Low-dimensional quantum gases out of equilibrium, Minneapolis, May 2012

Real-Space RG for dynamics of random spin chains and many-body localization

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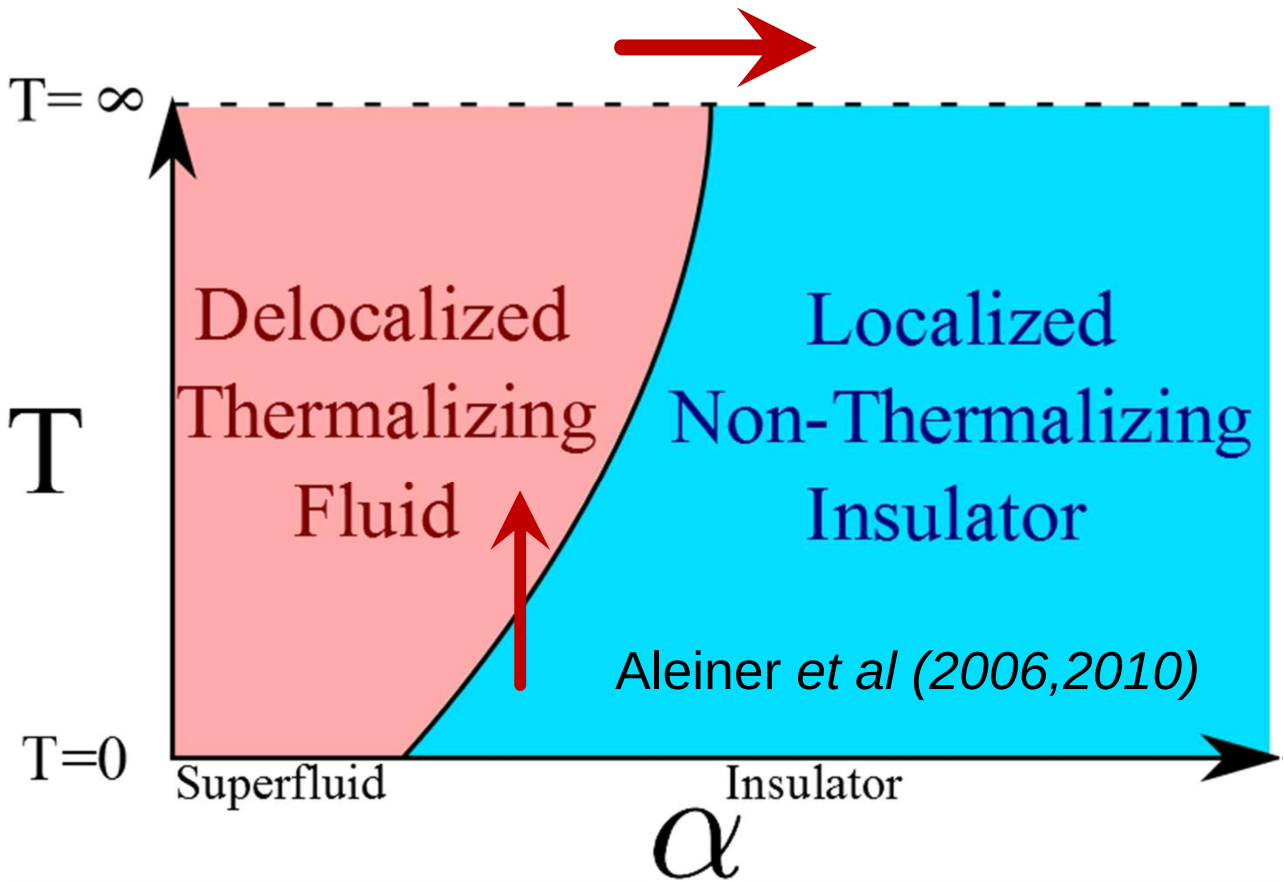
See: Ronen Vosk and E.A. arXiv:1205.0026

Thanks: D. Huse, A. Polkovnikov
G. Refael, Y. Kafri, J. E. Moore, F. Pollmann

Minerva-Weizmann program

Many-Body Localization



If the model has bounded spectrum, one can attempt to drive the transition at inifinite temperature

Oganesyan and Huse *(2007), Pal and Huse (2010)*

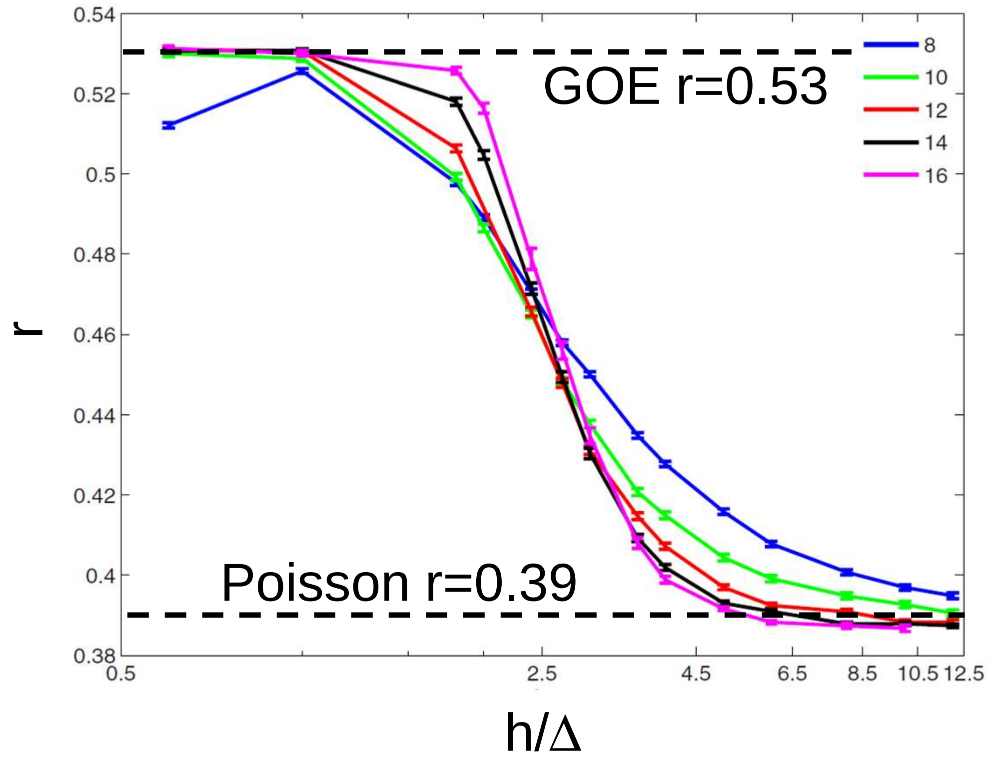
Disordered Spin Chains

A. Pal and D. Huse, Physical Review B **82**, 1 (2010)

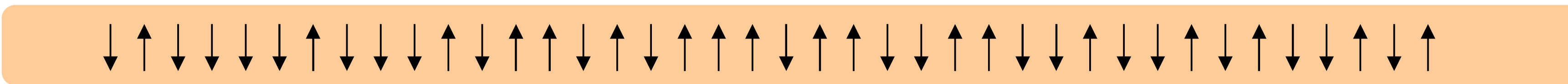
$$H = \frac{1}{2}\sum_{ij}\left(S_i^+ S_j^- + \mathrm{H.c}\right) + \Delta \sum_{ij} S_i^z S_j^z + \sum_i h_i S_i^z \qquad h_i \in [-h, h]$$

= interacting fermions: $$H = \frac{1}{2}\sum_{ij}\left(a_i^\dagger a_j + \mathrm{H.c.}\right) + \sum_i h_i n_i + \Delta \sum_{ij} n_i n_j$$

Ratio of adjacent energy gaps from exact diagonalization of 16 sites:



Thermalization and dynamics of entanglement entropy

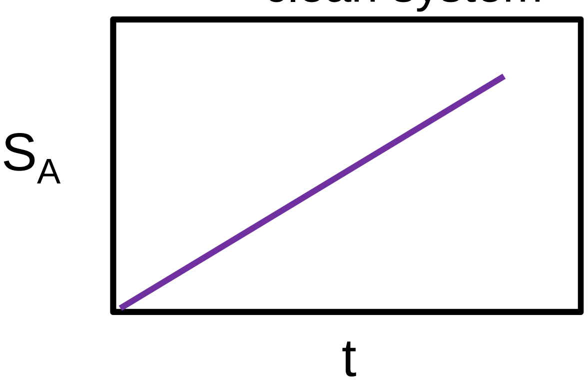


$$e^{-iHt}\,|\,\Psi_0\,\rangle \qquad H = \frac{1}{2}\sum_{ij}\left(S_i^+ S_j^- + \mathrm{H.c}\right) + \Delta\sum_{ij} S_i^z S_j^z + \sum_i h_i S_i^z$$

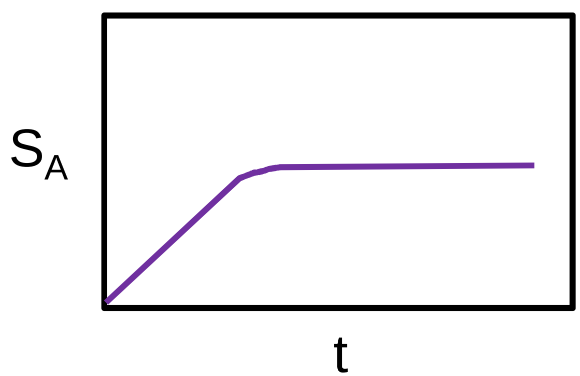
$$h_i \in [-h, h]$$

Von-Neuman entropy generated in the dynamics: $S_A(t) = -Tr\left[\rho_A(t)\ln\rho_A(t)\right]$

clean system



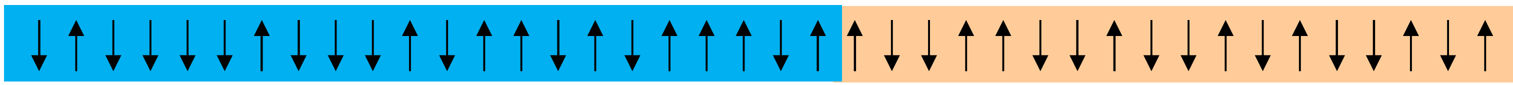
localized system (?)



$S_{saturation} \sim \xi_{localization}$

Bounded entanglement allows efficient numerics (using DMRG).
Approach transition from the localized side?

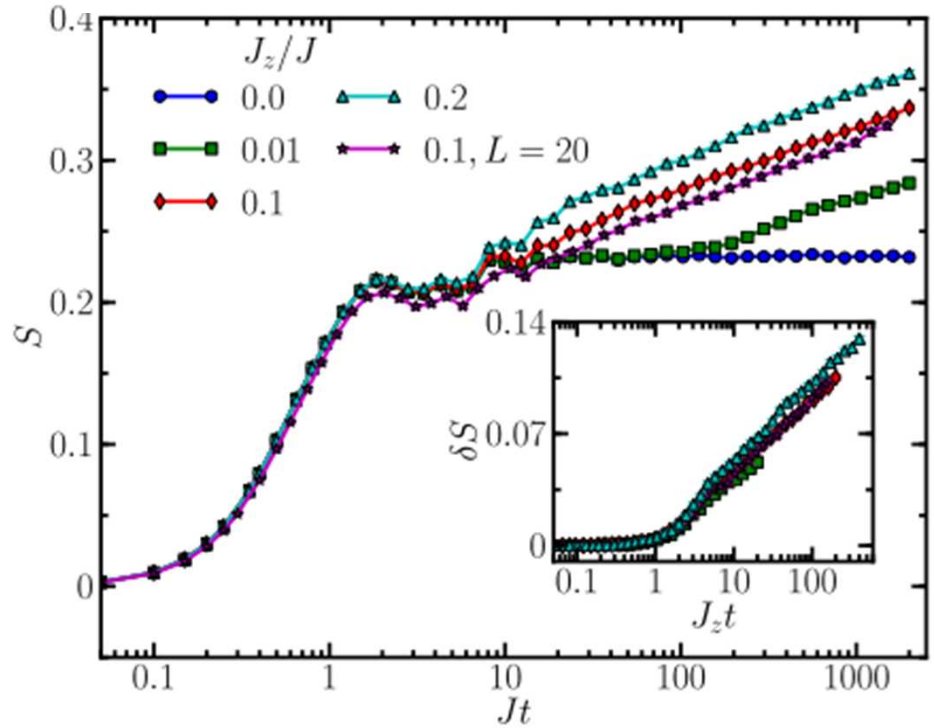
Entanglement dynamics: numerics



$$H = \frac{1}{2}\sum_{ij}\left(S_i^+ S_j^- + \text{H.c}\right) + J_z \sum_{ij} S_i^z S_j^z + \sum_i h_i S_i^z \qquad h_i \in [-h, h]$$

log(t) increase seen in the interacting disordered model.

Bardarson et. al. arXiv:1202.5532

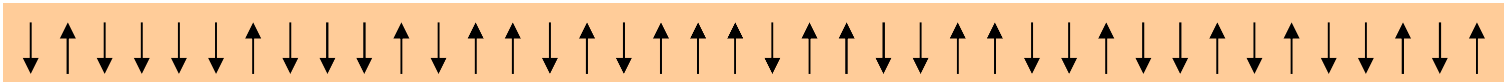


See also earlier numerical studies:
De Chiara et. al. J. Stat. Mech (2006);
Znidaric et. al. PRB (2008)

Outline

- Real space RG for quantum time evolution in strong disorder.
  Basic idea and scheme
- Application: random spin chain quenched from AFM state.
  Flow to Infinite randomness fixed point
- Evolution of entanglement entropy and number fluctuations.
  Non-thermalization and asymptotic GGE
- Basin of attraction of the infinite randomness fixed point.
  Criterion for the Many-body localization transition?

Real space RG for the dynamics, general scheme



Model: $$H = \frac{1}{2}\sum_{ij} J_{ij}\left(S_i^+ S_j^- + S_i^- S_j^+ + 2\Delta_i S_i^z S_j^z\right)$$

Basic idea:

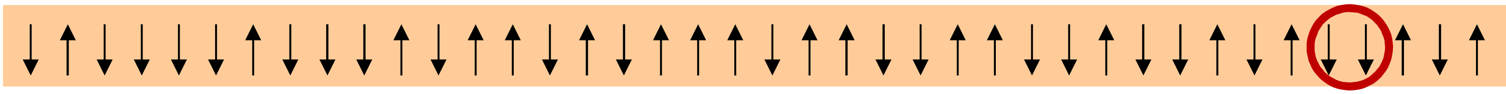
large local separation of scales (disorder)

solve the local fast time evolution exactly

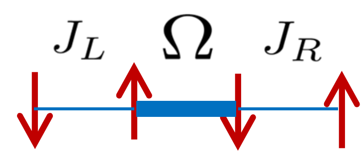
Compute effect on rest of chain perturbatively

Related to but somewhat different philosophy than RSRG that targets the ground state (Dasgupta & Ma 1980, D. Fisher 1994, …)

Real space RG for the dynamics, general scheme

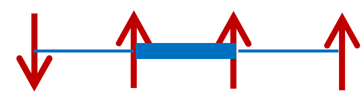


1. Choose pairs of spins coupled by the largest J=Ω.



These pairs perform rapid oscillations
(frequency Ω) if initially anti-aligned.

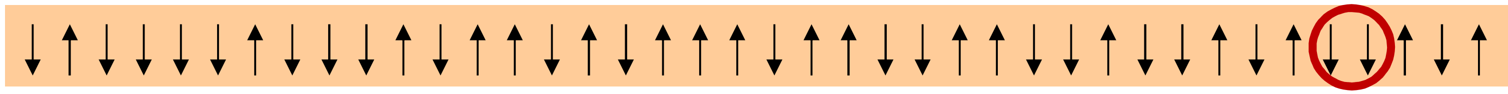
Or remain stuck if initially parallel



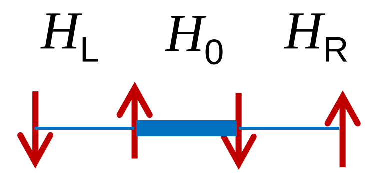
That is all we have at time scale $t \approx \Omega^{-1}$

all other spins are essentially frozen!

Real space RG for the dynamics, general scheme



2. Compute effective dynamics at times t>>Ω^{-1} (eliminating frequencies of order Ω)



$$\rho(t) = \left[U_I^\dagger \rho_0 U_I\right]_{\Omega^{-1}} = e^{iH_{eff}t}\rho_0 e^{-iH_{eff}t}$$

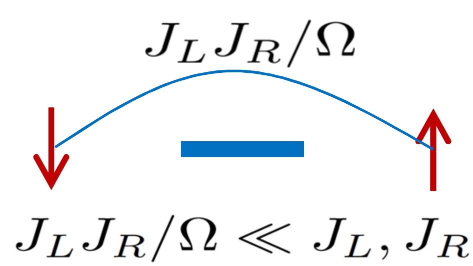
2nd order expansion of U in the interaction picture w.r.t H_0

Average over rapid oscillations

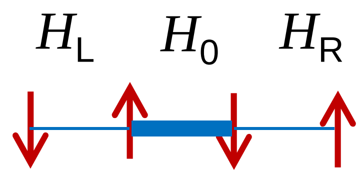
H_{eff} depends both on H and on initial state !

$$\rho_0 = |\psi_0^p\rangle\langle\psi_0^p|\,\rho_0^R$$

Example: result in simplest case (Δ=0):



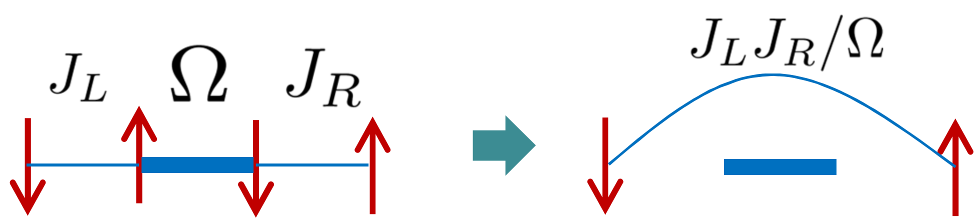
Perturbation expansion of the evolution operator:



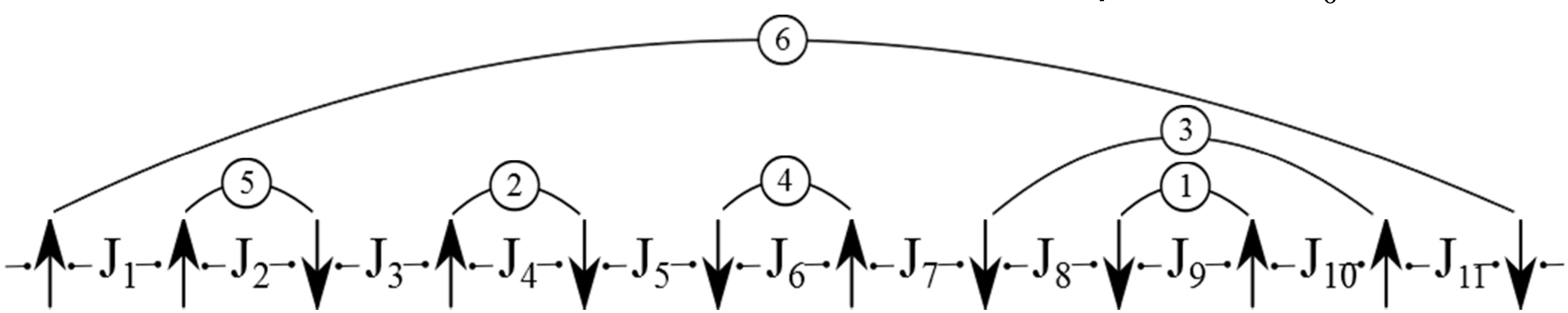
$$U_I = 1 - \frac{i}{\hbar}\int_0^t dt_1 e^{\frac{i}{\hbar}H_0 t_1}\left(H_R + H_L\right)e^{-\frac{i}{\hbar}H_0 t_1}$$

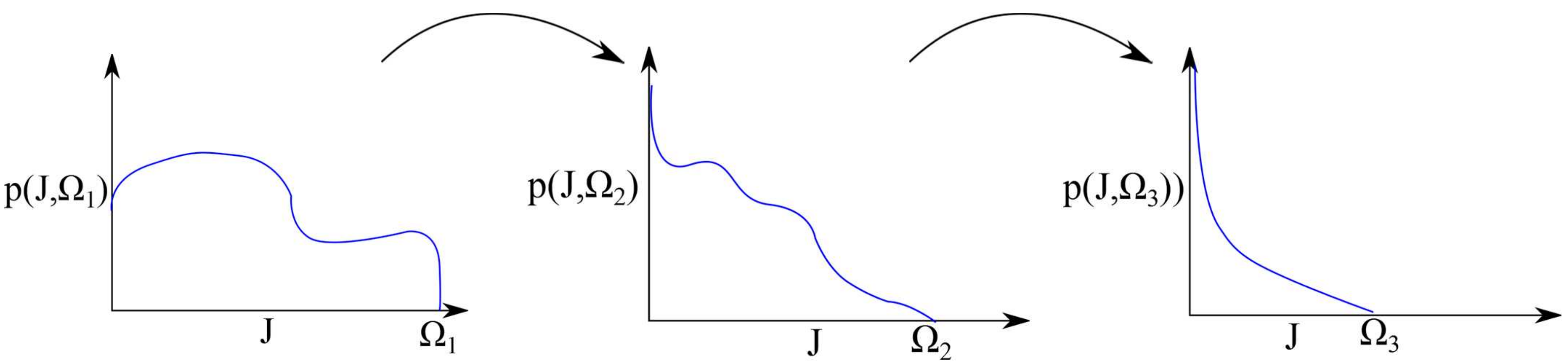
$$- \frac{1}{\hbar^2}\int_0^t dt_1 \int_0^{t_1} dt_2 e^{\frac{i}{\hbar}H_0 t_1}\left(H_R + H_L\right)e^{-\frac{i}{\hbar}H_0 t_1} e^{\frac{i}{\hbar}H_0 t_2}\left(H_R + H_L\right)e^{-\frac{i}{\hbar}H_0 t_2}$$

Real space RG for dynamics, general scheme



To compute the evolution up to time $t \approx \Omega^{-1}$, successively eliminate all pairs precessing at frequencies Ω_0 down to Ω.

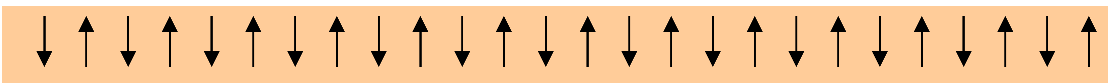




The RG decimation steps for our model

$$H = \frac{1}{2}\sum_{ij} J_{ij}\left(S_i^+ S_j^- + S_i^- S_j^+ + 2\Delta_i S_i^z S_j^z\right)$$

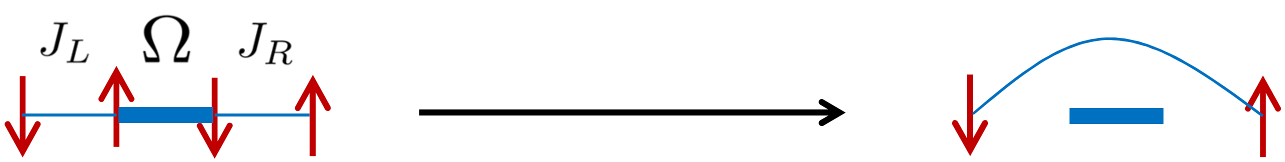
$\Delta_i << 1$



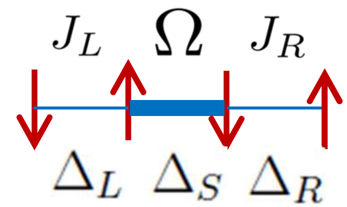
Take z-antiferromagnetic initial state
(Then we never have strong bonds with aligned spins)

<u>Simplest case Δ=0 (non-interacting):</u>

$$H_{eff} = \frac{J_L J_R}{2\Omega}(S_L^+ S_R^- + \mathrm{H.c.})$$



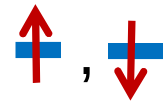
The RG decimation step for Δ>0



Need to keep track of a new spin on the strong bond

$$H_{\text{eff}} = \frac{J_L J_R}{2\Omega(1-\Delta_S^2)}\left(S_L^+ S_R^- + S_L^- S_R^+\right)$$

$$+\frac{\Delta_S J_L J_R}{2\Omega(1-\Delta_S^2)}\left[S_L^+ S_R^- + S_L^- S_R^+ - \frac{\Delta_L \Delta_R}{\Delta_S}(1-\Delta_S^2) S_L^z S_R^z\right] S_n^z$$



$$= |\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle\right)$$

The new spin initially points along x or –x therefore the evolution is a superposition of the dynamics given an up-spin on the bond and the dynamics with a down-spin:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}H_{\text{eff}}t}\frac{1}{\sqrt{2}}\left(|\uparrow_n\rangle|\psi_0^R\rangle \pm |\downarrow_n\rangle|\psi_0^R\rangle\right)$$

$$H_{\text{eff}} \approx \frac{J_L J_R}{2\Omega}\left(1 \pm \frac{\Delta_S}{2}\right)\left(S_L^+ S_R^- + S_L^- S_R^+ \mp 2\frac{\Delta_L \Delta_R}{4} S_L^z S_R^z\right)$$

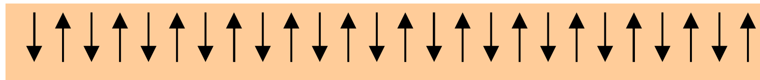
This generates entanglement between decimated bond and the nearby spins after a time

$$t_{\text{ent}} = \frac{2\Omega}{J_L J_R \Delta_S}$$

But no effect on subsequent renormalization of coupling constants!

$$\tilde{J} \approx J_L J_R / \Omega \qquad |\tilde{\Delta}| \approx |\Delta_L||\Delta_R|/4$$

Flow of distributions for initial Neel state



Scaling variables:

$$\Gamma = \ln(\Omega_0/\Omega) = \ln(\Omega_0 t)$$

$$\zeta = \ln(\Omega/J) \qquad \beta = -\ln|\Delta|$$

RG rules:

$$\zeta_L + \zeta_R \to \tilde{\zeta}$$

$$\beta_L + \beta_R - \ln 4 \to \tilde{\beta}$$

Flow equations for distributions:

$$\frac{\partial \rho(\zeta)}{\partial \Gamma} = \frac{\partial \rho(\zeta)}{\partial \zeta} + \rho(0) \int_0^\infty d\zeta_L d\zeta_R \delta(\zeta - \zeta_L - \zeta_R) \rho(\zeta_L) \rho(\zeta_R)$$

$$\frac{\partial f(\beta)}{\partial \Gamma} = \rho(0) \int_0^\infty d\beta_L d\beta_R \delta(\beta - \beta_L - \beta_R + \ln 4) f(\beta_L) f(\beta_R) - f(\beta)\rho(0)$$

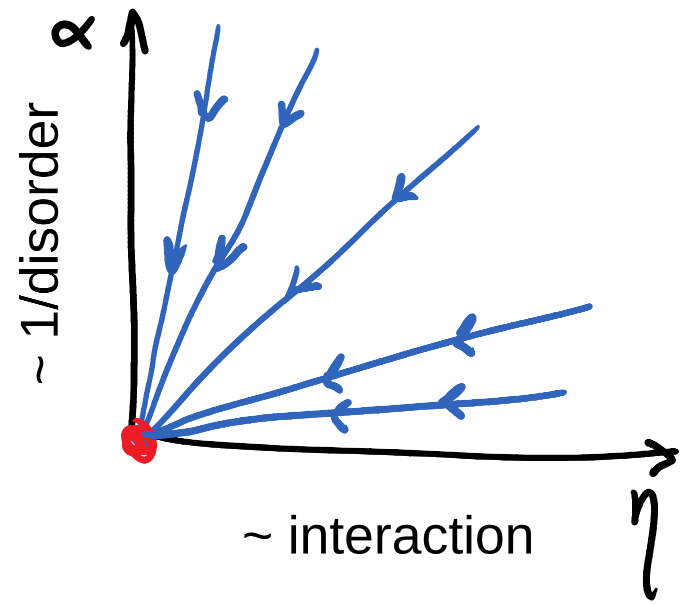
Solution of the flow equations

$$\rho(\zeta,\Gamma) = \alpha(\Gamma) e^{-\alpha(\Gamma)\zeta} \qquad \frac{d\alpha}{d\Gamma} = -\alpha^2 \qquad \alpha(\Gamma) = \frac{1}{\Gamma + \alpha_0^{-1}}$$

$$f(\beta,\Gamma) = \eta(\Gamma) e^{-\eta(\Gamma)\beta} \qquad \frac{d\eta}{d\Gamma} = -\alpha\eta \qquad \eta(\Gamma) = \frac{\eta_0}{a_0\Gamma + 1}$$

Or in the original variables:

$$P(J) = \frac{\alpha}{\Omega}(J/\Omega)^{\alpha-1} \qquad F(\Delta) = \eta|\Delta|^{\eta-1}$$

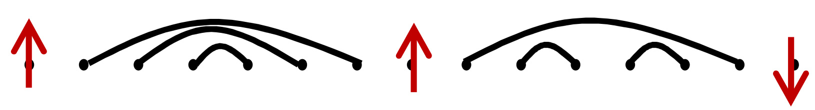


Flow to an infinite randomness fixed point!

Like the "Random singlet" phase of spin chains
(Dasgupta & Ma 80, Bhatt & Lee 82, Fisher 94)
Here oscillating pairs play the role of singlets

Relation between frequency (or time) scale and length scale:

$$L(\Gamma) = (\alpha_0\Gamma + 1)^2 \approx \alpha_0^2 \ln^2(\Omega_0 t)$$

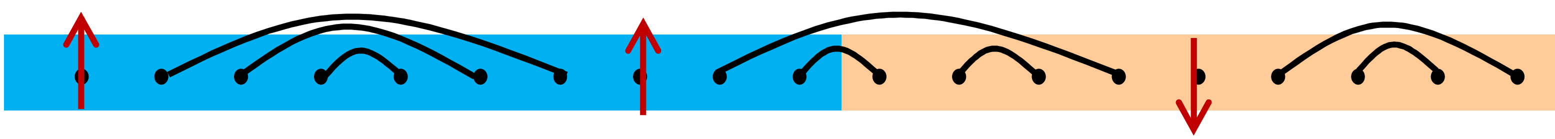


(Distance between remaining spins at that scale)

Evolution of the entanglement entropy

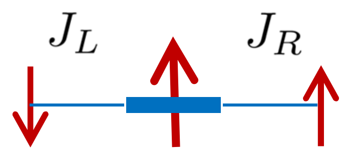
<u>Simplest case Δ=0 (“non interacting”):</u> Only intra-pair entanglement

Compute entanglement entropy by counting the number of decimated bonds that cut the interface.
Each decimated bond crossing the interface contributes ~log2.
(As in the ground state of random singlet phase – Refael & Moore PRL 2004)



$$S_{ent} \sim \int_0^{\Gamma} \alpha(\Gamma')d\Gamma' = \ln(\Gamma + \alpha_0^{-1}) = \ln\left(\ln(\Omega_0 t) + \alpha_0^{-1}\right)$$

Evolution of the entanglement entropy (Δ>0)



A bond eliminated at t_1 builds entanglement with neighbors only at a later time $t=t_1+t_{ent}$.

$$t_{\text{ent}} = \frac{2\Omega}{J_L J_R \Delta_S}$$

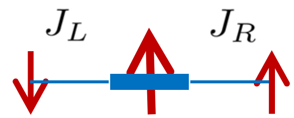
The interaction generates entanglement only after a delay time from the start of time evolution

$$t_{\text{delay}} \approx \frac{2\Omega_0}{J_0^2 \Delta_0} = \left(\frac{2\Omega_0}{J_0}\right)\frac{1}{J_0^z}$$

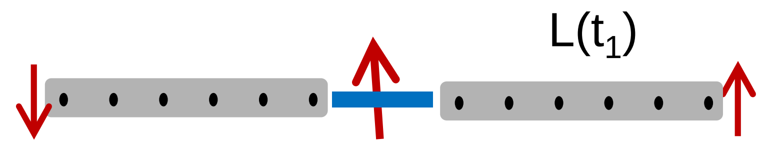
How much entanglement is generated?

Evolution of the entanglement entropy (Δ>0)

Entanglement measured at time t originates from pairs eliminated at earlier time t_1



Remaining spins at t_1 are separated by decimated clusters of length $L(t_1)$



By the time $t=t_1+t_{ent}$ that these spins become entangled the decimated clusters between them must also be entangled

$$S(t) \approx L(t_1) = \Big(\alpha_0 \ln(\Omega t_1) + 1\Big)^2$$

Using the relation between t_1 and t :

$$t = t_1 + t_{\rm ent} = t_1 \left(1 + \frac{2\Omega_1^2}{J_L J_R \Delta_S}\right) \approx t_1 \frac{2\Omega_1^2}{J_1^2 \Delta_1}$$

and the solutions of the flow equations, we have:

$$S(t) = \frac{1}{2}\Big(1 + \eta \ln\left(t/t_{\rm delay}\right)\Big)^2 \theta(t - t_{\rm delay})$$

$$\eta = \frac{-1}{\ln\Big((J_0/\Omega_0)^3 \Delta_0\Big)}$$

Evolution of the entanglement entropy (Δ>0)

$$S(t) = \frac{1}{2}\left(1 + \eta \ln\left(t/t_{\text{delay}}\right)\right)^2 \theta(t - t_{\text{delay}})$$

$$\eta = \frac{-1}{\ln\left((J_0/\Omega_0)^3 \Delta_0\right)}$$

After initial delay: $t_{\text{delay}} = \frac{2\Omega_0}{J_0 J_0^z}$

Crossover from log growth:

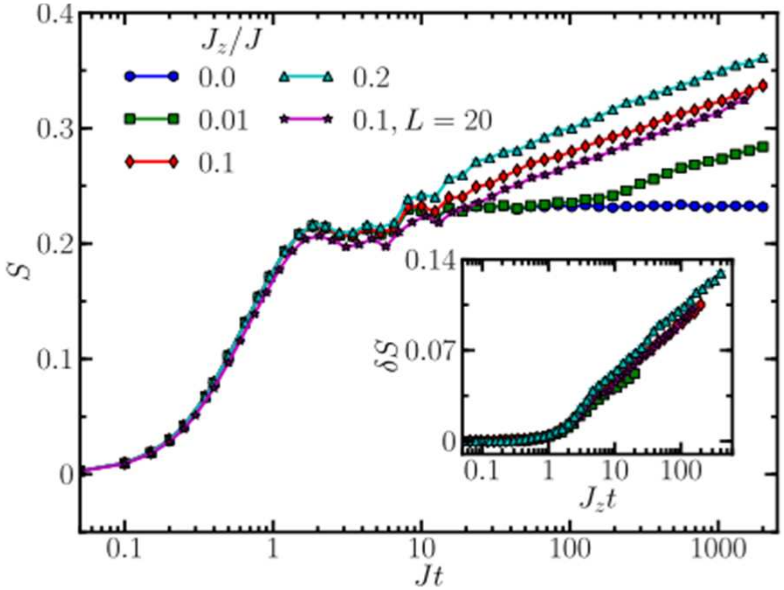
$$S(t) \sim \eta \ln(t/t_{\text{delay}}) \qquad t < t_*$$

to:

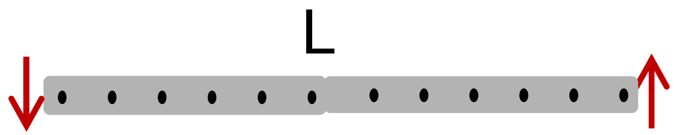
$$S(t) \sim \frac{1}{2}\eta^2 \ln^2(t/t_{\text{delay}}) \qquad t > t_*$$

$t_*/t_{\text{delay}} = \Omega_0^3/(J_0^2 J_0^z)$ >>1

Bardarson et. al. arXiv:1202.5532



Saturation of entanglement entropy in a finite system

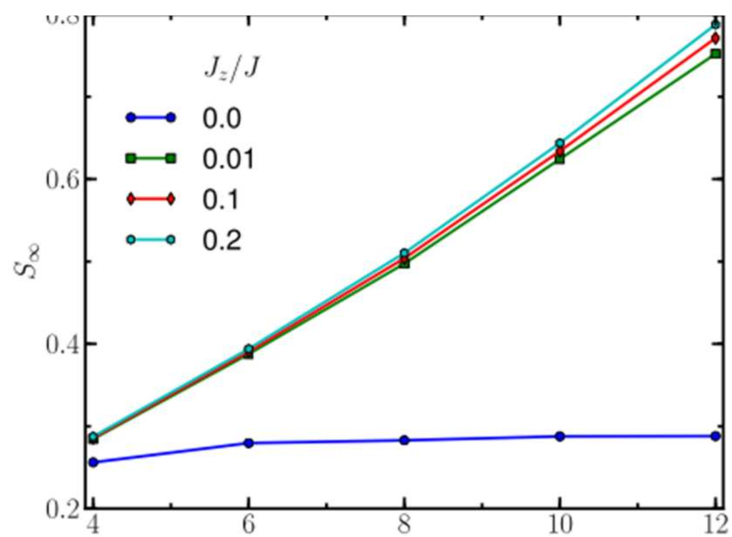


Saturation time: $t(L) = \Omega_0^{-1} e^{\Gamma(L)} \approx \Omega_0^{-1} e^{\sqrt{L}/\alpha_0}$

Entropy saturates to an extensive value: S(L) ~ L

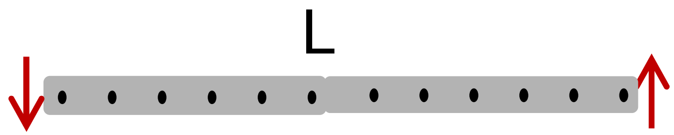
Bardarson et. al. arXiv:1202.5532

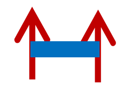
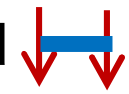
In agreement with the numerical results:



Saturation value is not the expected thermalized value S(L)= L ln2. Why?

Emergent conservation laws

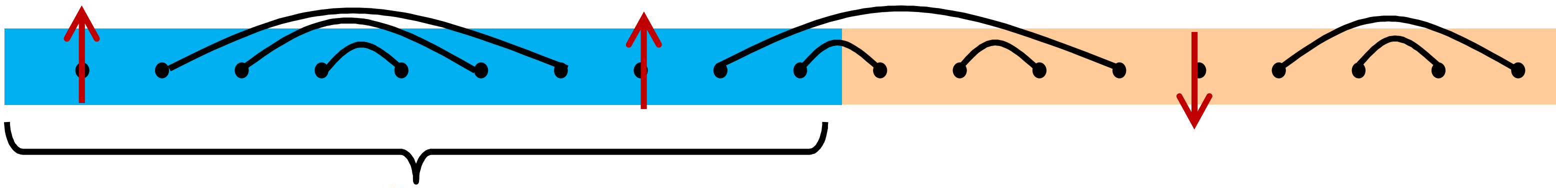


In every decimated pair of spins the states  and  are never populated therefore S(L)<(L/2)ln2

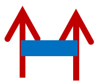
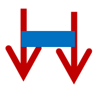
More generally $I_p = (S_1^z S_2^z)_p$ are approximate constants of motion (asymptotically exact for long distance pairs)

Many-body localization (non thermalization) $\stackrel{?}{=}$ emergent GGE

Evolution of particle number fluctuations



$$\langle \delta N^2(t) \rangle = ?$$

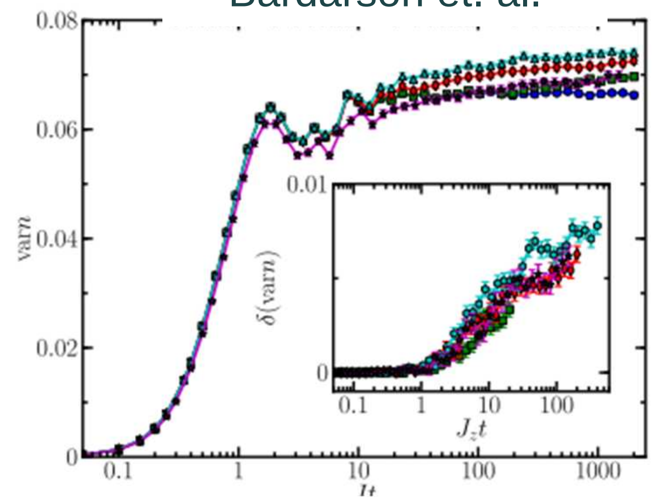
Since the  and  states of decimated pairs are not populated, only pairs that intersect the interface contribute to $\langle \delta N^2(t) \rangle$

$$\langle \delta N^2(t) \rangle = \int_0^{\Gamma} (t) d\Gamma' \alpha(\Gamma') = \ln\left(\ln(\Omega_0 t) + \alpha_0^{-1}\right)$$

Much slower than entanglement growth
and independent of interaction!

Saturates to a non-extensive value
in a finite system: $\langle \delta N^2(\infty) \rangle \sim \ln L$

Bardarson et. al.

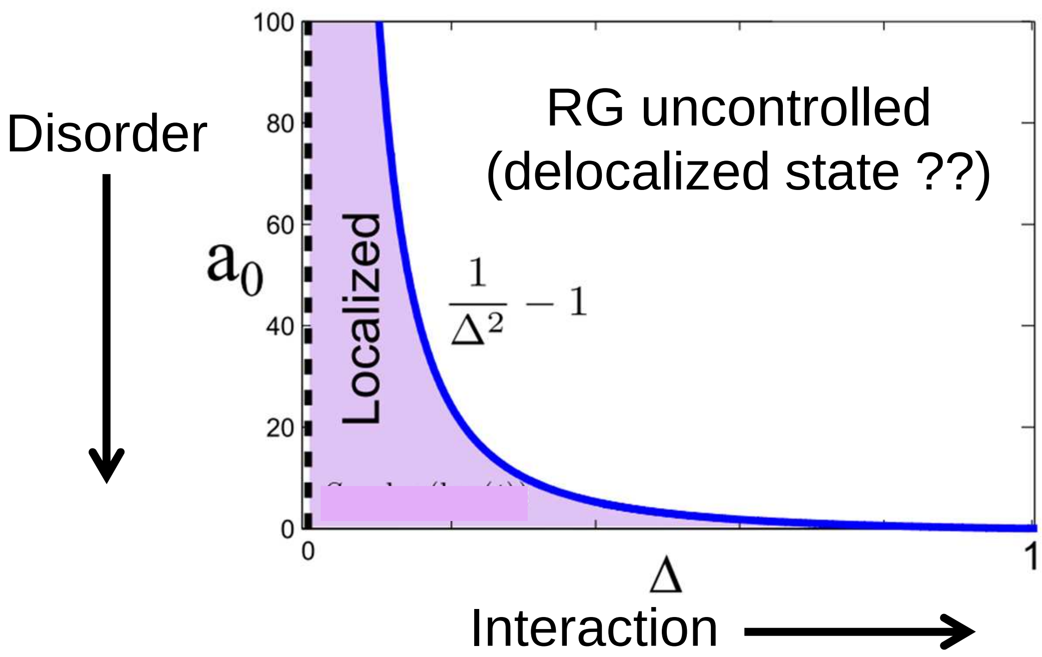


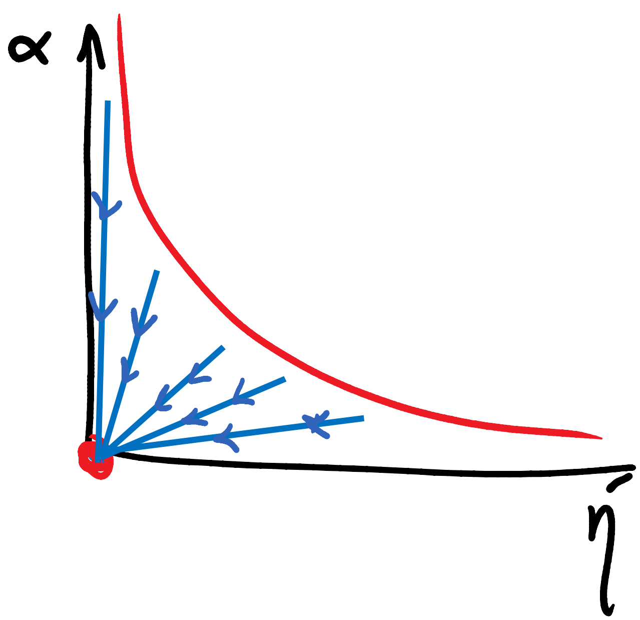
Phase diagram - Extent of the localized state

A criterion for initial conditions that lead to the localized fixed point can be found from the RG rule: $\tilde{J} = \dfrac{J_L J_R}{\Omega(1-\Delta_S^2)}$

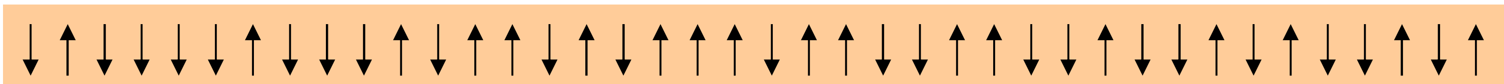
In order to flow to increasing randomness the typical J must decrease in the process. Therefore demand:

$$\frac{J_{typ}^2}{\Omega(1-\bar{\Delta}^2)} < J_{typ} \quad \Rightarrow \quad \alpha < \frac{1}{\Delta^2} - 1$$

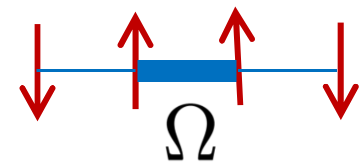




Generalize to random initial state



RG rule for a strong bond connecting parallel spins:



Keep track of a new spin: $= |\uparrow\uparrow\rangle$ $= |\downarrow\downarrow\rangle$

$$H_{eff} \approx -\frac{J_L J_R}{2\Omega(1-\Delta_S^2)}(S_L^+ S_R^- + \mathrm{H.c.}) + J_L \Delta_L S_L^z S_n^z + J_R S_R^z S_n^z$$
$$-\frac{J_L J_R \Delta_S}{2\Omega(1-\Delta_S^2)}\left(S_L^+ S_R^+ S_n^- + S_L^- S_R^- S_n^+\right)$$

Generates slow switching between the $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ states.
But the $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ states of the pair are not populated.

➡ The operators $S_1^z S_2^z$ of decimated pairs are asymptotic constants of motion if the system still flows to infinite randomness.

But the flow is complicated by the switching term (last term) and the generated interaction between the new spin and its neighbors.

RG flow for random initial state and random Zeeman fields neglecting resonances

$$\frac{\partial\rho(\zeta)}{\partial\Gamma} = \frac{\partial\rho(\zeta)}{\partial\zeta} + (\rho(0) + g(1)) \int_0^\infty d\zeta_L \int_0^\infty d\zeta_R \delta(\zeta - \zeta_L - \zeta_R)\rho(\zeta_L)\rho(\zeta_R) - \rho(\zeta)g(1)$$

$$\frac{\partial f(|\Delta|)}{\partial\Gamma} = p_1\rho(0) \int_0^1 d|\Delta|_L \int_0^1 d|\Delta|_R \delta(|\Delta| - |\Delta|_L|\Delta|_R) f(|\Delta|_L) f(|\Delta|_R)$$

$$+ (p_0\rho(0) + g(1,\Gamma))\,\delta(\Delta) - f(\Delta)\,(\rho(0) + g(1))$$

$$\frac{\partial g(\tilde{h})}{\partial\Gamma} = \tilde{h}\frac{\partial g(\tilde{h})}{\partial\tilde{h}} - g(\tilde{h})\,(g(1) - 1)\,. \qquad \tilde{h} = h/\Omega$$

Solved by the Ansatz:

Gradual condensation

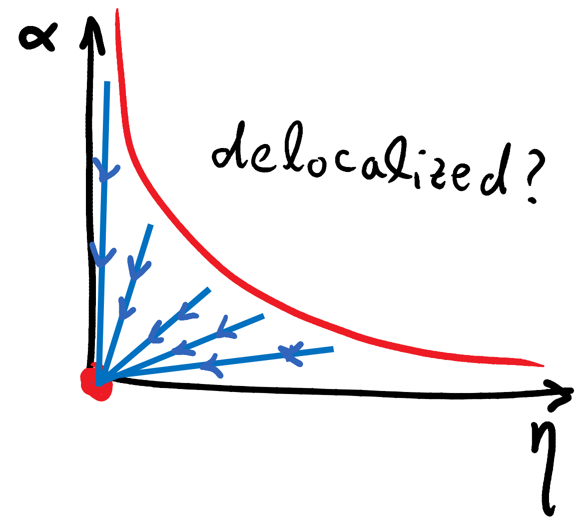
$$\rho(\zeta,\Gamma) = a(\Gamma)e^{-a(\Gamma)\zeta},$$

$$f(|\Delta|,\Gamma) = (1 - b_2(\Gamma))\, b_1(\Gamma)|\Delta|^{b_1(\Gamma)-1} + b_2(\Gamma)\delta(|\Delta|)$$

$$g(\tilde{h},\Gamma) = c\tilde{h}^{c-1}.$$

Flow to infinite rand. in J (peaked at small J) and large Local Zeeman fields

Summary



- Formulated RG for dynamics
  of random spin chains

- Many-body localized state found
  for *xxz* chain with initial Neel state.
  identified as infinite randomness fixed point

- Entanglement growth: $S(t) = \frac{1}{2}\Big(1 + \eta \ln\left(t/t_{\text{delay}}\right)\Big)^2 \theta(t - t_{\text{delay}})$

- Particle number fluctuations: $\langle \delta N^2(t) \rangle \sim \ln\ln(\Omega_0 t)$

- CDW (Neel) order parameter: $m_s \sim 1/(\alpha_0 \ln(\Omega_0 t) + 1)^2$

- Non thermal steady state can be understood as Generalized Gibbs
  ensemble with the asymptotic conserved quantities: $(S_1^z S_2^z)_{\text{pair}}$

Outlook / questions

- Nature of the steady state for generic initial conditions and generic disorder (allow local Zeeman fields)

- Critical point controling the many-body localization transition?