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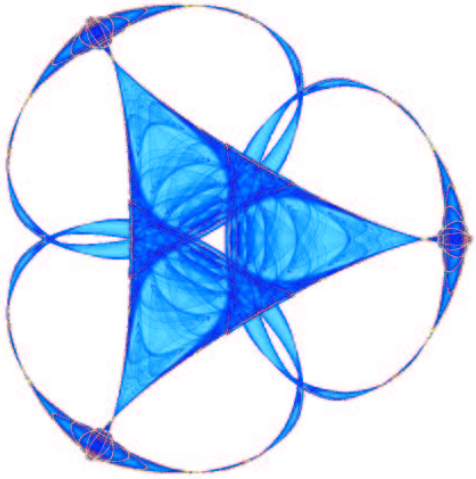

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NON-OVERLAPPING DOMAIN DECOMPOSITION METHOD AND NODAL FINITE ELEMENT METHOD

Y. BOUBENDIR*, A. BENDALI †, AND M.B. FARES ‡

Abstract. The non-overlapping domain decomposition method is an efficient approach for solving time harmonic scattering wave problems. It is used here to reduce large size systems solution to that of several systems of small size and to construct efficient procedures to couple finite element and boundary element methods. The lack of a satisfactory treatment of the so-called cross-points, nodes being shared by more than two domains, prevents one from taking advantage of the simplicity of the standard finite element method and the effectiveness of a domain decomposition procedure at the same time. A new approach overcoming this difficulty is introduced. It mainly consists in keeping a strong coupling at cross-points by enforcing a strong continuity requirement at these points for both trial and the test functions.

Key words. Helmholtz equation, non-overlapping domain decomposition method, coupling BEM-FEM, cross-points.

1. Introduction. Several methods have been devised in the last couple of years to solve the large size linear systems arising from the discretization of time harmonic scattering problems (see for example [26, 14, 19, 12, 22]). This is primarily because of the oscillatory character of the solution which accordingly requires resorting to very refined meshes. In addition, the lack of strong coercive properties of the underlaying equations, as compared for instance to those occurring in structural mechanics problems, seriously damages the efficiency of the usual solvers. This partly explains why various domain decomposition techniques have been proposed to deal with such a class of problems (e.g., [15, 19, 28, 12, 22]). The aim of this paper is to contribute to this circle of techniques. We devise a new approach for the cross-points, that is, points being shared by more that two subdomains, reducing the domain decomposition procedure to a simple and efficient iterative method for solving the nodal equations. We then show that this method adapts easily to the coupling BEM-FEM algorithm. In addition, we introduce an algorithm called "evanescent modes damping algorithm" [9, 8], in order to suitably treat the evanescent part of the solution and thus improve the convergence of the domain decomposition method.

The techniques presented in this paper are described in the framework of the treatment of the Helmholtz equation using the non-overlapping domain decomposition method originally introduced by P.-L. Lions [25] for solutions of the Laplace equation. It was subsequently extended to time harmonic wave propagation problems by B. Després [16, 18]. In this way, we can illustrate the several advantages it owns. It reduces the large size system solution to that of several small size ones. It also allows one to construct a coupling of algorithms between

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different methods of discretization, such as the finite and the boundary element methods, in order to solve scattering problems involving non-homogeneous materials. The FEM deals with the finite part of the domain enclosing any heterogeneousness of the scatterer, independently from the BEM which efficiently tackles the equations describing the propagation of the wave in an infinite homogeneous medium. In contrast, the common coupling methods yield systems of very large size having a matrix with sparse and dense blocks [23, 11] which are thus generally hard to solve. The separation of the solution in the finite part from that in the infinite one allows us to use special procedures to deal with the infinite region and again use non-overlapping domain decomposition in the finite part. A difficulty however appears at the level of the cross-points which are points common to several interfaces and thus to several subdomains. B. Després [15] and Collino, Ghanemi and Joly [12] have used a mixed finite element method to avoid nodes at cross-points. The disadvantage of this approach is that it imposes a special discretization which can be hard to develop for general systems. Another way to proceed, mainly met in the context of FETI methods (see e.g. [10]), consists of treating a cross-point like any other point on the interface. It results in an overdetermined system for the matching conditions at the cross-points and an underdetermined one for the auxiliary interface unknowns that join the formulation in these techniques.

The essential idea on which our method is based consists of keeping the finite elements unknowns and equations related to the cross-points. In other words, a strong coupling is maintained for the degrees of freedom carried by the cross-points for both the unknowns and the testing functions. In this way no nodal value is introduced for the interface unknowns at these points. In contrast to a strict domain decomposition method, the local problems remain coupled at these nodes. However, since their number is relatively small, even when compared to the size of the local problems, a Schur complement procedure deals with the coupling as a simple post-processing completing each iteration. Stability and convergence results of this algorithm are proved in [3, 4, 8].

The lack of coerciveness of the Helmholtz equation, already mentioned, has given rise to new developments on Després' algorithm with, as an objective, the improvement of the convergence of the related iterative procedure. Such an improvement is based on the construction of more adapted transmission conditions on the artificial interfaces [12, 22] than those initially proposed by Després. In this respect, we have introduced a modification of these conditions to construct an algorithm, named "the evanescent modes damping algorithm" [9, 8], which, while having much better convergence properties, consists of a simple modification of the parameter involved in Després' initial algorithm.

This paper is composed of three parts. First, we start by describing the non-overlapping domain decomposition method applied to a simple model problem. In the second part, we present its actual use within the framework of a nodal finite element discretization. We propose a new approach to overcome the difficulty arising from the cross-points. We conclude this part by some numerical results which validate this method and confirm the efficiency of the evanescent

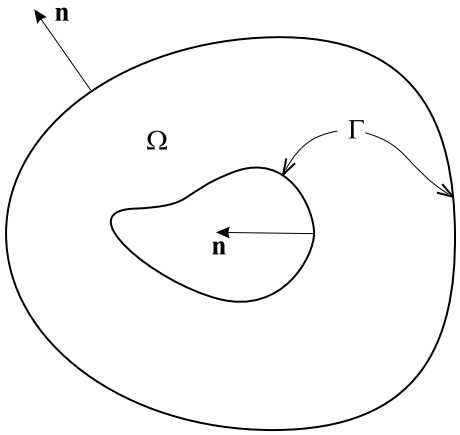

FIG. 1. *Initial domain.*

modes damping algorithm. Finally, in the last part, we show how the domain decomposition method makes it possible to couple finite and boundary elements for solving the scattering problems. In this context, this procedure has two advantages. As already mentioned, the first one concerns the uncoupling, at the level of each iteration, of the finite and the boundary element solution. The second one is much more unforeseen but perhaps as important as the former. The direct coupling procedure may generate spurious solutions at some frequencies. It is generally hard in practice to remove such spurious modes. The domain decomposition procedure appears as a simple and natural way to eliminate such flaws. Since in this algorithm an iteration is performed by computing the trace on every interface of the solution of a boundary-value problem set on each subdomain, we show in addition how to adapt the integral equation method to obtain these quantities. The most simple implementation of the domain decomposition algorithm can be seen as a solving procedure based on successive approximations of the solution to a fixed point problem posed on the interfaces of the decomposition in subdomains. As well-known (e.g., [17, 21]), the solution of the interface problem can be greatly improved by using a Krylov iterative method instead of the successive approximations procedure. Morever, the latter may diverge for a bad choice for the parameter destined to damp the evanescent modes while this choice is without influence on the former method. Numerical results are given to validate these procedures.

2. The basic domain decomposition method.

2.1. A model problem in the scattering of time harmonic waves. In this section, we focus on the description of the domain decomposition procedure. In this respect, we limit ourselves to consider the following simple two-dimensional problem relative to the Helmholtz equation posed on the bounded domain Ω (fig-

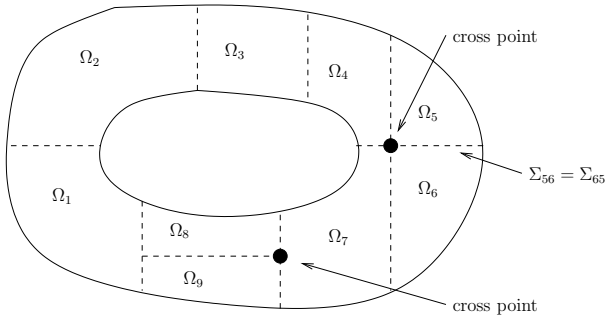

FIG. 2. *Partition of the initial domain.*

ure 1)

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ \partial_{\mathbf{n}} u - iku = f & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where $\Delta := \partial^2_{x_1} + \partial^2_{x_2}$ is the usual laplacian in two dimensions, $\mathbf{n}$ is the outward unit normal of the boundary of Ω (see figure 1), $k > 0$ is the wave number and f is a directly given function by an incident wave $f := -(\partial_{\mathbf{n}} u^{\text{inc}} - iku^{\text{inc}})$. The problem (2.1) is well posed in the usual Sobolev space $H^1(\Omega)$ (see for instance, [15]).

Precisely, we consider on Γ a terminating approximate condition based on the Sommerfeld radiation condition [1, 20, 7, 13]. For the sake of simplicity, we chose not to consider another kind of boundary condition somewere else on Γ. The method proposed in this paper is independent of this choice. We could also consider Dirichlet or Newmann condition, for example, in some interior part of Γ as long as problem (2.1) remains well-posed [8]. A more general boundary condition will be considered for the problem posed on the unbounded domain (section 4).

2.2. Non-overlapping domain decomposition method. The first step of the non-overlapping domain decomposition method consists of splitting the domain Ω into several subdomains Ω_i, $i = 1, \ldots, N$, such that

- $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$, $i = 1, \ldots, N$,
- $\Omega_i \cap \Omega_j = \emptyset$, if $i \neq j$, $i, j = 1, \ldots, N$,
- $\partial\Omega_i \cap \partial\Omega_j = \overline{\Sigma}_{ij} = \overline{\Sigma}_{ji}$, $i, j = 1, \ldots, N$, defines the artificial interface (figure 2) separating Ω_i from Ω_j as long as its interior Σ_{ij} is not empty.

In the second step, we write an equivalent system to problem (2.1) by setting in each subdomain Ω_i the following problem

$$\begin{cases} \Delta u_i + k^2 u_i = 0 \quad \text{in } \Omega_i, \\ \partial_{\mathbf{n}_i} u_i - iku_i = f_i \quad \text{on } \Gamma_i = \partial\Omega_i \cap \Gamma, \end{cases} \tag{2.2a}$$

$$u_i = u_j \quad \text{on } \Sigma_{ij}, \tag{2.2b}$$

$$\partial_{\mathbf{n}_i} u_i = -\partial_{\mathbf{n}_j} u_j \quad \text{on } \Sigma_{ij}, \tag{2.2c}$$

where $u_i = u|_{\Omega_i}$, $f_i = f|_{\Gamma_i}$ and $\mathbf{n}_i$ (resp. $\mathbf{n}_j$) the outward unit normal of the boundary of Ω_i (resp. Ω_j). The boundary condition on Γ_i does not take place if the interior of $\partial\Omega_i \cap \Gamma$ is the empty set. Specifically, our work in this paper is focused on the method combining the continuity conditions (2.2b)-(2.2c) under the form

$$\partial_{\mathbf{n}_i} u_i + \eta u_i = -\partial_{\mathbf{n}_j} u_j + \eta u_j \quad \text{on } \Sigma_{ij}, \tag{2.3}$$

$$\partial_{\mathbf{n}_j} u_j + \eta u_j = -\partial_{\mathbf{n}_i} u_i + \eta u_i \quad \text{on } \Sigma_{ij}, \tag{2.4}$$

where

$$\eta = -ik(1 + i\chi), \quad \chi \geq 0. \tag{2.5}$$

Initially, this kind of non-overlapping domain decomposition methods based on Robin boundary conditions was introduced by by P.-L. Lions [25] in the context of the Laplace equation, and was adapted to wave propagation problems by B. Després [15]. By taking into account the transmission conditions (2.3)-(2.4), the solution is then reduced to an iterative procedure, where each iteration is done by solving the local problems

$$\begin{cases} \Delta u_i^{(n+1)} + k^2 u_i^{(n+1)} = 0 \quad \text{in } \Omega_i, \\ \partial_{\mathbf{n}_i} u_i^{(n+1)} - iku_i^{(n+1)} = f_i \quad \text{on } \Gamma_i, \end{cases} \tag{2.6a}$$

$$\partial_{\mathbf{n}_i} u_i^{(n+1)} + \eta u_i^{(n+1)} = g_{ij}^{(n)}, \quad \text{on } \Sigma_{ij}, \tag{2.6b}$$

where

$$g_{ij}^{(n)} = -\partial_{\mathbf{n}_j} u_j^{(n)} + \eta u_j^{(n)}, \tag{2.7}$$

and transmitting information

$$g_{ji}^{(n+1)} = -\partial_{\mathbf{n}_i} u_i^{(n+1)} + \eta u_i^{(n+1)} = -g_{ij}^{(n)} + 2\eta u_i^{(n+1)}, \tag{2.8}$$

through the artificial interfaces. Finally let us note that choosing η as in (2.5), the problem (2.6) is well posed in $H^{(1)}(\Omega_i)$ [15].

The goal of this paper is not to discuss the convergence analysis of this algorithm. However, this question is so significant in the case of the Helmholtz equation that we have preferred to give some indications. The algorithm of B. Després

[15] is based on the transmission conditions (2.3)-(2.4) with $\chi = 0$, i.e. $\eta = -ik$. We have shown in [8, 9] that, for some special geometries making it possible to separate the propagative and evanescent parts of the wave, only the propagative part of the wave is damped, and have hence explained why the residual stagnates after some iterations. Actually, this flaw has already been brought out in [17] where the author has proposed to use a relaxation procedure to overcome the stagnation. We have shown that considering $\chi > 0$, produces an algorithm, named "evanescent modes damping algorithm", more efficient than the relaxed one. In [12], Collino, Ghanemi and Joly have theoretically shown that choosing $\eta := -ikS$ with S a suitable symmetric positive definite operator, necessarily non local, results in a contracting iteration operator with a constant independent of the mesh. However, from both the study relative to a special geometry [9, 8] and from several numerical experiments we have concluded that the resulting procedure does not bring any real improvement as compared to the "evanescent modes damping algorithm".

3. Non-overlapping domain decomposition method and nodal finite element method. In general, when we split the initial domain Ω, we generate cross-points which are points belonging to more than two subdomains (figure 2). Difficulties appear with the mathematical analysis of the boundary problems and the treatment of the degrees of freedom carried by these points during the iterative process [12, 8]. We propose a new approach which consists of preserving the finite element equation at the level of these points, i.e., taking a common value for the degrees of freedom located on the nodes at the junction of several subdomains. This gives an iterative algorithm of resolution different from that given by a classical domain decomposition method. We will see however that this difference is very slight. We refer to [8, 3, 4] for details concerning the mathematical analysis (stability and convergence) of this algorithm. We give here the various steps of this method.

Let $\mathcal{T}^h$ be a nondegenerate triangular mesh of Ω and consider a $\mathbb{P}_1$ -continuous finite element space noted X^h. The discrete associated problem of (2.1) is defined as follows

$$\left\{ \begin{array}{l} u^h \in X^h,\ \forall v^h \in X^h, \\ a(u^h, v^h) = Lv^h, \end{array} \right. \tag{3.1}$$

where

$$a(u^h, v^h) := \int_\Omega (\nabla u^h \cdot \nabla v^h - k^2 u^h v^h)\, d\Omega - ik \int_\Gamma u^h v^h \, d\Gamma,$$
$$Lv^h := \int_\Gamma f v^h \, d\Gamma.$$

Let us assume now that $\mathcal{T}^h$ is compatible with the domain decomposition in the sense that it induces a mesh $\mathcal{T}_i^h$ on each subdomain, and introduce X_i^h as the $\mathbb{P}_1$-continuous functions approximation space of $H^1(\Omega_i)$. Any function $v_i^h \in X_i^h$

can be written as

$$v_i^h = v_{iI}^h + \sum_{j \in \Lambda_i} v_{ij}^h + v_c^h,$$

where the indices of v_{ij}^h refer to the artificial interface Σ_{ij} reduced to the interior points only, i.e. without cross-points, and where

- all the nodal values of v_{iI}^h are zero on $\cup_{j \in \Lambda_i} \overline{\Sigma}_{ij}$ with Λ_i representing the set of numbers j of the subdomains Ω_j separated from Ω_i by the artificial interface Σ_{ij},
- all the nodal values of v_{ij}^h are zero except for those related to nodes on Σ_{ij},
- all the nodal values of v_c^h are zero except for the nodes corresponding to the cross-points.

In some way, w_{iI}^h and w_c^h can be, and are, identified to a function in X^h while the functions w_{ij}^h make it possible to relax the continuity requirement at the interfaces. We will denote by X_c^h the subspace of X^h spanned by the w_c^h.

To introduce the domain decomposition method, we consider the "broken" space X_B^h spanned by the functions w^h that can be written in an unique manner as

$$w^h = \sum_{i=1}^{N} \left(w_{iI}^h + \sum_{j \in \Lambda_i} w_{ij}^h \right) + w_c^h. \tag{3.2}$$

The finite element space X^h appears as the subspace of X_B^h consisting of those w^h that are continuous at the nodal points on Σ_{ij}. This continuity is expressed by a matching condition which is at the heart of the domain decomposition method. Let us notice also that the forms a and L can be written as follows

$$a(u^h, v^h) = \sum_{i=1}^{N} a_i(u_i^h, v_i^h), \qquad Lv^h = \sum_{i=1}^{N} L_i v_i^h, \tag{3.3}$$

where

$$u_i^h := u^h|_{\Omega_i},$$
$$a_i(u_i^h, v_i^h) := \int_{\Omega_i} (\nabla u_i^h \cdot \nabla v_i^h - k^2 u_i^h v_i^h)\, d\Omega_i - ik \int_{\Gamma_i} u_i^h v_i^h \, d\Gamma_i,$$
$$L_i v_i^h := \int_{\Gamma_i} f_i v_i^h \, d\Gamma_i.$$

Then, we show [8, 3, 4] that the problem (3.1) is equivalent to the system

$$\begin{array}{l} a_i\left(u_{iI}^h+\sum_{j\in\Lambda_i}u_{ij}^h+u_c^h,v_{iI}^h\right)=L_iv_{iI}^h,\quad \forall v_{iI}^h\in X_i^h,\quad i=1,\ldots,N,\\ \left\{\begin{array}{l} a_i\left(u_{iI}^h+\sum_{\ell\in\Lambda_i}u_{i\ell}^h+u_c^h,v_{ij}^h\right)\\ \qquad +a_j\left(u_{jI}^h+\sum_{\ell\in\Lambda_j}u_{j\ell}^h+u_c^h,v_{ji}^h\right)=L_iv_{ij}^h+L_jv_{ji}^h\\ c_{ij}^h\left(u_{ij}^h,v_{ij}^h\right)=c_{ji}^h\left(u_{ji}^h,v_{ji}^h\right)\\ \qquad\text{for all } v_{ij}^h\in X_i^h \text{ and } v_{ji}^h\in X_j^h,\ v_{ij}^h=v_{ji}^h \text{ on } \Sigma_{ij},\ \forall\,\Sigma_{ij},\end{array}\right.\\ \sum_{i=1}^N a_i\left(u_{iI}^h+\sum_{\ell\in\Lambda_i}u_{i\ell}^h+u_c^h,v_c^h\right)=\sum_{i=1}^N L_iv_c^h,\qquad \forall v_c^h\in X_c^h,\end{array}$$

where $c_{ij}^h=c_{ji}^h$ is a bilinear form ensuring the match of the traces u_{ij}^h and u_{ji}^h on Σ_{ij}. We will assume that it is the L^2 inner product on Σ_{ij} in the numerical results paragraph. The match of the normal derivatives traces is implicit since we work with variational forms. Finally, we relax the conditions on the artificial interfaces starting from the domain decomposition method of Lions-Després [25, 15], and obtain the final system

$$\left\{\begin{array}{l} \left.\begin{array}{l} a_i\left(u_{iI}^h+\sum_{\ell\in\Lambda_i}u_{i\ell}^h+u_c^h,v_{iI}^h\right)=L_iv_{iI}^h,\quad \forall v_{iI}^h\in X_i^h,\\ a_i\left(u_{iI}^h+\sum_{\ell\in\Lambda_i}u_{i\ell}^h+u_c^h,v_{ij}^h\right)+\eta c_{ij}^h\left(u_{ij}^h,v_{ij}^h\right)=\\ \qquad L_iv_{ij}^h+c_{ij}^h\left(g_{ij}^h,v_{ij}^h\right),\ \forall v_{ij}^h\in X_i^h,\ j\in\Lambda_i,\end{array}\right\}\quad i=1,\ldots,N,\\ \sum_{i=1}^N a_i\left(u_{iI}^h+\sum_{\ell\in\Lambda_i}u_{i\Sigma_\ell}^h+u_c^h,v_c^h\right)=\sum_{i=1}^N L_iv_c^h\quad \forall v_c^h\in X_c^h,\end{array}\right. \tag{3.4}$$

where g_{ij}^h are the boundary data on the artificial interfaces allowing us to perform an iteration.

The equation system (3.4) can be written in matrix form

$$\begin{bmatrix} A_{11} & & & & A_{1c}\\ & A_{22} & & & A_{2c}\\ & & \ddots & & \vdots\\ & & & A_{NN} & A_{Nc}\\ A_{c1} & A_{c2} & \ldots & A_{cN} & A_{cc}\end{bmatrix}\begin{bmatrix} u_1\\ u_2\\ \vdots\\ u_N\\ u_c\end{bmatrix}=\begin{bmatrix} f_1\\ f_2\\ \vdots\\ f_N\\ f_c\end{bmatrix}, \tag{3.5}$$

where A_{ii}, A_{ic}, A_{ci}, $i=1,\ldots,N$, are matrices related to the discretization on Ω_i and f_i the right hand side of each problem. The index "c" represents the cross-points parts describing the slight problem coupling at the level of these points. However, a Schur complement solution deals with the coupling as a simple post-processing completing each iteration.

To solve system (3.5), we start by replacing u_i in the last equation by

$$u_i=(A_{ii})^{-1}\left(f_i-A_{ic}u_c\right), \tag{3.6}$$

and obtain

$$\left(A_{cc}-\sum_{i=1}^N A_{ci}(A_{ii})^{-1}A_{ic}\right)u_c=f_c-\sum_{i=1}^N A_{ci}(A_{ii})^{-1}f_i. \tag{3.7}$$

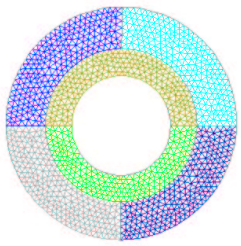

FIG. 3. *Decomposition of the initial domain into 6 subdomains*

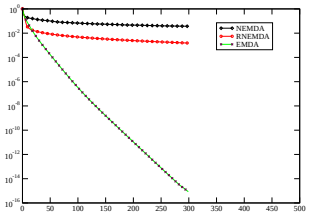

FIG. 4. *Residual.*

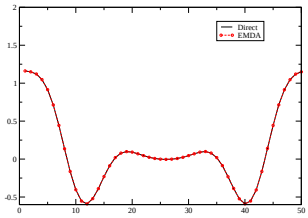

FIG. 5. *Comparison of the solutions.*

The size of the system (3.7) is small since it is given by the number of cross-points number. Practically, to solve it we proceed as follows

- we perform a LU factorization on each matrix A_{ii}, $i = 1, \ldots, N$.
- we then do a forward backward sweep to compute quantities $(A_{ii})^{-1} A_{ic}$ and $(A_{ii})^{-1} f_i$, $i = 1, \ldots, N$.
- finally, we solve the system (3.7).

After the computation of u_C, it is enough to redo a forward backward sweep on the matrices $A^{(i)}$ to compute u_i, $i = 1, \ldots, N$. It is almost the same procedure in comparison with a pure domain decomposition method.

3.1. Numerical results. We present numerical results obtained from the implementation of the above algorithm. In particular, we compare the resulting solution with that given by a direct finite element solution of the initial problem. This enables us to show that the resolution of the initial problem amounts to applying an iterative method. Specifically, we will consider two configurations: the first concerns a partition into 6 subdomains (figure 3) and the second into 24 subdomains (figure 6). Let us consider also $k = \pi$ and a mesh of 12 points per wavelength. We denote by EMDA the "evanescent modes damping algorithm" with $\chi = 0.5$ (see equation (2.5)), by NEMDA the Després's algorithm, i.e $\chi = 0$, by RNEMDA the relaxed algorithm of Després's with the relaxation parameter equal to 0.5, and by Direct the direct solution.

The plots represented in figures 4 and 7 (respectively related to the partitions into 6 and 24 subdomains) show the residual decreasing of the method. It confirms the robustness of the EMDA as compared to the NEMDA and RNEMDA which stagnates after some iterations. The plots in figures 5 and 8, represent the accurate

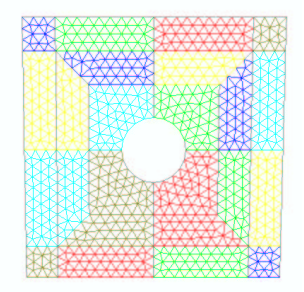

FIG. 6. *Decomposition of the initial domain into 24 subdomains.*

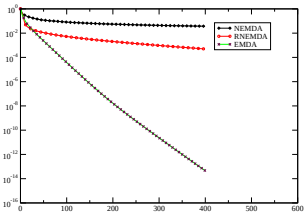

FIG. 7. *Residual.*

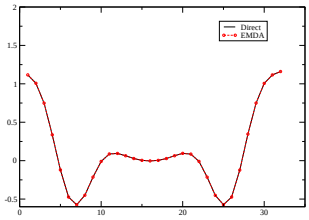

FIG. 8. *Comparison of the solutions.*

solution resulting in two configurations, 6 and 24 subdomains, as it compares with the solution obtained by Direct. Indeed, we obtain a relative error of order 10^{-12}. Let us note that it is not necessary to reach 10^{-16} to obtain a satisfactory solution.

4. Coupling the finite element and the boundary element methods. We develop in this section a coupling BEM-FEM method based on the domain decomposition method presented above. Since the BEM deals with the exterior problem (posed in an unbounded domain) and the FEM with the interior one (posed in a bounded domain), in order to show the main advantage of the method we replace the previous model problem (2.1) by the following one.

We consider the scattering two-dimensional time-harmonic wave by a possibly imperfectly conducting cylinder represented by a closed curve Γ of the plane covered by an heterogeneous dielectric occupying the domain Ω, bounded externally by a closed curve Σ. The wave propagates in the unbounded domain, noted Ω_0^∞, limited by Σ. We denote by $\mathbf{n}$ (resp. $\mathbf{n}_0$) the outward unit normal to the boundary of Ω (resp. Ω_0^∞) as depicted in figure 9 . If u and u_0 are the unique component of the magnetic field related to the total wave in Ω and Ω_0^∞

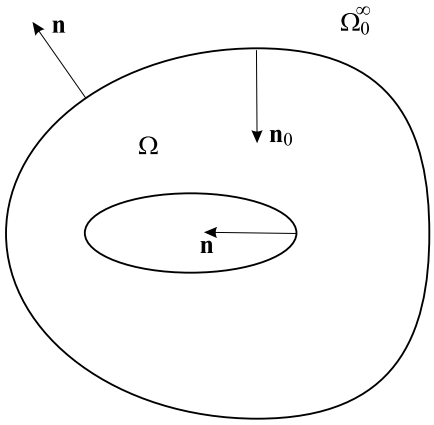

FIG. 9. *A typical geometry.*

respectively, the problem can be stated as follows

$$\begin{cases} \nabla \cdot (\dfrac{1}{\varepsilon}\nabla u) + k^2 \dfrac{n^2}{\varepsilon} u = 0 \quad \text{in } \Omega, \\ \Delta u_0 + k^2 u_0 = 0 \quad \text{in } \Omega_0^\infty, \\ \varepsilon^{-1}\partial_{\mathbf{n}} u + \beta u = 0 \quad \text{on } \Gamma, \\ \lim\limits_{|x|\to+\infty} |x|^{1/2} \left(\nabla(u_0 - u^{\text{inc}}) \cdot \dfrac{x}{|x|} - ik(u_0 - u^{\text{inc}}) \right) = 0, \end{cases} \tag{4.1}$$

where k is the wave number, n and ε are respectively the index and the relative permittivity of the dielectric medium filling Ω and β a constant with an imaginary part $\Im m(\beta) < 0$. Finally, the domains Ω_0^∞ and Ω are separated by an interface Σ on which the transmission conditions are set

$$u_0 - u = 0 \quad \text{on } \Sigma, \tag{4.2}$$

$$\partial_{\mathbf{n}_0} u_0 + \varepsilon^{-1}\partial_{\mathbf{n}} u = 0 \quad \text{on } \Sigma. \tag{4.3}$$

Under the following hypotheses on n and ε, which can be varying functions in $L^\infty(\Omega_1)$,

$$\Re e(\varepsilon) \text{ and } \Re e(n) \geq 1, \quad \Im m(\varepsilon) \text{ and } \Im m(n) \geq 0,$$

the problem (4.1)–(4.3) is well posed [30, 24].

The method of coupling that we propose consists first of all of uncoupling the exterior problem in Ω_0^∞ from the interior one in Ω using the method of decomposition domain, i.e., rewrite the conditions as follows

$$\varepsilon^{-1}\partial_{\mathbf{n}} u + \eta u = -\partial_{\mathbf{n}_0} u_0 + \eta u_0 \quad \text{on } \Sigma, \tag{4.4}$$

$$\partial_{\mathbf{n}_0} u_0 + \eta u_0 = -\varepsilon^{-1}\partial_{\mathbf{n}} u + \eta u \quad \text{on } \Sigma. \tag{4.5}$$

FIG. 10. *Cross point treatment by the coupling method*

Then, we apply the same domain decomposition algorithm, exposed in section 2, to solve the problem in Ω. Precisely, in each iteration we solve the above discrete versions of the problems

$$\begin{cases} \nabla \cdot (\dfrac{1}{\varepsilon_i} \nabla u_i^{(n+1)}) + k^2 \dfrac{n_i^2}{\varepsilon_i} u_i^{(n+1)} = 0 \quad \text{in } \Omega_i, \\ \varepsilon_i^{-1} \partial_{\mathbf{n}_i} u_i^{(n+1)} + \beta u_i^{(n+1)} = 0 \quad \text{on } \Gamma_i, \end{cases} \tag{4.6a}$$

$$\varepsilon_i^{-1} \partial_{\mathbf{n}_i} u_i^{(n+1)} + \eta u_i^{(n+1)} = g_i^{(n)} \quad \text{on } \Sigma_{ij}, \tag{4.6b}$$

and the following problem, posed in an unbounded domain, by a BEM

$$\begin{cases} \Delta u_0^{(n+1)} + k^2 u_0^{(n+1)} = 0 \quad \text{in } \Omega_0^\infty, \\ \lim\limits_{|x| \to +\infty} |x|^{1/2} \left(\nabla (u_0^{(n+1)} - u^{\text{inc}}) \cdot \dfrac{x}{|x|} - ik (u_0^{(n+1)} - u^{\text{inc}}) \right) = 0, \end{cases} \tag{4.7a}$$

$$\partial_{\mathbf{n}_0} u_0^{(n+1)} + \eta u_0^{(n+1)} = g_0^{(n)} \quad \text{on } \Sigma, \tag{4.7b}$$

where $i = 1, \ldots, N$, $j = 0, \ldots, N$ with N denoting the number of subdomains in Ω, $\mathbf{n}_i$ is the outward unit normal to the boundary of Ω_i, $\Gamma_i = \partial\Omega_i \cap \Gamma$, $\varepsilon_i = \varepsilon|_{\Omega_i}$, $\varepsilon_0 = 1$, $n_i = n|_{\Omega_i}$, η is given by (2.5) and form the quantities to be transmitted through the interfaces

$$g_{ij}^{(n+1)} = -\varepsilon_j^{-1} \partial_{\mathbf{n}_j} u_j^{(n)} + \eta u_j^{(n)}, \quad \text{on } \Sigma_{ij},$$

$g_0^{(n+1)}$ being defined in some special way from the $g_{0j}^{(n+1)}$.

The algorithm is similar and uses the same methods as explained previously. Differences lie in the treatment of the common cross points of two interior subdomains and a common interface with the exterior domain. The choice which we propose consists in taking only one value on the common nodes to the interior subdomains and a distinct value on the nodes belonging to the exterior domain (see figure 10). Thus, we keep the same finite element system as previously (3.5). We always take a common value for the degrees of freedom located on the cross points while the external cross points will belong to the nodes on which the quantities are exchanged. In short, each finite element and integral equation system has only one value to exchange at each iteration.

The above domain decomposition method makes it possible to deal separately with the solution in domain Ω_i, $i = 1, \ldots, N$, and the solution in domain

Ω_0^∞ . This allows one to use of the most adapted solution procedure for each of the boundary-value problems. For instance, the solution in Ω_0^∞ can be done through a boundary element method coupled with a fast procedure like the FMM (e.g., [26]). For nonhomogeneous dielectrics, the solution in Ω_i, $i = 1, \ldots, N$, can be obtained only through a finite element method. The domain decomposition method hence appears as an efficient procedure to couple the two solution processes.

The equation (2.8) shows that performing an iteration is reduced to computing the solution for each problem on the artificial interfaces only. The boundary element method is particularly adapted to evaluating this quantity without having to solve the boundary-value problem (4.7) in all of Ω_0^∞.

4.1. The boundary integral equation. Here, we adopt the approach developed in [29, 6, 2] for solving boundary-value problems related to an impedance boundary. Consider the problem

$$\left\{ \begin{array}{l} \Delta u_0 + k^2 u_0 = 0, \quad \text{in } \Omega_0, \\ \partial_{\mathbf{n}} u_0 + \eta u_0 = g, \quad \text{on } \Sigma, \\ \lim\limits_{|x|\to+\infty} |x|^{1/2} \left(\nabla(u_0 - u^{\rm inc}) \cdot \dfrac{x}{|x|} - ik(u_0 - u^{\rm inc}) \right) = 0, \end{array} \right. \tag{4.8}$$

where, to shorten the notation, we have set $\mathbf{n} := \mathbf{n}_0$.

The integral representation of u_0 is written in the following form

$$u_0(x) = u^{\rm inc}(x) + \int_\Sigma G_0(x,y) p(x)\, d\Sigma(y) - \int_\Sigma \partial_{\mathbf{n}_y} G_0(x,y) \lambda(x)\, d\Sigma(y) \;\; x \in \Omega_0,$$

in terms of a single- and a double-layer potential respectively created by the unknown densities p and λ relatively to the kernel

$$G_0(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \quad H_0^{(1)} := J_0 + iY_0,$$

expressed by means of Bessel J_0 and Neumann fonctions Y_0 of order 0. After solving the following variational problem involving a supplemental unknown ℓ

$$\left\{ \begin{array}{l} a(\{\lambda,p\},\{\lambda',p'\}) + b(\ell,\{\lambda',p'\}) = \int_\Sigma f\lambda' d\Sigma, \quad \forall \{\lambda',p'\}, \\ b(\ell',\{\lambda,p\}) = 0, \quad \forall \ell', \end{array} \right. \tag{4.9}$$

one obtains

$$u_0|_\Sigma = \ell + \lambda/2, \tag{4.10}$$

where the bilinear and linear forms are defined by

$$Vp(x) := \int_\Sigma G_0(x,y)p(y)\,d\Sigma(y), \quad N\lambda(x) := -\int_\Sigma \partial_{\mathbf{n}_y} G_0(x,y)\lambda(y)\,d\Sigma(y),$$
$$D\lambda(x) := -\partial_s V(\partial_s\lambda)(x) - k^2 V(\lambda\boldsymbol{\tau})(x)\cdot\boldsymbol{\tau}_x,$$
$$a(\{\lambda,p\},\{\lambda',p'\}) := \int_\Sigma \{D\lambda\lambda' - N\lambda p' - Np\lambda' - Vpp'\}\,d\Sigma,$$
$$b(\ell,\{\lambda',p'\}) := \int_\Sigma \ell\,(p'+\eta\lambda')\,d\Sigma,$$
$$f := g - (\partial_{\mathbf{n}} u^{\text{inc}} + \eta u^{\text{inc}}),$$

where $\boldsymbol{\tau}$ represents the unit tangent to Σ obtained by rotating $\mathbf{n}$ by $\pi/2$ counter-clockwise and s the curvilinear abscissa in the same orientation.

Meshing Σ in a polygonal curve, still denoted by Σ, with vertices on the exact curve and approximating every unknown and test function by a continuous function linear on each segment, we are lead to solve the following linear system

$$\begin{bmatrix} D & -N^{\mathrm{T}} & \eta M \\ -N & -V & M \\ \eta M & M & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ p \\ \ell \end{bmatrix} = \begin{bmatrix} Mf \\ 0 \\ 0 \end{bmatrix} \tag{4.11}$$

Now λ, p, ℓ and f are the colum-wise vectors formed by the nodal values of the respective functions at the vertices, D, N, V, Z and M are the matrices representing the bilinear forms respectively associated by the scalar product of $L^2(\Sigma)$ to operators D, N, V, η and the identity relatively to the nodal values. For example, masse matrix M is defined through the following relation

$$\int_\Sigma \ell p'\,d\Sigma = \begin{bmatrix} p'_1 & \cdots & p'_N \end{bmatrix} M \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_N \end{bmatrix}$$

where $\ell_1,\ldots,\ell_N$ and and $p'_1,\ldots,p'_N$ are the respective nodal values of ℓ and p'.

The technique used computes the matrix M according to a lumping process in order to obtain a diagonal mass matrix. The unknowns p and ℓ can be directly eliminated out of the previous system (4.11) resulting in a single equation for λ

$$\lambda = \left(D + \eta N^T + \eta N - \eta^2 S\right)^{-1} Mf. \tag{4.12}$$

The solution consists then of computing

$$\ell = M^{-1}(N - \eta V)\lambda \tag{4.13}$$

to obtain $u|_\Sigma$ from (4.10).

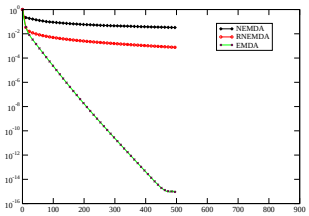

FIG. 11. *Residual.*

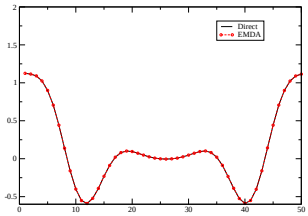

FIG. 12. *Comparison of the solutions.*

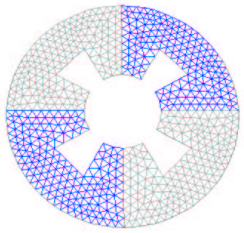

FIG. 13. *Partition into 4 subdomains.*

4.2. Numerical results. We take again the same domain decomposition depicted in figure 3 for the interior domain. More precisely, we have used 7 subdomains: 6 interior subdomains and the exterior one. We consider also the same mesh and the same wave number. Figures 11 and 12 display respectively the residual decreasing of the iterative procedure and the comparison of the solutions. The behavior of the iterative method is similar to that observed when using a domain decomposition method only by finite element method (see figures 11 and 4). However, the relative error of the solution obtained by the domain decomposition method with respect to the direct BEM-FEM solution is of order 10^{-6}, whereas the relative error, as we have shown, is of order 10^{-12} in the context of a finite element method only. Indeed, in the case of a pure finite element method, we can show that the domain decomposition method is exactly an iterative procedure to solve the discrete initial problem. In the context of BEM-FEM algorithm, a particular choice is required to obtain the same result [5]. Nevertheless, the relative errors with respect to the exact solution is of order 2% for the BEM-FEM iterative method and of order 3.5% for the FEM iterative one.

Another numerical result is related to a scattering problem by non-homogeneous material. Consider the domain decomposition represented in the figure 13: 4 interior subdomains and the exterior one. This geometry represents a severe case. Indeed, observe some angular points on the boundaries at each subdomain which could severely disrupt the computation of the solution.

In each interior subdomain Ω_i, we fixe ε_i, which gives us the index of the medium in each subdomain. We consider a mesh of 14 points per wavelength and $k = \pi$. Plots in figures 14 and 15 describe respectively the convergence of the

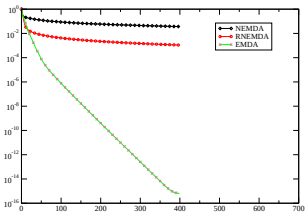

FIG. 14. *Residual.*

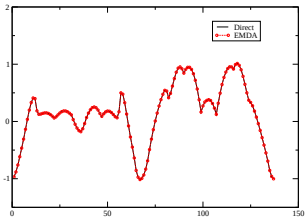

FIG. 15. *Comparison of the solutions.*

method and the satisfactory calculation of the solution. The relative error is of order 10^{-6}, which shows that the domain decomposition algorithm treating the cross-points presented in this paper is a robust and efficient method.

4.3. Using a Krylov method for the interface problem. In this paragraph, we limit the decomposition to just two subdomains: the interior and the exterior one. The domain decomposition procedure is thus nothing else than a way to couple the FEM and the BEM without having to solve a system which involves dense and sparse blocks at the same time. It is also clear that the algorithm can be viewed as the solution of the fixed point problem

$$\left\{ \begin{array}{l} g_0 = -g_1 + 2\eta u^h|_\Sigma, \\ g_1 = -g_0 + 2\eta u_0^h|_\Sigma, \end{array} \right. \tag{4.14}$$

where u^h and u_0^h are the respective FEM and BEM solutions.

From several numerical experiments, we have observed that the iterations relative to the successive approximations always congerge for $0 \leq \chi \leq 1$ and that $\chi = 0.5$ seems to be the optimal choice. However,for $\chi = 10$ which clearly appears as an irrelevant value, the residual grows without limit as depicted in figure 16.

A widespread idea in domain decomposition method (e.g., [17, 21]), is to solve the interface problem (4.14) using a Krylov method instead of the simple successive approximations procedure. The figure 17 which depicts the reduction of the residual when using the simple successive approximations and the GMRES solver (e.g., [27]) shows that the resulting procedure is not only more efficient but it yields a robust method which converges independently of the parameter χ.

5. Conclusion. In this paper, we have essentially introduced a new domain decomposition method using nodal finite element method to solve the cross-points problem. This approach does not require any additional implementation effort as compared to a pure domain decomposition domain. It only requires the utilization of a cheap Schur complement procedure as a simple post-processing completing each iteration. We showed that the method adapts easily for the coupling BEM-FEM. In addition, we validated the evanescent modes damping algorithm which consists of a simple modification of the algorithm of B. Després.

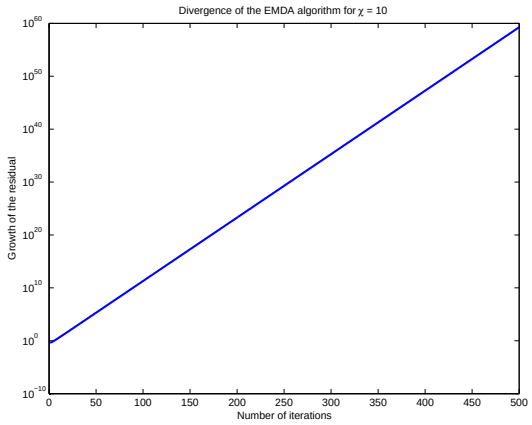

FIG. 16. *Divergence of the successive approximations for* $\chi = 10$.

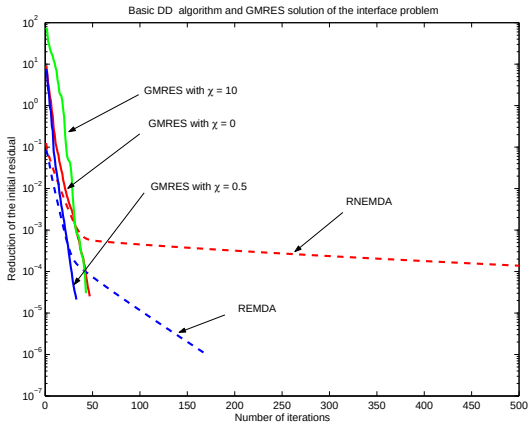

FIG. 17. *Reduction of the residual for successive approximations and GMRES solver.*

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