PRICES AND BARGAINING IN COOPERATIVE GAMES

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ABSTRACT

This paper introduces a new solution concept for cooperative games without sidepayments, called the aspiration bargaining set, and proves that this set is nonempty for a wide, and economically interesting, class of games. The aspiration bargaining set is based on familiar economic notions, and allows for the prediction of the endogeneous formation of coalitions and the division of payoffs within coalitions.

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1. INTRODUCTION

In many situations in Economics, there are gains from mutual cooperation and conflict over how these gains should be shared. Examples include the allocation of goods in a pure trade economy, the determination of levels of goods and jurisdiction structures in a local public good economy, and the determination of profit shares and firm formation in a coalition production economy. For each of these situations we would like to predict both the coalitions which are likely to form and the rewards that agents would receive for their participation.

Consider, for instance, the problem of allocating goods in a pure trade economy. Gains from trade make formation of various trading coalitions profitable. Each potential trading coalition must decide on the terms of trade within the coalition (and therefore on the resulting utilities of its members). Since the desirability of forming each potential coalition depends on the utilities its members receive from the coalition relative to the utilities they could obtain from other coalitions, one important factor in the negotiations in a given coalition is the trades (and utilities) its members can obtain in other coalitions. Once the negotiations over the trades are complete, agents form those trading coalitions that give them their highest utilities. Our approach to cooperative games allows us to model these situations and predict both the payoffs players receive for their coalitional participation and the coalitions that are likely to form.

We model these situations as characteristic function games without side-payments (NTU games). We view the characteristic function game as a collection of interrelated bargaining problems. The characteristic function specifies the set of attainable utilities available to each potential coalition. In each coalition players negotiate over which attainable utility vector the coalition would select if that coalition were to form. We think of negotiations proceeding in each coalition by each player presenting a payoff demand or "price" for his coalitional participation. A player's price is the minimum utility level he will accept for his participation in the coalition. The bargaining that takes place in each coalition is then bargaining over these prices. Since each agent can actually form at most one coalition, the desirability of forming a particular coalition depends on the prices its members would receive relative to the prices they could obtain from other coalitions. Each coalition's bargaining problem cannot be solved in isolation because the price a player can bargain for in each coalition will depend on the prices he could obtain from other coalitions. In principle, an agent could demand a different price from each potential coalition. However, Bennett (1985) showed that under mild assumptions, Nash (and other) bargaining solutions to the multilateral bargaining problem that arises from characteristic function games have the property that the price each player charges for his coalitional participation is actually the same for each of his potential coalitions.

In this paper we assume that each player sets a (single) price for his participation in any coalition. We use a bargaining criterion to decide which prices players will find acceptable. (Our bargaining criterion has some of the same intuition as the Bargaining Set of Aumann and Maschler (1965).) We call these prices "bargaining aspirations," and call the set of such prices the "aspiration bargaining set."

A number of solution concepts for NTU games have been proposed, the most prominant of which are the core, the NTU value (Shapley, 1969), the Harsanyi bargaining solution (1963, 1977), and the ordinal and cardinal bargaining sets (Asscher, 1976, 1977). These solution concepts take a fundamentally different approach to NTU games than the approach taken here. Each determines, based on fairness or stability notions, the division of gains within a fixed coalition (or fixed family of coalitions). These solution concepts seem appropriate for modelling situations (such as arbitration problems) in which one coalition (or a family of coalitions) has formed before the bargaining over payoff distributions begins. Our approach is appropriate for situations in which players negotiate over their payoffs before committing themselves to particular coalitions. We believe that our approach is a natural one for many economic applications. To quote from Shapley and Shubik (1972, p. 116), discussing two-sided markets: "A prudent 'economic' man playing this game would be loath to enter a partnership for a stated share of the proceeds until he had satisfied himself that more favorable terms could not be obtained elsewhere. We can imagine that each player would set a price on his participation, and that no contracts would be signed until the prices on both sides of each partnership formed are in harmony..." The central question we address is to determine which prices are reasonable prices for players to charge for their coalitional participation.

Our model also allows us to predict in a natural way which coalitions players will find desirable to form. Players set prices for their coalitional participation in light of their opportunities in all coalitions. Once players select prices for their coalitional participation, only those coalitions which can afford to pay their members' prices can possibly form. (A coalition can afford its members prices if there is a utility vector in its set of attainable utility vectors whose components are at least as large as its members' prices.) In principle, even when players charge the same price for their coalitional participation in every coalition, agents could still disagree as to which coalitions are the most desirable ones since some coalitions may be able to pay more than the prices its players charge. However, whenever the price vector is a bargaining aspiration, no player can obtain more than his price in any coalition he can form, given the prices of other players. This means that at any bargaining aspiration price vector, potential coalitions can be divided into those that can exactly afford to pay its members their prices (these are the desirable coalitions) and those that can't. Since the players agree on which coalitions can afford their members' prices, whenever one player wants to form one of these coalitions, all of his partners also find forming this coalition at least as desirable as any of their other potential coalitions. In this way, whenever bargaining leads players to select prices that are bargaining aspirations, players agree on which potential coalitions are desirable to form.

In a recent paper Hart and Kurz (1983) proposed a model of endogeneous coalition formation for sidepayment games which takes a distinctly different approach to predicting which coalitions form. They use a variant of the Shapley value due to Owen to obtain a payoff distribution for each partition of the player set into coalitions. Based on various rules by which players can leave their "current" coalitions to form others, Hart and Kurz determine which coalition structures are stable. One can interpret this approach to coalition formation in the following way. Suppose that players know in advance the payoff distributions for each possible coalition structure. They would then find desirable those coalition structures which yield them their highest payoffs. Unfortunately, there may be no coalition structure that all the players find desirable. Indeed, there may be no coalition structure such that all the players in any one coalition would agree that their coalition is the most desirable coalition. Because of these conflicts of interest, it is not clear how one coalition structure or another might form. In the Hart and Kurz model, payoff distributions do not respond to conflict among agents over which coalitions or coalition structures should form. It seems to us that this approach ignores the role of payoff distributions as decision variables. One should expect payoff distributions to respond to the bargaining over coalition structures. In our model, these conflicts of interest do not arise: whenever one player finds a particular coalition desirable, his partners will also find that coalition desirable. This is so precisely because payoffs in each coalition do adjust to reflect opportunities outside the coalition.

Since we assume that each player can actually form only one coalition at a time, when the NTU game is played, a partition of players into coalitions eventually forms. An outcome of the game is a partition of the player set into coalitions and a payoff distribution which is feasible for these coalitions. Our solution concept predicts the prices players charge for their coalitional participation and the set of coalitions players find desirable to form. A price vector is sometimes, but not always, a payoff distribution for some partition of coalitions. In particular, a bargaining aspiration may not be an imputation because the coalition of the whole may not be able to afford its members' prices. When prices are not feasible payoff distributions for a partition of coalitions, it means that not all players can realize their prices simultaneously. Bargaining aspirations, however, can be used to predict which outcomes would be realized in the play of the game.

The paper is organized in the following way. In section 2 we present the formal model, the existence theorem for bargaining aspirations, and a series of examples. We discuss bargaining aspirations for pure exchange economies and prove that every competitive equilibrium allocation is a bargaining aspiration in section 3. In section 4 we contrast the aspiration bargaining set with the classical bargaining set of Aumann and Maschler. In section 5, we discuss the transition from prices to outcomes. Finally, proofs of the theorems are collected in section 6.

2. NTU GAMES, ASPIRATIONS AND THE ASPIRATION BARGAINING SET

By a game in characteristic function form with nontransferable utility (an NTU game or nonsidepayment game) we mean a pair $\langle N, V\rangle$ where $N = \{1, \ldots, n\}$, (the set of players) is a nonempty set and V (the characteristic function) is a function which assigns to each nonempty subset S of N (a coalition) a compact subset V(S) of $\mathbb{R}^S_+$ which contains the origin and is comprehensive (i.e., if $x \in V(S)$, $y \in \mathbb{R}^S_+$ and $y_i \leq x_i$ for each $i \in S$, then $y \in V(S)$). We are particularly interested in NTU games for which each of the sets V(S) satisfies a slightly stronger property: if $x \in V(S)$, $y \in \mathbb{R}^S_+$, $y_i \leq x_i$ for each $i \in S$ and $y \neq x$, then $y \in \operatorname{int} V(S)$, the interior of V(S) with respect to $\mathbb{R}^S_+$. It seems natural to say that the sets V(S) are strongly comprehensive if they enjoy this property; if each of the sets V(S) is strongly comprehensive, we will say that $\langle N, V\rangle$ is a strongly comprehensive NTU game.* We do not require that the game $\langle N, V\rangle$ be superadditive or that the payoff sets be convex.

If x, y are in $\mathbb{R}^S$ we write $x \leq y$ if $x_i \leq y_i$ for each $i \in S$; we write $x < y$ if $x \leq y$ and $x \neq y$, and $x \ll y$ if $x_i < y_i$ for each i. If T is a subset of S, then by x^T we mean the restriction of x to T (thinking of vectors in $\mathbb{R}^S$ as functions from S to $\mathbb{R}$).

As is usual, we interpret a vector in V(S) to be an allocation of utility to the players of S which the coalition can achieve without the cooperation of players not in S. Comprehensiveness simply allows for free disposal of utility. Strong comprehensiveness asserts that for each feasible utility allocation for a coalition, any player receiving a nonzero utility can improve the allocation for other players by sacrificing some of his own utility. This will be the case, for example, whenever

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* The formalization given above is slightly different from some other formalizations in the literature. It is sometimes posited that V(S) should lie in $\mathbb{R}^S$ rather than $\mathbb{R}^S_+$, but the vectors which are not in the positive orthant generally play no role. Alternatively, it is sometimes posited that V(S) should lie in $\mathbb{R}^N$, but then only the projection into $\mathbb{R}^S_+$ plays a role. We have adopted our formalization because it is natural for our purposes. The arguments in our proofs can be adapted for either of these other formalizations.

there is a perfectly divisible commodity which is always desirable and freely transferable (e.g., money). When V(S) is strongly comprehensive the weak and strong Pareto efficient boundaries of V(S) coincide. That is, if x is in V(S) and there is a vector y in V(S) such that $y > x$ then there exists a vector z in V(S) such that $z \gg x$. Strong comprehensiveness is not primarily a technical assumption: it has implications for the bargaining process. Strong comprehensiveness guarantees that if a player "merits" a payoff increase, he can in fact be given an increase.*

In characteristic function games each coalition has available to it a set of attainable utility vectors from which it must select a single utility vector. Players bargain in the various coalitions over their payoffs. Since each player can eventually form only one coalition, the opportunity cost of forming a specific coalition is the utility that player could, instead, obtain from another coalition. When negotiating payoffs, each coalition does take into account the opportunity costs of its players in the negotiations over itspayoffs in the situations we wish to model. For these situations, from the results of Bennett (1985), we know that, under mild conditions, no matter which bargaining solutions coalitions use to solve their bargaining problems, <u>every</u> solution to such a "multilateral bargaining problem" can be achieved by each player setting and bargaining over a single price (rather than a coalition-specific price) for his coalitional participation. In this paper we study the bargaining over payoffs in coalitions by studying the bargaining over the prices players charge for their coalitional participation.

The solution concept we propose is based on some fundamental notions of how players set "reasonable" prices for their coalitional participation. Each player sets his price with some knowledge or expectation of the prices that other players demand. We expect rational players to raise their prices whenever there would be leftover payoff in any coalition; we therefore expect solution prices to be as large as possible. We call a price vector x <u>maximal</u> if there does not exist a coalition S and a vector y^S in V(S) such that $y^S > x^S$. We also expect that at any solution price vector, no player will price himself out of the market by setting his price so

* Every NTU game is either already strongly comprehensive or else can be approximated as closely as desired by a strongly comprehensive game.

high that no coalition can afford it along with those of its other members. (We say that the coalition S can afford the price vector x if x^S is in V(S).) We say that at x, player i's price is realizable if there exists a coalition that contains player i such that x^S is in V(S). If at x every player's price is realizable we call x realizable. We call vectors that are both realizable and maximal "aspirations" and look for solutions among price vectors that are aspirations.

Formally, a price vector x is an aspiration for the game $\langle N, V\rangle$ if x is both realizable and maximal. That is:

1. Realizability: for each $i \in N$, there is a coalition $S \subseteq N$ with $i \in S$ and $x^S \in V(S)$;
2. Maximality: there does not exist a coalition S and a vector $y^S \in V(S)$ such that $x^S < y^S$.

The set of aspirations for TU (sidepayment) games is defined in Bennet (1983a). Several authors have proposed cooperative solution concepts for sidepayment games that operate on the space of aspirations (see Bennett (1983a) for a review of this literature). Reinhardt Selten (1980) presented a noncooperative model of the play of characteristic function games for a broad class of sidepayment games and showed that aspirations are perfect equilibria of the resulting recursive game.

The following result gives us the information we need about the set of aspirations. For the proof, see Section 6.

THEOREM 2.1: For every strongly comprehensive NTU game $\langle N, V\rangle$, the set $A\langle N, V\rangle$ of aspirations is a continuous retraction of a compact convex subset of $\mathbb{R}^N_+$. In particular, $A\langle N, V\rangle$ is nonempty, compact, and connected. Moreover, every continuous map $f: A\langle N, V\rangle \rightarrow A\langle N, V\rangle$ has a fixed point.

Not all aspirations represent price vectors likely to be realized in actual play. For example, for each player in any given game, there is always at least one aspiration which limits that player's payoff to the maximum utility level he can obtain on his own. If such a player's participation in other coalitions is essential, our intuition about bargaining suggests that such an aspiration is unlikely to be an agreed-upon price vector. To eliminate these and other unlikely price vectors we introduce a bargaining criterion on aspirations; aspirations which satisfy this criterion are called bargaining aspirations.

To introduce this bargaining criterion, consider a proposed price vector x, a coalition S which can afford this price, and two players i, j in S. Suppose that every coalition which can afford the price x and contains player i also contains player j. Then player i cannot achieve his payoff without the cooperation of player j. If, on the other hand, player j belongs to a coalition which can afford x and does not contain player i, then player j can achieve his payoff without the cooperation of player i. Player i is thus in a vulnerable position vis-a-vis player j (and we say that player i is vulnerable at the price x): player j could demand a greater share of the profits from forming the coalition S, and threaten not to agree to cooperate in the coalition S otherwise. Since such a threat would entail a real loss to player i (without loss to player j), player i would be well-advised to accede to such a demand. The price vector x is therefore not stable under this kind of bargaining. Aspirations at which no player is vulnerable (and which therefore are stable under this kind of bargaining) are called bargaining aspirations. Before giving a formal definition, we need some additional notation.

If $\langle N, V \rangle$ is an NTU game, and $x \in \mathbb{R}_+^N$, we define the generating collection $GC(x)$ to be the set of coalitions $S \subseteq N$ such that $x^S \in V(S)$. For some values of x, the generating collection $GC(x)$ may be empty. If $S \in GC(x)$ we sometimes say that S can afford x. For $i \in N$, we write $GC_i(x)$ for the set of coalitions in $GC(x)$ which contain i.

If $x \in A\langle N, V \rangle$, we say that player i is vulnerable (to player j) at x if $GC_i(x) \subset GC_j(x)$ but $GC_j(x) \not\subset GC_i(x)$ (equivalently, $GC_i(x) \subsetneqq GC_j(x)$). That is, player i is vulnerable to player j at x if every coalition which can afford x and contains i also contains j (so that i needs the cooperation of j) but there is at least one coalition which can afford x and contains j but not i (so that j does not need the cooperation of i). Let us note a simple fact: vulnerability is a transitive relation on players. That is, if i is vulnerable to j at x (so that $GC_i(x) \subsetneqq GC_j(x)$ and j

is vulnerable to k at x (so that $GC_j(x) \subsetneq GC_k(x)$) then i is vulnerable to k at x (because $GC_i(x) \subsetneq GC_k(x)$).

If no player is vulnerable at x, we say that x is a bargaining aspiration. We write $AB\langle N, V\rangle$ for the set of bargaining aspirations; we sometimes refer to $AB\langle N, V\rangle$ as the aspiration bargaining set.

THEOREM 2.2: For every strongly comprehensive NTU game $\langle N, V\rangle$, the set $AB\langle N, V\rangle$ of bargaining aspirations is nonempty.

We refer the reader to Section 6 for the proof.

The aspiration bargaining set defined here for NTU games coincides with the aspiration bargaining set (Bennett, 1983c) and the set of partnered aspirations (Bennett, 1983a) for the class of TU (sidepayment) games.*

Given that the prices players charge for their coalitional participation are bargaining aspirations, we turn our attention to determining the corresponding outcomes of the game. Since players can form only one coalition "at a time," an outcome of the game is a pair $(z, \mathfrak{J})$ consisting of a partition $\mathfrak{J}$ of the player set N into disjoint coalitions (whose union is N), and a vector $z \in \mathbf{R}_+^N$ with the property that $z^S \in V(S)$ for each $S \in \mathfrak{J}$. We call $\mathfrak{J}$ a coalition structure and z a payoff distribution which is feasible for the coalition structure $\mathfrak{J}$.

We discuss at length the transition from prices to outcomes in Section 5; in this section we discuss only the easy case of simple games. We call a game simple if every two profitable coalitions overlap and no superset of a profitable coalition is also profitable. This means that no two coalitions can simultaneously form and obtain a positive profit and that once one coalition has formed no profit can be gained by adding members. Formally, the game $\langle N, V\rangle$ is a simple game if whenever S and T are coalitions with both $V(S) \neq \{0\}$ and $V(T) \neq \{0\}$, then $S \cap T \neq \emptyset$

* Recently Binmore (1985) formulated a noncooperative extensive form game model of the play of three player NTU games where the coalition of the whole earns nothing. The price vectors corresponding to the perfect equilibria of the extensive form game are always contained in the closure of the set of bargaining aspirations.

and $(N - S) \cap T \neq \emptyset$. For ease of notation we let $\langle N, V \rangle$ be 0-normalized (i.e., $V(i) = \{0\}$ for all i).

If $\langle N, V \rangle$ is a simple game we can specify an outcome by merely specifying the single nontrivial coalition that forms, call it S, and a payoff distribution, z, for all the players in N such that the players in S obtain a payoff distribution which is feasible for S and the players not in S earn nothing. Formally, an outcome of the game $\langle N, V \rangle$ is a pair (z, S) such that z^S is in V(S) and $z_i = 0$ if i is not in S.

If x is a bargaining aspiration for the simple game $\langle N, V \rangle$, we say that an outcome (z, S) is consistent with x if all members of S obtain their prices in x, i.e., $z^S = x^S$.* We frequently refer to such an outcome, (z, S), as a bargaining aspiration outcome or bargaining outcome.

At times we wish to compare bargaining aspiration outcomes with core outcomes or other solutions that operate on superadditive games. For each game $\langle N, V \rangle$ let $\langle N, V^* \rangle$ be the superadditive cover of $\langle N, V \rangle$.** For simple games the bargaining aspirations and bargaining aspiration outcomes of a game and its superadditive cover are identical. In general, for every pair of games $\langle N, V \rangle$ and $\langle N, V^* \rangle$, the set of bargaining aspirations is the same, and, if (z, S) is an outcome for $\langle N, V^* \rangle$ then (z, S) is an outcome for $\langle N, V \rangle$ or else there exist disjoint subsets of S, call then S^k for $k = 1, \ldots, K$, such that (z^k, S^k) is an outcome for $\langle N, V \rangle$, where z^k is the restriction of z to S^k.

Three examples are presented to make these ideas clearer. We begin with a simple production problem.

EXAMPLE 2.3: Consider the problem of 4 agents who can produce a single good with market value of $1 per unit. Production requires three agents to run the necessary machinery—two agents can produce nothing and a fourth adds nothing. The agents have different skill levels, with correspondingly different productivities, given by

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* From the definition of aspiration it follows that z^S is Pareto optimal for S.

** In the superadditive cover game $V^*(S)$ contains all the vectors in V(S) and in addition all the vectors which can be attained by a partition of the subsets of S.

the vector $w = (10, 20, 30, 40)$. For example, the coalition $[1,2,3]$ can produce 60 units.

This situation is most naturally modeled as a simple (sidepayment) game $\langle N, V\rangle$ with $N = \{1,2,3,4\}$ and V given by

$$V(i,j,k) = \left\{(x_i, x_j, x_k) \in \mathbf{R}_+^{\{i,j,k\}} \mid x_i + x_j + x_k \leq w_i + w_j + w_k\right\}$$

$$V(S) = \{0\} \text{ for all other coalitions } S.$$

The unique bargaining aspiration for this game is $x = (10, 20, 30, 40)$, which coincides with the productivity vector w. Thus each player should demand a price equal to his productivity. Since each of the nontrivial coalitions can afford the price vector x, we expect that any of them might form and devide its profit according to the price vector x. The predicted bargaining outcomes are: $((10, 20, 30, 0), [1,2,3])$, $((10, 20, 0, 40), [1,2,4])$, $((10, 0, 30, 40), [1,3,4])$, $((0, 20, 30, 40), [2,3,4])$.

The bargaining aspiration predictions may seem almost trivial, given the structure of payoffs. However, other solution concepts make less intuitive predictions for this economy. Let $\langle N, V^*\rangle$ be the superadditive cover of this game (recall that this does not change the bargaining aspiration predictions). The core of this game is empty. The Shapley value, $90/108\ (21, 25, 29, 33)$, gives each agent almost an equal share of the output while the nucleolus, $(7.5, 17.5, 27.5, 37.5)$ taxes each player 2.5 units of output in order to make the payoff vector feasible. The fundamental reason these solution concepts predict payoffs that differ from agents' productivities is that imputation-based solutions are seeking reasonable allocations of payoff for the coalition of the whole, whether or not this coalition would actually form. Since in the superadditive cover game, N can produce only 90 units, its members cannot be paid according to their productivities. By focusing on prices instead of on payoff distributions for a given coalition structure, bargaining aspirations make sensible predictions of both prices and outcomes.

The next example points out that bargaining aspirations may predict that certain coalitions, although potentially profitable, will not form.

EXAMPLE 2.4: We let N = 1, 2, 3, 4, 5 and let V be given by:

$$V[1,2,3] = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leqslant 20\}$$

$$V[2,3,4] = \{(x_2, x_3, x_4) : x_2 + x_3 + x_4 \leqslant 20\}$$

$$V[3,4,5] = \{(x_3, x_4, x_5) : x_3 + x_4 + 2x_5 \leqslant 20\}$$

$$V[1,4,5] = \{(x_1, x_4, x_5) : x_1 + x_4 + 3x_5 \leqslant 20\}$$

$$V[1,2,5] = \{(x_1, x_2, x_5) : x_1 + x_2 + 2x_5 \leqslant 20\}$$

$$V[1,3,5] = \{(x_1, x_3, x_5) : x_1 + x_3 + x_5 \leqslant 12\}$$

$$V(S) = \{0\} \text{ for all other coalitions } S .$$

This simple game has the unique bargaining aspiration $x = (4, 8, 8, 4, 4)$. The coalitions [1, 2, 3], [2, 3, 4], [3, 4, 5], [1, 4, 5], [1, 2, 5] can all afford this price, but the coalition [1, 3, 5] cannot. Thus the bargaining outcomes which are predicted are:

((4, 8, 8, 0, 0), [1, 2, 3]), ((0, 8, 8, 4, 0), [2, 3, 4]),

((0, 0, 8, 4, 4), [3, 4, 5]), ((4, 0, 0, 4, 4), [1, 4, 5]),

((4, 8, 0, 0, 4), [1, 2, 5]),

but it is predicted that the coalition [1, 3, 5] will <u>not</u> form.

EXAMPLE 2.5: Roth's Example. For each real number p, $0 \leqslant p < 1/2$, we define a three-person game as follows:

$$V([1]) = V([2]) = V([3]) = \{0\} ,$$

$$V([12]) = \{(x_1, x_2) \mid 0 \leqslant x_1 \leqslant 1/2,\ 0 \leqslant x_2 \leqslant 1/2\} ,$$

$$V([13]) = \{(x_1, x_3) \mid 0 \leqslant x_1 \leqslant p,\ 0 \leqslant x_3 \leqslant 1-p\} ,$$

$$V([23]) = \{(x_2, x_3) \mid 0 \leqslant x_2 \leqslant p,\ 0 \leqslant x_3 \leqslant 1-p\} ,$$

V([123]) = convex comprehensive hull of the points:

(1/2, 1/2, 0), (p, 0, 1 - p), (0, p, 1 - p) .

Consider the game for any value of p strictly less than 1/2. By forming the coalition [1, 2] players 1 and 2 can each obtain 1/2. By forming a two-player coalition with player 3, players 1 and 2 would each obtain p which is less than 1/2. Clearly, it is in both 1 and 2's best interest to form the coalition [1, 2] to obtain the payoff distribution (1/2, 1/2) or else to form the coalition [1, 2, 3] if player 3 will agree to the payoff distribution (1/2, 1/2, 0).

Given the very clear incentives in the game, it is not surprising that the unique bargaining aspiration for any p in the given range is (1/2, 1/2, 0). The two outcomes consistent with this aspiration are (1/2, 1/2, 0) for the coalition [1, 2] and (1/2, 1/2, 0) for the coalition [1, 2, 3].

Two traditional solution concepts for NTU games make a less obvious prediction of the payoffs. Roth (1980) showed that the NTU value (Shapley, 1969) and the Harsanyi bargaining solution (Harsanyi (1963), (1967)) both predict the payoff distribution of (1/3, 1/3, 1/3) for all games with the parameter p in the interval $1/4 < p < 1/2$. (See, for example, Roth (1980) for a clear definition of both the NTU value and the Harsanyi bargaining solution.)

3. PURE EXCHANGE ECONOMIES

In this section, we show that utility vectors corresponding to competitive equilibria of pure exchange economies are always bargaining aspirations. This might suggest that core imputations are also always bargaining aspirations (since competitive equilibria are in the core, and the core is a generalization of the set of competitive equilibria). However, we give an example to show that this is not so, even in the setting of pure exchange economies. The example we give is chosen to point out that some core allocations are unlikely outcomes of free trade. We also discuss an example due to Shafer (1980), for which the allocations corresponding to the NTU value and the Harsanyi bargaining solution are unlikely outcomes of free trade. By contrast, the allocations corresponding to bargaining aspirations are natural ones for the economy.

Consider an exchange ecomomy $\mathcal{E}$ with a set of consumers $N = \{1, 2, \ldots, n\}$ and M completely divisible commodities. For simplicity, we assume that the consumption set of each consumer is the positive cone $\mathbf{R}_+^M$. Each consumer is characterized by a utility function $u_i : \mathbf{R}_+^M \to \mathbf{R}_+^M$ (which we assume to be continuous, strictly monotone and strictly quasiconcave) and an initial endowment $\omega_i \in \mathbf{R}_+^M$. An <u>allocation</u> for $\mathcal{E}$ is an n-tuple $g = (g_1, \ldots, g_n)$ of vectors in $\mathbf{R}_+^M$; g is <u>feasible</u> if $\sum_{i \in N} g_i = \sum_{i \in N} \omega_i$. As usual, an allocation g is <u>competitive</u> if it is feasible and there is a price π (a linear functional on $\mathbf{R}^M$ such that:

$$\pi g_i \leq \pi\omega_i \qquad \text{for each } i$$

and if

$$u_i(h_i) > u_i(g_i) \quad ,$$

then

$$\pi h_i > \pi \omega_i \quad .$$

We call the pair (g, π) a <u>competitive equilibrium</u>, we call π an <u>equilibrium price</u>, and we call $(u_1(g_1), \ldots, u_n(g_n))$ a <u>utility allocation.</u>

To the exchange economy $\mathcal{E}$ we associate a game $\langle N, V_{\mathcal{E}} \rangle$ in the usual way. That is, for each S in N, we let $V_{\mathcal{E}}(S)$ be the subset of $\mathbf{R}_{+}^{S}$ consisting of all attainable utility vectors for the coalition (i.e., $x \in V_{\mathcal{E}}(S)$ if and only if there is an s-tuple ($s = |S|$), $(g_1, \ldots, g_s)$ in $\mathbf{R}_{+}^{M}$, such that $\sum_{i \in S} g_i = \sum_{i \in S} \omega_i$ and $u_i(g_i) \geq x_i$ for each i in S.

THEOREM 3.1: If $\mathcal{E}$ is an exchange economy as above and $(g_1, \ldots, g_n)$ is a competitive allocation of $\mathcal{E}$, then the utility allocation $(u_1(g_1), \ldots, u_n(g_n))$ is a bargaining aspiration.

For the class of pure trade economies the core is nonempty and contains the competitive utility allocations. However, as we argue in the following example, not every point in the core necessarily corresponds to reasonable trades for players to agree to. We show that such a core point is not a bargaining aspiration.

EXAMPLE 3.2: Consider a two-person, two-good economy with utility functions:

$$u_1(g_1, g_2) = u_2(g_1, g_2) = \sqrt{g_1} + \sqrt{g_2}$$

and initial endowments:

$$\omega_1 = (1,0) \quad \text{and} \quad \omega_2 = (0,1) \ .$$

The allocation $g^* = ((1/4,\ 1/4),\ (3/4,\ 3/4))$ corresponds to the utility allocation $u^* = (1, \sqrt{3})$ which is in the core of $V_{\mathcal{E}}$, but is not a bargaining aspiration. To see why g^* is not a bargaining aspiration notice that agent 2 is absorbing all the gains from trade, leaving agent 1 no better off than at his initial endowment. It is not at all clear that player 1 would ever agree to trade to achieve this allocation. The allocation g^* is not a competitive allocation; the unique competitive allocation for this economy is $g = ((1/2,\ 1/2),\ (1/2,\ 1/2))$, which does correspond to a bargaining aspiration utility allocation.

The next example, taken from Shafer (1980), is a pure trade economy for which the bargaining aspirations and the bargaining aspiration allocations select the intuitively plausible outcomes while both the NTU value allocation and the Harsanyi value allocation select intuitively unlikely outcomes.

EXAMPLE 3.2: Shafer (1980) presented this example* of a pure exchange economy. There are three agents labelled 0, 1, and 2 and two goods labelled x and y. The agents have the following utility functions and initial endowments:

$$u_0(x_0, y_0) = \frac{1}{2}(x_0 + y_0) \qquad \omega_0 = (\varepsilon, \varepsilon) ,$$

$$u_1(x_1, y_1) = (x_1 y_1)^{1/2} \qquad \omega_1 = (1-\varepsilon, 0) ,$$

$$u_2(x_2, y_2) = (x_2 y_2)^{1/2} \qquad \omega_2 = (0, 1-\varepsilon) .$$

The resulting characteristic function game is given by:

$$V(0) = \left\{u \in \mathbb{R}^1_+ \,\middle|\, u \leqslant \varepsilon\right\}$$

$$V(1) = V(0) = \{0\}$$

For $j = 1$ or 2:

$$V(0,j) = \left\{(u_0, u_j) \in \mathbb{R}^2_+ \,\middle|\, (u_0, u_j) \leqslant \left(\frac{1-\alpha}{2}, (\alpha\varepsilon)^{1/2}\right) \text{ for } \varepsilon \leqslant \alpha \leqslant 1 \text{ or } (u_0, u_j) \leqslant \left(\frac{1+\varepsilon-2\alpha}{2}, \alpha\right) \text{ for } 0 \leqslant \alpha \leqslant \varepsilon \right\} ,$$

$$V(1,2) = \left\{(u_1, u_2) \in \mathbb{R}^2_+ \,\middle|\, u_1 + u_2 \leqslant 1 - \varepsilon\right\}$$

$$V(0,1,2) = \left\{(u_0, u_1, u_2) \in \mathbb{R}^3_+ \,\middle|\, u_0 + u_1 + u_2 \leqslant 1\right\} .$$

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* For simplicity we consider the case from Example 2 of Shafer (1980) where $p = 0$, $\beta = 1$ and $\varepsilon < 1/2$. The same qualitative results hold for his other parameter values.

In this economy there are gains from trade in any two-agent coalition. Agents 1 and 2 can together exhaust the gains from trade. Since there are utility vectors in $V(1,2)$ which give both agents a higher utility than either can earn with player 0, we expect agents 1 and 2 to form a trading coalition leaving agent 0 to consume his initial endowment.

For this economy, there is a range of bargaining aspirations and corresponding allocations. Every bargaining aspiration corresponds to trade only between agents 1 and 2, with agent 0 consuming only his initial endowment. Given the utility functions of agents 1 and 2, efficiency requires that each of their final allocations contain equal amounts of the two goods. The nonvulnerability condition implies that agent 0's component of any bargaining aspiration must be ε (his utility for his initial endowment), and puts a bound on the asymmetry of the terms of trade between agents 1 and 2. The result is that the set of bargaining aspirations is:

$$AB\langle N,V\rangle = \left\{(\varepsilon,\alpha,\beta) \,\middle|\, \alpha+\beta = 1-\varepsilon,\ \alpha > (\varepsilon - 2\varepsilon^2),\ \beta > (\varepsilon - 2\varepsilon^2)\right\}.$$

For this economy the bargaining aspirations allocations are the reasonable outcomes of trade. By contrast, Shafer showed that the allocation of goods corresponding to the NTU value (the NTU value allocation) gives agent 0 more of every good then he has in his initial endowment. This would mean that somehow in trade agent 0 obtains goods without giving up any goods. Such an outcome seems implausible in an economy with free trade. This and other Shafer examples show that this anomaly is not an isolated problem. Shafer's examples make clear that the value allocation should not be interpreted as a reasonable outcome of free trade for pure exchange economies. Hart (1983) showed that the allocation corresponding to the Harsanyi bargaining solution is also subject to the same anomaly: agent 0 obtains more of each good in the Harsanyi allocation than in his initial endowment (but less than he does in the NTU value allocation).

Remark: Shafer describes this pure exchange economy as a sidepayment game (the value of a coalition is the maximum total utility the coalition can obtain for some distribution of the coalition's total initial endowment) while we analyze the corresponding NTU game. (Only the payoffs of the coalitions [0, 1] and [0, 2] differ between the TU and NTU versions.) Shafer suggests (p. 473) that the sidepayment game may not adequately reflect the worth of the various coalitions. In particular, it overestimates the utility player 1 (or 2) could obtain if he formed a coalition with player 0 and player 0 were willing to accept the utility of his initial endowment. Although the sidepayment game doesn't correctly represent the opportunities of the coalitions [0, 1] and [0, 2], the bargaining aspirations of the sidepayment game don't make the same mistakes as the NTU value or Harsanyi value allocation. Even in the sidepayment version, player 0 never receives more goods than in his initial endowment. The bargaining aspiration allocations give qualitatively the same results as for the NTU case; player 0 doesn't trade, while 1 and 2 trade so as to obtain utilities higher than they could receive with player 0 (given that player 0 receives the utility of his initial endowment). The effect of using the TU values for the coalitions [0, 1] and [0, 2] is to overestimate the utilities that 1 and 2 could obtain in these coalitions. This results in a narrower range of bargaining aspirations than in the NTU case.

4. THE BARGAINING SET AND THE ASPIRATION BARGAINING SET

The bargaining notion underlying the aspiration bargaining set is reminiscent of the bargaining notion underlying the bargaining set of Aumann and Maschler. In this section we present two examples to make clear the distinction between these solution concepts.

Aumann and Maschler (1964) introduced the bargaining set (defined below) for sidepayment games. Although the bargaining set for sidepayment games is nonempty, for NTU games the bargaining set is sometimes empty. Asscher (1976, 1977) extended the definition of bargaining set (to allow chains of objections and counterobjections), called the resulting solution concepts the cardinal and ordinal bargaining sets, and proved the existence of both solutions for NTU games. Since for TU games all three bargaining sets coincide, to point out the crucial differences between the various bargaining sets and the aspiration bargaining set we consider only TU games.

The bargaining set is based on the notion of a justified objection. A player has an objection against another player at one outcome if the player can earn more in another coalition, without the cooperation of the other, while paying his partners at least as much as they would earn in their original coalitions. The second player has a counterobjection against the first if he can maintain his payoff in another coalition, without the cooperation of the first player, while paying his partners at least as much as they could otherwise earn in either their original coalitions or in the objection. If a player has no counterobjection the objection is called a justified objection. If at some outcome, one player has a justified objection against another, then the second player is in a relatively weak bargaining position and would presumably agree to a higher payoff for the first player. An outcome is in the bargaining set if no player has a justified objection against another.

Formally, let $\langle N, v\rangle$ be a TU game and let $(z, \mathcal{J})$ be an outcome for $\langle N, v\rangle$. (Recall that since $(z, \mathcal{J})$ is an outcome, $z(S) \leq v(S)$ for every S in $\mathcal{J}$.) If i and j belong to the same coalition S in $\mathcal{J}$, an <u>objection</u> of i against j is a pair (w, U) where U is a coalition conatining i but not containing j and $w \in \mathbb{R}^U$ is a payoff distribution for U (i.e., $w(U) \leq v(U)$) that satisfies:

$$w_i > z_i \ , \ \text{for player } i$$

$$w_k \geqslant z_k \ , \ \text{for all } k \in U \ .$$

A counterobjection to the objection (w, U) is a pair (w', U') where U' is a coalition containing player j but not containing player i and $w' \in \mathbb{R}^{U'}$ is a payoff distribution for U' such that

$$w'_j \geqslant z_j$$

$$w'_k \geqslant w_k \ , \ \text{for } k \in U' \cap U$$

$$w_k \geqslant z_k \ , \ \text{for } k \in U' \ .$$

If I has an objection against j for which there is no counterobjection, then the objection is called a justified objection. The payoff distribution z for the coalition structure $\mathcal{J}$ is in the bargaining set if z is an individually rational payoff distribution for $\mathcal{J}$ (i.e., $z_i \geqslant v(i)$ and $z(S) = v(S)$ for every $S \in \mathcal{J}$) and no player has a justified objection against another player at $(z, \mathcal{J})$.

In the bargaining set, a player has a justified objection against another if the first player can earn more without the second player while the second cannot maintain his payoff without the first. In the aspiration bargaining set a player is vulnerable to another player if the first player can maintain his payoff without the second while the second cannot maintain his payoff without the first. The fundamental difference between the two solution concepts concerns the amounts that must be paid to partners in their alternative coalitions. The aspiration bargaining set requires that partners be paid the prices they demand for their coalitional participation while in an objection, the bargaining set requires that partners be paid as much as they earn in the given outcome and in a counterobjection partners must be paid as much as they would earn in either the original outcome or in the objection.

This difference over the payoffs that must be offered to potential partners results in different predictions of final outcomes. We present two examples to clarify these differences. In the first example, the bargaining set admits unintuitive payoff distributions as solutions while the aspiration bargaining set picks only the intuitively correct answer. In the second example, predicted payoffs of the two solution concepts are disjoint and are each intuitive answers for different situations.

EXAMPLE 4.1: A 13-player Voting Game. Consider a 13-player majority rule voting game, $N = \{1, \ldots, 13\}$. Any coalition with at least 7 players is a winning coalition. Let the value of any winning coalition be 1. The characteristic function is given by $v(S) = 1$ if $|S| \geqslant 7$ and $v(S) = 0$ otherwise.

The unique bargaining aspiration for the game assigns a price of 1/7 to each player. Given these prices we can predict that one of the minimal winning coalitions will form and equally divide the available utility. In particular, if the coalition $[1, \ldots, 7]$ formed the unique payoff distribution consistent with this bargaining aspiration is $(1/7, \ldots, 1/7, 0, \ldots, 0)$ for the coalition structure $\{[1, \ldots, 7], [8], \ldots, [13]\}$. This outcome is also in the bargaining set of this game. However, the bargaining set for the coalition structure $\{[1, \ldots, 7], [8], \ldots, [13]\}$ also contains counterintuitive payoff distributions, including $(1/4, 1/4, 1/4, 1/4, 0, \ldots, 0)$. In this payoff distribution three of the players who are contributing their votes obtain nothing for their cooperation even though they could instead form a different winning coalition without the cooperation of any of the "greedy" players. To see why this payoff distribution is nevertheless in the bargaining set, suppose player 5 objected to player 4. The most potent objection player 5 can make gives 5 only an ε payment while equally dividing the remaining payoff among all the players with 0 payoff. The resulting objection is $(0, 0, 0, 0, \varepsilon, (1-\varepsilon)/8, \ldots, (1-\varepsilon)/8)$ for the coalition structure $\{[1], \ldots, [4], [5, \ldots, 13]\}$. Player 4, however, has a counterobjection because he can maintain his payoff of 1/4 without the cooperation of 5 by forming a coalition with 6 of the players with the lowest payoff, paying them each 1/8. Since player 4 can maintain his payoff without the cooperation of 5, player 5's objection is not justified. Since there are no justified objections against it, $(1/4, 1/4, 1/4, 1/4, 0, \ldots, 0)$ is in the bargaining set.

EXAMPLE 4.2: Consider the sidepayment game $\langle N, v\rangle$ where $N = \{1,2,3,4,5\}$ and v is given by

$$v(1,2,3) = v(2,3,4) = v(3,4,5) = v(1,4,5) = v(1,2,5) = 6$$

$$v(S) = 0 \text{ otherwise.}$$

This game is symmetric under cyclic permutations of the players. The unique bargaining aspiration for this game reflects this symmetry and assigns a price of 2 to each player. Selection of this price means that whichever of the profitable coalitions forms, the total payoff of 6 units will be divided equally among the three participants. Thus, we would predict the outcome of this game to be $(2,2,2,0,0)$, $\{[1,2,3],[4],[5]\}$ or some cyclic permutation of this outcome.

By contrast, for each of the five possible profitable coalition structures, the bargaining set consists of a single payoff distribution in which the "middle" player of the nontrivial coalition receives nothing at all, e.g., for $\{[1,2,3],[4],[5]\}$ the unique payoff distribution in the bargaining set is $(3,0,3,0,0)$.

The difference in predicted payoffs points out the difference between the solution concepts. The aspiration bargaining set picks out the correct prices for players to demand for their coalitional participation in the bargaining <u>before</u> coalitions form. The bargaining set selects reasonable payoffs for players to demand <u>given</u> that the coalition structure has <u>already</u> formed before the bargaining over payoffs begins. This makes a significant difference in the strategic position of player 2 in the coalition structure $\{[1,2,3], [4], [5]\}$. If player 2 negotiates his price before agreeing to form $[1,2,3]$, the symmetry of the problem dictates a symmetric payoff distribution while, if the coalition $[1,2,3]$ forms before the bargaining begins, player 2 is in a weaker bargaining position since player 1's profitable alternative without 2 involves players 4 and 5 who are currently receiving nothing (and who would therefore presumably be willing to settle for nearly nothing), while player 2's only profitable alternative involves player 3 who must be paid at least the (positive) amount he is currently earning. Given the structure of payoffs, player 2 cannot defend any positive payment for himself against a justified objection. Player 2 is well advised to not agree to form $[1,2,3]$ until his partners make a binding agreement on their payoff distribution.

5. FROM PRICES TO OUTCOMES

In Section 2, we discussed how prices determine outcomes for simple games. In simple games, once a bargaining aspiration is chosen and one coalition (from its generating collection) forms, no profitable coalition formation is possible for the remaining players. In general NTU games, after a bargaining aspiration is chosen and one or more coalitions (in its generating collection) form, the remaining players may still have available some profitable possibilities for coalition formation ("second-best" coalitions). These are coalitions in which the remaining players can obtain payoffs which are positive, but not as high as their prices. When there are additional gains to be realized from further coalition formation, we expect that further coalition formation will indeed take place. Final outcomes will result only when there are no further gains to be realized from coalition formation. In this section we model coalition formation in general NTU games.

Traditional solution concepts don't need to be concerned with the rules of coalition formation because traditional solution concepts fix the coalition structure before looking for payoff distributions. Imputation solutions—such as the value and the core—look for reasonable or stable payoff distributions for the coalition of the whole, whether or not the coalition of the whole is likely to form. Solutions such as the bargaining set(s) and kernel find reasonable payoff distributions for each given coalition structure. By first fixing the coalition structure, these solution concepts avoid the question of how the particular coalition structure arises. Since we model coalition formation endogenously, the rules that govern coalition formation can and do make a difference in the final outcomes.

In traditional solution concepts, stability of payoff distributions is derived from the "reasonableness" of the payoff distribution, given the coalition structure and the opportunities of players in other coalitions (the notion of reasonableness differs across solution concepts). In the approach taken here, payoff distributions (and coalition structures) are stable because the agreements to form coalitions are binding contracts—players make their best deals before agreeing to form coalitions for specified shares of the profits, and such agreements are binding.

Recall that by an outcome of a game $\langle N,V\rangle$ we mean a pair $(z,\mathcal{J})$ where is a coalition structure, i.e., a partition of N into disjoint coalitions) and z is a feasible payoff distribution for $\mathcal{J}$ (i.e., $z^S \in V(S)$ for each $S \in \mathcal{J}$). Informally, an outcome is a specification of which coalitions actually form, and how coalitional profits are divided within these coalitions.

We believe that bargaining aspirations are the reasonable prices for players to charge for their coalitional participation. Bargaining aspirations, however, depend on the opportunities of players in all their profitable coalitions. Once a bargaining aspiration is selected and one or more coalitions in its generating collection actually forms, the situation changes in two fundamental ways. Since the agreement to form coalitions (for a specified share of the profits) is a binding agreement, some coalitions can no longer form—the coalitions whose formation would require dissolution of an existing coalition. Moreover, since the set of possible coalitions has changed, the prices players can charge for their participation in the remaining coalitions will also change. In order to analyze subsequent coalition formation, we derive a new game which represents the remaining possibilities, and use bargaining aspirations in this new game to determine the prices players should charge for their participation in this subsequent coalition formation. The following two examples should clarify our approach.

EXAMPLE 5.1: Consider the game $\langle N,V\rangle$ with $N = \{1,2,3\}$ and V given by:

$$V[i] = \{0\} \quad \text{for each } i$$

$$V[1,2] = \{(x_1,x_2) \in \mathbf{R}^2_+ \mid x_1 \leq 4,\ x_2 \leq 4\}$$

$$V[1,3] = \{(x_1,x_3) \in \mathbf{R}^2_+ \mid x_1 \leq 4,\ x_3 \leq 4\}$$

$$V[2,3] = \{(x_2,x_3) \in \mathbf{R}^2_+ \mid x_2 \leq 4,\ x_3 \leq 4\}$$

$$V[1,2,3] = \{(x_1,x_2,x_3) \in \mathbf{R}^2_+ \mid x_1 + x_2 + x_3 \leq 9\}.$$

This game has a unique bargaining aspiration, namely $x = (4,4,4)$. The only coalitions that can afford x are [1,2], [1,3] and [2,3], so we expect one of these coalitions to form, in which case both members of this coalition obtain payoffs of 4 while the "left-out" player obtains nothing. Of course such an outcome is not efficient, in the sense that not all the gains from coalition formation have been extracted (the coalition of the whole can obtain a total of 9). We would expect, therefore, that—if possible under the rules of play—the two players forming the initial coalition (say 1 and 2) would bargain with the third player over the remaining 1 unit. We would then expect—and our model predicts—that the coalition of the whole would eventually form, and that the final payoff distribution would be something like $(4+a, 4+b, 1-a-b)$ for some positive parameters a, b with $a+b < 1$. Note that the core of this game consists precisely of the closure of this set of payoffs and its permutations.

If it were not possible, under the rules of play, for the two players forming the initial coalition to bargain with the third player for the remaining 1 unit (if the game were played by a single round of sealed bids, for example) we would expect the final outcome to consist of the payoff distribution $(4, 4, 0)$ for the coalition structure $\{[1, 2], [3]\}$ (or some permutation). Of course, such an outcome is not efficient, but under these rules of play we would not necessarily expect the outcome to be efficient. Notice that although some outcomes may be inefficient, the prices that generate them never are. In effect, we always have <u>ex ante</u> efficiency; under some rules of play we should not expect <u>ex post</u> efficiency.

EXAMPLE 5.2: Consider the sidepayment game $\langle N, v\rangle$ with $N = \{1, 2, 3\}$ and v given by:

$$v[i] = 0 \quad \text{for each } i$$

$$v[i, j] = 4 \quad \text{for each } i \neq j$$

$$v[1, 2, 3] = 6 \quad .$$

The unique bargaining aspiration for this game is $x = (2, 2, 2)$, and its generating collection is $\{[1, 2], [1, 3], [2, 3], [1, 2, 3]\}$. Notice that if $[1, 2]$ forms the outcome will be $((2, 2, 0), \{[1, 2], [3]\})$, while if $[1, 2, 3]$ forms the outcome will be $((2, 2, 2), \{[1, 2, 3]\})$. It may be argued that one should always expect to see the coalition $[1, 2, 3]$ form, since forming this coalition exhausts all gains from coalition formation. Note, however, that players 1 and 2 receive no higher payoff working with player 3 than without player 3, so players 1 and 2 are indifferent between forming the coalitions [1,2] and [1,2,3], given that they will receive a payoff of 2 each in either case. Since a player receives the same utility from forming any coalition (to which he belongs) in the generating collection, the player is precisely indifferent to forming any of these coalitions. (If players 1 and 2 would prefer that player 3 received a positive payoff, so long as their own payoffs were unaffected, this should certainly be reflected in their characteristic functions, so that the payoff for the coalition [1,2,3] should be greater than 6.) Which of the coalitions in the generating collection actually form depends on circumstances omitted from the characteristic function. For example, if the game were played with the agents sitting at a table, we would expect the coalition [1,2,3] to form, but if the game were played in a decentralized environment and players 1 and 2 met first, there is no reason to suppose that they would "wait" for player 3. In fact, they will be better off if they do not do so, since they will then be in a position to bargain with player 3 for the remaining 2 units. We would then expect, if the coalition [1,2,3] eventually formed, to see a payoff distribution like $(2+a, 2+b, 2-a-b)$ for some positive numbers a, b with $a+b<2$.

In the following, we model coalition formation where the rules of the game (as dictated by the situation being modelled) allow coalitions, once formed, to add players or merge with other coalitions. The mechanism of coalition formation we describe has a simple underlying idea. Agreements to form coalitions (for stated shares of their profits) are binding agreements. Thus, once one or more coalitions have formed, the formation of certain other coalitions is no longer possible, because it would require dissolving one or more of the existing coalitions. After one or more coalitions have formed, what remains to be bargained for—in the coalitions that are possible and profitable—are the additional gains from coalition formation; that is, the utility vectors which give to each member of the coalition at least the utility that he has already obtained through coalition formation up to that point. In order to analyze the bargaining over the remaining gains from coalition formation, we construct a "reduced" game which represents these additional gains for each possible coalition. For this reduced game, we determine which coalitions form and the payoffs which players receive by analyzing its bargaining aspirations. Continuing in this way, we eventually arrive at a final outcome for the game which exhausts all potential gains from coalition formation.

In what follows, it will be convenient to assume that the game is 0-normalized; i.e., $V(i) = \{0\}$ for each i —the general case can be treated with only very small modifications.

To formalize these ideas, we begin with the initial stage $t = 0$. At this stage, no coalition formation has taken place and no gains have been extracted from coalition formation. To be consistent with later notation, we write $\mathfrak{J}(0)$ for the existing coalition structure, $z(0)$ for the payoff which players in the game have already guaranteed themselves, and set $\langle N, V_0 \rangle = \langle N, V \rangle$. At this initial stage, of course, $\mathfrak{J}(0) = \{[1], \ldots, [n]\}$, and $z(0) = (0, \ldots, 0)$. Bargaining at this stage results in the choice of some bargaining aspiration $x(0)$ for the game $\langle N, V \rangle = \langle N, V_0 \rangle$ and the formation of some collection $\mathcal{S}(0)$ of coalitions from the generating collection of $x(0)$. By forming coalitions in $\mathcal{S}(0)$, players in these coalitions obtain their components of $x(0)$; others obtain nothing. This yields an outcome after the first stage of $(z(1), \mathfrak{J}(1))$, where $z(1)$ is the payoff vector which gives to each member of any of the coalitions in $\mathcal{S}(0)$ their component of $x(0)$ and to each remaining player the payoff 0, and where $\mathfrak{J}(1)$ is the coalition structure consisting of the set $\mathcal{S}(0)$ and the remaining singleton coalitions (if any). This is, of course, exactly a formal description of what happens in a simple game, but since there may be additional gains to be obtained from coalition formation, further stages are possible.

At stage $t = 1$, the coalitions in the coalition structure $\mathfrak{J}(1)$ have already formed, and the players have guaranteed themselves their respective components of the payoff vector $z(1)$. Coalitions which are <u>not</u> unions of coalitions in $\mathfrak{J}(1)$ can no longer form, since formation of any such coalition would require breaking of a binding agreement. Coalitions in N which are unions of coalitions in $\mathfrak{J}(1)$ can form, but will only form if they allocate their members at least what they have already obtained in z(1). We construct a reduced game $\langle N, V_1 \rangle$, which reflects the remaining bargaining possibilities, by setting:

$$V_1(S) = \left\{x^S \in \mathbf{R}_+^S \,\middle|\, x^S + z(1)^S \in V_0(S)\right\} \text{ if } S \text{ is a union of coalitions in } \mathcal{J}(1) \text{ and } z(1)^S \in V_0(S)$$

$$V_1(S) = \{0\} \text{ otherwise.}$$

The reduced game $\langle N, V_1 \rangle$ represents the potential gains from coalition formation that remain after the first stage of bargaining.

After bargaining (over the additional gains from coalition formation represented by the game $\langle N, V_1 \rangle$), players choose a new bargaining aspiration x(1) and form a new collection of coalitions $\mathcal{S}(1)$ that can afford x(1). Notice that if the restriction of x(1) to any of these coalitions is not identically zero, then—by construction of the reduced game—the coalition must be a union of coalitions in $\mathcal{J}(1)$. We can—and will—ignore all other coalition formation at this point, since it achieves only the 0 payoff. Write z(2) for the corresponding payoff vector, which gives to each member of a coalition in $\mathcal{S}(1)$ his component of x(1) and to each remaining player the payoff 0. Notice that z(2) is the additional payoff for this further coalition formation. At the end of this stage, therefore, the coalitions which have formed are those in $\mathcal{S}(1)$, together with coalitions in $\mathcal{J}(1)$ which are not contained in any member of $\mathcal{S}(1)$; write $\mathcal{J}(2)$ for this set of coalitions. The payoff distribution to this point—that is, the total payoff obtained by the players at the end of this stage, is just the sum of the payoffs obtained at each stage, namely $z(1) + z(2)$.

We may now simply continue in this way for as many stages as are necessary. After stage $t-1$, we have a reduced game $\langle N, V_{t-1} \rangle$, a coalition structure $\mathcal{J}(t)$ and a payoff vector z(t) which is feasible for that coalition structure. We construct a reduced game $\langle N, V_t \rangle$ in the same way we constructed $\langle N, V_1 \rangle$ from $\langle N, V_0 \rangle$, and proceed as above. The process terminates when, at the end of some stage t^*, there are no more gains to be extracted from coalition formation. At this stage every coalition in the reduced game has 0 as its only attainable vector. Let $\mathcal{J}(t^*)$ be the coalition structure at t^* and let $z^* = z(1) + \ldots + z(t^*)$ be the total payoff players have obtained through coalition formation at all stages. The final outcome is then just $(z^*, \mathfrak{J}(t^*))$.

6. PROOFS OF THE THEOREMS

We begin this section by setting up some machinery which we use in the proofs of Theorem 2.1 (which describes the set of aspirations) and Theorem 2.2 (which asserts the existence of bargaining aspirations).

We define certain auxiliary functions. Fix a strongly comprehensive NTU game $\langle N, V\rangle$. For each $k \in N = \{1, 2, \ldots, n\}$ and each $x \in \mathbb{R}^N_+$, we set:

$$\gamma_k(x) = \sup\{t : (x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n) \text{ is feasible for player } k\} .$$

Notice that $\gamma_k(x)$ is well-defined and nonnegative (since each of the sets $V(S)$ is compact and $0 \in V(\{k\})$. Let $\phi_k(x)$ be the vector obtained from x by replacing x_k by $\gamma_k(x)$; i.e., $\phi_k(x) = (x_1, \ldots, x_{k-1}, \gamma_k(x), x_{k+1}, \ldots, x_n)$. Thus ϕ_k is a map from $\mathbb{R}^N_+$ onto itself, for each k.

The maps ϕ_k enjoy the following properties:

(i) $\phi_k(x)$ is realizable for player k and unblocked for player k;

(ii) if x is realizable for player k, then $\phi_k(x) \geqslant x$;

(iii) if x is maximal for player k,* then $\phi_k(x) \leqslant x$;

(iv) if x is maximal for player ℓ, then $\phi_k(x)$ is unblocked for player ℓ (whether $\ell = k$ or $\ell \neq k$).

The first three of these properties are easily checked. To see that the fourth is true, write $y = \phi_k(x)$ and suppose that y is not unblocked for player ℓ. Then, because of strong comprehensiveness, there is a coalition S with $\ell \in S$ and $y^S \in \operatorname{int} V(S)$. If $k \in S$, this implies that y is not unblocked for player k, contradicting property (i). On the other hand, if $k \notin S$ then $y^S = x^S$ (since applying ϕ_k to x alters at most the k-th component), so $x^S \in \operatorname{int} V(S)$ and this contradicts the assumption that x is unblocked for player ℓ.

We now define maps $\Phi : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ and $\Phi^2 : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ by setting (the ordering is only for convenience):

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* We say x is <u>maximal</u> for player k if there does not exist a coalition S with $k \in S$, and a vector $y^S \in V(S)$ with $y^S \gg x^S$.

$$\Phi = \phi_n \circ \phi_{n-1} \circ \dots \phi_2 \circ \phi_1$$

$$\Phi^2 = \Phi \circ \Phi \quad .$$

(Recall that $N = \{1, \dots, n\}$ so that n is the number of players.)

Having defined these functions, we now turn to the proof proper of Theorem 2.1.

PROOF OF THEOREM 2.1: We are going to show that Φ^2 is a continuous retraction of $\mathbf{R}_+^N$ onto $A\langle N,V\rangle$; i.e., Φ^2 is continuous, $\Phi^2(x) \in A\langle N,V\rangle$ for each $x \in \mathbf{R}_+^N$ and $\Phi^2(y) = y$ for each $y \in A\langle N,V\rangle$. The other parts of Theorem 2.1 will follow easily from these facts.

Let us verify the assertions about Φ^2 in the reverse order. Firstly, if $y \in A\langle N,V\rangle$ then y is realizable and unblocked for all players. By properties (ii), (iii) above we see that $\phi_k(y) = y$, for each k. Hence, $\Phi^2(y) = \phi_n \circ \dots \circ \phi_1 \circ \phi_n \circ \dots \circ \phi_1(y) = y$. Secondly, for each $x \in \mathbf{R}_+^N$ we see by repeated applications of (i) and (iv) that $\Phi(x)$ is unblocked for all players. By (i) and (iv) again, $\phi_1 \circ \Phi(x)$ is realizable for player 1 and unblocked for all players. By (iii), $\phi_2 \circ \phi_1 \circ \Phi(x) \leqslant \phi_1 \circ \Phi(x)$, so $\phi_2 \circ \phi_1 \circ \Phi(x)$ is also realizable for player 1. Continuing in this way, we see that $\Phi \circ \Phi(x) = \Phi^2(x)$ is realizable and unblocked for all players; i.e., $\Phi^2(x) \in A\langle N,V\rangle$.

It remains only to verify the continuity of Φ^2. To this end, for each player k in N and each coalition $S \subseteq N$ with k in S, define maps $\gamma_k^S : \mathbf{R}_+^N \to \mathbf{R}_+$ by:

$$\gamma_k^S(x) = 0 \ , \quad \text{if } (x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n)^S \notin V(S) \ ;$$

$$\gamma_k^S(x) = \sup\{t : (x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n)^S \in V(S)\} \quad \text{otherwise.}$$

Clearly, $\gamma_k(x) = \max_S \gamma_k^S(x)$, so continuity of each of the functions γ_k^S entails continuity of each of the functions γ_k. Continuity of the functions γ_k entails continuity of the maps ϕ_k and hence of Φ and Φ^2.

Thus to establish the continuity of Φ^2, we need to establish the continuity of each γ_k^S. Fix k, S and a vector x in $\mathbf{R}_+^N$, and let $\{x^\ell\}$ be a sequence in $\mathbf{R}_+^N$ which converges to x. Since V(S) is compact, the sequence $\{\gamma_k^S(x^\ell)\}$ is bounded; passing to a subsequence and renumbering if necessary, we may assume that $\{\gamma_k^S(x^\ell)\}$ converges to some number γ. We are to show that $\gamma = \gamma_k^S(x)$. To this end, we define vectors in $\mathbf{R}_+^N$ by:

$$y^\ell = (x_1^\ell, \ldots, x_{k-1}^\ell, \gamma_k^S(x^\ell), x_{k+1}^\ell, \ldots, x_n^\ell), \quad \text{for each } \ell ;$$

$$y = (x_1, \ldots, x_{k-1}, \gamma, x_{k+1}, \ldots, x_n) ;$$

$$z = (x_1, \ldots, x_{k-1}, \gamma_k^S(x), x_{k+1}, \ldots, x_n) .$$

Suppose that $\gamma > \gamma_k^S(x)$. In particular then, $\gamma > 0$ so that $\gamma_k^S(x^\ell) > 0$ for all sufficiently large ℓ. Clearly, the sequence $\{y^\ell\}$ converges to y; since V(S) is closed, this means that $y^S \in V(S)$. By definition, this means that $\gamma_k^S(x) \geqslant \gamma$, a contradiction. On the other hand, let us suppose that $\gamma < \gamma_k^S(x)$. In particular, this means that $\gamma_k^S(x) > 0$ so $z^S \in V(S)$. Since $y \leqslant z$ and $\gamma < \gamma_k^S$, strong comprehensiveness of V(S) implies that y lies in the interior of V(S) relative to $\mathbf{R}_+^N$. But then, for sufficiently large ℓ, y^ℓ will also lie in the interior of V(S) which contradicts the definition of $\gamma_k^S(x^\ell)$. We conclude that $\gamma = \gamma_k^S(x)$, so that γ_k^S is indeed continuous. That completes the proof of all the assertions about the map Φ^2.

Since $A\langle N, V\rangle$ is the range of a continuous mapping defined on $\mathbf{R}_+^N$, it is nonempty and connected. To see that it is compact, note that, since each of the sets V(S) is compact, we may find a positive number M such that $V(S) \subseteq [0, M]^S$ for every S. It is then easily checked that $A\langle N, V\rangle \subseteq [0, M]^N$, so that $A\langle N, V\rangle = \Phi^2([0, M]^N)$, and in particular, $A\langle N, V\rangle$ is compact.

It remains to show that every continuous map of $A\langle N, V\rangle$ to itself has a fixed point. Let $f: A\langle N, V\rangle \rightarrow A\langle N, V\rangle$ be a continuous map. As was already noted, we may find a positive number M so that $A\langle N, V\rangle = \Phi^2([0, M]^N) \subseteq [0, M]^N$.

Thus the composition $f \circ \Phi^2$ may be regarded as a map of $[0, M]^N$ into itself. By the Brouwer Fixed Point Theorem, $f \circ \Phi^2$ has a fixed point x in $[0, M]^N$. On the other hand, since the range of f is a subset of $A\langle N, V\rangle$ it must be that $x \in A\langle N, V\rangle$. Since Φ^2 fixes every point of $A\langle N, V\rangle$ we obtain $x = f \circ \Phi^2(x) = f(x)$; i.e., x is a fixed point of f. This completes the proof of Theorem 2.1. ∎

In order to prove that the aspiration bargaining set is not empty, we will isolate a certain subset of $A\langle N, V\rangle$ which has some of the flavor of the kernel of a game with transferable utility.

For any game $\langle N, V\rangle$ and for each pair i, j of distinct players in N, define a function

$$\sigma_{ij} : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$$

by setting

$$\sigma_{ij}(x) = \text{minimum dist}(x^S, V(S))$$

where the minimum is taken over all coalitions $S \subseteq N$ such that $i \in S$ but $j \notin S$ and dist is the Euclidean distance. Notice that the distance from a point to a closed set depends continuously on the point, so that σ_{ij} is a continuous function of x. It is convenient to set $\sigma_{ii}(x) = 0$ for each i, x. We also define functions

$$\alpha_i : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$$

$$\beta_i : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$$

by setting

$$\alpha_i(x) = \frac{1}{n} \sum_{j \in N} \sigma_{ij}(x)$$

$$\beta_i(x) = \frac{1}{n} \sum_{j \in N} \sigma_{ji}(x) .$$

Evidently, both α_i and β_i are also continuous functions of x.

We now define $K\langle N,V\rangle$ to be the set of points x in $A\langle N,V\rangle$ such that $\alpha_i(x) = \beta_i(x)$ for each $i \in N$. Since $A\langle N,V\rangle$ is a closed set and the functions α_i, β_i are continuous, the set $K\langle N,V\rangle$ is certainly closed; we will eventually prove that it is nonempty.

Although we do not need to give the set $K\langle N,V\rangle$ any literal interpretation—it is merely a technical device—we suggest a guide for the intuition. Let x be a vector in $A\langle N,V\rangle$. To say that $\sigma_{ij}(x)$ is nonzero is to say that no coalition containing i but not j is in GC(x). Thus, player i could be damaged by the failure of player j to cooperate; in some sense, player i is indebted to player j. The magnitude of $\sigma_{ij}(x)$ is a measure of the size of this debt. Thus, $\alpha_i(x)$ is a (normalized) measure of player i's total debts. On the other hand, $\beta_i(x)$ is a (normalized) measure of player i's total credits (i.e., the total of debts that others "owe" to player i). Thus, $x \in K\langle N,V\rangle$ exactly when each player's total credits and total debts exactly balance.

The relevance of the set $K\langle N,V\rangle$ to our purpose is that it is contained in the aspiration bargaining set. Before proving this, we recall some elementary notions from graph theory.

A <u>directed graph</u> is a pair (U,D) where U is a finite nonempty set (the set of <u>vertices</u>) and D is a (possibly empty) set (the set of <u>directed edges</u>) of ordered pairs of distinct elements of U. We think of an ordered pair (u_1, u_2) in D as representing an arrow from u_1 to u_2. For example, in Figure 1, $U = \{1,2,3,4,5,6,7\}$ and $D = \{(1,2), (2,3), (3,1), (4,5), (4,6)\}$.

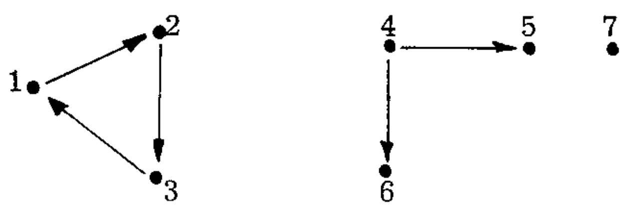

Figure 1

If (U,D) is a directed graph, we will say that a subset $\mathcal{C}$ of U is <u>connected</u> if for every two distinct vertices $a, b \in \mathcal{C}$, we can find vertices

$c_1(=a), c_2, \ldots, c_k(=b)$ with each $c_i \in \mathcal{C}$ and such that for all i with $1 \leqslant i \leqslant k-1$, either (c_i, c_{i+1}) is in D or (c_{i+1}, c_i) is in D. In Figure 1, the connected sets are: $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$, $\{4, 5\}$, $\{4, 6\}$, $\{4, 5, 6\}$ and all singletons. (Notice that we do not require that we can get from a to b by following in the direction of the arrows—we may follow some arrows forward and some backward.) An end of the graph is a vertex u such that there is no $w \in U$ with $(u, w) \in D$; i.e., no arrows come out of u, but arrows may go into u. In Figure 1, the vertices 5, 6, 7 are ends. In general there may be no ends, but if the graph has no loops (= cycles) then it always has ends (start somewhere and follow the directed arrows as far as passible—arriving at an end).

The following lemma establishes the connection between $K\langle N,V\rangle$ and $AB\langle N,V\rangle$.

LEMMA 2.3: For every strongly comprehensive NTU game $\langle N,V\rangle$, the set $K\langle N,V\rangle$ is a subset of the aspiration bargaining set $AB\langle N,V\rangle$.

PROOF: Let x be a vector in $K\langle N,V\rangle$; by definition, x is in $A\langle N,V\rangle$. Suppose that x is not bargaining. We construct a directed graph G in the following way. The vertices of G are the players in N. There is an edge of G directed from the player a to the player b if $a \neq b$ and a is vulnerable to b at x. Since we have supposed that x is not bargaining, the graph G has at least one directed edge. Among all the connected sets of vertices of G, let $\mathcal{C}$ be one which has the largest number of elements ($\mathcal{C}$ has at least 2 elements since G has at least one edge) and let $\mathcal{E}$ be the set of ends of $\mathcal{C}$. That is, $\mathcal{E}$ is the set of players e in $\mathcal{C}$ such that some player in $\mathcal{C}$ is vulnerable to e, while e is not vulnerable to any other player in $\mathcal{C}$. (Maximality of $\mathcal{C}$ then guarantees that e is not vulnerable to any other player in N.) The set $\mathcal{E}$ of ends of $\mathcal{C}$ is not empty because, as we remarked earlier, vulnerability is a transitive relation and in particular, is acyclic, so that the graph G contains no cycles. (See Figure 2.)

Now let $e \in \mathcal{E}$, $f \notin \mathcal{E}$. We claim that $\sigma_{ef}(x) = 0$. For suppose that $\sigma_{ef}(x)$ were strictly positive. Then every coalition in GC(x) which contains e also contains f.

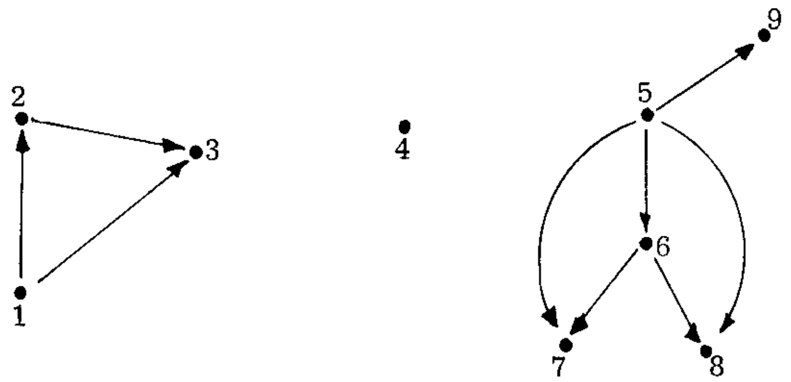

Figure 2: A graph of vulnerabilities: $\zeta = \{5, 6, 7, 8, 9\}$ and $\xi = \{7, 8, 9\}$.

Since e is an end, we know that e is not vulnerable to f, so that every coalition in GC(x) which contains f must also contain e. Moreover, since e is an end, there is a player g in ζ who is vulnerable to e. Since e, f belong to exactly the same coalitions in GC(x), it follows that g is also vulnerable to f. In particular, f belongs to ζ. Since f is not in ξ, there is a player h in ζ to whom f is vulnerable. But then e is also vulnerable to h, which contradicts our assumption that e is an end. We conclude that $\sigma_{ef}(x) = 0$, as claimed.

We next show that the ends have a nonnegative net credit:

$$n \sum_{e \in \xi} \left(\beta_e(x) - \alpha_e(x)\right) = \sum_{e \in \xi} \sum_{i \in N} \sigma_{ie}(x) - \sum_{e \in \xi} \sum_{i \in N} \sigma_{ei}(x)$$

$$= \sum_{e \in \xi} \sum_{i \in \xi} \sigma_{ie}(x) - \sum_{e \in \xi} \sum_{i \in \xi} \sigma_{ei}(x) + \sum_{e \in \xi} \sum_{i \notin \xi} \sigma_{ie}(x) - \sum_{e \in \xi} \sum_{i \notin \xi} \sigma_{ei}(x)$$

$$= \sum_{e \in \xi} \sum_{i \in \xi} \left[\sigma_{ie}(x) - \sigma_{ei}(x)\right] + \sum_{e \in \xi} \sum_{i \notin \zeta} \sigma_{ie}(x) - \sum_{e \in \xi} \sum_{i \notin \zeta} \sigma_{ei}(x) .$$

Of these last three double sums, the first is zero by symmetry, while the third is zero since we have just shown that all its terms are zero. The second double sum consists of nonnegative terms and so is certainly nonnegative. Moreover, for each end e, there is some player j in ζ who is vulnerable to e, so that $\sigma_{je}(x) > 0$.

Thus the second double sum is strictly positive, whence $n \sum_{n \in \mathcal{E}} (\beta_e(x) - \alpha_e(x))$ is also strictly positive. On the other hand, since $x \in K\langle N, V\rangle$, we have that $\alpha_i(x) = \beta_i(x)$ for every i and in particular that $\sum_{e \in \mathcal{E}} (\beta_e(x) - \alpha_e(x)) = 0$. This contradiction shows that x was indeed bargaining, as desired. This completes the proof of Lemma 2.3. ∎

We can now use Lemma 2.3 and a fixed-point argument to show that the aspiration bargaining set is not empty.

PROOF OF THEOREM 2.2: In view of Lemma 2.3, $K\langle N,V\rangle$ is a subset of $AB\langle N,V\rangle$ so it certainly suffices to prove that $K\langle N,V\rangle$ is not empty. We will do this by showing that $K\langle N,V\rangle$ is the set of fixed points of a certain continuous mapping of $A\langle N,V\rangle$ to itself.

We have constructed a retraction $\Phi: \mathbb{R}_+^N \rightarrow A\langle N,V\rangle$. We define a map $\Psi: A\langle N,V\rangle \rightarrow \mathbb{R}_+^N$ by writing $\Psi(x) = y$ where

$$y_i = x_i - \alpha_i(x) + \beta_i(x) \ .$$

(Notice that $\alpha_i(x) \leqslant x_i$ and that $\beta_i(x) \geqslant 0$ so that $y_i \geqslant 0$ and y does indeed belong to $\mathbb{R}_+^N$.) The composition $\Phi^2 \circ \Psi: A\langle N,V\rangle \rightarrow A\langle N,V\rangle$ is a continuous map, and by Theorem 2.1 has a fixed point. If we show that $K\langle N,V\rangle$ is the set of fixed points of $\Phi^2 \circ \Psi$, we can, in particular, conclude that $K\langle N,V\rangle$ and $AB\langle N,V\rangle$ are nonempty, as desired. It is clear that all points of $K\langle N,V\rangle$ are fixed by $\Phi^2 \circ \Psi$, so we need only show that every fixed point of $\Phi^2 \circ \Psi$ lies in $K\langle N,V\rangle$.

So let x be a fixed point of $\Phi^2 \circ \Psi$ and suppose that $x \notin K\langle N,V\rangle$. Write $y = \Psi(x)$ and $z = \Phi(y)$. As noted in the construction of Φ, z is maximal and $x = \Phi(z) \leqslant z$. We will see that this leads to a contradiction.

Notice first of all that

$$\sum_{i \in N} y_i = \sum_{i \in N} \left(x_i - \alpha_i(x) + \beta_i(x)\right)$$

$$= \sum_{i \in N} x_i - \frac{1}{n} \sum_{i \in N} \sum_{j \in N} \left(\sigma_{ij}(x) - \sigma_{ji}(x)\right)$$

$$= \sum_{i \in N} x_i \quad .$$

Since $x \notin K\langle N, V\rangle$, we have $x \neq y$. From the above equalities, it follows that at least one component of y is strictly less than the corresponding component of x. Letting ℓ be the index of the last such component we obtain that $y_\ell < x_\ell$ but $y_i \geqslant x_i$ for $i \geqslant \ell+1$. Recall that $\Phi = \phi_n \circ \ldots \circ \phi_1$, and that applying ϕ_i to any vector alters at most the i-th component. So if we write $a = \phi_{\ell-1} \circ \ldots \circ \phi_1(y)$ and $b = \phi_\ell(a) = \phi_\ell \circ \ldots \circ \phi_1(y)$, we see that $a_i = b_i = z_i$ for $1 \leqslant i \leqslant \ell-1$ and that $b_\ell = z_\ell$. Moreover, the vector b is realizable for player ℓ so there is a coalition $S \subseteq N$ with $\ell \in S$ and $b^S \in V(S)$. We want to see which players in N can belong to S.

Suppose that $i \in S$ and $1 \leqslant i \leqslant \ell-1$. Since a is obtained from y by altering at most the first $\ell-1$ components, and b is obtained from a by altering only the ℓ-th component, we have that $a_\ell = y_\ell$ and that $a_j = b_j$ for $j \neq \ell$. By our choice of ℓ, $y_\ell < x_\ell$. We have already noted that $x < z$ and $z_\ell = b_\ell$, so we obtain $a_\ell = y_\ell < x \leqslant z_\ell = b_\ell$ and $a < b$. Since $b^S \in V(S)$ and $i \in S$, it follows that $a^S < b^S$; strong comprehensiveness of $V(S)$ then implies that a^S lies in the interior of $V(S)$. But this means that a is not maximal for player i, contrary to the construction of the maps ϕ_k. We conclude that $S \subseteq \{\ell, \ell+1, \ldots, n\}$.

Now from our choice of ℓ we know that $y_i \geqslant x_i$ for $i \geqslant \ell+1$. Suppose that $y_j > x_j$ for some $j \in S$. Since $b_\ell = z_\ell \geqslant x_\ell$ and $b_i = y_i \geqslant x_i$ for $i + \ell+1$, we conclude as before that x^S belongs to the interior of $V(S)$. But this contradicts the fact that the vector x is maximal (since it is in $A\langle N, V\rangle$). We conclude that $y_j = x_j$ for every $j \in S$, $j \neq \ell$, and hence that $\alpha_j(x) = \beta_j(x)$ for every such j.

We now obtain the following equalities (the first one follows from the fact that $\alpha_s(x) - \beta_s(x) = 0$ for $s \in S$, $s \neq \ell$):

$$\alpha_\ell(x) - \beta_\ell(x) = \sum_{s \in S} \left(\alpha_s(x) - \beta_s(x)\right)$$

$$= \frac{1}{n} \sum_{s \in S} \sum_{t \in N} \left(\sigma_{st}(x) - \sigma_{ts}(x)\right)$$

$$= \frac{1}{n} \sum_{s \in S} \sum_{t \in S} \left(\sigma_{st}(x) - \sigma_{ts}(x)\right) + \frac{1}{n} \sum_{s \in S} \sum_{t \notin S} \sigma_{st}(x) -$$

$$- \frac{1}{n} \sum_{s \in S} \sum_{t \notin S} \sigma_{ts}(x) .$$

Of the last three double sums, the first is zero by symmetry, while the second is zero since the definition of $\sigma_{st}(x)$ guarantees that it will be zero whenever $s \in S$, $t \notin S$ and $x^S \in V(S)$. We conclude that $\alpha_\ell(x) - \beta_\ell(x) \leqslant 0$. However, $y_\ell < x_\ell$ so that $\alpha_\ell(x) - \beta_\ell(x) > 0$. This contradiction tells us that the fixed point x of the map $\Phi^2 \circ \Psi$ must indeed belong to $K\langle N, V\rangle$, as asserted. In particular, $K\langle N, V\rangle$ is nonempty; since $K\langle N, V\rangle$ is a subset of $AB\langle N, V\rangle$, by Lemma 2.3, the proof is complete. ∎

PROOF OF THEOREM 3.1: Let $g^* = (g_1^*, \ldots, g_n^*)$ be a competitive allocation and π an equilibrium price. Write $x^* = (u_1(g_1^*), \ldots, u_n(g_n^*))$ for the utility vector corresponding to g. It is well-known that x^* is in the core of V_ξ and hence is both realizable and maximal and therefore an aspiration. To see that x is bargaining it suffices to show that for any coalition $S \subseteq N$, if $x^S \in V_\xi(S)$ then $x^{N-S} \in V_\xi(N-S)$.

To this end, assume that $x^S \in V_\xi(S)$. Then there are vectors $h_j \in \mathbb{R}_+^M$ (for each $j \in S$) such that $\sum_{j \in S} h_j = \sum_{j \in S} \omega_j$ and $u_j(h_j) \geq x_j$ for each j. (So that $(h_j)_{j \in S}$

represents a reallocation of the resources of the coalition S which realizes utility levels as high as x_j for each $j \in S$. Since g^* is a competitive allocation and $u_j(h_j) \geq u_j(g_j^*)$ for each $j \in S$ it follows that $\pi h_j \geq \pi g_j^* = \pi\omega_j$ for each j. Since $\sum_{j \in S} h_j = \sum_{j \in S} \omega_j$, we conclude that $\pi h_j = \pi g_j = \pi\omega_j$ for each j, and hence that $u_j(h_j) = u_j(g_j^*)$ for each j. If for some agent k, $h_k \neq g_k$ then $u_k\left(\frac{1}{2}(h_k + g_k^*)\right) > u_k(g_k^*)$ —by strict quasiconcavity; since $\pi\left(\frac{1}{2}(h_k + g_k^*)\right) = \frac{1}{2}\left(\pi(h_k + g_k^*)\right) = \pi(\omega_k)$, this would contradict the equilibrium nature of π. We conclude that $g_j^* = h_j$ for each $j \in S$. Thus $\sum_{j \in S} g_j^* = \sum_{j \in S} h_j = \sum_{j \in S} \omega_j$, whence $\sum_{j \notin S} g_j^* = \sum_{j \notin S} \omega_j$. In other words, this means that, at the allocation g^*, members of S trade among themselves and members of N - S also trade among themselves. As a consequence, we conclude that $x^{N-S} \in V_\xi(N-S)$, as asserted, and this certainly implies that x is a bargaining aspiration. This completes the proof of Theorem 3.1. ∎

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