

**A WELL-POSED BOUNDARY VALUE PROBLEM
FOR SUPERCRITICAL FLOW
OF VISCOELASTIC FLUIDS OF MAXWELL TYPE**

By

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This paper is dedicated to Daniel D. Joseph
on the occasion of his 60th birthday

**A WELL-POSED BOUNDARY VALUE PROBLEM
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Abstract. For a class of viscoelastic fluids with differential constitutive laws of Maxwell type, we investigate the existence and uniqueness of steady flows. We consider small perturbations of uniform flow transverse to a strip. A well-posed boundary value problem is formulated for the case when the velocity of the fluid exceeds the speed of propagation of shear waves.

Key words. viscoelastic fluids, boundary conditions, change of type

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1. Introduction. While the study of existence and uniqueness results for steady flows of Newtonian fluids is well advanced, relatively little is known about viscoelastic fluids with memory. For such fluids, the nature of boundary conditions leading to well-posed problems is in general different from the Newtonian case. There are two reasons for this:

1. The memory of the fluid implies that what happens in the domain under consideration is dependent on the deformation history of the fluid before it entered the domain. Information about this deformation history must therefore be given in the form of boundary conditions at inflow boundaries. The precise nature of such inflow conditions is dependent on the constitutive relation; for example, fluids of Maxwell type [4] are different from fluids of Jeffreys type [5].
2. For fluids of Maxwell type, there is a change of type in the governing equations when the velocity of the fluid exceeds the propagation speed of shear waves (cf. [1], [2], [7], [8]). This necessitates a change in the nature of boundary conditions. If boundary conditions which would be correct in the subcritical case are imposed in a supercritical situation, the problem becomes ill-posed in a similar fashion as the Dirichlet problem for the wave equation (see [6]).

In the following, v denotes the velocity, p the pressure, \mathbf{T} the extra stress tensor, ρ the density and f a given body force. The equation of motion is

$$(1) \quad \rho(v \cdot \nabla)v = \operatorname{div} \mathbf{T} - \nabla p + f,$$

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and the incompressibility condition is

$$(2) \quad \operatorname{div} v = 0.$$

We assume a Maxwell-type constitutive relation of the following form:

$$(3) \quad \begin{aligned} & ((v \cdot \nabla) + \lambda)T_{ij} - \frac{\partial v_i}{\partial x_k} T_{kj} - T_{ik} \frac{\partial v_j}{\partial x_k} + P_{ik}(\mathbf{T}) \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k} \right) + P_{jk}(\mathbf{T}) \left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \right) \\ & + g_{ij}(\mathbf{T}) = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \end{aligned}$$

Here λ and μ are positive constants, and the matrix-valued functions \mathbf{P} and \mathbf{g} are assumed to be smooth; moreover, \mathbf{P} , \mathbf{g} and the first derivatives of \mathbf{g} vanish at $\mathbf{T}=\mathbf{0}$. Equation (3) includes a number of popular rheological models (cf. e.g. [3]).

The domain on which we want to solve (1)-(3) is the strip bounded by the planes $x_1 = 0$ and $x_1 = 1$. In the x_2 - and x_3 -directions, we assume periodicity with periods L and M . The solutions we seek are small perturbations of the uniform flow $v = (V, 0, 0)$, $p = 0$, $\mathbf{T} = 0$. The given body force and the imposed boundary conditions are assumed to satisfy smallness conditions consistent with this. In [4], we considered this problem under the assumption that $\rho V^2 < \mu$. A well-posed boundary value problem was obtained by prescribing the velocities at both boundaries plus additional stress conditions at the inflow boundary $x_1 = 0$. In two dimensions it is possible to prescribe the diagonal components T_{11} and T_{22} . In three dimensions a correct choice of inflow stress conditions was obtained as follows. We expand each stress component in a Fourier series, e.g.

$$(4) \quad T_{11}(0, x_2, x_3) = \sum_{k,l} t_{11}^{kl} \exp(2\pi i(kx_2/L + lx_3/M)).$$

Then one can, for example, prescribe the following inflow conditions:

$$(5) \quad \begin{aligned} & t_{11}^{kl}, t_{22}^{kl}, t_{13}^{kl}, t_{33}^{kl} \text{ if } |k| \gg |l|, \\ & t_{11}^{kl}, t_{22}^{kl}, t_{12}^{kl}, t_{33}^{kl} \text{ if } |l| \gg |k|, \\ & t_{11}^{kl}, t_{13}^{kl}, t_{23}^{kl}, t_{33}^{kl} \text{ if } |k| \text{ and } |l| \text{ are comparable,} \\ & t_{11}^{kl}, t_{22}^{kl}, t_{23}^{kl}, t_{33}^{kl} \text{ if } k = l = 0. \end{aligned}$$

If, on the other hand, $\rho V^2 > \mu$, then this choice of boundary conditions does not lead to a well-posed problem [6]. We shall show that, in this case, one can prescribe the following conditions: the inflow stresses as above, the normal velocity at both boundaries, plus the vorticity and its normal derivative (in two dimensions), or, respectively, the second and third components of the vorticity and their normal derivatives (in three dimensions) at the inflow boundary. The analysis for two space dimensions will be carried out in Section 2; the modifications needed for three dimensions will be discussed in Section 3.

2. The two-dimensional case. We apply the operation $(v \cdot \nabla) + \lambda + (\nabla v)^T$ to the equation of motion (1) and we use equation (3) to reexpress $((v \cdot \nabla) + \lambda)\mathbf{T}$. After some algebra, this yields an equation of the following form (written in components)

$$\begin{aligned}
\rho((v \cdot \nabla) + \lambda)(v \cdot \nabla)v_i &= \mu\Delta v_i + (T_{kj} - P_{kj}(\mathbf{T}))\frac{\partial^2 v_i}{\partial x_j \partial x_k} \\
&\quad - P_{ik}(\mathbf{T})\frac{\partial^2 v_k}{\partial x_j^2} - P_{jk}(\mathbf{T})\frac{\partial^2 v_k}{\partial x_i \partial x_j} - \frac{\partial q}{\partial x_i} \\
(6) \quad &+ ((v \cdot \nabla) + \lambda)f_i + \frac{\partial v_j}{\partial x_i} f_j + h_i(v, \nabla v, \mathbf{T}, \nabla \mathbf{T}).
\end{aligned}$$

Here we have set $q = ((v \cdot \nabla) + \lambda)p$. The term h is a complicated expression which we do not write out explicitly; it contains only quadratic and higher order terms.

Next we introduce a streamfunction-vorticity formulation. We set

$$(7) \quad v_1 = -\frac{\partial \psi}{\partial x_2}, \quad v_2 = \frac{\partial \psi}{\partial x_1}, \quad \zeta = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2},$$

so that the incompressibility condition (2) is automatically satisfied and

$$(8) \quad \Delta \psi = \zeta.$$

We take the curl of equation (6), which results in

$$\begin{aligned}
\rho((v \cdot \nabla) + \lambda)(v \cdot \nabla)\zeta &= \mu\Delta \zeta + (T_{kj} - P_{kj}(\mathbf{T}))\frac{\partial^2 \zeta}{\partial x_j \partial x_k} - P_{22}(\mathbf{T})\frac{\partial^2 \zeta}{\partial x_1^2} - P_{11}(\mathbf{T})\frac{\partial^2 \zeta}{\partial x_2^2} \\
(9) \quad &+ (P_{12}(\mathbf{T}) + P_{21}(\mathbf{T}))\frac{\partial^2 \zeta}{\partial x_1 \partial x_2} + r(v, \nabla v, \nabla^2 v, \mathbf{T}, \nabla \mathbf{T}, \nabla^2 \mathbf{T}, f, \nabla f, \nabla^2 f).
\end{aligned}$$

Here r is again a complicated expression which we do not write out explicitly.

In the following, we shall solve (1)-(3) subject to the following boundary conditions:

$$\begin{aligned}
\psi &= -Vx_2 + \psi_0, \text{ on } x_1 = 0, \quad \psi = -Vx_2 + \psi_1, \text{ on } x_1 = 1, \\
\zeta &= \zeta_0, \text{ on } x_1 = 0, \quad \frac{\partial \zeta}{\partial x_1} = \eta_0, \text{ on } x_1 = 0, \\
(10) \quad &T_{11} = t_1, \text{ on } x_1 = 0, \quad T_{22} = t_2 \text{ on } x_1 = 0.
\end{aligned}$$

We note that prescribing ψ on both boundaries is equivalent to prescribing the normal velocity on these boundaries as well as the total flow rate in the x_2 -direction.

We denote by H^s the space of all functions on the strip $0 \leq x_1 \leq 1$ which are periodic with period L in the x_2 -direction and have s derivatives which are square integrable over one period. Sobolev spaces of periodic functions depending only on x_2 are denoted by $H^{(s)}$. The corresponding norms are denoted by $\|\cdot\|_s$ and $\|\cdot\|_{(s)}$. Moreover, $\|\cdot\|_{k,l}$ denotes the norm in $W^{k,\infty}([0, 1]; H^{(l)})$.

The goal of this section is the following existence and uniqueness result:

THEOREM. Assume that $\|f\|_4, \|\psi_0\|_{\langle 9/2 \rangle}, \|\psi_1\|_{\langle 9/2 \rangle}, \|\zeta_0\|_{\langle 3 \rangle}, \|\eta_0\|_{\langle 2 \rangle}, \|t_1\|_{\langle 3 \rangle}$ and $\|t_2\|_{\langle 3 \rangle}$ are sufficiently small. Then there is a solution of (1)-(3) which satisfies the boundary conditions (10) and the regularity $\psi + Vx_2 \in H^5, \mathbf{T} \in H^3$. Moreover, this solution is the only one for which $\|\psi + Vx_2\|_5$, and $\|\mathbf{T}\|_3$ are small.

The construction of the solution is based on an iterative scheme. As a starting value for the iteration we use the uniform flow

$$(11) \quad \psi^0 = -Vx_2, \quad \zeta^0 = 0, \quad \mathbf{T}^0 = 0.$$

Given ψ^n, ζ^n and \mathbf{T}^n , we define v^n by

$$(12) \quad v_1^n = -\frac{\partial \psi^n}{\partial x_2}, \quad v_2^n = \frac{\partial \psi^n}{\partial x_1}.$$

Next, we determine \mathbf{T}^{n+1} from the equation

$$(13) \quad \begin{aligned} & ((v^n \cdot \nabla) + \lambda)T_{ij}^{n+1} - \frac{\partial v_i^n}{\partial x_k} T_{kj}^n - T_{ik}^n \frac{\partial v_j^n}{\partial x_k} + P_{ik}(\mathbf{T}^n) \left(\frac{\partial v_k^n}{\partial x_j} + \frac{\partial v_j^n}{\partial x_k} \right) + P_{jk}(\mathbf{T}^n) \left(\frac{\partial v_k^n}{\partial x_i} + \frac{\partial v_i^n}{\partial x_k} \right) \\ & + g_{ij}(\mathbf{T}^n) = \mu \left(\frac{\partial v_i^n}{\partial x_j} + \frac{\partial v_j^n}{\partial x_i} \right), \end{aligned}$$

subject to the following initial conditions at $x_1 = 0$:

$$(14) \quad T_{11}^{n+1} = t_1, \quad T_{22}^{n+1} = t_2,$$

$$(15) \quad \rho \operatorname{curl} ((v^n \cdot \nabla)v^n) = \operatorname{curl} (\operatorname{div} \mathbf{T}^{n+1} + f),$$

Then we determine ζ^{n+1} from the initial-value problem

$$(16) \quad \begin{aligned} & \rho((v^n \cdot \nabla) + \lambda)(v^n \cdot \nabla)\zeta^{n+1} = \mu \Delta \zeta^{n+1} + (T_{kj}^{n+1} - P_{kj}(\mathbf{T}^{n+1})) \frac{\partial^2 \zeta^{n+1}}{\partial x_j \partial x_k} \\ & - P_{22}(\mathbf{T}^{n+1}) \frac{\partial^2 \zeta^{n+1}}{\partial x_1^2} - P_{11}(\mathbf{T}^{n+1}) \frac{\partial^2 \zeta^{n+1}}{\partial x_2^2} + (P_{12}(\mathbf{T}^{n+1}) + P_{21}(\mathbf{T}^{n+1})) \frac{\partial^2 \zeta^{n+1}}{\partial x_1 \partial x_2} \\ & + r(v^n, \nabla v^n, \nabla^2 v^n, \mathbf{T}^{n+1}, \nabla \mathbf{T}^{n+1}, \nabla^2 \mathbf{T}^{n+1}, f, \nabla f, \nabla^2 f). \end{aligned}$$

$$(17) \quad \zeta^{n+1} = \zeta_0, \quad \text{on } x_1 = 0, \quad \frac{\partial \zeta^{n+1}}{\partial x_1} = \eta_0, \quad \text{on } x_1 = 0.$$

Finally, we obtain ψ^{n+1} from the Dirichlet problem

$$\Delta\psi^{n+1} = \zeta^{n+1},$$

$$(18) \quad \psi^{n+1} = -Vx_2 + \psi_0, \text{ on } x_1 = 0, \quad \psi^{n+1} = -Vx_2 + \psi_1, \text{ on } x_1 = 1.$$

We define

$$(19) \quad X(M) = \{(\psi, \zeta, \mathbf{T}) \mid \|\psi + Vx_2\|_5 + \|\zeta\|_3 + \|\mathbf{T}\|_3 \leq M\}.$$

The space $X(M)$ is complete under the metric

$$(20) \quad d((\psi, \zeta, \mathbf{T}), (\hat{\psi}, \hat{\zeta}, \hat{\mathbf{T}})) = \|\psi - \hat{\psi}\|_4 + \|\zeta - \hat{\zeta}\|_2 + \|\mathbf{T} - \hat{\mathbf{T}}\|_2.$$

We choose M small relative to 1, but sufficiently large relative to the norms of the prescribed data. In order to prove the theorem, it is sufficient to show that the mapping defined by the iteration (12)-(18) is a contraction in $X(M)$. We begin by showing that the iteration maps $X(M)$ into itself. Let us assume that $(\psi^n, \zeta^n, \mathbf{T}^n)$ lies in $X(M)$.

We first discuss the solution of (13)-(15). A rearrangement of (15) yields

$$(21) \quad \frac{\partial^2}{\partial x_1 \partial x_2} (T_{22}^{n+1} - T_{11}^{n+1}) + \left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) T_{12}^{n+1} = \rho \operatorname{curl} ((v^n \cdot \nabla)v^n) - \operatorname{curl} f.$$

We can use (13) to express x_1 -derivatives of the stresses, i.e.

$$(22) \quad \frac{\partial}{\partial x_1} T_{ij}^{n+1} = \frac{1}{v_1^n} \left[-v_2^n \frac{\partial}{\partial x_2} T_{ij}^{n+1} - \lambda T_{ij}^{n+1} \dots \right].$$

After substituting (22) in (21), we obtain an ODE from which we can determine T_{12}^{n+1} at the inflow boundary $x_1 = 0$. We denote this boundary value by t^{n+1} . The following estimate is immediate

$$(23) \quad \|t^{n+1}\|_{\langle 3 \rangle} \leq C(\|t_1\|_{\langle 3 \rangle} + \|t_2\|_{\langle 3 \rangle} + \|w^n\|_4 + \|w^n\|_4 \|\mathbf{T}^n\|_3 + \|\mathbf{T}^n\|_3^2).$$

Here we have set $w^n = v^n - (V, 0)$. After determining t^{n+1} , we have a full set of initial conditions to solve (13). Using standard energy estimates for hyperbolic equations (see [4] for some more details) we obtain a unique solution which satisfies an estimate of the form

$$(24) \quad \begin{aligned} & \|\mathbf{T}^{n+1}\|_{3,0} + \|\mathbf{T}^{n+1}\|_{2,1} + \|\mathbf{T}^{n+1}\|_{1,2} + \|\mathbf{T}^{n+1}\|_{0,3} \\ & \leq C(\|t_1\|_{\langle 3 \rangle} + \|t_2\|_{\langle 3 \rangle} + \|t^{n+1}\|_{\langle 3 \rangle} + \|w^n\|_4 + \|\mathbf{T}^n\|_3 \|w^n\|_4 + \|\mathbf{T}^n\|_3^2). \end{aligned}$$

From equation (13), we can see that an expression like the one on the right hand side of (24) also provides an upper bound for $\|(v^n \cdot \nabla)\mathbf{T}^{n+1}\|_3$.

For the solution of the initial-value problem (16), (17), one readily obtains the estimate

$$(25) \quad \|\zeta^{n+1}\|_{0,2} + \|\zeta^{n+1}\|_{1,1} + \|\zeta^{n+1}\|_{2,0} \leq C(\|\zeta_0\|_{\langle 2 \rangle} + \|\eta_0\|_{\langle 1 \rangle} + \|r^n\|_{1,0} + \|r^n\|_{0,1}).$$

This is insufficient because we need to estimate third order derivatives of ζ^{n+1} . Results available in the literature would require the existence of a higher order derivative of r^n either with respect to x_1 or with respect to x_2 . We cannot use such an assumption because of the dependence of r on second derivatives of \mathbf{T}^{n+1} . However, because of the bound for $\|(v^n \cdot \nabla)\mathbf{T}^{n+1}\|_3$, we can get bounds on $\|(v^n \cdot \nabla)r^n\|_1$. Instead of differentiating (16) with respect to either x_1 or x_2 , which is what is usually done, we can apply the operation $(v^n \cdot \nabla)$ to it. By doing this and deriving energy estimates in the usual fashion, we obtain an estimate of the form

$$(26) \quad \|\zeta\|_3 \leq C(\|\zeta_0\|_{\langle 3 \rangle} + \|\eta_0\|_{\langle 2 \rangle} + \|r^n\|_{0,1} + \|r^n\|_{1,0} + \|(v^n \cdot \nabla)r^n\|_1).$$

By taking into account the form of r , we can estimate the last three terms in (26) by a constant times

$$(27) \quad \|f\|_4 + (\|w^n\|_4 + \|\mathbf{T}^{n+1}\|_{3,0} + \|\mathbf{T}^{n+1}\|_{2,1} + \|\mathbf{T}^{n+1}\|_{1,2} + \|\mathbf{T}^{n+1}\|_{0,3} + \|(v^n \cdot \nabla)\mathbf{T}^{n+1}\|_3)^2.$$

Finally, from (18) we immediately obtain

$$(28) \quad \|\psi^{n+1} + Vx_2\|_5 \leq C(\|\psi_0\|_{\langle 9/2 \rangle} + \|\psi_1\|_{\langle 9/2 \rangle} + \|\zeta^{n+1}\|_3).$$

The claim that the iteration maps $X(M)$ into itself now follows easily by combining the estimates (23)-(28).

The derivation of estimates to show that the mapping defined by the iteration is a contraction is fairly straightforward, and we shall only demonstrate one step. From (16), (17) we obtain

$$\begin{aligned} & \rho((v^n \cdot \nabla) + \lambda)(v^n \cdot \nabla)(\zeta^{n+1} - \zeta^n) = \mu\Delta(\zeta^{n+1} - \zeta^n) \\ & + (T_{kj}^{n+1} - P_{kj}(\mathbf{T}^{n+1}))\frac{\partial^2(\zeta^{n+1} - \zeta^n)}{\partial x_j \partial x_k} - P_{22}(\mathbf{T}^{n+1})\frac{\partial^2(\zeta^{n+1} - \zeta^n)}{\partial x_1^2} \\ & - P_{11}(\mathbf{T}^{n+1})\frac{\partial^2(\zeta^{n+1} - \zeta^n)}{\partial x_2^2} + (P_{12}(\mathbf{T}^{n+1}) + P_{21}(\mathbf{T}^{n+1}))\frac{\partial^2(\zeta^{n+1} - \zeta^n)}{\partial x_1 \partial x_2} \\ & - \rho\left[\left((v^n \cdot \nabla) + \lambda\right)(v^n \cdot \nabla) - \left((v^{n-1} \cdot \nabla) + \lambda\right)(v^{n-1} \cdot \nabla)\right]\zeta^n \end{aligned}$$

$$\begin{aligned}
& +[(T_{kj}^{n+1} - P_{kj}(\mathbf{T}^{n+1})) - (T_{kj}^n - P_{kj}(\mathbf{T}^n))] \frac{\partial^2 \zeta^n}{\partial x_j \partial x_k} \\
& - [P_{22}(\mathbf{T}^{n+1}) - P_{22}(\mathbf{T}^n)] \frac{\partial^2 \zeta^n}{\partial x_1^2} - [P_{11}(\mathbf{T}^{n+1}) - P_{11}(\mathbf{T}^n)] \frac{\partial^2 \zeta^n}{\partial x_2^2} \\
(29) \quad & + [(P_{12}(\mathbf{T}^{n+1}) + P_{21}(\mathbf{T}^{n+1})) - (P_{12}(\mathbf{T}^n) + P_{21}(\mathbf{T}^n))] \frac{\partial^2 \zeta^n}{\partial x_1 \partial x_2} + r^n - r^{n-1},
\end{aligned}$$

$$(30) \quad \zeta^{n+1} - \zeta^n = 0, \text{ on } x_1 = 0, \quad \frac{\partial(\zeta^{n+1} - \zeta^n)}{\partial x_1} = 0, \text{ on } x_1 = 0.$$

Energy estimates now yield

$$\begin{aligned}
& \|\zeta^{n+1} - \zeta^n\|_{0,2} + \|\zeta^{n+1} - \zeta^n\|_{1,1} + \|\zeta^{n+1} - \zeta^n\|_{2,0} \\
(31) \quad & \leq C(\|\zeta^n\|_3 (\|v^n - v^{n-1}\|_3 + \|\mathbf{T}^{n+1} - \mathbf{T}^n\|_2) + \|r^n - r^{n-1}\|_{0,0} + \|(v^n \cdot \nabla)(r^n - r^{n-1})\|_0).
\end{aligned}$$

We note that a bound on $\|\zeta^n\|_3$ has already been obtained. In order to deal with the last term in (31), we note that

$$(32) \quad (v^n \cdot \nabla)(r^n - r^{n-1}) = (v^n \cdot \nabla)r^n - (v^{n-1} \cdot \nabla)r^{n-1} + ((v^n - v^{n-1}) \cdot \nabla)r^{n-1}.$$

By taking into account the form of r and equation (13), it is easy to estimate the terms on the right hand side of (32).

3. Modifications in three dimensions. The basic iteration scheme used to construct solutions and the function spaces chosen for the analysis will be as in the two-dimensional case, and we shall therefore confine the following discussion to those points where modifications are needed. One of these changes is that $\zeta = \text{curl } v$ is now a vector, and the equation analogous to (9), written in components, is

$$\begin{aligned}
\rho((v \cdot \nabla) + \lambda)(v \cdot \nabla)\zeta_i & = \mu \Delta \zeta_i + (T_{kj} - P_{kj}(\mathbf{T})) \frac{\partial^2 \zeta_i}{\partial x_j \partial x_k} - \epsilon_{mli} \epsilon_{krs} P_{mk}(\mathbf{T}) \frac{\partial^2 \zeta_s}{\partial x_l \partial x_r} \\
(33) \quad & + r_i(v, \nabla v, \nabla^2 v, \mathbf{T}, \nabla \mathbf{T}, \nabla^2 \mathbf{T}, f, \nabla f, \nabla^2 f).
\end{aligned}$$

This is a system of PDE's for the components of ζ , and in order to make it symmetric hyperbolic, we add the assumption that the matrix \mathbf{P} is symmetric. In the iteration, we use the following equation which is analogous to (16):

$$\rho((v^n \cdot \nabla) + \lambda)(v^n \cdot \nabla)\zeta_i^{n+1} = \mu \Delta \zeta_i^{n+1} + (T_{kj}^{n+1} - P_{kj}(\mathbf{T}^{n+1})) \frac{\partial^2 \zeta_i^{n+1}}{\partial x_j \partial x_k}$$

$$(34) \quad -\epsilon_{mli}\epsilon_{krs}P_{mk}(\mathbf{T}^{n+1})\frac{\partial^2\zeta_s^{n+1}}{\partial x_l\partial x_r} + r_i(v^n, \nabla v^n, \nabla^2 v^n, \mathbf{T}^{n+1}, \nabla \mathbf{T}^{n+1}, \nabla^2 \mathbf{T}^{n+1}, f, \nabla f, \nabla^2 f).$$

The velocity can be determined in terms of the vorticity if we prescribe the normal velocity on both boundaries and the mean flux in the y - and z -directions. Unfortunately, we shall not be able to guarantee that all the iterates satisfy $\operatorname{div} \zeta^{n+1} = 0$, and hence we cannot simply use the equation $\operatorname{curl} v^{n+1} = \zeta^{n+1}$. Let Π denote the orthogonal projection (in L^2) onto the subspace

$$(35) \quad V = \{\zeta \in L^2 \mid \operatorname{div} \zeta = 0, \int_0^L \int_0^M \zeta_1(\cdot, x_2, x_3) dx_3 dx_2 = 0\}.$$

The set of equations determining v^{n+1} is

$$(36) \quad \begin{aligned} \operatorname{curl} v^{n+1} &= \Pi \zeta^{n+1}, \quad \operatorname{div} v^{n+1} = 0, \\ v_1^{n+1} &= V + a(x_2, x_3), \quad \text{at } x_1 = 0, \quad v_1^{n+1} = V + b(x_2, x_3) \quad \text{at } x_1 = 1, \\ \int_0^1 \int_0^L \int_0^M v_2^{n+1} dx_3 dx_2 dx_1 &= \alpha, \quad \int_0^1 \int_0^L \int_0^M v_3^{n+1} dx_3 dx_2 dx_1 = \beta. \end{aligned}$$

Here the numbers α and β as well as the functions a and b are prescribed and $|\alpha| + |\beta| + \|a\|_{\langle 7/2 \rangle} + \|b\|_{\langle 7/2 \rangle}$ is assumed to be small. Moreover, we have to assume the compatibility condition

$$(37) \quad \int_0^L \int_0^M a(x_2, x_3) dx_3 dx_2 = \int_0^L \int_0^M b(x_2, x_3) dx_3 dx_2.$$

It can easily be shown along the same lines as in Section 2 that by combining (34), (36) with (13) and (small) inflow data for \mathbf{T} , ζ and $\partial\zeta/\partial x_1$, we obtain a convergent iteration. However, there are two problems:

1. It is not guaranteed that the limit of the iteration satisfies $\zeta = \operatorname{curl} v$, or, equivalently, $\zeta = \Pi \zeta$.
2. It is not guaranteed that the original equation of motion (1) holds. This is because in proceeding from (1) to (6) we have applied the operation $(v \cdot \nabla) + \lambda + (\nabla v)^T$. In order to reverse this step and go from (6) to (1), we have to assume that (1) holds on the inflow boundary $x_1 = 0$ (cf. [4]). In two dimensions we imposed this condition as equation (15).

In order to remove these two difficulties, we must restrict the inflow data; i.e., only part of these data can be prescribed, and the rest have to be determined at each step of the iteration.

We take the divergence of equation (33). If ζ were equal to $\text{curl } v$, we would get 0 (to see this, recall that (33) was derived by taking the curl of (6)). Hence we find (we set $\text{curl } v = \omega$)

$$(38) \quad \frac{\partial}{\partial x_i} \left[\rho((v \cdot \nabla) + \lambda)(v \cdot \nabla)(\zeta_i - \omega_i) - \mu \Delta(\zeta_i - \omega_i) \right. \\ \left. - (T_{kj} - P_{kj}(\mathbf{T})) \frac{\partial^2(\zeta_i - \omega_i)}{\partial x_j \partial x_k} + \epsilon_{mli} \epsilon_{krs} P_{mk}(\mathbf{T}) \frac{\partial^2(\zeta_s - \omega_s)}{\partial x_l \partial x_r} \right] = 0.$$

After some algebra, this yields

$$(39) \quad \left[\rho((v \cdot \nabla) + \lambda)(v \cdot \nabla) - \mu \Delta - (T_{kj} - P_{kj}(\mathbf{T})) \frac{\partial^2}{\partial x_j \partial x_k} \right] \text{div } \zeta \\ = D^2(v, \nabla v, \nabla^2 v, \mathbf{T}, \nabla \mathbf{T})(\zeta - \omega).$$

Here D^2 is a second order differential operator with coefficients depending on the arguments indicated. Let d_1 denote the value of $\text{div } \zeta$ at $x_1 = 0$, and let d_2 denote the value of $\frac{\partial}{\partial x_1} \text{div } \zeta$ at $x_1 = 0$. From (39), we obtain the estimate

$$(40) \quad \|\text{div } \zeta\|_{0,1} + \|\text{div } \zeta\|_{1,0} \leq C(\|d_1\|_{\langle 1 \rangle} + \|d_2\|_{\langle 0 \rangle}) + (\|w\|_4 + \|\mathbf{T}\|_3) \|\zeta - \omega\|_2.$$

As before, w denotes $v - (V, 0, 0)$. We note that $\|w\|_4 + \|\mathbf{T}\|_3$ is small. Moreover, (36) yields the estimate

$$(41) \quad \|\zeta - \omega\|_2 \leq C \|\text{div } \zeta\|_1 + \left| \int_0^L \int_0^M \zeta_1(0, x_2, x_3) dx_3 dx_2 \right|.$$

Inflow conditions are now handled as follows. We prescribe arbitrary data for ζ_2 , ζ_3 and their normal derivatives. The initial datum for $\frac{\partial \zeta_1}{\partial x_1}$ is then determined by requiring that $\text{div } \zeta = 0$ at $x_1 = 0$. Finally, the initial datum for ζ_1 cannot be determined a priori, but must be computed at each step of the iteration. We require that, at $x_1 = 0$,

$$(42) \quad \frac{\partial^2 \zeta_1^{n+1}}{\partial x_1^2} + \frac{\partial^2 \zeta_2^{n+1}}{\partial x_2 \partial x_1} + \frac{\partial^2 \zeta_3^{n+1}}{\partial x_3 \partial x_1} = s_{n+1},$$

where s_{n+1} is a constant to be determined. We then solve (42) for $\partial^2 \zeta_1^{n+1} / \partial x_1^2$ and substitute into the first equation of (34). This yields an elliptic problem from which we

can uniquely determine the inflow datum for ζ_1^{n+1} up to an arbitrary constant, as well as the constant s_{n+1} . Finally, the arbitrary constant in ζ_1^{n+1} is fixed by the requirement that

$$(43) \quad \int_0^L \int_0^M \zeta_1^{n+1}(0, x_2, x_3) dx_3 dx_2 = 0.$$

For the limit of the iteration, this obviously insures that $d_1 = 0$, $d_2 = s$, and $\int_0^L \int_0^M \zeta_1(0, x_2, x_3) dx_3 dx_2 = 0$. To determine the constant s , we take the first equation of (33), set $x_1 = 0$, and integrate over x_2 and x_3 . If ζ is replaced by $\omega = \text{curl } v$, we obtain an expression which vanishes identically (recall that (33) was derived by taking the curl of (6)). Hence we find

$$(44) \quad \int_0^L \int_0^M \left[\rho((v \cdot \nabla) + \lambda)(v \cdot \nabla)(\zeta_1 - \omega_1) - \mu \Delta(\zeta_1 - \omega_1) \right. \\ \left. - (T_{kj} - P_{kj}(\mathbf{T})) \frac{\partial^2(\zeta_1 - \omega_1)}{\partial x_j \partial x_k} \right. \\ \left. + \epsilon_{ml1} \epsilon_{krs} P_{mk}(\mathbf{T}) \frac{\partial^2(\zeta_s - \omega_s)}{\partial x_l \partial x_r} \right] (0, x_2, x_3) dx_3 dx_2 = 0.$$

Next we integrate by parts in all terms which involve second order derivatives of $\zeta - \omega$ such that one of the differentiations is with respect to x_2 or x_3 . This leads to terms which can be estimated by a constant times $(\|\mathbf{T}\|_3 + \|w\|_4) \|\zeta - \omega\|_2$. The term which remains is the integral of

$$(45) \quad \rho v_1^2 \frac{\partial^2}{\partial x_1^2} (\zeta_1 - \omega_1) + \lambda \rho v_1 \frac{\partial}{\partial x_1} (\zeta_1 - \omega_1) - \mu \frac{\partial^2}{\partial x_1^2} (\zeta_1 - \omega_1) \\ - (T_{11} - P_{11}(\mathbf{T})) \frac{\partial^2}{\partial x_1^2} (\zeta_1 - \omega_1).$$

We now note that

$$(46) \quad \frac{\partial}{\partial x_1} (\zeta_1 - \omega_1) = -\frac{\partial}{\partial x_2} (\zeta_2 - \omega_2) - \frac{\partial}{\partial x_3} (\zeta_3 - \omega_3), \\ \frac{\partial^2}{\partial x_1^2} (\zeta_1 - \omega_1) = -\frac{\partial^2}{\partial x_2 \partial x_1} (\zeta_2 - \omega_2) - \frac{\partial^2}{\partial x_3 \partial x_1} (\zeta_3 - \omega_3) + s.$$

By using this, we obtain again terms which can, after an integration by parts, be estimated by a constant times $(\|\mathbf{T}\|_3 + \|w\|_4) \|\zeta - \omega\|_2$, plus s times the integral of $\rho v_1^2 - \mu - T_{11} + P_{11}(\mathbf{T})$.

As a result, s can be estimated by a constant times $(\|\mathbf{T}\|_3 + \|w\|_4)\|\zeta - \omega\|_2$. In conjunction with (40) and (41) this yields that $\operatorname{div} \zeta$ is indeed zero.

To make sure that (1) is satisfied, we proceed as in [4]. At each step of the iteration, q^{n+1} is determined by the relation

$$(47) \quad \nabla q^{n+1} = \Sigma \left[((v^n \cdot \nabla) + \lambda + (\nabla v^n)^T)(\operatorname{div} \mathbf{T}^n + f - \rho(v^n \cdot \nabla)v^n) \right].$$

Here Σ is the orthogonal projection of L^2 onto the subspace of vectorfields with vanishing curl. From (47), q^{n+1} is determined up to an arbitrary constant; we may fix this constant by requiring that

$$(48) \quad \int_0^1 \int_{-L/2}^{L/2} \int_{-M/2}^{M/2} q^{n+1}(x_1, x_2, x_3) dx_3 dx_2 dx_1 = 0.$$

We note that q^{n+1} is not necessarily periodic in the x_2 and x_3 -directions, but may contain a part which is linear in x_2 and x_3 . At $x_1 = 0$, we impose the condition

$$(49) \quad \rho(v^n \cdot \nabla)v^n = \operatorname{div} \mathbf{T}^{n+1} - \nabla p^{n+1} + f.$$

This condition, in conjunction with (13) and the equation

$$(50) \quad (v^n \cdot \nabla)p^{n+1} + \lambda p^{n+1} = q^{n+1}$$

can be used to express some of the inflow data for \mathbf{T} in terms of others. For details we refer to [4]. Specifically we can prescribe the stress components specified in (5) and solve for the rest.

Let us summarize the iteration scheme. We prescribe the following boundary data a priori: the normal velocities on both boundaries and the total flux in the x_2 - and x_3 -directions according to (35); the second and third components of the vorticity and their normal derivatives at the inflow boundary,

$$(51) \quad \begin{aligned} \zeta_2(0, x_2, x_3) &= \zeta_2^0(x_2, x_3), \quad \zeta_3(0, x_2, x_3) = \zeta_3^0(x_2, x_3), \\ \frac{\partial \zeta_2}{\partial x_1}(0, x_2, x_3) &= \zeta_2^1(x_2, x_3), \quad \frac{\partial \zeta_3}{\partial x_1}(0, x_2, x_3) = \zeta_3^1(x_2, x_3), \\ \frac{\partial \zeta_1}{\partial x_1}(0, x_2, x_3) &= -\frac{\partial \zeta_2^0}{\partial x_2}(x_2, x_3) - \frac{\partial \zeta_3^0}{\partial x_3}(x_2, x_3); \end{aligned}$$

and the inflow stresses according to (5). We denote this prescribed part of the stress by \mathbf{T}_p . We start the iteration by setting $\mathbf{T} = 0$, $\zeta = 0$, $v = (V, 0, 0)$. At each step of the iteration, we first determine q^{n+1} from (47), (48). Then we calculate the inflow boundary value of \mathbf{T}^{n+1} from (5), (49), (50) and (13). We can now determine \mathbf{T}^{n+1} from (13). Next we use (42), (43) and (34) to determine the inflow value of ζ_1^{n+1} . Then we determine ζ^{n+1} from (34) and v^{n+1} from (36).

The existence theorem thus obtained is the following:

THEOREM: Assume that $\|f\|_4$, $\|a\|_{(7/2)}$, $\|b\|_{(7/2)}$, $\|\zeta_2^0\|_{(3)}$, $\|\zeta_3^0\|_{(3)}$, $\|\zeta_2^1\|_{(2)}$, $\|\zeta_3^1\|_{(2)}$, $\|\mathbf{T}_p\|_{(3)}$, $|\alpha|$ and $|\beta|$ are sufficiently small. Assume, moreover, that the matrix function \mathbf{P} has symmetric values. Then there is a solution of (1-3) which satisfies the boundary conditions given by (5), (35) and (51) and the regularity $v \in H^4$, $\mathbf{T} \in H^3$. Moreover, this solution is the only one for which $\|v - (V, 0, 0)\|_4$ and $\|\mathbf{T}\|_3$ are small.

REFERENCES

- [1] D.D. JOSEPH, M. RENARDY AND J.C. SAUT, *Hyperbolicity and change of type in the flow of viscoelastic fluids*, Arch. Rat. Mech. Anal., 87 (1985), pp. 213–251.
- [2] M. LUSKIN, *On the classification of some model equations for viscoelasticity*, J. Non-Newt. Fluid Mech., 16 (1984), pp. 3–11.
- [3] J.G. OLDROYD, *Non-Newtonian effects in steady motion of some idealized elastico-viscous liquids*, Proc. Roy. Soc. London, A 245 (1958), pp. 278–297.
- [4] M. RENARDY, *Inflow boundary conditions for steady flow of viscoelastic fluids with differential constitutive laws*, Rocky Mt. J. Math., 18 (1988), pp. 445–453.
- [5] *Recent advances in the mathematical theory of steady flows of viscoelastic fluids*, J. Non-Newt. Fluid Mech., 29 (1988), pp. 11–24.
- [6] *Boundary conditions for steady flows of non-Newtonian fluids*, Proc. Xth Int. Congr. Rheology (ed. P.H.T. Uhlherr), Vol. 2, Sydney, 1988, pp. 202–204.
- [7] I.M. RUTKEVICH, *The propagation of small perturbations in a viscoelastic fluid*, J. Appl. Math. Mech., 34 (1970), pp. 35–50.
- [8] J.S. ULTMAN AND M.M. DENN, *Anomalous heat transfer and a wave phenomenon in dilute polymer solutions*, Trans. Soc. Rheol., 14 (1970), pp. 307–317.

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