A ROBUSTNESS PROPERTY OF THE TWO-SIDED T-TEST

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Abstract

Let R^n be Euclidean n-space and let $\mathfrak{O}(n)$ be the group of $n \times n$ orthogonal matrices. Consider $\mathfrak{F}_0 = \{f | f$ is a density on R^n, $f(x) = f(gx)$ for $g \in \mathfrak{O}(n)\}$. For $\mu \in R^1$, let $\mathfrak{F}_1(\mu) = \{f | f$ is a density on R^n, $f(x) = q(\|x - \mu e\|^2)$ where q is a decreasing function on $(0, \infty)$ and $e' = (1,1,\ldots,1)/\sqrt{n}\}$ and let $\mathfrak{F}_2(\mu) = \{f | f \in \mathfrak{F}_1(\mu)$ and q is convex on $(0, \infty)\}$. Let $X \in R^n$ have a density h. For testing $H_0: h \in \mathfrak{F}_0$ versus $H_2: h \in \{\mathfrak{F}_2(\mu), \mu \neq 0\}$ it is shown that the two-sided t-test is uniformly most powerful unbiased (UMPU). It is also conjectured that the two sided t-test is UMPU for testing $H_0: h \in \mathfrak{F}_0$ versus $H_1: h \{\mathfrak{F}_1(\mu), \mu \neq 0\}$.

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§ 1. Introduction and Summary

Throughout this paper, R^n will denote Euclidean n-space. To describe the results in this paper, we need to define three sets of probability densities (with repsect to Lebesgue measure) on R^n. First, with $\|x\|^2 = \sum_1^n x_i^2$, let

$$(1.1)\quad \mathfrak{F}_0 = \left\{ f \;\middle|\; \begin{array}{ll} f(x) \geq 0, & \int_{R^n} f(x)dx = 1 \\ f(x) = q(\|x\|^2), \; q & \text{on } [0, \infty) \end{array} \right\} .$$

Let $e' = (1,1\ldots1)'/\sqrt{n} \in R^n$ and let μ be a real number. Then let

$$(1.2)\quad \mathfrak{F}_1(\mu) = \left\{ f \;\middle|\; \begin{array}{l} f(x) = q(\|x-\mu e\|^2) \geq 0, \; \int_{R^n} f(x)dx = 1 , \\ q \text{ is a non-increasing function on } [0, \infty) \end{array} \right\}$$

and

$$(1.3)\quad \mathfrak{F}_2(\mu) = \left\{ f \;\middle|\; \begin{array}{l} f(x) = q(\|x-\mu e\|^2) \geq 0, \; \int_{R^n} f(x)dx = 1, \\ q \text{ is a non-increasing convex function on } [0, \infty) \end{array} \right\} .$$

Suppose $X \in R^n$ is a random vector with a density h. Let $T = e'X/\|X\|$ and $W = \|X\|^2$. For the problem of testing $H_0{:}h \in \mathfrak{F}_0$ versus the alternative $H_1{:}h \in \{\mathfrak{F}_1(\mu);\ \mu > 0\}$, Kariya and Eaton (1975) showed that the test which rejects H_0 for large values of T is a uniformly most powerful (UMP) test. Of course, this test is equivalent to rejecting for large values of the usual one-sample t-statistic. In Section 2, we show that the test which rejects for large values of T^2 is UMP unbiased for testing $H_0{:}h \in \mathfrak{F}_0$ versus $H_2{:}h \in \{\mathfrak{F}_2(\mu) | \mu \neq 0\}$. Section 3 is concerned with a discussion of the conjecture that rejecting for large values of T^2 gives a test which is UMP unbiased for testing $H_0:h \in \mathfrak{F}_0$ versus $H_1:h \in \{\mathfrak{F}_1(\mu) | \mu \neq 0\}$.

§ 2. A Property of the two sided t-test

The notation of Section 1 will be used in this section. We will consider the problem of testing $H_0:h \in \mathfrak{F}_0$ versus the alternative $H_2:h \in \{\mathfrak{F}_2(\mu) | \mu \neq 0\}$ at level α, $0 < \alpha < 1$. Let $\mathfrak{D}_\alpha$ be the class of test functions which are unbiased for the above testing problem.

Lemma 2.1: The pair (T,W) is a complete sufficient statistic for the family $\{\mathfrak{F}_1(\mu) | \mu \in R^1\}$. Further, W is a complete sufficient statistic for the family $\mathfrak{F}_0$.

Proof: Both of the sufficiency assertions follow immediately from the Factorization Theorem. The completeness of (T,W) follows by noting that; (i) if $X \sim N_n(\mu e, \sigma^2 I_n)$, $\sigma^2 > 0$, then the density of X is in $\mathfrak{F}_1(\mu)$; (ii) (T,W) is complete for the set of distributions in (i); and (iii) the joint distribution of (T,W), under any distribution in $\{\mathfrak{F}_1(\mu) | \mu \in R^1\}$, is absolutely continuous with respect to the distribution of (T, W) under (i). The completeness of W under H_0 follows by similar considerations.

Lemma 2.2: Under H_0, T has a density on $[-1,1]$ given by

$$(2.1) \quad r_0(t) = 2[\mathcal{B}e(\tfrac{1}{2}, \tfrac{n-3}{2})]^{-1}(1-t^2)^{\frac{n-3}{2}}$$

and W has a density on $(0, \infty)$ given by

$$(2.2) \quad r_1(\omega) = q(\omega) \frac{(\Gamma(\frac{1}{2}))^n}{\Gamma(\frac{n}{2})} \omega^{n/2-1} .$$

Further, under H_0, T and W are independent.

Proof: Under H_0, $X/\|X\|$ has a uniform distribution on $\{x | x \in R^n, \|x\| = 1\}$ so $X_1^2/\|X\|^2$ has a beta distribution with parameters $\frac{1}{2}$ and $\frac{n-3}{2}$. That T has the density (2.1) is now clear. The density of W is derived by changing to polar coordinates in R^n. Since the density of T does not depend on the particular distribution in H_0 and since W is a complete sufficient statistic under H_0, the independence of T and W under H_0 follows from a result due to Basu (1955). This completes the proof.

Define the probability measure λ_0 on $[-1,1]$ by

$$(2.3) \qquad \lambda_0(dt) = r_0(t)dt$$

and define the measure μ_0 on $(0, \infty)$ by

$$(2.4) \qquad \mu_0(d\omega) = \frac{(\Gamma(\frac{1}{2}))^n}{\Gamma(\frac{n}{2})} \omega^{\frac{n}{2}-1} d\omega \ .$$

Lemma 2.3: If X has a density $h \in \mathcal{F}_1(\mu)$, then the joint density of T and W with respect to $\lambda_0 \times \mu_0$ when $h(x) = q(\|x-\mu e\|^2)$ is

$$(2.5) \qquad g(t,\omega;\mu) = q(\omega - 2\sqrt{\omega}\, t\mu + \mu^2) \ .$$

Proof: This follows from Lemma 2.2 and an application of Proposition 7.39 in Eaton (1972).

Now, set

$$(2.6) \qquad k(t;\mu,\omega) = \frac{g(t,\omega;\mu)}{\int g(t,\omega;\mu)\lambda_0(dt)}$$

so $k(t;\mu,\omega)$ is the conditional density of T given W with respect to λ_0. Note that $k(t;0,\omega) = 1$. Let $\mathcal{E}_0^T$ denote expectation under H_0

with respect to λ_0.

Lemma 2.4: If ϕ is a test function in $\mathcal{S}_\alpha$, then

$$(2.7) \qquad \mathcal{E}_0^T \phi(T, W) = \alpha \qquad \text{a.e. } (W)$$

and

$$(2.8) \qquad \mathcal{E}_0^T T \phi(T, W) = 0 \qquad \text{a.e. } (W) .$$

Proof: Since $\phi \in \mathcal{S}_\alpha$, $\mathcal{E}_h \phi \geq \alpha$ for all $h \in \{\mathcal{F}_2(\mu) | \mu \neq 0\}$ and $\mathcal{E}_h \phi \leq \alpha$ for all $h \in \mathcal{F}_0$. Hence $\mathcal{E}_h \phi = \alpha$ for all $h \in \mathcal{F}_0$, by a simple continuity argument. Thus $\mathcal{E}_W[\mathcal{E}_0^T(\varphi(T, W) - \alpha)|W] = 0$. Since W is a complete statistic under H_0, (2.7) follows. (2.8) follows by assuming $X \sim N_n(\mu e, \sigma^2 I_n)$ and arguing as in Lehmann (1959), Chapter 4. This completes the proof.

Let $\tilde{\mathcal{S}}_\alpha$ denote the set of test functions which satisfy (2.7) and (2.8). Also, define the test function ϕ_0 by

$$(2.9) \qquad \phi_0(T) = \begin{cases} 1 & \text{if } |T| > c \\ 0 & \text{if } |T| \leq c \end{cases}$$

where c is chosen so that $\mathcal{E}_0^T \phi_0 = \alpha$.

Theorem 2.1: If $\phi \in \tilde{\mathcal{S}}_\alpha$, then

$$(2.10) \qquad \mathcal{E}_h \phi_0 \geq \mathcal{E}_h \phi$$

for all $h \in \{\mathcal{F}_2(\mu) | \mu \in R^1\}$.

Proof: If $h \in \mathcal{F}_2(0)$, then equality holds in (2.10). Fix $h(x) = q(\|x-\mu e\|^2)$ where q is convex and non-increasing on $[0, \infty)$

so $h \in \mathfrak{F}_2(\mu)$. For a fixed value of W, consider the problem of testing $H_0:\mu = 0$ versus $H_1:\mu = \mu_0 \neq 0$. Applying the Generalized Neyman-Pearson Lemma (Lehmann (1959)), the supremum of $\mathcal{E}_{\mu_0}(\phi(T,W)|W)$ over the set $\tilde{\mathfrak{D}}_\alpha$ is achieved by test functions of the form

$$(2.11)\qquad \phi_1(t) = \begin{cases} 1 \text{ if } k(t,\mu_0,\omega) > c_1 + c_2 t \\ 0 \text{ if } k(t;\mu_0,\omega) \le c_1 + c_2 t \end{cases}$$

where $k(t;\mu_0,\omega)$ is given by (2.6) and c_1 and c_2 are chosen so that ϕ_1 satisfies (2.7) and (2.8). But, since q is convex, $k(t;\mu_0,\omega) - c_2 t$ is a convex function of t. Thus ϕ_1 can be written as

$$(2.12)\qquad \phi_1(t) = \begin{cases} 0 & \text{if } a \le t \le b \\ 1 & \text{otherwise} \end{cases}$$

where a and b are chosen so ϕ_1 satisfies (2.7) and (2.8). However, the only values of a and b which make ϕ_1 satisfy (2.7) and (2.8) are $-a = b = c$ where c is defined in (2.9). Thus for each fixed value of W, ϕ_0 maximizes $\mathcal{E}_{\mu_0}(\phi(T, W)|W)$ over the set $\tilde{\mathfrak{D}}_\alpha$. Thus, if $\phi \in \tilde{\mathfrak{D}}_\alpha$, $\mathcal{E}_{\mu_0}(\phi_0(T, W)|W) \ge \mathcal{E}_{\mu_0}(\phi(T, W)|W)$ a.e. (W). Integrating with respect to W then yields $\mathcal{E}_h\phi_0 \ge \mathcal{E}_h\phi$ for all $\phi \in \tilde{\mathfrak{D}}_\alpha$. Since ϕ_0 did not depend on the particular $h \in \mathfrak{F}_2(\mu_0)$, (2.10) holds. This completes the proof.

<u>Theorem 2.2:</u> The test ϕ_0 of (2.9) is UMP unbiased for testing $H_0: h \in \mathfrak{F}_0$ versus $H_1: h \in \{\mathfrak{F}_2(\mu) | \mu \neq 0\}$.

<u>Proof:</u> Since $\mathfrak{D}_\alpha \subseteq \tilde{\mathfrak{D}}_\alpha$, the result follows immediately from Theorem 2.1.

We note that the proof of Theorem 2.1 gives a result which is substantially stronger than that in Theorem 2.2. The proof of Theorem 2.1 shows that ϕ_0 actually maximizes the conditional power (for W fixed) over all tests which satisfy (2.7) and (2.8) at the particular W.

§ 3. A conjectured result.

Based on a fair amount of work, we conjecture that the test ϕ_0 defined in (2.9) is UMP unbiased for testing $H_0:h \in \mathfrak{F}_0$ versus $H_1:h \in \{\mathfrak{F}_1(\mu)|\mu \neq 0\}$. The method of proof used in section 2 fails completely for this conjecture since the proof of Theorem 2.1 no longer holds. In fact we have reason to believe that Theorem 2.1 is in fact false with $\{\mathfrak{F}_2(\mu)|\mu \neq 0\}$ replaced by $\{\mathfrak{F}_1(\mu)|\mu \neq 0\}$.

A main difficulty in attempting to establish the conjecture is getting a reasonable analytic description of what unbiasedness means. Of course, (2.7) and (2.8) must hold, but these conditions are not nearly sufficient to imply unbiasedness.

Let $q^*(y) = c^* I_{[0,1]}(y)$ where $I_{[0,1]}$ denotes the indicator function of $[0, 1]$ and c^* is a constant so that $\int_{R^n} q^*(\|x\|^2)dx = 1$. Let

$$\mathfrak{F}_3 = \{f|f(x) = \frac{1}{\sigma^n} q^*(\frac{\|x-\mu e\|^2}{\sigma^2}), \quad \sigma > 0, \ \mu \in R^1, \ \mu \neq 0\} .$$

It is not difficult to show that: If ϕ_0 is UMP unbiased for testing $H_0:h \in \mathfrak{F}_0$ versus $H_1:h \in \mathfrak{F}_3$, then the conjectured result is true. This follows from the fact that every decreasing q (as in $\mathfrak{F}_1(\mu)$) on $(0, \infty)$ can be represented as $q(y) = \int \frac{1}{\sigma^n} q^*(y/\sigma)F(d\sigma)$ where F is a probability distribution function on $(0, \infty)$. However, this observation has not helped us much in establishing the conjecture.

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