HEAD-MEDIA INTERACTION IN MAGNETIC RECORDING

By

Avner Friedman

and

Bei Hu

IMA Preprint Series # 1363

November 1995

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455

HEAD-MEDIA INTERACTION IN MAGNETIC RECORDING

AVNER FRIEDMAN* AND BEI HU†

Abstract. The head-tape interaction in magnetic recording is modelled by a coupled system of a second order differential equation for the pressure and a fourth order differential equation for the tape deflection. There is also the constraint that the spacing between the head and tape remains positive. In this paper, we study the stationary 1-d case: we establish the existence of a smooth solution and a boundary layer phenomenon observed both numerically and experimently. The 2-d case is briefly discussed.

1. The model. The motion of magnetic media entrains air in between the head and media that forms a thin air film separating the head and the media. Figure 1 shows the one-dimensional head-media interaction system

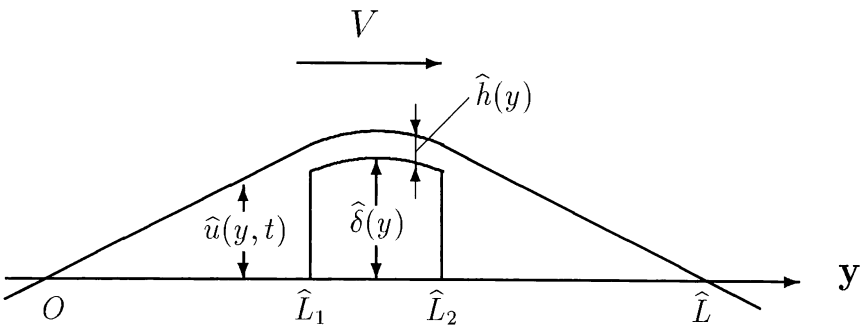

Figure 1.

Here

y = length parameter,
$\widehat{u}$ = deflection of the tape,
$\widehat{\delta}$ = the profile of the head,

and

$\widehat{h}$ = the spacing between head and tape.

---

* Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, MN 55455. Partially supported by National Science Foundation Grant DMS # 94-01251.

† Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556. Partially supported by National Science Foundation Grant DMS # 92-24935.

This problem was studied numerically by several authors; see [1] [2] [6] and the references therein.

The mathematical model for the air pressure $\widehat{p}$ between the head and the tape is based on the modified Reynolds equation, which is the lubrication approximation to the Navier-Stokes equations (taking slip into account)

$$12\frac{\partial(\widehat{p}\widehat{h})}{\partial t} + 6V\frac{\partial(\widehat{p}\widehat{h})}{\partial y} - \frac{6\lambda p_a}{\mu}\frac{\partial}{\partial y}\Big(\widehat{h}^2\frac{\partial \widehat{p}}{\partial y}\Big) - \frac{\partial}{\partial y}\Big(\frac{\widehat{h}^3\widehat{p}}{\mu}\frac{\partial \widehat{p}}{\partial y}\Big) = 0, \qquad \widehat{L}_1 < y < \widehat{L}_2.$$

Here

$\mu =$ air viscosity,
$\lambda =$ mean free path length at ambient pressure,
$p_a =$ atmospheric pressure,
$V =$ tape velocity.

The tape deflection $\widehat{u}$ satisfies the elasticity equation

$$\rho\frac{\partial^2\widehat{u}}{\partial t^2} + 2\rho V\frac{\partial^2\widehat{u}}{\partial y\partial t} - (T - \rho V^2)\frac{\partial^2\widehat{u}}{\partial y^2} + EI\frac{\partial^4\widehat{u}}{\partial y^4} = (\widehat{p} - p_a)\chi_{[\widehat{L}_1,\widehat{L}_2]}(y), \quad 0 < y < \widehat{L},$$

and

$$\widehat{u}(y,t) = \widehat{\delta}(y) + \widehat{h}(y,t).$$

Here

$\rho =$ mass density of the tape,
$T =$ tape tension,
$E =$ Young's modulus of the tape,
$I =$ second moment of inertia of the tape section.

Typical magnitudes of the above quantities are:

$$\begin{aligned}&\mu = 2\times 10^{-5}Kg/m\cdot sec, \quad \lambda = 6\times 10^{-8}m, \quad p_a = 8\times 10^4 N/m^2,\\ &V = 2.5m/sec, \quad \rho = 4\times 10^{-2}Kg/m^2, \quad T = 3\times 10^2 N/m,\\ &EI = 2\times 10^{-5}N\cdot m, \quad \widehat{L} = 10^{-1}m, \quad \widehat{L}_2 - \widehat{L}_1 = 10^{-2}m,\end{aligned}$$

where N =Newton. The spacing $\widehat{h}$ is of order of magnitude of $10^{-6}m$. For more details on the model see [1] and [4; Chap. 3].

We shall consider the steady state and nondemensionalize the system by introducing

$$\begin{aligned}&x = \frac{y}{10^{-2}}, \quad p(x) = \frac{\widehat{p}(y)}{p_a}, \quad h(x) = \frac{\widehat{h}(y)}{10^{-6}},\\ &u(x) = \frac{\widehat{u}(y)}{10^{-6}}, \quad \delta(x) = \frac{\widehat{\delta}(y)}{10^{-6}}.\end{aligned}$$

We then obtain the following system:

$$\frac{\partial(ph)}{\partial x} - \varepsilon\frac{\partial}{\partial x}\Big(\alpha h^2\frac{\partial p}{\partial x} + \beta h^3 p\frac{\partial p}{\partial x}\Big) = 0, \quad L_1 < x < L_2, \tag{1.1}$$

$$-\frac{\partial^2 u}{\partial x^2} + \eta\frac{\partial^4 u}{\partial x^4} = K(p-1)\chi_{[L_1,L_2]}, \quad 0 < x < L, \tag{1.2}$$

$$u(x) = h(x) + \delta(x), \quad h(x) > 0 \quad \text{if } L_1 \le x \le L_2, \tag{1.3}$$

where

$$\varepsilon = O(10^{-2}), \quad \beta = O(1), \quad \alpha = O(10^{-1}), \quad \eta = O(10^{-3}),$$
$$L_2 - L_1 = O(1), \quad L = O(10), \quad K = O(10^4).$$

The system is supplemented by the boundary conditions:

$$p = 1 \quad \text{at } x = L_1 \text{ and } x = L_2, \tag{1.4}$$

$$u = \frac{\partial u}{\partial x} = 0 \quad \text{at } x = 0 \text{ and } x = L. \tag{1.5}$$

We assume throughout this paper that $\delta(x)$ is in $C^2[L_1, L_2]$ and

$$\begin{gathered}\delta(x) > 0, \quad \delta''(x) < 0 \text{ if } L_1 \le x \le L_2, \\ \frac{\delta(L_1)}{L_1} < \delta'(L_1).\end{gathered} \tag{1.6}$$

THEOREM 1.1. *There exist positive constants ε^*, η^* such that if $0 < \varepsilon < \varepsilon^*$, $0 < \eta < \eta^*$, then the system (1.1)–(1.6) has a classical solution (p, h, u) with p, h in $C^\infty[L_1, L_2]$, and $p > 0$, $h > 0$ in $[L_1, L_2]$.*

This result will be proved in Sections 2 – 5. In Section 6 we shall study the boundary layer behavior of the solution as $\varepsilon \to 0$, $\eta \to 0$. Finally, in Section 7 we discuss the two-dimensional case.

2. The special case $\eta = 0$, $\varepsilon = 0$. In the special case $\eta = 0$, $\varepsilon = 0$ the system (1.1)–(1.3) reduces to

$$\frac{\partial(ph)}{\partial x} = 0, \quad h = u - \delta > 0 \text{ if } L_1 \le x \le L_2, \tag{2.1}$$

$$-\frac{\partial^2 u}{\partial x^2} = K(p-1)\chi_{[L_1,L_2]} \text{ if } 0 < x < L. \tag{2.2}$$

Since we are dealing with a singular perturbation, only some of the boundary conditions in (1.4), (1.5) are preserved. As will be shown later on, the correct boundary conditions for (2.1), (2.2) are:

$$p = 1 \text{ at } x = L_1, \tag{2.3}$$

$$u = 0 \text{ at } x = 0 \text{ and } x = L. \tag{2.4}$$

From (2.1), we deduce that $p \cdot h$ =constant= C; since $p(L_1) = 1$,

$$C = h(L_1) = u(L_1) - \delta(L_1),$$

and (2.2) becomes

$$-u_{xx} = K\Big(\frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1\Big)\chi_{[L_1,L_2]}. \tag{2.5}$$

To solve (2.5), (2.4) we use the shooting method: We solve (2.5) subject to the initial conditions

$$u(0) = 0,\ u_x(0) = k \quad (> 0) \tag{2.6}$$

and vary the parameter k. Any solution of (2.5), (2.6) is continued as long as it remains larger than $\delta(x)$. We shall denote this solution (extended to its maximal existence interval) by $u(x, k)$. If $k \le \delta(L_1)/L_1$, then the solution does not exist for $x > L_1$; hence we shall always take $k > \delta(L_1)/L_1$.

Lemma 2.1. *If $k \ge \delta'(L_1)$, then $u_x(x, k) \ge 0$ for all $x \ge 0$.*

Proof. It suffices to prove the lemma for $k > \delta'(L_1)$. Introduce the linear function $v(x) = kx$. Then $u - v = (u - v)_x = 0$ at $x = L_1$. Also $u_{xx}(L_1 + 0, k) = 0$ and

$$u_{xxx}(L_1 + 0, k) = \frac{K}{u(L_1, k) - \delta(L_1)}(k - \delta'(L_1)) > 0$$

so that $u_{xx}(x, k) > 0$ in some interval $L_1 < x < L_1 + \varepsilon_1$, and consequently $(u - v)_x > 0$, $(u - v) > 0$ if $L_1 < x < L_1 + \varepsilon_1$. Let x_0 denote the largest number $\le L_2$ such that $(u - v)_x > 0$, $(u - v) > 0$ if $L_1 < x < x_0$. If $x_0 = L_2$ then the lemma follows. Suppose then that $x_0 < L_2$. Then $u_x(x_0, k) = v_x(x_0)$ and

$$u_{xx}(x_0, k) \le v_{xx}(x_0) = 0.$$

However, since $u(x_0, k) > v(x_0)$ and $u(L_1, k) = v(L_1)$,

$$u_{xx}(x_0, k) = -K\Big(\frac{u(L_1, k) - \delta(L_1)}{u(x_0, k) - \delta(x_0)} - 1\Big) > -K\Big(\frac{v(L_1) - \delta(L_1)}{v(x_0) - \delta(x_0)} - 1\Big)$$

whereas, by (1.6)

$$v(x_0) - \delta(x_0) > v(L_1) - \delta(L_1) > 0.$$

It follows that $u_{xx}(x_0, k) > 0$, a contradiction. □

Lemma 2.2. *Let $\dfrac{\delta(L_1)}{L_1} < k < \delta'(L_1)$. Then*

$$u_{xx}(x, k) < 0 \quad \text{if } 0 < x - L_1 < \varepsilon_1 \tag{2.7}$$

for some small $\varepsilon_1 > 0$*, and*

(2.8) $u_{xx}(x,k)$ *will change sign at most once in the interval* (L_1, L_2).

Proof. Since $u_{xx}(L_1,k) = 0$ and

$$u_{xxx}(L_1+0,k) = \frac{K}{u(L_1,k)-\delta(L_1)}(k-\delta'(L_1)) < 0,$$

(2.7) follows. To prove (2.8) suppose that $u_{xx}(x,k)$ changes sign and let $\overline{x}$ denote the first zero of $u_{xx}(x,k)$ in the interval (L_1, L_2), i.e.,

$$\begin{aligned} &u_{xx}(x,k) < 0 \quad \text{if } L_1 < x < \overline{x} < L_2, \\ &u_{xx}(\overline{x},k) = 0. \end{aligned}$$

From the differential equation for u it follows that

$$\begin{aligned} &u(x,k) - \delta(x) < u(L_1,k) - \delta(L_1) \quad \text{if } L_1 < x < \overline{x}, \\ &u(\overline{x},k) - \delta(\overline{x}) = u(L_1,k) - \delta(L_1). \end{aligned}$$

Hence

$$u_x(\overline{x},k) \geq \delta'(\overline{x}).$$

Since also

$$u_{xx}(\overline{x},k) = 0 > \delta''(\overline{x}),$$

it follows that

$$u_x(x,k) > \delta'(x) \quad \text{if } 0 < x - \overline{x} < \sigma$$

for some $\sigma > 0$, and therefore

$$u(x,k) - \delta(x) > u(\overline{x},k) - \delta(\overline{x}) = u(L_1,k) - \delta(L_1) \quad \text{(by (2.5) and } u_{xx}(\overline{x},k) = 0),$$

so that, by the differential equation for u,

$$u_{xx}(x,k) > 0 \quad \text{if } 0 < x - \overline{x} < \sigma.$$

Since $\delta'' < 0$, we can extend the solution beyond $x = \overline{x} + \sigma$; furthermore, as long as u_{xx} remains positive, we have $u_x(x,k) - \delta'(x) > 0$ and

$$u(x,k) - \delta(x) > u(L_1,k) - \delta(L_1),$$

and so, by the differential equation, u_{xx} will remain uniformly positive. $\square$

LEMMA 2.3. *Suppose*

$$\frac{\delta(L_1)}{L_1} < k_1 < k_2 < \delta'(L_1)$$

and

$$u_{xx}(x, k_1) \le 0 \quad \text{for } 0 \le x < \overline{x} \qquad \text{and some } L_1 < \overline{x} \le L_2. \tag{2.9}$$

Then

$$u(x, k_2) - u(L_1, k_2) > u(x, k_1) - u(L_1, k_1) \quad \text{for } L_1 < x < \overline{x}, \tag{2.10}$$

and

$$u_x(x, k_2) > u_x(x, k_1) \quad \text{for } 0 < x < \overline{x}. \tag{2.11}$$

Proof. Since (2.10) is a consequence of (2.11) whereas (2.11) is already valid for $0 \le x \le L_1$, it remains to prove that (2.11) holds for all x in $(L_1, \overline{x})$. If this is not the case then there exists a point x^* in $(L_1, \overline{x})$ such that

$$\begin{aligned} u(x, k_2) - u(L_1, k_2) > u(x, k_1) - u(L_1, k_1), \quad & L_1 < x < x^*, \\ u_x(x, k_2) > u_x(x, k_1), \quad & 0 \le x < x^* \end{aligned} \tag{2.12}$$

and

$$u_x(x^*, k_2) = u_x(x^*, k_1).$$

It follows that $u_{xx}(x^*, k_2) \le u_{xx}(x^*, k_1) \le 0$, or, from the differential equation,

$$\frac{u(L_1, k_2) - \delta(L_1)}{u(x^*, k_2) - \delta(x^*)} \ge \frac{u(L_1, k_1) - \delta(L_1)}{u(x^*, k_1) - \delta(x^*)}.$$

Combining this with the first inequality in (2.12) for $x = x^*$, we find that

$$\frac{u(x^*, k_1) - \delta(x^*)}{u(L_1, k_1) - \delta(L_1)} > 1.$$

This implies, by (2.5), that $u_{xx}(x^*, k_1) > 0$, which is a contradiction. $\square$

LEMMA 2.4. *Let* $\dfrac{\delta(L_1)}{L_1} < k < \delta'(L_1)$. *If*

$$u_x(x, k) \ge -C \quad \text{for } L_1 < x < \overline{x} \qquad \text{and some } \overline{x} \in (L_1, L_2), \tag{2.13}$$

then $u(\overline{x}, k) > \delta(\overline{x})$ *and hence the solution* $u(x, k)$ *can be extended beyond* $x = \overline{x}$.

Proof. Since $-u_{xx} \geq -K$ in (L_1, L_2), $u_{xx} \leq K$ and u_x is bounded from above. Under the assumption (2.13) it then follows that $u(\overline{x}-0,k)$ exists; further, by (2.13),

$$u(x,k) \leq u(\overline{x},k) + C(\overline{x}-x) \quad \text{if } L_1 < x < \overline{x}.$$

This implies that

$$-u_{xx}(x,k) \geq K\Big(\frac{u(L_1,k)-\delta(L_1)}{C(\overline{x}-x)+u(\overline{x},k)-\delta(x)} - 1\Big) \quad \text{for } L_1 < x < \overline{x}.$$

We need to show that $u(\overline{x},k) > \delta(\overline{x})$. Suppose $u(\overline{x},k) = \delta(\overline{x})$. Then, since C may be assumed to be larger than $\sup|\delta'|$,

$$\begin{aligned} u_x(\overline{x}-0,k) - u_x(x,k) &= \int_x^{\overline{x}-0} u_{xx}(\xi,k)d\xi \\ &\leq -K\int_x^{\overline{x}-0}\Big(\frac{u(L_1,k)-\delta(L_1)}{C(\overline{x}-\xi)+\delta(\overline{x})-\delta(\xi)} - 1\Big)d\xi = -\infty \end{aligned}$$

for $L_1 < x < \overline{x}$, which is a contradiction to (2.13). □

We define

$$\overline{k} \equiv \inf\Big\{k > 0;\ \inf_{L_1 \leq x \leq L_2}[u(x,\tilde{k}) - \delta(x)] > 0 \quad \forall\ k \leq \tilde{k} < \delta'(L_1)\Big\}.$$

Then

$$\frac{\delta(L_1)}{L_1} < \overline{k} < \delta'(L_1),$$

where the second inequality follows from Lemma 2.1.

LEMMA 2.5. *There holds:*

$$u(x,\overline{k}) > \delta(x) \quad \text{if } L_1 \leq x < L_2, \tag{2.14}$$

$$u(L_2,\overline{k}) = \delta(L_2), \tag{2.15}$$

i.e., the maximal existence interval for $u(x,\overline{k})$ *is* $0 \leq x < L_2$.

Remark 2.1. Since u_{xx} has a fixed sign near $x = L_2$, Lemma 2.4 and (2.15) imply that

$$u_x(L_2-0,\overline{k}) = -\infty. \tag{2.16}$$

Proof. Suppose (2.14) is not true. Then there exists an $x^* \in (L_1, L_2)$ such that

$$\begin{aligned} &u(x,\overline{k}) > \delta(x) \quad \text{if } L_1 \leq x < x^*, \\ &u(x^*,\overline{k}) = \delta(x^*). \end{aligned}$$

As in Remark 2.1

$$u_x(x^*-0,\overline{k}) = -\infty$$

and so for any large positive number N there is an $x^{**} < x^*$ ($x^* - x^{**}$ small) such that $u_x(x^{**}, \overline{k}) < -3N$; then also

$$u_x(x^{**}, \overline{k} + \sigma) < -2N$$

if σ is positive and small enough. Since $u_{xx} \le K$, we deduce that

$$u_x(x, \overline{k} + \sigma) < -N$$

for all $x^{**} < x < L_2$. Taking N large enough we find that the curve $y = u(x, \overline{k} + \sigma)$ intersects the curve $y = \delta(x)$ at some point between x^{**} and L_2 (in fact, near x^*), and this contradicts the definition of $\overline{k}$.

Next, if (2.15) is not true, i.e., if $u(L_2, \overline{k}) > \delta(L_2)$ then by continuity we have that $u(x, \overline{k} - \sigma)$ exists for all $0 < x \le L_2$, for any σ positive and small enough, and

$$\inf_{L_1 \le x \le L_2} [u(x, \overline{k} - \sigma) - \delta(x)] > 0;$$

this however contradicts the definition of $\overline{k}$. ☐

From (2.16), we have, for any $N \gg 1$,

$$u_x(L_2 - \overline{\delta}, \overline{k}) < -3N$$

if $\overline{\delta}$ is small enough and, by the continuous dependence of the solution on k,

$$u_x(L_2 - \overline{\delta}, \overline{k} + \sigma) < -2N$$

if $\sigma > 0$ and small. Since $u_{xx} \le K$ it follows that $u_x(x, \overline{k} + \sigma) < -N$ if $x > L_2 - \overline{\delta}$, and therefore

$$u(L, \overline{k} + \sigma) < 0. \tag{2.17}$$

The inequality

$$u(L_2, \overline{k} + \sigma) - \delta(L_2) < (\overline{k} + \sigma)L_1 - \delta(L_1)$$

is true for $\sigma = 0$ and by continuity also for $\sigma > 0$ and sufficiently small. It implies (by the differential equation for u) that

$$u_{xx}(L_2, \overline{k} + \sigma) < 0 \tag{2.18}$$

and therefore, by Lemma 2.2,

$$u_{xx}(x, \overline{k} + \sigma) \le 0 \quad \text{if } 0 \le x < L_2. \tag{2.19}$$

We now define

$$\begin{aligned}k^* &= \sup\{k;\ \ k \in (\bar{k}, \delta'(L_1)) \text{ and } u(L,k) \le 0\},\\ k_* &= \inf\{k;\ \ k \in (\bar{k}, \delta'(L_1)) \text{ and } u(L,k) \ge 0\}.\end{aligned}$$

Lemma 2.1 and (2.17) show that k^* and k_* are well defined, and then, clearly, $u(x,k^*)$, $u(x,k_*)$ are solutions of (2.5), (2.4).

LEMMA 2.6. *For any solution $u(x)$ of (2.5), (2.4),*

$$u(x,k_*) \le u(x) \le u(x,k^*), \quad 0 < x < L. \tag{2.20}$$

Proof. We shall show that any two non-identical solutions $u_1(x)$ and $u_2(x)$ cannot intersect; this clearly implies the assertion (2.20), since $k_* \le k \le k^*$ if $u(x) = u(x,k)$.

If $u_1(L_1) = u_2(L_1)$ then obviously $u_1 \equiv u_2$. Hence we may suppose that $u_1(L_1) \ne u_2(L_1)$. Consider for definiteness the case $u_1(L_1) > u_2(L_1)$. Then

$$\begin{aligned}-(u_1)_{xx} &= K\Big(\frac{u_1(L_1)-\delta(L_1)}{u_1(x)-\delta(x)} - 1\Big)\chi_{[L_1,L_2]}\\ &\ge K\Big(\frac{u_2(L_1)-\delta(L_1)}{u_1(x)-\delta(x)} - 1\Big)\chi_{[L_1,L_2]},\end{aligned}$$

or

$$-(u_1)_{xx} + F(u_1,x) \ge 0$$

where

$$F(s,x) = -K\Big(\frac{u_2(L_1)-\delta(L_1)}{s-\delta(x)} - 1\Big)\chi_{[L_1,L_2]}.$$

Since $-(u_2)_{xx} + F(u_2,x) = 0$ and

$$\frac{\partial F(s,x)}{\partial s} > 0,$$

the maximum principle shows that $u_1(x) > u_2(x)$ for $0 < x < L$. □

We summarize the main results of this section:

THEOREM 2.7. *(i) There exists at least one classical solution to the system (2.1)–(2.4); (ii) No two distinct solutions of (2.5), (2.4) can intersect, and every solution $u(x)$ satisfies (2.20); (iii) For every solution $u(x)$, $u_{xx}(L_1+\xi) < 0$ for ξ positive and small, and u_{xx} can change sign at most once in the interval $L_1 < x < L_2$.*

Remark 2.2. If

$$u_{xx}(x,k_*) \le 0 \quad \text{for } L_1 < x < L_2, \tag{2.21}$$

then the solution $u(x)$ is unique. Indeed, if $k > k_*$, then $u(L_1, k) > u(L_1, k_*)$ and, by (2.10) of Lemma 2.3, $u(L_2, k) > u(L_2, k_*)$. Since, by (2.11) of Lemma 2.3, also $u_x(L_2, k) \geq u_x(L_2, k_*)$, it follows that $u(L, k) > 0$.

Remark 2.3. If

$$\delta'(L_2) \geq -\frac{\delta(L_2)}{L - L_2} \tag{2.22}$$

then (2.21) is valid, and so uniqueness holds. Indeed, if (2.21) is not valid, then

$$\begin{aligned} u_{xx}(x, k_*) &\leq 0 \quad \text{if } L_1 < x < \widehat{x}, \\ u_{xx}(x, k_*) &\geq 0 \quad \text{if } \widehat{x} < x < L_2 \end{aligned}$$

for some $\widehat{x} \in (L_1, L_2)$. Using the differential equation we see that the values of $u(x, k_*) - \delta(x)$ at $x < \widehat{x}$ are smaller than the values at $x > \widehat{x}$, so that

$$u_x(\widehat{x}, k_*) \geq \delta'(\widehat{x})$$

and hence

$$u_x(L_2, k_*) \geq u_x(\widehat{x}, k_*) \geq \delta'(\widehat{x}) > \delta'(L_2) \geq -\frac{\delta(L_2)}{L - L_2}.$$

But this implies that $u(L, k_*) > 0$, which is a contradiction.

Remark 2.4. For any solution of (2.5), (2.4), we have the bound

$$u(L_1) - \delta(L_1) < L_1\delta'(L_1) - \delta(L_1). \tag{2.23}$$

Indeed, if (2.23) is not true, then writing $u(x) = u(x, k)$ we have $k = u(L_1)/L_1 \geq \delta'(L_1)$, which contradicts Lemma 2.1.

We end this section by proving a uniqueness theorem for symmetric heads.

THEOREM 2.8. *Suppose, in addition to (1.6), that*

$$\delta(x) = \delta(2L_m - x) \quad \textit{for } L_m < x \leq L_2 \tag{2.24}$$

for some L_m such that

$$\frac{L_1 + L_2}{2} \leq L_m < L_2, \qquad L - L_m \leq L_m. \tag{2.25}$$

Then the solution to the system (2.1)-(2.4) is unique.

Proof. According to Remark 2.2, it suffices to prove that $u_{xx}(x, k_*) \leq 0$ for $L_1 < x < L_2$. We first claim that

$$u_{xx}(x, k_*) \leq 0 \quad \text{for } L_1 < x \leq L_m. \tag{2.26}$$

In fact, if this is not true, then by Lemma 2.2, there exists a point $\widehat{x} \in (L_1, L_m]$ such that

$$u_{xx}(x, k_*) < 0 \quad \text{for } L_1 < x < \widehat{x}, \tag{2.27}$$

$$u_{xx}(x, k_*) > 0 \quad \text{for } \widehat{x} < x < L_2. \tag{2.28}$$

Notice that the assumption (2.24) implies that $\delta'(L_m) = 0$. Using the differential equation, we obtain

$$u_x(\widehat{x}, k_*) \geq \delta'(\widehat{x}) \geq \delta'(L_m) = 0.$$

By (2.28) we then have $u_x(x, k_*) > 0$ for all $x > \widehat{x}$, which is a contradiction.

Note that under the assumption (2.25), $2L_m - L \geq 0$ and $2L_m - L_2 \geq L_1$. We next claim that

$$u(x, k_*) \geq u(2L_m - x, k_*) \quad \text{for } 2L_m - L < x < L_m. \tag{2.29}$$

To prove this, we let

$$v(x) = u(x, k_*) - u(2L_m - x, k_*) \quad \text{for } 2L_m - L < x < L_m.$$

Using (2.24) and (2.26), we find that $v(x)$ satisfies

$$\begin{aligned} &-v_{xx} + c(x)v(x) \geq 0 \quad \text{for } 2L_m - L < x < L_m, \\ &v(2L_m - L) \geq 0, \qquad v(L_m) = 0, \end{aligned}$$

where

$$c(x) = \frac{K[u(L_1, k_*) - \delta(L_1)]}{[u(x, k_*) - \delta(x)][u(2L_m - x, k_*) - \delta(x)]}\chi_{[L_1, L_m]} \geq 0.$$

Therefore the maximum principle implies that $v(x) \geq 0$, which is the assertion (2.29). Using (2.29) in (2.5) we get $u_{xx}(x, k_*) \geq u_{xx}(2L_m - x, k_*)$ for $2L_m - L < x < L_m$, and then, by (2.26), $u_{xx}(2L_m - x, k_*) \leq 0$ for $L_1 < x < L_m$. Consequently, altogether, $u_{xx}(x, k_*) \leq 0$ for $L_1 < x < L_2$. □

3. A variational inequality for $\varepsilon = 0$, $\eta = 0$. We shall need to recast the solution of (2.1)–(2.4) (or rather ((2.5), (2.4)) as a solution of a variational inequality. We first note that from the results of §2 ((2.17)–(2.19)) it follows that if $0 < k - \overline{k} \ll 1$, then

$$u_{xx}(x, k) < 0 \quad \text{for } L_1 < x \leq L_2 - 0$$

and $u(L, k) < 0$. It follows that if $\widehat{\alpha}$ is positive and sufficiently small, then there exists a solution of

$$g_{xx} = -K\Big(\frac{g(L_1) - \delta(L_1)}{g(x) - \delta(x)} - 1\Big)\chi_{[L_1, L_2]}, \quad 0 < x < L, \tag{3.1}$$

$$g(0) = -\widehat{\alpha}, \qquad g(L) < 0 \tag{3.2}$$

and

$$u(x, k_*) - g(x) \geq m > 0 \quad \text{for } 0 < x < L, \tag{3.3}$$

where m is a constant; in fact, $g'(0) = k$ where $0 < k - \overline{k} \ll 1$. We introduce a truncation of the linear function s,

$$f(s) = \begin{cases} s & \text{if } s \leq 1 + L_1\delta'(L_1) - \delta(L_1) \\ 1 + L_1\delta'(L_1) - \delta(L_1) & \text{if } s > 1 + L_1\delta'(L_1) - \delta(L_1), \end{cases}$$

and the variational inequality

$$\begin{aligned} & -u_{xx} \geq K\Big(\frac{f(u(L_1) - \delta(L_1))}{u(x) - \delta(x)} - 1\Big)\chi_{[L_1,L_2]}, \\ & u(x) \geq g(x), \\ & (u-g)\Big\{ - u_{xx} - K\Big(\frac{f(u(L_1) - \delta(L_1))}{u(x) - \delta(x)} - 1\Big)\chi_{[L_1,L_2]}\Big\} = 0 \end{aligned} \tag{3.4}$$

for $0 < x < L$, with the boundary conditions

$$u(0) = u(L) = 0. \tag{3.5}$$

Lemma 3.1. *Any solution $u(x)$ of (3.4), (3.5) satisfies*

$$\begin{aligned} u(x) - g(x) \geq m > 0 \quad & \text{if } L_1 \leq x \leq L_2, \\ u(L_1) - \delta(L_1) \leq L_1\delta'(L_1) - \delta(L_1); & \end{aligned} \tag{3.6}$$

and, consequently, it is also a solution of (2.5), (2.4).

Proof. Since $u(L_1) \geq g(L_1)$ and $f(s)$ is monotone in s, we have $f(u(L_1) - \delta(L_1)) \geq f(g(L_1) - \delta(L_1)) = g(L_1) - \delta(L_1)$; the last equation follows from (3.3) and Remark 2.4. It follows that

$$-u_{xx} \geq K\Big(\frac{g(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1\Big)\chi_{[L_1,L_2]}.$$

Comparing with (3.1) and using the strong maximum principle, we conclude that $u(x) > g(x)$ for $0 \leq x \leq L$ and therefore u satisfies the equation

$$-u_{xx} = K\Big(\frac{f(u(L_1) - \delta(L_1))}{u(x) - \delta(x)} - 1\Big)\chi_{[L_1,L_2]}.$$

Using the proof for Lemma 2.1, we conclude that $u(L_1) \leq L_1\delta'(L_1)$ and so the second inequality of (3.6) is satisfied. Hence u is a solution of (2.5), (2.4) and then of course

$$u(x) \geq u(x, k_*) \geq g(x) + m \qquad \text{by (3.3).} \qquad \square$$

In Remark 4.1 we shall explain why the truncation was needed.

4. A variational system for small ε and η.

To avoid the difficulty of possible degeneracy when h is near zero, we first impose in (1.1), (1.2) the restriction that $u(x) \geq g(x)$ (recall that $g(x) > \delta(x)$ for $L_1 \leq x \leq L_2$). The boundary condition for p will also be modified for technical reasons. Thus we consider the system

$$\frac{\partial(ph)}{\partial x} - \varepsilon \frac{\partial}{\partial x}\Big(\alpha h^2 \frac{\partial p}{\partial x} + \beta h^3 p \frac{\partial p}{\partial x}\Big) = 0, \quad L_1 < x < L_2, \tag{4.1}$$

$$\begin{aligned} &-\frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^4 u}{\partial x^4} \geq K(p-1)\chi_{[L_1,L_2]}, \quad 0 < x < L, \\ &u(x) \geq g(x), \quad 0 < x < L, \\ &[u(x) - g(x)]\Big\{ -\frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^4 u}{\partial x^4} - K(p-1)\chi_{[L_1,L_2]}\Big\} = 0, \quad 0 < x < L, \end{aligned} \tag{4.2}$$

and

$$h(x) = u(x) - \delta(x), \quad L_1 \leq x \leq L_2 \tag{4.3}$$

with the boundary conditions

$$p(L_1) = p(L_2) = \frac{f(h(L_1))}{h(L_1)}, \tag{4.4}$$

$$u = u_x = 0 \quad \text{at } x = 0, \quad x = L. \tag{4.5}$$

Let

$$G = \Big\{ p \in C[L_1, L_2]; \ 0 \leq p \leq 1 + \frac{1 + L_1\delta'(L_1) - \delta(L_1)}{l} \Big\} \tag{4.6}$$

where

$$l = \min_{L_1 \leq x \leq L_2} [g(x) - \delta(x)] > 0.$$

For each $p \in G$ we solve the variational inequality (4.2) with boundary conditions (4.5). By general theory, u is uniquely determined as the minimizer of

$$\int_0^L \{u_x^2 + \eta u_{xx}^2 - K(p-1)\chi_{[L_1,L_2]}\} dx$$

subject to the boundary condition (4.5), and consequently

$$\int_0^L u_x^2 dx \leq C, \quad C \text{ independent of } p \text{ and } \eta. \tag{4.7}$$

We shall need the following stronger estimate:

LEMMA 4.1. *The following estimate holds:*

$$|u_x(x)| \leq C, \quad 0 \leq x \leq L \tag{4.8}$$

where C is a constant independent of p and η.

Proof. The proof relies on the fact that $g \in W^{2,\infty}(0,L)$. We also note that $u \in H^2(0,L)$ so that $u \in C^1[0,L]$. At any point x_0 where $u(x_0) = g(x_0)$, we have

$$|u_x(x_0)| = |g_x(x_0)| \leq C.$$

Thus it remains to estimate u_x in the open set $\{u > g\}$, which consists of intervals $\{x_1 < x < x_2\}$. If at least one end point is 0 or L, then $x_2 - x_1 \geq c > 0$, c independent of η, and so

$$\frac{x_2 - x_1}{\sqrt{\eta}} \geq 3 \tag{4.9}$$

if η is small enough.

We shall first consider the case where $0 < x_1 < x_2 < L$. Then

$$u(x_i) = g(x_i), \quad u_x(x_i) = g_x(x_i) \quad \text{for } i = 1, 2, \tag{4.10}$$

and

$$-\frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^4 u}{\partial x^4} = q(x) \quad \text{if } x_1 < x < x_2, \tag{4.11}$$

where $q(x) = K(p-1)\chi_{[L_1,L_2]}$. Multiplying (4.11) by $u - g$ and integrating over (x_1, x_2), we obtain

$$\begin{aligned}
&\int_{x_1}^{x_2} u_x^2 dx + \eta \int_{x_1}^{x_2} u_{xx}^2 dx \\
= &\int_{x_1}^{x_2} u_x g_x dx + \eta \int_{x_1}^{x_2} u_{xx} g_{xx} dx + \int_{x_1}^{x_2} q(u-g) dx \\
\leq &\frac{1}{2}\Big(\int_{x_1}^{x_2} u_x^2 dx + \int_{x_1}^{x_2} g_x^2 dx + \eta \int_{x_1}^{x_2} u_{xx}^2 dx + \eta \int_{x_1}^{x_2} g_{xx}^2 dx\Big) \\
&+ C(x_2 - x_1) \int_{x_1}^{x_2} |u_x - g_x| dx,
\end{aligned}$$

so that

$$\eta \int_{x_1}^{x_2} u_{xx}^2 dx \leq C(x_2 - x_1).$$

Hence

$$\begin{aligned}
|u_x(x)| &\leq |g_{xx}(x_1)| + \int_{x_1}^{x_2} |u_{xx} - g_{xx}| dx \\
&\leq C + \int_{x_1}^{x_2} |u_{xx}| dx \\
&\leq C + (x_2 - x_1)^{1/2} \Big(\int_{x_1}^{x_2} u_{xx}^2 dx\Big)^{1/2} \\
&\leq C + \frac{C(x_2 - x_1)}{\sqrt{\eta}}
\end{aligned}$$

and (4.8) follows provided $(x_2 - x_1)/\sqrt{\eta} < 3$.

It remains to consider the case (4.9). Let

$$s(x) = v(x) - \int_{x_1}^{x} \int_{x_1}^{\tau} q(\xi) d\xi d\tau \tag{4.12}$$

where $v(x)$ satisfies

$$\begin{aligned} -v + \eta v_{xx} = \eta q, \quad x_1 < x < x_2, \\ v(x_1) = v(x_2) = 0. \end{aligned} \tag{4.13}$$

Thus $-s_{xx} + \eta s_{xxxx} = q(x)$ so that the solution u has the form

$$\begin{aligned} u(x) = C_1 + C_2 x + C_3 \exp\Big(-\frac{x - x_1}{\sqrt{\eta}}\Big) + C_4 \exp\Big(-\frac{x_2 - x}{\sqrt{\eta}}\Big) + s(x), \\ x_1 < x < x_2. \end{aligned} \tag{4.14}$$

By the maximum principle

$$\|v\|_{L^\infty(x_1,x_2)} \le \eta \|q\|_{L^\infty(x_1,x_2)}$$

and, therefore, from the differential equation in (4.13),

$$\|v_{xx}\|_{L^\infty(x_1,x_2)} \le C \|q\|_{L^\infty(x_1,x_2)}.$$

Since $v(x_1) = v(x_2) = 0$, there is a point $\xi \in (x_1, x_2)$ such that $v_x(\xi) = 0$, and consequently

$$\|v_x\|_{L^\infty(x_1,x_2)} \le C(x_2 - x_1) \|q\|_{L^\infty(x_1,x_2)}. \tag{4.15}$$

Using the last two estimates on v, we deduce from (4.12) that

$$\begin{aligned} &|s(x_i)| + |s_x(x_i)| \le C, \\ &|s(x_2) - s(x_1)| + |s_x(x_2) - s_x(x_1)| \le C(x_2 - x_1). \end{aligned}$$

From (4.10) and the fact that $g \in W^{2,\infty}$ it follows that u satisfies the same estimates, and thus the function $w(x) = u(x) - s(x)$ also satisfies

$$\begin{aligned} &|w(x_i)| + |w_x(x_i)| \le C, \\ &|w(x_2) - w(x_1)| + |w_x(x_2) - w_x(x_1)| \le C(x_2 - x_1). \end{aligned} \tag{4.16}$$

We need to choose the constants C_i in (4.14) such that

$$\begin{aligned} C_1 + C_2 x_1 + C_3 + C_4 \exp\Big(-\frac{x_2 - x_1}{\sqrt{\eta}}\Big) &= w(x_1), \\ C_2 + \frac{C_3}{\sqrt{\eta}} + \frac{C_4}{\sqrt{\eta}} \exp\Big(-\frac{x_2 - x_1}{\sqrt{\eta}}\Big) &= w_x(x_1), \\ C_1 + C_2 x_2 + C_3 \exp\Big(-\frac{x_2 - x_1}{\sqrt{\eta}}\Big) + C_4 &= w(x_2), \\ C_2 - \frac{C_3}{\sqrt{\eta}} \exp\Big(-\frac{x_2 - x_1}{\sqrt{\eta}}\Big) + \frac{C_4}{\sqrt{\eta}} &= w_x(x_2). \end{aligned}$$

Eliminating C_1 and C_2, we obtain

$$\begin{aligned} -\frac{C_3+C_4}{\sqrt{\eta}} &= \frac{w_x(x_1)-w_x(x_2)}{1-\gamma}, \\ \frac{C_3-C_4}{\sqrt{\eta}} &= \frac{1}{\frac{1}{2}(1+\gamma)-\frac{1}{\beta}(1-\gamma)}\Big\{\frac{w(x_1)-w(x_2)}{x_1-x_2}-\frac{1}{2}[w_x(x_1)+w_x(x_2)]\Big\}, \end{aligned}$$

where $\gamma = e^{-\beta}$, $\beta = (x_2-x_1)/\sqrt{\eta} \geq 3$. Since the right-hand sides are bounded independently of η (by (4.16)), $C_3/\sqrt{\eta}$ and $C_4/\sqrt{\eta}$ are also bounded independently of η. We can now easily check that also C_1 and C_2 are bounded independently of η, and then the bound (4.8) for $x_1 < x < x_2$ follows by (4.14) upon recalling (4.12) and the estimate (4.15).

The above proof is valid also if $x_1 = 0$ or $x_2 = L$, and the proof of the lemma is thus complete. ☐

Having solved the variational inequality (4.2), (4.5) for u, given $p \in G$, we set $h = u - \delta$, and recall that

$$h(x) \geq g(x) - \delta(x) \geq l > 0, \quad L_1 \leq x \leq L_2. \tag{4.17}$$

Lemma 4.2. *Given h as above, there exists a unique positive solution $\widehat{p}$ of (4.1), (4.4).*

Proof. By integration, $\widehat{p}$ must satisfy

$$\begin{aligned} &\widehat{p}h - \varepsilon(\alpha h^2 + \beta h^3 \widehat{p})\widehat{p}_x = \lambda, \quad L_1 \leq x \leq L_2, \\ &\widehat{p}(L_1) = \frac{f(h(L_1))}{h(L_1)} \quad (\leq 1), \end{aligned} \tag{4.18}$$

where λ is a constant that clearly determines $\widehat{p}$ uniquely. Thus it remains to show that λ can be uniquely determined such that

$$\widehat{p}(L_2) = \frac{f(h(L_1))}{h(L_1)} \tag{4.19}$$

If $\lambda \leq 0$, then $\widehat{p}_x > 0$ and so $\widehat{p}(L_2) > \widehat{p}(L_1)$ and (4.19) cannot be satisfied. If λ is positive and large then $\widehat{p}_x$ is negative and large and so $\widehat{p}$ cannot remain positive throughout the interval $[L_1, L_2]$. Denoting the solution of (4.18) by $\widehat{p}(x,\lambda)$, we let

$$\lambda_0 = \inf\{\lambda;\ \ \lambda > 0,\ \ \widehat{p}(x,\lambda) \geq 0 \quad \text{for } L_1 \leq x \leq L_2\}.$$

Then $\widehat{p}(x,\lambda_0) \geq 0$ and $\widehat{p}(x,\lambda_0)$ must vanish somewhere in the interval $\{L_1 < x \leq L_2\}$. It cannot vanish at an interior point x_0, for otherwise also $\widehat{p}_x(x,\lambda_0) = 0$ so that (by the differential equation in (4.18)) $\lambda_0 = 0$, a contradiction. Thus

$$\widehat{p}(x,\lambda_0) > 0 \quad \text{if } L_1 \leq x < L_2, \quad \widehat{p}(L_2,\lambda_0) = 0.$$

We claim:

$$\text{(4.20)} \qquad \text{if } 0 < \lambda_1 < \lambda_2 \le \lambda_0 \quad \text{then } \widehat{p}(x,\lambda_1) > \widehat{p}(x,\lambda_2) \quad \text{for } L_1 < x \le L_2.$$

Indeed, since $\widehat{p}_x(L_1,\lambda_1) > \widehat{p}_x(L_1,\lambda_2)$, the asserted inequality is valid for all x near $x = L_1$. If the assertion is not true then there is a smallest number x_0 such that

$$\widehat{p}(x,\lambda_1) > \widehat{p}(x,\lambda_2) \quad \text{if } L_1 \le x < x_0, \qquad \text{and } \widehat{p}(x_0,\lambda_1) = \widehat{p}(x_0,\lambda_2).$$

Hence $\widehat{p}_x(x_0,\lambda_1) \le \widehat{p}_x(x_0,\lambda_2)$ and, by (4.18), $\lambda_1 \ge \lambda_2$, a contradiction.

Setting

$$\widehat{\lambda} = \inf\{\lambda;\ \ 0 < \lambda < \lambda_0,\ \ \widehat{p}(L_2,\lambda) < \frac{f(h(L_1))}{h(L_1)}\}$$

it follows from (4.20) that $\widehat{p}(x,\widehat{\lambda})$ is the unique positive solution (4.1), (4.4). □

We define the mapping T on G by

$$\text{(4.21)} \qquad (Tp)(x) = \widehat{p}(x,\widehat{\lambda})$$

and wish to prove that T has a fixed point, which is then the solution of the variational system (4.1)–(4.5).

Let C^* be any positive constant such that

$$\text{(4.22)} \qquad C^* \ge \sup_{L_1 \le x \le L_2} (|h(x)| + |h_x(x)|) \quad \text{and } \frac{l}{C^*+1} < \frac{1}{L_2 - L_1},$$

where l is the positive constant in (4.17); by Lemma 4.1, C^* can be chosen independently of $p \in G$ and η.

Lemma 4.3. *If ε is positive and small enough, then T maps G into itself, and*

$$\text{(4.23)} \qquad |\widehat{\lambda} - f(h(L_1))| \le C\varepsilon,$$

where C is a constant independent of ε and η.

Proof. We shall first establish the bound

$$\text{(4.24)} \qquad \widehat{p}_x(L_1,\widehat{\lambda}) \ge -\frac{C^*+1}{l}.$$

Introduce the differential operator

$$\text{(4.25)} \qquad \mathcal{L}(q) = \frac{\partial}{\partial x}(qh) - \varepsilon\frac{\partial}{\partial x}[(\alpha h^2 + \beta h^3 q)\frac{\partial q}{\partial x}]$$

and the function

$$v(x) = \frac{f(h(L_1))}{h(L_1)} - \frac{C^*+1}{l}(x - L_1) - \sigma, \quad L_1 \le x < \widehat{x}$$

for any small $\sigma > 0$, where $\widehat{x}$ is such that $v(\widehat{x}) = 0$. By (4.22), $L_1 < \widehat{x} < L_2$. By direct computation,

$$\begin{aligned}\mathcal{L}(v) &= -\frac{C^*+1}{l}h(x) + v(x)h_x(x) + \frac{C^*+1}{l}\varepsilon\frac{\partial}{\partial x}(\alpha h^2 + \beta h^3 v) \\ &\leq -1 + \frac{C^*+1}{l}\varepsilon\frac{\partial}{\partial x}(\alpha h^2 + \beta h^3 v) \\ &< 0, \quad L_1 \leq x \leq \widehat{x}\end{aligned}$$

if ε is small enough depending only on C^*, l. We claim that for any $0 \leq \lambda \leq \widehat{\lambda}$,

$$\widehat{p}(x,\lambda) > v(x) \quad \text{if } L_1 < x \leq \widehat{x}. \tag{4.26}$$

Indeed, this is clearly true for $\lambda = 0$. If (4.25) does not hold for some $\lambda^* \in (0, \widehat{\lambda}]$, we take the smallest such λ^* and then

$$\widehat{p}(x^*, \lambda^*) = v(x^*)$$

must hold for some $x^* \in [L_1, \widehat{x}]$, whereas

$$\widehat{p}(x, \lambda^*) \geq v(x) \quad \text{elsewhere .}$$

Since $\widehat{p}(x, \lambda^*) > v(x)$ at both $x = L_1$ and $x = \widehat{x}$, x^* lies in the open interval $(L_1, \widehat{x})$, so that

$$\widehat{p}_x(x^*, \lambda^*) = v_x(x^*), \quad \widehat{p}_{xx}(x^*, \lambda^*) \geq v_{xx}(x^*).$$

Consequently

$$(\mathcal{L}\widehat{p})(x^*, \lambda^*) \leq (\mathcal{L}v)(x^*) < 0,$$

a contradiction.

Taking $\sigma \downarrow 0$ in (4.26) we deduce that

$$\widehat{p}(x, \widehat{\lambda}) \geq \frac{f(h(L_1))}{h(L_1)} - \frac{C^*+1}{l}(x - L_1)$$

and, since $\widehat{p}(L_1, \widehat{\lambda}) = f(h(L_1))/h(L_1)$,

$$\widehat{p}_x(L_1, \widehat{\lambda}) \geq -\frac{C^*+1}{l}. \tag{4.27}$$

From (4.18) we then obtain

$$\widehat{\lambda} - f(h(L_1)) \leq -\varepsilon h^2(L_1)(\alpha + \beta h(L_1))\frac{C^*+1}{l} \leq C\varepsilon. \tag{4.28}$$

We can now estimate $\widehat{p}(x, \widehat{\lambda})$ from above. If the maximum is attained at the boundary point, then

$$\max_{L_1 \le x \le L_2} \widehat{p}(x, \widehat{\lambda}) = \frac{f(h(L_1))}{h(L_1)} \le 1$$

whereas if it occurs at an interior point $\tilde{x}$, then $\widehat{p}_x(\tilde{x}, \widehat{\lambda}) = 0$ and, by (4.18), $\widehat{p}(\tilde{x}, \widehat{\lambda})h(\tilde{x}) = \widehat{\lambda}$. Using the upper bound on $\widehat{\lambda}$ obtained in (4.28), we conclude that

$$\max_{L_1 \le x \le L_2} \widehat{p}(x, \widehat{\lambda}) \le \frac{\widehat{\lambda}}{l} \le \frac{1 + L_1\delta'(L_1) - \delta(L_1)}{l} + \frac{C\varepsilon}{l}.$$

Thus if ε is chosen small enough so that $C\varepsilon < l$, then T maps G into itself.

To complete the proof of the lemma it remains to prove that

$$\widehat{\lambda} - f(h(L_1)) \ge -C\varepsilon. \tag{4.29}$$

We shall compare $\widehat{p}(x, \widehat{\lambda})$ with the function

$$w(x) = \frac{f(h(L_1))}{h(L_1)} + \frac{C^* + 1}{l}(x - L_1) + \sigma, \quad L_1 \le x \le L_2$$

where σ is any small positive constant. We have $\mathcal{L}(w) > 0$ in $L_1 < x < L_2$ and $w > 1$ at $x = L_1$, $x = L_2$. If λ is large, then $w(x) > \widehat{p}(x, \lambda)$ in the largest interval $L_1 \le x < x_\lambda$ where $\widehat{p}(x, \lambda)$ is positive. By decreasing λ and arguing as before we deduce that

$$\widehat{p}_x(L_1, \widehat{\lambda}) \le \frac{C^* + 1}{l}$$

which yields (by (4.18)) the bound (4.29). □

For each $\varepsilon > 0$, the $\widehat{p}(x, \widehat{\lambda})$ form a bounded set in $C^{2+\alpha}[L_1, L_2]$ and thus T maps G (endowed with the $C[L_1, L_2]$ norm) into a compact subset of G. It is also easily seen that T is a continuous map. Invoking the Schauder fixed point theorem we conclude that T has a fixed point. Thus we have proved:

Theorem 4.4. *If ε is sufficiently small and, say, $0 < \eta < 1$, then there exists a classical solution of (4.1)–(4.5), such that*

$$p(x) > 0, \quad h(x) = u(x) - \delta(x) \ge l > 0 \quad \text{if } L_1 \le x \le L_2, \tag{4.30}$$

$$|u_x(x)| \le C \quad \text{if } 0 \le x \le L, \tag{4.31}$$

and

$$|\lambda - f(h(L_1))| \le C\varepsilon \tag{4.32}$$

where λ is defined as in (4.18) with $\widehat{p} = p$ and C is a constant independent of ε and η.

Remark 4.1. From the proof of Lemma 4.3, it is clear that the truncation on the boundary condition for p in (4.4) is necessary. Without the truncation, (4.28) becomes

$$\widehat{\lambda} \le h(L_1) + C\varepsilon;$$

and then the maximum norm of p can only be estimated in terms of $Ch(L_1)$; this does not allow us to conclude that T maps G into itself, even if ε is small.

5. Existence for small ε and η.

In this section we prove Theorem 1.1 by establishing:

LEMMA 5.1. *There exist $\varepsilon^* > 0$, $\eta^* > 0$ such that if*

$$0 < \varepsilon < \varepsilon^*, \quad 0 < \eta < \eta^*, \tag{5.1}$$

then the solution (p, h, u) of (4.1)–(4.5) established in §4 is a solution asserted in Theorem 1.1.

Proof. It suffices to show that

$$u(x) > g(x) \quad \text{if } 0 < x < L \tag{5.2}$$

and

$$h(L_1) \le 1 + L_1\delta'(L_1) - \delta(L_1) \tag{5.3}$$

for ε and η positive and sufficiently small.

Suppose either (5.2) or (5.3) is not satisfied for a sequence $(\varepsilon_j, \eta_j) \to (0, 0)$; we shall derive a contradiction. We denote the p, h, u and λ corresponding to (ε_j, η_j) by p_j, h_j, u_j, λ_j, respectively. We may assume that

$$\begin{aligned} &\lambda_j \to \lambda, \\ &p_j \to p \quad \text{weakly } * \text{ in } L^\infty(L_1, L_2), \\ &u_j \to u \quad \text{uniformly in } [0, L], \\ &h_j \to h \quad \text{uniformly in } [L_1, L_2], \end{aligned}$$

and

$$\left|\frac{\partial}{\partial x} u_j\right| \le C \quad \text{in } [0, L]. \tag{5.4}$$

For any $\zeta \in C_c^\infty(L_1, L_2)$ we have, by integration by parts,

$$\int_{L_1}^{L_2} (p_j h_j - \lambda_j)\zeta dx = -\varepsilon \int_{L_1}^{L_2} \Big[p_j \frac{\partial}{\partial x}(\alpha h_j^2 \zeta) + \frac{1}{2} p_j^2 \frac{\partial}{\partial x}(\beta h_j^3 \zeta)\Big] dx$$

Taking $j \to \infty$ and using (4.32), we obtain

$$\int_{L_1}^{L_2} \Big[ph - \frac{f(h(L_1))}{h(L_1)}\Big]\zeta dx = 0.$$

Hence

$$p(x)h(x) = \frac{f(h(L_1))}{h(L_1)} \quad \text{for } L_1 \le x \le L_2.$$

Next, multiplying the differential inequality for u_j by $\zeta \in C_c^\infty(0,L)$, $\zeta \geq 0$ and integrating by parts, we find that u is a solution of the variational inequality with $\eta = 0$ and, by Lemma 3.1,

$$\begin{gathered} u(x) - g(x) \geq m > 0, \quad L_1 \leq x \leq L_2, \\ h(L_1) \leq L_1 \delta'(L_1) - \delta(L_1). \end{gathered} \tag{5.5}$$

(Actually u is initially only known to be in $C^{0,1}$ (by (5.4)); but by standard regularity for variational inequalities (e.g. [4; Chap 1]) u is in $W^{2,p}$ (for any $p > 1$) and satisfies (3.4) a.e.)

The inequality (5.5) implies that (5.2) and (5.3) hold for u_j, h_j if j is large enough, which is a contradiction. $\square$

6. Boundary layer behavior. Denote the solution asserted in Theorem 1.1 by $(p^{\varepsilon,\eta}, h^{\varepsilon,\eta}, u^{\varepsilon,\eta})$ and the corresponding λ by $\lambda^{\varepsilon,\eta}$. We take a sequence (ε_j, η_j) such that

$$\begin{aligned} &\lambda_j = \lambda^{\varepsilon_j,\eta_j} \longrightarrow \lambda^0, \\ &p_j = p^{\varepsilon_j,\eta_j} \longrightarrow p^0 \quad \text{weakly} * \text{ in } L^\infty(L_1, L_2), \\ &u_j = u^{\varepsilon_j,\eta_j} \longrightarrow u^0 \quad \text{uniformly in } [0, L]. \end{aligned}$$

Then also

$$h_j = h^{\varepsilon_j,\eta_j} \longrightarrow h^0 \quad \text{uniformly in } [L_1, L_2].$$

By Lemma 4.1,

$$\left| \frac{\partial}{\partial x} u_j \right| \leq C \quad \text{in } [0, L]. \tag{6.1}$$

Since the boundary condition $p_j(L_2) = 1$ is lost in the limit, we want to study the behavior of p_j near $x = L_2$.

Making a change of variables

$$y = \frac{x - L_2}{\varepsilon}, \quad \tilde{p}_j(y) = p_j(x),$$

it is clear that

$$\begin{gathered} \tilde{p}_j(y) h_j(L_2 + \varepsilon y) - [\alpha(h_j(L_2 + \varepsilon y))^2 + \beta(h_j(L_2 + \varepsilon y))^3 \tilde{p}_j(y)] \frac{\partial \tilde{p}_j}{\partial y} = \lambda_j \\ \text{for } -\frac{L_2 - L_1}{\varepsilon} < y < 0, \end{gathered} \tag{6.2}$$

$$\tilde{p}_j(0) = 1. \tag{6.3}$$

From the standard ODE theory it follows that $\partial\widetilde{p}_j/\partial y$ is uniformly bounded in any bounded interval $[-N,0]$, $\widetilde{p}_j \to \widetilde{p}$ uniformly in any such interval, and $\widetilde{p}$ satisfies:

$$\begin{aligned} &\widetilde{p}(y)h^0(L_2) - [\alpha(h^0(L_2))^2 + \beta(h^0(L_2))^3\widetilde{p}(y)]\frac{\partial\widetilde{p}}{\partial y} = h^0(L_1) \\ &\qquad\qquad\qquad\qquad\qquad \text{for } -\frac{L_2-L_1}{\varepsilon} < y < 0, \\ &\widetilde{p}(0) = 1. \end{aligned} \tag{6.4}$$

This system has a unique solution:

If $h^0(L_1) = h^0(L_2)$ then $\widetilde{p}(y) \equiv 1$.

If $h^0(L_1) \neq h^0(L_2)$ then $\widetilde{p}(y)$ is given by

$$y - \beta h_2^2\widetilde{p} - (\beta h_1^2 + \alpha h_2)\log|h_1 - h_2\widetilde{p}| = C_0 \tag{6.5}$$

where

$$h_1 = h^0(L_1), \quad h_2 = h^0(L_2), \tag{6.6}$$

and C_0 is the constant which is evaluated by taking $y = 0$, $\widetilde{p} = 1$ in (6.5).

If we take other sequences (ε_j, η_j), we may possibly obtain other limits p^0, h^0, u^0, but the limit $\widetilde{p}(y)$ depends only on the constants in (6.6). (Notice that if we have a symmetric head as in Theorem 2.8, then the limit is unique, and we have the convergence as $(\varepsilon,\eta) \to (0,0)$, not just on subsequences).

The above result shows that

$$p_j(x) \sim \widetilde{p}\Big(\frac{x-L_2}{\varepsilon}\Big) \quad \text{in } [L_2 - C\varepsilon, L_2] \tag{6.7}$$

for any constant C, and this describes the boundary layer near $x = L_2$ for (ε,η) small.

The boundary layer phenomenon for general differential equations was widely studied in the literature; see for example [10]. From [10] it follows that near $x = L_1$ the functions $p_j(x)$ are uniformly converging to $p^0(x)$, so that $p^0(L_1) = 1$.

As for the $u_j(x)$, from (6.1) we deduce that the $u_j(x)$ are uniformly continuous in $0 \le x \le L$ and $u^0(L_1) = u^0(L_2) = 0$. However the boundary conditions $\partial u_j/\partial x = 0$ at $x = 0$ and $x = L$ disappear in the limit. The derivatives $\partial u_j/\partial x$ must therefore exhibit a boundary layer behavior at $x = 0$ and $x = L$. However what is more important for the head-tape interaction is the behavior of the $u_j(x)$, or the spacing $h_j(x)$, at $x = L_2$. Although there is no boundary layer for the $u_j(x)$ at $x = L_2$, the fact that K in (1.2) is large tends to magnify the oscillatory behavior of p near $x = L_2$ and to create a *boundary layer-like* behavior for the h_j.

Conclusion. Both experiments and numerical simulations have established boundary layer behavior for the head-tape problem at $x = L_2$ [1][5] for a range of physical parameters ε, η; these parameters are small, e.g., $10^{-2} \sim 10^{-3}$. In the present paper we give rigorous mathematical proofs of existence of solutions and of the same type of boundary layer behavior provided the parameters ε, η are sufficiently small.

7. The two-dimensional case. In this section we consider the 2-d model of head-tape interaction. The profile of the head is a function $\delta(x,y)$, and the steady state system, after nondimensionalizing, takes the form

$$\frac{\partial(ph)}{\partial x} - \varepsilon\nabla(\alpha h^2\nabla p + \beta h^3 p\nabla p) = 0 \quad \text{in } S_1, \tag{7.1}$$

$$-\frac{\partial^2 u}{\partial x^2} + \eta\Delta^2 u = K(p-1)\chi_{S_1} \quad \text{in } S_2, \tag{7.2}$$

$$u(x,y) = h(x,y) + \delta(x,y) \quad \text{in } S_1 \tag{7.3}$$

where

$$S_1 = \{(x,y);\ L_1 < x < L_2,\ 0 < y < b\},$$
$$S_2 = \{(x,y);\ 0 < x < L,\ 0 < y < b\}, \quad b = O(1),$$

with the boundary conditions

$$p = 1 \quad \text{on } \partial S_1, \tag{7.4}$$

$$u = u_x = 0 \quad \text{on } x = 0 \text{ and } x = L, \tag{7.5}$$

$$u_{yy} + \sigma u_{xx} = 0, \quad u_{yyy} + (2-\sigma)u_{xxy} = 0 \quad \text{on } y = 0 \text{ and } y = b, \tag{7.6}$$

where σ is the Poisson ratio, $0 < \sigma < 1/2$.

The boundary conditions for u are those of a plate clamped at $x = 0$ and $x = L$ and free at $y = 0$ and $y = b$; see [9].

Given p, the function u is determined as a minimizer of the energy functional (see [7; Chap. 11])

$$\int_{S_2} \Big\{\eta(\Delta u)^2 + 2\eta(1-\sigma)\Big[\Big(\frac{\partial^2 u}{\partial x\partial y}\Big)^2 - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}\Big] + \Big(\frac{\partial u}{\partial x}\Big)^2 - 2Ku(p-1)\chi_{S_1}\Big\}dxdy$$

subject to the boundary conditions (7.5); the differential equation (7.2) and the boundary conditions (7.6) are necessary optimality conditions.

In the limit case $\varepsilon = \eta = 0$ the system reduces to (cf. (2.4))

$$-u_{xx} = K\Big(\frac{u(L_1,y) - \delta(L_1,y)}{u(x,y) - \delta(x,y)} - 1\Big)\chi_{S_1} \quad \text{in } S_2$$

and, with the boundary conditions

$$u(0,y) = u(L,y) = 0,$$

it can be solved by the shooting method, for each value y.

For $\varepsilon > 0$, $\eta > 0$ we can set up a variational system (analogous to (4.1)–(4.5)). We still have the bound

$$\int_{S_2} u_x^2 \le C$$

but we do not know whether $\int u_y^2 \leq C$, much less whether $|\nabla u| \leq C$, and thus we are unable to carry out rigorously the analysis of §5. On the other hand one can proceed formally to derive boundary layer behaviors for p near $x = L_2$ and near $y = 0$ and $y = b$, assuming that u remains continuous there; the boundary layers near $y = 0$ and $y = b$ are of the form given in [10; p. 302]. For numerical results we refer to [8].

REFERENCES

[1] B. BHUSHAN, *Tribology and Mechanics of Magnetic Storage Devices*, Springer-Verlag, New York 1990.

[2] A. BURGDORFER, *The influence of the molecular mean free path on the performance of hydrodynamic gas lubricated bearings*, ASME J. Basic Engineering, 81(1959), 94–100.

[3] A. FRIEDMAN, *Variational Principles and Free Boundary Problems*, Wiley Interscience, New York 1982.

[4] A. FRIEDMAN, *Mathematics in Industrial Problems, Part 7*, IMA Volume 67, Springer-Verlag, New York 1994.

[5] W.A. GROSS, L.A. MATSCH, V. CASTELLI, A. ESHEL, J.M. VOHN AND M. WILDMANN, *Fluid Film lubrication*, Wiley Interscience, New York 1980.

[6] C.A. LACEY AND F.E. TALKE, *A tightly coupled numerical foil bearing solution*, IEEE Trans. on Magnetics, 26(1990) 3039–3093.

[7] L.D. LANDAU AND E.M. LIFSHITZ, *Theory of Elasticity*, Pergamon Press, London, 1959.

[8] K. ONO, N. KODAMA AND S. MICHIMURA, *A new numerical analysis method for two-dimensional foil-bearing problems based on inverse analysis concept*, ISME International Journal, Ser. III, 34(1991), 82–61.

[9] S. TIMOSHENKO AND S. WOINOWSKY-KRIEGER, *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

[10] M.I. VISHIK AND L.A. LYUSTERNIK, *Regular degeneration and boundary layer for linear differential equations with small parameter*, Amer. Math. Soc. Translations, Ser. 2, Vol. 20, Providence, RI (1962), 239–364.

Recent IMA Preprints

#	Author/s	Title
1276	**Z. Chen & R.E. Ewing**	Stability and convergence of a finite element method for reactive transport in ground water
1277	**Z. Chen & Do Y. Kwak**	The analysis of multigrid algorithms for nonconforming and mixed methods for second order elliptic problems
1278	**Z. Chen**	Expanded mixed finite element methods for quasilinear second order elliptic problems II
1279	**M.A. Horn & W. Littman**	Boundary control of a Schrödinger equation with nonconstant principal part
1280	**R.E. Ewing, Yu.A. Kuznetsov, R.D. Lazarov & S. Maliassov**	Substructuring preconditioning for finite element approximations of second order elliptic problems. I. Nonconforming linear elements for the Poisson equation in a parallelepiped
1281	**S. Maliassov**	Substructuring preconditioning for finite element approximations of second order elliptic problems. II. Mixed method for an elliptic operator with scalar tensor
1282	**V. Jakšić & C.-A. Pillet**	On model for quantum friction II. Fermi's golden rule and dynamics at positive temperatures
1283	**V. M. Malkin**	Kolmogorov and nonstationary spectra of optical turbulence
1284	**E.G. Kalnins, V.B. Kuznetsov & W. Miller, Jr.**	Separation of variables and the XXZ Gaudin magnet
1285	**E.G. Kalnins & W. Miller, Jr.**	A note on tensor products of q-algebra representations and orthogonal polynomials
1286	**E.G. Kalnins & W. Miller, Jr.**	q-algebra representations of the Euclidean, pseudo-Euclidean and oscillator algebras, and their tensor products
1287	**L.A. Pastur**	Spectral and probabalistic aspects of matrix models
1288	**K. Kastella**	Discrimination gain to optimize detection and classification
1289	**L.A. Peletier & W.C. Troy**	Spatial patterns described by the Extended Fisher-Kolmogorov (EFK) equation: Periodic solutions
1290	**A. Friedman & Y. Liu**	Propagation of cracks in elastic media
1291	**A. Friedman & C. Huang**	Averaged motion of charged particles in a curved strip
1292	**G. R. Sell**	Global attractors for the 3D Navier-Stokes equations
1293	**C. Liu**	A uniqueness result for a general class of inverse problems
1294	**H-O. Kreiss**	Numerical solution of problems with different time scales II
1295	**B. Cockburn, G. Gripenberg, S-O. Londen**	On convergence to entropy solutions of a single conservation law
1296	**S-H. Yu**	On stability of discrete shock profiles for conservative finite difference scheme
1297	**H. Behncke & P. Rejto**	A limiting absorption principle for separated Dirac operators with Wigner Von Neumann type potentials
1298	**R. Lipton B. Vernescu**	Composites with imperfect interface
1299	**E. Casas**	Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations
1300	**G.R. Sell**	References on dynamical systems
1301	**J. Zhang**	Swelling and dissolution of polymer: A free boundary problem
1302	**J. Zhang**	A nonlinear nonlocal multi-dimensional conservation law
1303	**M.E. Taylor**	Estimates for approximate solutions to acoustic inverse scattering problems
1304	**J. Kim & D. Sheen**	A priori estimates for elliptic boundary value problems with nonlinear boundary conditions
1305	**B. Engquist & E. Luo**	New coarse grid operators for highly oscillatory coefficient elliptic problems
1306	**A. Boutet de Monvel & I. Egorova**	On the almost periodicity of solutions of the nonlinear Schrödinger equation with the cantor type spectrum
1307	**A. Boutet de Monvel & V. Georgescu**	Boundary values of the resolvent of a self-adjoint operator: Higher order estimates
1308	**S.K. Patch**	Diffuse tomography modulo Graßmann and Laplace
1309	**A. Friedman & J.J.L. Velázquez**	Liouville type theorems for fourth order elliptic equationsin a half plane
1310	**T. Aktosun, M. Klaus & C. van der Mee**	Recovery of discontinuities in a nonhomogeneous medium
1311	**V. Bondarevsky**	On the global regularity problem for 3-dimensional Navier-Stokes equations
1312	**M. Cheney & D. Isaacson**	Inverse problems for a perturbed dissipative half-space
1313	**B. Cockburn, D.A. Jones & E.S. Titi**	Determining degrees of freedom for nonlinear dissipative equations
1314	**B. Engquist & E. Luo**	Convergence of a multigrid method for elliptic equations with highly oscillatory coefficients
1315	**L. Pastur & M. Shcherbina**	Universality of the local eigenvalue statistics for a class of unitary invariant random matrix ensembles
1316	**V. Jakšić, S. Molchanov & L. Pastur**	On the propagation properties of surface waves
1317	**J. Nečas, M. Ružička & V. Šverák**	On self-similar solutions of the Navier-Stokes equations
1318	**S. Stojanovic**	Remarks on $W^{2,p}$-solutions of bilateral obstacle problems
1319	**E. Luo & H-O. Kreiss**	Pseudospectral vs. Finite difference methods for initial value problems with discontinuous coefficients
1320	**V.E. Grikurov**	Soliton's rebuilding in one-dimensional Schrödinger model with polynomial nonlinearity
1321	**J.M. Harrison & R.J. Williams**	A multiclass closed queueing network with unconventional heavy traffic behavior

1322 **M.E. Taylor**, Microlocal analysis on Morrey spaces
1323 **C. Huang**, Homogenization of biharmonic equations in domains perforated with tiny holes
1324 **C. Liu**, An inverse obstacle problem: A uniqueness theorem for spheres
1325 **M. Luskin**, Approximation of a laminated microstructure for a rotationally invariant, double well energy density
1326 **Rakesh & P. Sacks**, Impedance inversion from transmission data for the wave equation
1327 **O. Lafitte**, Diffraction for a Neumann boundary condition
1328 **E. Sobel, K. Lange, J.R. O'Connell & D.E. Weeks**, Haplotyping algorithms
1329 **B. Cockburn, D.A. Jones & E.S. Titi**, Estimating the number of asymptotic degrees of freedom for nonlinear dissipative systems
1330 **T. Aktosun**, Inverse Schrödinger scattering on the line with partial knowledge of the potential
1331 **T. Aktosun & C. van der Mee**, Partition of the potential of the one-dimensional Schrödinger equation
1332 **B. Engquist & E. Luo**, Convergence of the multigrid method with a wavelet coarse grid operator
1333 **V. Jakšić & C.-A. Pillet**, Ergodic properties of the Spin-Boson system
1334 **S.K. Patch**, Recursive solution for diffuse tomographic systems of arbitrary size
1335 **J.C. Bronski**, Semiclassical eigenvalue distribution of the non self-adjoint Zakharov-Shabat eigenvalue problem
1336 **J.C. Cockburn**, Bitangential structured interpolation theory
1337 **S. Kichenassamy**, The blow-up problem for exponential nonlinearities
1338 **F.A. Grünbaum & S.K. Patch**, How many parameters can one solve for in diffuse tomography?
1339 **R. Lipton**, Reciprocal relations, bounds and size effects for composites with highly conducting interface
1340 **H.A. Levine & J. Serrin**, A global nonexistence theorem for quasilinear evolution equations with dissipation
1341 **A. Boutet de Monvel & R. Purice**, The conjugate operator method: Application to DIRAC operators and to stratified media
1342 **G. Michele Graf**, Stability of matter through an electrostatic inequality
1343 **G. Avalos**, Sharp regularity estimates for solutions of the wave equation and their traces with prescribed Neumann data
1344 **G. Avalos**, The exponential stability of a coupled hyperbolic/parabolic system arising in structural acoustics
1345 **G. Avalos & I. Lasiecka**, A differential Riccati equation for the active control of a problem in structural acoustics
1346 **G. Avalos**, Well-posedness for a coupled hyperbolic/parabolic system seen in structural acoustics
1347 **G. Avalos & I. Lasiecka**, The strong stability of a semigroup arising from a coupled hyperbolic/parabolic system
1348 **A.V. Fursikov**, Certain optimal control problems for Navier-Stokes system with distributed control function
1349 **F. Gesztesy, R. Nowell & W. Pötz**, One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics
1350 **F. Gesztesy & H. Holden**, On trace formulas for Schrödinger-type operators
1351 **X. Chen**, Global asymptotic limit of solutions of the Cahn-Hilliard equation
1352 **X. Chen**, Lorenz equations, Part I: Existence and nonexistence of homoclinic orbits
1353 **X. Chen**, Lorenz equations Part II: "Randomly" rotated homoclinic orbits and chaotic trajectories
1354 **X. Chen**, Lorenz equations, Part III: Existence of hyperbolic sets
1355 **R. Abeyaratne, C. Chu & R.D. James**, Kinetics of materials with wiggly energies: Theory and application to the evolution of twinning microstructures in a Cu-Al-Ni shape memory alloy
1356 **C. Liu**, The Helmholtz equation on Lipschitz domains
1357 **G. Avalos & I. Lasiecka**, Exponential stability of a thermoelastic system without mechanical dissipation
1358 **R. Lipton**, Heat conduction in fine scale mixtures with interfacial contact resistance
1359 **V. Odisharia & J. Peradze**, Solvability of a nonlinear problem of Kirchhoff shell
1360 **P.J. Olver, G. Sapiro & A. Tannenbaum**, Affine invariant edge maps and active contours
1361 **R.D. James**, Hysteresis in phase transformations
1362 **A. Sei & W. Symes**, A note on consistency and adjointness for numerical schemes
1363 **A. Friedman & B. Hu**, Head-media interaction in magnetic recording
1364 **A. Friedman & J.J.L. Velázquez**, Time-dependent coating flows in a strip, part I: The linearized problem
1365 **X. Ren & M. Winter**, Young measures in a nonlocal phase transition problem
1366 **K. Bhattacharya & R.V. Kohn**, Elastic energy minimization and the recoverable strains of polycrystalline shape-memory materials
1367 **G.A. Chechkin**, Operator pencil and homogenization in the problem of vibration of fluid in a vessel with a fine net on the surface
1368 **M. Carme Calderer & B. Mukherjee**, On Poiseuille flow of liquid crystals
1369 **M.A. Pinsky & M.E. Taylor**, Pointwise Fourier inversion: A wave equation approach
1370 **D. Brandon & R.C. Rogers**, Order parameter models of elastic bars and precursor oscillations
1371 **H.A. Levine & B.D. Sleeman**, A system of reaction diffusion equations arising in the theory of reinforced random walks
1372 **B. Cockburn & P.-A. Gremaud**, A priori error estimates for numerical methods for scalar conservation laws. Part II: Flux-splitting monotone schemes on irregular Cartesian grids