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SOME NECESSARY CONDITIONS AT AN INTERNAL BOUNDARY FOR MINIMIZERS IN FINITE ELASTICITY

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1. Introduction

Consider a deformed body that contains surfaces across which the material properties are not continuous such as a twin boundary in a crystal, a boundary between shear bands in a metal or polymer, or a boundary at which two distinct materials have been joined. Assuming that the body is elastic at equilibrium it is of interest to determine constitutive restrictions upon the stored-energy density that are consequences of the observed stability of the body.

In this paper we address the problem of finding pointwise algebraic conditions upon the elasticity tensors $\mathbf{C}^+(\mathbf{x}_0, \nabla\mathbf{f}^+(\mathbf{x}_0))$ and $\mathbf{C}^-(\mathbf{x}_0, \nabla\mathbf{f}^-(\mathbf{x}_0))$ that are necessary for a continuous, piecewise smooth deformation $\mathbf{f}$ to be a weak relative minimizer, *i.e.*, a local minimizer of the energy in the C^1-topology. In particular we show that the Legendre-Hadamard condition, Agmon's condition, and two new conditions: if for some vector $\underset{\sim}{e}$

$$\underset{\sim}{e}\otimes\underset{\sim}{n}_0\cdot\underset{\sim}{C}_0^+[\underset{\sim}{e}\otimes\underset{\sim}{n}_0] = \underset{\sim}{e}\otimes\underset{\sim}{n}_0\cdot\underset{\sim}{C}_0^-[\underset{\sim}{e}\otimes\underset{\sim}{n}_0] = 0$$

then

$$\underset{\sim}{C}_0^+[\underset{\sim}{e}\otimes\underset{\sim}{n}_0] = \underset{\sim}{C}_0^-[\underset{\sim}{e}\otimes\underset{\sim}{n}_0];$$

and if for some vector $\underset{\sim}{a}$

$$\underset{\sim}{a}\otimes\underset{\sim}{t}_0\cdot\underset{\sim}{C}_0^+[\underset{\sim}{a}\otimes\underset{\sim}{t}_0] = \underset{\sim}{a}\otimes\underset{\sim}{t}_0\cdot\underset{\sim}{C}_0^-[\underset{\sim}{a}\otimes\underset{\sim}{t}_0] = 0$$

then

$$\underset{\sim}{C}_0^+[\underset{\sim}{a}\otimes\underset{\sim}{t}_0]\underset{\sim}{n}_0 = \underset{\sim}{C}_0^-[\underset{\sim}{a}\otimes\underset{\sim}{t}_0]\underset{\sim}{n}_0$$

are consequences of the infinitesimal stability[1] of the deformation $\underset{\sim}{f}$. Here $\underset{\sim}{n}_0 = \underset{\sim}{n}(\underset{\sim}{x}_0)$ is the unit normal and $\underset{\sim}{t}_0$ is a tangent vector to a surface $\mathcal{S}$ across which the elasticity tensor and/or deformation gradient suffer a discontinuity while $\nabla\underset{\sim}{f}^{\pm}(\underset{\sim}{x}_0)$ are the limits of the deformation gradients and $\underset{\sim}{C}_0^{\pm} = \underset{\sim}{C}^{\pm}(\underset{\sim}{x}_0,\nabla\underset{\sim}{f}^{\pm}(\underset{\sim}{x}_0))$ are the limits of the elasticity tensors as the point $\underset{\sim}{x}_0$ is approached from opposite sides of the surface $\mathcal{S}$.

Remark. Agmon's condition (cf. (3.1)-(3.2)) can be interpreted as a dynamic stability requirement for linear elastodynamics[2] since the failure of this condition yields a nontrivial, continuous $\underset{\sim}{u}\colon \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ that satisfies

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[1] That is, the nonnegativity of the second variation of the energy.

[2] Hayes & Rivlin [7] have noted that the Legendre-Hadamard condition can also be viewed as a dynamic stability requirement for linear elastodynamics. See also Rivlin [11], Truesdell & Noll [13, Section 73] and Simpson & Spector [12].

$$\operatorname{div} \underset{\sim}{C}_0^{\pm}[\nabla \underset{\sim}{u}] = \underset{\sim}{u}_{tt} \text{ in } \mathcal{H}_{\pm} ,$$
$$\underset{\sim}{C}_0^{+}[\nabla \underset{\sim}{u}^{+}]\underset{\sim}{n}_0 = \underset{\sim}{C}_0^{-}[\nabla \underset{\sim}{u}^{-}]\underset{\sim}{n}_0 \text{ on } \partial\mathcal{H}_{\pm} ,$$

where

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \exp(\alpha t)\mathrm{Re}\{\underset{\sim}{z}(\underset{\sim}{x}\cdot\underset{\sim}{n}_0)\exp(i\underset{\sim}{x}\cdot\underset{\sim}{t}_0)\},$$

$\alpha > 0$, $\underset{\sim}{z}$ is bounded, and $\underset{\sim}{t}_0$ is tangent and $\underset{\sim}{n}_0$ is normal to the boundary of the two complementary half-spaces $\mathcal{H}_+$ and $\mathcal{H}_-$. Thus the failure of Agmon's condition implies that the rest state, $\underset{\sim}{u} = \underset{\sim}{0}$, is unstable since there are initial data, arbitrarily close to $\underset{\sim}{0}$ that give rise to a solution growing exponentially in time.

In Section 7 we apply our results to a boundary at which two linearly elastic isotropic materials have been joined. Here the elasticity tensors are given by

$$\underset{\sim}{C}_0^{\pm}[\underset{\sim}{H}] = \mu^{\pm}(\underset{\sim}{H}+\underset{\sim}{H}^T)+\lambda^{\pm}(\operatorname{trace} \underset{\sim}{H})\underset{\sim}{I}.$$

In particular, we show that given the strong ellipticity of $\underset{\sim}{C}_0^{+}$ and $\underset{\sim}{C}_0^{-}$ (i.e., given $\mu^{\pm} > 0$ and $2\mu^{\pm}+\lambda^{\pm} > 0$)

$$\mu^{-} \geq \frac{-\mu^{+}(\mu^{+}+\lambda^{+})}{3\mu^{+}+\lambda^{+}} , \qquad \mu^{+} \geq \frac{-\mu^{-}(\mu^{-}+\lambda^{-})}{3\mu^{-}+\lambda^{-}}$$

are necessary conditions for the reference configuration to be infinitesimally stable.

James [8] and Gurtin [6] have considered the problem of determining constitutive restrictions upon the stored-energy density that are necessary for a continuous, piecewise smooth deformation to be a strong relative minimizer, i.e., a local minimizer of the energy in the C^0-topology. In particular, James has noted that for a smooth stored-energy density the Maxwell relation must be satisfied at every point on the surface where the deformation gradient is discontinuous while Gurtin has obtained a new integral inequality[3] (see (4.4)) that the stored-energy density must satisfy. Our results make crucial use of Gurtin's inequality.

We note that the analysis in this paper is essentially the same as the analysis of Simpson & Spector [12] who derive pointwise algebraic conditions on the elasticity tensor at an (external) boundary that are necessary for a deformation to be a weak relative minimizer. Both the mechanics of internal boundaries and the precise results obtained are sufficiently different to lead us to believe that the results in this paper are of interest.

Finally, we note that we have not been able to determine if the four necessary conditions that we have obtained are also sufficient for the infinitesimal stability of even a continuous, piecewise homogeneous deformation of a piecewise homogeneous body. In particular, we have been unable to find algebraic conditions that are necessary and sufficient for

$$\int_{\mathcal{H}\mathcal{B}_+} \nabla \underset{\sim}{u} \cdot \underset{\sim}{C}_0^+[\nabla \underset{\sim}{u}] d\underset{\sim}{x} + \int_{\mathcal{H}\mathcal{B}_-} \nabla \underset{\sim}{u} \cdot \underset{\sim}{C}_0^-[\nabla \underset{\sim}{u}] d\underset{\sim}{x} \geq 0$$

whenever $\underset{\sim}{u} \in C_0^1(\mathcal{B}, \mathbb{R}^3)$. (Here $\mathcal{H}\mathcal{B}_+$ and $\mathcal{H}\mathcal{B}_-$ are two complementary

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[3] This inequality is similar to Morrey's [10] quasiconvexity condition.

half-balls that comprise the unit ball $\mathcal{B}$.) However in the special case when $\mathbf{C}_0^+$ and $\mathbf{C}_0^-$ are strongly-elliptic Agmon's condition alone is necessary and sufficient for the above inequality.

2. Notations

We let

Lin := space of all linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$ with inner product and norm

$$\mathbf{G}\cdot\mathbf{H} := \text{trace}\,(\mathbf{G}\mathbf{H}^T), \quad |\mathbf{G}|^2 := \mathbf{G}\cdot\mathbf{G},$$

where $\mathbf{H}^T$ denotes the transpose of $\mathbf{H}$. We write

$$\text{Lin}^+ := \{\mathbf{H} \in \text{Lin} : \det \mathbf{H} > 0\},$$

where det denotes the determinant. Given any $\mathbf{M} \in$ Lin we extend its domain of definition to $\mathbb{C}^n$ and write $\mathbf{M}\mathbf{z} := \mathbf{M}\mathbf{x} + i\,\mathbf{M}\mathbf{y}$ where $\mathbf{x} := (\mathbf{z}+\bar{\mathbf{z}})/2$ and $\mathbf{y} := -i(\mathbf{z}-\bar{\mathbf{z}})/2$. We denote by $\mathbf{a}\otimes\mathbf{b}$ the tensor product of any two vectors $\mathbf{a},\mathbf{b} \in \mathbb{C}^n$; in components

$$(\mathbf{a}\otimes\mathbf{b})_{ij} = a_i b_j.$$

We write ∇ and div for the gradient and divergence operators in $\mathbb{R}^n$: for a vector field $\mathbf{u}$, $\nabla\mathbf{u}$ is the tensor field with components $(\nabla\mathbf{u})_{ij} = \partial u_i/\partial x_j$; for a tensor field $\mathbf{S}$, div $\mathbf{S}$ is the vector field with components $\sum_j \partial S_{ij}/\partial x_j$. Given any function $\Phi(\mathbf{a},\mathbf{b},\ldots,\mathbf{c})$ with vector or tensor arguments we write, _e.g._, $\partial\Phi/\partial\mathbf{a}$ for the partial Fréchet derivative of Φ with respect to $\mathbf{a}$ holding the remaining arguments fixed.

Throughout this paper $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ will denote a bounded open region with C^1 boundary, $\partial\Omega$. Further we assume that

$$\Omega = \Omega_+ \cup \mathcal{S} \cup \Omega_-, \qquad \Omega_+ \cap \Omega_- = \emptyset,$$

where Ω_+ and Ω_- are open regions with C^1 boundaries and

$$\mathcal{S} = \overline{\Omega}_+ \cap \overline{\Omega}_- \cap \Omega$$

is a C^1 surface that separates the subregions Ω_+ and Ω_-. Further we assume that

$$\partial\Omega = \mathcal{D} \cup \Gamma, \tag{2.1}$$

where $\mathcal{D}$ and Γ are disjoint and Γ is relatively open.

Given any $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$ we write

$$\mathcal{B} := B_r(\mathbf{x}_0) := \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y}-\mathbf{x}_0| < r\}$$

for the ball of radius r centered at $\mathbf{x}_0$. We denote by $L^2(\mathcal{B},\mathbb{R}^n)$ the space (of equivalence classes) of square-integrable vector-valued functions on $\mathcal{B}$ and define

$$H_0^1(\mathcal{B},\mathbb{R}^n) := \text{completion of } \{\mathbf{u} \in C_0^\infty(\mathcal{B},\mathbb{R}^n) : \|\mathbf{u}\|_1 < \infty\}$$

with respect to the norm

$$\|\mathbf{u}\|_1^2 := \int_{\mathcal{B}} |\mathbf{u}(\mathbf{x})|^2 d\mathbf{x} + \int_{\mathcal{B}} |\nabla\mathbf{u}(\mathbf{x})|^2 d\mathbf{x}.$$

3. The Constitutive Relation

We consider a body that, for convenience, we identify with the region $\overline{\Omega}$ that it occupies in a fixed reference configuration. A (two-phase) *deformation* of the body is a member of the space

$$\mathrm{Def} := \{\mathbf{f} \in C^0(\overline{\Omega},\mathbb{R}^n) : \mathbf{f}^\pm \in C^2(\overline{\Omega}_\pm,\mathbb{R}^n),\ \det \nabla\mathbf{f}^\pm > 0\},$$

where

$$\underset{\sim}{f}^{+} : \overline{\Omega}_{+} \to \mathbb{R}^{n}, \quad \underset{\sim}{f}^{-} : \overline{\Omega}_{-} \to \mathbb{R}^{n}$$

denote the restrictions of $\underset{\sim}{f}$ to $\overline{\Omega}_{+}$ and $\overline{\Omega}_{-}$, respectively.

We assume that the subbodies $\overline{\Omega}_{-}$ and $\overline{\Omega}_{+}$ are each hyperelastic with continuous response functions $W^{\pm} : \overline{\Omega}_{\pm} \times \mathrm{Lin} \to \mathbb{R} \cup \{+\infty\}$. $W^{\pm}$ gives the *stored energy*

$$W^{\pm}(\underset{\sim}{x}, \nabla \underset{\sim}{f}^{\pm}(\underset{\sim}{x}))$$

at any point $\underset{\sim}{x} \in \overline{\Omega}_{\pm}$ when the body is deformed by $\underset{\sim}{f} \in \mathrm{Def}$.

We further assume that $W^{\pm}$ restricted to $\overline{\Omega}_{\pm} \times \mathrm{Lin}^{+}$ are C^{2} and finite-valued. The derivatives

$$\underset{\sim}{S}^{\pm}(\underset{\sim}{x}, \underset{\sim}{F}) = \frac{\partial}{\partial \underset{\sim}{F}} W^{\pm}(\underset{\sim}{x}, \underset{\sim}{F}), \quad \underset{\sim}{C}^{\pm}(\underset{\sim}{x}, \underset{\sim}{F}) := \frac{\partial^{2}}{\partial \underset{\sim}{F}^{2}} W^{\pm}(\underset{\sim}{x}, \underset{\sim}{F})$$

are called the (Piola-Kirchhoff) *stress* and the *elasticity tensor*, respectively. To simplify the notation we will occasionally drop the superscripts + and −; thus W, $\underset{\sim}{S}$, and $\underset{\sim}{C}$ will denote the stored energy, stress, and the elasticity tensor, respectively.

Given $\underset{\sim}{x}_{0} \in \overline{\Omega}$ and $\underset{\sim}{f} \in \mathrm{Def}$ we write $\underset{\sim}{C}_{0}^{\pm} := \underset{\sim}{C}^{\pm}(\underset{\sim}{x}_{0}, \nabla \underset{\sim}{f}^{\pm}(\underset{\sim}{x}_{0}))$. We say that $\underset{\sim}{C}_{0}$ satisfies the *Legendre-Hadamard condition* provided that

$$\underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underset{\sim}{C}_{0}[\underset{\sim}{a} \otimes \underset{\sim}{b}] \geq 0$$

whenever $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^{n}$. We say that $\underset{\sim}{C}_{0}$ is *strongly elliptic* provided that

$$\underset{\sim}{a} \otimes \underset{\sim}{b} \cdot \underset{\sim}{C}_{0}[\underset{\sim}{a} \otimes \underset{\sim}{b}] > 0$$

whenever $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^{n}$ with $\underset{\sim}{a} \neq \underset{\sim}{0}$ and $\underset{\sim}{b} \neq \underset{\sim}{0}$.

For $\underset{\sim}{x}_0 \in \mathcal{S}$ we write $\underset{\sim}{n}_0 = \underset{\sim}{n}(\underset{\sim}{x}_0)$ for the unit normal to $\mathcal{S}$ that points into Ω_- and $\mathcal{H}_\pm$ for the half-spaces

$$\mathcal{H}_+ := \{\underset{\sim}{y} \in \mathbb{R}^n : (\underset{\sim}{x}_0-\underset{\sim}{y})\cdot\underset{\sim}{n}_0 > 0\},\ \mathcal{H}_- := \{\underset{\sim}{y} \in \mathbb{R}^n : (\underset{\sim}{x}_0-\underset{\sim}{y})\cdot\underset{\sim}{n}_0 < 0\}.$$

The remark in the introduction then motivates us to consider the problem: Find $\underset{\sim}{w} \in C^0(\mathbb{R}^n,\mathbb{C}^n) \cap C^2(\overline{\mathcal{H}}_+,\mathbb{C}^n) \cap C^2(\overline{\mathcal{H}}_-,\mathbb{C}^n)$ that satisfies

$$\begin{aligned} \operatorname{div} \underset{\sim}{C}_0^\pm[\nabla\underset{\sim}{w}] &= \alpha^2\underset{\sim}{w} \text{ in } \mathcal{H}_\pm, \\ \underset{\sim}{C}_0^+[\nabla\underset{\sim}{w}^+]\underset{\sim}{n}_0 &= \underset{\sim}{C}_0^-[\nabla\underset{\sim}{w}^-]\underset{\sim}{n}_0 \text{ on } \partial\mathcal{H}\ , \end{aligned} \tag{3.1}$$

where $\alpha > 0$ and $\underset{\sim}{w}^+$ and $\underset{\sim}{w}^-$ are the restrictions of $\underset{\sim}{w}$ to $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively.

We are specifically interested in whether or not there are solutions of (3.1) that are <u>bounded</u> <u>exponentials</u>, that is, functions

$$\underset{\sim}{w}(\underset{\sim}{y}) = \underset{\sim}{z}((\underset{\sim}{x}_0-\underset{\sim}{y})\cdot\underset{\sim}{n}_0)\exp(i(\underset{\sim}{y}-\underset{\sim}{x}_0)\cdot\underset{\sim}{t}) \tag{3.2}$$

for some $\underset{\sim}{t} \in \mathbb{R}^n$ that is <u>tangent</u> to $\partial\mathcal{H}$ ($\underset{\sim}{t}\cdot\underset{\sim}{n}_0 = 0$) and some $\underset{\sim}{z} \in C^0(\mathbb{R},\mathbb{C}^n) \cap C^2([0,\infty),\mathbb{C}^n) \cap C^2((-\infty,0],\mathbb{C}^n)$ satisfying $\sup_s |\underset{\sim}{z}(s)| < +\infty$.

We say that the triple $(\underset{\sim}{C}_0^+,\underset{\sim}{C}_0^-,\underset{\sim}{n}_0)$ satisfies *Agmon's condition*[4] *at an internal boundary* provided that for every real $\alpha > 0$ the only bounded exponential solution of (3.1) is $\underset{\sim}{w} = \underset{\sim}{0}$.

<u>Remark</u>. Suppose that (3.1) has a bounded exponential solution $\underset{\sim}{w}_\alpha$ for some $\alpha > 0$. Then (3.1) has such a solution for <u>all</u> $\alpha > 0$.

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[4] <u>Cf</u>. <u>Agmon</u> [1, p. 123].

The required solution is given by

$$\mathbf{w}_\beta(\mathbf{x}) = \mathbf{w}_\alpha\left(\frac{\beta}{\alpha}(\mathbf{x}-\mathbf{x}_0) + \mathbf{x}_0\right).$$

4. The Energy. Minimizers

We assume that in each deformation $\mathbf{f}$ the environment exerts a body force $\mathbf{b}_{\mathbf{f}}(\mathbf{x})$ on the points $\mathbf{x} \in \bar{\Omega}$ and a surface force $\mathbf{s}_{\mathbf{f}}(\mathbf{x})$ on the points $\mathbf{x} \in \Gamma$ (*cf*. (2.1)). We assume further that there are potentials $\beta \in C^2(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R})$ and $\sigma \in C^2(\bar{\Gamma} \times \mathbb{R}^n, \mathbb{R})$ such that

$$\mathbf{b}_{\mathbf{f}}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{f}}\beta(\mathbf{x},\mathbf{f}(\mathbf{x})), \quad \mathbf{s}_{\mathbf{f}}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{f}}\sigma(\mathbf{x},\mathbf{f}(\mathbf{x}))$$

for any $\mathbf{f} \in \mathrm{Def}$. We let

$$E(\mathbf{f}) := \int_{\Omega_+} W^+(\nabla\mathbf{f}^+) + \int_{\Omega_-} W^-(\nabla\mathbf{f}^-) - \int_{\Omega} \beta(\mathbf{f}) - \int_{\Gamma} \sigma(\mathbf{f}) \tag{4.1}$$

denote the total energy when the body is deformed by $\mathbf{f}$.

We are interested in a deformation $\mathbf{f}$ that minimizes the total energy among certain classes of competing deformations, *i.e.*,

$$E(\mathbf{f}) \leq E(\mathbf{f}+\mathbf{u}) \tag{4.2}$$

for certain variations

$$\mathbf{u} \in \mathrm{Var} := \{\mathbf{u} \in C^1(\bar{\Omega},\mathbb{R}^n) : \mathbf{u} = \mathbf{0} \text{ on } \mathcal{D}\}.$$

Definition. Let $\mathbf{f} \in C^0(\bar{\Omega},\mathbb{R}^n)$ be such that $\mathbf{f}^\pm \in C^1(\bar{\Omega}_\pm,\mathbb{R}^n)$ and $E(\mathbf{f}) < +\infty$. We say that $\mathbf{f}$ is a

(i) strong local minimizer provded that there is an $\epsilon > 0$ such that (4.2) is satisfied by all $\mathbf{u} \in \mathrm{Var}$ with

$$\sup_{\mathbf{x}\in\Omega} |\mathbf{u}(\mathbf{x})| < \epsilon;$$

(ii) weak local minimizer provided that there is an $\epsilon > 0$ such that (4.2) is satisfied by all $\underset{\sim}{u} \in \mathrm{Var}$ with

$$\sup_{\underset{\sim}{x} \in \Omega} |\underset{\sim}{u}(\underset{\sim}{x})| + \sup_{\underset{\sim}{x} \in \Omega} |\nabla \underset{\sim}{u}(\underset{\sim}{x})| < \epsilon.$$

Given any $\underset{\sim}{x}_0 \in \Omega \backslash \mathcal{S}$, $\underset{\sim}{f} \in \mathrm{Def}$, and $r > 0$ we define $\underset{\sim}{F}_0 := \nabla \underset{\sim}{f}(\underset{\sim}{x}_0)$,

$$\underset{\sim}{f}_0(\underset{\sim}{y}) := \underset{\sim}{F}_0(\underset{\sim}{y} - \underset{\sim}{x}_0) + \underset{\sim}{f}(\underset{\sim}{x}_0), \qquad W_0(\underset{\sim}{G}) := W(\underset{\sim}{x}_0, \underset{\sim}{G})$$

for any $\underset{\sim}{G} \in \mathrm{Lin}^+$ and

$$\underset{\sim}{y} \in \mathcal{B} := \{\underset{\sim}{y} \in \mathbb{R}^n : |\underset{\sim}{y} - \underset{\sim}{x}_0| < r\}.$$

We interpret $\underset{\sim}{f}_0$ as a homogeneous deformation of a homogeneous body that occupies the region $\mathcal{B}$ in a homogeneous reference configuration.

If $\underset{\sim}{x}_0 \in \mathcal{S}$ we write $\underset{\sim}{F}_0^+ := \nabla \underset{\sim}{f}^+(\underset{\sim}{x}_0)$, $\underset{\sim}{F}_0^- := \nabla \underset{\sim}{f}^-(\underset{\sim}{x}_0)$,

$$\underset{\sim}{f}_0^+(\underset{\sim}{y}) := \underset{\sim}{F}_0^+(\underset{\sim}{y} - \underset{\sim}{x}_0) + \underset{\sim}{f}(\underset{\sim}{x}_0), \qquad W_0^+(\underset{\sim}{G}) := W^+(\underset{\sim}{x}_0, \underset{\sim}{G}),$$
$$\underset{\sim}{f}_0^-(\underset{\sim}{z}) := \underset{\sim}{F}_0^-(\underset{\sim}{z} - \underset{\sim}{x}_0) + \underset{\sim}{f}(\underset{\sim}{x}_0), \qquad W_0^-(\underset{\sim}{G}) := W^-(\underset{\sim}{x}_0, \underset{\sim}{G})$$

for any $\underset{\sim}{G} \in \mathrm{Lin}^+$,

$$\underset{\sim}{y} \in \mathcal{HB}_+ := \{\underset{\sim}{y} \in \mathcal{B} : (\underset{\sim}{x}_0 - \underset{\sim}{y}) \cdot \underset{\sim}{n}_0 > 0\},$$
$$\underset{\sim}{z} \in \mathcal{HB}_- := \{\underset{\sim}{z} \in \mathcal{B} : (\underset{\sim}{x}_0 - \underset{\sim}{z}) \cdot \underset{\sim}{n}_0 < 0\}.$$

We interpret the pair $(\underset{\sim}{f}_0^+, \underset{\sim}{f}_0^-)$ as a piecewise homogeneous deformation of the piecewise homogeneous body that occupies the region $\mathcal{B}$ in a piecewise homogeneous reference configuration.

<u>Proposition 4.1</u>. <u>Let $\underset{\sim}{f}$ be a strong local minimizer. Then,</u>[5] <u>for every</u> $\underset{\sim}{x}_0 \in \Omega\backslash\mathcal{S}$,

$$\int_{\mathcal{B}} W_0(\underset{\sim}{F}_0)d\underset{\sim}{y} \le \int_{\mathcal{B}} W_0(\underset{\sim}{F}_0+\nabla\underset{\sim}{u}(\underset{\sim}{y}))d\underset{\sim}{y} \qquad (4.3)^6$$

<u>for all</u> $\underset{\sim}{u} \in C_0^1(\mathcal{B},\mathbb{R}^n)$ <u>such that</u> $W_0(\underset{\sim}{F}_0+\nabla\underset{\sim}{u}(\cdot))$ <u>is uniformly bounded on</u> $\overline{\mathcal{B}}$. <u>Moreover,</u>[7] <u>for every</u> $\underset{\sim}{x}_0 \in \mathcal{S}$,

$$\begin{bmatrix} \int_{\mathcal{HB}_+} W_0^+(\underset{\sim}{F}_0^+)d\underset{\sim}{y} \\ + \\ \int_{\mathcal{HB}_-} W_0^-(\underset{\sim}{F}_0^-)d\underset{\sim}{y} \end{bmatrix} \le \begin{bmatrix} \int_{\mathcal{HB}_+} W_0^+(\underset{\sim}{F}_0^+ +\nabla\underset{\sim}{u}(\underset{\sim}{y}))d\underset{\sim}{y} \\ + \\ \int_{\mathcal{HB}_-} W_0^-(\underset{\sim}{F}_0^- +\nabla\underset{\sim}{u}(\underset{\sim}{y}))d\underset{\sim}{y} \end{bmatrix} \qquad (4.4)$$

<u>for all</u> $\underset{\sim}{u} \in C_0^1(\mathcal{B},\mathbb{R}^n)$ <u>such that</u> $W_0^\pm(\underset{\sim}{F}_0^\pm+\nabla\underset{\sim}{u}(\cdot))$ <u>are uniformly bounded on</u> $\overline{\mathcal{HB}}_\pm$, <u>respectively</u>.

In other words, in order for $\underset{\sim}{f}$ to be a strong local minimizer each of the homogeneous deformations $\underset{\sim}{f}_0(\underset{\sim}{y}) = [\nabla\underset{\sim}{f}(\underset{\sim}{x}_0)](\underset{\sim}{y}-\underset{\sim}{x}_0) + \underset{\sim}{f}(\underset{\sim}{x}_0)$, $\underset{\sim}{x}_0 \in \Omega\backslash\mathcal{S}$, must be a <u>global</u> minimizer for the pure displacement problem on a ball composed of the material at $\underset{\sim}{x}_0$ and each of the piecewise homogeneous deformations $(\underset{\sim}{f}_0^+,\underset{\sim}{f}_0^-)$

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[5]This result is essentially due to <u>Meyers</u> [9, pp. 128-131]. See <u>Ball</u> [2, Theorem 2.1] or <u>Gurtin</u> [6, pp. 9-10] for a direct proof when W is finite-valued and <u>Ball</u> & <u>Marsden</u> [3] for a proof under our hypotheses.

[6]This inequality is <u>Morrey's</u> [10] <u>quasiconvexity</u> condition.

[7]This result is due to <u>Gurtin</u> [6, pp. 9-10]. See <u>Ball</u> & <u>Marsden</u> [3] for a proof under our hypotheses.

$(\underset{\sim}{f}_0^+(\underset{\sim}{y}) = [\nabla \underset{\sim}{f}^+(\underset{\sim}{x}_0)](\underset{\sim}{y}-\underset{\sim}{x}_0) + \underset{\sim}{f}(\underset{\sim}{x}_0),\ \underset{\sim}{f}_0^-(\underset{\sim}{y}) = [\nabla \underset{\sim}{f}^-(\underset{\sim}{x}_0)](\underset{\sim}{y}-\underset{\sim}{x}_0) + \underset{\sim}{f}(\underset{\sim}{x}_0))$, $\underset{\sim}{x}_0 \in \mathcal{S}$, must be a <u>global</u> minimizer for the pure displacement problem on a pair of half-balls each of which is composed of one of the two materials at $\underset{\sim}{x}_0$.

We now suppose that $\underset{\sim}{f} \in \mathrm{Def}$ is a <u>weak</u> local minimizer. Then, for every $\underset{\sim}{u} \in \mathrm{Var}$

$$0 \leq \varphi(t) := E(\underset{\sim}{f}+t\underset{\sim}{u})-E(\underset{\sim}{f})$$

provided t is sufficiently small. Thus $\dot{\varphi}(0) = 0$ and $\ddot{\varphi}(0) \geq 0$. If $\underset{\sim}{f}$ is C^2 on $\Omega^+ \cup \Omega^-$ then one can show that $\underset{\sim}{f}$ is equilibrated, that is

$$\operatorname{div} \underset{\sim}{S}(\nabla \underset{\sim}{f}) + \underset{\sim}{b}_{\underset{\sim}{f}} = \underset{\sim}{0} \text{ in } \Omega^+ \cup \Omega^-,$$

$$\underset{\sim}{S}(\nabla \underset{\sim}{f})\underset{\sim}{N} = \underset{\sim}{s}_{\underset{\sim}{f}} \text{ on } \Gamma,$$

$$\underset{\sim}{S}^+(\nabla \underset{\sim}{f}^+)\underset{\sim}{n} = \underset{\sim}{S}^-(\nabla \underset{\sim}{f}^-)\underset{\sim}{n} \text{ on } \mathcal{S},$$

where $\underset{\sim}{N} = \underset{\sim}{N}(\underset{\sim}{x})$ is the outward unit normal to $\partial\Omega$ at $\underset{\sim}{x}$. In addition (whether or not $\underset{\sim}{f}$ is C^2) the second variation of E at $\underset{\sim}{f}$ must be nonnegative, <u>i</u>.<u>e</u>.,

$$0 \leq \int_{\Omega^+} \nabla\underset{\sim}{u}\cdot\underset{\sim}{C}^+[\nabla\underset{\sim}{u}] + \int_{\Omega^-} \nabla\underset{\sim}{u}\cdot\underset{\sim}{C}^-[\nabla\underset{\sim}{u}] - \int_{\Omega} \underset{\sim}{u}\cdot\underset{\sim}{B}\underset{\sim}{u} - \int_{\Gamma} \underset{\sim}{u}\cdot\underset{\sim}{\Sigma}\underset{\sim}{u} \tag{4.5}$$

for all $\underset{\sim}{u} \in \mathrm{Var}$, where

$$\underset{\sim}{B}(\underset{\sim}{x},\underset{\sim}{f}(\underset{\sim}{x})) := \frac{\partial^2}{\partial \underset{\sim}{f}^2}\beta(\underset{\sim}{x},\underset{\sim}{f}(\underset{\sim}{x})), \quad \underset{\sim}{\Sigma}(\underset{\sim}{x},\underset{\sim}{f}(\underset{\sim}{x})) := \frac{\partial^2}{\partial \underset{\sim}{f}^2}\sigma(\underset{\sim}{x},\underset{\sim}{f}(\underset{\sim}{x})).$$

If we now define

$$\hat{W}^{\pm}(\underset{\sim}{x},\underset{\sim}{U}) := \underset{\sim}{U}\cdot\underset{\sim}{C}^{\pm}(\underset{\sim}{x},\nabla\underset{\sim}{f}^{\pm}(\underset{\sim}{x}))[\underset{\sim}{U}],$$

$$\hat{\beta}(\underset{\sim}{x},\underset{\sim}{v}) := \underset{\sim}{v}\cdot\underset{\sim}{B}(\underset{\sim}{x},\underset{\sim}{f}(\underset{\sim}{x}))\underset{\sim}{v}, \qquad \hat{\sigma}(\underset{\sim}{x},\underset{\sim}{v}) := \underset{\sim}{v}\cdot\underset{\sim}{\Sigma}(\underset{\sim}{x},\underset{\sim}{f}(\underset{\sim}{x}))\underset{\sim}{v},$$

$$\hat{E}(\underset{\sim}{u}) := \int_{\Omega^+}\hat{W}^+(\nabla\underset{\sim}{u}) + \int_{\Omega^-}\hat{W}^-(\nabla\underset{\sim}{u}) - \int_{\Omega}\hat{\beta}(\underset{\sim}{u}) - \int_{\Gamma}\hat{\sigma}(\underset{\sim}{u}),$$

we find that (4.5) is equivalent to $\hat{E}(\underset{\sim}{0}) \le \hat{E}(\underset{\sim}{u})$ for all $\underset{\sim}{u} \in \text{Var}$. Thus the function $\underset{\sim}{0}$ is a strong local minimizer of $\hat{E}$. Therefore by Proposition 4.1, the density of $C_0^1(\mathscr{B},\mathbb{R}^n)$ in $H_0^1(\mathscr{B},\mathbb{R}^n)$, and the continuity of the resulting quadratic form on $H_0^1(\mathscr{B},\mathbb{R}^n)$ we arrive at the following result.

Proposition 4.2. Let $\underset{\sim}{f} \in \text{Def}$ be a weak local minimizer. Then, for every $\underset{\sim}{x}_0 \in \mathscr{S}$,

$$0 \le \int_{\mathscr{H}\mathscr{B}_+} \nabla\underset{\sim}{u}(\underset{\sim}{y})\cdot\underset{\sim}{C}_0^+[\nabla\underset{\sim}{u}(\underset{\sim}{y})]d\underset{\sim}{y} + \int_{\mathscr{H}\mathscr{B}_-} \nabla\underset{\sim}{u}(\underset{\sim}{y})\cdot\underset{\sim}{C}_0^-[\nabla\underset{\sim}{u}(\underset{\sim}{y})]d\underset{\sim}{y} \tag{4.6}$$

for all $\underset{\sim}{u} \in H_0^1(\mathscr{B},\mathbb{R}^n)$. Here $\underset{\sim}{C}_0^\pm := \underset{\sim}{C}^\pm(\underset{\sim}{x}_0,\nabla\underset{\sim}{f}^\pm(\underset{\sim}{x}_0))$.

5. Necessary Conditions

Theorem 1. Let $\underset{\sim}{f} \in \text{Def}$ be a weak local minimizer. Then, for every $\underset{\sim}{x}_0 \in \mathscr{S}$ $(= \bar{\Omega}_+ \cap \bar{\Omega}_- \cap \Omega)$,

(i) $\underset{\sim}{C}_0^+$ and $\underset{\sim}{C}_0^-$ satisfy the Legendre-Hadamard condition;

(ii) $(\underset{\sim}{C}_0^+,\underset{\sim}{C}_0^-,\underset{\sim}{n}_0)$ satisfies Agmon's condition;

(iii) if, for some $\underset{\sim}{a} \in \mathbb{R}^n$,

$$\underset{\sim}{a}\otimes\underset{\sim}{n}_0\cdot\underset{\sim}{C}_0^+[\underset{\sim}{a}\otimes\underset{\sim}{n}_0] = \underset{\sim}{a}\otimes\underset{\sim}{n}_0\cdot\underset{\sim}{C}^-[\underset{\sim}{a}\otimes\underset{\sim}{n}_0] = 0$$

then

$$\underset{\sim}{C}_0^+[\underset{\sim}{a}\otimes\underset{\sim}{n}_0] = \underset{\sim}{C}_0^-[\underset{\sim}{a}\otimes\underset{\sim}{n}_0];$$

(iv) if, for some $\underset{\sim}{a} \in \mathbb{R}^n$,

$$\underset{\sim}{a}\otimes\underset{\sim}{t}_0\cdot\underset{\sim}{C}_0^+[\underset{\sim}{a}\otimes\underset{\sim}{t}_0] = \underset{\sim}{a}\otimes\underset{\sim}{t}_0\cdot\underset{\sim}{C}_0^-[\underset{\sim}{a}\otimes\underset{\sim}{t}_0] = 0$$

then

$$\underset{\sim}{C}_0^+[\underset{\sim}{a}\otimes\underset{\sim}{t}_0]\underset{\sim}{n}_0 = \underset{\sim}{C}_0^-[\underset{\sim}{a}\otimes\underset{\sim}{t}_0]\underset{\sim}{n}_0 \; .$$

Here, $\underset{\sim}{C}_0^\pm := \underset{\sim}{C}^\pm(\underset{\sim}{x}_0,\nabla\underset{\sim}{f}^\pm(\underset{\sim}{x}_0))$ $(\underset{\sim}{C}^\pm = \partial^2 W^\pm/\partial(\nabla\underset{\sim}{f}^\pm)^2)$, $\underset{\sim}{n}_0 := \underset{\sim}{n}(\underset{\sim}{x}_0)$ the unit normal to $\mathscr{S}$ that points into Ω_-, and $\underset{\sim}{t}_0$ is tangent to $\mathscr{S}$ at $\underset{\sim}{x}_0$.

Proof.[8] We first apply Proposition 4.2 to conclude that (4.6) is satisfied at every $\underset{\sim}{x}_0 \in \mathscr{S}$. The Legendre-Hadamard condition is then a well-known consequence of (4.6).[9] We show that (ii)-(iv) also follow from (4.6). In order to simplify the notation we first change coordinates so that $\underset{\sim}{x}_0 = \underset{\sim}{0}$.

(4.6) ⇒ (ii). Let $b \in C^\infty(\mathbb{R},\mathbb{R})$ satisfy $b(s) = 1$ for $s \le r/3$ and $b(s) = 0$ for $s \ge 2r/3$. Define $\varphi(\underset{\sim}{x}) := b(|\underset{\sim}{x}|)$. Then given $\underset{\sim}{w} \in C^0(\mathbb{R}^n,\mathbb{C}^n) \cap C^2(\overline{\mathscr{H}}_+,\mathbb{C}^n) \cap C^2(\overline{\mathscr{H}}_-,\mathbb{C}^n)$ the functions $\underset{\sim}{u} := \mathrm{Re}\{\varphi\underset{\sim}{w}\}$ and $\underset{\sim}{v} := \mathrm{Im}\,\{\varphi\underset{\sim}{w}\}$ are contained in $H_0^1(\mathscr{B},\mathbb{R}^n)$ and a straightforward computation (using the symmetry of $\underset{\sim}{C}_0^{\pm}$) shows that

$$\begin{aligned}\int_{\mathscr{H}\mathscr{B}_\pm} (\nabla\underset{\sim}{u}\cdot\underset{\sim}{C}_0^{\pm}[\nabla\underset{\sim}{u}]+\nabla\underset{\sim}{v}\cdot\underset{\sim}{C}_0^{\pm}[\nabla\underset{\sim}{v}]) &= \int_{\mathscr{H}\mathscr{B}_\pm} \nabla(\varphi\overline{\underset{\sim}{w}})\cdot\underset{\sim}{C}_0^{\pm}[\nabla(\varphi\underset{\sim}{w})] \\ &= \int_{\mathscr{H}\mathscr{B}_\pm} (\overline{\underset{\sim}{w}}\otimes\nabla\varphi\cdot\underset{\sim}{C}_0^{\pm}[\underset{\sim}{w}\otimes\nabla\varphi] + \mathrm{Re}\{\nabla(\varphi^2\overline{\underset{\sim}{w}})\cdot\underset{\sim}{C}_0^{\pm}[\nabla\underset{\sim}{w}]\})\end{aligned} \tag{5.1}$$

and, by the divergence theorem,

$$\int_{\mathscr{H}\mathscr{B}_\pm} \nabla(\varphi^2\overline{\underset{\sim}{w}})\cdot\underset{\sim}{C}_0^{\pm}[\nabla\underset{\sim}{w}] = -\int_{\mathscr{H}\mathscr{B}_\pm} \varphi^2\overline{\underset{\sim}{w}}\cdot\mathrm{div}\;\underset{\sim}{C}_0^{\pm}[\nabla\underset{\sim}{w}] + \int_{\partial(\mathscr{H}\mathscr{B}_\pm)} \varphi^2\overline{\underset{\sim}{w}}\cdot\underset{\sim}{C}_0^{\pm}[\nabla\underset{\sim}{w}]\underset{\sim}{N}^{\pm}, \tag{5.2}$$

where $\underset{\sim}{N}^{\pm}$ is the outward unit normal to $\partial(\mathscr{H}\mathscr{B}_\pm)$.

---

[8] This proof is almost identical to the proof of Theorem 2 in [12] where necessary conditions at an (external) boundary for a minimizer are obtained.

[9] Cf., e.g., Truesdell & Noll [13, Section 68bis].

We now suppose that Agmon's condition is not satisfied and let $\underset{\sim}{w}_1$ be a bounded exponential solution of (3.1) with $\alpha = 1$ (see the remark following the definition of Agmon's condition). Then $\underset{\sim}{w}_\alpha(\underset{\sim}{x}) := \underset{\sim}{w}_1(\alpha\underset{\sim}{x})$ will be a bounded exponential solution of (3.1) for any $\alpha > 0$. Moreover, $|\underset{\sim}{w}_\alpha(\underset{\sim}{x})|$ is uniformly bounded independently of α.

We note that $\underset{\sim}{N}^+ = \underset{\sim}{n}_0$ and $\underset{\sim}{N}^- = -\underset{\sim}{n}_0$ on $\partial\mathcal{H}$. Thus by (3.1), (5.1), and (5.2) we find that there is a constant $k > 0$ such that

$$\int_{\mathcal{HB}_+} (\nabla\underset{\sim}{u}_\alpha\cdot\underset{\sim}{C}_0^+[\nabla\underset{\sim}{u}_\alpha]+\nabla\underset{\sim}{v}_\alpha\cdot\underset{\sim}{C}_0^+[\nabla\underset{\sim}{v}_\alpha]) + \int_{\mathcal{HB}_-} (\nabla\underset{\sim}{u}_\alpha\cdot\underset{\sim}{C}_0^-[\nabla\underset{\sim}{u}_\alpha]+\nabla\underset{\sim}{v}_\alpha\cdot\underset{\sim}{C}_0^-[\nabla\underset{\sim}{v}_\alpha])$$
$$\le k - \alpha^2\int_{\mathcal{B}} \varphi^2|\underset{\sim}{w}_\alpha|^2 \le k - \alpha^2\int_{B_{r/3}} |\underset{\sim}{w}_\alpha|^2, \tag{5.3}$$

where $B_{r/3}$ is the ball of radius $r/3$ centered at $\underset{\sim}{x}_0$.

We next note that $\underset{\sim}{w}_\alpha$ is of the form (3.2) and hence $|\underset{\sim}{w}_\alpha(\underset{\sim}{x})| = |\underset{\sim}{z}_\alpha(s)|$. Thus

$$\int_{B_{r/3}} |\underset{\sim}{w}_\alpha(\underset{\sim}{x})|^2 d\underset{\sim}{x} \ge (2r/3n)^{n-1}\int_{-r/3n}^{r/3n} |\underset{\sim}{z}_\alpha(s)|^2 ds$$
$$= (2r/3n)^{n-1}\int_{-r/3n}^{r/3n} |\underset{\sim}{z}_1(\alpha s)|^2 ds$$
$$= \alpha^{-1}(2r/3n)^{n-1}\int_{-\alpha r/3n}^{\alpha r/3n} |\underset{\sim}{z}_1(\sigma)|^2 d\sigma.$$

If we combine the last inequality with (5.3) we conclude that, for $\alpha \ge 3n$,

$$\left[\begin{array}{l} \displaystyle\int_{\mathcal{HB}_+} \{\nabla\underset{\sim}{u}_\alpha\cdot\underset{\sim}{C}_0^+[\nabla\underset{\sim}{u}_\alpha]+\nabla\underset{\sim}{v}_\alpha\cdot\underset{\sim}{C}_0^+[\nabla\underset{\sim}{v}_\alpha] \\ \displaystyle + \int_{\mathcal{HB}_-} \{\nabla\underset{\sim}{u}_\alpha\cdot\underset{\sim}{C}_0^-[\nabla\underset{\sim}{u}_\alpha]+\nabla\underset{\sim}{v}_\alpha\cdot\underset{\sim}{C}_0^-[\nabla\underset{\sim}{v}_\alpha] \end{array}\right] \le k-\alpha(2r/3n)^{n-1}\int_{-r}^{r} |\underset{\sim}{z}_1(\sigma)|^2 d\sigma.$$

Therefore, by choosing α sufficiently large, we can find a function in $H_0^1(\mathcal{B},\mathbb{R}^n)$ for which (4.6) is not satisfied.

(4.6) $\Rightarrow$ (iv). Suppose that $\mathbf{a} \in \mathbb{R}^n$ satisfies

$$\mathbf{a}\otimes\mathbf{t}_0\cdot\mathbf{C}_0^+[\mathbf{a}\otimes\mathbf{t}_0] = \mathbf{a}\otimes\mathbf{t}_0\cdot\mathbf{C}_0^-[\mathbf{a}\otimes\mathbf{t}_0] = 0, \tag{5.4}$$

where $\mathbf{t}_0 \perp \mathbf{n}_0$ and $|\mathbf{t}_0| = 1$. Let $\mathbf{b} \in \mathbb{R}^n$. Choose coordinates so that $\mathbf{n}_0 = (-1,0,0,\dots,0)$, $\mathbf{t}_0 = (0,1,0,0,\dots,0)$ and write $\mathbf{x} = (s,t,\mathbf{z})$, where $s,t \in \mathbb{R}$ and $\mathbf{z} \in \mathbb{R}^{n-2}$. Let $\varphi,\psi \in C^\infty(\mathbb{R},\mathbb{R})$ and $\rho \in C^\infty(\mathbb{R}^{n-2},\mathbb{R})$ satisfy (recall that r is the radius of the ball $\mathcal{B}$)

$$\left.\begin{aligned}
\int_{-1}^{1} |\dot\psi(t)|^2 dt &= 1, \quad \psi(t) = 0 \quad \text{for } |t| \geq 1,\\
\varphi(0) &= 1, \quad \varphi(s) = 0 \quad \text{for } |s| \geq 1,\\
\int_{|\mathbf{z}|\leq r} |\rho(\mathbf{z})|^2 d\mathbf{z} &= 1, \quad \rho(\mathbf{z}) = 0 \quad \text{for } |\mathbf{z}| \geq r/3.
\end{aligned}\right\} \tag{5.5}$$

For $\epsilon > 0$, $\delta > 0$, and $\beta > 0$ define

$$\mathbf{u}_{\delta,\epsilon}^{\beta}(s,t,\mathbf{z}) = \rho(\mathbf{z})[\mathbf{a}\delta^{-1}\varphi(\tfrac{\epsilon s}{\beta})\psi(\tfrac{t}{\beta}) + \mathbf{b}\delta\varphi(\tfrac{s}{\beta})\dot\psi(\tfrac{t}{\beta})]$$

Then $\mathbf{u}_{\delta,\epsilon}^{\beta} \in C_0^\infty(\overline{\mathcal{B}},\mathbb{R}^n)$ (provided $\beta < r$ and $\epsilon > \beta/r$) and

$$\nabla\mathbf{u}_{\delta,\epsilon}^{\beta} = \begin{bmatrix} \beta^{-1}\rho(\mathbf{a}\otimes\mathbf{n}_0\delta^{-1}\epsilon\dot\varphi\psi + \mathbf{b}\otimes\mathbf{n}_0\delta\dot\varphi\dot\psi) \\ + \beta^{-1}\rho(\mathbf{a}\otimes\mathbf{t}_0\delta^{-1}\varphi\dot\psi + \mathbf{b}\otimes\mathbf{t}_0\delta\varphi\ddot\psi) \\ + (\mathbf{a}\delta^{-1}\varphi\psi + \mathbf{b}\delta\varphi\dot\psi)\otimes\nabla\rho \end{bmatrix}. \tag{5.6}$$

Now, by (4.6),

$$0 \leq \int_{|\mathbf{z}|\leq r} d\mathbf{z} \int_{-\beta}^{\beta} dt \left[\int_0^\beta \nabla\mathbf{u}_{\delta,\epsilon}^{\beta}\cdot\mathbf{C}_0^+[\nabla\mathbf{u}_{\delta,\epsilon}^{\beta}]ds + \int_{-\beta}^{0}\nabla\mathbf{u}_{\delta,\epsilon}^{\beta}\cdot\mathbf{C}_0^-[\nabla\mathbf{u}_{\delta,\epsilon}^{\beta}]ds\right].$$

We make the change of variables $\sigma = s/\beta$ and $\tau = t/\beta$ (and hence $dsdt = \beta^2 d\sigma d\tau$) and then let $\beta \to 0^+$ in the last inequality to conclude, with the aid of (5.4), (5.5), and (5.6), that (assuming $\epsilon < 1$)

$$
\begin{aligned}
&\mathbf{b}\otimes\mathbf{n}_0\cdot\mathbf{C}_0^{+}[\mathbf{b}\otimes\mathbf{n}_0]\delta^2\int_0^1\dot{\varphi}^2+\mathbf{a}\otimes\mathbf{n}_0\cdot\mathbf{C}_0^{+}[\mathbf{a}\otimes\mathbf{n}_0]\delta^{-2}\epsilon^2\int_0^{1/\epsilon}|\dot{\varphi}(\sigma\epsilon)|^2 d\sigma\int_{-1}^1\psi^2\\
&\quad+\mathbf{b}\otimes\mathbf{n}_0\cdot\mathbf{C}_0^{-}[\mathbf{b}\otimes\mathbf{n}_0]\delta^2\int_{-1}^0\dot{\varphi}^2+\mathbf{a}\otimes\mathbf{n}_0\cdot\mathbf{C}_0^{-}[\mathbf{a}\otimes\mathbf{n}_0]\delta^{-2}\epsilon^2\int_{-1/\epsilon}^0|\dot{\varphi}(\sigma\epsilon)|^2 d\sigma\int_{-1}^1\psi^2\\
&\quad+\mathbf{b}\otimes\mathbf{t}_0\cdot\mathbf{C}_0^{+}[\mathbf{b}\otimes\mathbf{t}_0]\delta^2\int_0^1\varphi^2\int_{-1}^1\ddot{\psi}^2+2\mathbf{b}\otimes\mathbf{n}_0\cdot\mathbf{C}_0^{+}[\mathbf{a}\otimes\mathbf{t}_0]\int_0^1\dot{\varphi}(\sigma)\varphi(\sigma\epsilon)d\sigma\\
&\quad+\mathbf{b}\otimes\mathbf{t}_0\cdot\mathbf{C}_0^{-}[\mathbf{b}\otimes\mathbf{t}_0]\delta^2\int_{-1}^0\varphi^2\int_{-1}^1\ddot{\psi}^2+2\mathbf{b}\otimes\mathbf{n}_0\cdot\mathbf{C}_0^{-}[\mathbf{a}\otimes\mathbf{t}_0]\int_{-1}^0\dot{\varphi}(\sigma)\varphi(\sigma\epsilon)d\sigma\\
&\quad-2\mathbf{b}\otimes\mathbf{t}_0\cdot\mathbf{C}_0^{+}[\mathbf{a}\otimes\mathbf{n}_0]\epsilon\int_0^1\dot{\varphi}(\sigma\epsilon)\varphi(\sigma)d\sigma\\
&\quad-2\mathbf{b}\otimes\mathbf{t}_0\cdot\mathbf{C}_0^{-}[\mathbf{a}\otimes\mathbf{n}_0]\epsilon\int_{-1}^0\dot{\varphi}(\sigma\epsilon)\varphi(\sigma)d\sigma\geq 0.
\end{aligned}
\tag{5.7}
$$

In deriving the above inequality we have made use of the identities

$$\int_{-1}^1\psi(\tau)\dot{\psi}(\tau)d\tau=\int_{-1}^1\dot{\psi}(\tau)\ddot{\psi}(\tau)d\tau=0,\quad\int_{-1}^1\psi(\tau)\ddot{\psi}(\tau)d\tau=-1,$$

which are consequences of $(5.5)_1$.

We note that the change of variables $y = \sigma\epsilon$ yields

$$
\begin{aligned}
\epsilon^2\int_0^{1/\epsilon}|\dot{\varphi}(\sigma\epsilon)|^2 d\sigma &= \epsilon\int_0^1|\dot{\varphi}(y)|^2 dy,\\
\epsilon^2\int_{-1/\epsilon}^0|\dot{\varphi}(\sigma\epsilon)|^2 d\sigma &= \epsilon\int_{-1}^0|\dot{\varphi}(y)|^2 dy,
\end{aligned}
\tag{5.8}
$$

and by the bounded convergence theorem and $(5.5)_2$ we find that

$$
\begin{aligned}
\lim_{\epsilon\to 0^+}\int_0^1\dot{\varphi}(\sigma)\varphi(\sigma\epsilon)d\sigma &= \varphi(0)\int_0^1\dot{\varphi}(y)dy=-1,\\
\lim_{\epsilon\to 0^+}\int_{-1}^0\dot{\varphi}(\sigma)\varphi(\sigma\epsilon)d\sigma &= \varphi(0)\int_{-1}^0\dot{\varphi}(y)dy=+1.
\end{aligned}
\tag{5.9}
$$

Finally, we note that φ and $\dot{\varphi}$ are bounded and let $\epsilon \to 0^+$ followed by $\delta \to 0^+$ in (5.7) to conclude, with the aid of (5.8) and (5.9), that

$$2\underset{\sim}{b}\otimes\underset{\sim}{n}_0\cdot\underset{\approx}{C}_0^-[\underset{\sim}{a}\otimes\underset{\sim}{t}_0]-2\underset{\sim}{b}\otimes\underset{\sim}{n}_0\cdot\underset{\approx}{C}_0^+[\underset{\sim}{a}\otimes\underset{\sim}{t}_0] \geq 0$$

and hence that

$$\underset{\sim}{b}\cdot(\underset{\approx}{C}_0^-[\underset{\sim}{a}\otimes\underset{\sim}{t}_0]\underset{\sim}{n}_0-\underset{\approx}{C}_0^+[\underset{\sim}{a}\otimes\underset{\sim}{t}_0]\underset{\sim}{n}_0) \geq 0.$$

Since $\underset{\sim}{b}$ is arbitrary we find that

$$\underset{\approx}{C}_0^+[\underset{\sim}{a}\otimes\underset{\sim}{t}_0]\underset{\sim}{n}_0 = \underset{\approx}{C}_0^-[\underset{\sim}{a}\otimes\underset{\sim}{t}_0]\underset{\sim}{n}_0 .$$

(4.6) ⇒ (iii). The variations $\underset{\sim}{u}^{\beta}_{\delta,\epsilon}$ that were used to prove that (iv) is a consequence of (4.6) will also yield this implication upon letting $\beta \to 0^+$ followed by $\epsilon \to +\infty$ and then $\delta \to 0^+$. See the proof of Theorem 2 in [12] for the details. □

6. Sufficient Conditions

In this section we briefly consider the problem of finding algebraic conditions on $\underset{\approx}{C}_0^+$ and $\underset{\approx}{C}_0^-$ that are sufficient for the nonnegativity of the second variation of the energy of a piecewise homogeneous body that occupies the region $\mathfrak{B}$ in a piecewise homogeneous reference configuration; _i_._e_., conditions to ensure that

$$0 \leq \int_{\mathcal{H}\mathfrak{B}_+} \nabla\underset{\sim}{u}(\underset{\sim}{y})\cdot\underset{\approx}{C}_0^+[\nabla\underset{\sim}{u}(\underset{\sim}{y})]d\underset{\sim}{y} + \int_{\mathcal{H}\mathfrak{B}_-} \nabla\underset{\sim}{u}(\underset{\sim}{y})\cdot\underset{\approx}{C}_0^-[\nabla\underset{\sim}{u}(\underset{\sim}{y})]d\underset{\sim}{y} \qquad (6.1)$$

whenever $\underset{\sim}{u} \in H_0^1(\mathfrak{B},\mathbb{R}^n)$.

We do not know if conditions (i)-(iv) of Theorem 1 are sufficient for (6.1) to be satisfied. However we would like to note that the techniques of [4, 12] can be used to show that when $\mathbf{C}_0^+$ and $\mathbf{C}_0^-$ are strongly elliptic Agmon's condition alone implies that (6.1) is satisfied by all $\mathbf{u} \in H_0^1(\mathscr{B},\mathbb{R}^n)$.

Theorem 2. *Suppose that $\mathbf{C}_0^+$ and $\mathbf{C}_0^-$ are strongly elliptic. Then the following are equivalent:*

(a) (6.1) *is satisfied by all* $\mathbf{u} \in H_0^1(\mathscr{B},\mathbb{R}^n)$;

(b) $(\mathbf{C}_0^+,\mathbf{C}_0^-,\mathbf{n})$ *satisfies Agmon's condition;*

(c) *If* $\mathbf{t} \perp \mathbf{n}_0$ *and* $\mathbf{z} \in C^0(\mathbb{R},\mathbb{C}^n) \cap C^2([0,\infty),\mathbb{C}^n) \cap C^2((-\infty,0],\mathbb{C}^n)$ *is any bounded solution of*

$$\left.\begin{aligned} -\mathbf{M}^+\ddot{\mathbf{z}}+i[\mathbf{N}_t^++(\mathbf{N}_t^+)^T]\dot{\mathbf{z}} + \mathbf{P}_t^+\mathbf{z} &= \mathbf{0} \text{ on } (0,\infty), \\ -\mathbf{M}^-\ddot{\mathbf{z}}+i[\mathbf{N}_t^-+(\mathbf{N}_t^-)^T]\dot{\mathbf{z}} + \mathbf{P}_t^-\mathbf{z} &= \mathbf{0} \text{ on } (-\infty,0) \end{aligned}\right\} \tag{6.2}$$

then

$$\bar{\mathbf{z}}(0)\cdot[(\mathbf{M}^+\dot{\mathbf{z}}(0^+)-i\mathbf{N}_t^+\mathbf{z}(0))-(\mathbf{M}^-\dot{\mathbf{z}}(0^-)-i\mathbf{N}_t^-\mathbf{z}(0))] \leq 0. \tag{6.3}$$

Here

$$\begin{gathered} \mathbf{M}^\pm\mathbf{a} := \mathbf{C}_0^\pm[\mathbf{a}\otimes\mathbf{n}_0]\mathbf{n}_0, \quad \mathbf{N}_t^\pm\mathbf{a} := \mathbf{C}_0^\pm[\mathbf{a}\otimes\mathbf{t}]\mathbf{n}_0, \\ \mathbf{P}_t^\pm\mathbf{a} := \mathbf{C}_0^\pm[\mathbf{a}\otimes\mathbf{t}]\mathbf{t}. \end{gathered} \tag{6.4}$$

Remark. Agmon's condition can be difficult to verify for particular consitutive relations. We have found that condition (c) of Theorem 2 is sometimes simpler to check.

7. An Internal Boundary Between Two Isotropic Materials

We let

$$\mathbf{C}_0^{\pm}[\mathbf{H}] := \mu^{\pm}(\mathbf{H}+\mathbf{H}^T)+\lambda^{\pm}(\mathbf{H}\cdot\mathbf{I})\mathbf{I} \tag{7.1}$$

and consider the problem of finding necessary and sufficient conditions on $\mu^{\pm}$ and $\lambda^{\pm}$ so that (6.1) is satisfied whenever $\mathbf{u} \in H_0^1(\mathcal{B},\mathbb{R}^n)$.

Lemma 7.1. *Let (6.1) be satisfied by all $\mathbf{u} \in H_0^1(\mathcal{B},\mathbb{R}^n)$ where $\mathbf{C}_0^{\pm}$ are given by (7.1). Then*

$$\mu^{\pm} \geq 0, \qquad 2\mu^{\pm}+\lambda^{\pm} \geq 0. \tag{7.2}$$

Proof. We let

$$W^{\pm}(\mathbf{x},\mathbf{U}) := \mathbf{U}\cdot\mathbf{C}_0^{\pm}[\mathbf{U}]$$

and apply Theorem 1 to conclude that $\mathbf{C}_0^{+}$ and $\mathbf{C}_0^{-}$ each satisfy the Legendre-Hadamard condition. Thus

$$\mu^{\pm}|\mathbf{a}|^2|\mathbf{b}|^2+(\mu^{\pm}+\lambda^{\pm})(\mathbf{a}\cdot\mathbf{b})^2 \geq 0$$

for every $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$. In particular we let $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ satisfy $\mathbf{a}\cdot\mathbf{b} = 0$ to get $(7.2)_1$, while the choice $\mathbf{a} = \mathbf{b} \neq \mathbf{0}$ yields $(7.2)_2$. □

Proposition 7.2. *Let $\mathbf{C}_0^{\pm}$ be given by (7.1). Suppose that*

$$\mu^{\pm} > 0, \qquad 2\mu^{\pm}+\lambda^{\pm} > 0. \tag{7.3}$$

Then necessary and sufficient conditions for (6.1) to be satisfied for all $\mathbf{u} \in H_0^1(\mathcal{B},\mathbb{R}^n)$ are

$$\mu^{-} \geq \frac{-\mu^{+}(\mu^{+}+\lambda^{+})}{3\mu^{+}+\lambda^{+}}, \qquad \mu^{+} \geq \frac{-\mu^{-}(\mu^{-}+\lambda^{-})}{3\mu^{-}+\lambda^{-}}. \tag{7.4}$$

Proof. A standard result[10] shows that (7.3) is equivalent to the strong ellipticity of $\underset{\sim}{C}_0^{\pm}$. We will show that (7.4) is equivalent to condition (c) of Theorem 2 and the desired result will follow from Theorem 2.

For simplicity we suppose that n = 3 and choose coordinates so that $\underset{\sim}{n} = (-1,0,0)$ and $\underset{\sim}{t} = (0,\tau,0)$ where $\tau \geq 0$. Then (6.4) and (7.1) imply that

$$\left.\begin{aligned} \underset{\sim}{M}\underset{\sim}{a} &= \mu\underset{\sim}{a}+(\mu+\lambda)(\underset{\sim}{a}\cdot\underset{\sim}{n})\underset{\sim}{n}, \\ \underset{\sim}{N}_{\underset{\sim}{t}}\underset{\sim}{a} &= \mu(\underset{\sim}{a}\cdot\underset{\sim}{n})\underset{\sim}{t}+\lambda(\underset{\sim}{a}\cdot\underset{\sim}{t})\underset{\sim}{n}, \\ \underset{\sim}{P}_{\underset{\sim}{t}}\underset{\sim}{a} &= \mu\tau^2\underset{\sim}{a}+(\mu+\lambda)(\underset{\sim}{a}\cdot\underset{\sim}{t})\underset{\sim}{t}, \end{aligned}\right\} \tag{7.5}$$

and hence (6.2) reduces to

$$\left.\begin{aligned} -(2\mu+\lambda)\ddot{z}_1-i\tau(\mu+\lambda)\dot{z}_2+\mu\tau^2 z_1 &= 0, \\ -\mu\ddot{z}_2-i\tau(\mu+\lambda)\dot{z}_1+(2\mu+\lambda)\tau^2 z_2 &= 0, \\ -\ddot{z}_3+\tau^2 z_3 &= 0. \end{aligned}\right\} \tag{7.6}$$

We first note that if $\tau = 0$ then the general bounded solution to (7.6) is $\underset{\sim}{z}(s) = \underset{\sim}{c}$ for all $s \in \mathbb{R}$ and hence we conclude, with the aid of $(7.5)_2$, that (6.3) is satisfied.

We next note that when $\mu+\lambda = 0$ (and $\tau > 0$) the general bounded solution to (7.6) is

$$\underset{\sim}{z}(s) = \underset{\sim}{c}\, \exp(-\tau|s|), \tag{7.7}$$

while when $\mu+\lambda \neq 0$ (and $\tau > 0$) the general bounded solution to

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[10] Cf., e.g., Gurtin [5, p. 86].

(7.6) is

$$\mathbf{z}(s) = \begin{cases} c_1^+\mathbf{z}_+^1(s)+c_2^+\mathbf{z}_+^2(s) + c_3^+\mathbf{z}^3(s), & s \geq 0 \\ c_1^-\mathbf{z}_-^1(s)+c_2^-\mathbf{z}_-^2(s) + c_3^-\mathbf{z}^3(s), & s \leq 0 \end{cases} \tag{7.8}$$

$$\begin{aligned} \mathbf{z}_\pm^1(s) &= (\pm i,1,0)\ \exp(\mp\tau s), \quad \mathbf{z}^3(s) = (0,0,1)\ \exp(-\tau|s|), \\ \mathbf{z}_\pm^2(s) &= \mp(\mu^\pm+\lambda^\pm)\tau s\ \mathbf{z}_\pm^1(s) + (\mp i\mu^\pm, 2\mu^\pm+\lambda^\pm, 0)\ \exp(\mp\tau s). \end{aligned} \tag{7.9}$$

<u>Case</u> <u>I</u>. $\mu^+ + \lambda^+ = \mu^- + \lambda^- = 0$. Then (7.5) and (7.7) imply that

$$\begin{aligned} \mathbf{M}^\pm\dot{\mathbf{z}}(0^\pm) &= \mp\mu^\pm\tau\mathbf{c}, \\ \mathbf{N}_{\mathbf{t}}^\pm\mathbf{z}(0) &= -\mu^\pm[c_1\mathbf{t}+c_2\tau\mathbf{n}], \end{aligned}$$

and hence (6.3) reduces to

$$(\mu^+ + \mu^-)|\mathbf{c}|^2 \geq i(\mu^+ - \mu^-)(c_1\bar{c}_2 - c_2\bar{c}_1).$$

Since μ^+ and μ^- are positive it is easy to show that the above inequality is always satisfied.

<u>Main</u> <u>Case</u>. $\mu^+ + \lambda^+ \neq 0$ and $\mu^- + \lambda^- \neq 0$. Then (7.5) and (7.8)-(7.9) imply that

$$\mathbf{z}(0^\pm) = \mathbf{D}^\pm\mathbf{c}^\pm, \qquad \mathbf{M}^\pm\dot{\mathbf{z}}(0^\pm) = -\mathbf{E}^\pm\mathbf{c}^\pm,$$

$$\mathbf{D}^\pm := \begin{bmatrix} \pm i & \mp i\mu^\pm & 0 \\ 1 & 2\mu^\pm+\lambda^\pm & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}^\pm := \tau\begin{bmatrix} i(2\mu^\pm+\lambda^\pm) & i\lambda^\pm(2\mu^\pm+\lambda^\pm) & 0 \\ \pm\mu^\pm & \pm\mu^\pm(3\mu^\pm+2\lambda^\pm) & 0 \\ 0 & 0 & \pm\mu^\pm \end{bmatrix}.$$

We note that $\mathbf{z}$ is continuous at $s = 0$. Therefore if we let

$$\mathbf{c}^+ := (\mathbf{D}^+)^{-1}\mathbf{z}(0), \qquad \mathbf{c}^- := (\mathbf{D}^-)^{-1}\mathbf{z}(0)$$

we find that (6.3) is equivalent to

$$\bar{\mathbf{z}}(0)\cdot[\mathbf{G}^{+}-\mathbf{G}^{-}]\mathbf{z}(0) \geq 0, \qquad (7.10)$$

$$\mathbf{G}^{\pm} := [\mathbf{E}^{\pm}(\mathbf{D}^{\pm})^{-1}+i\mathbf{N}_{t}^{\pm}].$$

Next, a simple calculation shows that

$$(\mathbf{D}^{\pm})^{-1} = \frac{1}{i(3\mu^{\pm}+\lambda^{\pm})}\begin{bmatrix} \pm(2\mu^{\pm}+\lambda^{\pm}) & i\mu^{\pm} & 0 \\ \mp 1 & i & 0 \\ 0 & 0 & i(3\mu^{\pm}+\lambda^{\pm}) \end{bmatrix}$$

and hence that

$$\mathbf{G}^{\pm} = \frac{\tau\mu^{\pm}}{3\mu^{\pm}+\lambda^{\pm}}\begin{bmatrix} \pm 2(2\mu^{\pm}+\lambda^{\pm}) & 2\mu^{\pm}i & 0 \\ -2\mu^{\pm}i & \pm 2(2\mu^{\pm}+\lambda^{\pm}) & 0 \\ 0 & 0 & \pm(3\mu^{\pm}+\lambda^{\pm}) \end{bmatrix}.$$

Since (7.3) implies that $(\mathbf{G}^{+}-\mathbf{G}^{-})_{jj} > 0$ $(j = 1,2,3)$, (7.10) will be satisfied for all initial values $\mathbf{z}(0)$ if and only if

$$\det(\mathbf{G}^{+}-\mathbf{G}^{-}) \geq 0. \qquad (7.11)$$

However

$$\det(\mathbf{G}^{+}-\mathbf{G}^{-}) = 4\tau^{3}(\mu^{+}+\mu^{-})\left[\mu^{+} + \frac{\mu^{-}(\mu^{-}+\lambda^{-})}{3\mu^{-}+\lambda^{-}}\right]\left[\mu^{-} + \frac{\mu^{+}(\mu^{+}+\lambda^{+})}{3\mu^{+}+\lambda^{+}}\right]$$

and hence we conclude, with the aid of (7.3), that (7.11) is equivalent to (7.4).

<u>Case</u> <u>III</u>. $\mu^{+}+\lambda^{+} = 0$ and $\mu^{-}+\lambda^{-} \neq 0$. We let $\mathbf{D}^{-}$, $\mathbf{E}^{-}$, and $\mathbf{G}^{-}$ be given by the previous case. If we use (7.7) and the fact that $\lambda^{+} = -\mu^{+}$ we find that

$$\mathbf{D}^{+} = \mathbf{I}, \quad \mathbf{E}^{+} = \tau\mu^{+}\mathbf{I},$$

and hence that

$$\mathbf{G}^{+} = \tau\mu^{+}\begin{bmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Clearly $(\mathbf{G}^{+}-\mathbf{G}^{-})_{jj} > 0$ $(j = 1,2,3)$ and

$$\det(\mathbf{G}^{+}-\mathbf{G}^{-}) = 4\tau^{3}\mu^{-}(\mu^{+}+\mu^{-})\left[\mu^{+} + \frac{\mu^{-}(\mu^{-}+\lambda^{-})}{3\mu^{-}+\lambda^{-}}\right].$$

Thus the desired result follows by the reasoning used in the previous case.

Case IV. $\mu^{+}+\lambda^{+} \neq 0$ and $\mu^{-}+\lambda^{-} = 0$. Same as Case III. □

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