the associated local system on $\mathrm{Conf}_n(\mathbb{C}^\times)$. Since $\mathcal{L}$ trivializes on the open cells of the Fox–Neuwirth stratification of $\mathrm{Conf}_n(\mathbb{C}^\times)$, as graded groups the cellular chain complex with local coefficients $C_*(\mathrm{Conf}_*(\mathbb{C}^\times) \cup \{\infty\}, \{\infty\}; \mathcal{L})$ is isomorphic to $D(n,1)_* \otimes L$. To show their isomorphism as chain complexes, we need to study the differential; in particular, we must incorporate the braid action on L.

Recall that in the induced representation discussed in Section 3.2, we have a full set of representatives $\{\alpha_i\}_{i=1}^{n+1}$ in the braid group B_{n+1} for the left cosets of $B_{n,1}$. Define the map $\eta_i : B_{n+1} \to B_{n,1} \subset B_{n+1}$ by sending a to $\alpha_{\underline{a}(i)}^{-1} a \alpha_i$, i.e., the image of a is exactly the element $b_i \in B_{n,1}$ chosen in the proof of Proposition 3.2.2. This map is not a group homomorphism; however, observe that $\eta_i(ab) = \eta_{\underline{b}(i)}(a)\eta_i(b)$, as the result of the definition of this induced B_{n+1}-action. We may then define the differential of $D(n,1)_* \otimes L$ by

$$\begin{aligned} d[(\lambda, i, j) \otimes \ell)] = &\sum_{m=1}^{i-2} (-1)^{m-1} \left[(\rho^m, i-1, j) \otimes \sum_{\gamma_m} (-1)^{|\gamma_m|} \eta_\iota(\widetilde{\gamma_m})(\ell) \right] \\ &+ \sum_{m=i+1}^{q-n} (-1)^{m-1} \left[(\rho^m, i, j) \otimes \sum_{\gamma_m} (-1)^{|\gamma_m|} \eta_\iota(\widetilde{\gamma_m})(\ell) \right] \\ &+ (-1)^{i-2} \sum_{h=0}^{\lambda_{i-1}} \left[(\rho^{i-1}, i-1, j+h) \otimes \sum_{\gamma_{i-1,h}} (-1)^{|\gamma_{i-1,h}|} \eta_\iota(\widetilde{\gamma_{i-1,h}})(\ell) \right] \\ &+ (-1)^{i-1} \sum_{h=0}^{\lambda_{i+1}} \left[(\rho^{i}, i, j+h) \otimes \sum_{\gamma_{i,h}} (-1)^{|\gamma_{i,h}|} \eta_\iota(\widetilde{\gamma_{i,h}})(\ell) \right] \end{aligned}$$

where $\iota = j+1+\sum_{m=1}^{i-1} \lambda_m$ is the overall position of the fixed point z_1 in the configuration (λ, i, j); $\gamma_{i-1,h}$ runs over all $((\lambda_{i-1}, h), \lambda_i, j)$-shuffles; $\gamma_{i,h}$ runs over all $(\lambda_i, (\lambda_{i+1}, h), j)$-shuffles; and γ_m runs over all $(\lambda_m, \lambda_{m+1})$-shuffles for all $m \neq i-1, i$. The lift $\widetilde{\gamma_m}$ (defined similarly for $\widetilde{\gamma_{i-1,h}}$ and $\widetilde{\gamma_{i,h}}$) in this differential is the lift of the shuffle γ_m to the copy

$B_{\lambda_m+\lambda_{m+1}} \leq B_{n+1}$ consisting of braids that are only nontrivial on the $\lambda_m+\lambda_{m+1}$ strands starting with the $\lambda_1+\cdots+\lambda_{m-1}+1^{\text{st}}$.

Theorem 5.4.1. *There is an isomorphism*

$$H_*(\mathrm{Conf}_n(\mathbb{C}^\times)\cup\{\infty\},\{\infty\};\mathcal{L}) \cong H_*(D(n,1)_*\otimes L).$$

Proof. Our argument will follow the outline of the proof of Theorem 3.3 of [40]. Let $\widetilde{D(n,1)}_*$ be the cellular chain complex of the universal cover on $\mathrm{Conf}_n(\mathbb{C}^\times)$ obtained by lifting the Fox–Neuwirth cells. It suffices to describe an identification

$$\widetilde{D(n,1)}_q \cong \mathbb{Z}\{(((\lambda_1,\ldots,\lambda_{q-n+1}),i,j),b)|b\in B_{n,1}\}$$

as right $B_{n,1}$-representations which gives the desired description of the differentials.

The top dimensional cells of $\widetilde{D(n,1)}_*$ occur when $q=2n$ and have the general form $(((1,\ldots,1),i,0),b)$. Consider the codimension-1 faces of this cell obtained by combining the m^{th} and $m+1^{\text{st}}$ columns, i.e., putting the m^{th} and $m+1^{\text{st}}$ points on the same vertical line. There are two main outcomes of this operation: either the m^{th} point lies below the $m+1^{\text{st}}$ point, or vice versa. Each of these are divided into subcases, depending on whether the fixed point z_1 is involved. Recall that for any configuration in $\mathrm{Conf}_n(\mathbb{C}^\times)$, the lexicographic order of points in the configuration is obtained by indexing them from bottom to top for each subsequent column starting with the leftmost one. We then label the braid element of a face based on its effect on the lexicographic order of points in the configuration: if the lexicographic order is preserved, we apply $\eta_i(\mathrm{id})=\mathrm{id}$ to b on the left:

Case	m^{th} point below $m+1^{\text{st}}$ point
$m<i-1$	$(((1,\ldots,1,2^{(m)},1,\ldots,1),i-1,0),b)$
$m=i-1$	$(((1,\ldots,1,2^{(i-1)},1,\ldots,1),i-1,1),b)$
$m=i$	$(((1,\ldots,1,2^{(i)},1,\ldots,1),i,0),b)$
$m>i$	$(((1,\ldots,1,2^{(m)},1,\ldots,1),i,0),b)$

where $(1,\ldots,1,2^{(m)},1,\ldots,1)$ denotes the composition of $n+1$ where the only non-1 part is $\lambda_m = 2$. Otherwise, we apply $\eta_i(\widetilde{\gamma})$ where γ is the corresponding permutation:

Case	m^{th} point above $m+1^{\text{st}}$ point
$m < i-1$	$(((1,\ldots,1,2^{(m)},1,\ldots,1),i-1,0),\eta_i(\sigma_m)b)$
$m = i-1$	$(((1,\ldots,1,2^{(i-1)},1,\ldots,1),i-1,0),\eta_i(\sigma_{i-1})b)$
$m = i$	$(((1,\ldots,1,2^{(i)},1,\ldots,1),i,1),\eta_i(\sigma_i)b)$
$m > i$	$(((1,\ldots,1,2^{(m)},1,\ldots,1),i,0),\eta_i(\sigma_m)b)$

Note that this choice of labelling is consistent with the right action of $B_{n,1}$.

More generally, the cell $((\lambda,i,j),b)$ corresponds to the face of $(((1,\ldots,1),\iota,0),b)$ obtained by putting points into columns according to the configuration λ while preserving the lexicographic order of points. Here the number $\iota = j+1+\sum_{m=1}^{i-1}\lambda_m$ is the overall position of z_1 in the configuration. However, if we arrange the face so that the lexicographic order is altered by a permutation γ, we need to multiply the element of $B_{n,1}$ in the cell's label on the left with $\eta_\iota(\widetilde{\gamma})$. Note that this labelling system is compatible with the decomposition of braid elements into generators precisely because $\eta_i(ab) = \eta_{\underline{b}(i)}(a)\eta_i(b)$ for any $a,b \in B_{n+1}$.

It follows from this labelling system that the face maps of the complex $\widetilde{D(n,1)}_*$ are given by

$$d_m((\lambda,i,j),b) = \begin{cases} \displaystyle\sum_{\gamma_m}(-1)^{|\gamma_m|}((\rho^m,i-1,j),\eta_\iota(\widetilde{\gamma_m})b) & m < i-1 \\ \displaystyle\sum_{h=0}^{\lambda_{i-1}}\sum_{\gamma_{i-1,h}}(-1)^{|\gamma_{i-1,h}|}((\rho^{i-1},i-1,j+h),\eta_\iota(\widetilde{\gamma_{i-1,h}})b) & m = i-1 \\ \displaystyle\sum_{h=0}^{\lambda_{i+1}}\sum_{\gamma_{i,h}}(-1)^{|\gamma_{i,h}|}((\rho^i,i,j+h),\eta_\iota(\widetilde{\gamma_{i,h}})b) & m = i \\ \displaystyle\sum_{\gamma_m}(-1)^{|\gamma_m|}((\rho^m,i,j),\eta_\iota(\widetilde{\gamma_m})b) & m > i. \end{cases}$$

Notice that the sums over h in the face maps d_{i-1} and d_i result from the fact that the number j of points below the fixed point is a meaningful index of these cells. The signs in the formula come from the orientations of the cells. The differential $d : \widetilde{D(n,1)}_q \to \widetilde{D(n,1)}_{q-1}$ is given by the alternating sum of the face maps: $d = \sum_{m=1}^{q-n}(-1)^{m-1}d_m$.

Given a $B_{n,1}$-representation L, we want to give a description of the chain complex $\widetilde{D(n,1)}_* \otimes_{\mathbb{Z}B_{n,1}} L$ and its differential. Observe that we may identify $((\lambda,i,j),b)\otimes\ell$ with $((\lambda,i,j),1)\otimes b(\ell)$, hence there is a natural isomorphism of $\mathbf{k}$-modules $\widetilde{D(n,1)}_* \otimes_{\mathbb{Z}B_{n,1}} L \cong D(n,1)_* \otimes L$. Furthermore, this identification when applied to the face maps and the differential results in the desired formula for the differential of $D(n,1)_* \otimes L$; for instance, when $m < i-1$,

$$\begin{aligned} d_m(((\lambda,i,j),b)\otimes\ell) &= \sum_{\gamma_m}(-1)^{|\gamma_m|}((\rho_m,i-1,j),\eta_\iota(\widetilde{\gamma_m})b)\otimes\ell \\ &= \sum_{\gamma_m}(-1)^{|\gamma_m|}((\rho_m,i-1,j),1)\otimes\eta_\iota(\widetilde{\gamma_m})b(\ell). \end{aligned}$$

This concludes our proof of the theorem. □

This result gives another tool to compute the homology of $B_{n,1}$ with twisted coefficients in a similar manner as Corollary 5.2.3; however, the geometric construction of the chain complex $D(n,1)_* \otimes L$ offers a more intuitive view of the computation than the induced representation $\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L)$ does.

Corollary 5.4.2. *There is an isomorphism*

$$H_*(B_{n,1};L) \cong H_{2n-*}(D(n,1)_* \otimes L^*)^*.$$

Proof. Dualize over $\mathbf{k}$ both sides of Theorem 5.4.1 with coefficients in $\mathcal{L}^*$, apply the universal coefficient theorem and Poincaré duality to the dual of the left side, and invoke the fact that $\pi_1(\mathrm{Conf}_n(\mathbb{C}^\times)) = B_{n,1}$ to complete our proof. □

This result combined with Corollary 5.2.3 shows that there is a quasi-isomorphism between the complexes $D(n,1)_* \otimes L^*$ and $C(n+1)_* \otimes \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L^*)$. As we hinted earlier, these chain complexes are in fact isomorphic, which evidences a close relationship between the cellular stratification we constructed for $\mathrm{Conf}_n(\mathbb{C}^\times)$ and the induced representation.

Proposition 5.4.3. *For any $B_{n,1}$-representation L, there is an isomorphism of chain complexes*

$$D(n,1)_* \otimes L \cong C(n+1)_{*+2} \otimes \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L).$$

Proof. Define a chain map $D(n,1)_*\otimes L \to C(n+1)_{*+2}\otimes \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L)$ by sending $(\lambda, i, j)\otimes\ell$ to $\lambda \otimes \widetilde{\alpha_\iota}(\ell)$ where $\iota = j + 1 + \sum_{m=1}^{i-1} \lambda_m$. Conversely, given an element $\lambda \otimes t \in C(n+1)_{*+2} \otimes \mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L)$, observe that t must be an element of $\widetilde{\alpha_\iota} L$ for some $1 \le \iota \le n+1$. Let i be the largest integer such that $j' := \iota - \sum_{m=1}^{i-1} \lambda_m > 0$, then there is a chain map $C(n+1)_{*+2}\otimes\mathrm{Ind}_{B_{n,1}}^{B_{n+1}}(L) \to D(n,1)_*\otimes L$ that sends $\lambda\otimes t$ to $(\lambda, i, j'-1)\otimes\widetilde{\alpha_\iota}^{-1}(t)$. These maps are evidently inverses as maps of graded **k**-modules, hence the chain complexes are isomorphic. □

At this point, we have introduced two different methods to compute the homology of $B_{n,1}$ with twisted coefficients. The case of $B_{n,1} \cong B_n(\mathbb{C}^\times)$ offers an interesting intersection between two families of braid subgroups: mixed braid groups and surface braid groups on punctured genus-0 surfaces. As hinted by the discussion in Section 3.3, the algebraic approach in Section 5.2, which relies on understanding the induced representation of the subgroups, is more suitable for studying the homology of mixed braid groups. However, for braid subgroups such as $B_n(\mathbb{C}_m)$ whose index is infinite, it is challenging to analyze the induced representation. The geometric method demonstrated in this section, which relies on developing an explicit model for the space $\mathrm{Conf}_n(\mathbb{C}^\times)$, offers a more natural approach to study braid groups on the plane with more punctures.

One additional benefit of this geometric argument is that it provides an explicit connection between the homological algebra objects developed in Section 5.1 and the Fox–Neuwirth cellular stratification of $\mathrm{Conf}_n(\mathbb{C}^\times)$. Recall that we constructed two algebraic objects to express the homology of $B_{n,1}$: the free resolution $F_*(M, A)$ of an A-bimodule M and the $\mathfrak{A}$-bimodule $\mathfrak{M}$ over a quantum shuffle algebra $\mathfrak{A}$. These objects capture two key aspects of the cellular stratification of $\mathrm{Conf}_n(\mathbb{C}^\times)$: the configuration of the vertical columns and the configuration of points on the imaginary axis.

The $\mathfrak{A}$-bimodule $\mathfrak{M}$ provides an algebraic analogue of the imaginary axis in a configuration in $\mathrm{Conf}_n(\mathbb{C}^\times)$. The single copy of W^* in every summand $(V_\epsilon^*)^{\otimes j}\otimes W^*\otimes(V_\epsilon^*)^{\otimes q-j-1}$ of $\mathfrak{M}$ decorates the fixed point at the origin, while all other points in the configuration are decorated by V. Meanwhile, the chain complex $F_*(M, A)$ mirrors the structure of the vertical columns in the stratification of $\mathrm{Conf}_*(\mathbb{C}^\times)$. Given the choices of the algebra $A = \mathfrak{A}(V_\epsilon^*)$ and the bimodule $M = \mathfrak{M}(V_\epsilon^*, W^*)$, the single copy of M in each summand of the graded module $F_*(M, A)$ represents the imaginary axis in the configuration, while column-combining operations are encoded in the multiplication of both the algebra and the bimodule. These observations offer a more intuitive perspective on the construction of our algebraic objects, which was not obvious from the first approach involving the induced representation.

Chapter 6

Twisted homology of genus-0 surface braid groups

In this chapter, we will develop a framework to compute the cellular homology of $\mathrm{Conf}_n(\mathbb{C}_m)$ with coefficients in local systems. As our main topological result, we will identify the homology of $B_n(\mathbb{C}_m)$ with coefficients arising from mixed-braided vector spaces (V, W) with the cohomology of bimodules $\mathfrak{M}(V_\epsilon^*, W_\epsilon^*)$ over the quantum shuffle algebra $\mathfrak{A}(V_\epsilon^*)$. As an application, we will compute the homology of $B_n(\mathbb{C}_m)$ and in turn prove a vanishing range for the homology of $B_{n,m}$ with certain one-dimensional twisted coefficients. The content of this chapter is extracted from Section 4 of [54].

6.1 Twisted cellular chain complex of $\mathrm{Conf}_n(\mathbb{C}_m)$

We will generalize the topological construction in the previous section.

Let $\mathcal{I} = (i_1, \dots, i_m)$ where $1 \le i_1 < \cdots < i_m \le n+m$, and $\alpha_\mathcal{I}$ the associated (n,m)-shuffle. Define a function of sets $\eta_\mathcal{I} : B_{n+m} \to B_{n,m}$ by sending a to $\widetilde{\underline{a}_{\alpha_\mathcal{I}}}^{-1} a \widetilde{\alpha_\mathcal{I}}$, i.e., the unique element b such that $a\widetilde{\alpha_\mathcal{I}} = \widetilde{\alpha}' b$ for some $\alpha' \in \mathrm{Sh}(n,m)$. This map is not a group homomorphism; however, it satisfies the composition relation $\eta_\mathcal{I}(ba) = \eta_{\underline{a}(\mathcal{I})}(b)\eta_\mathcal{I}(a)$. When restricting the domain to only braids with pairwise parallel i_j^{th} and i_k^{th} strands for all distinct $i_j, i_k \in \mathcal{I}$, we observed that the range is in fact $B_n(\mathbb{C}_m)$. Since our braids always arise from lifting shuffles that preserve the order on the set $\mathcal{I}$ of overall positions of the fixed points in a configuration, this assumption applies to our discussion below.

Let L be a representation of $B_n(\mathbb{C}_m)$, and $\mathcal{L}$ be the associated local system over $\mathrm{Conf}_n(\mathbb{C}_m)$. Since $\mathcal{L}$ trivializes on the open cells of the Fox–Neuwirth stratification of $\mathrm{Conf}_n(\mathbb{C}_m)$ constructed in Section 4.3, the twisted cellular chain complex $C_*(\mathrm{Conf}_*(\mathbb{C}_m)\cup\{\infty\},\{\infty\};\mathcal{L})$ is isomorphic to $D(n,m)_*\otimes L$ as graded groups. The differential of $D(n,m)_*\otimes L$ which incorporates the braid action on L is defined by

$$\begin{aligned} d[(\lambda,I,J)\otimes\ell)] \quad &= \sum_{i=1}^{i_1-2}(-1)^{i-1}\left[(\rho^i,I_1,J)\otimes\sum_{\gamma_i}(-1)^{|\gamma_i|}\eta_{\mathcal{I}}(\widetilde{\gamma_i})(\ell)\right] \\ &+\sum_{k=1}^{m-1}\sum_{i=i_k+1}^{i_{k+1}-2}(-1)^{i-1}\left[(\rho^i,I_{k+1},J)\otimes\sum_{\gamma_i}(-1)^{|\gamma_i|}\eta_{\mathcal{I}}(\widetilde{\gamma_i})(\ell)\right] \\ &+\sum_{i=i_m+1}^{q-n+m-1}(-1)^{i-1}\left[(\rho^i,I,J)\otimes\sum_{\gamma_i}(-1)^{|\gamma_i|}\eta_{\mathcal{I}}(\widetilde{\gamma_i})(\ell)\right] \\ &+\sum_{k=1}^{m}(-1)^{i_k-2}\sum_{h=0}^{\lambda_{i_k-1}}\left[(\rho^{i_k-1},I_k,J_{k,h})\otimes\sum_{\gamma_{i_k-1,h}}(-1)^{|\gamma_{i_k-1,h}|}\eta_{\mathcal{I}}(\widetilde{\gamma_{i_k-1,h}})(\ell)\right] \\ &+\sum_{k=1}^{m}(-1)^{i_k-1}\sum_{h=0}^{\lambda_{i_k+1}}\left[(\rho^{i_k},I_{k+1},J_{k,h})\otimes\sum_{\gamma_{i_k,h}}(-1)^{|\gamma_{i_k,h}|}\eta_{\mathcal{I}}(\widetilde{\gamma_{i_k,h}})(\ell)\right] \end{aligned}$$

where $\mathcal{I}=(\iota_1,\ldots,\iota_m)$ with $\iota_k:=j_k+1+\sum_{i=1}^{i_k-1}\lambda_i$ the overall position of z_k in the configuration (λ,I,J); $\gamma_{i_k-1,h}$ runs over all $((\lambda_{i_k-1},h),\lambda_{i_k},j_k)$-shuffles; $\gamma_{i_k,h}$ runs over all $(\lambda_{i_k},(\lambda_{i_k+1},h),j_k)$-shuffles; and γ_i runs over all $(\lambda_i,\lambda_{i+1})$-shuffles for all $i\neq i_k-1,i_k$. The lift $\widetilde{\gamma_i}$ (defined similarly for $\widetilde{\gamma_{i_k-1,h}}$ and $\widetilde{\gamma_{i_k,h}}$) in this differential is the lift of the shuffle γ_i to the copy $B_{\lambda_i+\lambda_{i+1}}\leq B_{n+m}$ consisting of braids that are only nontrivial on the $\lambda_i+\lambda_{i+1}$ strands starting with the $\lambda_1+\cdots+\lambda_{i-1}+1^{\mathrm{st}}$. We prove the main structural theorem of this chapter below.

Theorem 6.1.1. *There is an isomorphism*

$$H_*(\mathrm{Conf}_n(\mathbb{C}_m)\cup\{\infty\},\{\infty\};\mathcal{L})\cong H_*(D(n,m)_*\otimes L).$$

Proof. Our argument will generalize of the proof of Theorem 5.4.1. Let $\widetilde{D(n,m)}_*$ be the cellular chain complex of the universal cover on $\mathrm{Conf}_n(\mathbb{C}_m)$ obtained by lifting the

Fox–Neuwirth cells. It suffices to describe an identification

$$\widetilde{D(n,m)}_q \cong \mathbb{Z}\{(((\lambda_1,\ldots,\lambda_{q-n+m}),I,J),b)|b\in B_n(\mathbb{C}_m)\}$$

as right $B_n(\mathbb{C}_m)$-representations which gives the desired description of the differentials.

The top dimensional cells of $\widetilde{D(n,m)}_*$ occur when $q = 2n$ and have the general form $(((1,\ldots,1),I,(0,\ldots,0)),b)$ for some $I = (i_1,\ldots,i_m)$. Consider the codimension-1 faces of this cell obtained by combining the i^{th} and $i+1^{\text{st}}$ columns, i.e., putting the i^{th} and $i+1^{\text{st}}$ points on the same vertical line. There are two main outcomes of this operation: either the i^{th} point lies below the $i+1^{\text{st}}$ point, or vice versa. Each of these are divided into subcases, depending on whether a fixed point is involved. Recall that for any configuration in $\text{Conf}_n(\mathbb{C}_m)$, the lexicographic order of points in the configuration is obtained by indexing them from bottom to top for each subsequent column starting with the leftmost one. We then label the braid element of a face based on its effect on this order of points in the configuration as follows. If the lexicographic order is preserved, we apply $\eta_I(\text{id}) = \text{id}$ to b on the left; the codimension-1 faces in this case are labelled by:

Case	i^{th} point lies below $i+1^{\text{st}}$ point
$1 \le i < i_1 - 1$	$(((1,\ldots,1,2^{(i)},1,\ldots,1),I_1,(0,\ldots,0)),b)$
$i_k < i < i_{k+1}-1$ $(1\le k<m)$	$(((1,\ldots,1,2^{(i)},1,\ldots,1),I_{k+1},(0,\ldots,0)),b)$
$i_m < i < n+m$	$(((1,\ldots,1,2^{(i)},1,\ldots,1),I,(0,\ldots,0)),b)$
$i = i_k - 1$ $(1\le k\le m)$	$(((1,\ldots,1,2^{(i_k-1)},1,\ldots,1),I_k,(0,\ldots,0,1^{(k)},0,\ldots,0)),b)$
$i = i_k$ $(1\le k\le m)$	$(((1,\ldots,1,2^{(i_k)},1,\ldots,1),I_{k+1},(0,\ldots,0)),b)$

where $(1,\ldots,1,2^{(i)},1,\ldots,1)$ denotes the composition of $n+m$ where the only non-1 part is $\lambda_i = 2$, and $(0,\ldots,0,1^{(k)},0,\ldots,0)$ denotes an m-tuple where the only nonzero entry is 1 at the k^{th} coordinate. On the other hand, if the lexicographic order changes, we apply $\eta_I(\widetilde{\gamma})$ where γ is the corresponding permutation. The labelling system in this case is given by

Case	i^{th} point lies above $i+1^{\text{st}}$ point
$1 \leq i < i_1 - 1$	$(((1,\ldots,1,2^{(i)},1,\ldots,1), I_1, (0,\ldots,0)), \eta_I(\sigma_i)b)$
$i_k < i < i_{k+1} - 1$ $(1 \leq k < m)$	$(((1,\ldots,1,2^{(i)},1,\ldots,1), I_{k+1}, (0,\ldots,0)), \eta_I(\sigma_i)b)$
$i_m < i < n+m$	$(((1,\ldots,1,2^{(i)},1,\ldots,1), I, (0,\ldots,0)), \eta_I(\sigma_i)b)$
$i = i_k - 1$ $(1 \leq k \leq m)$	$(((1,\ldots,1,2^{(i_k-1)},1,\ldots,1), I_k, (0,\ldots,0)), \eta_I(\sigma_{i_k-1})b)$
$i = i_k$ $(1 \leq k \leq m)$	$(((1,\ldots,1,2^{(i_k)},1,\ldots,1), I_{k+1}, (0,\ldots,0,1^{(k)},0,\ldots,0)), \eta_I(\sigma_{i_k})b)$

Note that this choice of labelling is consistent with the right action of $B_n(\mathbb{C}_m)$.

More generally, a generic cell $((\lambda, I, J), b)$ corresponds to the face of the top dimensional cell $(((1,\ldots,1), \mathcal{I}, (0,\ldots,0)), b)$ obtained by putting points into columns according to the configuration λ while preserving the lexicographic order of points; here $\mathcal{I} = (\iota_1, \ldots, \iota_m)$ records the overall positions of the fixed points in the configuration, i.e., $\iota_k = j_k + 1 + \sum_{i=1}^{i_k-1} \lambda_i$ for $1 \leq k \leq m$. However, if we arrange the face so that the lexicographic order is altered by a permutation γ, we need to multiply the element of $B_n(\mathbb{C}_m)$ in the cell's label on the left with $\eta_{\mathcal{I}}(\widetilde{\gamma})$. Note that this labelling system is compatible with the decomposition of braid elements into generators precisely because $\eta_{\mathcal{I}}(ba) = \eta_{\underline{a}(\mathcal{I})}(b)\eta_{\mathcal{I}}(a)$ for any $a, b \in B_{n+m}$.

It follows from this labelling system that the face maps of the complex $\widetilde{D(n,m)}_*$ are given by

$$d_i((\lambda, I, J), b) = \begin{cases} \displaystyle\sum_{\gamma_i} (-1)^{|\gamma_i|}((\rho^i, I_1, J), \eta_{\mathcal{I}}(\widetilde{\gamma_i})b) & 1 \leq i < i_1 - 1 \\ \displaystyle\sum_{\gamma_i} (-1)^{|\gamma_i|}((\rho^i, I_{k+1}, J), \eta_{\mathcal{I}}(\widetilde{\gamma_i})b) & i_k < i < i_{k+1} - 1 \\ \displaystyle\sum_{\gamma_i} (-1)^{|\gamma_i|}((\rho^i, I, J), \eta_{\mathcal{I}}(\widetilde{\gamma_i})b) & i_m < i \leq q - n + m - 1 \\ \displaystyle\sum_{h=0}^{\lambda_{i_k-1}} \sum_{\gamma_{i_k-1,h}} (-1)^{|\gamma_{i_k-1,h}|}((\rho^{i_k-1}, I_k, J_{k,h}), \eta_{\mathcal{I}}(\widetilde{\gamma_{i_k-1,h}})b) & i = i_k - 1 \\ \displaystyle\sum_{h=0}^{\lambda_{i_k+1}} \sum_{\gamma_{i_k,h}} (-1)^{|\gamma_{i_k,h}|}((\rho^{i_k}, I_{k+1}, J_{k,h}), \eta_{\mathcal{I}}(\widetilde{\gamma_{i_k,h}})b) & i = i_k. \end{cases}$$

The signs in the formula come from the orientations of the cells. The differential $d : \widetilde{D(n,m)}_q \to \widetilde{D(n,m)}_{q-1}$ is given by the alternating sum of the face maps: $d = \sum_{i=1}^{q-n+m-1}(-1)^{i-1}d_i$.

Given a $B_n(\mathbb{C}_m)$-representation L, we want to give a description of the chain complex $\widetilde{D(n,m)}_* \otimes_{\mathbb{Z}B_n(\mathbb{C}_m)} L$ and its differential. Observe that we may identify $((\lambda, I, J), b) \otimes \ell$ with $((\lambda, I, J), 1) \otimes b(\ell)$, hence there is a natural isomorphism of $\mathbf{k}$-modules $\widetilde{D(n,m)}_* \otimes_{\mathbb{Z}B_n(\mathbb{C}_m)} L \cong D(n,m)_* \otimes L$. Furthermore, this identification when applied to the face maps and the differential results in the desired formula for the differential of $D(n,m)_* \otimes L$; for instance, when $1 \leq i < i_1 - 1$,

$$\begin{aligned} d_i(((\lambda, I, J), b) \otimes \ell) &= \sum_{\gamma_i} (-1)^{|\gamma_i|}((\rho_i, I_1, J), \eta_{\mathcal{I}}(\widetilde{\gamma_i})b) \otimes \ell \\ &= \sum_{\gamma_i} (-1)^{|\gamma_i|}((\rho_i, I_1, J), 1) \otimes \eta_{\mathcal{I}}(\widetilde{\gamma_i})b(\ell). \end{aligned}$$

This concludes our proof of the theorem. □

This result provides a general framework for computing the cellular homology of $\text{Conf}_n(\mathbb{C}_m)$ with local coefficients. An easy observation is that the cellular homology of

$\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}$ concentrates only in certain degrees.

Corollary 6.1.2. *For an arbitrary local system $\mathcal{L}$ over $\mathrm{Conf}_n(\mathbb{C}_m)$,*

$$H_j(\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}, \{\infty\}; \mathcal{L}) = 0$$

for all $j < n$ or $j > 2n$.

Proof. Given a cell $e_{(\lambda,I,J)}$ of dimension $n + l(\lambda) - m$, the length of the partition λ varies from m to $n+m$, and hence $n \leq \dim(e_{(\lambda,I,J)}) \leq 2n$. It follows that there are no cells of dimension $j < n$ or $j > 2n$, so the homology vanishes at these degrees. □

6.2 Algebraic analog of column configurations

In this section, we will develop and study a chain complex that will capture the structure of the vertical columns in our cellular stratification of $\mathrm{Conf}_n(\mathbb{C}_m)$, in particular when specialized to the cellular chain complex of $\mathrm{Conf}_n(\mathbb{C}_m)$ with coefficients in the local system associated with the representation of $B_n(\mathbb{C}_m)$ on $V^{\otimes n} \otimes W^{\otimes m}$ discussed in Sections 3.1 and 3.3.

Definition 6.2.1. Given an associative $\mathbf{k}$-algebra A and a tuple $\mathcal{M} = (M_1, \ldots, M_m)$ of A-bimodules, the *chain complex* $F_*(\mathcal{M}, A)$ is defined at degree $q \geq m$ by

$$F_q(\mathcal{M}, A) = \bigoplus_{\mathcal{I}} A^{\otimes i_1 - 1} \otimes M_1 \otimes A^{\otimes i_2 - i_1 - 1} \otimes M_2 \otimes \cdots \otimes M_m \otimes A^{\otimes q - i_m}$$

where $\mathcal{I} = (i_1, \ldots, i_m)$ for $1 \leq i_1 < \cdots < i_m \leq q$. The face maps d_i for $1 \leq i \leq q-1$ are given by

$$d_i(a_1 \otimes \cdots \otimes a_{i_k - 1} \otimes \mu_k \otimes a_{i_k+1} \otimes \cdots \otimes a_q) =$$
$$\begin{cases} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q & i \neq i_k - 1, i_k \\ a_1 \otimes \cdots \otimes a_{i_k - 1}\mu_k \otimes \cdots \otimes a_q & m = i_k - 1 \notin \mathcal{I} \\ a_1 \otimes \cdots \otimes \mu_k a_{i_k+1} \otimes \cdots \otimes a_q & m = i_k \text{ and } i_k + 1 \notin \mathcal{I} \\ 0 & \{i, i+1\} \subseteq \mathcal{I} \end{cases}$$

and the differential is $d = \sum_{i=1}^{q-1} (-1)^{i-1} d_i$.

The last case of the face maps guarantees that there will always be m A-bimodules present in every summand of $F_*(\mathcal{M}, A)$.

Let I be the augmentation ideal of A, and let $F_*(\mathcal{M}, I)$ denote the chain complex obtained by replacing all copies of A in $F_*(\mathcal{M}, A)$ with I. If the bimodules M_k are graded, for an element $f = a_1 \otimes \cdots \otimes \mu_k \otimes \cdots \otimes a_q$ with a_i homogeneous elements of A of degree $\deg(a_i)$, we may define the degree of f to be $\deg(f) := \sum_k \deg(\mu_k) + \sum_{i \neq i_k} \deg(a_i)$. The differential in $F_*(\mathcal{M}, I)$ strictly preserves the degree of elements, hence we may define the split subcomplex generated by homogeneous elements of $F_*(\mathcal{M}, I)$ of degree precisely n, denoted by $F_{*,n}(\mathcal{M}, I)$.

Observe that this chain complex is a multi-module version of the chain complex $F_*(M, A)$ (Definition 5.1.2). A key observation made in the proof of Theorem 5.1.4 where we compute the homology of $F_*(M, I)$ is that $F_*(M, I)$ is isomorphic to the subcomplex

$$Z_*(M, I) := \bigoplus_{q \geq 1} \bigoplus_{i=1}^{q} M \otimes I^{\otimes q-i} \otimes \mathbf{k} \otimes I^{\otimes i-1}$$

of the Hochschild chain complex $CH_*(A, M)$. We will develop analogs of these chain complexes that involve multiple A-bimodules (i.e., $m \geq 2$) and mimic this strategy to study the homology of $F_*(\mathcal{M}, I)$.

Definition 6.2.2. Given an ordered set $\mathcal{M} = (M_1, \ldots, M_m)$ of A-bimodules, *the multi-module Hochschild chain complex* $CH_*(A, \mathcal{M})$ is defined degree-wise by

$$CH_q(A, \mathcal{M}) = \bigoplus_{\mathcal{I}} M_1 \otimes A^{\otimes i_2 - 2} \otimes M_2 \otimes A^{\otimes i_3 - i_2 - 1} \otimes \cdots \otimes M_m \otimes A^{\otimes q - i_m}$$

for $\mathcal{I} = (i_1, i_2, \ldots, i_m)$ where $1 = i_1 < i_2 < \cdots < i_m \leq q$, with face maps

$$d_i(\mu_1 \otimes a_2 \otimes \cdots \otimes a_{i_k-1} \otimes \mu_{i_k} \otimes a_{i_k+1} \otimes \cdots \otimes a_q) = \begin{cases} \mu_1 a_2 \otimes a_3 \otimes \cdots \otimes a_{i_k-1} \otimes \mu_{i_k} \otimes a_{i_k+1} \otimes \cdots \otimes a_q & i = 1 \\ \mu_1 \otimes a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q & i \neq i_k - 1, i_k \\ \mu_1 \otimes a_2 \otimes \cdots \otimes a_{i_k-1}\mu_k \otimes \cdots \otimes a_q & i = i_k - 1 \notin \mathcal{I} \\ \mu_1 \otimes a_2 \otimes \cdots \otimes \mu_k a_{i_k+1} \otimes \cdots \otimes a_q & i = i_k \text{ and } i_k + 1 \notin \mathcal{I} \\ a_q\mu_1 \otimes a_2 \otimes \cdots \otimes a_{i_k-1} \otimes \mu_{i_k} \otimes a_{i_k+1} \otimes \cdots \otimes a_{q-1} & i = q \notin \mathcal{I} \\ 0 & \{i, i+1\} \subseteq \mathcal{I} \end{cases}$$

and differential $d = \sum_{i=1}^{q} (-1)^{i-1} d_i$.

We may define a chain map $f_* : F_*(\mathcal{M}, I) \to CH_*(A, \mathcal{M})$ that sends $a_1 \otimes \cdots \otimes a_{i_k-1} \otimes \mu_k \otimes a_{i_k+1} \otimes \cdots \otimes a_q$ to

$$(-1)^{q(i_1-1)} \mu_1 \otimes a_{i_1+1} \otimes \cdots \otimes a_{i_k-1} \otimes \mu_k \otimes a_{i_k+1} \otimes \cdots \otimes a_q \otimes 1 \otimes a_1 \otimes \cdots \otimes a_{i_1-1}.$$

Note that for all $i \neq i_k$, $\deg(a_i) > 0$ since each a_i is in the augmentation ideal I of A. It follows that there is precisely one occurrence of the unit in the image of an arbitrary element of $F_*(\mathcal{M}, I)$ under this map, and its position is determined by the indices q and i_1. We deduce that this map is injective. The isomorphic image of $F_*(\mathcal{M}, I)$ via this map forms a subcomplex $Z_*(\mathcal{M}, I)$ of the multi-module Hochschild chain complex which in degree q has the form

$$Z_q(\mathcal{M}, I) = \bigoplus_{\mathcal{I}} M_1 \otimes I^{\otimes i_2 - i_1 - 1} \otimes M_2 \otimes \cdots \otimes M_m \otimes I^{\otimes q - i_m} \otimes \mathbf{k} \otimes I^{\otimes i_1 - 1}.$$

It suffices to compute the homology of this chain complex, which will be our focus for the rest of this subsection.

The case $m = 2$. The key observation is that the chain complex $Z_*(\mathcal{M}, I)$ is isomorphic to the totalization of an m-simplicial object. We give the treatment for the case $m = 2$ as an example.

Construct a double complex $C_{*,*}$ (Figure 6.1) by setting

$$C_{p,q} := Z_q(B_{p-1}(M_1, I, M_2), I) = \bigoplus_{i=1}^{q}(M_1 \otimes I^{\otimes p-1} \otimes M_2) \otimes I^{\otimes q-i} \otimes \mathbf{k} \otimes I^{\otimes i-1}.$$

The vertical face maps are given by

$$d_j^v((\mu_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes \mu_p) \otimes b_1 \otimes \cdots \otimes b_q) =$$
$$\begin{cases} (\mu_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes \mu_p b_1) \otimes b_2 \otimes \cdots \otimes b_q & j = 0 \\ (\mu_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes \mu_p) \otimes b_1 \otimes \cdots \otimes b_j b_{j+1} \otimes \cdots \otimes b_q & 1 \leq j \leq q-1 \\ (b_q\mu_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes \mu_p) \otimes b_1 \otimes \cdots \otimes b_{q-1} & j = q \end{cases}$$

and the vertical differential is $d^v = \sum_{j=0}^{q} d_j^v$. Similarly, the horizontal face maps are given by

$$d_j^h((\mu_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes \mu_p) \otimes b_1 \otimes \cdots \otimes b_q) =$$
$$\begin{cases} (\mu_0 a_1 \otimes a_2 \otimes \cdots \otimes a_{p-1} \otimes \mu_p) \otimes b_1 \otimes \cdots \otimes b_q & j = 0 \\ (\mu_0 \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{p-1} \otimes \mu_p) \otimes b_1 \otimes \cdots \otimes b_q & 1 \leq j \leq p-2 \\ (\mu_0 \otimes a_1 \otimes \cdots \otimes a_{p-2} \otimes a_{p-1}\mu_p) \otimes b_1 \otimes \cdots \otimes b_q & j = p-1 \end{cases}$$

and the horizontal differential is $d^h = \sum_{j=0}^{p-1} d_j^h$.

Observe that the associated total complex $\mathrm{Tot}(C)_*$ of $C_{*,*}$ at degree n has the form

$$\begin{aligned} \mathrm{Tot}(C)_n &= \bigoplus_{p+q=n} C_{p,q} = \bigoplus_{p+q=n} (M_1 \otimes I^{\otimes p-1} \otimes M_2) \otimes I^{\otimes q-i} \otimes \mathbf{k} \otimes I^{\otimes i-1} \\ &= Z_n((M_1, M_2), I). \end{aligned}$$

For the differential at degree n, we have

$$\begin{aligned} d &= d^h + (-1)^p d^v = d_0^h - d_1^h + \cdots + (-1)^{p-1} d_{p-1}^h + (-1)^p (d_0^v - d_1^v + \cdots + (-1)^q d_q^v) \\ &= d_1^Z - d_2^Z + \cdots + (-1)^{p-1} d_p^Z + (-1)^p d_{p+1}^Z + \cdots + (-1)^n d_{n+1}^Z = d^Z \end{aligned}$$

thus indeed $Z_*((M_1, M_2), I)$ is isomorphic to the total complex of the double complex $C_{*,*}$. Notice that the horizontal filtering of $C_{*,*}$ is by the reduced bar complex

$$
\begin{array}{ccccccc}
 & & \cdots & & \cdots & & \\
 & & \downarrow d^v & & \downarrow d^v & & \\
 & & (M_1 \otimes I \otimes M_2) \otimes I^{\otimes 2} \otimes \mathbf{k} & & (M_1 \otimes M_2) \otimes I^{\otimes 2} \otimes \mathbf{k} & & \\
\cdots & \xrightarrow{d^h} & \oplus(M_1 \otimes I \otimes M_2) \otimes I \otimes \mathbf{k} \otimes I & \xrightarrow{d^h} & \oplus(M_1 \otimes M_2) \otimes I \otimes \mathbf{k} \otimes I & \longrightarrow & 0 \\
 & & \oplus(M_1 \otimes I \otimes M_2) \otimes \mathbf{k} \otimes I^{\otimes 2} & & \oplus(M_1 \otimes M_2) \otimes \mathbf{k} \otimes I^{\otimes 2} & & \\
 & & \downarrow d^v & & \downarrow d^v & & \\
\cdots & \xrightarrow{d^h} & (M_1 \otimes I \otimes M_2) \otimes I \otimes \mathbf{k} & \xrightarrow{d^h} & (M_1 \otimes M_2) \otimes I \otimes \mathbf{k} & \longrightarrow & 0 \\
 & & \oplus(M_1 \otimes I \otimes M_2) \otimes \mathbf{k} \otimes I & & \oplus(M_1 \otimes M_2) \otimes \mathbf{k} \otimes I & & \\
 & & \downarrow d^v & & \downarrow d^v & & \\
\cdots & \xrightarrow{d^h} & (M_1 \otimes I \otimes M_2) \otimes \mathbf{k} & \xrightarrow{d^h} & (M_1 \otimes M_2) \otimes \mathbf{k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

Figure 6.1: Double complex $C_{*,*}$.

$B_{*-1}(M_1, I, M_2)$, whereas the vertical filtering is by the complex $Z_*(M, I)$. Recall that the complex $Z_*(M, I)$ is isomorphic to $F_*(M, I)$, whose homology is given by $H_*(Z_*(M, I)) \cong H_*(F_*(M, I)) \cong \left(M \overset{L}{\underset{A^e}{\otimes}} \mathbf{k} \right) [-1]$ (shifted in homological degree by -1). It follows that

Proposition 6.2.3. $H_*(Z_*((M_1, M_2), I)) \cong \left(\left(M_1 \overset{L}{\underset{A}{\otimes}} M_2 \right) \overset{L}{\underset{A^e}{\otimes}} \mathbf{k} \right) [-2]$.

General case. We can generalize the construction above for any $m \geq 3$. Let $\mathcal{M} = (M_1, \ldots, M_m)$. Instead of a bisimplicial object $C_{*,*}$, we construct an m-simplicial object $C_{*,\ldots,*}$ which for each m-tuple $(p_1, \ldots, p_m)$ is given by

$$
\begin{aligned}
C_{p_1,\ldots,p_m} &:= Z_{p_m}(B_{p_{m-1}-1}((\ldots B_{p_2-1}(B_{p_1-1}(M_1, I, M_2), I, M_3) \ldots), I, M_m), I) \\
&= \bigoplus_{i=1}^{p_m} ((\ldots((M_1 \otimes I^{\otimes p_1 - 1} \otimes M_2) \otimes I^{\otimes p_2 - 1} \otimes M_3) \otimes \ldots) \otimes I^{\otimes p_m - 1} \otimes M_m) \\
&\qquad \otimes I^{\otimes p_m - i} \otimes \mathbf{k} \otimes I^{\otimes i-1}.
\end{aligned}
$$

The face maps in the i^{th} direction are defined similarly to the horizontal mappings in the previous case if $1 \leq i \leq m-1$ and to the vertical mappings if $i = m$. From this specification, we have an analogous observation about the filtering of the m-complex constructed from $C_{*,\ldots,*}$: the filtering in the i^{th} direction for $1 \leq i \leq m-1$ is by a reduced bar complex (with degree shift -1), whereas that in the m^{th} direction is by the complex $Z_*(M, I)$ for some M. The totalization of $C_{*,\dots,*}$ is given degree-wise by $\mathrm{Tot}(C)_n = \bigoplus_{\sum p_i = n} C_{p_1,\dots,p_m} = Z_n(\mathcal{M}, I)$, which implies

Theorem 6.2.4. *For $\mathcal{M} = (M_1, \dots, M_m)$,*

$$H_*(F_*(\mathcal{M}, I)) \cong \left\{ \left(\left(\dots \left(\left(M_1 \overset{L}{\underset{A}{\otimes}} M_2 \right) \overset{L}{\underset{A}{\otimes}} M_3 \right) \dots \right) \overset{L}{\underset{A}{\otimes}} M_m \right) \overset{L}{\underset{A^{\mathrm{e}}}{\otimes}} \mathbf{k} \right\} [-m].$$

6.3 Homology with coefficients arising from mixed-braided vector spaces

Let (V, W, τ, φ) be a left-separable mixed-braided vector space. Observe that the twisted mixed-braided vector space $(V_\epsilon, W_\epsilon, \tau, \varphi_\epsilon)$ and its dual $(V_\epsilon^*, W_\epsilon^*, \tau^*, \varphi_\epsilon^*)$ are also left-separable with $\varphi_\epsilon := -\varphi$ and $\varphi_\epsilon^* := -\varphi^*$, respectively.

Let $\mathfrak{A} = \mathfrak{A}(V_\epsilon)$ and $\mathfrak{M} = \mathfrak{M}(V_\epsilon, W_\epsilon)$, and let $\mathfrak{I}$ be the augmentation ideal of $\mathfrak{A}$. The following proposition shows the relationship between the algebraic structures we developed in the previous section and the cellular chain complex of configuration spaces with twisted coefficients.

Proposition 6.3.1. *Let $\mathcal{M} = (\mathfrak{M}_1, \dots, \mathfrak{M}_m)$ where each $\mathfrak{M}_k = \mathfrak{M}$. Then there is an isomorphism of chain complexes*

$$F_{*,n+m}(\mathcal{M}, \mathfrak{I}) \cong D(n, m)_{*+n-m} \otimes (V^{\otimes n} \otimes W^{\otimes m}).$$

Proof. Observe that $F_{q,n+m}(\mathfrak{M}, \mathfrak{I})$ consists of all spaces of the form

$$V_\epsilon^{\otimes \lambda_1} \otimes \dots \otimes V_\epsilon^{\otimes \lambda_{i_k - 1}} \otimes (V_\epsilon^{\otimes j_k} \otimes (W_k)_\epsilon \otimes V_\epsilon^{\otimes \lambda_{i_k} - j_k - 1}) \otimes V_\epsilon^{\otimes \lambda_{i_k + 1}} \otimes \dots \otimes V_\epsilon^{\lambda_q}$$

where $\sum \lambda_i = n + m$. This is an ordered partition λ of $n + m$ with q parts labelled by an element of $V^{\iota_1 - 1} \otimes W_1 \otimes V^{\otimes \iota_2 - \iota_1 - 1} \otimes W_2 \otimes \dots \otimes W_m \otimes V^{\otimes n+m-\iota_m}$, where $\iota_k = j_k + 1 + \sum_{i=1}^{i_k - 1} \lambda_i$ is the position of the factor W_k in the tensor product; furthermore, there is at most one number ι_k in every part of the partition λ. These data about the partition λ are in one-to-one correspondence with an element of $D(n, m)_{q+n-m}$, whereas the labelling element is identified via the isomorphism $\varphi_{\mathcal{I},n+m} : V^{\otimes n} \otimes W^{\otimes m} \to V^{\iota_1 - 1} \otimes$

$W_1 \otimes V^{\otimes \iota_2 - \iota_1 - 1} \otimes W_2 \otimes \cdots \otimes W_m \otimes V^{\otimes n+m-\iota_m}$ where $\mathcal{I} = (\iota_1, \dots, \iota_m)$. Hence there is an isomorphism of $\mathbf{k}$-modules between $F_{q,n+m}(\mathcal{M}, \mathfrak{I})$ and $D(n,m)_{q+n-m} \otimes (V^{\otimes n} \otimes W^{\otimes m})$.

There are two main pieces of data in the boundary of an element $\lambda \otimes \ell$ in the chain complex $D(n,m)_* \otimes (V^{\otimes n} \otimes W^{\otimes m})$: the coarsening ρ^i of λ and the signed sum over all $(\lambda_i, \lambda_{i+1})$-shuffles of the actions of their lifts on ℓ. Both are encapsulated in the differential of $F_{*,n+m}(\mathcal{M}, \mathfrak{I})$: the coarsening ρ^i is encoded in the choice of two multiplied elements, and the sum of the braid actions is contained in the quantum shuffle product of $\mathfrak{A}$ or the multiplication of $\mathfrak{M}_k$ by $\mathfrak{A}$. Observe that in the differential of $F_{*,n+m}(\mathcal{M}, \mathfrak{I})$, the braids act only on a tensor subfactor $V^{\otimes j_k} \otimes W_k \otimes V^{\otimes \lambda_{i_k} - j_k - 1}$ of $V^{\iota_1 - 1} \otimes W_1 \otimes V^{\otimes \iota_2 - \iota_1 - 1} \otimes W_2 \otimes \cdots \otimes W_m \otimes V^{\otimes n+m-\iota_m}$, whereas the corresponding braids act on the isomorphic image $V^{\otimes n} \otimes W^{\otimes m}$ of this full factor in the differential of $D(n,m)_{*+n-m} \otimes (V^{\otimes n} \otimes W^{\otimes m})$. These braid actions match precisely due to the commutativity of Diagram $\mathscr{E}$ (Figure 3.3), which is equivalent to the left-separability of the mixed-braided vector space $(V_\epsilon, W_\epsilon, \tau, \varphi_\epsilon)$ by Proposition 3.3.5. The signs coming from ϵ encode the boundary orientations on cells in the Fox–Neuwirth model. Via these identifications, the differentials of $F_{*,n+m}(\mathcal{M}, \mathfrak{I})$ and $D(n,m)_{*+n-m} \otimes (V^{\otimes n} \otimes W^{\otimes m})$ are precisely the same formula, which shows their isomorphism as chain complexes. □

Combining this with Theorem 6.1.1 and Theorem 6.2.4, we get a formula for the relative homology of the 1-point compactification of $\mathrm{Conf}_n(\mathbb{C}_m)$:

Corollary 6.3.2. *Let $\mathcal{L}$ be the local system over $\mathrm{Conf}_n(\mathbb{C}_m)$ associated with the $B_n(\mathbb{C}_m)$-representation on $V^{\otimes n} \otimes W^{\otimes m}$. Then there is an isomorphism*

$$\begin{aligned} &H_*(\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}, \{\infty\}; \mathcal{L}) \\ &\quad \cong \left\{ \left(\left(\cdots \left(\left(\mathfrak{M}_1 \overset{L}{\underset{\mathfrak{A}}{\otimes}} \mathfrak{M}_2 \right) \overset{L}{\underset{\mathfrak{A}}{\otimes}} \mathfrak{M}_3 \right) \cdots \right) \overset{L}{\underset{\mathfrak{A}}{\otimes}} \mathfrak{M}_m \right) \overset{L}{\underset{\mathfrak{A}^e}{\otimes}} \mathbf{k} \right\} [-n][n+m] \end{aligned}$$

where the first bracket denotes the shift in homological degree and the second the internal degree part.

In Section 5.3, we showed that $\mathfrak{M}(V,W)$ is a free left (or right) $\mathfrak{A}(V)$-module in a special case, namely when $V = W = \mathbf{k}$ with the braidings given by multiplication by units. It turns out that this is true in general:

Proposition 6.3.3. *Given a left-separable mixed-braided vector space* (V, W)*,* $\mathfrak{M}(V, W)$ *is a free left* $\mathfrak{A}(V)$*-module.*

Proof. Let $\mathfrak{A} = \mathfrak{A}(V)$ and $\mathfrak{M} = \mathfrak{M}(V, W)$. Define a filtration of $\mathfrak{M}$ by

$$F_p\mathfrak{M} = \bigoplus_{b \geq p,\, a \geq 0} V^{\otimes a} \otimes W \otimes V^{\otimes b}.$$

for every $p \geq 0$. Then the p-graded piece is given by

$$\mathfrak{M}^{\mathrm{gr}}_{*,p} = F_p\mathfrak{M}/F_{p+1}\mathfrak{M} = \bigoplus_{a \geq 0} V^{\otimes a} \otimes W \otimes V^{\otimes p}.$$

Observe that whenever we left-multiply an element x of $\mathfrak{A}$ with an element

$$[v_1|\dots|v_a|w|v_{a+2}|\dots|v_{a+p+1}] \in V^{\otimes a} \otimes W \otimes V^{\otimes p} \subset \mathfrak{M}^{\mathrm{gr}}_{*,p},$$

only the elements in the product resulting from shuffles that preserve the number p of copies of V on the right of W survive in $\mathfrak{M}^{\mathrm{gr}}_{*,p}$. Thus, the multiplication is precisely the quantum shuffle product of x with $[v_1|\dots|v_a]$ in the first a tensor copies of V, tensoring with the identity on $W \otimes V^{\otimes p}$. Therefore, each graded piece $\mathfrak{M}^{\mathrm{gr}}_{*,p}$ is a free left $\mathfrak{A}$-module, and so is the associated graded $\mathfrak{M}^{\mathrm{gr}}_{*,*} = \bigoplus_{p \geq 0} \mathfrak{M}^{\mathrm{gr}}_{*,p}$. In particular, $\mathfrak{M}^{\mathrm{gr}}_{*,p} \cong \mathfrak{A}_* \otimes (W \otimes V^{\otimes p})$, hence

$$\mathfrak{M}^{\mathrm{gr}}_{*,*} = \bigoplus_{p \geq 0} \mathfrak{M}^{\mathrm{gr}}_{*,p} \cong \mathfrak{A}_* \otimes \left(\bigoplus_{p \geq 0} W \otimes V^{\otimes p} \right).$$

By convention, we define the bigrading of an element of $\mathfrak{A}_q \otimes (W \otimes V^{\otimes p})$ to be $(q+p+1, p)$.

Define a map $\alpha : \mathfrak{M}^{\mathrm{gr}}_{*,*} \to \mathfrak{M}_*$ by sending

$$[a_1|\dots|a_q] \otimes [w|v_1|\dots|v_p] \mapsto [a_1|\dots|a_q] \star [w|v_1|\dots|v_p]$$

where $\star$ is the multiplication on $\mathfrak{M}$. We claim that α is an isomorphism of graded left $\mathfrak{A}$-modules, when considering $\mathfrak{M}^{\mathrm{gr}}_{*,*}$ as an $\mathfrak{A}$-module with respect to the first grading. The fact that α is a module homomorphism comes directly from the fact that the multiplication on $\mathfrak{M}$ is associative. Since the grading is preserved, it is a graded $\mathfrak{A}$-module homomorphism. It suffices to prove that α is an isomorphism on each degree.

To show injectivity, suppose we have $\alpha(m_1) = \alpha(m_2)$. Then the pieces of $\alpha(m_1)$ and $\alpha(m_2)$ with the smallest number p of copies of V on the right of W must be exactly the same, say $\sum[a_1|\dots|a_q|w|v_1|\dots|v_p]$. This can only come from $m = \sum[a_1|\dots|a_q] \otimes [w|v_1|\dots|v_p] \in \mathfrak{M}^{\mathrm{gr}}_{q+p+1,p}$, so m_1 and m_2 must agree on this part. Let $m_1' = m_1 - m$ and $m_2' = m_2 - m$. Now observe that the smallest number of copies of V on the right of W in $\alpha(m_1') = \alpha(m_1) - \alpha(m)$ and $\alpha(m_2') = \alpha(m_2) - \alpha(m)$ is now at least $p+1$; furthermore, the pieces of this second grading must agree. Repeat the process above till it terminates, and observe that $m_1 = m_2$.

To show surjectivity, let $m \in \mathfrak{M}$. Again let $\sum[a_1|\dots|a_q|w|v_1|\dots|v_p]$ be the piece of m with the minimal p, then this could come from $m' = \sum[a_1|\dots|a_q] \otimes [w|v_1|\dots|v_p] \in \mathfrak{M}^{\mathrm{gr}}_{q+p+1,p}$. Take $m_1 = m - \alpha(m')$, then the smallest number of copies of V on the right of W in m_1 must be at least $p+1$. Repeat this process till it terminates, and it is clear that we then obtain m as an image under α. $\square$

We are ready to prove the main result of this section.

Theorem 6.3.4. *For $\mathfrak{A} = \mathfrak{A}(V_\epsilon^*)$ and $\mathfrak{M}_i = \mathfrak{M}(V_\epsilon^*, W_\epsilon^*)$ for every $1 \le i \le m$, there is an isomorphism*

$$H_*(B_n(\mathbb{C}_m); V^{\otimes n} \otimes W^{\otimes m}) \cong \mathrm{Ext}^{n-*,n+m}_{\mathfrak{A}^{\mathrm{e}}} \left(\mathfrak{M}_1 \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_2 \underset{\mathfrak{A}}{\otimes} \dots \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_m, \mathbf{k} \right).$$

Proof. Let $\mathcal{L}$ be the local system associated to the representation of $B_n(\mathbb{C}_m)$ on $V^{\otimes n} \otimes W^{\otimes m}$, then the dual local system $\mathcal{L}^*$ corresponds to the representation on $(V^*)^{\otimes n} \otimes (W^*)^{\otimes m}$ of $B_n(\mathbb{C}_m)$. Since each $\mathfrak{M}_i$ is free as a left $\mathfrak{A}$-module by Proposition 6.3.3, Corollary 6.3.2 applied to $\mathcal{L}^*$ is simplified to

$$\begin{aligned} H_*(\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}, \{\infty\}; \mathcal{L}^*) &\cong \left\{ \left(\mathfrak{M}_1 \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_2 \underset{\mathfrak{A}}{\otimes} \dots \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_m \right) \overset{L}{\underset{\mathfrak{A}^{\mathrm{e}}}{\otimes}} \mathbf{k} \right\} [-n][n+m] \\ &= \mathrm{Tor}^{\mathfrak{A}^{\mathrm{e}}}_{*-n,n+m} \left(\mathfrak{M}_1 \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_2 \underset{\mathfrak{A}}{\otimes} \dots \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_m, \mathbf{k} \right). \end{aligned}$$

By applying the universal coefficient theorem and Poincaré duality to the dual over $\mathbf{k}$ of the left hand side, we have

$$\begin{aligned} H_*(\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}, \{\infty\}; \mathcal{L}^*)^* &\cong H^*(\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}, \{\infty\}; \mathcal{L}) \\ &\cong H_c^*(\mathrm{Conf}_n(\mathbb{C}_m); \mathcal{L}) \\ &\cong H_{2n-*}(\mathrm{Conf}_n(\mathbb{C}_m); \mathcal{L}), \end{aligned}$$

so

$$\begin{aligned} H_*(\mathrm{Conf}_n(\mathbb{C}_m); \mathcal{L}) &\cong H_{2n-*}(\mathrm{Conf}_n(\mathbb{C}_m) \cup \{\infty\}, \{\infty\}; \mathcal{L}^*)^* \\ &\cong \mathrm{Tor}^{\mathfrak{A}^e}_{(2n-*)-n,n+m}\left(\mathfrak{M}_1 \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_2 \underset{\mathfrak{A}}{\otimes} \dots \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_m, \mathbf{k}\right)^* \\ &\cong \mathrm{Ext}^{n-*,n+m}_{\mathfrak{A}^e}\left(\mathfrak{M}_1 \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_2 \underset{\mathfrak{A}}{\otimes} \dots \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_m, \mathbf{k}\right). \end{aligned}$$

□

6.4 Homology with one-dimensional coefficients

We revisit Example 3.1.8 when $\mathbf{k}$ is a field of characteristic 0. Let the mixed-braided vector space (V, W, τ) be composed of one-dimensional $\mathbf{k}$-vector spaces $V = \mathbf{k}$ and $W = \mathbf{k}\{w\}$, with braidings σ_V, σ_W, and τ on $\mathbf{k}$ given by multiplications by q, u, and p respectively, for some $p, q, u \in \mathbf{k}^\times$. It is separable with the separated braiding $\varphi := s\varphi'$ for some $s \in \mathbf{k}^\times$, where φ' simply permutes two tensor factors. The action of the mixed braid group $B_{n,m}$ on $T_{q,u,p} = V^{\otimes n} \otimes W^{\otimes m} \cong \mathbf{k}$ is given by $\sigma_{i<n} \mapsto q$, $\sigma_{i>n} \mapsto u$, and $\tau_n \mapsto p$. This restricts to an action of $B_n(\mathbb{C}_m)$ on a representation $L_{q,p} \cong \mathbf{k}$ given by $\sigma_i \mapsto q$ for all $1 \leq i \leq n-1$ and $\theta_{n,n+k} = \sigma_{n+k-1} \dots \sigma_{n+1} \tau_n \sigma_{n+1}^{-1} \dots \sigma_{n+k-1}^{-1} \mapsto u^{k-1} p (u^{-1})^{k-1} = p$ for $1 \leq k \leq m$. Let $\mathcal{T}_{q,u,p}$ and $\mathcal{L}_{q,p}$ denote the local systems associated to the representations $T_{q,u,p}$ and $L_{q,p}$, respectively.

In Section 5.3, we illustrated the computation of the homology of $B_n(\mathbb{C}^\times) \cong B_{n,1}$ with similar coefficients as a special case. We will now expand these findings to compute $H_*(B_n(\mathbb{C}_m); L_{q,p})$ for some special values of p and q. Let $\Gamma := \Gamma_{-q^{-1}}[x]$. Recall that each $\mathfrak{M}_i \cong \mathfrak{M}(V_\epsilon^*, W_\epsilon^*)$ is free as a left (or right) Γ-module with respect to a basis

$\mathcal{Y}_i = \{y_1, y_2, \dots\}$. Thus, Theorem 6.3.4 reduces to

$$\begin{aligned} H_*(B_n(\mathbb{C}_m); L_{q,p}) &\cong \operatorname{Ext}^{n-*,n+m}_{\Gamma^e}\left(\mathfrak{M}_1 \underset{\Gamma}{\otimes} \mathfrak{M}_2 \underset{\Gamma}{\otimes} \dots \underset{\Gamma}{\otimes} \mathfrak{M}_m, \mathbf{k}\right) \\ &\cong \operatorname{Ext}^{n-*,n+m}_{\Gamma}\left(\mathbf{k}\mathcal{Y}_1 \underset{\Gamma}{\otimes} \mathbf{k}\mathcal{Y}_2 \underset{\Gamma}{\otimes} \dots \underset{\Gamma}{\otimes} \mathbf{k}\mathcal{Y}_m, \mathbf{k}\right) \\ &\cong \left\{(\mathbf{k}\mathcal{Y}_1 \otimes \mathbf{k}\mathcal{Y}_2 \otimes \dots \otimes \mathbf{k}\mathcal{Y}_m) \overset{L}{\underset{\Gamma}{\otimes}} \mathbf{k}\right\}^*_{n-*,n+m}. \end{aligned}$$

Recall that in Section 5.3, when we fully computed $H_*(B_{n,1}; L_{q,p})$ for all $p, q \in \mathbf{k}^\times$, the output was divided into cases depending on whether $-q$ is a root of unity and whether p is a power of $-q$. For the purpose of the arithmetic application in the next chapter, we assume $-q$ is a primitive r^{th} root of unity and perform computations when $p = -q$ and when p is not a power of $-q$.

Case 1: $p = -q$. By Proposition 5.3.2, $\Gamma = \mathbf{k}[x_1]/x_1^r \otimes \Gamma[x_r]$, so it suffices to study the right multiplication of the generators y_n by x_1 and x_r. Let $\Lambda_r := \mathbf{k}[x_1]/x_1^r$ denote the degree-r truncated polynomial algebra in variable x_1. If $\mathbf{k}$ has characteristic 0, there is an isomorphism $\Gamma[x_r] \cong \mathbf{k}[x_r]$. Consider the multiplication by x_r. In this case, we have

$$y_n x_r = \frac{1}{(-s)^r}\left[\left\lceil\frac{n}{r}\right\rceil \prod_{k=0}^{r-1}\left(p - \frac{1}{(-q)^{n-1+k}}\right)\right] y_{n+r}$$

in $\mathbf{k}\mathcal{Y}$. Since the power of $-q$ in the product cycles through r consecutive values, we see that $y_n x_r = 0$ if and only if p is a power of $-q$. Here, we have $p = -q$, so $\mathbf{k}\mathcal{Y}$ is trivial as a $\mathbf{k}[x_r]$-module. On the other hand,

$$y_n x_1 = \frac{1 - (-q)^n}{-s(1+q)}\left(p - \frac{1}{(-q)^{n-1}}\right) y_{n+1} = \frac{1 - (-q)^n}{-s(1+q)}\left(-q - \frac{1}{(-q)^{n-1}}\right) y_{n+1}$$

vanishes iff r divides n. It follows that $\mathbf{k}\mathcal{Y}$ is freely generated by $\{y_1, y_{r+1}, \dots\}$ as a

Λ_r-module, so we have

$$\begin{aligned}
\mathbf{k}\mathcal{Y}_1 \otimes \cdots \otimes \mathbf{k}\mathcal{Y}_m &\cong \mathbf{k}\{y_1, y_2, \dots\} \otimes \cdots \otimes \mathbf{k}\{y_1, y_2, \dots\} \otimes \mathbf{k}\{y_1, y_{r+1}, \dots\}[x_1]/x_1^r \\
&\cong \bigoplus_{a_1,\dots,a_{m-1}\geq 1, a_m \geq 0} \Lambda_r\{y_{a_1} \otimes \cdots \otimes y_{a_{m-1}} \otimes y_{ra_m+1}\} \\
&\cong \bigoplus_{a_1,\dots,a_{m-1}\geq 1, a_m \geq 0} \Sigma^{a_1+\cdots+a_{m-1}+ra_m+1}\Lambda_r
\end{aligned}$$

as Λ_r-modules. We then have

$$\begin{aligned}
(\mathbf{k}\mathcal{Y}_1 \otimes \mathbf{k}\mathcal{Y}_2 \otimes \cdots \otimes \mathbf{k}\mathcal{Y}_m) \overset{L}{\underset{\Gamma}{\otimes}} \mathbf{k} &\cong \left(\bigoplus_{a_1,\dots,a_{m-1}\geq 1, a_m \geq 0} \Sigma^{a_1+\cdots+a_{m-1}+ra_m+1}\Lambda_r \right) \overset{L}{\underset{\Lambda_r \otimes \mathbf{k}[x_r]}{\otimes}} \mathbf{k} \\
&\cong \bigoplus_{a_1,\dots,a_{m-1}\geq 1, a_m \geq 0} \Sigma^{a_1+\cdots+a_{m-1}+ra_m+1}\mathbf{k} \overset{L}{\underset{\mathbf{k}[x_r]}{\otimes}} \mathbf{k} \\
&\cong \bigoplus_{a_1,\dots,a_{m-1}\geq 1, a_m \geq 0} \Sigma^{a_1+\cdots+a_{m-1}+ra_m+1}\Lambda[z_r]
\end{aligned}$$

for some $z_r \in \mathrm{Tor}_{1,r}$, i.e.,

$$\left\{ (\mathbf{k}\mathcal{Y}_1 \otimes \mathbf{k}\mathcal{Y}_2 \otimes \cdots \otimes \mathbf{k}\mathcal{Y}_m) \overset{L}{\underset{\Gamma}{\otimes}} \mathbf{k} \right\}_{j,n} = \begin{cases} \mathbf{k}^{\oplus P(n+r-1)} & \text{for } j = 0 \\ \mathbf{k}^{\oplus P(n-1)} & \text{for } j = 1 \\ 0 & \text{else,} \end{cases}$$

where $P(n)$ is the number of compositions $(a_1, \dots, a_{m-1}, ra_m)$ of n $(a_i \geq 1)$, which can be computed explicitly as

$$P(n) = \sum_{a=1}^{\lfloor \frac{n}{r} \rfloor} P_{m-1}(n - ra) = \sum_{a=1}^{\lfloor \frac{n}{r} \rfloor} \binom{n - ra - 1}{m - 2}$$

where the partition function $P_k(n) = \binom{n-1}{k-1}$ counts the number of k-part compositions of n. It follows that:

Theorem 6.4.1.

$$H_j(B_n(\mathbb{C}_m); L_{q,p}) = \begin{cases} \mathbf{k}^{\oplus P(n+m+r-1)} & \text{for } j = n \\ \mathbf{k}^{\oplus P(n+m-1)} & \text{for } j = n-1 \\ 0 & \text{else.} \end{cases}$$

In particular, $H_j(B_n(\mathbb{C}_m); L_{q,p}) = 0$ for all $j \leq n-2$.

Recall that there is a fiber sequence

$$\mathrm{Conf}_n(\mathbb{C}_m) \to \mathrm{Conf}_{n,m}(\mathbb{C}) \to \mathrm{Conf}_m(\mathbb{C})$$

of Fadell–Neuwirth [43], which induces a Lyndon–Hochschild–Serre spectral sequence that computes

$$H_i(\mathrm{Conf}_m(\mathbb{C}); H_j(\mathrm{Conf}_n(\mathbb{C}_m); \mathcal{L}_{q,p})) \Rightarrow H_{i+j}(\mathrm{Conf}_{n,m}(\mathbb{C}); \mathcal{T}_{q,u,p}).$$

Since $H_j(\mathrm{Conf}_n(\mathbb{C}_m); \mathcal{L}_{q,p}) = 0$ for all $j \leq n-2$, it follows that $H_j(\mathrm{Conf}_{n,m}(\mathbb{C}); \mathcal{T}_{q,u,p}) = 0$ for all $j \leq n-2$. Now assume $u = q$ (e.g., $q = u = 1$, $p = -1$). Repeat this analysis with the fiber sequence

$$\mathrm{Conf}_m(\mathbb{C}_n) \to \mathrm{Conf}_{n,m}(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C}),$$

we get the spectral sequence

$$H_i(\mathrm{Conf}_n(\mathbb{C}); H_j(\mathrm{Conf}_m(\mathbb{C}_n); \mathcal{L}_{u,p})) \Rightarrow H_{i+j}(\mathrm{Conf}_{n,m}(\mathbb{C}); \mathcal{T}_{q,u,p}).$$

Since $H_j(\mathrm{Conf}_m(\mathbb{C}_n); \mathcal{L}_{u,p}) = 0$ for all $j \leq m-2$, it follows that $H_j(\mathrm{Conf}_{n,m}(\mathbb{C}); \mathcal{T}_{q,u,p}) = 0$ for all $j \leq m-2$. Thus,

Corollary 6.4.2. $H_j(\mathrm{Conf}_{n,m}(\mathbb{C}); \mathcal{T}_{q,u,p}) = 0$ *for all* $j \leq \max(n,m) - 2$.

Case 2: p is not a power of $-q$. In this case, we have shown that $\mathbf{k}\mathcal{Y} \cong \Sigma^1\Gamma$, thus

$$\begin{aligned}(\mathbf{k}\mathcal{Y}_1 \otimes \mathbf{k}\mathcal{Y}_2 \otimes \cdots \otimes \mathbf{k}\mathcal{Y}_m) \overset{L}{\underset{\Gamma}{\otimes}} \mathbf{k} &\cong \left(\bigoplus_{a_1,\ldots,a_{m-1}\geq 1} \Sigma^{a_1+\cdots+a_{m-1}+1}\Gamma \right) \overset{L}{\underset{\Gamma}{\otimes}} \mathbf{k} \\ &\cong \bigoplus_{a_1,\ldots,a_{m-1}\geq 1} \Sigma^{a_1+\cdots+a_{m-1}+1}\mathbf{k}.\end{aligned}$$

In particular, we have

$$\left\{ (\mathbf{k}\mathcal{Y}_1 \otimes \mathbf{k}\mathcal{Y}_2 \otimes \cdots \otimes \mathbf{k}\mathcal{Y}_m) \overset{L}{\underset{\Gamma}{\otimes}} \mathbf{k} \right\}_{j,n} = \begin{cases} \mathbf{k}^{\oplus P_{m-1}(n-1)} & \text{for } j = 0 \\ 0 & \text{else,} \end{cases}$$

therefore:

Theorem 6.4.3. $H_j(\mathrm{Conf}_n(\mathbb{C}_m); \mathcal{L}_{q,p}) = \begin{cases} \mathbf{k}^{\oplus \binom{n+m-2}{m-2}} & \textit{for } j = n \\ 0 & \textit{else.} \end{cases}$

With this, we obtain a slightly better vanishing range for the homology of $\mathrm{Conf}_{n,m}(\mathbb{C})$ in this case, again when $u = q$.

Corollary 6.4.4. $H_j(\mathrm{Conf}_{n,m}(\mathbb{C}); \mathcal{T}_{q,u,p}) = 0$ *for all* $j \leq \max(n, m) - 1$.

Chapter 7

Character sums of the resultant

In this chapter, we will relate the resultant of pairs of monic squarefree coprime polynomials to local systems on bicolor configuration spaces, and prove an upper bound on the character sums

$$F_\chi(n,m,q) = \sum_{(f,g)\in \mathrm{Conf}_{n,m}(\mathbb{F}_q)} \chi(\mathcal{R}(f,g))$$

for any nontrivial character $\chi : \mathbb{F}_q \to \mathbb{C}$ by leveraging our new topological computations. The content of this chapter is extracted from Section 5 of [54].

Recall that on $\mathrm{Conf}_{n,m}$, the resultant can be interpreted as a map $\mathcal{R} : \mathrm{Conf}_{n,m} \to \mathbb{A}^1 \setminus \{0\}$. In the analytic topology over $\mathbb{C}$, the map $\mathcal{R} : \mathrm{Conf}_{n,m}(\mathbb{C}) \to \mathbb{C}^\times$ is given by the product formula

$$\mathcal{R}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \prod_{i,j}(x_i - y_j)$$

where $x_1, \ldots, x_n, y_1, \ldots, y_m$ correspond to the roots of two squarefree coprime polynomials. Recall that the fundamental group $B_{n,m}$ of $\mathrm{Conf}_{n,m}(\mathbb{C})$ has three types of generators: blue-blue crossings, red-red crossings, and blue-red wraparounds. Consider the following representative of a blue-blue crossing in $\pi_1(\mathrm{Conf}_{n,m}(\mathbb{C}))$: let x be a configuration with two blue points $x_i = -1, x_{i+1} = 1$, and all other points far away, and let α be the loop in $\mathrm{Conf}_{n,m}(\mathbb{C})$ based at x given by an $e^{\pi i}$-rotation on x_i, x_{i+1}, and identity on the rest. Observe that each term in the product formula of the resultant has a negligible deviation as the input varies on α, so in particular, the loop $\mathcal{R}\alpha$ is contractible. A similar observation can be made for a red-red crossing. Meanwhile, for a representative of a blue-red wraparound, let x now be a configuration with a blue point $x_i = 1$, a red point $y_j = 0$, and all other points far away, and let α be the loop in $\mathrm{Conf}_{n,m}(\mathbb{C})$ given by an $e^{2\pi i}$-rotation on x_i and identity on the rest. In this case, almost every term in the product formula of the resultant again does not change much as the input varies on α, except for the term $x_i - y_j = e^{2\pi t i}$ for $t \in [0,1]$. This implies that the loop $\mathcal{R}\alpha$ travels around the puncture of $\mathbb{C}^\times$ precisely once. From these analyses, the homomorphism of fundamental groups $\mathcal{R}_* : B_{n,m} \to \mathbb{Z}$ induced by the resultant map sends the generators σ_i of $B_{n,m}$ to 0 ($1 \le i \le n+m-1, i \ne n$) and τ_n to 1, i.e., counting the winding number of blue strands wrapping around red strands. This map agrees with the induced homomorphism on étale fundamental groups $\mathcal{R}_* : \widehat{B_{n,m}} \to \widehat{\mathbb{Z}}$, i.e., the diagram

$$\begin{array}{ccc} \widehat{B_{n,m}} & \xrightarrow{\mathcal{R}_*} & \widehat{\mathbb{Z}} \\ \uparrow & & \uparrow \\ B_{n,m} & \xrightarrow{\mathcal{R}_*} & \mathbb{Z} \end{array}$$

commutes.

Recall that the *Kummer sheaf* $\mathcal{L}_\chi$ associated to a character χ of $\mathbb{F}_q$ is the rank-1 local system on $\mathbb{A}^1_{\mathbb{F}_q} \setminus \{0\}$ defined by means of the Lang isogeny, with a special property that $\mathrm{tr}(\mathrm{Frob}_q | (\mathcal{L}_\chi)_x) = \chi(x)$ for all $x \in \mathbb{F}_q^\times$ (see, e.g., [76]). By pulling back this sheaf by the resultant map $\mathcal{R} : \mathrm{Conf}_{n,m} \to \mathbb{A}^1 \setminus \{0\}$, we obtain a rank-1 local system $\mathcal{R}^*\mathcal{L}_\chi$ on $\mathrm{Conf}_{n,m}$ such that $\mathrm{tr}(\mathrm{Frob}_q | (\mathcal{R}^*\mathcal{L}_\chi)_{(f,g)}) = \chi(\mathcal{R}(f,g))$ for all $(f,g) \in \mathrm{Conf}_{n,m}$. It follows that the character sum $F_\chi(n,m,q)$ can be written as

$$F_\chi(n,m,q) = \sum_{(f,g) \in \mathrm{Conf}_{n,m}(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}_q | (\mathcal{R}^*\mathcal{L}_\chi)_{(f,g)})$$

and thus, by the Grothendieck–Lefschetz trace formula with twisted coefficients, can be approached by studying the cohomology groups of $\mathrm{Conf}_{n,m}$ with coefficients in $\mathcal{R}^*\mathcal{L}_\chi$.

Theorem 7.0.1. *Let χ be a nontrivial character of $\mathbb{F}_q$. Then*

$$|F_\chi(n,m,q)| \le 2^{2n+2m-1} \frac{q^{n+m+1-\max(n,m)/2} - 1}{\sqrt{q} - 1}$$

if χ is quadratic, and

$$|F_\chi(n,m,q)| \leq 2^{2n+2m-1}\frac{q^{n+m+(1-\max(n,m))/2}-1}{\sqrt{q}-1}$$

otherwise.

Proof. Observe that the local system $\mathcal{R}^*\mathcal{L}_\chi$ on $\mathrm{Conf}_{n,m}$ is associated with the representation of its étale fundamental group $\phi : \widehat{B_{n,m}} \xrightarrow{\mathcal{R}_*} \widehat{\mathbb{Z}} \to \mathbb{Z}/(q-1)\mathbb{Z} \cong \mathbb{F}_q^\times \xrightarrow{\chi} \mathbb{C}^\times$. The corresponding local system on $\mathrm{Conf}_{n,m}(\mathbb{C})$ is therefore associated with the homomorphism $B_{n,m} \to \mathbb{C}^\times$ that sends the generators σ_i to 1 for $1 \leq i \leq n+m-1, i \neq n$, and τ_n to a primitive d^{th} root of unity ξ, where $d = |\chi|$. This is precisely the representation of $B_{n,m}$ on $T_{1,1,\xi}$ discussed in Section 6.4, i.e., $\mathcal{R}^*\mathcal{L}_\chi = \mathcal{T}_{1,1,\xi}$. By Poincaré duality and Artin's comparison theorem, we have

$$\begin{aligned}\dim H^{2n+2m-i}_{c,ét}(\mathrm{Conf}_{n,m};\mathcal{R}^*\mathcal{L}_\chi) &= \dim H^i_{ét}(\mathrm{Conf}_{n,m};\mathcal{R}^*\mathcal{L}_\chi)\\ &= \dim H^i(\mathrm{Conf}_{n,m}(\mathbb{C});\mathcal{R}^*\mathcal{L}_\chi).\end{aligned}$$

From our topological computations in Section 6.4, it follows that

$$\dim H^i(\mathrm{Conf}_{n,m}(\mathbb{C});\mathcal{R}^*\mathcal{L}_\chi) = 0$$

for all $i \leq \max(n,m)-2$ when $d=2$ by Corollary 6.4.2, and for $i \leq \max(n,m)-1$ when $d>2$ by Corollary 6.4.4. For other degrees, we give a bound on the dimension of the cohomology group by directly bounding the number of $(2n+2m-i)$-dimensional cells of $\mathrm{Conf}_{n,m}$; that is,

$$\dim H^{2n+2m-i}_{c,ét}(\mathrm{Conf}_{n,m};\mathcal{R}^*\mathcal{L}_\chi) \leq \binom{n+m}{n}\binom{n+m-1}{i} \leq 2^{2n+2m-1}$$

where the first binomial coefficient is the number of coloring choices of n blue and m red points, and the second is the number of compositions of $n+m$ of length $n+m-i$.

Let $\rho : X_\chi \to \mathrm{Conf}_{n,m}$ be the d-fold cover of $\mathrm{Conf}_{n,m}$ associated with $\phi : \widehat{B_{n,m}} \to \mathrm{im}(\phi) = \mu_d$, where $d = |\chi|$. Since X_χ is a finite-sheeted cover of $\mathrm{Conf}_{n,m}$, there is an injection

$$H^*(\mathrm{Conf}_{n,m};\mathcal{R}^*\mathcal{L}_\chi) \hookrightarrow H^*(X_\chi;\mathbb{Q}_\ell).$$

The eigenvalues of the geometric Frobenius on the right hand side are bounded by Deligne's bounds, which thus also apply to the left hand side. Letting μ be the smallest number such that $\dim H^i(\mathrm{Conf}_{n,m}(\mathbb{C}); \mathcal{R}^*\mathcal{L}_\chi) \neq 0$, by the Grothendieck–Lefschetz trace formula with twisted coefficients we then have

$$\begin{aligned}
|F_\chi(n,m,q)| &= \left| \sum_{(f,g)\in \mathrm{Conf}(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}_q | (\mathcal{R}^*\mathcal{L}_\chi)_{(f,g)}) \right| \\
&= \left| \sum_{i=0}^{2n+2m} (-1)^i \mathrm{tr}(\mathrm{Frob}_q | H^{2n+2m-i}_{c,\acute{e}t}(\mathrm{Conf}_{n,m}; \mathcal{R}^*\mathcal{L}_\chi)) \right| \\
&\leq \sum_{i=0}^{2n+2m} q^{n+m-i/2} \dim H^{2n+2m-i}_{c,\acute{e}t}(\mathrm{Conf}_{n,m}; \mathcal{R}^*\mathcal{L}_\chi) \\
&= \sum_{i=\mu}^{2n+2m} q^{n+m-i/2} \dim H^{2n+2m-i}_{c,\acute{e}t}(\mathrm{Conf}_{n,m}; \mathcal{R}^*\mathcal{L}_\chi) \\
&\leq 2^{2n+2m-1} \sum_{i=\mu}^{2n+2m} q^{n+m-i/2} \\
&= 2^{2n+2m-1} \frac{(\sqrt{q})^{2n+2m-\mu+1} - 1}{\sqrt{q} - 1} \\
&= 2^{2n+2m-1} \frac{q^{n+m+(1-\mu)/2} - 1}{\sqrt{q} - 1}.
\end{aligned}$$

The statement of the theorem then follows immediately from the homological vanishing range asserted by Corollaries 6.4.2 and 6.4.4. □

Notice that the factor $2^{2n+2m-1}$ resulted from our very crude bound on the dimensions of $H^*_{c,\acute{e}t}(\mathrm{Conf}_{n,m}; \mathcal{R}^*\mathcal{L}_\chi)$, whose computation may significantly improve the bound for small q. For large q however, this factor is negligible, and our bound on $|F_\chi(n,m,q)|$ for any nontrivial character χ approximates $q^{n+m+(1-\max(n,m))/2}$.

The number of monic squarefree degree-n polynomials over $\mathbb{F}_q$ is classically known to be $q^n(1-1/q)$ (see, e.g., [27]). The average of a character χ of the resultant over all pairs of monic squarefree polynomials of degrees n and m is therefore bounded by

$$\begin{aligned}\frac{|F_\chi(n,m,q)|}{q^{n+m}(1-1/q)^2} &\leq \frac{1}{q^{n+m}(1-1/2)^2}\cdot 2^{2n+2m-1}\frac{q^{n+m+1-\max(n,m)/2}-1}{\sqrt{q}/2}\\ &\leq 4q^{-n-m}\cdot 2^{2n+2m+1}q^{n+m+(1-\max(n,m))/2}\\ &= 2^{2n+2m+3}q^{(1-\max(n,m))/2}\end{aligned}$$

which approaches 0 when n or m approaches ∞ if $q \gtrsim 2^8$, and at the rate of approximately $q^{(1-\max(n,m))/2}$ when q is large. Qualitatively,

Corollary 7.0.2. *For a sufficiently large q, the asymptotic average of a nontrivial character of the resultant over pairs of monic squarefree polynomials over $\mathbb{F}_q$ approaches 0 as the degree of either or both polynomials grows indefinitely.*

Chapter 8

Hurwitz spaces over punctured curves and Malle's conjecture

In this final chapter, we outline a proof of Theorem 1.4.4. Our proof strategy mimics that detailed in Section 8 of [40].

We first recall the setup. Fix a finite set $\mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_k)$ of primes in $\mathbb{F}_q[t]$. For every $1 \leq i \leq k$, let m_i be the degree of $\mathfrak{p}_i$, and let $m := \sum_{i=1}^k m_i$. Fix a finite group $G \subseteq S_d$ and a finite set $\mathbf{c} = (c_0, c_1, \ldots, c_k)$ of conjugacy invariant subsets of G. Let $N_G^{\mathfrak{p},\mathbf{c}}(\mathbb{F}_q(t), X)$ be the number of isomorphism classes of degree-d extensions L of $\mathbb{F}_q(t)$ with Galois group G, discriminant $|\Delta(L/\mathbb{F}_q(t))| < X$, local monodromy at the prime $\mathfrak{p}_i$ contained in c_i for every $1 \leq i \leq k$, and all other local monodromy elements contained in c_0. The data $\{\mathfrak{p}, c_1, \ldots, c_k\}$ specify a divisor $D \subset \mathbb{A}^1_{\mathbb{F}_q}$ and the ramification on D. Then, there is a bijection between the set of such field extensions (up to isomorphism) and a union of sets $\mathscr{S}_n$ of G-covers of $X_D := \mathbb{A}^1_{\mathbb{F}_q} \setminus D$ with n branched points of local monodromy type c_0 and the prescribed ramification on D. For simplicity, assume that all elements of c_0 have the same index. Let $a = a(G, c_0) = \mathrm{ind}(c_0)^{-1}$, then the discriminant is given by

$$|\Delta| = \prod_{j=1}^{n} q^{\mathrm{ind}(c_0)} \prod_{i=1}^{k} \left(q^{\mathrm{ind}(c_i)}\right)^{m_i} = rq^{n/a}$$

where $r(G, \mathfrak{p}, \mathbf{c}) = \prod_{i=1}^k \left(q^{\mathrm{ind}(c_i)}\right)^{m_i}$ is a constant. So, the assumption $|\Delta| = rq^{n/a} < X$ is equivalent to an upper bound on the number of branched points $n < a \log_q(X/r)$. In more generality, the formula above represents only the dominant term of the discriminant; however, a careful analysis of the discriminant yields a similar upper bound on the number of branched points.

For each n, there is a *Hurwitz moduli stack* $\mathrm{Hn}^{\mathfrak{p},\mathbf{c}}_{G,n}$ that parameterizes the set $\mathscr{S}_n$. By an analog of Theorem 8.4 of [40], we have

$$N^{\mathfrak{p},\mathbf{c}}_G(\mathbb{F}_q(t), X) = |Z_G| \sum_{n<a\log_q(X/r)} |\mathrm{Hn}^{\mathfrak{p},\mathbf{c}}_{G,n}(\mathbb{F}_q)|$$

where Z_G denotes the center of G. Using the Grothendieck–Lefschetz trace formula and Deligne's bounds on the eigenvalues of the geometric Frobenius, it suffices to control the compactly supported étale cohomology of $\mathrm{Hn}^{\mathfrak{p},\mathbf{c}}_{G,n}$. By analogs of Proposition 8.6 and Theorem 8.7 of [40], these cohomology groups can be bounded by the homology of $\mathrm{Hur}^{\mathfrak{p},\mathbf{c}}_{G,n}$, a variant of *Hurwitz spaces* over $X_D(\mathbb{C})$ that we will construct in the next paragraph. We remark that in order to consider both the $\mathbb{F}_q$-points and $\mathbb{C}$-points of X_D, we need to choose a suitable integral model of X_D. A natural choice is to regard each irreducible polynomial over $\mathbb{F}_q$ that defines a prime in D as having coefficients in $\mathbb{Z}$ by the function $\mathbb{F}_q \to \mathbb{Z}$ which sends (a mod q) to the representative of a that lies between 0 and $q-1$. The product of these polynomials forms a degree-m polynomial that is squarefree over $\mathbb{Z}$ and thus over $\mathbb{C}$. Hence, regardless of the chosen integral model of X_D, there is always a homeomorphism $X_D(\mathbb{C}) \cong \mathbb{C}_m$.

For each $0 \le i \le k$, recall that the map $\sigma_{i,i} : c_i \times c_i \to c_i \times c_i$ given by $\sigma_{i,i}(g,h) = (h, h^{-1}gh)$ induces an action of the braid group B_n on the finite set $c_i^{\times n}$, called the Hurwitz action, such that the standard generator σ_j of B_n acts by $\mathrm{id}^{\times j-1} \times \sigma_{i,i} \times \mathrm{id}^{\times n-j-1}$. Observe that this formula also defines isomorphisms $\sigma_{i,j} : c_i \times c_j \to c_j \times c_i$ for all $0 \le i, j \le k$; for convenience, we simply denote any braiding given by this formula by σ. Furthermore, by abuse of notation, denote any map of the form $\mathrm{id}^{\times i-1} \times \sigma \times \mathrm{id}^{\times n-i-1}$ regardless of n by σ_i. Recall that the genus-0 surface braid group $B_n(\mathbb{C}_m)$ can be generated as a subgroup of B_{n+m} by the generators $\sigma_1, \dots, \sigma_{n-1}$ of B_{n+m}, and the generators $\theta_{nj} = \sigma_n^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_n$ for $n+1 \le j \le n+m$ of the pure braid group $PB_{n+m} \subset B_{n+m}$. Then, there is an action of $B_n(\mathbb{C}_m)$ on $c_0^{\times n} \times c_1^{\times m_1} \times \cdots \times c_k^{\times m_k}$ given by $\sigma_i \mapsto \sigma_i$ for $1 \le i \le n-1$, and $\theta_{nj} \mapsto \sigma_n^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_n$ for $n+1 \le j \le n+m$.

Definition 8.0.1. The n^{th} *topological Hurwitz space* associated with the data $(G, \mathfrak{p}, \mathbf{c})$

is the quotient

$$\mathrm{Hur}^{\mathfrak{p},\mathbf{c}}_{G,n} := \widetilde{\mathrm{Conf}}_n(\mathbb{C}_m) \times_{B_n(\mathbb{C}_m)} (c_0^{\times n} \times c_1^{\times m_1} \times \cdots \times c_k^{\times m_k})$$

where $\widetilde{\mathrm{Conf}}_n(\mathbb{C}_m)$ is the universal cover of $\mathrm{Conf}_n(\mathbb{C}_m)$ and the action of the surface braid group is diagonal.

Thus, the final piece in our blueprint for proving Theorem 1.4.4 is an upper bound on the Betti numbers of $\mathrm{Hur}^{\mathfrak{p},\mathbf{c}}_{G,n}$, analogous to Theorem 1.4.3:

Theorem 8.0.2. *For some constants $C(G,\mathfrak{p},\mathbf{c})$, $B(G,c_0)$, and the constant $e(G,c_0)$ appearing in Theorem 1.4.3,*

$$\mathrm{rk}\, H_j(\mathrm{Hur}^{\mathfrak{p},\mathbf{c}}_{G,n};\mathbf{k}) \le Cn^{e+m-1}B^j.$$

We detail the proof of this result below. For each $0 \le i \le k$, let $V_i = \mathbf{k}c_i$ be the braided vector space spanned by elements of c_i with the braiding $\sigma_{i,i} : V_i^{\otimes 2} \cong \mathbf{k}[c_i^{\times 2}] \to \mathbf{k}[c_i^{\times 2}] \cong V_i^{\otimes 2}$ given by extending the Hurwitz action linearly. Similarly, extending the map $\sigma : c_i \times c_j \to c_j \times c_i$ linearly over $\mathbf{k}$ produces an isomorphism $\sigma_{i,j} : V_i \otimes V_j \to V_j \otimes V_i$ for all $0 \le i, j \le k$; we ambiguously use the same notation σ for both the maps of finite sets and vector spaces. Then, there is an action of $B_n(\mathbb{C}_m)$ on $V_0^{\otimes n} \otimes V_1^{\otimes m_1} \otimes \cdots \otimes V_k^{\otimes m_k}$ given by extending linearly the action of $B_n(\mathbb{C}_m)$ on the basis $c_0^{\times n} \times c_1^{\times m_1} \times \cdots \times c_k^{\times m_k}$. Observe that there is an isomorphism

$$H_*(\mathrm{Hur}^{\mathfrak{p},\mathbf{c}}_{G,n};\mathbf{k}) \cong H_*(B_n(\mathbb{C}_m); V_0^{\otimes n} \otimes V_1^{\otimes m_1} \otimes \cdots \otimes V_k^{\otimes m_k}).$$

We now explain how Theorem 6.3.4 can be modified to study these homology groups. It is easy to check that each pair of braided vector spaces (V_i, V_j) forms a left-separable mixed-braided vector space when equipped with a mixed braiding σ^2 and a separated braiding σ. For convenience, set $V := V_0$, $W_1 = \cdots = W_{m_1} := V_1$, $W_{m_1+1} = \cdots = W_{m_1+m_2} := V_2$, etc. Recall that for any braided vector space V, V_ϵ^* denotes the dual vector space V^* with the braiding dual to that of V and twisted by a sign.

Proposition 8.0.3. *Let $\mathfrak{A} = \mathfrak{A}(V_\epsilon^*)$ and $\mathfrak{M}_i = \mathfrak{M}(V_\epsilon^*, (W_i)_\epsilon^*)$ for every $1 \le i \le m$.*

Then

$$H_*(B_n(\mathbb{C}_m); V^{\otimes n} \otimes W_1 \otimes \cdots \otimes W_m) \cong \mathrm{Ext}^{n-*,n+m}_{\mathfrak{A}^e}\left(\mathfrak{M}_1 \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_2 \underset{\mathfrak{A}}{\otimes} \cdots \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_m, \mathbf{k}\right).$$

Proof. It suffices to prove a modified version of Corollary 6.3.2 where the $\mathfrak{A}$-bimodules $\mathfrak{M}_i$ are allowed to arise from different pairs $(V^*_\epsilon, (W_i)^*_\epsilon)$. The proof of this corollary is the culmination of Theorem 6.1.1, Proposition 6.3.1, and Theorem 6.2.4. The only modification we need to introduce is to the proof of Proposition 6.3.1, which originally relied upon the commutativity of Diagram $\mathscr{E}$ (Figure 3.3). In the setting of this chapter, this sufficient condition is simplified to the following desired property of the pairs $(V^*_\epsilon, (W_i)^*_\epsilon)$: for any $1 \le i \le m$, any permissible $1 \le j \le q \le n+m$, and $a \in B_q$, the maps

$$\begin{array}{c} Y^{\otimes p} \otimes V^{\otimes j-1} \otimes W_i \otimes V^{\otimes q-j} \otimes Y^{\otimes n+m-p-q} \\ {\scriptstyle \mathrm{id}^{\otimes p} \otimes a \otimes \mathrm{id}^{\otimes n+m-p-q}} \downarrow \; \downarrow {\scriptstyle a'} \\ Y^{\otimes p} \otimes V^{\otimes a(j)-1} \otimes W_i \otimes V^{\otimes q-a(j)} \otimes Y^{\otimes n+m-p-q} \end{array}$$

agree, where each Y denotes a copy of V or any $W_{\neq i}$, a' is the natural inclusion of a into the copy $B_q \le B_{n+m}$ consisting of braids that are only nontrivial on the q strands starting with the $p+1^{\text{st}}$, a acts on $V^{\otimes j-1} \otimes W_i \otimes V^{\otimes q-j}$, and a' acts on $Y^{\otimes p} \otimes V^{\otimes j-1} \otimes W_i \otimes V^{\otimes q-j} \otimes Y^{\otimes n+m-p-q}$. When $a = \sigma_r \in B_q$ is a standard generator, a acts by $\mathrm{id}^{\otimes r-1} \otimes \sigma \otimes \mathrm{id}^{\otimes q-r}$ and a' acts by $\mathrm{id}^{\otimes p+r-1} \otimes \sigma \otimes \mathrm{id}^{\otimes n+m-p-r}$, so indeed the maps above agree. Therefore, this always holds. □

We will prove an upper bound on the rank of these homology groups by leveraging the upper bound on the cohomology of $\mathfrak{A}$ in Theorem 1.4.3.

Proof of Theorem 8.0.2. By Proposition 8.0.3, it suffices to show

$$\mathrm{rk}\,\mathrm{Ext}^{n-j,n+m}_{\mathfrak{A}^e}\left(\mathfrak{M}_1 \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_2 \underset{\mathfrak{A}}{\otimes} \cdots \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_m, \mathbf{k}\right) \le C n^{e+m-1} B^j.$$

Let $\mathfrak{N}_i := \bigoplus_{p \ge 0} (W_i)^*_\epsilon \otimes (V^*_\epsilon)^{\otimes p}$, i.e., $\mathfrak{M}_i \cong (\mathfrak{M}_i)^{\mathrm{gr}} \cong \mathfrak{A} \otimes \mathfrak{N}_i$ by Proposition 6.3.3. It follows that

$$\mathrm{Ext}^{n-j,n+m}_{\mathfrak{A}^e}\left(\mathfrak{M}_1 \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_2 \underset{\mathfrak{A}}{\otimes} \cdots \underset{\mathfrak{A}}{\otimes} \mathfrak{M}_m, \mathbf{k}\right) \cong \mathrm{Ext}^{n-j,n+m}_{\mathfrak{A}^{\mathrm{op}}}\left(\mathfrak{N}_1 \otimes \mathfrak{N}_2 \otimes \cdots \otimes \mathfrak{N}_m, \mathbf{k}\right).$$

Filtering by degree, we get

$$\mathrm{Ext}^{n-j,n+m}_{\mathfrak{A}^{\mathrm{op}}}(\mathfrak{N}_1 \otimes \mathfrak{N}_2 \otimes \cdots \otimes \mathfrak{N}_m, \mathbf{k}) \Leftarrow \bigoplus_{a+b=n+m} (\mathfrak{N}_1 \otimes \mathfrak{N}_2 \otimes \cdots \otimes \mathfrak{N}_m)_a \otimes \mathrm{Ext}^{n-j,b}_{\mathfrak{A}^{\mathrm{op}}}(\mathbf{k}, \mathbf{k})$$

thus

$$\mathrm{rk}\ \mathrm{Ext}^{n-j,n+m}_{\mathfrak{A}^{\mathrm{op}}}(\mathfrak{N}_1 \otimes \mathfrak{N}_2 \otimes \cdots \otimes \mathfrak{N}_m, \mathbf{k}) \leq \bigoplus_{a+b=n+m} \mathrm{rk}\,(\mathfrak{N}_1 \otimes \mathfrak{N}_2 \otimes \cdots \otimes \mathfrak{N}_m)_a \cdot\ \mathrm{rk}\ \mathrm{Ext}^{n-j,b}_{\mathfrak{A}^{\mathrm{op}}}(\mathbf{k}, \mathbf{k}).$$

By Theorem 1.4.3,

$$\mathrm{rk}\ \mathrm{Ext}^{n-j,b}_{\mathfrak{A}^{\mathrm{op}}}(\mathbf{k}, \mathbf{k}) \leq C_1 b^{e(G,c_0)-1} B_1^{j+b-n} \leq C_1(n+m)^{e-1} B_1^{j+b-n} \leq C_2 n^{e-1} B_1^{j+b-n}$$

for some constants $C_2(G, c_0, m) = (1+m)^{e-1} C_1(G, c_0)$ and $B_1(G, c_0)$. Meanwhile,

$$\mathrm{rk}\,(\mathfrak{N}_1 \otimes \mathfrak{N}_2 \otimes \cdots \otimes \mathfrak{N}_m)_a = \binom{a-m+m-1}{m-1} (\mathrm{dim} V_\epsilon^*)^{a-m} \prod_{i=1}^{m} \mathrm{dim}(W_i)_\epsilon^*$$

where the binomial coefficient results from the number of ordered partitions of $a-m$ into m non-negative integers, i.e., the number of ways to distribute the internal degree a among m graded modules $\mathfrak{N}_i$. Now observe that

$$\binom{a-1}{m-1} = \frac{(a-1)!}{(m-1)!(a-m)!} = \frac{(a-1)\dots(a-m+1)}{(m-1)!} \leq \frac{(n+m-1)^{m-1}}{(m-1)!} \leq C_3 n^{m-1}$$

where $C_3(m) = m^{m-1}/((m-1)!)$.

Putting all of the above together, we get

$$\mathrm{rk}\ H_j(\mathrm{Hur}^{\mathfrak{p},\mathbf{c}}_{G,n}; \mathbf{k}) \leq \bigoplus_{a+b=n+m} C_3 n^{m-1} (\mathrm{dim} V_\epsilon^*)^{a-m} \prod_{i=1}^{m} \mathrm{dim}(W_i)_\epsilon^* \cdot C_2 n^{e-1} B_1^{j+b-n}.$$

Let $B = \max\{\dim V_\epsilon^*, B_1\}$, then

$$\begin{aligned}\text{rk } H_j(\text{Hur}^{\mathfrak{p},\mathbf{c}}_{G,n};\mathbf{k}) &\leq \left(C_3C_2\prod_{i=1}^{m}\dim(W_i)^*_\epsilon\right)\bigoplus_{a+b=n+m} n^{m-1}B^{a-m}\cdot n^{e-1}B^{j+b-n}\\ &= \left(C_3C_2\prod_{i=1}^{m}\dim(W_i)^*_\epsilon\right)n^{e+m-2}B^j\bigoplus_{a+b=n+m}1 \leq Cn^{e+m-1}B^j\end{aligned}$$

for some constants $C(G,\mathfrak{p},\mathbf{c})$ and $B(G,c_0)$. □

We now return to the arithmetic setting to complete the proof of Theorem 1.4.4.

Theorem 8.0.4. *There exist constants $C(G,\mathfrak{p},\mathbf{c})$ and $Q(G,c_0)$ such that for all $q > Q$ and prime to $|G|$,*

$$|\text{Hn}^{\mathfrak{p},\mathbf{c}}_{G,n}(\mathbb{F}_q)| \leq Cn^{e+m-1}q^n$$

where $e = e(G,c_0)$ is the constant appearing in Theorem 1.4.3.

Proof. By the Grothendieck–Lefschetz trace formula, Deligne's bounds on the eigenvalues of the geometric Frobenius, and Theorem 8.0.2, we have

$$\begin{aligned}\frac{|\text{Hn}^{\mathfrak{p},\mathbf{c}}_{G,n}(\mathbb{F}_q)|}{q^n} &= q^{-n}\sum_{j=0}^{2n}(-1)^j\text{tr}(\text{Frob}|H^{2n-j}_{\text{c,et}}(\text{Hn}^{\mathfrak{p},\mathbf{c}}_{G,n}/\overline{\mathbb{F}}_q,\mathbb{Q}_\ell))\\ &\leq \sum_{j=0}^{2n} q^{-j/2}\text{rk } H^{2n-j}_{\text{c,et}}(\text{Hn}^{\mathfrak{p},\mathbf{c}}_{G,n}/\overline{\mathbb{F}}_q,\mathbb{Q}_\ell)\\ &\leq \sum_{j=0}^{2n} q^{-j/2}\text{rk } H_j(\text{Hur}^{\mathfrak{p},\mathbf{c}}_{G,n},\mathbf{k})\\ &\leq \sum_{j=0}^{2n} q^{-j/2}C'n^{e+m-1}B^j = C'n^{e+m-1}\sum_{j=0}^{2n}(q^{-1/2}B)^j\\ &\leq C'n^{e+m-1}\sum_{j=0}^{\infty}(q^{-1/2}B)^j \leq Cn^{e+m-1}\end{aligned}$$

where the last inequality holds, for example, when $C = 2C'$ and $q \geq 4B^2$. □

Theorem 8.0.5. *There exist constants $C(G,\mathfrak{p},\mathbf{c})$ and $Q(G,c_0)$ such that for all $q > Q$*

coprime to $|G|$ *and all* $X > 0$*,*

$$N_G^{\mathfrak{p},\mathbf{c}}(\mathbb{F}_q(t), X) \leq C X^{a(G,c_0)} (\log X)^{e(G,c_0)+m-1}$$

where $a(G, c_0) = [\min_{g\in c_0} \mathrm{ind}(g)]^{-1}$ *is the constant predicted by Malle's conjecture and* $e(G, c_0)$ *is the constant appearing in Theorem 1.4.3.*

Proof. By Theorem 8.0.4, for all $q > Q(G, c_0)$, we have

$$\begin{aligned} N_G^{\mathfrak{p},\mathbf{c}}(\mathbb{F}_q(t), X) &= |Z_G| \sum_{n<a\log_q(X/r)} |\mathrm{Hn}_{G,n}^{\mathfrak{p},\mathbf{c}}(\mathbb{F}_q)| \leq |Z_G| \sum_{n<a\log_q(X/r)} C_1 n^{e+m-1} q^n \\ &\leq |Z_G| C_1 [a \log_q(X/r)]^{e+m-1} \sum_{n<a\log_q(X/r)} q^n \\ &\leq C_2 (\log X)^{e+m-1} q^{a\log_q(X/r)} \sum_{n<a\log_q(X/r)} q^{-n} \\ &\leq C_2 (\log X)^{e+m-1} (X/r)^a \sum_{n=0}^{\infty} q^{-n} \\ &= (C_2 r^{-a}) X^a (\log X)^{e+m-1} \frac{1}{1-1/q} \leq C X^a (\log X)^{e+m-1} \end{aligned}$$

for some constant $C(G, \mathfrak{p}, \mathbf{c}) = C_2 r^{-a}(1 - 1/Q)^{-1}$. □

References

[1] T. Akita and Y. Liu. Second mod 2 homology of Artin groups. *Algebr. Geom. Topol.*, 18(1):547–568, 2018.

[2] B. H. An and K. H. Ko. A family of representations of braid groups on surfaces. *Pacific J. Math.*, 247(2):257–282, 2010.

[3] V. I. Arnol'd. The cohomology ring of the group of dyed braids. *Mat. Zametki*, 5:227–231, 1969.

[4] V. I. Arnol'd. On some topological invariants of algebraic functions. *Trans. Moscow Math. Soc.*, 21:30–52, 1970.

[5] E. Artin. Theorie der Zöpfe. *Abh. Math. Sem. Univ. Hamburg*, 4(1):47–72, 1925.

[6] E. Artin. Theory of braids. *Ann. of Math. (2)*, 48:101–126, 1947.

[7] J. C. Baez. Hochschild homology in a braided tensor category. *Trans. Amer. Math. Soc.*, 344(2):885–906, 1994.

[8] P. Bellingeri and E. Godelle. Positive presentations of surface braid groups. *J. Knot Theory Ramifications*, 16(9):1219–1233, 2007.

[9] J. Bergström, A. Diaconu, D. Petersen, and C. Westerland. Hyperelliptic curves, the scanning map, and moments of families of quadratic L-functions. *arXiv preprint arXiv:2302.07664*, 2023.

[10] M. Bhargava. The density of discriminants of quartic rings and fields. *Ann. of Math. (2)*, 162(2):1031–1063, 2005.

[11] M. Bhargava. The density of discriminants of quintic rings and fields. *Ann. of Math. (2)*, 172(3):1559–1591, 2010.

[12] M. Bhargava, A. Shankar, and X. Wang. Geometry-of-numbers methods over global fields I: Prehomogeneous vector spaces. *arXiv preprint arXiv:1512.03035*, 2015.

[13] M. Bhargava, A. Shankar, and X. Wang. Squarefree values of polynomial discriminants I. *Invent. Math.*, 228(3):1037–1073, 2022.

[14] A. Bianchi and A. Stavrou. Homology of configuration spaces of surfaces modulo an odd prime. *arXiv preprint arXiv:2307.08664*, 2023.

[15] S. J. Bigelow. Braid groups are linear. *J. Amer. Math. Soc.*, 14(2):471–486, 2001.

[16] J. S. Birman. *Braids, links, and mapping class groups.* Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974.

[17] C.-F. Bödigheimer and F. R. Cohen. Rational cohomology of configuration spaces of surfaces. In *Algebraic topology and transformation groups (Göttingen, 1987)*, volume 1361 of *Lecture Notes in Math.*, pages 7–13. Springer, Berlin, 1988.

[18] R. Boyd. The low-dimensional homology of finite-rank Coxeter groups. *Algebr. Geom. Topol.*, 20(5):2609–2655, 2020.

[19] L. Brantner, J. Hahn, and B. Knudsen. The Lubin-Tate theory of configuration spaces: I. *arXiv preprint arXiv:1908.11321*, 2019.

[20] F. Callegaro. The homology of the Milnor fiber for classical braid groups. *Algebr. Geom. Topol.*, 6:1903–1923, 2006.

[21] F. Callegaro and I. Marin. Homology computations for complex braid groups. *J. Eur. Math. Soc. (JEMS)*, 16(1):103–164, 2014.

[22] F. Callegaro, D. Moroni, and M. Salvetti. Cohomology of affine Artin groups and applications. *Trans. Amer. Math. Soc.*, 360(8):4169–4188, 2008.

[23] F. Callegaro, D. Moroni, and M. Salvetti. Cohomology of Artin groups of type A_n, B_n and applications. In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 85–104. Geom. Topol. Publ., Coventry, 2008.

[24] A. Cayley. *On the Theory of Elimination*, volume 1 of *Cambridge Library Collection - Mathematics*, page 370–374. Cambridge University Press, 2009.

[25] M. Chen and A. Zhang. Mod p homology of unordered configuration spaces of p points in parallelizable surfaces. *Proc. Amer. Math. Soc.*, 152(5):2239–2248, 2024.

[26] W. Chen. Homology of braid groups, the Burau representation, and points on superelliptic curves over finite fields. *Israel J. Math.*, 220(2):739–762, 2017.

[27] T. Church, J. S. Ellenberg, and B. Farb. Representation stability in cohomology and asymptotics for families of varieties over finite fields. In *Algebraic topology: applications and new directions*, volume 620 of *Contemp. Math.*, pages 1–54. Amer. Math. Soc., Providence, RI, 2014.

[28] P. L. Clark and P. Pollack. Reciprocity by resultant in $k[t]$. *Enseign. Math.*, 65(1-2):101–116, 2019.

[29] F. Cohen. Cohomology of braid spaces. *Bull. Amer. Math. Soc.*, 79:763–766, 1973.

[30] F. Cohen. Homology of $\Omega^{(n+1)}\Sigma^{(n+1)}X$ and $C_{(n+1)}X$, $n > 0$. *Bull. Amer. Math. Soc.*, 79:1236–1241 (1974), 1973.

[31] J. Crisp. Injective maps between Artin groups. In *Geometric group theory down under (Canberra, 1996)*, pages 119–137. de Gruyter, Berlin, 1999.

[32] B. Datskovsky and D. J. Wright. Density of discriminants of cubic extensions. *J. Reine Angew. Math.*, 386:116–138, 1988.

[33] H. Davenport and H. Heilbronn. On the density of discriminants of cubic fields. II. *Proc. Roy. Soc. London Ser. A*, 322(1551):405–420, 1971.

[34] C. De Concini, C. Procesi, and M. Salvetti. Arithmetic properties of the cohomology of braid groups. *Topology*, 40(4):739–751, 2001.

[35] C. De Concini, C. Procesi, M. Salvetti, and F. Stumbo. Arithmetic properties of the cohomology of Artin groups. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):695–717, 1999.

[36] G. C. Drummond-Cole and B. Knudsen. Betti numbers of configuration spaces of surfaces. *J. Lond. Math. Soc. (2)*, 96(2):367–393, 2017.

[37] G. Duchamp, A. Klyachko, D. Krob, and J.-Y. Thibon. Noncommutative symmetric functions. III. Deformations of Cauchy and convolution algebras. volume 1, pages 159–216. 1997. Lie computations (Marseille, 1994).

[38] J. S. Ellenberg. Legendre symbols and secondary stability. 2021. Talk at *BIRS Workshop: Cohomology of Arithmetic Groups: Duality, Stability, and Computations.*

[39] J. S. Ellenberg and A. Landesman. Homological stability for generalized Hurwitz spaces and Selmer groups in quadratic twist families over function fields. *arXiv preprint arXiv:2310.16286*, 2023.

[40] J. S. Ellenberg, T. Tran, and C. Westerland. Fox-Neuwirth-Fuks cells, quantum shuffle algebras, and Malle's conjecture for function fields. *arXiv preprint arXiv:1701.04541*, 2023.

[41] J. S. Ellenberg and A. Venkatesh. The number of extensions of a number field with fixed degree and bounded discriminant. *Ann. of Math. (2)*, 163(2):723–741, 2006.

[42] J. S. Ellenberg, A. Venkatesh, and C. Westerland. Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields. *Ann. of Math. (2)*, 183(3):729–786, 2016.

[43] E. Fadell and L. Neuwirth. Configuration spaces. *Math. Scand.*, 10:111–118, 1962.

[44] B. Farb and J. Wolfson. Topology and arithmetic of resultants, II: The resultant=1 hypersurface. *Algebr. Geom.*, 4(3):337–352, 2017. With an appendix by Christophe Cazanave.

[45] B. Farb, J. Wolfson, and M. M. Wood. Coincidences between homological densities, predicted by arithmetic. *Adv. Math.*, 352:670–716, 2019.

[46] R. Fox and L. Neuwirth. The braid groups. *Math. Scand.*, 10:119–126, 1962.

[47] E. V. Frenkel. Cohomology of the commutator subgroup of the braid group. *Funktsional. Anal. i Prilozhen.*, 22(3):91–92, 1988.

[48] D. B. Fuks. Cohomology of the braid group mod 2. *Funkcional. Anal. i Priložen.*, 4(2):62–73, 1970.

[49] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.

[50] V. Ginzburg. Lectures on noncommutative geometry. *arXiv preprint math/0506603*, 2005.

[51] C. Giusti and D. Sinha. Fox-Neuwirth cell structures and the cohomology of symmetric groups. In *Configuration spaces*, volume 14 of *CRM Series*, pages 273–298. Ed. Norm., Pisa, 2012.

[52] V. V. Gorjunov. The cohomology of braid groups of series C and D and certain stratifications. *Funktsional. Anal. i Prilozhen.*, 12(2):76–77, 1978.

[53] D. Grinberg and V. Reiner. Hopf algebras in combinatorics. *arXiv preprint arXiv:1409.8356v7*, 2020.

[54] A. T. N. Hoang. Fox–Neuwirth cells, quantum shuffle algebras, and character sums of the resultant. *arXiv preprint arXiv:2308.01410*, 2023.

[55] A. T. N. Hoang. Fox–Neuwirth cells, quantum shuffle algebras, and the homology of type-B Artin groups. *Math. Z.*, 306(4):Paper No. 57, 2024.

[56] M. Kapranov and V. Schechtman. Shuffle algebras and perverse sheaves. *Pure Appl. Math. Q.*, 16(3):573–657, 2020.

[57] M. Kapranov, O. Schiffmann, and E. Vasserot. Spherical Hall algebra of $\overline{\mathrm{Spec}(\mathbb{Z})}$. In *Homological mirror symmetry and tropical geometry*, volume 15 of *Lect. Notes Unione Mat. Ital.*, pages 153–196. Springer, Cham, 2014.

[58] J. Klüners. A counterexample to Malle's conjecture on the asymptotics of discriminants. *C. R. Math. Acad. Sci. Paris*, 340(6):411–414, 2005.

[59] J. Klüners. The asymptotics of nilpotent Galois groups. *Acta Arith.*, 204(2):165–184, 2022.

[60] B. Knudsen. Betti numbers and stability for configuration spaces via factorization homology. *Algebr. Geom. Topol.*, 17(5):3137–3187, 2017.

[61] P. Koymans and C. Pagano. On Malle's conjecture for nilpotent groups. *Trans. Amer. Math. Soc. Ser. B*, 10:310–354, 2023.

[62] D. Krammer. Braid groups are linear. *Ann. of Math. (2)*, 155(1):131–156, 2002.

[63] R. J. Lawrence. Homological representations of the Hecke algebra. *Comm. Math. Phys.*, 135(1):141–191, 1990.

[64] V. Lebed. Homologies of algebraic structures via braidings and quantum shuffles. *J. Algebra*, 391:152–192, 2013.

[65] R. J. Lemke Oliver and F. Thorne. Upper bounds on number fields of given degree and bounded discriminant. *Duke Math. J.*, 171(15):3077–3087, 2022.

[66] Y. Liu, M. M. Wood, and D. Zureick-Brown. A predicted distribution for Galois groups of maximal unramified extensions. *Invent. math.*, 2024.

[67] J.-L. Loday. Cup-product for Leibniz cohomology and dual Leibniz algebras. *Math. Scand.*, 77(2):189–196, 1995.

[68] G. Malle. On the distribution of Galois groups. *J. Number Theory*, 92(2):315–329, 2002.

[69] G. Malle. On the distribution of Galois groups. II. *Experiment. Math.*, 13(2):129–135, 2004.

[70] S. Manfredini. Some subgroups of Artin's braid group. volume 78, pages 123–142. 1997. Special issue on braid groups and related topics (Jerusalem, 1995).

[71] I. Marin. Homology computations for complex braid groups II. *J. Algebra*, 477:540–547, 2017.

[72] N. S. Markaryan. Homology of braid groups with nontrivial coefficients. *Mat. Zametki*, 59(6):846–854, 960, 1996.

[73] R. Masri, F. Thorne, W.-L. Tsai, and J. Wang. Malle's conjecture for $G \times A$, with $G = S_3, S_4, S_5$. *arXiv preprint arXiv:2004.04651*, 2020.

[74] J. Miller, P. Patzt, D. Petersen, and O. Randal-Williams. Uniform twisted homological stability. *arXiv preprint arXiv:2402.00354*, 2024.

[75] J. Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958.

[76] B. C. Ngo. Perverse sheaves and fundamental lemmas. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 217–250. Amer. Math. Soc., Providence, RI, 2017.

[77] M. Palmer and A. Soulié. Polynomiality of surface braid and mapping class group representations. *arXiv preprint arXiv:2302.08827*, 2023.

[78] O. Randal-Williams and N. Wahl. Homological stability for automorphism groups. *Adv. Math.*, 318:534–626, 2017.

[79] M. Rosso. Quantum groups and quantum shuffles. *Invent. Math.*, 133(2):399–416, 1998.

[80] P. Salvatore. Configuration spaces on the sphere and higher loop spaces. *Math. Z.*, 248(3):527–540, 2004.

[81] M. Salvetti. Topology of the complement of real hyperplanes in $\mathbf{C}^N$. *Invent. Math.*, 88(3):603–618, 1987.

[82] M. Salvetti. The homotopy type of Artin groups. *Math. Res. Lett.*, 1(5):565–577, 1994.

[83] M. Salvetti. Cohomology of Coxeter groups. volume 118, pages 199–208. 2002. Arrangements in Boston: a Conference on Hyperplane Arrangements (1999).

[84] W. Sawin. General multiple dirichlet series from perverse sheaves. *arXiv preprint arXiv:2209.13072*, 2022.

[85] W. Sawin and M. Shusterman. Möbius cancellation on polynomial sequences and the quadratic Bateman-Horn conjecture over function fields. *Invent. Math.*, 229(2):751–927, 2022.

[86] W. Sawin and M. Shusterman. On the Chowla and twin primes conjectures over $\mathbb{F}_q[T]$. *Ann. of Math. (2)*, 196(2):457–506, 2022.

[87] R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[88] D.-M. Tong, S.-D. Yang, and Z.-Q. Ma. A new class of representations of braid groups. *Comm. Theoret. Phys.*, 26(4):483–486, 1996.

[89] S. Türkelli. Connected components of Hurwitz schemes and Malle's conjecture. *J. Number Theory*, 155:163–201, 2015.

[90] F. V. Vainshtein. The cohomology of braid groups. *Funktsional. Anal. i Prilozhen.*, 12(2):72–73, 1978.

[91] V. A. Vassiliev. *Complements of discriminants of smooth maps: topology and applications*, volume 98 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by B. Goldfarb.

[92] V. V. Vershinin. Homology of braid groups and their generalizations. In *Knot theory (Warsaw, 1995)*, volume 42 of *Banach Center Publ.*, pages 421–446. Polish Acad. Sci. Inst. Math., Warsaw, 1998.

[93] V. V. Vershinin. Generalizations of braids and homology. *Sovrem. Mat. Prilozh.*, (19, Topol. i Nekommut. Geom.):90–104, 2004.

[94] J. Wang. Malle's conjecture for $S_n \times A$ for $n = 3, 4, 5$. *Compos. Math.*, 157(1):83–121, 2021.

[95] J. H. Wang. On the braid groups for $\mathbf{R}P^2$. *J. Pure Appl. Algebra*, 166(1-2):203–227, 2002.

[96] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[97] M. M. Wood. Asymptotics for number fields and class groups. In *Directions in number theory*, volume 3 of *Assoc. Women Math. Ser.*, pages 291–339. Springer, [Cham], 2016.

[98] D. J. Wright. Distribution of discriminants of abelian extensions. *Proc. London Math. Soc. (3)*, 58(1):17–50, 1989.