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SINGULAR STRUCTURE OF THE FUNDAMENTAL SOLUTION OF THE TRANSPORT EQUATION

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ABSTRACT. The structure of singularities of the fundamental solution of the transport equation with smooth coefficients is studied. As it has been found out, this structure is more diverse than at the hyperbolic equations. The existence of the singularities inside "a characteristic cone" has allowed us to solve the problem of reconstruction of local characteristics of tissue using the information on flows of radiation on the boundary. The process of interaction is of a complex nature and here we are limited by the simple model.

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1. Introduction. In recent years, beginning with the Hadamard article [1], the problem of description of the singularities of the fundamental solution (FS) of PDE has attracted the attention of the mathematicians and this area has always been fruitful for connection with various branches of mathematics.

Consider the equation

$$L[u] \equiv \sum_{i,j=1}^{n} g^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} a^i(x)\frac{\partial u}{\partial x^i} + c(x)u = 0 \tag{0.1}$$

for strictly hyperbolic operator L. Denote by V_n the associated quasi-Riemannian space with metric tensor defined by functions $g_{i,j}(x)$, where

$$\sum_{k=1}^{n} g^{ik} g_{kj} = \delta^i_j.$$

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The Hadamard construction in modern terminology yields for this operator the following representation of the fundamental solution $E(x,y,t)$ of this operator for sufficiently small t.

$$E(x,y,t) = \sum_{k=\frac{3-m}{2}}^{N} \delta^{(-k)}(\Gamma)\, g_k(x,y) + R_N(x,y,t)$$

where $\delta^{(j)}$ is a j - derivation of the Dirac delta function, $\Gamma = t^2 - d^2(x,y)$ is a Minkovsky distance between points (x,y,t) and $(x,y,0)$, $d(x,y)$ is the distance between point x and y and $g_k(x,y)$ are the smooth functions defined by a certain "transport" equation. Note that all singularities are concentrated on characteristic cone $\Gamma = 0$. This series represents $E(x,y,t)$ in the sense that the difference $R_N(x,y,t)$ can be made arbitrarily smooth in the neighborhood of the characteristic cone if N is chosen large enough. The structure of the fundamental solution of the conjugate to (0.1) equation

$$(0.2)\quad \mathrm{L}^*[v] \equiv \sum_{i,j=1}^{n} \frac{\partial^2(g^{ij}v)}{\partial x^i \partial x^j} - \sum_{i=1}^{n} \frac{\partial(b^i v)}{\partial x^i} + c(x)u$$

$$= \sum_{ij=1}^{n} g^{ij} \frac{\partial^2 v}{\partial x^i \partial x^j} + \sum_{i=1}^{n} \hat{b}^i \frac{\partial v}{\partial x^i} + \hat{c}(x)u = 0$$

for even n gives the answer to the question: is (0.1) a Huygens' equation.

Definition. *The equation satisfies the Huygens' principle if the value of the solution of the Cauchy problem with arbitrary initial data, depends at each point x_0 on the initial data prescribed only on the intersection of the manifold S with the characteristic conoid having vertex at x_0 (in Hadamard's terminology it is Huygens principle in the "narrow sense").*

The Hadamard criterion. *The equation (0.1) is a Huygens' equation if and only if the elementary (fundamental) solution of the conjugate equation is free of the logarithmic term.*

On the other hand the Huygens principle is closely connected with the Painleve property which is defined both for linear and non-linear equations and is the main test for integrability of non-linear equation by inverse scattering method [2]:

Proposition. *The equation conjugate to (0.1) has the Painleve property if and only if the (0.2) is a Huygens' equation.*

Hence we can consider the Painleve property as a generalization of the Huygens principle on non-linear PDE. The connection between the spectrum and development of singularities of the fundamental solution of the hyperbolic equation is used very successfully in the so called "hyperbolic equation method" [3] for finding asymptotical behavior of the spectral function of an elliptical operator on the Riemannian manifold.

The Hadamard method of construction of singularities of the fundamental solution for hyperbolic equations was continued on elliptic and parabolic equations [4]. In the last case we have only one singular point $t = 0, x = y$ of the fundamental solution and one can receive an asymptotical representation for $t \to +0$

$$E(x,y,t) \cong \left(\frac{1}{4\pi t}\right)^{n/2} \exp(-d^2(x,y)/4t) \sum_{j \geq 0} v_j(x,y) t^j .$$

Fundamental solution for elliptic equation has only one singular point $x = y$. The next representation one can find in John [5]

$$E(x,y) = |x-y|^{m-n} \sum_{\alpha=0}^{\infty} C_\alpha \left(y, \frac{x-y}{|x-y|}\right) |x-y|^\alpha + w(x,y) \ln |y| .$$

Here $w(x,y)$ is a regular analytic solution of the equation $Lw = 0$ and $C_\alpha(x^0, \xi)$ are the analytic functions. Further investigations of the structure of the fundamental solution were connected with the lacunas problem and with the description of the development of singularities of Pseudo DO.

Note that terms $g_k(x,y), v_j(x,y), w(x,y)$ of this representation contain significant information about the coefficients of these equations. This fact was applied for solving inverse problems in scattering particles theory [6], in particular for "X-ray tomography with scattering problem" [7].

Some new result, one can find in [13-14]. These results may be considered as an adaptation of the classical tomography method [12] to the scattering media.

The next interesting representative of equations without quasilocal properties is the transport equation which is a linearized Boltzmann equation and is used for describing the process of particle diffusion in a material body [8-10]. This research has been initiated by the desire to understand the singularity structure of solutions of this equation

$$(0.3) \quad \frac{1}{v}\frac{\partial U}{\partial t} + \Omega \cdot \nabla_x U + \sigma(x) U = \frac{\sigma_s(x)}{4\pi}\int_{\mathbb{S}^2} \eta(x,\Omega,\Omega^\star) U(x,t,\Omega^\star) d\Omega^\star + Q(x,t,\Omega).$$

A phase space $\{x,t,\Omega\}$ is a 6-dimensional continuum, three independent coordinates are used to represent position x at moment t and two to indicate direction Ω of the particle (Ω is a unit vector). Ω. The solution $\partial U(x,t,\Omega)$ is a 6-dimensional continuum, three independent coordinates are used to represent position x at moment t and two to indicate direction Ω of the particle (Ω is a unit vector).

Here $Q(x,t,\Omega)$ means the physical source density; $\sigma(x)$, $\sigma_s(x)$ are total and scatter macroscopic cross sections (coefficients), respectively:

$$(0.4) \qquad \sigma(x) = \sigma_a(x) + \sigma_s(x), \qquad \sigma(x), \sigma_a(x), \sigma_s(x) \geq 0.$$

$\sigma_a(x)$ is the absorption macroscopic cross section, $\eta(x,\Omega,\Omega^\star)$ denotes the probability that a particle entering in collision at the direction $\Omega^\star$ leaves it in the direction Ω. We call this function the scatter indicatrix. It is normalized to number one:

$$(0.5) \qquad \frac{1}{4\pi}\int_{\mathbb{S}^2} \eta(x,\Omega^\star,\Omega) d\Omega^\star = 1.$$

Usually for the fundamental solutions of PDE the asymptotics for small and large Γ is considered separately. For the transport equation we shall show, having isolated the regular part from $L^\infty(R^4 \times S^2)$, that the singular part of the FS contains two delta-like distributions, one concentrated on the ray

$$r = x^0 + vt\Omega^0, \quad t \geq 0,$$

in R^4 and one on the 4-dimensional portion of the surface Γ in $R^4 \times S^2$ and it also contains two singular terms having power-logarithmic and logarithmic estimates. The estimates obtained for the coefficients preceding the singularities

$$0 \leq \psi_n \leq \beta^n \exp(-\gamma vt), \quad n = 0, 1, 2,$$

$$\beta = \frac{v}{4\pi} \sup_{\Omega^* \in S^2} \|\sigma_s \eta\|_\infty, \quad \gamma = \inf_{x \in D} \sigma_a(x),$$

make it possible to estimate the singularity of once and twice scattered radiation generated by a monodirected dot source in a distant area. Note that in the paper [11] the time-independent case was considered.

The paper is organized as follows. In Section 2 we establish the notation, give some definitions needed in the sequel and prove the basic facts about the initial value problem for equation (0.3). In Section 3 we introduce the definitions concerning the fundamental solution and give it's Dyson-Philips [16] and Feynman-Kac representations. In Section 4 we prove two auxiliary lemmas and in Section 5 we prove the representation theorem for the FS.

2. Transport equation. Let $\mathcal{D}$ be a test function space of all C_0^∞ functions with compact support in $\mathbb{R}^4 \times \mathbb{S}^2$. Further on we shall call the pair(system) $(\sigma, \sigma_s\eta)$ *admissible* if:

i) The functions $\sigma_s(x), \sigma(x)$ are non negative of $\mathcal{D}$ and $\eta(x, \Omega, \Omega^*)$ are non negative of C_0^∞.

ii) The system under consideration is *subcritical* [9,10]:

$$0 \leq \sigma_s(x) \leq \sigma(x), \quad x \in \mathbb{R}^3.$$

Let B be the closed cone $B = \{x \in \mathbb{R}^3, t \in \overline{\mathbb{R}^1_+} : vt \geq |x - x^0|\}$ and $K^+ = B \times \mathbb{S}^2$ be the region in

$$K^+(\tau) = K^+ \cap \{0 \leq t \leq \tau\}, D(\tau u) = \left\{|x - x^0| < v\tau\right\} \subset \mathbb{R}^3$$

$$\text{and } K \equiv K^+(T), \ D \equiv D(T)$$

for some constant $T > 0$ fixed throughout the paper.

We denote by

$$\beta = \frac{v}{4\pi} \sup_{\Omega^* \in S^2} \|\sigma_s \eta\|_\infty,$$

here $\| \ \ \|_\infty$ is the norm on $L^\infty(K)$

$$\|f\|_\infty = \operatorname*{ess\,sup}_K |f(x, t, \Omega)|.$$

We shall need the following conditions:

(UA) There is a convex, connected $D_0 \subset D$, diam $D_0 = R_0$

so that D_0, for some $T_1 \leq T$ and $\gamma = const$

$$0 < \gamma \leq \sigma_a(x), \quad x \in \overline{D_0}$$

(uniform absorption),

(WS) $$\sup_{(x,t,\Omega)\in K} \int_\alpha^t \rho(x, x - v(t-\tau)\Omega)dt < \frac{1}{v\beta}$$

(weak scattering),

$$\alpha = \frac{|x - x^0 - vt\Omega|^2}{2v(vt - \Omega \cdot (x - x^0))}.$$

and $\rho(x,y)$ is the usual optical path

$$\rho(x,y) = \exp\left(-|x-y| \int_0^1 \sigma(x + s(y-x))ds \right).$$

Note that if i) is valid, then

$$\sup_{(x,t,\Omega)\in K} \int_\alpha^t \rho(x, x - v(t-\tau)\Omega)dt \leq T.$$

Then the Cauchy problem for transport equation

$$\frac{1}{v}\frac{\partial U}{\partial t} + \Omega \cdot \nabla_x U + \sigma(x)U = \lambda \frac{\sigma_s(x)}{4\pi} \int_{\mathbb{S}^2} \eta(x,\Omega,\Omega^\star)U(x,t,\Omega^\star)d\Omega^\star + Q(x,t,\Omega) \tag{1.1}$$

$$U(x,t,\Omega)|_{t<0} \equiv 0 \tag{1.2}$$

will be easily transformed into integral equation:

$$\begin{aligned} U(x,t,\Omega) = \lambda \frac{v}{4\pi} \int_0^t & d\tau \rho(x - v(t-\tau)\Omega, x)\sigma_s(x - v(t-\tau)\Omega) \\ & \times \left(\int_{\mathbb{S}^2} \eta(x - v(t-\tau)\Omega, \Omega, \Omega')U(x - v(t-\tau)\Omega, \tau, \Omega')d\Omega' \right. \\ & \left. + Q(x - v(t-\tau)\Omega, \tau, \Omega) \right) \end{aligned} \tag{1.3}$$

Equation (1.3) can be written in the form

$$U(x,t,\Omega) = \lambda \mathrm{K}\,\mathrm{S}\,U(x,t,\Omega) + \mathrm{K}\,Q(x,t,\Omega) \tag{1.4}$$

where S, K denote the integral operators

$$\mathrm{S}\ \varphi(x,t,\Omega) \equiv \frac{\sigma_s(x)}{4\pi} \int_{\mathbb{S}^2} \eta(x,\Omega,\Omega^\star)\varphi(x,t,\Omega^\star)d\Omega^\star$$

$$\mathrm{K}\ \varphi(x,t,\Omega) \equiv$$

$$\theta(t) \int_0^t v\,d\tau \exp\left(- \int_{x - v(t-\tau)\Omega}^x \sigma(x)dl \right) \varphi(x - v(t-\tau)\Omega, \tau, \Omega)$$

defined on $L^\infty(K)$.

We need the following

Lemma 2.1. *Suppose* $\rho(t) \in L_{+}^{loc}(\mathbb{R})$. *Then for any* $x, x^0 \in \mathbb{R}^3, \Omega \in \mathbb{S}^2$

$$\int_0^t \theta(v\tau - |x - x^0 - v(t-\tau)\Omega|)\rho(\tau)d\tau = \theta(vt - |x - x^0|)\int_\alpha^t \rho(\tau)d\tau,$$

where

$$\alpha = \frac{|x - x^0 - vt\Omega|^2}{2v(vt - \Omega\cdot(x - x^0))}.$$

and we also denote by

$$\theta(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

the Heavisides function.

Proof. Setting

$$D_\tau(x^0,\Omega) = \{x \in \mathbb{R}^3 | \theta(v\tau - |x - x^0 - v(t-\tau)\Omega|) = 1\}.$$

we have that $D_\tau(x^0,\Omega) \subseteq D_{\tau'}(x^0,\Omega)$ for $0 \leq \tau \leq \tau' \leq t$.

The following properties of the domain $D_t(x^0,\Omega)$ are easily verified.

For any $x \in D_t(x^0,\Omega)$ there exits unique α , $0 \leq \alpha \leq t$ such that $x \in D_\tau(x^0,\Omega)$ for $\tau \leq \alpha$ and $x \in D_\tau(x^0,\Omega)$ for $\tau > \alpha$.

Value of α is calculated from condition $|\Omega| = 1$ and equality

$$v\alpha - |x - x^0 - v(t-\alpha)\Omega| = 0.$$

i.e.

$$(v\alpha)^2 = |x - x^0 - vt\Omega|^2 + 2v\alpha((x - x^0 - vt\Omega)\cdot\Omega) + (v\alpha)^2.$$

This has completed the proof. The following statement will be useful

Theorem 2.1. *Let* $(\sigma, \sigma_s\eta)$ *define admissible system and obey (WS).Then the integral equation*

$$U(x,t,\Omega) = \mathrm{K}\,\mathrm{S}U(x,t,\Omega) + P(x,t,\Omega) \qquad (1.5)$$

has a unique solution $U(x,t,\Omega) \in L^\infty(K)$ *for any* $P(x,\Omega) \in L^\infty(K)$.

Proof. Note that the integral operator K S is well defined as operator K S $: L^\infty(K) \mapsto L^\infty(K)$.

It is clear that if (i) is obeyed, then S $: L^\infty(K) \mapsto L^\infty(K)$. Since $L^\infty(K) \subset L^1(K)$ for any $u \in L^\infty(K)$

$$\|u\|_1 = \int_0^T dt \int_{\mathbb{S}^2} d\Omega \int_{|x-x^0|\leq vt} dx |u(x,t,\Omega)| \leq \infty.$$

Hence we have

$$\begin{aligned}
&\left| \int_0^T dt \int_{\mathbb{S}^2} d\Omega \int_{|x-x^0|\leq vt} dx u(x,t,\Omega) \right| = \left| \int_{\mathbb{S}^2} d\Omega \int_K dx dt u(x,t,\Omega) \right| \\
&\leq \left| \int_{\mathbb{S}^2} d\Omega \int_{P(\Omega)} d\sigma \int_{\tau_0}^{\tau_1} u(x - v(t-\tau)\Omega, \tau, \Omega) d\tau \right| \\
&\leq \int_{\mathbb{S}^2} d\Omega \int_{P(\Omega)} d\sigma \int_{\tau_0}^{\tau_1} |u(x - v(t-\tau)\Omega, \tau, \Omega)| d\tau \\
&= \|u\|_1.
\end{aligned}$$

Here $P(\Omega)$ is a projection of the domain B on the plane in $\mathbb{R}^4$ is orthogonal to the vector $(\Omega, 1/v)$, and $r_{1,2} = r_{1,2}(\Omega, p)$, $p \in P(\Omega)$ are the points of intersection the of corresponding ray and ∂B.

From Fubini-Tonelli theorem the integral

$$\int_{\tau_0}^{\tau_1} u(x - v(t-\tau)\Omega, \tau, \Omega) d\tau$$

exists almost everywhere and hence

$$\mathrm{K} \,:\, L^\infty(K) \mapsto L^\infty(K).$$

Estimate the norm of K S. We have from lemma 2.1 for any function $u \in L^{\infty}(K)$

$$\begin{aligned}
&\|\mathrm{K\,S}\,u\|_{\infty} \\
&= \operatorname*{ess\ sup}_{K}\left|\frac{v}{4\pi}\int_0^t d\tau \int_{\mathbb{S}^2} d\Omega^\star \eta(x - v(t-\tau)\Omega, \Omega, \Omega^\star)\rho(x - v(t-\tau)\Omega, x)\right. \\
&\left.\times\ \sigma_s(x - v(t-\tau)\Omega)\theta(v\tau - |x - x^0 - v(t-\tau)\Omega)u(x - v(t-\tau)\Omega, \tau, \Omega^\star)\right| \\
&\le v\beta\|u\|_{\infty}\int_0^t d\tau\theta(v\tau - |x - x^0 - v(t-\tau)\Omega|)\rho(x - v(t-\tau)\Omega, x) \\
&\le v\beta\|u\|_{\infty} \sup_{(x,t,\Omega)\in \mathrm{K}} \int_{\alpha}^{t} \rho(x, x - v(t-\tau)\Omega)dt
\end{aligned}$$

and provided that (WS)

$$\|\,\mathrm{K\,S}\,\|_{\infty} < 1. \tag{1.6}$$

We proove the lemma by constructing the Neumann series solution for equation (1.5)

$$q(x,t,\Omega) = \sum_{n=0}^{\infty} q_n(x,t,\Omega)$$

where

$$q_0(x,t,\Omega) = p, \quad q_n(x,t,\Omega) = \mathrm{K\,S}\ q_{n-1}(x,t,\Omega), \quad n > 0.$$

The condition (1.6) guarantees the convergence of the this series in $L^{\infty}(K)$. This completes the proof of the lemma.

3. Fundamental solution. A distribution $f \in \mathcal{D}'$ on $\mathbb{R}^4 \times \mathbb{S}^2$ is a continuous linear functional defined on the test function space $\mathcal{D}$ [15]. We write (f, φ) for the action of a distribution f on a test function φ. Two distributions f, g, not necessarily with the same domain, are said to be equal on the domain G if $(f,\varphi) = (g,\varphi)$ for all $\varphi \in \mathcal{D}(G)$. Let $\delta(\Omega - \Omega^0) \in \mathcal{D}'(\mathbb{S}^2)$ denote the delta function on $\mathbb{S}^2$ concentrated in point $\Omega^0 \in \mathbb{S}^2$, i.e

$$\big(\delta(\Omega - \Omega^0), \phi(\Omega)\big) = \phi(\Omega^0), \quad \phi(\Omega) \in \mathcal{D}(\mathbb{S}^2).$$

The distribution from $\mathcal{D}'$ satisfying the equation (1.3) with

$$Q^0(x,t,\Omega;x^0,\Omega^0) = \delta(x-x^0)\delta(t)\delta(\Omega-\Omega^0)$$

is called the *fundamental solution* of the time dependent transport equation.

In fact the Born representation

$$(3.1)\qquad E = \sum_{n=0}^{\infty} \lambda^n E_n = \sum_{n=0}^{N} \lambda^n (\mathrm{K\,S})^n E_0 + R_N(\lambda), \qquad E_0 = \mathrm{K}Q^0, \qquad \lambda \in C.$$

with $R_m(\lambda)$ N-th order remainder of this solution will be investigated.

Denote by

$$\begin{aligned} x^1 &= x^0 + vt_0\Omega^0, \\ x^k &= x^{k-1} + v(t_{k-1}-t_{k-2})\Omega^{k-1}, k = \overline{2,n}, \\ x^{n+1} &= x^n + v(t-t_{n-1})\Omega. \end{aligned}$$

Here

$$\Omega^n = \Omega, \quad t_n = t, \quad dz = dz_1 \cdots dz_n, \quad dz_k = d\Omega^k dt_k, \quad \Re = [0,t]\times \mathbb{S}^2.$$

Theorem 3.1. *The members of the expansion (3.1)*

$$E_n(x,t,\Omega;x^0,\Omega^0) = (\mathrm{K\,S})^n \mathrm{K} Q^0(x,t,\Omega;x^0,\Omega^0), \quad n \geq 0.$$

have the following Dyson-Phillips representation:

$$(E_0(x,t,\Omega;x^0,\Omega^0),\phi(x,t,\Omega)) = \int_0^\infty v dt \rho(x^0,x^1)\phi(x^1,t,\Omega^0);$$

$$(E_n(x,t,\Omega;x^0,\Omega^0),\phi(x,t,\Omega)) = \int_0^\infty v dt \int_{\mathbb{S}^2} d\Omega \ \mathrm{P}_n[\phi], \quad n > 0;$$

$$\mathrm{P}_n[\phi] = \frac{(v/4\pi)^n}{n!}\int_{\Re^{n-1}} dz \int_0^t dt_0 \ \mathrm{T}\left\{\prod_{k=0}^{k=n} V_k(t_0,t_1,\cdots,t_k)\right\}\varphi(x^{n+1},t,\Omega),$$

for $\phi = \phi(x,t,\Omega) \in \mathcal{D}$. *Here*

$$V_0(t_0) = \rho(x^0, x^1),$$

$$V_k(t_0,t_1,\cdots,t_k) = \rho(x^k, x^{k+1})\sigma(x^k)\eta(x^k,\Omega^k,\Omega^{k-1}), \quad k>0,$$

and we denote the chronological ordering operator by

$$\mathrm{T}\left\{H(t_0,t_1,\cdots,t_n)\right\} = \sum_{\alpha\in\mathfrak{S}_{n+1}} H(t_{\pi(0)},t_{\pi(1)},\cdots,t_{\pi(n)}\theta(t_{\pi(0)}-t_{\pi(1)})\cdots\theta(t_{\pi(n-1)}-t_{\pi(n)}),$$

here $\mathfrak{S}_{n+1}$ *- the permutation group.*

Note that we can rewrite the representation for $E_n(x,t,\Omega;x^0,\Omega^0)$ as Feynmann-Kac formula. Really,

$$E_n(x,t,\Omega;x^0,\Omega^0) = \frac{(v/4\pi)^n}{n!}\int_{\mathfrak{R}^{n-1}} dz \int_0^t dt_0\ \mathrm{T}\left\{\prod_{k=0}^{k=n} V_k(t_0,t_1,\cdots,t_k)\right\}\delta(x-x^{n+1}),$$

Denote by $E_t^0(q,q^0) \equiv E(x,t,\Omega;x^0,\Omega^0)$ the fundamental solution transport equation with

$\sigma_a(x) \equiv 0$. Here $q=(x,\Omega), q^{'} = (x^0,\Omega^0)$ are two points in phase space $\mathbb{R}^3\times\mathbb{S}^2$.

It is easy to show that this solution has the following properties:

i. $E_t^0(q,q^{'}) \geq 0$;

ii. $\int E_t^0(q,q^{'})dq = \int E_t^0(q,q^{'})dq^{'} = 1$;

iii. $E_{t+s}^0(q,q^{'}) = \int E_t^0(q,r)E_s^0(r,q^{'})dr$.

For any $t\geq 0$ and $q^{'}$ let μ_t be the probabilistic Borel measure on $\mathbb{R}^3\times\mathbb{S}^2$ defined by the formula

$$\mu(B) = \int_B E_t^0(q,q^{'})dq$$

where B is an arbitrary Borel subset of $\mathbb{R}^3\times\mathbb{S}^2$. The measures μ_t have compact supports, namely

$$supp\ \mu_t \subset \{(x,\Omega) : |x-x_0| \leq vt, |\Omega| = 1\}$$

Actually, we are going to study the singularities of this measure. We shall prove that this measure has the following structure of singularities:

$$\mu_t = \delta_1 + \delta_2 + \sum_{n=1}^{n=3} \beta_n + \gamma$$

Where δ_1, δ_2 are the singular distributions, β_n are the distributions with the integrable kernel and γ are the a.e. bounded functions.

For the research of singularities of this measure we shall first construct analogue of Winiers measure on a trajectory space. For this purpose we shall designate, through $W_n^t(x^0, \Omega^0; x, \Omega, t)$, the set of broken lines of length vt,with n-parts leaving a point x_0 in the direction of the vector Ω^0 and included in a trailer point x in the direction Ω. We also denote by

$$\int_0^t \sigma_a(x_n(v\tau))d\tau$$

the integral of $\sigma_a(x)$ along these broken lines $x_n(\zeta) \in W_n^t(x^0, \Omega^0; x, \Omega, t)$.

The properties (i-iii) give us an opportunity to define a Wiener measure $dW^t(q, q^{'})$ on

$$W^t(q, q^{'}) \equiv \bigcup_{n>0} W_n^t(x^0, \Omega^0; x, \Omega, t)$$

Let the complete measure of the set $W^t(q, q^{'})$ be assumed to be equal to $E_t^0(q, q^{'})$, and measure of the set Q of trajectories, getting at the moment of the time t_1 in the subset I_1 be defined by the expression:

$$\int_{I_1} dq_1 E_{t_1}^0(q, q^\star) E_{t-t_1}^0(q^\star, q^{'}) dq^\star$$

In a general case we shall consider the subsets of trajectories in W^t points of which in n fixed moment of time t_i belong to Borel subsets $I_i \subset \mathbb{R}^3$. Such subsets in W^t

are referred to as the cylindrical sets, and the measure of this set is defined by the expression:

$$\int_{I_1} dq_1 \int_{I_2} dq_2 \int_{I_n} dq_n E^0_{t_1}(q, q_1) E^0_{t_2 - t_1}(q_1, q_2) ... E^0_{t_n - t_{n-1}}(q_n, q_{n-1})$$

We formalate the basic existance theorem.

Lemma 3.1. *The measure dW^t_n is countably additive on the cylindrical subset W^t_n of the space W^t and has a unique continuation dW^t on the σ-algabra of Borel subsets of W^t.*

The following theorem is an analogue of the well known statement.

Theorem 3.2. *(Feynman-Kac formula.)*

$$E^0_t(q, q^0) \equiv E(x, t, \Omega; x^0, \Omega^0) = \int \exp\left(\int_0^t \sigma_a(x(v\tau))d\tau\right) dW^t$$

Proof. For $n = 0$ we have

$$\begin{aligned}
&(E_0(x, t, \Omega; x^0, \Omega^0), \phi(x, t, \Omega)) = (\mathrm{K}Q(x, t, \Omega; x^0, \Omega^0), \phi(x, t, \Omega)) \\
&= \left(v\theta(t)\rho(x^0, x^0 + vt\Omega^0)\delta(x - x^0 - vt\Omega^0)\delta(\Omega - \Omega^0), \phi(x, t, \Omega)\right) \\
&= \int_0^\infty v dt \rho(x^0, x^1)\phi(x^1, t, \Omega^0).
\end{aligned}$$

Similarly for $n = 1$.

$$\begin{aligned}
&(E_1(x, t, \Omega; x^0, \Omega^0), \phi(x, t, \Omega)) = (\mathrm{K}\,\mathrm{S}E_0(x, t, \Omega; x^0, \Omega^0), \phi(x, t, \Omega)) \\
&= \frac{v}{4\pi}\left(\mathrm{K}\theta(t)\rho(x^0, x^0 + vt\Omega^0)\delta(x - x^0 - vt\Omega^0)\sigma_s(x)\eta(x, \Omega, \Omega^0), \phi(x, t, \Omega)\right) \\
&= \frac{v^2}{4\pi}(\theta(t)\int_0^t d\tau \rho(x - v(t - \tau)\Omega, x)\delta(x - v(t - \tau)\Omega - x^0 - v\tau\Omega^0) \\
&\times \rho(x^0, x^0 + v\tau\Omega^0)\sigma_s(x - v(t - \tau)\Omega)\eta(x - v(t - \tau)\Omega, \Omega, \Omega^0), \phi(x, t, \Omega)) \\
&= \frac{v^2}{4\pi}\int_0^\infty dt \int_{\mathbb{S}^2} d\Omega \int_0^t dt_0 \rho(x^0, x^1)\sigma_s(x^1)\eta(x^1, \Omega, \Omega^0)\rho(x^1, x^2)\phi(x^2, t, \Omega).
\end{aligned}$$

As

$$\int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_1} H(t_0, t_1, \cdots, t_n) dt_0$$

$$= \frac{1}{(n+1)!} \int_0^t dt_n \int_0^t dt_{n-1} \int_0^t dt_{n-2} \cdots \int_0^t \mathrm{T}\left\{H(t_0, t_1, \cdots, t_n)\right\} dt_0 .$$

Let us have for $n > 0$

$$(E_n(x,t,\Omega;x^0,\Omega^0), \phi(x,t,\Omega)) = \frac{v}{n!}\left(\frac{v}{4\pi}\right)^n \int_0^\infty dt \int_{\mathbb{S}^2} d\Omega \int_{\Re^{n-1}} dz \int_0^t dt_0$$

$$\times \mathrm{T}\left\{\prod_{k=0}^{k=n} V_k(t_0,t_1,\cdots,t_k)\right\} \varphi(x^{n+1},t,\Omega)$$

$$= \left(\frac{v}{4\pi}\right)^n \int_{\mathbb{R}^3} dx \int_0^\infty v dt \int_{\mathbb{S}^2} d\Omega \underbrace{\int_0^t dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2}}_{n-1} dt_1 \int_0^{t_1} dt_0$$

$$\times \int_{\mathbb{S}^2} d\Omega^1 \cdots \int_{\mathbb{S}^2} d\Omega^{n-1} \left\{\prod_{k=0}^{k=n} V_k(t_0,t_1,\cdots,t_k)\right\} \delta(x - x^{n+1}) \phi(x,t,\Omega)$$

Introducing the new variables $\Omega \to \Omega^n, t \to t_n$ we get finally for $\phi(x,t,\Omega) \in \mathcal{D}$

$$(E_{n+1}(x,t,\Omega;x^0,\Omega^0), \phi(x,t,\Omega)) = (x,t,\Omega;x^0,\Omega^0), \phi(x,t,\Omega))$$

$$= \left(\frac{v}{4\pi}\right)^{n+1} \int_0^\infty v dt \int_{\mathbb{S}^2} d\Omega \underbrace{\int_0^t dt_n \int_0^{t_n} dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2}}_{n} dt_1$$

$$\times \int_0^{t_1} dt_0 \int_{\mathbb{S}^2} d\Omega^1 \cdots \int_{\mathbb{S}^2} d\Omega^n \left\{\prod_{k=0}^{k=n+1} V_k(t_0,t_1,\cdots,t_k)\right\} \phi(x^{n+2},t,\Omega)$$

$$= \frac{v}{(n+1)!}\left(\frac{v}{4\pi}\right)^{n+1} \int_0^\infty dt \int_{\mathbb{S}^2} d\Omega \underbrace{\int_0^t dt_n \cdots \int_0^t}_{n} dt_1 \int_0^t dt_0$$

$$\times \underbrace{\int_{\mathbb{S}^2} d\Omega^1 \cdots \int_{\mathbb{S}^2}}_{n} d\Omega^n \, \mathrm{T}\left\{\prod_{k=0}^{k=n+1} V_k(t_0,t_1,\cdots,t_k)\right\} \phi(x^{n+2},t,\Omega)$$

$$= \frac{1}{(n+1)!}\left(\frac{v}{4\pi}\right)^{n+1} \int_0^\infty dt \int_{\mathbb{S}^2} d\Omega \int_{[0,t]\times\mathbb{S}^2} dt_n d\Omega^n \cdots \int_{[0,t]\times\mathbb{S}^2} dt_1 d\Omega^1$$

$$\times T \int_0^t v dt_0 \left\{\prod_{k=0}^{k=n+1} V_k(t_0,t_1,\cdots,t_k)\right\} \phi(x^{n+2},t,\Omega)$$

$$= \frac{v}{(n+1)!}\left(\frac{v}{4\pi}\right)^{n+1}\int_0^\infty dt\int_{\mathbb{S}^2} d\Omega\int_{\Re^n} dz\int_0^t dt_0$$
$$\times \mathrm{T}\left\{\prod_{k=0}^{k=n+1} V_k(t_0,t_1,\cdots,t_k)\right\}\phi(x^{n+2},t,\Omega).$$

The proof is completed.

4. Auxiliary lemmas.

For the proof of the main theorem we need several lemmas.

Lemma 4.1.

$$I_1(x-x^0,t,\Omega^0)\equiv\int_0^t dt_0\int_{\mathbb{S}^2}\delta(x-x^2)d\Omega^1=\frac{2}{v}\frac{\theta(vt-|x-x_0|)}{|x-x^0-vt\Omega^0|^2}.$$

Proof. For distribution

$$I_1(x-x^0,t,\Omega^0)=\int\limits_0^t dt_0\int\limits_{\mathbb{S}^2}\delta(x-x^0-vt_0\Omega^0-v(t-t_0)\Omega^1)d\Omega^1$$

and any test function $\phi(x,t,\Omega)\in\mathcal{D}$ we have

$$(I_1,\phi)=\int_0^\infty dt\int_{\mathbb{S}^2}d\Omega\phi(x^0+vt_0\Omega^0+v(t-t_0)\Omega^1,t,\Omega)d\Omega^1$$
$$=\int_0^\infty dt\int_{\mathbb{S}^2}d\Omega\int\limits_0^t\frac{dt_0}{v^2(t-t_0)^2}\int\limits_{O_1}\phi(x,t,\Omega)dO_1$$

Here O_1 -is a sphere with radius $v(t-t_0)$ centered in point $x^0+vt_0\Omega^0\in\mathbb{R}^3_x$.

Denote by

$$I_2=O_1\phi(x,t,\Omega)dO_1$$

Now we can take the coordinate system Y with center at point x^0 such as $\Omega^0=(0,1,0)$. Introducing new variables $z_1=y_1$, $z_2=-(y_2-vt)$, $z_3=y_3$ in this new coordinate system Y we get

$$I_2=\int\limits_0^t\frac{dt_0}{(vt_0)^2}\int\limits_{O_2}\phi(z,t,\Omega)dO_2.$$

Here O_2 -is a sphere with radius vt_0 placed in point $(0, vt^0, 0)$. It is an equation of the form:

$$z_2(z_1^2 + z_2^2 + z_3^2)^{-2} = (2vt_0)^{-1}.$$

In the polar coordinate system

$$z_1 = \rho \cos\psi \cos\varphi, \quad z_2 = \rho\cos\psi \sin\varphi, \quad z_3 = \rho \sin\psi$$

we may rewrite this equation as

$$\rho^2 = 2vt_0 z_2 \quad \text{ or } \rho = 2vt_0 \cos\psi \sin\varphi.$$

In (z_1, z_2, z_3) coordinate system this equation is reduced to:

$$\begin{aligned} z_1 &= 2vt_0 \cos^2\psi \sin\varphi \cos\varphi, \\ z_2 &= 2vt_0 \cos^2\psi \sin^2\varphi, \\ z_3 &= 2vt_0 \cos\psi \sin\varphi \sin\psi. \end{aligned}$$

Area element dO_2 is equal to

$$dO_2 = \left| \frac{\partial r}{\partial \psi} \times \frac{\partial r}{\partial \varphi} \right| d\psi d\varphi, \quad r(\varphi, \psi) = (z_1, z_2, z_3).$$

Hence

$$\begin{aligned} I_2 &= \int\limits_0^t \frac{dt_0}{(vt_0)^2} \int\limits_{O_3} \phi(z, t, \Omega) dO_2 \\ &= \int_0^t \int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \frac{\partial r}{\partial \psi} \times \frac{\partial r}{\partial \varphi} \right| (vt_0)^{-2} \phi(z, t, \Omega) d\psi d\varphi dt_0. \end{aligned}$$

The Jacobian of the map

$$(\psi, \varphi, t_0) \longrightarrow (z_1, z_2, z_3)$$

is equal to

$$\left|\frac{\partial(z_1,z_2,z_3)}{\partial(\psi,\varphi,t_0)}\right| = \left(\frac{\partial r}{\partial\varphi}\times\frac{\partial r}{\partial\psi}\right)\cdot\left(\frac{\partial r}{\partial t_0}\right) = \left|\frac{\partial r}{\partial\varphi}\times\frac{\partial r}{\partial\psi}\right|\cdot\left|\frac{\partial r}{\partial t_0}\right|\cdot\cos\theta.$$

But

$$\left|\frac{\partial z}{\partial t_0}\right| = \frac{\partial|z|}{\partial t_0} = \frac{\partial\rho}{\partial t_0} = 2v\cos\psi\sin\varphi, \qquad \cos\theta = \cos\psi\sin\varphi,$$

$$|z|^2 = 4(vt_0)^2\cos^2\psi\sin^2\varphi.$$

Hence

$$\left|\frac{\partial(z_1,z_2,z_3)}{\partial(\psi,\varphi,t_0)}\right| = 2v\left|\frac{\partial r}{\partial\varphi}\times\frac{\partial r}{\partial\psi}\right|\cos^2\psi\sin^2\varphi$$

and

$$\left|\frac{\partial(\psi,\varphi,t_0)}{\partial(z_1,z_2,z_3)}\right| = 2t_0|z|^{-1}\left|\frac{\partial r}{\partial\varphi}\times\frac{\partial r}{\partial\psi}\right|^{-1}.$$

Return to (z_1,z_2,z_3) variable in the last integral.

$$\begin{aligned} I_2 &= \int_0^t\int_0^\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\frac{\partial r}{\partial\psi}\times\frac{\partial r}{\partial\varphi}\right|(vt_0)^{-2}\phi(z,t,\Omega)d\psi d\varphi dt_0 \\ &= \frac{2}{v}\underset{D_{vt}}{\int\int\int}\frac{\phi(z,t,\Omega)}{|z|^2}dz_1dz_2dz_3 \\ &= \frac{2}{v}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta(vt-|(z_1,z_2-vt,z_3)|)\frac{\phi(z,t,\Omega)}{|z|^2}dz_1dz_2dz_3. \end{aligned}$$

Here D_{vt} is an interior of the sphere with radius vt centered at point $(0,vt,0)$.

Return to the old variable

$$\begin{aligned} &= \frac{2}{v}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\theta(vt-|y|)}{y_1^2+(y_2-vt)^2+y_3^2}\phi(y,t,\Omega)d)dy_1dy_2dy_3 \\ &= \frac{2}{v}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\theta(vt-|x-x^0|)}{|x-x^0-vt\Omega^0|^2}\phi(x,t,\Omega)dx_1dx_2dx_3 \end{aligned}$$

Finally

$$(I_1,\phi) = \frac{2}{v}\int_0^\infty dt\int_{\mathbb{S}^2}d\Omega\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\theta(vt-|x-x^0|)}{|x-x^0-vt\Omega^0|^2}\phi(x,t,\Omega)dx_1dx_2dx_3,$$

i.e.

$$I_1(x,t,\Omega) = \int\limits_0^t dt_0 \int\limits_{S^2} \delta(x - x^0 - vt_0\Omega^0 - v(t-t_0)\Omega^1)d\Omega^1 = \frac{2}{v}\frac{\theta(vt - |x - x_0|)}{|x - x_0 - vt\Omega^0|^2}$$

as required.

Lemma 4.2. *For any* $x, y \in \mathbb{R}^3$

$$\int \frac{d\tau}{|x+\tau y|^\alpha} = \begin{cases} |y|^{-1}\operatorname{arsh}\left((\tau+b)/a\right) \ , \ \alpha = 1 \\ |x\times y|^{-1}\operatorname{arctg}\left((\tau+b)/a\right) \ , \ \alpha = 2, \end{cases}$$

here $a = |x\times y||y|^{-2}, \quad b = (x\cdot y)|y|^{-2}$.

Proof.

$$|x+\tau y|^\alpha == \left(\tau^2|y|^2 + 2\tau(x\cdot y) + |x|^2\right)^{\frac{\alpha}{2}} =$$

$$|y|^\alpha\left(\tau^2 + 2\tau(x\cdot y)|y|^{-2} + |x|^2|y|^{-2}\right)^{\alpha/2}$$

$$= |y|^\alpha\left((\tau+b)^2 + |y|^{-4}(|x|^2|y|^2 - (x\cdot y)^2)\right)^{\alpha/2}$$

But

$$|x|^2|y|^2 - (x\cdot y)^2 = |x|^2|y|^2(1-\cos\varphi^2) = |x\times y|^2.$$

Hence

$$\int \frac{d\tau}{|x+\tau y|^\alpha} = \frac{1}{|y|^\alpha}\int \frac{d\tau_1}{(\tau_1^2 + |y|^{-4}|x\times y|^2)^{\alpha/2}} = \frac{1}{|y|^\alpha}\int \frac{d\tau_1}{(\tau_1^2 + a^2)^{\alpha/2}}$$

$$= \begin{cases} |y|^{-1}\operatorname{arsh}\left((\tau+b)/a\right) \ , \ \alpha = 1 \\ |x\times y|^{-1}\operatorname{arctg}\left((\tau+b)/a\right) \ , \ \alpha = 2. \end{cases}$$

The proof is completed.

Lemma 4.3. *There are constants* $M_{1,2,3} > 0$ *such that for any* $x, x^0 \in \mathbb{R}^3, t \in \mathbb{R}^+, \Omega^0, Z \in \mathbb{S}^2, a \geq 4$

$$I_3 \equiv \int_{\mathbb{S}^2} \frac{d\Omega}{|(x - x^0 - vt\Omega)\times(\Omega - \Omega^0)|} \leq \frac{M_1}{|x - x^0 - vt\Omega^0|},$$

$$I_4 \equiv \int_{\mathbb{S}^2} \ln\left(\frac{a}{|(Z-\Omega^0)\times(\Omega-\Omega^0)|}\right)\frac{d\Omega}{|\Omega-\Omega^0|} \leq M_2 \ln\left(\frac{a}{|Z-\Omega^0|}\right) + M_3.$$

Proof. At first we note that the first integral doesn't depend on a choice of coordinate system. Hence, it is enough to prove it for the coordinate system with the center in point x^0 such as the vector Ω^0 considered as a vector from $\mathbb{R}^3_x$ has the form $\Omega^0 = \omega \equiv (0,1,0)$.

In the polar coordinates terms (ρ,φ,ψ) $z_1 = \rho\cos\psi\cos\varphi$, $z_2 = \rho\cos\psi\sin\varphi$, $z_3 = \rho\sin\psi$ the equation of the unit sphere O_1 with center in point $(0,-1,0)\in\mathbb{R}^3_x$ may be written as

$$\rho^2 = -2z_2, z_2 \leq 0, \text{ or } \rho = -2\cos\psi\sin\varphi, \quad \psi\in\left(\frac{\pi}{2},-\frac{\pi}{2}\right), \quad \varphi\in(\pi,2\pi).$$

Calculate the area element of this sphere

$$dO_1 = \left|\frac{\partial z}{\partial\psi}\times\frac{\partial z}{\partial\varphi}\right| d\psi d\varphi, \quad z(\varphi,\psi) = (z_1,z_2,z_3).$$

Then

$$\left|\frac{\partial z}{\partial\psi}\times\frac{\partial z}{\partial\varphi}\right|^2 = \begin{vmatrix} -2\sin 2\psi\sin^2\varphi & 2\sin\varphi\cos 2\psi \\ 2\cos^2\psi\sin 2\varphi & \sin 2\psi\cos\varphi \end{vmatrix}^2$$

$$+\begin{vmatrix} 2\sin\varphi\cos 2\psi & -\sin 2\psi\sin 2\varphi \\ \sin 2\psi\cos\varphi & 2\cos^2\psi\cos 2\varphi \end{vmatrix}^2 + \begin{vmatrix} \sin 2\psi\sin 2\varphi & 2\sin 2\psi\sin^2\varphi \\ 2\cos^2\psi\cos 2\varphi & 2\cos^2\psi\sin 2\varphi \end{vmatrix}^2.$$

Setting

$$B(\varphi,\psi) = \Big((16\sin\varphi\cos\varphi\cos^2\psi)^2 + (4\cos 2\psi\cos 2\varphi + 8\sin^2\psi\cos^2\varphi)^2$$

$$+(8\sin 2\psi\sin\varphi\cos^2\varphi)^2\Big)^{\frac{1}{2}}$$

we finally have

$$dO_1 = B(\varphi,\psi)\cos^2\psi|\sin\varphi|.$$

Note that for $y = x - x^0 - vt\Omega^0$

$$|(x - x^0 - vt\Omega)\times(\Omega-\Omega^0)| = |y\times(\Omega-\Omega^0)|.$$

Then

$$I_3 \equiv \int\limits_{S^2} \frac{d\Omega}{|(x - x^0 - vt\Omega) \times (\Omega - \Omega^0)|} = \int\limits_{S^2} \frac{d(\Omega - \omega)}{|y \times (\Omega - \omega)|} = \int\limits_{O_1} \frac{dO_1}{|y \times (z)|}$$

$$= \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{\pi}^{2\pi} \frac{B(\varphi, \psi) \cos^2 \psi |\sin \varphi|}{|y||\rho|| \sin \theta(\varphi, \psi)|} d\varphi d\psi = \frac{1}{|y|} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{\pi}^{2\pi} \frac{B(\varphi, \psi) \cos \psi}{2| \sin \theta(\varphi, \psi)|} d\varphi d\psi.$$

Here $\theta(\varphi, \psi)$ is an angle between vectors $y, z = z(\varphi, \psi)$.

As $0 \leq B(\varphi, \psi) \leq C$ for all φ, ψI_3 is majored by

$$I_3 \leq \frac{C}{|y|} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{\pi}^{2\pi} \frac{\cos \psi}{2| \sin \theta(\varphi, \psi)|} d\varphi d\psi \leq \frac{1}{|y|} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{0}^{2\pi} \frac{B(\varphi, \psi) \cos \psi}{2| \sin \theta(\varphi, \psi)|} d\varphi d\psi$$

$$= \frac{C}{2|y|} \int_{S^2} \frac{d\Omega}{|\Omega \times \Omega^*|},$$

$$\Omega^* = \frac{y}{|y|}.$$

The last integral does not depend on the choice of a coordinate system, hence we can take it such that $\Omega^* = (0, 0, 1)$. Then

$$I_3 \leq \frac{C}{2|y|} \int_{S^2} \frac{d\Omega}{\sin(\frac{\pi}{2} - \psi)} = \frac{C}{2|y|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} d\varphi d\psi = \frac{C\pi^2}{|y|}.$$

For the second integral in similar way we have

$$I_4 \equiv \int_{S^2} \ln\left(\frac{a}{|(Z - \Omega^0) \times (\Omega - \Omega^0)|}\right) \frac{d\Omega}{|\Omega - \Omega^0|}$$

$$= \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{\pi}^{2\pi} |\rho|^{-1} B(\varphi, \psi) \cos^2 \psi |\sin \varphi| \left(\ln\left(\frac{a}{|Z - \Omega^0|}\right) - \ln|\rho| - \ln|\sin \theta|\right) d\varphi d\psi$$

$$\leq M_2 \ln\left(\frac{a}{|Z - \Omega^0|}\right) + \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{\pi}^{2\pi} |\ln(\cos \psi \sin \varphi)| \, B(\varphi, \psi) \cos \psi d\varphi d\psi$$

$$+ \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_{\pi}^{2\pi} |\ln(|\sin \theta|)| \, B(\varphi, \psi) \cos \psi d\varphi d\psi$$

$$\leq M_2 \ln\left(\frac{a}{|Z - \Omega^0|}\right) + M + M \int_{S^2} |\ln(|\Omega \times \Omega^*|)| \, d\Omega$$

$$\leq M_2 \ln\left(\frac{a}{|Z - \Omega^0|}\right) + M_3, \quad M_{2,3} = const > 0.$$

This completes the proof.

5. The structure of the fundamental solution. The equation of the line in $\mathbb{R}^3_x$ passing through points $x^0+vt\Omega$ and $x^0+vt\Omega^0$ for fixed $t>0, \Omega\in\mathbb{S}^2$ has the form

$$(x-x^0-vt\Omega)\times(\Omega-\Omega^0)=0.$$

Denote by $L(t,\Omega,x^0,\Omega^0)$ the closed segment of this line lying between these points. Let us consider two segments in $\mathbb{R}^3_x$, $x=x^0+v\tau\Omega$, $x=x^0+v\tau\Omega^0, \tau\in[0,t]$. We also denote by $\Delta_\Omega(t,\Omega,x^0\Omega^0)$ the part of the plane in $\mathbb{R}^3_x$ bounded by these three segments. This construction defines in K a 4 dimensional surface $\Gamma^+=\{x,t,\Omega,(x-x^0-vt\Omega)\times(\Omega-\Omega^0)=0\}\cap\{K\}$. We also introduce the function:

$$\Gamma=\Gamma(x,t,\Omega,x^0,\Omega^0)=|(x-x^0-vt\Omega^0)\times(\Omega-\Omega^0)|.$$

Now we define $x^*(x,\Omega,x^0,\Omega^0)$ as the point of intersection of the ray $r=x^0+\Omega^0\tau, \tau\in[0,t]$ and the ray $r=x-\Omega\tau$ for any point $x\in\Delta_\Omega$.

For fixed x^0,Ω^0 we denote by

$$Q^+=\{x,t,\Omega\in K,(x-x^0-vt\Omega)\cdot(\Omega-\Omega^0)<0\},\quad Q^-=K\setminus Q^+,$$

and

$$\theta_Q\equiv\theta_Q(x,t,\Omega)=\begin{cases}1, & (x,t,\Omega)\in Q^+\\ 0, & (x,t,\Omega)\in Q^-\end{cases}$$

We denote by $\delta=\delta(\Gamma^+)$ the Dirac delta function concentrated on 4 dimensional surface Γ^+. It affects a test function $\phi(x,t,\Omega)\in\mathcal{D}$ as follows

$$(\delta(\Gamma^+),\phi)=\int\limits_{\mathbb{R}^1_+}dt\int\limits_{\mathbb{S}^2}d\Omega\int\limits_{L(t,\Omega,x^0,\Omega^0)}\frac{dl}{|\Omega-\Omega^0|}\phi(x,t,\Omega).$$

One may view $\delta(x-x^0-vt\Omega^0)\delta(\Omega-\Omega^0)$ as a direct product distributions $\delta(x-x^0-vt\Omega^0)\in\mathcal{D}'(\mathbb{R}^4)$ and $\delta(\Omega-\Omega^0)\in\mathcal{D}'(\mathbb{S}^2)$ for any $\Omega^0\in\mathbb{S}^2, x^0\in\mathbb{R}^3$.

We divide the main result into three statements:

Theorem 5.1. *(Hierarchy of singularities.) The fundamental solution $E(x,t,\Omega)$ of an admissible system can be represented as follows:*

$$E(x,t,\Omega)=\sum_{n=0}^{n=3}S_n(x,t,\Omega)\psi_n+\theta(vt-|x-x^0|)g(x,t,\Omega) \tag{$*$}$$

where:

$$S_0=v\theta(t)\delta(x-x^0-vt\Omega^0)\delta(\Omega-\Omega^0),$$

$$S_1=\delta(\Gamma^+),$$

$$S_2=\pi\theta(vt-|x-x^0|)\,\frac{1}{\Gamma},$$

$$S_3=\frac{\theta(vt-|x-x^0|)}{|\Omega-\Omega^0|}\,\ln\left(\frac{ct}{\Gamma}\right),$$

$$S_4=\theta(vt-|x-x^0|)\left(\ln\left(\frac{c}{|x-x^0-vt\Omega|}\right)+\theta_Q\ln\left(\frac{c|\Omega-\Omega^0|}{\Gamma}\right)\right),$$

$$\psi_0=v\rho(x^0,x^0+vt\Omega^0),$$

$$\psi_1=\frac{v}{4\pi}\rho(x^0,x^*)\rho(x^*,x)\sigma_s(x^*)\eta(x^*,\Omega,\Omega^0),$$

$$\psi_2=\varphi(x,t,\Omega)\,\mathrm{arctg}\,\Phi(x,t,\Omega)$$

for some $\Phi(x,t,\Omega)(see(5.2)below), c=4(1+\sqrt{2})v$ and

$$\varphi(x,t,\Omega),g(x,t,\Omega),\psi_3(x,t,\Omega),\psi_4(x,t,\Omega)\in L^\infty(K).$$

Moreover, if (UA) is valid and, in addition, if

$$\eta(x,\Omega,\Omega^\star)>0 \text{ for } \Omega,\Omega^\star\in\mathbb{S}^2, |x^0-x|<vT_1 \tag{5.1}$$

then $0<\varphi(x,t,\Omega)$ on $\operatorname{int}K^+(T_1)$.

Theorem 5.2. *(Decay properties.) Moreover, if (UA) is valid then there exists M independent of T_1 such that*

$$0\le\psi_n(x,t,\Omega)\le\beta^n\exp(-\gamma vt),\quad \text{on } K^+(T_1),\quad n=0,2$$

For each point of the boundary we build a cone

$$K_1(x^0, x^1; x, t) = \{(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+ : |x - x^1| + |x^1 - x^0| = vt\}.$$

It is easy to see, that the area

$$\mathcal{K}_1(x^0; x, t) = \bigcap_{x^1 \in \partial D} Int\ K_1(x^1, x^0)$$

is a cone. We shall similarly define the cone $\mathcal{K}_n(x^0; x, t)$

$$K_n(x^0, x^1, x^2, ..., x^n; x, t) = \{(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+ : |x - x^n| + \sum_{k=1}^{k=n} |x^k - x^{k-1}| = vt\}.$$

$$\mathcal{K}_n(x^0; x, t) = \bigcap_{x^1, x^2, ..., x^n \in \partial D} nt\ K_n(x^0, x^1, x^2, ..., x^n)$$

It easy to see that

$$\mathcal{K}_4(x^0; x, t) \subset \mathcal{K}_3(x^0; x, t) \subset \mathcal{K}_2(x^0; x, t) \subset \mathcal{K}_1(x^0; x, t) \subset \mathcal{K}_0(x^0; x, t).$$

Denote by $\chi(\mathcal{D})-$ the characteristic function of a set $\mathcal{D}$.

Theorem 5.3. *(Back fronts of singularities.) The terms $\psi_n(x, t, \Omega; x^0, \Omega^0)$ in (*) have the next representation:*

$$psi_n(x, t, \Omega; x^0, \Omega^0) = (1 - \chi(\mathcal{K}_n(x^0; x, t)))phi_n(x, t, \Omega; x^0, \Omega^0)$$

for some bounded functions $\phi_n(x, t, \Omega; x^0, \Omega^0)$

Proof. For the first term from direct calculation we have:

$$E_0(x, t, \Omega, x^0, \Omega^0) = \mathrm{K}\delta(t)\delta(x - x^0)\delta(\Omega - \Omega^0)$$

$$= v\theta(t)\rho(x^0, x)\delta(x - x^0 - vt\Omega^0)\delta(\Omega - \Omega^0)$$

and

$$0 < \rho(x^0, x^0 + vt\Omega) \leq \exp(-\gamma vt) \quad x, t \in K(T_1).$$

Consider the second term

$$E_1(x,t,\Omega,x^0,\Omega^0) = \mathrm{K}\,\mathrm{S}E_0(x,t,\Omega,x^0,\Omega^0).$$

From the theorem 3.1 it then follows that for $\phi(x,t,\Omega) \in D(K)$

$$\big(E_1(x,t,\Omega,x^0,\Omega^0)\ ,\ \phi(x,t,\Omega)\big) = \frac{v^2}{4\pi}\int\limits_0^\infty dt \int\limits_{\mathbb{S}^2} d\Omega \int\limits_0^t dt_0$$

$$\rho(x^0, x^0+vt_0\Omega^0)\rho(x^0+vt_0\Omega^0, x^0+v(t-t_0)\Omega+vt_0\Omega^0)$$

$$\times\,\sigma_s(x^0+vt_0\Omega^0)\eta(x^0+vt_0\Omega^0,\Omega,\Omega^0)\phi(x^0+vt_0\Omega^0+v(t-t_0)\Omega,t,\Omega).$$

In the last integral we replace the variables

$$vt_0(\Omega^0-\Omega) = l\vec{e}, \quad dt_0 = \frac{dl}{v|\Omega-\Omega^0|}$$

here $\vec{e}$ - is a unit directing vector of $L(t,\Omega,x^0,\Omega^0)$.

For any point $x = x^0+v(t-t_0)\Omega+vt_0\Omega^0 \in L(t,\Omega)$ the point $x^* = x^0+vt^*\Omega^0$ is defined as we mentioned above.

Then

$$\begin{aligned}
&\big(E_1(x,t,\Omega,x^0,\Omega^0)\ ,\ \phi(x,t,\Omega)\big)\\
&= \frac{v^2}{4\pi}\int\limits_0^\infty dt \int\limits_{\mathbb{S}^2} d\Omega \int\limits_{L(t,\Omega)} \frac{dl}{v|\Omega-\Omega^0|}\rho(x^*, x^0+v(t-t_0)\Omega+vt_0\Omega^0)\\
&\times\,\sigma_s(x^*)\eta(x^*,\Omega,\Omega^0)\exp\left(-\int_{x^0}^{x^*}\sigma(x)dl\right)\phi(x,t,\Omega)dl\\
&= \big(\delta(\Gamma^+)\psi_1(x,t,\Omega)\ ,\ \phi(x,t,\Omega)\big)\,.
\end{aligned}$$

i.e.

$$E_1(x,t,\Omega,x^0,\Omega^0) = \frac{v}{4\pi}\delta(\Gamma^+)\rho(x^0,x^*)\rho(x^*,x)\sigma_s(x^*)\eta(x^*,\Omega,\Omega^0).$$

Note that

$$\begin{aligned}
0 \le \psi_1 &= \frac{v}{4\pi}\rho(x^0,x^*)\rho(x^*,x)\sigma_s(x^*)\eta(x^*,\Omega,\Omega^0)\\
&\le \beta\exp(-\gamma|x^0-x^*|)\exp(-\gamma|x^*-x|) = \beta\exp(-\gamma vt).
\end{aligned}$$

Now we shall calculate the third term

$$E_2(x,t,\Omega,x^0,\Omega^0) = \mathrm{K}\,\mathrm{S}E_1(x,t,\Omega',x^0,\Omega^0).$$

The interchange of the order of integration is reduced by Fubini theorem. Then from the theorem 3.1:

$$\begin{aligned} I_2 &= \big(E_2(x,t,\Omega,x^0,\Omega^0)\ ,\ \phi(x,t,\Omega)\big) = \big(\mathrm{K}\,\mathrm{S}E_1(x,t,\Omega,x^0,\Omega^0)\ ,\ \phi(x,t,\Omega)\big) \\ &= \frac{v^3}{16\pi^2}\int_0^\infty dt \int_{\mathbb{S}^2} d\Omega \int_0^t dt_1 \int_0^{t_1} dt_0 \int_{\mathbb{S}^2} d\Omega^1 \rho(x^0,x^1)\rho(x^1,x^2)\rho(x^2,x^3) \\ &\times \sigma_s(x^1)\eta(x^1,\Omega^1,\Omega^0)\sigma_s(x^2)\eta(x^2,\Omega,\Omega^1)\phi(x^3,t,\Omega) \end{aligned}$$

where

$$\begin{aligned} x^1 &= x^0 + vt_0\Omega^0 \\ x^2 &= x^0 + vt_0\Omega^0 + v(t_1-t_0)\Omega^1 \\ x^3 &= x^0 + vt_0\Omega^0 + v(t_1-t_0)\Omega^1 + v(t-t_1)\Omega. \end{aligned}$$

It follows from lemmas 2.1, 4.1

$$\begin{aligned} I_2 &= \frac{v^2}{8\pi^2}\int_0^\infty dt \int_{\mathbb{S}^2} d\Omega \int_{\mathbb{R}^3} dx\phi(x,t,\Omega) \int_0^t dt_1 \frac{\theta(vt_1 - |x-x^0-v(t-t_1)\Omega|)}{|x-x^0-v(t-t_1)\Omega - vt_1\Omega^0|^2} \\ &\times \rho(x^0,x^1)\rho(x^1,x^2)\rho(x^2,x^3) \times \sigma_s(x^1)\eta(x^1,\Omega^1,\Omega^0)\sigma_s(x^2)\eta(x^2,\Omega,\Omega^1) \\ &= \frac{v^2}{8\pi^2}\int_0^\infty dt \int_{\mathbb{S}^2} d\Omega \int_{\mathbb{R}^3} dx\phi(x,t,\Omega)\theta(vt-|x-x^0|) \\ &\times \int_\alpha^t dt_1 \frac{\rho(x^0,x^1)\rho(x^1,x^2)\rho(x^2,x^3)\sigma_s(x^1)\eta(x^1,\Omega^1,\Omega^0)\sigma_s(x^2)\eta(x^2,\Omega,\Omega^1)}{|x-x^0-v(t-t_1)\Omega - vt_1\Omega^0|^2}. \end{aligned}$$

Here Ω^1, t_1 depend on x. Now using lemma 4.3 for non negative, locally integrable kernel of this distribution we have almost everywhere in K the next estimate on

$K^+(T_1)$:

$$0 \le \frac{v^2}{8\pi^2} \int_{\alpha}^{t} dt_1 \frac{\rho(x^0,x^1)\rho(x^1,x^2)\rho(x^2,x^3)\sigma_s(x^1)\eta(x^1,\Omega^1,\Omega^0)\sigma_s(x^2)\eta(x^2,\Omega,\Omega^1)}{|x - x^0 - v(t-t_1)\Omega - vt_1\Omega^0|^2}$$

$$\le 2\beta^2 \int_{\alpha}^{t} dt_1 \frac{\exp(-\gamma v t)}{|x - x^0 - v(t-t_1)\Omega - vt_1\Omega^0|^2} = 2\beta^2 \frac{\exp(-\gamma v t)}{|(x - x^0 - vt\Omega) \times (\Omega - \Omega^0)|}$$

$$\operatorname{arctg}\left(\frac{t_1 + b}{a}\right)\Bigg|_{\alpha}^{t}.$$

Here

$$a = \frac{|(x - x^0 - vt\Omega) \times (\Omega - \Omega^0)|}{v^2|\Omega - \Omega^0|^2} \quad , \quad b = \frac{(x - x^0 - vt\Omega)\cdot(\Omega - \Omega^0)}{v^2|\Omega - \Omega^0|^2}.$$

Hence, observing that

$$\operatorname{arctg}\ x - \ \operatorname{arctg}\ y = \operatorname{arctg} \frac{x - y}{1 + xy}$$

we obtain

$$E_2(x,t,\Omega;x^0,\Omega^0) = S_2(x,t,\Omega)\psi_2(x,t,\Omega) = \frac{\pi\theta(vt - |x - x_0|)\psi_2(x,t,\Omega;x^0,\Omega^0)}{|(x - x^0 - vt\Omega) \times (\Omega - \Omega^0)|}$$

where

$$0 \le \psi_2(x,t,\Omega) \le \beta^2 \exp(-\gamma v t), \quad x,t,\Omega \in K^+(T_1), \text{ for any } \Omega^0 \in \mathbb{S}^2,$$

$$S_2(x,t,\Omega) \in L^{loc}(K^+), \quad \psi_2(x,t,\Omega) \in L^{\infty}(K) \quad \text{for any } \Omega^0 \in \mathbb{S}^2$$

and

$$\Phi = \frac{t - \alpha}{a + a^{-1}(t+b)(\alpha + b)}. \tag{5.2}$$

The assumptions (5.1) now imply that

$$0 < E_2(x,t,\Omega;x^0,\Omega^0)$$

on $K^+(T_1)$. Finally we have on$K$

$$E_2(x,t,\Omega;x^0,\Omega^0)90=\theta(vt-|x-x^0|)\frac{1}{|(x-x^0-vt\Omega)\times(\Omega-\Omega^0)|}\psi_2(x,t,\Omega;x^0,\Omega^0),$$

where

$$\psi_2(x,t,\Omega)=\varphi(x,t,\Omega)\operatorname{arctg}\Phi$$

Φ,α as above, $\varphi(x,t,\Omega)\in L^\infty(K)$ and, moreover, when condition (5.1) is satisfied then $0<\varphi(x,t,\Omega)$ on $K^+(T_1)$. It is obvious, that for calculating the singularities in

$$E_3(x,t,\Omega;x^0,\Omega^0)=\mathrm{K}\,\mathrm{S}E_2(x,t,\Omega;x^0,\Omega^0),$$

we should investigate $E_3(x,t,\Omega;x^0,\Omega^0)=\mathrm{K}\,\mathrm{S}\;S_2(x,t,\Omega;x^0,\Omega^0)$. From lemma 4.3 we have on K

$$\begin{aligned}0\le \mathrm{S}S_2(x,t,\Omega;x^0,\Omega^0)&=\frac{\sigma_s(x)}{4\pi}\int_{\mathbb{S}^2}\frac{\theta(vt-|x-x^0|)\eta(x,\Omega,\Omega^*)}{|(x-x^0-vt\Omega^*)\times(\Omega^*-\Omega^0)|}d\Omega^*\\&\le\frac{M_1M^2}{4\pi}\frac{\theta(vt-|x-x^0|)}{|x-x^0-vt\Omega^0|}\end{aligned}$$

Hence

$$\begin{aligned}0&\le E_3(x,t,\Omega;x^0,\Omega^0)\le\frac{M_1M^9}{4\pi}\,\mathrm{K}\,\frac{\theta(vt-|x-x_0|)}{|x-x_0-vt\Omega^0|}\\&=\frac{M_1M^9}{4\pi}\int_0^t\rho(x-x^0-v(t-\tau)\Omega,x-x^0)\frac{\theta(v\tau-|x-x_0-v(t-\tau)\Omega|)}{|x-x_0-v(t-\tau)\Omega-v\tau\Omega^0|}d\tau\\&\le\frac{M_1M^9}{4\pi}\int_0^t\frac{\theta(v\tau-|x-x_0-v(t-\tau)\Omega|)}{|x-x_0-v(t-\tau)\Omega-v\tau\Omega^0|}d\tau\end{aligned}$$

As by lemma 4.1

$$\le\frac{M_1M^9}{4\pi}\theta(vt-|x-x^0|)\int_\alpha^t\frac{d\tau}{|x-x^0-v(t-\tau)\Omega-v\tau\Omega^0|}$$

we have by lemma 4.2

$$0\le I_3\equiv\int_\alpha^t\frac{d\tau}{|x-x^0-vt\Omega+v\tau(\Omega-\Omega^0)|}=\frac{1}{v|\Omega-\Omega^0|}\operatorname{arsh}\left(\frac{\tau+b}{a}\right)\Bigg|_\alpha^t$$

$$\leq \frac{1}{v|\Omega-\Omega^0|}\operatorname{arsh}\left(\frac{v\tau|\Omega-\Omega^0|^2+(x-x^0-vt\Omega)\cdot(\Omega-\Omega^0)}{|(x-x^0-vt\Omega)\times(\Omega-\Omega^0)|}\right)\Bigg|_0^t$$

$$=\frac{1}{v|\Omega-\Omega^0|}\left\{\operatorname{arsh}\left(\frac{(x-x^0-vt\Omega^0)\cdot(\Omega-\Omega^0)}{|(x-x^0-vt\Omega)\times(\Omega-\Omega^0)|}\right)-\operatorname{arsh}\left(\frac{(x-x^0-vt\Omega)\cdot(\Omega-\Omega^0)}{|(x-x^0-vt\Omega)\times(\Omega-\Omega^0)|}\right)\right\}$$

$$\leq\frac{1}{v|\Omega-\Omega^0|}\left\{\operatorname{arsh}\left(\frac{|(x-x^0-vt\Omega^0)\cdot(\Omega-\Omega^0)|}{|(x-x^0-vt\Omega)\times(\Omega-\Omega^0)|}\right)+\operatorname{arsh}\left(\frac{|(x-x^0-vt\Omega)\cdot(\Omega-\Omega^0)|}{|(x-x^0-vt\Omega)\times(\Omega-\Omega')|}\right)\right\}.$$

For $x,t,\Omega\in K$ we have

$$|(x-x^0-vt\Omega^0)\cdot(\Omega-\Omega^0)|=vt\left|\left(\frac{x-x^0}{vt}-\Omega^0\right)\cdot(\Omega-\Omega^0)\right|\leq 4vt.$$

Thus, on the other hand, by a similar estimation

$$|(x-x^0-vt\Omega)\cdot(\Omega-\Omega^0)|\leq 4vt,\quad (x,t,\Omega)\in K,\quad \Omega^0\in\mathbb{S}^2$$

we get finally for $x,t,\Omega\in K,\quad \Omega^0\in\mathbb{S}^2$

$$I_3\leq\frac{2}{v|\Omega-\Omega^0|}\operatorname{arsh}\left(\frac{4}{|((vt)^{-1}(x-x^0)-\Omega)\times(\Omega-\Omega^0)|}\right). \tag{5.3}$$

Let us note that the inequality

$$\operatorname{arsh}\frac{a}{|y|}=\ln\left(\frac{a}{|y|}+\sqrt{1+\frac{a^2}{y^2}}\right)\leq\ln\left(\frac{a+a\sqrt{2}}{|y|}\right),$$

which holds for $|y|\leq a,\quad a>0$ and inequality (5.3) implies

$$0\leq E_3(x,t,\Omega;x^0,\Omega^0)\leq\frac{M_1M^9}{4\pi}\frac{\theta(vt-|x-x^0|)}{v|\Omega-\Omega^0|}\ln\left(\frac{4(1+\sqrt{2})}{|((vt)^{-1}(x-x^0)-\Omega)\times(\Omega-\Omega^0)|}\right)$$

$$=\frac{M_1M^9}{4\pi}\frac{\theta(vt-|x-x^0|)}{v|\Omega-\Omega^0|}\ln\left(\frac{4(1+\sqrt{2})vt}{\Gamma}\right).$$

As in the previous case we have

$$E_3(x,t,\Omega;x^0,\Omega^0)=\psi_3(x,t,\Omega)S_3(x,t,\Omega),$$

$$S_3(x,t,\Omega)\in L^{loc}(K),\quad \psi_3(x,t,\Omega)\in L^\infty(K)\quad\text{for any }\Omega^0\in\mathbb{S}^2.$$

In order to estimate $E_4(x,t,\Omega;x^0,\Omega^0) = \mathrm{K\,S}\ E_3(x,t,\Omega;x^0,\Omega^0)$, we use the lemma 4.3.

$$\mathrm{S}\ S_3(x,t,\Omega;x^0,\Omega^0) = \int_{\mathbb{S}^2} \frac{1}{v|\Omega-\Omega^0|} \ln\left(\frac{4(1+\sqrt{2})}{|((vt)^{-1}(x-x^0)-\Omega)\times(\Omega-\Omega^0)|}\right) d\Omega$$

$$\le M_2 \ln\left(\frac{4(1+\sqrt{2})vt}{|x-x^0-vt\Omega^0|}\right) + M_3 \quad , \quad (x,t,\Omega)\in K,\ \Omega^0 \in \mathbb{S}^2.$$

Hence

$$E_4(x,t,\Omega;x^0,\Omega^0) \le M_4 \mathrm{K\,S}\ E_3(x,t,\Omega;x^0,\Omega^0) \le M_4\mathrm{K}\ \left(M_2 \ln\left(\frac{4(1+\sqrt{2})vt}{|x-x^0-vt\Omega^0|}\right) + M_3\right)$$

$$= M_5 \int_0^t \ln\left(\frac{4(1+\sqrt{2})v\tau}{|x-x^0-v(t-\tau)\Omega - v\tau\Omega^0|}\right) d\tau + M_6.$$

Consider the integral

$$I_4 \equiv \int_0^t \ln\left(\frac{c\tau}{|x-x^0-v(t-\tau)\Omega-v\tau\Omega^0|}\right) d\tau = \int_0^t \ln\left(\frac{c\tau}{|x-x^0+v\tau(\Omega-\Omega^0)-vt\Omega|}\right) d\tau.$$

$$= \int_0^t \ln\left(\frac{c\tau}{\sqrt{v^2|\Omega-\Omega^0|^2((\tau+b)^2+a^2)}}\right) d\tau.$$

Here

$$a = |(x-x^0-vt\Omega)\times(\Omega-\Omega^0)|v^2|\Omega-\Omega^0|^{-2}, \quad b = |(x-x^0-vt\Omega)\cdot(\Omega-\Omega^0)|v^2|\Omega-\Omega^0|^{-2}.$$

We consider two cases. First, let $b < 0$.Then$(\tau+b)^2 + a^2 \ge a^2$ and

$$I_4 \le \int_0^t \ln\frac{c\tau}{|\Omega-\Omega^0|a} d\tau = \ln\frac{c}{|\Omega-\Omega^0|a} + t\ln t - t.$$

Let $b \ge 0$. Then for $\tau \in [0,t]$ $\quad (\tau+b)^2+a^2 \ge b^2+a^2$ and

$$I_4 \le \int_0^t \ln\frac{c\tau}{\sqrt{a^2|\Omega-\Omega^0|^2+b^2|\Omega-\Omega^0|^2}} d\tau = \ln\frac{c}{\sqrt{a^2|\Omega-\Omega^0|^2+b^2|\Omega-\Omega^0|^2}} + t\ln t - t.$$

Note that

$$\sqrt{a^2v^2|\Omega-\Omega^0|^2+b^2v^2|\Omega-\Omega^0|^2} = |x-x^0-vt\Omega|$$

and as in the previous case we have

$$E_4(x,t,\Omega;x^0,\Omega^0) = \psi_4(x,t,\Omega)S_4(x,t,\Omega)$$

$$S_4(x,t,\Omega) \in L^{loc}(K), \quad \psi_4(x,t,\Omega) \in L^\infty(K) \quad \text{for any } \Omega^0 \in \mathbb{S}^2.$$

We have proved now for representation

$$E(x,t,\Omega;x^0,\Omega^0) = \sum_{n=0}^{4} E_n(x,t,\Omega;x^0,\Omega^0) + E_r(x,t,\Omega;x^0,\Omega^0)$$

that

$$E_0(x,t,\Omega;x^0,\Omega^0) = \mathrm{K}\ \delta(x-x^0)\delta(t)\delta(\Omega-\Omega^0),$$

$$E_k(x,t,\Omega;x^0,\Omega^0) = \mathrm{K}\,\mathrm{S}\ E_{k-1}(x,t,\Omega;x^0,\Omega^0), \quad k=1,2,3,4.$$

From theorem 2.1 it follows immediately that the integral equation

$$E_r(x,t,\Omega;x^0,\Omega^0) = \mathrm{K}\,\mathrm{S}E_r(x,t,\Omega;x^0,\Omega^0) + \mathrm{K}\,\mathrm{S}E_4(x,t,\Omega;x^0,\Omega^0)$$

has the solution

$$E_r(x,t,\Omega;x^0,\Omega^0) = \theta(vt-|x-x^0|)g(x,t,\Omega)$$

with $g(x,t,\Omega) \in L^\infty(K)$. We obtain the conclusion of the theorem.

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REFERENCES

1. J. Hadamard, *Le probleme de Cauchy et les equations aux derivees partielles lineaires hyperboliques*, Hermann, Paris (1932).
2. J. Weiss, M. Tabor, and G. Carnevale, *The Painleve property for partial differential equations*, J. Math. Phys. **24** (1983), 522–526.
3. L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218.
4. V.M. Babich, *Hadamard's ansatz, its analogies, generalizations, applications*, Algebra and Analysis **3** (1991), 1–37.
5. F. John, *Plane waves and spherical means*, New York (1955).
6. A.N. Bondarenko, *Structure of singularities of the fundamental solution of the transport equation and inverse problems in particles scattering theory*, Dokl. Akad. Nauk SSSR **322** (1992), 274–276.
7. V.S. Antiufeev and A.N. Bondarenko, *X-ray tomography in scattering media*, SIAM J. Appl. Math. **56, 2** (1996), 573–587.
8. B. Davison, *Neutron Transport Theory*, The Claderon Press, Oxford (1958).
9. J. Hejtmanek, *Scattering theory of the linear Boltzmannoperator*, Commun. Math. Phys. **43** (1975), 109–120.
10. B. Simon, *Existence of the scattering matrix for the linearized Boltzmann equation*, Commun. Math. Phys. **41** (1975), 99–108.
11. A.N. Bondarenko, *On the structure of the fundamental solution of the time-independent transport equation*, Preprint of Institute of Mathematics **31** (1996), 20.
12. F. Natterer, *The Mathematical of Computerized Topography*, B.G. Teubner, Stuttgart (1986).
13. O. Dorn, *Das inverse Transport Problem in der Laser Tomography*, Thesis, Universität Münster (1997).
14. M. Choulli and P. Stefanov, *Inverse scattering and inverse oundary value problems for the linear Boltrmann equation*, Comm. Pure and Appl. Diff. **21 (5,6)** (1996), 763–785.
15. A. Friedman, *Generalized Functions and Partial Differential Equations*, Prentice-Hall, Inc. (1963), 340 pp.
16. J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics.* (1964), 300 pp.

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