IMPLEMENTATION OF THE STRAIGHTENING ALGORITHM OF CLASSICAL INVARIANT THEORY

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IMA Preprint Series # 436

August 1988

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Abstract. The straightening algorithm for bracket polynomials or Young tableaux has many possible variations. We examine the choices that are involved in implementing a straightening algorithm, describe some particular variations which we found to be relatively efficient, and provide a comparison of their performances on a number of instances of the problem. We also draw connections to the more general Gröbner basis normal form algorithms. Finally, we use one of the variations to straighten an invariant of Turnbull and Young which specifies when ten points lie on a common quadric surface in projective three-space.

1. The Straightening Algorithm. Let $a, b, \ldots, z$ be N points (or unspecified vectors) in a vector space V of dimension r over an arbitrary field K. The First Fundamental Theorem of Invariant Theory [5],[4],[13] states that the invariants of the general linear group acting on V are the homogeneous polynomials in the *brackets*, which are determinants of r of the vectors. Homogeneity here refers to the usual $\mathbf{N}$ grading of polynomials. The invariants of the projective linear group are the bracket polynomials which satisfy a stronger homogeneity condition, namely that they are homogeneous with respect to the $\mathbf{N}^N$-grading induced on a bracket monomial M by the number of occurrences of each point in M. Of course, V may have infinite cardinality, but a bracket polynomial involves only a finite number of points, and for notational convenience we are assuming that they are always contained in the given set of points. Each bracket monomial will be abbreviated as a *tableau*, or rectangular $d \times r$ array of points, where each bracket has its points listed as one row of the tableau, and d is the $\mathbf{N}$-degree of the monomial.

The straightening algorithm is a procedure for writing any bracket polynomial, or linear combination of tableaux, as a linear combination of certain tableaux called *standard tableaux*. It is well known that the standard tableaux form a linear basis of the bracket polynomials. The reader is assumed to have some familiarity with the straightening algorithm; for an introduction see [10].

The following six items are required for the straightening algorithm.

1. A linear ordering on the points.

We will assume that alphabetical ordering on $a, b, \ldots, z$ has been chosen. The definition of standard tableau is stated in terms of this ordering: a tableau is standard if the points in each row are written in strictly increasing order from left to right, and that the points in each column are written in non-strictly increasing order from top to bottom. Actually, by anti-symmetry of the brackets, we may rewrite any non-zero tableau

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so that each row is in strictly increasing order, up to a sign change on coefficient of the entire tableau. We will henceforth assume that all tableaux are so rewritten whenever appropriate.

2. A linear ordering on the brackets, or rows.

The usual orderings are lexicographic or reverse lexicographic ordering induced by the order in item 1. We will now define these on arbitrary n-sequences of elements from a linearly ordered set S. We say that $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$ in *lexicographic order induced by the order on* S if there exists $i, 1 \leq i \leq n$, such that $x_i < y_i$ in S, and for all $j, 1 \leq j < i, x_j = y_j$. We say that $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$ in *reverse lexicographic order induced by the order on* S if there exists $i, 1 \leq i \leq n$, such that $x_i < y_i$ in S, and for all $j, i < j \leq n, x_j = y_j$. Since multiplication of brackets is commutative, the rows of a tableau may be interchanged freely, and we may assume that the rows are always written in order. For example, in lexicographic order, this implies that we may assume that every tableau has non-strictly increasing order in its first column, as well as strictly increasing rows.

3. A linear ordering on tableaux.

This is usually obtained by thinking of each tableau as a sequence of d rows, and using either lexicographic or dual reverse lexicographic order induced by the order on the rows in item 2. *Dual* here means that $(x_1, x_2, \ldots, x_n) > (y_1, y_2, \ldots, y_n)$ replaces $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$.

4. A class of syzygies which give a monotone improvement in the tableau ordering and which are sufficient to correct any violation of standardness.

A syzygy is a non-trivial bracket polynomial which is identically zero. For our purposes, a syzygy will always be solved for its greatest term (in the tableau ordering), and used to substitute for that term. The monotone improvement will guarantee termination of the straightening algorithm, because there are always a finite number of tableaux less than a given tableau.

The most commonly used syzygy for straightening algorithm purposes is the *van der Waerden syzygy*, which is

$$\sum sgn(\sigma)[x_1, x_2, \ldots, \sigma(x_q), \sigma(x_{q+1}), \ldots, \sigma(x_r)][\sigma(y_1), \sigma(y_2), \ldots, \sigma(y_q), y_{q+1}, \ldots, y_r] = 0.$$

The sum is over combinations σ of q of the symbols $y_1, \ldots, y_q, x_q, \ldots, x_r$ chosen for insertion into the first q places in the second bracket and the remaining $r - q + 1$ symbols inserted into the last $r - q + 1$ places in the first bracket. The sign of σ is just the usual sign of σ when σ is thought of as a permutation. If $x_q > y_q$, then $[x_1, \ldots, x_d][y_1, \ldots, y_d]$ is the greatest tableau in the syzygy in either lexicographic or dual reverse lexicographic order on tableaux, and, furthermore, the violation of standardness by x_q and y_q in column q is corrected in all other terms of the syzygy. The above syzygy will be abbreviated by the following underline notation:

$$\left[\begin{array}{ccccccc} x_1 & x_2 & \cdots & \underline{x_q} & \underline{x_{q+1}} & \underline{\cdots} & \underline{x_r} \\ \underline{y_1} & \underline{y_2} & \underline{\cdots} & \underline{y_q} & y_{q+1} & \cdots & y_r \end{array}\right].$$

That all such syzygies are identically zero is easily seen, for the syzygy itself may be regarded as an alternating multilinear form in at least $r+1$ vector variables from a vector space of dimension r.

Another useful syzygy is the *multiple syzygy*,

$$\begin{aligned} &[x_1, x_2, \ldots, x_{r-q}, x_{r-q+1}, \ldots, x_r][y_1, y_2, \ldots, y_r] \;- \\ &\sum [x_1, x_2, \ldots, x_{r-q}, y_{i_1}, \ldots, y_{i_q}][y_1, y_2, \ldots, y_r \leftarrow x_{r-q+1}, \ldots, x_r] \quad = \quad 0, \end{aligned}$$

where the arrow denotes substitution of $x_{r-q+1}, \ldots, x_r$ for $y_{i_1}, \ldots, y_{i_q}$ in order, and the sum is over all q-combinations $i_1 < i_2 < \ldots < i_q$ from $1, \ldots, r$. This syzygy will be abbreviated

$$\left[\begin{array}{ccccccc} x_1 & x_2 & \cdots & x_{r-q} & \underline{x_{r-q+1}} & \underline{\cdots} & \underline{x_r} \\ \underline{y_1} & \underline{y_2} & \underline{\cdots} & \underline{y_{r-q}} & \underline{y_{r-q+1}} & \underline{\cdots} & \underline{y_r} \end{array}\right].$$

Because of the special case of van der Waerden syzygy we are using, as above, the notation for van der Waerden and multiple syzygies will not coincide except in the particular situation of the last element of one bracket and the entire next bracket being underlined. However, in this case, the two types of syzygies are easily seen to coincide, and this syzygy is usually referred to as an *ordinary syzygy*. The ordinary syzygies are the well-known Grassmann-Plücker relations, and in general they can interchange any fixed element of one bracket with variable elements of another bracket.

5. A strategy to choose which violation of standardness in a given tableau T will be corrected.

One typical choice is to find the first pair of consecutive rows of T, starting from the top, which contain a violation, and choose the first column from the left containing a violation in those two rows. This will be referred to as the *first-first* strategy. Another choice is the analogous *last-last* strategy. Other strategies may involve choosing a violation in non-consecutive rows.

6 A strategy to choose which syzygy in item 4 will be used to correct the violation in item 5.

The straightening algorithm can now be described as a whole. The input bracket polynomial is first broken into its homogeneous components with respect to the $\mathbf{N}^N$-grading. The algorithm is then applied to each component separately. The terms of the homogeneous polynomial are sorted according to the ordering in item 3. The greatest non-standard tableau T is selected, and one of its violations corrected according to items 5 and 6. This means that a syzygy is applied which rewrites T as a linear combination of smaller tableaux, each of the same $\mathbf{N}^N$-grade as T. Thus the greatest non-standard tableau remaining is smaller than previously, and since there are only a finite number of tableaux of a given $\mathbf{N}^N$-grade, the algorithm must terminate, outputing a linear combination of standard tableaux.

As shown in [10], the straightening algorithm is a special case of the normal form algorithm with respect to a Gröbner basis. On the other hand, the terminology of standard and non-standard terms can be very useful when working with Gröbner bases. Items 1 and 2 correspond to the choice of an ordering on the variables in Gröbner basis theory. Two steps are needed to specify the ordering, since the variables are the brackets, which have another level of combinatorial structure, namely, the points. Item 3, the tableau order, is the admissable order on monomials. The choice of syzygies in Item 4 is simply the choice of a particular Gröbner basis, which is far from unique. A reduced Gröbner basis, which is unique, corresponds to a syzygy for each non-standard tableau of two rows, which expresses that tableau directly as a linear combination of standard tableaux. In the straightening algorithm, it is generally not practical to compute the reduced Gröbner basis ahead of time. The strategies in Items 5 and 6 amount to a strategy for an order in which to reduce terms in the normal form algorithm. As we shall see, there are many ways to choose such a strategy in general, and the choice may have a significant effect on the efficiency of the algorithm, even if a reduced Gröbner basis is used. This problem seems not to have been sufficiently studied in Gröbner basis theory.

2. Implementation of the Straightening Algorithm. There is a wide variety of choices in the orderings, syzygies, and strategies of a straightening algorithm. All must give the same output for a given input, since the standard tableaux form a linear basis for all tableaux. What effect do these different choices make on the efficiency of the algorithm?

First we observe that any bracket straightening algorithm is inherently exponential. The number of standard tableaux of a given shape is given by the well-known hook-length formula [6], which is factorial in the number of entries of the tableau. An input of a single non-standard tableau may require most of these standard tableaux to appear in its output.

We implemented a number of variations of the straightening algorithm, and will now describe some of the relatively more efficient ones. All were written in FORTRAN and run on an Apollo Domain 3000. All of them keep two separate stacks of tableaux, the first having both standard and non-standard tableaux sorted in tableau order, and the second having only standard tableaux which are greater in tableau order than any tableau in the first stack. The program then looks at the greatest tableau in the first stack, and if it is standard, moves it to the second stack, but if non-standard, applies the appropriate syzygy and returns all tableaux resulting from the syzygy to the first stack by a mergesort. All of the variations have a limit of 5000 tableaux in each of the two stacks. All the variations discussed below use lexicographic order on tableaux, since a variation using dual reverse lexicographic order may be converted to lexicographic order by rotating all tableaux by 180°, reversing the order on the points, switching from reverse lexicographic on rows to lexicographic or vice-versa, and suitably modifying the syzygies and strategies used. The variations discussed also use only lexicographic order on the rows, since a few variations using reverse lexicographic order on rows and lexicographic order on tableaux were found not to be competitive. The two simplest variations, using only van der Waerden syzygies, or else only multiple syzygies and lexicographic order on both rows and tableaux, were both found not to be competitive.

We now list five variations which were found to be relatively competitive. In the following descriptions, $a, b, \ldots$ are the first points in order, and $\ldots, y, z$ the last points. Let T be the tableau being currently examined, that is, the greatest tableau in the first stack.

A. If the first pair of consecutive rows of T which have a violation have one in the last column but not in the second column, use an ordinary syzygy on the last element of the former row and the entire latter row. If there are violations in the second and last columns, use an ordinary syzygy on the first element of the former row and the entire latter row. If there is no violation in the last column of the two rows, then correct the first violation in those two rows using a van der Waerden syzygy (a first-first strategy). For example,

$$\left[\begin{array}{cccc} a & b & \ldots & z \\ & & & \text{---} \\ b & c & \ldots & y \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array}\right],$$

$$\left[\begin{array}{cccc} a & d & \ldots & z \\ \text{---} & & & \\ b & c & \ldots & y \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array}\right].$$

B. Examine the first two consecutive rows of T which have a violation. Let the s-th column be the first column having a violation in those two rows. If the last column or two of the last three columns have a violation in those two rows, then use a multiple syzygy on the s-th through last points in the former row, and the entire latter row. Otherwise use a van der Waerden syzygy on the violation in the s-th column (a first-first strategy). For example,

$$\left[\begin{array}{ccccc} a & \ldots & k & \ldots & z \\ & & \text{---} & \text{---} & \text{---} \\ b & \ldots & j & \ldots & y \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \end{array}\right],$$

$$\begin{bmatrix} a & \cdots & \underline{k} & \underline{\cdots} & \underline{w} & \underline{x} & \underline{y} \\ \underline{b} & \underline{\cdots} & \underline{j} & \underline{\cdots} & \underline{u} & \underline{w} & \underline{z} \end{bmatrix}.$$

C. This is the same as B, except that if those two rows have a violation in the last column, say z over y, then the last column is searched for the least element situated below z, and a multiple syzygy is performed as in B on the row of z and that least element. Thus we may be correcting a violation in non-consecutive rows.

D. This is the same as C, except that only the last column non-consecutive row multiple syzygies and the van der Waerden syzygies are used, not the multiple syzygies when there are violations in two of the last three columns.

E. Find the last violation in the first pair of consecutive rows having a violation, and do van der Waerden syzygies.

In Table 1, we describe the performance of the five above variations on seventeen different inputs. For each input, we list r, the dimension or row length, d, the **N**-degree or column length, and N, the number of distinct points among the rd entries in each tableau. This is followed by the number of terms of the input, the number of terms of the output, and the performance of each of the five variations. This performance is measured by the number of syzygies performed times the average size of the first stack of tableaux. This number is approximately proportional to the execution time, excluding time for input, output, and an initial sort. An hour of execution time on the Apollo 3000 corresponded to approximately 2.2 million, a minute to about 30,000. In the table, "e" denotes that the stack size was exceeded, "m" denotes million and "k" denotes thousand. The best performance for each input is indicated in boldface.

We note that the relative performances of the five variations, as shown in Table 1, are quite erratic. The best overall were perhaps C and D, but they were beaten badly by each of the others on certain inputs. If time but not space were the only consideration, then a parallel implementation of all five of these and perhaps more variations would be the fastest straightening algorithm. We do not claim to be exhaustive in our investigation, and it is possible that better variations exist.

TABLE 1

Performance of Five Variations

	r	d	N	In	Out	A	B	C	D	E
1.	5	2	10	12	16	709	**185**	**185**	709	347
2.	4	3	12	1	6	24k	1557	**7**	**7**	22k
3.	5	3	12	1	115	13k	16k	9k	9k	**2.6k**
4.	3	6	9	9	0	**13k**	**13k**	21k	21k	52k
5.	5	4	15	1	2677	6m	**1.2m**	1.3m	1.5m	9m
6.	6	3	14	1	508	159k	115k	**109k**	147k	158k
7.	6	2	12	1	104	**603**	1206	1206	**603**	802
8.	5	4	13	1	78	4.6k	1.8k	**1.6k**	**1.6k**	6.6k
9.	5	4	14	1	314	95k	29k	**7.3k**	**7.3k**	65k
10.	3	6	10	2	27	**303**	**303**	**303**	**303**	372
11.	4	3	12	9	3	**171**	3.6k	7.5k	2.8k	240
12.	3	5	15	24	526	368k	368k	467k	467k	**365k**
13.	4	4	16	27	2111	e	28m	28m	**16m**	e
14.	3	6	10	2	245	75k	75k	**59k**	**59k**	79k
15.	8	3	14	1	60	3k	**181**	**181**	**181**	5k
16.	8	3	15	1	295	31k	9.7k	**9.2k**	14k	36k
17.	8	3	18	1	2531	e	**2.042m**	**2.042m**	2.044m	e

3. The Turnbull-Young Invariant. The Turnbull-Young invariant [12] is a 240-term bracket polynomial with $r = 5$, $d = 4$, and $N = 10$, which is zero precisely when the 10 points lie on a common quadric surface in projective 3-space. This polynomial was used as input for our variations C and D. The stack size was exceeded by C, but D produced the output after about two days. This output appears in Table 2; this is probably the first time that this invariant has been straightened. A numerical factor of 20 appears, as predicted by Turnbull and Young. A very interesting question is whether there exists a theorem analogous to Pascal's theorem for 6 points on a planar conic, which would give an incidence property of ten points equivalent to their lying on a common quadric. Such a property would amount to a Cayley factorization (see [15]) of the Turnbull-Young invariant, or of some bracket multiple of it. The straightened output of this invariant was input into another program which attempts to find such a Cayley factorization, by identifying certain points and re-straightening. This program exceeded the stack size in one of these later straightenings, but it had proceeded far enough that it appears very unlikely that the Turnbull-Young invariant can be Cayley factored directly. Sturmfels and Whiteley [11] show that some bracket multiple of any such invariant must be Cayley factorable, hence there exists a projective geometric incidence theorem. However, this theorem may so complex as to be uninteresting.

TABLE 2

Straightened form of the Turnbull-Young Invariant

[0136]	[0247]	[1258]	[3459]	[6789]	(-20)
[0135]	[0247]	[1268]	[3469]	[5789]	(20)
[0135]	[0246]	[1278]	[3479]	[5689]	(-20)
[0134]	[0257]	[1268]	[3569]	[4789]	(-20)
[0134]	[0256]	[1278]	[3579]	[4689]	(20)
[0134]	[0256]	[1258]	[3479]	[6789]	(-20)
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[0134]	[0246]	[1258]	[3579]	[6789]	(20)
[0134]	[0246]	[1257]	[3589]	[6789]	(-20)
[0134]	[0245]	[1268]	[3579]	[6789]	(-20)
[0134]	[0245]	[1267]	[3589]	[6789]	(20)
[0134]	[0245]	[1256]	[3789]	[6789]	(20)
[0134]	[0236]	[1258]	[4579]	[6789]	(-20)
[0134]	[0236]	[1257]	[4589]	[6789]	(20)
[0134]	[0235]	[1268]	[4579]	[6789]	(20)
[0134]	[0235]	[1267]	[4589]	[6789]	(-20)
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[0124]	[0357]	[1368]	[2569]	[4789]	(20)
[0124]	[0356]	[1378]	[2579]	[4689]	(-20)
[0124]	[0356]	[1358]	[2479]	[6789]	(20)
[0124]	[0356]	[1357]	[2489]	[6789]	(-20)
[0124]	[0346]	[1358]	[2579]	[6789]	(-20)
[0124]	[0346]	[1357]	[2589]	[6789]	(20)
[0124]	[0345]	[1368]	[2579]	[6789]	(20)
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[0124]	[0236]	[1357]	[4589]	[6789]	(20)
[0124]	[0235]	[1368]	[4579]	[6789]	(20)
[0124]	[0235]	[1367]	[4589]	[6789]	(-20)
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[0123]	[0145]	[2456]	[3789]	[6789]	(40)
[0123]	[0145]	[2378]	[4679]	[5689]	(20)
[0123]	[0145]	[2368]	[4679]	[5789]	(20)
[0123]	[0145]	[2368]	[4579]	[6789]	(40)

TABLE 2 Continued

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[0124]	[0135]	[2356]	[4789]	[6789]	(20)	[0123]	[0145]	[2357]	[4689]	[6789]	(20)
[0124]	[0135]	[2348]	[5679]	[6789]	(20)	[0123]	[0145]	[2356]	[4789]	[6789]	(-60)
[0124]	[0135]	[2347]	[5689]	[6789]	(-20)	[0123]	[0145]	[2348]	[5679]	[6789]	(40)
[0124]	[0135]	[2346]	[5789]	[6789]	(20)	[0123]	[0145]	[2346]	[5789]	[6789]	(60)
[0124]	[0135]	[2345]	[6789]	[6789]	(-20)	[0123]	[0145]	[2345]	[6789]	[6789]	(-60)
[0124]	[0134]	[2358]	[5679]	[6789]	(-20)	[0123]	[0137]	[2458]	[4569]	[6789]	(-20)
[0124]	[0134]	[2357]	[5689]	[6789]	(20)	[0123]	[0136]	[2458]	[4579]	[6789]	(20)
[0124]	[0134]	[2356]	[5789]	[6789]	(-40)	[0123]	[0136]	[2457]	[4589]	[6789]	(20)
[0123]	[0457]	[1468]	[2569]	[3789]	(-20)	[0123]	[0135]	[2467]	[4589]	[6789]	(-20)
[0123]	[0456]	[1478]	[2579]	[3689]	(20)	[0123]	[0135]	[2458]	[4679]	[6789]	(-20)
[0123]	[0356]	[1458]	[2479]	[6789]	(-20)	[0123]	[0134]	[2568]	[4579]	[6789]	(40)
[0123]	[0356]	[1457]	[2489]	[6789]	(20)	[0123]	[0134]	[2567]	[4689]	[5789]	(20)
[0123]	[0346]	[1457]	[2589]	[6789]	(-20)	[0123]	[0134]	[2467]	[5689]	[5789]	(-20)
[0123]	[0345]	[1468]	[2579]	[6789]	(20)	[0123]	[0134]	[2456]	[5789]	[6789]	(20)
[0123]	[0345]	[1458]	[2679]	[6789]	(-20)	[0123]	[0127]	[3468]	[4569]	[5789]	(20)
[0123]	[0345]	[1457]	[2689]	[6789]	(20)	[0123]	[0127]	[3458]	[4569]	[6789]	(20)
[0123]	[0345]	[1456]	[2789]	[6789]	(-20)	[0123]	[0126]	[3478]	[4579]	[5689]	(-20)
[0123]	[0257]	[1468]	[3469]	[5789]	(-20)	[0123]	[0126]	[3458]	[4579]	[6789]	(-20)
[0123]	[0257]	[1458]	[3469]	[6789]	(20)	[0123]	[0126]	[3457]	[4589]	[6789]	(-40)
[0123]	[0256]	[1478]	[3479]	[5689]	(20)	[0123]	[0125]	[3478]	[4679]	[5689]	(20)
[0123]	[0256]	[1457]	[3489]	[6789]	(-20)	[0123]	[0125]	[3468]	[4679]	[5789]	(20)
[0123]	[0247]	[1468]	[3569]	[5789]	(20)	[0123]	[0125]	[3467]	[4689]	[5789]	(-20)
[0123]	[0247]	[1368]	[4569]	[5789]	(-20)	[0123]	[0125]	[3467]	[4589]	[6789]	(60)
[0123]	[0247]	[1358]	[4569]	[6789]	(-20)	[0123]	[0124]	[3567]	[4689]	[5789]	(-20)
[0123]	[0246]	[1478]	[3579]	[5689]	(-20)	[0123]	[0124]	[3567]	[4589]	[6789]	(-60)
[0123]	[0246]	[1457]	[3589]	[6789]	(-20)	[0123]	[0124]	[3478]	[5679]	[5689]	(-20)
[0123]	[0246]	[1378]	[4579]	[5689]	(20)	[0123]	[0124]	[3468]	[5679]	[5789]	(-20)
[0123]	[0246]	[1358]	[4579]	[6789]	(20)	[0123]	[0124]	[3467]	[5689]	[5789]	(40)
[0123]	[0246]	[1357]	[4589]	[6789]	(40)	[0123]	[0124]	[3457]	[5689]	[6789]	(-20)
[0123]	[0245]	[1478]	[3679]	[5689]	(20)	[0123]	[0123]	[4567]	[4689]	[5789]	(20)
[0123]	[0245]	[1468]	[3679]	[5789]	(20)	[0123]	[0123]	[4567]	[4589]	[6789]	(60)

4. Acknowledgements. I wish to thank Bernd Sturmfels for programming an early version of the straightening algorithm, and for providing several helpful subroutines. I also thank Joel Stein for help in preparing the input of the Turnbull-Young invariant.

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