

**DIFFUSION IN NETWORK**

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**Abstract.** In this paper we study a diffusion in an orthogonal array of 2-dimensional channels as the width of the channels decreases to zero and the number of channels increases to infinity. It is shown that the limit is again a diffusion process and the diffusion coefficients are computed explicitly in terms of the relative sizes of the rectangles surrounded by the channels

**Introduction.** Many materials used in industry have granular structure. The granularity can occur at various length scales ranging from a few nanometers for amorphous material to tens of micrometers for grain structure. These scales are larger than the microscopic scale (atomic, molecular) and smaller than the macroscopic scale. One refers to these regimes as the mesoscopic regimes. Neither the methods of atomic chemistry and physics nor the methods of continuum physics are applicable at the mesoscopic level.

Of particular interest in semiconductor manufacturing is the doping of impurities in a polysilicon structure (such as the gate of a semiconductor device). This structure has the form shown in Figure 1 where the grains or crystals fill the space in irregular form; the black lines indicate the grains boundaries and their width, although small, is non-negligible.

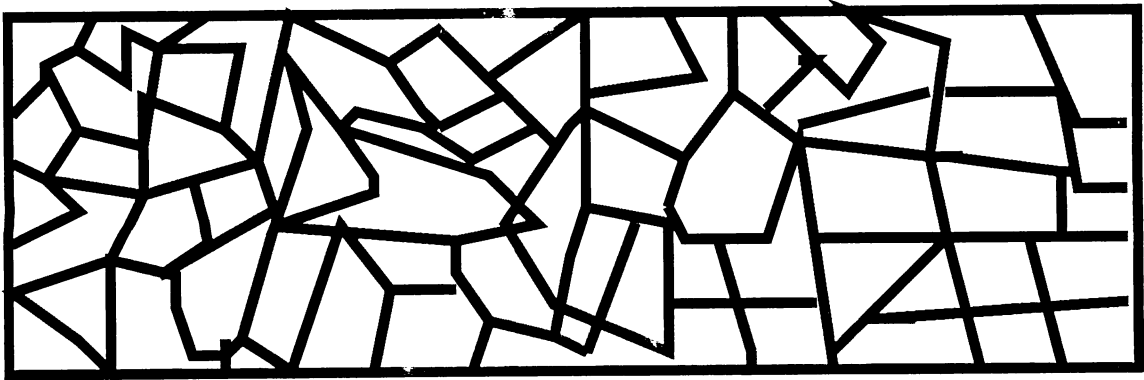


FIGURE 1

The actual boundary between two grains is  $5\text{\AA}$ , although the “effective” boundary might range up to  $100\text{\AA}$ . The grain size varies from  $1,000\text{\AA}$  to  $10,000\text{\AA}$ . The dopant diffuses preferentially through the grains boundaries [5] [6] [8], and diffusion through the grains bulk may (to a first

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approximation) be ignored. The grains boundaries constitute a complex network of channels, and one would like to determine, at least approximately, the density  $u$  of the diffusing dopant.

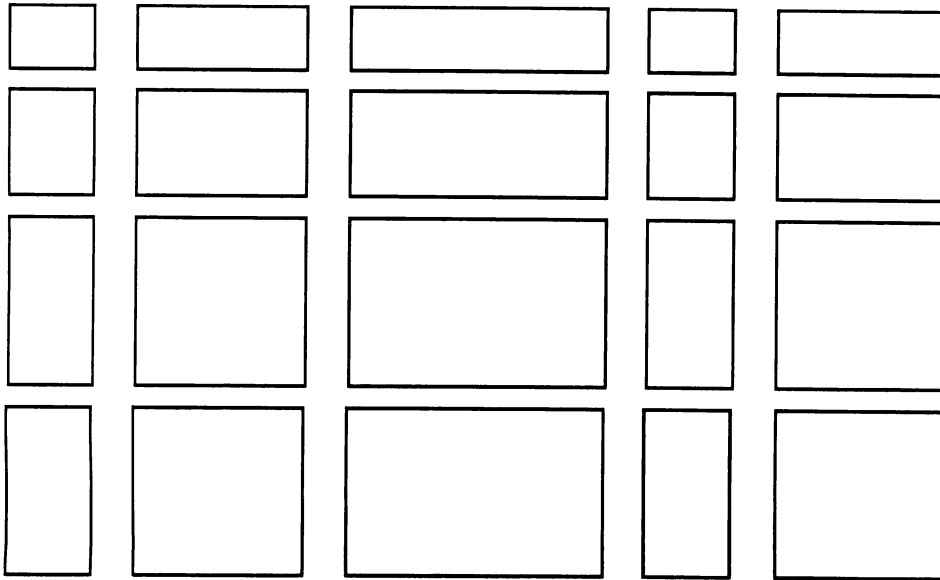


FIGURE 2

Figure 2 shows a simple 2-dimensional network with channels parallel to the  $x$ - and  $y$ -axis; the width of the channels is  $\beta_1$  in the  $y$ -direction and  $\beta_2$  in the  $x$ -direction. Notice that the grains size is not uniform. We can model the diffusion in the channels in two ways:

- (i) uniform diffusion in all directions with the stipulation that there is no diffusion across the grains boundaries, that is,  $\partial u / \partial n = 0$  on these boundaries, or
- (ii) uniform diffusion in all directions only at the intersections of channels, and diffusion only in the direction of the channel in the remaining portions of the network.

Since the boundary between grains is not sharply marked, it is actually not clear which of the two models provides a better description.

In this paper we concentrate on model (ii). Assuming that the material is a square  $\Omega$ , we impose a boundary condition

$$(0.1) \quad A \frac{\partial u}{\partial n} + u = F \quad (A \geq 0)$$

on the parts of  $\partial\Omega$  which bound the ends of the channels.

We wish to study this diffusion problem and determine the limit of  $u = u_\beta$  as  $\beta = (\beta_1, \beta_2) \rightarrow 0$ .

We shall consider the model (ii) for the non-periodic structure as exhibited in Figure 2. We shall assume that the grains in the  $x$ - and  $y$ -directions vary in accordance with functions  $f_1(x)$  and  $f_2(y)$ ; more precisely, the grain in the  $i$ -th row and  $j$ -th column is a rectangle with sides

$$\alpha_i = f_1((i+1)\beta_1) - f_1(i\beta_1) \quad \text{and} \quad \tilde{\alpha}_j = f_2((j+1)\beta_2) - f_2(j\beta_2)$$

respectively. One can perform a translation of all the intersections of channels so that they fill a rectangle  $\tilde{\Omega}_\beta$ ;  $\tilde{\Omega}_\beta$  converges to some rectangle  $\Omega_0$  as  $\beta \rightarrow 0$ . We shall prove that the solution  $u$ ,

restricted to  $\tilde{\Omega}_\beta$ , converges to the solution  $w$  of

$$(0.2) \quad \left( \frac{w_x}{1 + f'_1(x)} \right)_x + \left( \frac{w_y}{1 + f'_2(y)} \right)_y = 0 \quad \text{in } \tilde{\Omega}_0 ,$$

with the boundary conditions

$$(0.3) \quad \begin{aligned} \frac{A}{1 + f'_1(x)} \frac{\partial w}{\partial n} + w &= F_0 \quad \text{on the vertical sides of } \partial\tilde{\Omega}_0 , \\ \frac{A}{1 + f'_2(y)} \frac{\partial w}{\partial n} + w &= F_0 \quad \text{on the horizontal sides of } \partial\tilde{\Omega}_0 \end{aligned}$$

where  $\hat{F}_0$  is a limit of the  $F$ 's in (0.1) as  $\beta \rightarrow 0$  and  $F_0$  coincides with  $\hat{F}_0$  after a change of variables

$$(0.4) \quad x \rightarrow x + f_1(x) - f_1(0), \quad y \rightarrow y + f_2(y) - f_2(0)$$

which maps  $\tilde{\Omega}_0$  onto  $\Omega$ .

For the special periodic structure  $f_1(x) = \lambda x$ ,  $f_2(y) = \mu y$  the function  $w$  coincides with the solution obtained by the homogenization of model (i) (as computed using the formulas in [2]); for general  $f_1(x)$ ,  $f_2(y)$ , see Remark 7.1.

The method for proving our main results is entirely different from the various homogenization methods. For clarity of notation we shall first consider, in Sections 1–6, the special case of uniform size grains. The case of non-uniform size grains is studied in Section 7. Finally, in Section 8 we extend our results to the case where  $\Omega$  is a general 2-dimensional domain, and in Section 9 we briefly consider non-orthogonal networks.

**§1. The problem.** Let  $\Omega = \{0 < x < 1, 0 < y < 1\}$  and let  $\alpha, \beta$  be positive constants and  $N$  a positive integer such that

$$(1.1) \quad (N + 1)\alpha + N\beta = 1 .$$

Set

$$x_i = y_i = i\alpha + \left(i - \frac{1}{2}\right)\beta \quad (1 \leq i \leq N)$$

and introduce vertical and horizontal “channels” of width  $\beta$

$$I_i = \left(x_i - \frac{\beta}{2}, x_i + \frac{\beta}{2}\right) \times (0, 1), \quad J_i = (0, 1) \times \left(y_i - \frac{\beta}{2}, y_i + \frac{\beta}{2}\right) .$$

Set

$$I = \bigcup_{i=1}^N I_i, \quad J = \bigcup_{i=1}^N J_i .$$

The boundary  $\partial\Omega$  of  $\Omega$  consists of

$$\Gamma_1 \text{ on } y = 0, \quad \Gamma_2 \text{ on } x = 0, \quad \Gamma_3 \text{ on } y = 1 \text{ and } \Gamma_4 \text{ on } x = 1 .$$

Consider the following problem (P): Find

$$(1.2) \quad u \in C^1(\bar{I} \cup \bar{J})$$

such that

$$(1.3) \quad \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } J \setminus \bar{I},$$

$$(1.4) \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } I \setminus \bar{J},$$

$$(1.5) \quad \Delta u = 0 \quad \text{in } I \cap J$$

and

$$(1.6) \quad \frac{\partial u}{\partial n} + u = F_i \quad \text{on } \Gamma_i \cap \partial(I \cup J) \quad (1 \leq i \leq 4)$$

where  $n$  is the outward normal.

It follows that  $u(x, y)$  must be linear in  $x$  in  $J \setminus \bar{I}$  and linear in  $y$  in  $I \setminus \bar{J}$ . Set

$$I_i^0 = I_i \cap \left\{ 0 < y < y_1 - \frac{\beta}{2} \right\}, \quad I_i^N = I_i \cap \left\{ y_N + \frac{\beta}{2} < y < 1 \right\},$$

$$I_i^m = I_i \cap \left\{ y_m + \frac{\beta}{2} < y < y_{m+1} - \frac{\beta}{2} \right\}, \quad 1 \leq m \leq N-1.$$

The  $i$ -th vertical channel  $I_i$  consists of  $N$  components  $I_i \cap J_j$  ( $1 \leq j \leq N$ ) and the  $N+1$  regions  $I_i^m$  which do not belong to  $J$ . Similarly the  $i$ -th horizontal channel  $J_i$  consists of  $N$  components  $J_i \cap I_k$  ( $1 \leq k \leq N$ ) and the  $N+1$  regions

$$J_i^0 = J_i \cap \left\{ 0 < x < x_1 - \frac{\beta}{2} \right\}, \quad J_i^N = J_i \cap \left\{ x_N + \frac{\beta}{2} < x < 1 \right\},$$

$$J_i^m = J_i \cap \left\{ x_m + \frac{\beta}{2} < x < x_{m+1} - \frac{\beta}{2} \right\}, \quad 1 \leq i \leq N-1,$$

From (1.3), (1.4) we get

$$(1.7) \quad \frac{\partial u}{\partial x} = \frac{1}{\alpha} \left[ u \left( x_{m+1} - \frac{\beta}{2}, y \right) - u \left( x_m + \frac{\beta}{2}, y \right) \right] \quad \text{in } J_i^m,$$

$$(1.8) \quad \frac{\partial u}{\partial y} = \frac{1}{\alpha} \left[ u \left( x, y_{m+1} - \frac{\beta}{2} \right) - u \left( x, y_m + \frac{\beta}{2} \right) \right] \quad \text{in } I_i^m$$

for  $1 \leq i \leq N$  and  $1 \leq m \leq N-1$ . Further,

$$(1.9) \quad \frac{\partial u}{\partial y} = \frac{1}{\alpha} \left[ u \left( x, y_1 - \frac{\beta}{2} \right) - u(x, 0) \right] \quad \text{in } I_i^0.$$

Using the boundary condition (1.6) we get

$$(1.10) \quad \left( \frac{\partial u}{\partial n} + \frac{u}{1+\alpha} \right) \left( x, y_1 - \frac{\beta}{2} \right) = \frac{F_1(x)}{1+\alpha} \quad \text{on } \partial I_1^0 \cap \left\{ y = y_1 - \frac{\beta}{2} \right\}$$

Analogous boundary conditions hold on  $y = y_N + \frac{\beta}{2}$  and on  $x = x_1 - \frac{\beta}{2}$ ,  $x = x_N + \frac{\beta}{2}$ .

We introduce a notation for the four edges of  $I_i \cap J_j$ :

$$\begin{aligned}\Gamma_1^{ij} &= \partial(I_i \cap J_j) \cap \left\{ y = y_j - \frac{\beta}{2} \right\}, & \Gamma_3^{ij} &= \partial(I_i \cap J_j) \cap \left\{ y = y_j + \frac{\beta}{2} \right\}, \\ \Gamma_2^{ij} &= \partial(I_i \cap J_j) \cap \left\{ x = x_i - \frac{\beta}{2} \right\}, & \Gamma_4^{ij} &= \partial(I_i \cap J_j) \cap \left\{ x = x_i + \frac{\beta}{2} \right\}.\end{aligned}$$

Denote by  $u_{ij}$  the restriction of  $u$  to  $I_i \cap J_j$ . Then problem (P) can be reduced to the following problem:

$$(1.11) \quad \Delta u_{ij} = 0 \quad \text{in } I_i \cap J_j \quad (1 \leq i, j \leq N),$$

$$(1.12) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{u_{ij}}{1+\alpha} = \frac{F_1}{1+\alpha} & \text{on } \Gamma_1^{i1}, j=1, \\ \left( \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} \right) \left( x, y_j - \frac{\beta}{2} \right) = \frac{1}{\alpha} u_{i,j-1} \left( x, y_{j-1} + \frac{\beta}{2} \right) & \text{on } \Gamma_1^{ij}, 1 < j < N, \end{cases}$$

with similar conditions on  $\Gamma_3^{ij}, \Gamma_2^{ij}, \Gamma_4^{ij}$ .

To deal with this system it is convenient to translate the regions  $I_i \cap J_j$  parallel to the coordinate axes so that they fill a rectangle

$$\tilde{\Omega} = \{0 < x < N\beta, 0 < y < N\beta\}.$$

Set

$$\tilde{I}_i = ((i-1)\beta, i\beta) \times (0, N\beta), \quad \tilde{J}_j = (0, N\beta) \times ((j-1)\beta, j\beta) \quad 1 \leq i, j \leq N,$$

and introduce the edges  $\tilde{\Gamma}_\ell^{ij}$  of  $\tilde{I}_i \cap \tilde{J}_j$  as follows

$$\begin{aligned}\tilde{\Gamma}_1^{ij} &= \partial(\tilde{I}_i \cap \tilde{J}_j) \cap \{y = (j-1)\beta\}, & \tilde{\Gamma}_3^{ij} &= \partial(\tilde{I}_i \cap \tilde{J}_j) \cap \{y = j\beta\}, \\ \tilde{\Gamma}_2^{ij} &= \partial(\tilde{I}_i \cap \tilde{J}_j) \cap \{x = (i-1)\beta\}, & \tilde{\Gamma}_4^{ij} &= \partial(\tilde{I}_i \cap \tilde{J}_j) \cap \{x = i\beta\}.\end{aligned}$$

We translate the independent variable so that the functions  $u_{ij}$  defined originally on  $I_i \cap J_j$  become functions (denoted again by  $u_{ij}$ ) defined on  $\tilde{I}_i \cap \tilde{J}_j$ . We can now state the original problem (P) in the following equivalent form:

*Problem ( $\tilde{P}$ ).* Find a family of functions  $u_{ij}$  ( $1 \leq i, j \leq N$ ) such that

$$(1.13) \quad u_{ij} \in C^1(\overline{\tilde{I}_i \cap \tilde{J}_j}),$$

$$(1.14) \quad \Delta u_{ij} = 0 \quad \text{in } \tilde{I}_i \cap \tilde{J}_j,$$

and the following boundary and interface conditions hold:

$$(1.15) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{1+\alpha} u_{ij} = \frac{1}{1+\alpha} \tilde{F}_1 & \text{on } \Gamma_2^{i1}, j=1, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} u_{i,j-1} & \text{on } \Gamma_2^{ij}, 1 < j < N, \end{cases}$$

$$(1.16) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{1+\alpha} u_{ij} = \frac{1}{1+\alpha} \tilde{F}_3 & \text{on } \Gamma_3^{iN}, j = N, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} u_{i,j+1} & \text{on } \Gamma_3^{ij}, 1 < j < N, \end{cases}$$

$$(1.17) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{1+\alpha} u_{ij} = \frac{1}{1+\alpha} \tilde{F}_2 & \text{on } \Gamma_2^{1j}, i = 1, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} u_{i-1,j} & \text{on } \Gamma_2^{ij}, 1 < i < N, \end{cases}$$

$$(1.18) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{1+\alpha} u_{ij} = \frac{1}{1+\alpha} \tilde{F}_4 & \text{on } \Gamma_4^{Nj}, i = N, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} u_{i+1,j} & \text{on } \Gamma_4^{ij}, 1 < i < N. \end{cases}$$

Here the  $\tilde{F}_\ell$  are defined by

$$(1.19) \quad \begin{aligned} \tilde{F}_\ell(i\beta + x) &= F_\ell\left(x_{i+1} - \frac{\beta}{2} + x\right) \quad \text{for } \ell = 1, 3 \quad (0 \leq i \leq N-1), \\ \tilde{F}_\ell(j\beta + y) &= F_\ell\left(y_{i+1} - \frac{\beta}{2} + y\right) \quad \text{for } \ell = 2, 4 \quad (0 \leq j \leq N-1). \end{aligned}$$

for  $0 \leq x \leq \beta$ ,  $0 \leq y \leq \beta$ . In the following sections we shall prove that problem  $(\tilde{P})$  has a unique solution and we shall determine the limit of the solution as  $\beta \rightarrow 0$ , while  $\alpha/\beta$  and  $N\beta$  converges to some positive numbers.

**§2. Uniqueness for  $(\tilde{P})$ .** Let  $\Omega = \{0 < x < 1, 0 < y < 1\}$  and introduce its four edges  $\Gamma_i$  as in Section 1. Consider a solution  $w \in W^{1,2}(\Omega)$  of

$$\begin{aligned} \Delta w &= 0 \quad \text{in } \Omega, \\ \frac{\partial w}{\partial n} &= f_\ell \quad \text{on } \Gamma_\ell \quad (1 \leq \ell \leq 4). \end{aligned}$$

**LEMMA 2.1.** *Let  $0 < \gamma < 1$ . (i) If  $f_\ell \in C^\gamma(\Gamma_\ell)$  for  $1 \leq \ell \leq 4$  then  $w \in C^{1+\gamma}(\overline{\Omega})$ . (ii) If  $f_1, f_2$  belong to  $C^{1+\gamma}(\Gamma_1)$  and  $C^{1+\gamma}(\Gamma_2)$ , respectively, and  $f_1'(0) = f_2'(0)$ , then  $w \in C^{2+\gamma}(\overline{\Omega} \cap B_{1/2}(0))$ , where  $B_r(O)$  is a ball of radius  $r$  with center at the origin  $O = (0, 0)$ .*

*Proof.* To prove (i) we only need to consider the smoothness of  $w$  near a corner, say near  $O$ ; the regularity elsewhere follows by standard elliptic theory. Extend  $f_2$  as  $C^\gamma$  function to  $\{x = 0, y < 0\}$  and choose a function  $w_0$  such that

$$\begin{aligned} \Delta w_0 &= 0 \quad \text{in } B_{2/3}(O) \cap \{x > 0\}, \\ \frac{\partial w_0}{\partial n} &= f_2 \quad \text{on } B_{2/3}(O) \cap \{x = 0\}; \end{aligned}$$

then  $w_0 \in C^{1+\gamma}(B_{1/2}(O) \cap \{x \geq 0\})$ . The function  $v = w - w_0$  satisfies

$$\begin{aligned} \Delta v &= 0 \quad \text{in} \quad B_{1/2}(O) \cap \Omega, \\ \frac{\partial v}{\partial x} &= 0 \quad \text{on} \quad B_{1/2}(O) \cap \Gamma_2, \\ \frac{\partial v}{\partial y} &= f_1 - \frac{\partial w_0}{\partial y} \equiv \tilde{f}_1 \in C^\gamma \quad \text{on} \quad B_{1/2}(O) \cap \Gamma_1. \end{aligned}$$

We extend  $v$  to  $x < 0$  by reflection:  $v(x, y) = v(-x, y)$ ; then  $\partial v / \partial y = \tilde{f}_1$  on  $y = 0$  where  $\tilde{f}_1$  is in  $C^\gamma$ . Therefore  $v \in C^{1+\gamma}(B_{1/2}(O) \cap y \geq 0)$ , which implies that  $w \in C^{1+\gamma}(B_{1/2}(O) \cap \Omega)$ .

To prove (ii) we observe that the function  $z = w_y$  satisfies

$$\begin{aligned} \Delta z &= 0 \quad \text{in} \quad \Omega, \\ z &= -f_1 \quad \text{on} \quad \Gamma_1, \quad \frac{\partial z}{\partial n} = f'_2 \quad \text{on} \quad \Gamma_2. \end{aligned}$$

Extend  $f'_2$  as a  $C^\gamma$  function to  $y < 0$  and take  $w_2$  in  $C^{1+\gamma}(B_{2/3}(O) \cap \{x \geq 0\})$  such that

$$(2.1) \quad \begin{aligned} \Delta w_1 &= 0 \quad \text{in} \quad B_{2/3}(O) \cap \{x > 0\}, \\ \frac{\partial w_1}{\partial n} &= f'_2 \quad \text{on} \quad B_{2/3}(O) \cap \{x = 0\}. \end{aligned}$$

Then the function  $\tilde{v} = z - w_1$  satisfies

$$(2.2) \quad \begin{aligned} \Delta \tilde{v} &= 0 \quad \text{in} \quad B_{1/2}(O) \cap \Omega, \\ \tilde{v} &= -f_1 - w_1 = \hat{f}_1 \quad \text{on} \quad B_{1/2}(O) \cap \Gamma_1, \\ \frac{\partial \tilde{v}}{\partial n} &= 0 \quad \text{on} \quad B_{1/2}(O) \cap \Gamma_2. \end{aligned}$$

We extend  $\tilde{v}$  to  $x < 0$  by reflection. The extended function  $\tilde{v}$  is harmonic in  $B_{1/2}(O) \cap \{y > 0\}$  and

$$\tilde{v}(x, 0) = \hat{f}(x) = \begin{cases} \hat{f}_1(x) & \text{if } x \geq 0 \\ \hat{f}_1(-x) & \text{if } x < 0. \end{cases}$$

By (2.1), (2.2) we have

$$\frac{d}{dx} \hat{f}_1(0) = -f'_1(0) - \frac{\partial w_1(0, 0)}{\partial x} = -f'_1(0) + f'_2(0) = 0$$

where we have used the compatibility assumption  $f'_1(0) = f'_2(0)$ . It follows that  $\hat{f} \in C^{1+\gamma}$  and therefore  $\tilde{v} \in C^{1+\gamma}(B_{1/2}(O) \cap \bar{\Omega})$ . Hence also  $w_y \in C^{1+\gamma}$  in  $B_{1/2}(O) \cap \Omega$ . Similarly we can prove that  $w_x \in C^{1+\gamma}$  in  $B_{1/2}(O) \cap \Omega$ , and the proof of the lemma is complete.

We return to problem  $(\tilde{P})$  and assume that

$$(2.3) \quad F_\ell \in L^\infty(\Gamma_\ell).$$

In the sequel we identify a solution  $\{u_{ij}\}$  of problem  $(\tilde{P})$  with the function  $u$  which takes the values  $u_{ij}$  on  $\tilde{I}_i \cap \tilde{J}_j$ .

LEMMA 2.2. Let  $u = \{u_{ij}\}$  be a solution to problem  $(\tilde{P})$ ,  $u \not\equiv \text{const}$ . Then  $u$  cannot take maximum or minimum at any interior point of  $\tilde{\Omega}$ .

*Proof.* Suppose  $u$  takes a maximum  $M$  at an interior point  $(x_0, y_0)$ :

$$(2.4) \quad u_{ij}(x, y) \leq u_{i_0 j_0}(x_0, y_0) = M \quad \text{for all } (x, y) \in \tilde{\Omega}.$$

Consider first the case where  $(x_0, y_0)$  is an interior point of the square  $\tilde{I}_{i_0} \cap \tilde{J}_{j_0}$ . Then, by the maximum principle,  $u_{i_0 j_0} \equiv M$  and, by the interface conditions,

$$u_{i_0+1, j_0} = M \quad \text{and} \quad \frac{\partial}{\partial n} u_{i_0+1, j_0} = 0 \quad \text{on } \tilde{\Gamma}_4^{i_0 j_0}.$$

But then, by the maximum principle, also  $u_{i_0+1, j_0} \equiv M$  in  $\tilde{I}_{i_0+1} \cap \tilde{J}_{j_0}$ . Similarly we deduce that  $u_{ij} \equiv M$  for all  $i, j$ , which is a contradiction. We conclude that  $(x_0, y_0)$  must lie on the boundary of some  $\tilde{I}_{i_0} \cap \tilde{J}_{j_0}$ , say in  $\tilde{\Gamma}_3^{i_0 j_0}$ , and  $u_{ij} \not\equiv M$  for any  $i, j$ .

If  $(x_0, y_0)$  is an interior point of  $\tilde{\Gamma}_3^{i_0 j_0}$  then  $\partial u_{i_0 j_0} / \partial n > 0$  at that point, so that  $u_{i_0 j_0}(x_0, y_0) < u_{i_0, j_0+1}(x_0, y_0)$ , a contradiction to (2.4). Therefore  $(x_0, y_0)$  must be an endpoint of  $\tilde{\Gamma}_3^{i_0 j_0}$ , say the right-end point. Since  $u_{i_0 j_0}$  takes its maximum on  $\tilde{\Gamma}_3^{i_0 j_0}$  at  $(x_0, y_0)$ ,

$$\lim_{x \nearrow x_0} \frac{\partial}{\partial x} u_{i_0 j_0}(x, y_0) \geq 0$$

so that, by the interface condition on  $\tilde{\Gamma}_4^{i_0 j_0}$ ,  $u_{i_0 j_0}(x_0, y_0) \leq u_{i_0+1, j_0}(x_0, y_0)$ . Recalling (2.4) we conclude that  $u_{i_0+1, j_0} = M$  at  $(x_0, y_0)$  and similarly

$$(2.5) \quad u_{i_0 j_0+1} = u_{i_0+1, j_0} = u_{i_0 j_0} = M \quad \text{at } (x_0, y_0).$$

By Lemma 2.1 (i)  $u_{i_0 j_0} \in C^{1+\gamma}$  in  $\overline{\tilde{I}_{i_0} \cap \tilde{J}_{j_0}}$ -neighborhood of  $(x_0, y_0)$ , and, similarly, the same regularity holds also for  $u_{i_0+1, j_0}$ ,  $u_{i_0, j_0+1}$ . Writing

$$(2.6) \quad \begin{aligned} \frac{\partial u_{i_0 j_0}}{\partial n} &= \frac{1}{\alpha} (u_{i_0, j_0+1} - u_{i_0 j_0}) \equiv f_3 \quad \text{on } \tilde{\Gamma}_3^{i_0 j_0}, \\ \frac{\partial u_{i_0 j_0}}{\partial n} &= \frac{1}{\alpha} (u_{i_0+1, j_0} - u_{i_0 j_0}) \equiv f_4 \quad \text{on } \tilde{\Gamma}_4^{i_0 j_0}, \end{aligned}$$

we conclude that  $f_3, f_4$  are in  $C^{1+\gamma}$ . Furthermore, by (2.5),  $\nabla u_{i_0 j_0}(x_0, y_0) = 0$ . Similarly

$$(2.7) \quad \nabla u_{i_0+1, j_0} = \nabla u_{i_0, j_0+1} = \nabla u_{i_0+1, j_0+1} = \nabla u_{i_0 j_0} = 0 \quad \text{at } (x_0, y_0).$$

This implies that  $f_3'(x_0) = f_4'(y_0) = 0$ . Hence, by Lemma 2.1 (ii),  $u_{i_0 j_0} \in C^{2+\gamma}$  in  $\overline{\tilde{I}_{i_0} \cap \tilde{J}_{j_0}}$ -neighborhood of  $(x_0, y_0)$ . From (2.6), (2.7) it follows that

$$(2.8) \quad \frac{\partial^2 u_{i_0 j_0}}{\partial x \partial y}(x_0, y_0) = 0.$$

For any direction  $\nu$ , pointing from  $(x_0, y_0)$  into  $\tilde{I}_{i_0} \cap \tilde{J}_{j_0}$ , we have

$$\frac{\partial^2 u_{i_0 j_0}}{\partial \nu^2} \leq 0 \quad \text{at } (x_0, y_0) .$$

Writing  $\nu = (\cos \theta, \sin \theta)$ ,  $-\frac{\pi}{2} < \theta < 0$ , we get

$$\cos \theta \frac{\partial^2 u_{i_0 j_0}}{\partial x^2} + \sin 2\theta \frac{\partial^2 u_{i_0 j_0}}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 u_{i_0 j_0}}{\partial y^2} \leq 0 .$$

Recalling that  $u_{i_0 j_0}$  is harmonic and that it satisfies (2.8), we get

$$(\cos^2 \theta - \sin^2 \theta) \frac{\partial^2 u_{i_0 j_0}}{\partial x^2} \leq 0 \quad \text{if } -\frac{\pi}{2} < \theta < 0 .$$

Consequently  $\partial^2 u_{i_0 j_0} / \partial x^2 = 0$  and thus all the second derivatives of  $u_{i_0 j_0}$  vanish at  $(x_0, y_0)$ .

Introduce the complex variable  $z$  with center at  $(x_0, y_0)$  such that  $\tilde{\Gamma}^{i_0 j_0}$  lies on the negative real axis. The function  $w(z) = u_{i_0 j_0}(z^{1/2})$  is harmonic in  $D = \{|z| < \delta, \text{Im} z > 0\}$  for some  $\delta > 0$  and takes its maximum  $M$  at  $z = 0$ . Further, since the first and second derivatives of  $u_{i_0 j_0}(x, y)$  vanish at  $(x_0, y_0)$ , the first derivatives of  $w(z)$  vanish at  $z = 0$ ; in particular,

$$\frac{\partial w}{\partial n} = 0 \quad \text{at } z = 0 ,$$

a contradiction to the maximum principle.

**THEOREM 2.3.** *If  $u = \{u_{ij}\}$  is a solution to problem  $(\tilde{P})$  then*

$$(2.9) \quad \|u\|_{L^\infty} \leq \max_{\ell} \{\|F_\ell\|_{L^\infty}\} .$$

Indeed, by Lemma 2.2, the maximum of  $u$  must be attained at some boundary point  $(x_0, y_0) \in \partial(\tilde{I}_{i_0} \cap \tilde{J}_{j_0}) \cap \Gamma_\ell$ . Then  $\partial u / \partial n \geq 0$  at  $(x_0, y_0)$  and, by the boundary conditions,  $u_{i_0 j_0} \leq F_\ell$  at that point. Similarly one can estimate the minimum of  $u$ .

**COROLLARY 2.4.** *There exists at most one solution to problem  $(\tilde{P})$ .*

**§3. Existence for  $(\tilde{P})$ .** Set

$$X = \prod_{i,j=1}^N C^{1+\gamma}(\overline{\tilde{I}_i \cap \tilde{J}_j})$$

and introduce the norm

$$\|u\|_X = \sum_{i,j=1}^N \|u_{ij}\|_{C^{1+\gamma}(\overline{\tilde{I}_i \cap \tilde{J}_j})} .$$

We shall assume:

$$(3.1) \quad F_\ell \in C^\gamma(\Gamma_\ell) \quad (1 \leq \ell \leq 4) .$$

THEOREM 3.1. If (3.1) holds then there exists a unique solution to problem  $(\tilde{P})$ , and

$$(3.2) \quad \|u\|_X < \infty$$

*Proof.* For any  $u \in X$  introduce  $v = \{v_{ij}\}$  as the solution to

$$(3.3) \quad \Delta v_{ij} = 0 \quad \text{in} \quad \tilde{I}_i \cap \tilde{J}_j$$

with the boundary conditions

$$(3.4) \quad \begin{cases} \frac{\partial v_{ij}}{\partial n} + \frac{1}{1+\alpha} v_{ij} = \frac{1}{1+\alpha} \tilde{F}_1 & \text{on} \quad \tilde{\Gamma}_1^{i1}, j=1, \\ \frac{\partial v_{ij}}{\partial n} + \frac{1}{\alpha} v_{ij} = \frac{1}{\alpha} u_{i,j-1} & \text{on} \quad \tilde{\Gamma}_1^{ij}, 1 < j \leq N, \end{cases}$$

$$(3.5) \quad \begin{cases} \frac{\partial v_{ij}}{\partial n} + \frac{1}{1+\alpha} v_{ij} = \frac{1}{1+\alpha} \tilde{F}_3 & \text{on} \quad \Gamma_3^{iN}, j=N, \\ \frac{\partial v_{ij}}{\partial n} + \frac{1}{\alpha} v_{ij} = \frac{1}{\alpha} u_{i,j+1} & \text{on} \quad \Gamma_3^{ij}, 1 \leq j < N, \end{cases}$$

$$(3.6) \quad \begin{cases} \frac{\partial v_{ij}}{\partial n} + \frac{1}{1+\alpha} v_{ij} = \frac{1}{1+\alpha} \tilde{F}_2 & \text{on} \quad \tilde{\Gamma}_2^{1j}, i=1, \\ \frac{\partial v_{ij}}{\partial n} + \frac{1}{\alpha} v_{ij} = \frac{1}{\alpha} u_{i-1,j} & \text{on} \quad \tilde{\Gamma}_2^{ij}, 1 < i \leq N, \end{cases}$$

$$(3.7) \quad \begin{cases} \frac{\partial v_{ij}}{\partial n} + \frac{1}{1+\alpha} v_{ij} = \frac{1}{1+\alpha} \tilde{F}_4 & \text{on} \quad \tilde{\Gamma}_4^{Nj}, i=N, \\ \frac{\partial v_{ij}}{\partial n} + \frac{1}{\alpha} v_{ij} = \frac{1}{\alpha} u_{i+1,j} & \text{on} \quad \tilde{\Gamma}_4^{ij}, 1 \leq i < N. \end{cases}$$

We write  $v = Tu$ . By Lemma 3.1

$$(3.8) \quad \|Tu\|_X \leq C \left\{ \sum_{i,j=1}^N \|u_{ij}\|_{C^\gamma(\overline{\tilde{I}_i \cap \tilde{J}_j})} + \sum_{\ell=1}^4 \|\tilde{F}_\ell\|_{C^\gamma} \right\}$$

where  $C$  is a constant depending on  $\alpha$  and  $\beta$ . Set

$$K = \max_{\ell} \|F_\ell\|_\infty$$

and introduce a subset  $Y$  of  $X$  by

$$Y = \left\{ u = \{u_{ij}\} \in X, \|u\|_{L^\infty} \leq K, \right. \\ \left. \|u\|_1 \equiv \sum_{i,j=1}^N \|u_{ij}\|_{C^1(\overline{\tilde{I}_i \cap \tilde{J}_j})} \leq M \right\}$$

where  $M$  is a positive constant.

By the maximum principle,

$$(3.9) \quad \|Tu\|_{L^\infty} \leq \max\{\|u\|_{L^\infty}, K\} = K.$$

From (3.8) and interpolation

$$(3.10) \quad \begin{aligned} \|Tu\|_1 &\leq \|Tu\|_X \leq C \left\{ \sum_{i,j=1}^N \|u_{ij}\|_{C^\gamma} + \sum_{\ell=1}^4 \|\tilde{F}_\ell\|_{C^\gamma} \right\} \\ &\leq C\delta \|u\|_1 + C(\delta) \sum_{i,j=1}^N \|u_{ij}\|_{L^\infty} + C \sum_{\ell=1}^4 \|\tilde{F}_\ell\|_{C^\gamma} \end{aligned}$$

for any small  $\delta > 0$ . Since the sum  $\sum \|u_{ij}\|_{L^\infty}$  is bounded by  $KN$ , taking  $\delta = 1/(2C)$  we deduce that

$$\|Tu\|_1 \leq \frac{1}{2} \|u\|_1 + C_1(\delta)$$

where  $C_1(\delta)$  is independent of  $u$ . Hence if  $M \geq 2C_1(\delta)$  then  $T$  maps  $Y$  into itself. From (3.10) we see that  $T$  is compact and continuous. Applying the Schauder fixed point theorem we conclude that  $T$  has a fixed point  $u$ , which is the asserted solution to problem  $(\tilde{P})$ .

**§4. Estimates independent of  $\alpha, \beta$ .** In this section we establish estimates on the solution  $u = \{u_{ij}\}$ , independently of  $\alpha$  and  $\beta$ . These estimates will be used in Section 5 to determine the limit behavior of the solutions as  $\beta \rightarrow 0$ ,  $\alpha \rightarrow 0$ ,  $N \rightarrow \infty$ . We shall assume that

$$(4.1) \quad \alpha = \lambda\beta \quad \text{where} \quad 0 < \lambda < \infty$$

and  $\lambda$  remains constant as  $\beta \rightarrow 0$ . For clarity we shall denote the domain  $\tilde{\Omega}$  by  $\tilde{\Omega}_\beta$  and the solution  $u$  by  $u_\beta$ , and write  $u_\beta = \{u_{ij}\}$ .

**LEMMA 4.1.** *There exists a constant  $C$  independent of  $\beta$  such that the solution  $u_\beta = \{u_{ij}\}$  satisfies:*

$$(4.2) \quad \sum_{i,j=1}^N \int_{\tilde{I}_i \cap \tilde{J}_j} |\nabla u_{ij}|^2 \leq C,$$

$$(4.3) \quad \sum_{i=1}^N \sum_{j=2}^N \int_{\tilde{\Gamma}_1^{ij}} |u_{ij} - u_{i,j-1}|^2 + \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{ij}} |u_{ij} - u_{i-1,j}|^2 \leq C\alpha.$$

*Proof.* By the divergence theorem and (1.13)–(1.18),

$$\begin{aligned} \int_{\tilde{I}_i \cap \tilde{J}_j} |\nabla u_{ij}|^2 &= \frac{1}{\alpha} \int_{\tilde{\Gamma}_1^{ij}} (u_{i,j-1} - u_{ij})u_{ij} + \frac{1}{\alpha} \int_{\tilde{\Gamma}_3^{ij}} (u_{i,j+1} - u_{ij})u_{ij} \\ &\quad + \frac{1}{\alpha} \int_{\tilde{\Gamma}_2^{ij}} (u_{i-1,j} - u_{ij})u_{ij} + \frac{1}{\alpha} \int_{\tilde{\Gamma}_4^{ij}} (u_{i+1,j} - u_{ij})u_{ij} \end{aligned}$$

for  $1 < i, j < N$ . For  $i = 1, N$  or  $j = 1, N$  we have similar results. Taking summation we get

$$\begin{aligned} & \sum_{i,j=1}^N \int_{\tilde{I}_i \cap \tilde{J}_j} |\nabla u_{ij}|^2 + \frac{1}{\alpha} \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{ij}} |u_{ij} - u_{i-1,j}|^2 \\ & + \frac{1}{\alpha} \sum_{i=1}^N \sum_{j=2}^N \int_{\tilde{\Gamma}_1^{ij}} |u_{ij} - u_{i,j-1}|^2 + \frac{1}{1+\alpha} \int_{\partial \tilde{\Gamma}} u^2 = \frac{1}{1+\alpha} \int_{\partial \tilde{\Omega}} u \tilde{F} \end{aligned}$$

where  $\tilde{F} = \tilde{F}_\ell$  on  $\tilde{\Gamma}_\ell$ , and this implies the assertions (4.2), (4.3).

For any small  $\delta > 0$ , let  $G_\delta$  denote the square in  $\tilde{\Omega}_\beta$  whose four sides are parallel to the corresponding sides of  $\tilde{\Omega}_\beta$  such that the distance between each pair is  $\delta$ .

LEMMA 4.2. *There exists a constant  $C$  independent of  $\beta$  and  $\delta$  such that, for any  $h = (h_1, h_2)$  with  $|h| \leq \delta/2$ ,  $\delta \geq 2\beta$ ,*

$$(4.4) \quad \int_{G_\delta} |u_\beta(x + h_1, y + h_2) - u_\beta(x, y)|^2 dx dy \leq C \delta^2 .$$

*Proof.* It will suffice to establish (4.4) for  $h = (h_1, 0)$ ,  $h_1 > 0$ . Consider first the case  $0 < h_1 \leq \beta$ . Then

$$\int_{G_\delta} |u_\beta(x + h_1, y) - u_\beta(x, y)|^2 = \sum_{i,j=1}^N \int_{G_\delta \cap \tilde{I}_i \cap \tilde{J}_j} |u_\beta(x + h_1, y) - u_{ij}(x, y)|^2 .$$

We proceed to estimate each term on the right-hand side. For  $(x, y) \in \tilde{I}_i \cap \tilde{J}_j$  and  $i \neq N$ ,

$$u_\beta(x + h_1, y) = \begin{cases} u_{ij}(x + h_1, y) & \text{if } (i-1)\beta < x < i\beta - h_1 , \\ u_{i+1,j}(x + h_1, y) & \text{if } i\beta - h_1 < x < i\beta , \end{cases}$$

so that

$$\begin{aligned} \int_{\tilde{I}_i \cap \tilde{J}_j} |u_\beta(x + h_1, y) - u_{ij}(x, y)|^2 &= \int_{(j-1)\beta}^{j\beta} dy \int_{(i-1)\beta}^{i\beta-h_1} |u_{ij}(x + h_1, y) - u_{ij}(x, y)|^2 dx \\ &+ \int_{(j-1)\beta}^{j\beta} dy \int_{i\beta-h_1}^{i\beta} |u_{i+1,j}(x + h_1, y) - u_{ij}(x, y)|^2 dx \equiv K_1 + K_2 . \end{aligned}$$

We have

$$\begin{aligned} K_1 &\leq \int_{(j-1)\beta}^{j\beta} dy \int_{(i-1)\beta}^{i\beta-h_1} \left[ \int_0^1 \frac{\partial}{\partial t} u_{ij}(x + th_1, y) dt \right]^2 dx \leq h_1^2 \int_{\tilde{I}_i \cap \tilde{J}_j} |\nabla u_{ij}|^2 , \\ K_2 &\leq 3 \int_{(j-1)\beta}^{j\beta} dy \int_{(i\beta-h_1)\beta}^{i\beta} \left\{ |u_{i+1,j}(x + h_1, y) - u_{i+1,j}(i\beta, y)|^2 \right. \\ &\quad \left. + |u_{i+1,j}(i\beta, y) - u_{ij}(i\beta, y)|^2 + |u_{ij}(i\beta, y) - u_{ij}(x, y)|^2 \right\} dx \end{aligned}$$

The integral of the integrand's middle term is bounded by  $h_1 \int_{\tilde{\Gamma}_4^{ij}} |u_{i+1,j} - u_{ij}|^2$  whereas the integral of each of the remaining two terms in the integrand can be estimated in the same way as  $K_1$ . We thus obtain

$$\begin{aligned} \int_{\tilde{I}_i \cap \tilde{J}_j} |u_\beta(x + h_1, y) - u_{ij}(x, y)|^2 &\leq 4h_1^2 \int_{\tilde{I}_i \cap \tilde{J}_j} |\nabla u_{ij}|^2 \\ &+ 3h_1^2 \int_{\tilde{I}_{i+1} \cap \tilde{J}_j} |\nabla u_{i+1,j}|^2 + 3h_1 \int_{\tilde{\Gamma}_4^{ij}} |u_{i+1,j} - u_{ij}|^2. \end{aligned}$$

For  $i = N$  we have

$$\int_{(\tilde{I}_i \cap \tilde{J}_j) \cap G_\delta} |u_\beta(x + h_1, y) - u_{ij}(x, y)|^2 \leq h_1^2 \int_{\tilde{I}_i \cap \tilde{J}_j} |\nabla u_{ij}|^2.$$

Summing over  $i, j$  and using Lemma 4.1 and (4.1), the assertion (4.4) follows.

Consider next the case  $h_1 > \beta$  and write  $h_1 = k\beta + \theta$ ,  $0 < \theta \leq \beta$ . For  $i < N - k$ , if  $(x, y) \in \tilde{I}_i \cap \tilde{J}_j$  then  $(i-1)\beta < x < i\beta$  and

$$u_\beta(x + h_1, y) = \begin{cases} u_{i+k,j}(x + h_1, y), & x < i\beta - \theta \\ u_{i+k+1,j}(x + h_1, y), & x > i\beta - \theta. \end{cases}$$

Hence

$$\begin{aligned} (4.5) \quad \int_{\tilde{I}_i \cap \tilde{J}_j} |u_\beta(x + h_1, y) - u_{ij}(x, y)|^2 &= \int_{(j-1)\beta}^{j\beta} dy \int_{(i-1)\beta}^{i\beta-\theta} |u_{i+k}(x + h_1, y) - u_{ij}(x, y)|^2 dx \\ &+ \int_{(j-1)\beta}^{j\beta} dy \int_{i\beta-\theta}^{i\beta} |u_{i+k+1,j}(x + h_1, y) - u_{ij}(x, y)|^2 dx \equiv \bar{K}_1 + \bar{K}_2. \end{aligned}$$

We can write

$$\begin{aligned} u_{i+k,j}(x + h_1, y) - u_{ij}(x, y) &= [u_{i+k,j}(x + h_1, y) - u_{i+k,j}((i+k-1)\beta, y)] \\ &+ \sum_{\ell=1}^k [u_{i+\ell,j}((i+\ell-1)\beta, y) - u_{i+\ell-1,j}((i+\ell-1)\beta, y)] \\ &+ \sum_{\ell=1}^k [u_{i+\ell-1,j}((i+\ell-1)\beta, y) - u_{i+\ell-1,j}((i+\ell-2)\beta, y)] \\ &+ [u_{ij}((i-1)\beta, y) - u_{ij}(x, y)] \end{aligned}$$

and using the inequality

$$(4.6) \quad \left( \sum_{\ell=1}^n a_\ell \right)^2 \leq n \sum_{\ell=1}^n a_\ell^2$$

we find that

$$\begin{aligned}
\bar{K}_1 &\leq 2(k+1) \int_{(j-1)\beta}^{j\beta} dy \int_{(i-1)\beta}^{i\beta-\theta} \left\{ |u_{ij}((i-1)\beta, y) - u_{ij}(x, y)|^2 \right. \\
&\quad \left. + |u_{i+k,j}(x+h_1, y) - u_{i+k,j}((i+k-1)\beta, y)|^2 \right\} dx \\
&+ 2(k+1) \sum_{\ell=1}^k \int_{(j-1)\beta}^{j\beta} dy \int_{(i-1)\beta}^{i\beta-\theta} |u_{i+\ell,j}((i+\ell-1)\beta, y) - u_{i+\ell-1,j}((i+\ell-1)\beta, y)|^2 dx \\
&+ 2(k+1) \sum_{\ell=1}^k \int_{(j-1)\beta}^{j\beta} dy \int_{(i-1)\beta}^{i\beta-\theta} |u_{i+\ell-1,j}((i+\ell-1)\beta, y) - u_{i+\ell-1,j}((i+\ell-2)\beta, y)|^2 dx \\
&= 2(k+1)(K_{11} + K_{12} + K_{13}) .
\end{aligned}$$

We can estimate  $K_{11}$  as  $K_1$  above and obtain

$$K_{11} \leq 2\beta^2 \int_{\tilde{I}_i \cap \tilde{J}_i} |\nabla u_{ij}|^2 + \int_{\tilde{I}_{i+k} \cap \tilde{J}_j} |\nabla u_{i+k,j}|^2 .$$

Similarly

$$K_{13} \leq \beta^2 \sum_{\ell=1}^k \int_{\tilde{I}_{i+\ell-1} \cap \tilde{J}_j} |\nabla u_{i+\ell-1,j}|^2 .$$

Hence

$$\bar{K}_1 \leq 4(k+2)\beta^2 \sum_{\ell=i}^{i+k} \int_{\tilde{I}_\ell \cap \tilde{J}_j} |\nabla u_{\ell j}|^2 + 2(k+2)\beta \sum_{\ell=i+1}^{i+k} \int_{\tilde{\Gamma}_2^{\ell j}} |u_{\ell j} - u_{\ell-1,j}|^2 .$$

Since  $\bar{K}_2$  can be estimated similarly to  $K_1$  above, we conclude from (4.5) that

$$\begin{aligned}
(4.7) \quad &\int_{\tilde{I}_i \cap \tilde{J}_j} |u_\beta(x+h_1, y) - u_\beta(x, y)|^2 \leq 8(k+2)\beta^2 \sum_{\ell=i}^{i+k+1} \int_{\tilde{I}_\ell \cap \tilde{J}_j} |\nabla u_{\ell j}|^2 \\
&+ 4(k+2)\beta \sum_{\ell=i+1}^{i+k+1} \int_{\tilde{\Gamma}_2^{\ell j}} |u_{\ell j} - u_{\ell-1,j}|^2
\end{aligned}$$

provided  $i < N - k$ .

Observe next that if  $i \geq N - k$  then  $\tilde{I}_i \cap \tilde{J}_j$  does not intersect  $G_\delta$  since

$$(i-1)\beta \geq (N-k-1)\beta = (N-1)\beta - k\beta = (N-1)\beta - h_1 + \theta \geq (N-1)\beta - \frac{\delta}{2} + \theta > N\beta - \delta .$$

Hence summing in (4.7) over all  $1 \leq i \leq N - k - 1$ ,  $1 \leq j \leq N$ , we get

$$\begin{aligned} \int_{G_\delta} |u_\beta(x + h_1, y) - u_\beta(x, y)|^2 &\leq 8(k + 2)^2 \beta^2 \sum_{i,j=1}^N \int_{\tilde{I}_i \cap \tilde{J}_j} |\nabla u_{ij}|^2 \\ &+ 4(k + 2)^2 \beta \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_{ij}^2} |u_{i-1,j} - u_{ij}|^2 . \end{aligned}$$

Recalling Lemma 4.1 we then conclude that the right-hand side of (4.4) is bounded by  $C(k + 2)^2 \beta^2 + C(k + 2)\beta\alpha$ . Since  $k\beta \leq h_1$  and  $\alpha \leq C\beta$  (by (4.1)), (4.4) follows.

Set

$$\tilde{\Omega}_0 = \left\{ 0 < x < \frac{1}{\lambda + 1}, 0 < y < \frac{1}{\lambda + 1} \right\} .$$

Recall, by (1.1), (4.1), that

$$(4.8) \quad ((N + 1)\lambda + N)\beta = 1 .$$

Hence,

$$(4.9) \quad \tilde{\Omega}_\beta \rightarrow \tilde{\Omega}_0 \quad \text{if } \beta \rightarrow 0 .$$

**THEOREM 4.3.** *For any sequence  $\beta = \tilde{\beta}_n$  converging to zero there exists a subsequence  $\beta = \beta_n$  and a function  $u$  in  $L^2(\tilde{\Omega}_0)$  such that*

$$(4.10) \quad \int_G |u_{\beta_n} - u|^2 \rightarrow 0$$

for any compact subset  $G \subset \tilde{\Omega}_0$ .

*Proof.* For any compact subset  $G \subset \tilde{\Omega}_0$

$$\int_G |u_\beta(x + h_1, y + h_2) - u_\beta(x, y)|^2 \leq C|h|^2 \quad (\text{by Lemma 4.2})$$

if  $|h|$  is small enough. Since furthermore the  $u_\beta$  are uniformly bounded, the set  $\{u_\beta\}$  is precompact in  $L^2(G)$ . The assertion of the theorem now follows by the diagonalization argument.

**§5. Homogenization.** Introduce the function

$$(u_\beta)_x = \{u_{ij,x}\} ,$$

i.e.,  $(u_\beta)_x$  is the function which coincides in each  $\tilde{I}_i \cap \tilde{J}_j$  with the derivative  $u_{ij,x}$ ; it is *not* the weak  $x$ -derivative of  $u_\beta$ . Similarly we introduce the function

$$(u_\beta)_y = \{u_{ij,y}\} .$$

and set  $\nabla u_\beta = ((u_\beta)_x, (u_\beta)_y)$ . By Lemma 4.1

$$(5.1) \quad \int_{\tilde{\Omega}_\beta} |\nabla u_\beta|^2 \leq C, \quad C \text{ independent of } \beta$$

It follows that the assertion of Theorem 4.3 can be supplemented by

$$(5.2) \quad (u_{\beta_n})_x \rightarrow u_x^*, \quad (u_{\beta_n})_y \rightarrow u_y^* \quad \text{weakly in } L^2(G)$$

for any compact subset  $G$  in  $\tilde{\Omega}_0$ , where  $u_x^*, u_y^*$  are functions in  $L^2(\tilde{\Omega}_0)$ , not necessarily weak derivatives of  $u$ .

In the sequel we shall need the following integration by parts formula: For any  $\xi \in C^\infty(\tilde{\Omega}_\beta)$ ,

$$(5.3) \quad \begin{aligned} \int_{\tilde{\Omega}_\beta} (u_\beta)_x \xi &= \sum_{i,j=1}^N \int_{\tilde{I}_i \cap \tilde{J}_j} u_{i,j,x} \xi = - \sum_{i,j=1}^N \int_{\tilde{I}_i \cap \tilde{J}_j} u_{i,j} \xi_x \\ &+ \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{ij}} (u_{i-1,j} - u_{i,j}) \xi + \sum_{j=1}^N \left[ \int_{\tilde{\Gamma}_4^{Nj}} u_{Nj} \xi - \int_{\tilde{\Gamma}_2^{1j}} u_{1j} \xi \right]. \end{aligned}$$

We also note that since  $\Delta u_{ij} = 0$  in  $\tilde{I}_i \cap \tilde{J}_j$ ,

$$\int_{\tilde{I}_i \cap \tilde{J}_j} \nabla u_{ij} \nabla \xi = \int_{\partial(\tilde{I}_i \cap \tilde{J}_j)} \frac{\partial u_{ij}}{\partial n} \xi$$

for any  $\xi \in C^\infty(\tilde{\Omega}_\beta)$ . Summing over  $i, j$  and using the interface and boundary conditions, we obtain

$$(5.4) \quad \int_{\tilde{\Omega}_\beta} \nabla u_\beta \cdot \nabla \xi + \frac{1}{1+\alpha} \int_{\partial \tilde{\Omega}_\beta} u_\beta \xi = \frac{1}{1+\alpha} \int_{\partial \tilde{\Omega}_\beta} \tilde{F} \xi.$$

Denote by  $\tilde{F}_\beta$  the function which coincides with the  $\tilde{F}_\ell$  ( $1 \leq \ell \leq 4$ ) on  $\partial \tilde{\Omega}_\beta$ . In this section we assume that

$$(5.5) \quad F_\ell \in C^\gamma(\Gamma_\ell) \quad (1 \leq \ell \leq 4).$$

Define a function  $F_0$  on  $\partial \tilde{\Omega}_0$  by

$$F_0(z) = F_\ell(z + \lambda z) \quad (1 \leq \ell \leq 4)$$

and extend it into  $\tilde{\Omega}_0$  as a  $C^\gamma$  function. From (1.19) we get

$$(5.6) \quad |\tilde{F}_\beta - F_0|_{L^\infty(\partial \tilde{\Omega}_\beta)} \leq C \beta^\gamma.$$

We introduce a “harmonic” conjugate  $v_\beta$  of  $u_\beta$  by

$$(5.7) \quad v_\beta(x, y) = - \int_0^x (u_\beta)_y(x, 0) dx + \int_0^y (u_\beta)_x(x, y) dy .$$

Since  $(u_\beta)_x$  is continuous in the  $x$ -direction and  $(u_\beta)_y$  is continuous in the  $y$ -direction, one can see that  $v_\beta$  is a well defined single-valued function, and that it is continuous in  $\overline{\tilde{\Omega}_\beta}$ . Furthermore,

$$(5.8) \quad \frac{\partial}{\partial x} v_\beta = -(u_\beta)_y \quad \text{if } (x, y) \in \overline{\tilde{I}_i \cap \tilde{J}_j} \setminus (\tilde{\Gamma}_2^{ij} \cap \tilde{\Gamma}_4^{ij}) ,$$

$$(5.9) \quad \frac{\partial}{\partial y} v_\beta = (u_\beta)_x \quad \text{if } (x, y) \in \overline{\tilde{I}_i \cap \tilde{J}_j} \setminus (\tilde{\Gamma}_1^{ij} \cap \tilde{\Gamma}_3^{ij}) ,$$

where the derivatives of  $v_\beta$  are the usual weak derivatives.

LEMMA 5.1. *There exists a subsequence of  $\beta_n$ , which we again denote by  $\beta_n$ , and a function  $v$  in  $W^{1,2}(\tilde{\Omega}_0)$  such that*

$$(5.10) \quad v_{\beta_n} \rightarrow v \quad \text{weakly in } W^{1,2}(G)$$

for any compact set  $G$  in  $\tilde{\Omega}_0$ .

*Proof.* By (5.8), (5.9) and Lemma 4.1,

$$(5.11) \quad \int_{\tilde{\Omega}_\beta} |\nabla v_\beta|^2 \leq C , \quad C \text{ independent of } \beta .$$

By (5.7)

$$v_\beta(x, 0) = - \int_0^x (u_\beta)_y(x, 0) dx = \frac{1}{1 + \alpha} \int_0^x (F_1 - u_\beta)(x, 0)$$

so that  $|v_\beta(x, 0)| \leq C$ . It follows that the  $W^{1,2}(\tilde{\Omega}_\beta)$ -norm of  $v_\beta$  is uniformly bounded, which implies the assertion of the lemma.

Combining (5.2), (5.8), (5.9) and Lemma 5.1, we see that

$$(5.12) \quad v_x = -u_y^* , \quad v_y = u_x^* .$$

LEMMA 5.2. *The limit  $u$  is harmonic in  $\tilde{\Omega}_0$ , it belongs to  $W^{1,2}(\tilde{\Omega}_0)$ , and*

$$(5.13) \quad \nabla u_\beta \rightarrow \frac{1}{1 + \lambda} \nabla u \quad \text{weakly in } L^2(G)$$

for any compact set  $G$  in  $\tilde{\Omega}_0$ .

*Proof.* For any  $\xi \in C_0^\infty(\tilde{\Omega}_0)$  we have, by (5.3),

$$(5.14) \quad \int_{\tilde{\Omega}_\beta} (u_\beta)_x \xi = - \int_{\tilde{\Omega}_\beta} u_\beta \xi_x + \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{ij}} (u_{i-1,j} - u_{i,j}) \xi .$$

if  $\beta$  is small enough. Since

$$-\frac{\partial u_{ij}}{\partial x} = \frac{1}{\alpha}(u_{i-1,j} - u_{ij}) \quad \text{on} \quad \tilde{\Gamma}_2^{ij},$$

we see that

$$(5.15) \quad \begin{aligned} \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{ij}} (u_{i-1,j} - u_{ij}) \xi &= -\alpha \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{ij}} \frac{\partial u_{ij}}{\partial x} \xi \\ &= -\lambda \beta \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{ij}} \frac{\partial v_\beta}{\partial y} \xi = \lambda \beta \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{ij}} v_\beta \frac{\partial \xi}{\partial y} \equiv K_\beta. \end{aligned}$$

We claim that

$$(5.16) \quad K_\beta \rightarrow \lambda \int_{\tilde{\Omega}_0} v \frac{\partial \xi}{\partial y} \quad \text{as} \quad \beta = \beta_n \rightarrow 0.$$

Indeed,

$$\begin{aligned} K_\beta &= \lambda \sum_{i,j=1}^N \int_{\tilde{I}_i \cap \tilde{J}_j} (v_\beta \xi_y)((i-1)\beta, y) dx dy \\ &= \lambda \sum_{i,j=1}^N \int_{\tilde{I}_i \cap \tilde{J}_j} [(v_\beta \xi_y)((i-1)\beta, y) - (v_\beta \xi_y)(x, y)] + \lambda \int_{\tilde{\Omega}_\beta} v_\beta \xi_y \equiv K_{\beta 1} + K_{\beta 2}. \end{aligned}$$

Clearly

$$K_{\beta 2} \rightarrow \int_{\tilde{\Omega}_0} v \xi_y \quad \text{as} \quad \beta = \beta_n \rightarrow 0,$$

and  $|K_{\beta 1}|$  can be estimated by

$$\begin{aligned} &\lambda \left| \sum_{i,j=1}^N \int_{(i-1)\beta}^{i\beta} dx \int_{(j-1)\beta}^{j\beta} \left[ \int_x^{(i-1)\beta} \frac{\partial}{\partial x} (v_\beta \xi_y)(s, y) ds \right] dy \right| \\ &\leq \lambda \beta \int_{\tilde{\Omega}_\beta} \left| \frac{\partial}{\partial x} (v_\beta \xi_y) \right| \rightarrow 0 \quad \text{as} \quad \beta \rightarrow 0 \end{aligned}$$

since  $v_\beta$  and  $(v_\beta)_x$  are uniformly bounded in  $L^2(\tilde{\Omega}_\beta)$ . Hence (5.16) holds.

Taking  $\beta = \beta_n \rightarrow 0$  in (5.14) and using (5.15), (5.16) we get

$$(5.17) \quad \int_{\tilde{\Omega}_0} u_x^* \xi = - \int_{\tilde{\Omega}_0} u \xi_x + \lambda \int_{\tilde{\Omega}_0} v \xi_y.$$

Since  $v \in W^{1,2}(\tilde{\Omega}_0)$ , it follows that the weak derivative  $u_x$  exists and

$$(5.18) \quad u_x = u_x^* + \lambda v_y = (\lambda + 1)u_x^*$$

by (5.12). Similarly the weak derivative  $u_y$  exists and

$$(5.19) \quad u_y = (\lambda + 1)u_y^* .$$

Finally, taking  $\xi \in C_0^\infty(\tilde{\Omega}_0)$  in (5.4) with  $\beta = \beta_n \rightarrow 0$ , we get

$$\int_{\tilde{\Omega}_0} (u_x^* \xi_x + u_y^* \xi_y) = 0$$

and, in view of (5.18), (5.19),  $\Delta u = 0$  in  $\tilde{\Omega}_0$ .

We next proceed to determine the boundary condition of  $u$  on  $\partial\tilde{\Omega}_0$ .

LEMMA 5.3. *Suppose  $\xi \in C^\infty(\tilde{\Omega}_0)$  with support in  $\left\{0 \leq x \leq \frac{1}{\lambda+1}, \delta \leq y \leq \frac{1}{\lambda+1} - \delta\right\}$  for some  $\delta > 0$ . Then, as  $\beta = \beta_n \rightarrow 0$ ,*

$$(5.20) \quad \int_{\tilde{\Gamma}_2} u_\beta \xi \rightarrow \int_{\tilde{\Gamma}_2^0} u \xi$$

where  $\tilde{\Gamma}_2, \tilde{\Gamma}_2^0$  are the intersections of  $x = 0$  with  $\partial\tilde{\Omega}_\beta$  and  $\partial\tilde{\Omega}_0$  respectively, and  $u|_{\tilde{\Gamma}_2^0}$  is defined in the sense of trace class.

*Proof.* Let  $\eta = \eta(x)$  be a  $C^\infty$  function such that  $\eta(0) = 1$ ,  $\eta(x) = 0$  for  $x$  in a neighborhood of  $x = \frac{1}{\lambda+1}$ . As in (5.3)

$$(5.21) \quad \begin{aligned} \int_{\tilde{\Gamma}_2} u_\beta \xi &= \int_{\tilde{\Gamma}_2} u_\beta \xi \eta = \sum_{j=1}^N \int_{(j-1)\beta}^{j\beta} (u_{1j} \xi \eta)(0, y) dy \\ &= - \int_{\tilde{\Omega}_\beta} \frac{\partial}{\partial x} (u_\beta \xi \eta) + \sum_{i=1}^{N-1} \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{i+1,j}} (u_{ij} - u_{i+1,j}) \xi \eta \\ &= - \int_{\tilde{\Omega}_\beta} [(u_\beta)_x \xi \eta + u_\beta (\xi \eta)_x] + \sum_{i=1}^{N-1} \sum_{j=1}^N \int_{\tilde{\Gamma}_2^{i+1,j}} (u_{ij} - u_{i+1,j}) \xi \eta . \end{aligned}$$

Using (5.16) (where  $K_\beta$  is defined by (5.15)) with  $\xi \eta$  instead of  $\xi$ ) we deduce that

$$\begin{aligned} \int_{\tilde{\Gamma}_2} u_\beta \xi &\rightarrow - \int_{\tilde{\Omega}_0} [u_x^* \xi \eta + u (\xi \eta)_x] + \lambda \int_{\tilde{\Omega}_0} v (\xi \eta)_y , \\ &= - \int_{\tilde{\Omega}_0} [u_x (\xi \eta) + u (\xi \eta)_x] \quad (\text{by (5.18)}) \\ &= \int_{\tilde{\Gamma}_2^0} u \xi . \end{aligned}$$

Similarly one can extend Lemma 5.3 to the other parts of the boundary.

We can now state the main result of this section.

THEOREM 5.4. Assume that (5.5) holds. Then the entire family  $u_\beta$  is convergent in  $L^2(G)$ , for any compact set  $G$  in  $\tilde{\Omega}_0$ , to a function  $u$  in  $W^{1,2}(\tilde{\Omega})$  where

$$(5.22) \quad \Delta u = 0 \quad \text{in} \quad \tilde{\Omega}_0$$

and

$$(5.23) \quad \frac{1}{\lambda+1} \frac{\partial u}{\partial n} + u = F_0 \quad \text{on} \quad \partial\tilde{\Omega}_0 .$$

*Proof.* Taking  $\beta_n \rightarrow 0$  in (5.4) and using Lemma 5.3, (5.6) and (5.13), we get

$$\int_{\tilde{\Omega}_0} \frac{1}{\lambda+1} \nabla u \cdot \nabla \xi + \int_{\partial\tilde{\Omega}_0} u \xi = \int_{\partial\tilde{\Omega}_0} F_0 \xi ,$$

i.e.,  $u$  is the solution of (5.22), (5.23). Since  $u$  is uniquely determined by (5.22), (5.23), the full family  $u_\beta$  is convergent to  $u$  in the sense of  $L^2_{\text{loc}}(\tilde{\Omega}_0)$ .

*Remark 5.1.* The results of the previous sections extend to the case where the boundary condition (1.6) is replaced by

$$(5.24) \quad A \frac{\partial u}{\partial n} + u = F_i \quad \text{on} \quad \Gamma_i \cap \partial(I \cap J) \quad (1 \leq i \leq 4)$$

where  $A > 0$ ; in particular, Theorem 5.4 remains valid with (5.23) replaced by

$$(5.25) \quad \frac{A}{\lambda+1} \frac{\partial u}{\partial n} + u = F_0 \quad \text{on} \quad \partial\tilde{\Omega}_0 .$$

*Remark 5.2.* The above results remain true if  $\alpha/\beta$  is not constant, but  $\alpha/\beta \rightarrow \lambda$  as  $\beta \rightarrow 0$ , where  $0 < \lambda < \infty$ .

**§6. The Dirichlet problem.** In this section we extend the results of the previous sections to the case where  $u$  satisfies Dirichlet data:

$$(6.1) \quad u = G_\ell \quad \text{on} \quad \Gamma_\ell \cap \partial(I \cap J) \quad (i \leq \ell \leq 4) .$$

Then from (1.9) we get

$$\left( \frac{\partial u}{\partial n} + \frac{u}{\alpha} \right) (x, y_1 - \frac{\beta}{2}) = \frac{G_1}{\alpha}$$

with similar relations on the other edges. Thus the problem (1.2)–(1.5), (6.1), which we shall call problem  $(P_D)$ , can be written in the following form (similar to (1.13)–(1.18)):

$$(6.2) \quad u_{ij} \in C^1(\overline{\tilde{I}_i \cap \tilde{J}_j}) ,$$

$$(6.3) \quad \Delta u_{ij} = 0 \quad \text{in} \quad \tilde{I}_i \cap \tilde{J}_j ,$$

$$(6.4) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} \tilde{G}_1 & \text{on } \Gamma_1^{i1}, \quad j = 1, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} u_{i,j-1} & \text{on } \tilde{\Gamma}_1^{ij}, \quad 1 < j \leq N, \end{cases}$$

$$(6.5) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} \tilde{G}_3 & \text{on } \tilde{\Gamma}_3^{iN}, \quad j = N, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} u_{i,j+1} & \text{on } \tilde{\Gamma}_3^{ij}, \quad 1 \leq j < N, \end{cases}$$

$$(6.6) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} \tilde{G}_2 & \text{on } \tilde{\Gamma}_2^{1j}, \quad i = 1, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} u_{i-1,j} & \text{on } \tilde{\Gamma}_2^{ij}, \quad 1 < j \leq N, \end{cases}$$

$$(6.7) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} \tilde{G}_4 & \text{on } \Gamma_4^{Nj}, \quad i = N, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha} u_{ij} = \frac{1}{\alpha} u_{i+1,j} & \text{on } \tilde{\Gamma}_4^{ij}, \quad 1 \leq i < N. \end{cases}$$

Here the  $\tilde{G}_\ell$  are defined by

$$(6.8) \quad \begin{aligned} \tilde{G}_\ell(i\beta + x) &= G_\ell \left( x_{i+1} - \frac{\beta}{2} + x \right) \quad \text{for } \ell = 1, 3 \quad (0 \leq i \leq N-1), \\ \tilde{G}_\ell(j\beta + y) &= G_\ell \left( y_{j+1} - \frac{\beta}{2} + y \right) \quad \text{for } \ell = 2, 4 \quad (0 \leq j \leq N-1) \end{aligned}$$

for  $0 \leq x \leq \beta$ ,  $0 \leq y \leq \beta$ . We shall refer to the system (6.2)–(6.8) as problem  $(\tilde{P}_D)$ .

The existence of a unique solution to problem  $(\tilde{P}_D)$  can be established precisely as for problem  $(\tilde{P})$  provided we assume that

$$(6.9) \quad G_i \in C^{1+\gamma}(\Gamma_i), \quad G_i = G_j \quad \text{at } \Gamma_i \cap \Gamma_j \quad (1 \leq i, j \leq 4).$$

Furthermore,

$$\|u\|_X < \infty.$$

We shall denote by  $\tilde{G}_\beta$  the function defined on  $\partial\Omega_\beta$  by the  $\tilde{G}_\ell$  in (6.8).

Define a function  $G_0$  on  $\partial\tilde{\Omega}_0$  by

$$G_0(z) = G_\ell(z + \lambda z) \quad (1 \leq \ell \leq 4).$$

We extend  $G_0$  into  $\overline{\tilde{\Omega}_0}$  as a  $C^{1+\gamma}$  function. From (6.9) we have

$$(6.10) \quad |\tilde{G}_\beta - G_0|_{L^\infty(\partial\tilde{\Omega}_\beta)} \leq C\beta.$$

LEMMA 6.1. Let  $w_\beta$  be the solution of

$$(6.11) \quad \begin{aligned} \Delta w_\beta &= 0 \quad \text{in } \tilde{\Omega}_\beta, \\ \alpha \frac{\partial w_\beta}{\partial n} + w_\beta &= \tilde{G} \quad \text{on } \partial \tilde{\Omega}_\beta. \end{aligned}$$

where  $\tilde{G}(z) = G_\ell(z/N\beta)$  on the corresponding edges of  $\partial \tilde{\Omega}_\beta$ . Then

$$(6.12) \quad \left| \frac{\partial w_\beta}{\partial n} \right| \leq C \quad \text{on } \partial \tilde{\Omega}_\beta$$

where  $C$  is a constant independent of  $\beta$ .

*Proof.* Introduce the solution  $v$  of

$$\Delta v = 0 \quad \text{in } \tilde{\Omega}_\beta, \quad v = \tilde{G} \quad \text{on } \partial \tilde{\Omega}_\beta.$$

Then

$$|\nabla v| \leq C \|\tilde{G}\|_{C^{1+\gamma}}.$$

The function  $\varphi = w_\beta - v$  satisfies:

$$\begin{aligned} \Delta \varphi &= 0 \quad \text{in } \tilde{\Omega}_\beta, \\ \alpha \frac{\partial \varphi}{\partial n} + \varphi &= -\alpha \frac{\partial v}{\partial n} \quad \text{on } \partial \tilde{\Omega}_\beta. \end{aligned}$$

By the maximum principle we deduce that

$$|\varphi| \leq \alpha \left| \frac{\partial v}{\partial n} \right|_{L^\infty} \leq C\alpha.$$

Consequently

$$\left| \frac{\partial w_\beta}{\partial n} \right| = \frac{1}{\alpha} |\tilde{G} - w_\beta| = \frac{1}{\alpha} |\varphi| \leq C.$$

LEMMA 6.2. The estimates (4.2), (4.3) are valid for the solution  $u = \{u_{ij}\}$  of problem  $(\tilde{P}_D)$ .

*Proof.* By Lemma 6.1

$$(6.13) \quad \int_{\tilde{\Omega}_\beta} |\nabla w_\beta|^2 = \int_{\partial \tilde{\Omega}_\beta} w_\beta \frac{\partial w_\beta}{\partial n} \leq C.$$

By the divergence theorem and (6.3)–(6.7) we get (cf. the proof of Lemma 4.1)

$$(6.14) \quad \begin{aligned} & \int_{\tilde{\Omega}_\beta} |\nabla(u_\beta)|^2 + \frac{1}{\alpha} \sum_{i=2}^N \sum_{j=1}^N \int_{\tilde{\Gamma}_{ij}^2} |u_{ij} - u_{i-1,j}|^2 \\ & + \frac{1}{\alpha} \sum_{i=1}^N \sum_{j=2}^N \int_{\tilde{\Gamma}_{ij}^1} |u_{ij} - u_{i,j-1}|^2 = \int_{\partial \tilde{\Omega}_\beta} u_\beta \frac{\partial u_\beta}{\partial n}. \end{aligned}$$

Similarly to (5.4),

$$(6.15) \quad \int_{\tilde{\Omega}_\beta} \nabla(u_\beta) \cdot \nabla w_\beta = \int_{\partial\tilde{\Omega}_\beta} \frac{\partial u_\beta}{\partial n} w_\beta = \int_{\partial\tilde{\Omega}_\beta} \frac{\partial u_\beta}{\partial n} (w_\beta - u_\beta) + \int_{\partial\tilde{\Omega}_\beta} \frac{\partial u_\beta}{\partial n} u_\beta .$$

Also

$$(6.16) \quad \begin{aligned} \int_{\partial\tilde{\Omega}_\beta} \frac{\partial u_\beta}{\partial n} (w_\beta - u_\beta) &= \frac{1}{\alpha} \int_{\partial\tilde{\Omega}_\beta} (\tilde{G}_\beta - u_\beta)(w_\beta - u_\beta) \\ &= \frac{1}{\alpha} \int_{\partial\tilde{\Omega}_\beta} (w_\beta - u_\beta)^2 + \frac{1}{\alpha} \int_{\partial\tilde{\Omega}_\beta} (\tilde{G}_\beta - w_\beta)(w_\beta - u_\beta) \\ &= \frac{1}{\alpha} \int_{\partial\tilde{\Omega}_\beta} (w_\beta - u_\beta)^2 + \frac{1}{\alpha} \int_{\partial\tilde{\Omega}_\beta} (\tilde{G}_\beta - \tilde{G})(w_\beta - u_\beta) + \frac{1}{\alpha} \int_{\partial\tilde{\Omega}_\beta} (\tilde{G} - w_\beta)(w_\beta - u_\beta) . \end{aligned}$$

We can now evaluate the right-hand side of (6.14):

$$\begin{aligned} \int_{\partial\tilde{\Omega}_\beta} u_\beta \frac{\partial u_\beta}{\partial n} &= \int_{\tilde{\Omega}_\beta} \nabla(u_\beta) \cdot \nabla w_\beta - \int_{\partial\tilde{\Omega}_\beta} \frac{\partial u_\beta}{\partial n} (w_\beta - u_\beta) \quad (\text{by (6.15)}) \\ &= \int_{\tilde{\Omega}_\beta} \nabla(u_\beta) \cdot \nabla w_\beta - \frac{1}{\alpha} \int_{\partial\tilde{\Omega}_\beta} (w_\beta - u_\beta)^2 \\ &\quad - \frac{1}{\alpha} \int_{\partial\tilde{\Omega}_\beta} (\tilde{G}_\beta - \tilde{G})(w_\beta - u_\beta) - \frac{1}{\alpha} \int_{\partial\tilde{\Omega}_\beta} (\tilde{G} - w_\beta)(w_\beta - u_\beta) . \end{aligned}$$

By (6.10) and Lemma 6.1 the right-hand side is bounded from above by

$$\int_{\tilde{\Omega}_\beta} |\nabla(u_\beta) \cdot \nabla w_\beta| + C \leq \frac{1}{2} \int_{\tilde{\Omega}_\beta} |\nabla(u_\beta)|^2 + 2 \int_{\tilde{\Omega}_\beta} |\nabla w_\beta|^2 + C .$$

Substituting this estimate on the right-hand side of (6.14), the estimates (4.2), (4.3) follow.

We can now proceed as in Section 5 and obtain the following result:

**THEOREM 6.3.** *If (6.9) holds then the solution  $u_\beta$  to  $(\tilde{P}_D)$  converges (as  $\beta \rightarrow 0$ ) in  $L^2(\Omega')$ , for any compact set  $\Omega'$  in  $\tilde{\Omega}_0$ , to the function  $u$  in  $W^{1,2}(\tilde{\Omega})$  of*

$$(6.17) \quad \Delta u = 0 \quad \text{in} \quad \tilde{\Omega}_0 ,$$

$$(6.18) \quad u = G_0 \quad \text{on} \quad \partial\tilde{\Omega}_0 .$$

*Remark 6.1.* One can establish the existence of a solution  $u_\beta$  to  $(\tilde{P}_D)$  even if (6.9) is replaced by  $G_i \in C^\gamma(\Gamma_i)$  ( $1 \leq i \leq 4$ ), but of course  $u_\beta$  will not belong to  $C^{1+\gamma}$  at the boundary.

**§7. The case of non-uniform grains.** In this section we consider the case where the grains are rectangular but not of uniform size. More precisely, the unit square  $\Omega$  is cut out by  $N_1$  vertical channels  $I_i$  of width  $\beta_1$  and  $N_2$  horizontal channels  $J_j$  of width  $\beta_2$ . We denote the separation between the channels  $I_i$  and  $I_{i+1}$  by  $\alpha_i$  and the separation between the channels  $J_j$  and  $J_{j+1}$  by  $\tilde{\alpha}_j$ . Thus

$$(7.1) \quad \begin{aligned} I_i &= \left( \sum_{\ell=0}^{i-1} \alpha_\ell + (i-1)\beta_1, \sum_{\ell=0}^{i-1} \alpha_\ell + i\beta_1 \right) \times (0, 1), \\ J_j &= (0, 1) \times \left( \sum_{\ell=0}^{j-1} \tilde{\alpha}_\ell + (j-1)\beta_2, \sum_{\ell=0}^{j-1} \tilde{\alpha}_\ell + j\beta_2 \right). \end{aligned}$$

As in Section 1 we set

$$I = \bigcup_{i=1}^{N_1} I_i, \quad J = \bigcup_{j=1}^{N_2} J_j.$$

The numbers  $\alpha_i, \tilde{\alpha}_j$  are given by monotone increasing functions  $f_1(x), f_2(y)$ :

$$(7.2) \quad \begin{aligned} \alpha_i &= f_1((i+1)\beta_1) - f_1(i\beta_1) \quad (0 \leq i \leq N_1), \\ \tilde{\alpha}_j &= f_2((j+1)\beta_2) - f_2(j\beta_2) \quad (0 \leq j \leq N_2). \end{aligned}$$

We assume:

$$(7.3) \quad f_i \in C^1[0, \infty), \quad c \leq f'_i \leq C \quad (C, c \text{ positive constants}).$$

The parameters  $\beta_1, \beta_2, N_1, N_2$  are assumed to be such that

$$(7.4) \quad \sum_{i=0}^{N_1} \alpha_i + N_1\beta_1 = 1 \quad \text{and} \quad \sum_{i=0}^{N_2} \tilde{\alpha}_i + N_2\beta_2 = 1,$$

so that the rectangles  $I_i \cap J_j$  extend up to the boundary of the unit square  $\Omega$ .

We define  $\theta_1, \theta_2$  uniquely by

$$(7.5) \quad \theta_i + f_i(\theta_i) - f_i(0) = 1,$$

and set

$$(7.6) \quad \tilde{\Omega}_0 = \{0 < x < \theta_1, 0 < y < \theta_2\}.$$

Consider now problem (P) ((1.2)–(1.6)) with the  $I_i, J_j$  defined by (7.1). This problem can be reduced to the following

*Problem ( $\tilde{P}$ ).* Find a family  $u = \{u_{ij}\}$  of functions  $u_{ij}$  ( $1 \leq i \leq N_1$ ,  $1 \leq j \leq N_2$ ) satisfying (1.13), (1.14) and the following interface and boundary conditions:

$$(7.7) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{1 + \tilde{\alpha}_0} u_{ij} = \frac{1}{1 + \tilde{\alpha}_0} \tilde{F}_1 & \text{on } \tilde{\Gamma}_1^{i1}, \quad j = 1, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\tilde{\alpha}_{j-1}} u_{ij} = \frac{1}{\tilde{\alpha}_{j-1}} u_{i,j-1} & \text{on } \tilde{\Gamma}_1^{ij}, \quad 1 < j \leq N_2, \end{cases}$$

$$(7.8) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{1 + \tilde{\alpha}_{N_2}} u_{ij} = \frac{1}{1 + \tilde{\alpha}_{N_2}} \tilde{F}_3 & \text{on } \tilde{\Gamma}_3^{iN_2}, \quad j = N_2, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\tilde{\alpha}_j} u_{ij} = \frac{1}{\tilde{\alpha}_j} u_{i,j+1} & \text{on } \tilde{\Gamma}_3^{ij}, \quad 1 \leq j < N_2, \end{cases}$$

$$(7.9) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{1 + \alpha_0} u_{ij} = \frac{1}{1 + \alpha_0} \tilde{F}_2 & \text{on } \tilde{\Gamma}_2^{1j}, \quad i = 1, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha_{i-1}} u_{ij} = \frac{1}{\alpha_{i-1}} u_{i-1,j} & \text{on } \tilde{\Gamma}_2^{ij}, \quad 1 < i \leq N_1, \end{cases}$$

$$(7.10) \quad \begin{cases} \frac{\partial u_{ij}}{\partial n} + \frac{1}{1 + \alpha_{N_1}} u_{ij} = \frac{1}{1 + \alpha_{N_1}} \tilde{F}_4 & \text{on } \tilde{\Gamma}_4^{N_1j}, \quad i = N_1, \\ \frac{\partial u_{ij}}{\partial n} + \frac{1}{\alpha_i} u_{ij} = \frac{1}{\alpha_i} u_{i+1,j} & \text{on } \tilde{\Gamma}_4^{ij}, \quad 1 \leq i < N_1. \end{cases}$$

Here the  $\tilde{F}_\ell$  are defined by translating the  $F_\ell$ , analogously to (1.19).

LEMMA 7.1. *If  $F_\ell \in C^\gamma(\Gamma_\ell)$  ( $\ell \leq \ell \leq 4$ ) then there exists a unique solution to problem ( $\tilde{P}$ ).*

The proof is the same as for Theorem 3.1.

We next consider the limit behavior of the solution  $u$  as  $\beta \rightarrow 0$ , where  $\beta = (\beta_1, \beta_2)$ . We shall write  $u = u_\beta$  and the rectangle  $\tilde{\Omega}$  over which  $u_\beta$  is defined by  $\tilde{\Omega}_\beta$ . Define a function  $F_0$  on  $\partial\tilde{\Omega}_\beta$  by

$$F_0 = \begin{cases} F_{0\ell}(x) = F_\ell(x + f_1(x) - f_1(0)) & (\ell = 1, 3), \\ F_{0\ell}(y) = F_\ell(y + f_2(y) - f_2(0)) & (\ell = 2, 4). \end{cases}$$

It is easily verified that

$$(7.11) \quad \tilde{F}_\ell \rightarrow F_{0\ell}, \quad \tilde{\Omega}_\beta \rightarrow \tilde{\Omega}_0 \quad \text{as } \beta \rightarrow 0.$$

The proof of Lemma 4.1 can easily be extended; it yields the estimates

$$(7.12) \quad \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_{\tilde{I}_i \cap \tilde{J}_j} |\nabla u_{ij}|^2 \leq C,$$

$$(7.13) \quad \sum_{i=1}^{N_1} \sum_{j=2}^{N_2} \frac{1}{\tilde{\alpha}_{j-1}} \int_{\tilde{\Gamma}_1^{ij}} |u_{ij} - u_{i,j-1}|^2 + \sum_{i=2}^{N_1} \sum_{j=1}^{N_1} \frac{1}{\alpha_{i-1}} \int_{\tilde{\Gamma}_2^{ij}} |u_{ij} - u_{i-1,j}|^2 \leq C.$$

These estimates can be used, as in Section 4, to establish Theorem 4.3 for the present case.

We next introduce the harmonic conjugate  $v_\beta$  (of  $u_\beta$ ) by (5.7). As in Section 5, the weak limits in (5.2) and (5.10) exist and (5.12) holds.

LEMMA 7.2. The function  $u$  belongs to  $W^{1,2}(\tilde{\Omega}_0)$  and

$$(7.14) \quad u_x = (1 + f'_1(x))u_x^* , \quad u_y = (1 + f'_2(y))u_y^* .$$

*Proof.* Proceeding as in (5.14) we need to evaluate the last term on the right-hand side, which is now

$$\begin{aligned} & \sum_{i=2}^{N_1} \sum_{j=1}^{N_2} \int_{\tilde{\Gamma}_2^{ij}} (u_{i-1,j} - u_{ij}) \xi = - \sum_{i=2}^{N_1} \sum_{j=1}^{N_2} \alpha_{i-1} \int_{\tilde{\Gamma}_2^{ij}} \frac{\partial u_{ij}}{\partial x} \xi \\ & = - \sum_{i=2}^{N_1} \sum_{j=1}^{N_2} \int_{\tilde{\Gamma}_2^{ij}} [f_1(i\beta_1) - f_1((i-1)\beta_1)] \frac{\partial v_\beta}{\partial y} \xi dy \\ & = \sum_{i=2}^{N_1} \sum_{j=1}^{N_2} \int_{\tilde{\Gamma}_2^{ij}} [f_1(i\beta_1) - f_1((i-1)\beta_1)] v_\beta \xi_y dy \\ & = \sum_{i=2}^{N_1} \sum_{j=1}^{N_2} \int_{\tilde{I}_i \cap \tilde{J}_j} \left[ \frac{f_1(i\beta_1) - f_1((i-1)\beta_1)}{\beta_1} \right] (v_\beta \xi_y)((i-1)\beta_1, y) dx dy \\ & \rightarrow \int_{\tilde{\Omega}_0} f'_1(x) v \xi_y dx dy \quad \text{if } \beta \rightarrow 0 . \end{aligned}$$

Consequently, taking  $\beta \rightarrow 0$  in the equation analogous to (5.14) we obtain (cf. (5.17))

$$\int_{\tilde{\Omega}_0} u_x^* \xi = - \int_{\tilde{\Omega}_0} u \xi_x + \int_{\tilde{\Omega}_0} f'_1(x) v \xi_y ,$$

which is the first assertion (7.14). The proof of the second assertion is similar.

We now state the main result of this section.

THEOREM 7.3. If (5.5) holds then the limit  $u$  of the  $u_\beta$ 's is the solution of

$$(7.15) \quad \left( \frac{u_x}{1 + f'_1(x)} \right)_x + \left( \frac{u_y}{1 + f'_2(y)} \right)_y = 0 \quad \text{in } \tilde{\Omega}_0 ,$$

$$(7.16) \quad \frac{1}{1 + f'_2(0)} \frac{\partial u}{\partial n} + u = F_{01} \quad \text{on } \tilde{\Gamma}_1^0 , \quad \frac{1}{1 + f'_2(\theta_2)} \frac{\partial u}{\partial n} + u = F_{03} \quad \text{on } \tilde{\Gamma}_3^0 ,$$

$$(7.17) \quad \frac{1}{1 + f'_1(0)} \frac{\partial u}{\partial n} + u = F_{02} \quad \text{on } \tilde{\Gamma}_2^0 , \quad \frac{1}{1 + f'_1(\theta_1)} \frac{\partial u}{\partial n} + u = F_{04} \quad \text{on } \tilde{\Gamma}_4^0 .$$

Here  $\tilde{\Gamma}_i^0$  ( $i = 1, 2, 3, 4$ ) are the sides of  $\partial\tilde{\Omega}_0$  which lie in  $y = 0$ ,  $x = 0$ ,  $y = \theta_2$  and  $x = \theta_1$  respectively.

Theorem 7.3 extends in obvious way to the boundary condition (5.24) with any  $A > 0$ . It also extends to the Dirichlet problem, by combining the methods of Section 6 with the method of proof of Theorem 7.3.

*Remark 7.1.* If  $f_1(x)$  and  $f_2(y)$  are piecewise linear then the diffusion model suggested in (i) (in the Introduction), after a change of variables, can be written “approximately” in the form

$$(7.18) \quad \operatorname{div}(a(X, X/\varepsilon)\nabla u) = 0 \quad \text{in } \Omega$$

where  $X = (x, y)$  and  $a(X, Y)$  is periodic in  $Y$  in the unit square and discontinuous in  $X$  at the points  $X_m$  where  $f_1'(x)f_2'(y)$  is discontinuous; near  $X = X_m$  the equation (7.18) is generally incorrect. Note that  $a = 0$  in the grains. The method of homogenization extends to diffusion equations of the form (7.18) (see [1; Chap. 1, Sec. 6]) provided  $a$  is uniformly positive. The method can probably be extended also to perforated domains (using ideas from [2]) and it would then yield the same limit (for  $\beta \rightarrow 0$ ) as in Theorem 7.3 (when  $f_1'(x)$  and  $f_2'(y)$  are piecewise constants). For model (i) with arbitrary non-periodic structure, one can only derive estimates on the effective diffusion matrix [3] [4] [7].

**§8. General domains.** Consider the case where  $\Omega$  is a convex domain, divided by vertical channels  $I_i$  and horizontal channels  $J_j$  as in Section 7. The distance between vertical channels is given by a function  $f_1(x)$  and, between horizontal channels, by a function  $f_2(y)$ . We assume that  $\Omega$  contains the origin, so that  $f_1(x)$ ,  $f_2(y)$  are defined for  $x$  and  $y$  both positive and negative.

We also assume that

$$(8.1) \quad c_1\beta \leq \operatorname{dist} \left( \bigcup_{i=1}^{N_1} \bigcup_{j=1}^{N_2} (I_i \cap J_j), \partial\Omega \right) \leq c_2\beta$$

where  $c_1, c_2$  are positive constants. This assumption is generically satisfied in the physical case, since the grains sizes are much larger than the width of the grains boundaries.

Consider the Dirichlet problem with data  $G$ . The problem can be reduced to a problem similar to  $(\tilde{P}_D)$  in a domain  $\tilde{\Omega}_\beta$  which is a collection of rectangles. As  $\beta \rightarrow 0$

$$\tilde{\Omega}_\beta \rightarrow \tilde{\Omega}_0$$

where  $\tilde{\Omega}_0$  is given by

$$(8.2) \quad \tilde{\Omega}_0 = \{(x, y) ; (x + f_1(x), y + f_2(y)) \in \Omega\} .$$

The methods of Sections 6, 7 easily extend to conclude that  $u_\beta$  is convergent to the solution  $u$  of (7.16) with  $u = \tilde{G}$  on  $\partial\tilde{\Omega}_0$ ; here  $\tilde{\Omega}_0$  is given by (8.2) and  $\tilde{G}$  is a “local average” of  $G$ .

Let us next consider the boundary value problem

$$(8.3) \quad \begin{aligned} A \frac{\partial u}{\partial x} + u &= F \quad \text{on the boundary of horizontal channels,} \\ A \frac{\partial u}{\partial y} + u &= G \quad \text{on the boundary of vertical channels.} \end{aligned}$$

In that case the problem can be studied precisely as for  $(\tilde{P})$ . To determine the limit problem as  $\beta \rightarrow 0$  we need to make the assumption that

$$(8.4) \quad (F_\beta, G_\beta) \cdot n_\beta \rightarrow H$$

in a weak sense, where  $F_\beta$  ( $G_\beta$ ) are the values of  $F$  ( $G$ ) shifted to the boundary  $\partial\tilde{\Omega}_\beta$ , and  $n_\beta$  is the normal to  $\partial\tilde{\Omega}_\beta$ ;  $H$  is defined on  $\partial\tilde{\Omega}_0$ . The conclusion is then that the limit function satisfies (7.15) in  $\tilde{\Omega}_0$  and the boundary condition

$$\frac{A}{1+f'_1} \frac{\partial u}{\partial x} \cos(x, n) + \frac{A}{1+f'_2} \frac{\partial u}{\partial y} \cos(y, n) + u = H \quad \text{on} \quad \partial\tilde{\Omega}_0 .$$

**9. Non-orthogonal networks.** The results of this paper can be extended to non-orthogonal channels of the form in Figure 3.

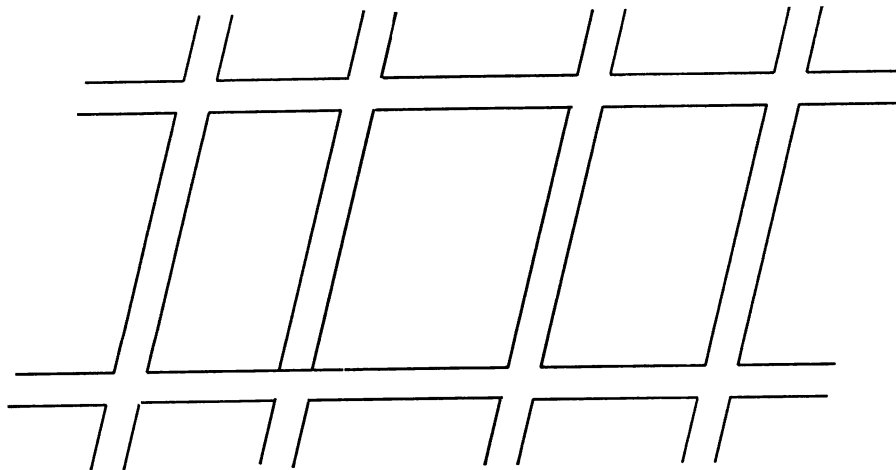


FIGURE 3

Denote the horizontal channels by  $J_j$  and the non-horizontal channels by  $I_i$ ; the  $I_i$  are in a direction, say,

$$\ell = \frac{(1, a)}{(1+a^2)^{1/2}} .$$

Equations (1.4), (1.5) are then replaced by

$$\begin{aligned} \frac{\partial^2 u}{\partial \ell^2} &= \quad \text{in} \quad I \setminus \bar{J} , \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \ell^2} &= 0 \quad \text{in} \quad I \cap J , \end{aligned}$$

respectively. Consider for simplicity the case where the grains are of uniform size and all sides are equal to  $\alpha$ , and the channels width are all equal to  $\beta$ . We modify the definition of (5.7):

$$v_\beta(x, y) = - \int_0^{x_0} (u_\beta)_\ell(x, 0) dx + \int_{\ell_{xy}} (u_\beta)_x d\ell$$

where  $\ell_{xy}$  is the line segment

$$\{(x', y'); y - y' = a(x - x'), 0 < y' < y\}$$

and  $x_0$  is determined by  $y = a(x - x_0)$ . Then, analogously to (5.8), (5.9) we get

$$(v_\beta)_x = -(u_\beta)_\ell \quad \text{and} \quad (v_\beta)_\ell = (u_\beta)_x .$$

Proceeding as before one can show that  $u_\beta \rightarrow u$  in  $L^2_{\text{loc}}$  and (cf. Theorem 6.3)

$$\begin{aligned} u_{xx} + u_{\ell\ell} &= 0 \quad \text{in} \quad \tilde{\Omega}_0 , \\ u &= F_0 \quad \text{on} \quad \partial\tilde{\Omega}_0 . \end{aligned}$$

Similarly one can extend the results to the Neumann problem and to non-uniform size grains with non-periodic structure.

For the periodic case the homogenization approach (model (i) in the Introduction) yields the same result, as can be computed from the formulas in [2].

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