

**NOTE ON THE BALANCE OF ENERGY  
AT A PHASE CHANGE INTERFACE**

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**IMA Preprint Series # 518**

April 1989

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Balance laws across interfaces between different materials or different phases of the same material have been derived by Delhaye [1974]. Film condensation and film boiling are examples of physical processes involving these interface conditions. The balance of energy used by Delhaye, his equation [7], is incomplete since he has omitted the contribution to the energy which arises from the interface and is proportional to the interfacial area. The interfacial contribution to the energy which appears in Delhaye's equation [7] is in the form

$$\frac{d}{dt} \int_{\Sigma} \rho_i (u_i + \frac{1}{2} v_p^2) d\Sigma, \quad (1)$$

where the subscript i stands for the interface,  $\Sigma$  ( $= A_i$  in the notation of Delhaye) is the area of the interface intersecting the material volume shown in figure 1,  $\rho_i$  is the area density of the interface,  $u_i$  is the internal energy per unit mass and  $v_p$  is the speed of a point on the interface. The interface term (1) vanishes for the massless interfaces ( $\rho_i=0$ ) which are considered in most of the applications. We make a verifiable claim that (1) should be supplemented by the time derivative of the classical surface energy

$$\frac{d}{dt} \int_{\Sigma} \sigma d\Sigma . \quad (2)$$

In the usual situation, where  $\rho_i=0$ , (2) is the only surface energy. This would be the case at the boundary of two fluids, between a fluid and a solid, and at a phase change boundary where  $\Sigma$  is the interface, say, where the density jumps.

It is of interest to see how the interfacial energy balance given by equation [7] of Delhaye is altered by the addition of (2) to the balance. Let  $\mathbf{x}=\chi(\xi, \eta, t)$  where  $(\xi, \eta)$  are time independent reference coordinates on the interface. If  $F(\mathbf{x}, t)=0$  is the equation for this interface, then

$$\frac{dF}{dt}(\mathbf{x}(\xi, \eta, t), t) = \frac{\partial F}{\partial t} + \mathbf{v}_{\Sigma} \cdot \nabla F = 0 \quad (3)$$

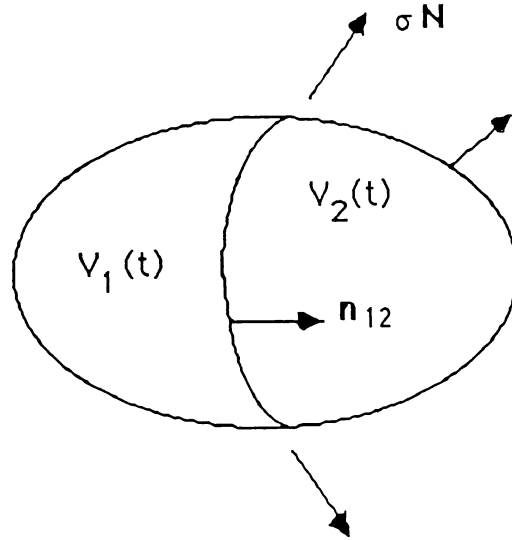
where  $\mathbf{v}_{\Sigma} = \partial\chi/\partial t$ . Hence

$$\mathbf{v}_{\Sigma} \cdot \mathbf{n}_{12} = -\frac{\partial F}{\partial t} / |\nabla F| \quad (4)$$

when  $\mathbf{n}_{12}$  is the normal of  $F$ , pointing from side 1 to side 2 in figure 1.

The mass balance at the interface gives rise to

$$\rho_1 \tilde{\mathbf{v}}_1 \cdot \mathbf{n}_{12} = \rho_2 \tilde{\mathbf{v}}_2 \cdot \mathbf{n}_{12} = -\dot{m}(\mathbf{x}, t) \quad (5)$$



**Figure 1.**  $V = V_1 \cup V_2$  is a material volume and  $\Sigma$  is an interface across which simple jumps are allowed; for and  $f$ ,  $[[f]] = f_1 - f_2$  is the jump of  $f$  at a point  $\mathbf{x}$  on  $\Sigma$ . The interface can sustain normal tractions on the boundary of  $V$ ,  $\mathbf{N}$  is a unit normal vector, normal to  $\partial\Sigma$ , in  $\Sigma$ ,  $\mathbf{n}_{12} \cdot \mathbf{N} = 0$  and  $\sigma$  is the interfacial tension.

where

$$\tilde{\mathbf{v}}_j \stackrel{\text{def}}{=} \mathbf{v}_j - \mathbf{v}_\Sigma, j = 1, 2 \quad (6)$$

is the velocity relative to the moving interface and  $\dot{m}$  is positive when there is a net transfer of mass from region 2 to region 1.

The momentum balance at the interface is given by

$$\dot{m} [[\mathbf{v}]] + [[\mathbf{T}]] \cdot \mathbf{n}_{12} = \nabla_{\text{II}} \sigma + 2H \sigma \mathbf{n}_{12} \quad (7)$$

where  $[[\cdot]] = (\cdot)_1 - (\cdot)_2$  is the usual notation for the jump across  $\Sigma$ ,  $\mathbf{T}$  is the stress (if  $\mathbf{T}$  is not symmetric, it should be replaced with the transpose of  $\mathbf{T}$ ),  $\nabla_{\text{II}}$  is the surface gradient,

$$\nabla_{\text{II}} \cdot \mathbf{n}_{12} = \text{div}_{\text{II}} \mathbf{n}_{12} = -2H = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad (8)$$

where  $H$  is the mean curvature and  $R_1$  and  $R_2$  the principal radii of curvature.

The energy balance for the material volume  $V$  shown in figure 1, when  $\Sigma$  is a massless interface, is given by

$$\frac{d\mathcal{E}}{dt} = \int_{\partial V} \{\mathbf{v} \cdot (\mathbf{T}\mathbf{n}) - \mathbf{q} \cdot \mathbf{n}\} da + \int_{\partial\Sigma} \sigma \mathbf{N} \cdot \mathbf{v}_\Sigma d\ell + \int_V \rho \mathbf{g} \cdot \mathbf{v} dV, \quad (9)$$

where  $\mathbf{g}$  is a body force per unit mass,  $\mathbf{v}$  is the velocity,  $\mathbf{q}$  the heat flux,  $\mathbf{n}$  is the outward normal on  $V$  and  $\mathbf{n} = \mathbf{N}$  is also on  $\Sigma$  and

$$\mathcal{E} = \int_V \rho \left\{ u + \frac{1}{2} |\mathbf{v}|^2 \right\} dv + \int_\Sigma \sigma d\Sigma. \quad (10)$$

Equation (9) states that the rate of change of the sum of internal, kinetic and surface energy is balanced by the sum of the power of the body force, the power of surface tractions  $\mathbf{T}\mathbf{n}$ , the heat flow into  $V$ , and the power of the surface tension.

Equations (9) and (10) can be extended to include more general problems of change of phase, for example to the growth of silicon crystals. For this type of extension, it is necessary to replace  $\sigma \mathbf{N}$  with a general traction vector on  $\partial\Sigma$ , allowing for effects of shear and bending as well as dilatation and  $\sigma$  would be interpreted as a strain energy per unit area.

To reduce (9) further, we eliminate the surface energy with the kinematic identity

$$\frac{d}{dt} \int_\Sigma \sigma d\Sigma = \int_{\partial\Sigma} \sigma \mathbf{v}_\Sigma \cdot \mathbf{N} d\ell + \int_\Sigma \left\{ \frac{\partial \sigma}{\partial t} - \mathbf{v}_\Sigma \cdot \nabla_{II} \sigma - 2H\sigma \mathbf{v}_\Sigma \cdot \mathbf{n}_{12} \right\} d\Sigma \quad (11)$$

Equation (11) is derived, for example, as equation (55.15) in Joseph [1976], when  $\mathbf{x} = \hat{\mathbf{x}}$  and  $\mathbf{v} = \hat{\mathbf{v}}$ . The partial time derivative is for fixed  $(\xi, \eta)$ , and therefore

$$\frac{\partial \sigma}{\partial t} \Big|_{\xi, \eta} = \frac{\partial \sigma}{\partial t} \Big|_{\mathbf{x}} + \mathbf{v}_\Sigma \cdot \nabla_{II} \sigma. \quad (12)$$

After combining (10) and (11), we let  $V$  tend to zero holding  $\Sigma$  fixed, and we find that

$$[[\mathbf{q}]] \cdot \mathbf{n}_{12} - [[\mathbf{v} \cdot \mathbf{T} \mathbf{n}_{12}]] + \mathbf{v}_\Sigma \cdot \nabla_{\text{II}} \sigma + 2H\sigma \mathbf{v}_\Sigma \cdot \mathbf{n}_{12} - \frac{\partial \sigma}{\partial t} = \dot{m} \left[ u + \frac{1}{2} |\mathbf{v}|^2 \right] . \quad (13)$$

This equation was derived by Joseph [1989] for the case in which the interface is a material surface  $\mathbf{v}_\Sigma = \mathbf{v}$  on  $\Sigma$ , as is the case for an interface between two fluids or between, say, a fluid and a solid across which no mass is transported. Equation (13) allows for mass transfer as well. When  $\sigma=0$ , (13) also expresses the energy balance across a viscous and heat conducting shock wave.

The energy balance (13) may be written relative to an observer moving with the interface

$$[[\mathbf{q}]] \cdot \mathbf{n}_{12} + [[\tilde{\mathbf{v}} \cdot \mathbf{T}]] \mathbf{n}_{12} = \frac{\partial \sigma}{\partial t} + \dot{m} \left[ u + \frac{1}{2} |\mathbf{v}|^2 \right] . \quad (14)$$

To derive (14), subtract the projection of (7) with  $\mathbf{v}_\Sigma$  from (11).

Acknowledgements; Joseph's work was supported by the Department of Energy, Office of Basic Energy Sciences, the National Science Foundation and the Army Research Office. Narain's work was supported by NASA grant NAG 3-711.

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